

Advancing the phenomenology of GeV-scale axionlike particles

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Searches for axionlike particles (ALPs) with masses in the GeV range are a central objective of present and future intensity-frontier experiments. Interpreting these searches demands a reliable description of ALP production and decay. The prescription currently adopted by the community depends on unphysical chiral-rotation parameters, wrongly describes the mass scaling of the ALP flux, and ignores mixing with heavy pseudoscalar resonances. We introduce the framework that treats GeV-scale ALP interactions self-consistently, includes their mixing with heavier excitations $\pi(1300)$, $\eta(1295)$, and $\eta(1440)$, and properly describes production channels. When applying our description to proton beam experiments, we find that existing bounds and projected sensitivities shift by up to an order of magnitude relative to earlier estimates. We further delineate the dominant theoretical uncertainties, which originate from the still-incomplete experimental knowledge of the light-meson spectrum.

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I. INTRODUCTION

Axionlike particles (ALPs) a are hypothetical pseudo-scalars inspired by the QCD axion [1–3]. Their interaction Lagrangian, defined at some scale $\Lambda > \Lambda_{\text{EW}}$, often includes the terms

$$\mathcal{L}_a \subset c_G \frac{\alpha_s}{4\pi} \frac{a}{f_a} G^{\mu\nu} \tilde{G}_{\mu\nu} + \frac{\partial_\mu a}{f_a} \sum_F c_F \bar{F} \gamma^\mu \gamma_5 F, \quad (1)$$

where c_G/f_a and c_F are the interaction constants, α_s is the QCD running coupling, $G_{\mu\nu}$ and $\tilde{G}^{\mu\nu}$ are the gluon field strength and its dual, respectively, and F are fermion fields.

The ALPs with mass in the GeV range naturally appear in various extensions of the Standard Model [4–7] and may have a relation to various cosmological phenomena, being, for example, a portal to light dark matter [8–11]. Because of this, they are subject of extensive phenomenological studies [12–24], while ALP searches are among the most important goals in the physics program of accelerator experiments. In particular, the models with $c_G = 1$, $c_F = 0$ and $c_G = 0$, $c_F = 1$ at the scale $\Lambda = 1$ TeV (further denoted by *gluonic* and *fermionic* ALPs, respectively) are considered as the

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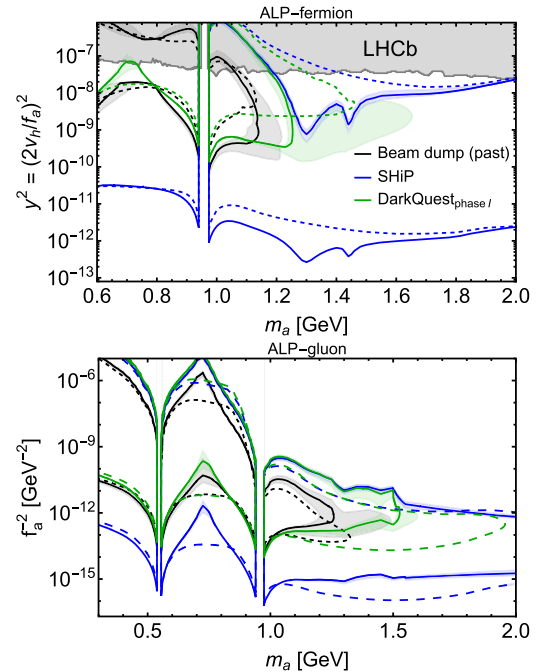


FIG. 1. The parameter space of fermionic (top panel) and gluonic ALPs (bottom panel). We show the constraints and sensitivities of CHARM [38], NuCal [39], BEBC [40], and NA62 [27] experiments (collectively marked as Beam dumps_{past}) and future DarkQuest [41] and SHiP [31,32] searches. The solid lines depict the parameter space computed using the ALP phenomenology described in this work, with shaded regions estimating the theoretical uncertainties in the ALP production, while the dashed lines correspond to the approach of Refs. [12–14]. The constraints and sensitivities have been computed using SensCalc [36] (see Refs. [42,43] for the details of the calculation).

benchmark models to compare the reach of various experiments [25,26].

The ALPs are searched for at currently running [23,27–30], approved [31–33], and proposed lifetime frontier experiments [17,25,26,34]. In particular, in the future SHiP experiment, ALPs may be copiously produced in collisions of a very intense proton beam with a thick target. Reconstructing their decay in a displaced decay volume allows for differentiating between the a particles with different coupling patterns [35]. Thus, it is essential to understand the phenomenology of the GeV-scale ALPs at accelerator experiments.

The descriptions of the ALP production at proton accelerators and their decay modes from Refs. [12–14] are incorporated in the event generators [13,36,37] and widely adopted by the community [26]. However, they suffer from inconsistencies and miss important interactions. In this study, we address these issues, providing for the first time the description that (i) is independent of the unphysical chiral rotation eliminating the gluon coupling in Eq. (1), (ii) properly describes the ALP production modes in deep inelastic collisions, and (iii) incorporates the ALP mixing with heavy pseudoscalar excitations $\pi^0(1300)$, $\eta(1295)$, and $\eta(1440)$. Using the obtained results, we refine the ALP parameter space (see Fig. 1), highlighting changes in the probed couplings compared to the previous description by up to an order of magnitude.

II. PHENOMENOLOGY OF ALPS: STATE OF THE ART

For the ALP masses $m_a \lesssim 1$ GeV, the description of their hadronic interactions in terms of quarks and gluons breaks down; instead, one needs to know how the a particles interact with various bound states, such as mesons and baryons. For the light ALPs $m_a \lesssim m_\pi$, these interactions may be obtained by matching the operator (1) and a modified chiral perturbation theory (ChPT) describing the interaction of ALPs with the pseudoscalar octet $P_8 = \pi, K, \eta$. To do this, one may first perform the chiral rotation of the quark fields:

$$q \rightarrow \exp \left[-i\gamma_5 c_G \kappa_q \frac{a}{f_a} \right] q, \quad (2)$$

where κ_q is a matrix satisfying the condition $\text{Tr}[\kappa_q] = 1$, chosen to be diagonal. This rotation converts the gluon coupling into the derivative coupling to the quark axial-vector current [44] (and also modifies the quark mass term), which can be translated to ChPT [18,45].

The resulting ChPT + ALP Lagrangian has quadratic nondiagonal terms between a and $P^0 = \pi^0, \eta$; i.e., there is a $P^0 - a$ mixing. The Lagrangian can be diagonalized by the linear transformation

$$P^0 \approx P_{\text{mass}}^0 + \theta_{P^0 a}(m_a) a + \dots, \quad (3)$$

where the modulus of the *mixing angle* $|\theta_{P^0 a}(m_a)| \ll 1$ everywhere except for the domain $m_a \simeq m_{P^0}$, where it gets resonantly enhanced.

The crucial next step is to perform backward matching—ensure that the hadronic interactions at the level of mesons smoothly repeat those described by the Lagrangian (1). In particular, the hadronic decay width of ALPs calculated exclusively using the interactions with mesons must saturate the perturbative QCD prediction obtained by calculating the width into two gluons.

The matching is performed at mass $m_a \gtrsim 1$ GeV. Relying solely on the pure ChPT is insufficient. First, ALPs have mixing with the pseudoscalar singlet η' , heavier excitations $P_h^0 = \pi^0(1300), \eta(1295), \eta(1440), \dots$, and axial-vector mesons $A^0 = a_1^0, f_1(1285)$, and $f_1(1415)$. It may heavily enhance the ALP widths around the masses of these excitations. Second, various other mesons (such as vector, scalar, and tensor resonances) may contribute to the interactions in this mass range [46]. Therefore, one has to include the interactions of a with these particles as well. An essential property of the resulting description is that it *must* predict the observables that are independent of the chiral rotation κ_q (modulo the unambiguous contribution $\text{Tr}[\kappa_q]$). This is achieved by summing the two κ_q -dependent contributions: from the diagonalization of the ALP-meson quadratic Lagrangian (see Fig. 2) and from the ALP-meson operators being cubic and quartic in fields.

To the best of our knowledge, no existing study addresses all these issues. On the one hand, Refs. [19,20,33,45,47,48] formulated the ALP interactions in a κ_q -invariant way. However, they considered only pure ChPT and the anomalous coupling of ALPs to photons. On the other hand, Refs. [12,14,49] added the phenomenological interactions of ALPs with vector, scalar, and tensor mesons and performed the matching of the ALP decay modes. However, in the case of nonzero c_G , the added interactions produced κ_q -dependent results: The contribution from higher-power operators was not included.

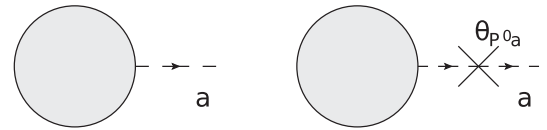


FIG. 2. Practical realization of the independence of the ALP production and decay rates of the unphysical chiral rotation (2) parametrized by the matrix κ_q . In the matrix element of an arbitrary process, the κ_q -dependent summands coming from n -field operators $n > 3$ (the left diagram) cancel the terms originating from the quadratic Lagrangian. The latter are induced by the mixing between a and pseudoscalar mesons $P^0 = \pi^0, \eta, \eta', \pi^0(1300), \dots$ and are parametrized by the mixing angles $\theta_{P^0 a}$ (the right diagram).

In addition, they miss the mixing of ALPs with P_h^0 and A^0 . These resonances are typically narrow and do not overlap, questioning the $\mathcal{O}(1)$ estimate of the uncertainty of the approach from Ref. [12], which is based on the $U(3)$ symmetry restoration of the ALP representation in the mass range $1 \text{ GeV} \lesssim m_a \lesssim 2 \text{ GeV}$.

Let us now consider the production of ALPs at proton accelerator experiments. The state-of-the-art approach from Ref. [13] describes the ALP production in deep inelastic scatterings and decays of light mesons by the product of the fluxes of π^0 , η , and η' mesons times the corresponding squared mixing angles $|\theta_{P^0 a}|^2$. It has been consequently incorporated in the event generators with long-lived particles (LLPs) [26,36,37]. Such a description is not only κ_q dependent—it also wrongly describes how the flux of ALPs depends on the ALP mass. In addition, it does not account for the theoretical uncertainty coming from different production modes.¹

III. OUR APPROACH

The method we develop² combines the self-consistent, κ_q -independent approach with the data-driven description of interactions of various mesons. Its technical details are summarized in Appendix B, while below we provide a summary.

We include the light pseudoscalar and scalar mesons, vector, and tensor mesons, similarly to how it is done in Refs. [12,49], and then extend the meson sector by adding the axial-vector A and heavy pseudoscalar mesons P_h (details are summarized in Sec. III A below).

Here and below, we consider the three-flavor setup, incorporating the η' meson in the pseudoscalar nonet matrix as in Refs. [12,47]. We calculate all the quantities in the

order $\mathcal{O}(\delta)$, where $\delta \equiv (m_d - m_u)/(m_d + m_u)$ is the isospin parameter, and $\mathcal{O}(\epsilon)$, where $\epsilon \equiv f_\pi/f_a \ll 1$ is the ALP dimensionless coupling, with $f_\pi \approx 93 \text{ MeV}$ being the pion decay constant. We incorporate the renormalization group flow from Ref. [19], describing the evolution of the fermionic coupling $c_F(Q, \Lambda)$ from the scale Λ , defining the Lagrangian (1), down to the scale of interest $Q \simeq 2 \text{ GeV}$.

To find the mixing (3) between the ALPs and mesons, we first diagonalize the quadratic part of the ALP-meson interactions; derivations and expressions for the mixing angles $\theta_{P^0 a}$, $\theta_{A^0 a}$, and $\theta_{P_h^0 a}$ may be found in Appendixes B 2 and B 5 b.

Let us now formulate the ALP interactions with various mesons in κ_q -independent fashion. The effective description of Standard Model interactions of such mesons is discussed in Refs. [51,52] (vector mesons and anomalous photon vertex), [53] (light scalar sector), [52] (tensor mesons), and [54–56] (vector, axial-vector, and heavy pseudoscalar excitations). On top of that, we will also need the $s \rightarrow d$ operator induced by the octet operator from [21] to describe the decays of kaons into ALPs.

The interactions are given in terms of either the manifestly $U(3)$ covariant objects $\Sigma = \exp[2i\mathcal{P}/f_\pi]$ or just \mathcal{P} , where $\mathcal{P} = \sum_P t_P P$ is the matrix of the pseudoscalar meson nonet. To ensure the κ_q -invariant results, we replace the Σ matrix with its transformed version

$$\Sigma \rightarrow \exp \left[-ic_G \kappa_q \frac{a}{f_a} \right] \cdot \Sigma \cdot \exp \left[-ic_G \kappa_q \frac{a}{f_a} \right] \quad (4)$$

and then expand the resulting Lagrangian in the powers of \mathcal{P} and a .³ As for the interactions written in terms of \mathcal{P} , let us utilize the Baker-Campbell-Hausdorff formula:

$$\exp \left[-ic_G \kappa_q \frac{a}{f_a} \right] \cdot \Sigma \cdot \exp \left[-ic_G \kappa_q \frac{a}{f_a} \right] \rightarrow \exp \left[i \frac{2}{f_\pi} (\mathcal{P}(x) - \epsilon c_G \kappa_q a) + \mathcal{O}(\epsilon^2) \right]. \quad (5)$$

To maintain the κ_q invariance, we have to replace (note the sign change)

$$\mathcal{P} \rightarrow \mathcal{P} + \epsilon c_G \kappa_q a. \quad (6)$$

The resulting Lagrangian, formulated in terms of (5), (6), and the diagonalization (3), provides κ_q -invariant description of all ALP interactions in the $\mathcal{O}(\epsilon)$ limit, thanks to the cancellation depicted in Fig. 2.

¹Reference [33] has partially disentangled different production mechanisms, leaving, however, unexplored theoretical uncertainties and important production modes such as the production in fragmentation.

²The approach is implemented in the *Mathematica* notebook from Ref. [14] and available online [50].

Importantly, using the “rotated” matrices (4) and (6) not only ensures the κ_q independence of the observables, but also induces the additional terms with the unambiguous part $\text{Tr}[\kappa_q] = 1$; therefore, it may lead to the cancellation or enhancement of the κ_q -independent piece.

The main limitation of our study is that we do not include the effects of higher-order ChPT terms on the ALP phenomenology. These effects may induce $< \mathcal{O}(1)$ corrections to $\theta_{P^0 a}$ [57,58], but the precise calculation of their impact is limited in light of undetermined parameters of the higher-order expansions. This error, however, is much

³A similar procedure has been adopted for the $s \rightarrow d$ and anomalous interactions with photons in [20,33] and for the Wess-Zumino-Witten terms in [24,33].

TABLE I. Properties of heavy pseudoscalar excitations from PDG [59]. Uncertainties are small for all the parameters except for the decay width of $\pi^0(1300)$. The asterisk in $\eta(1405/1475)^*$ means that these two states may be equally well described as the single state $\eta(1440)$ [56].

Resonance	$\eta(1295)$	$\pi^0(1300)$	$\eta(1405/1475)^*$	$\pi^0(1800)$
Mass [GeV]	1.294	1.3	1.41/1.48	1.81
Width [MeV]	55	200–600	50/96	215

smaller than the effect of the ALP interaction with heavy pseudoscalar excitations and subsequent potential uncertainty. We leave the detailed incorporation of the higher-order corrections for future work, stressing that the higher-order operators may be included within our approach similarly to other terms.

A. Heavy pseudoscalar excitations

The properties of heavy pseudoscalar excitations according to PDG [59] are summarized in Table I.

As we can see, the decay widths of most of them are comparable to or smaller than the width of the ρ meson. As a result, the resonances are not overlapping, and their effects must be incorporated into the ALP phenomenology explicitly. In contrast, the past studies, following Ref. [12], used the monotonic form factor entering any process with the ALP, claiming an $\mathcal{O}(1)$ uncertainty in the resulting description.

To describe the interactions of heavy mesons, we utilize the extended linear sigma model (ELSM) framework from Ref. [55], which includes the following particles that have mixing with the ALPs: neutral pseudoscalar

$$P_h^0 = \pi^0(1300), \eta(1295), \eta(1440) \quad (7)$$

and axial-vector mesons

$$A^0 = a_1^0, f_1(1285), f_1(1415). \quad (8)$$

More details may be found in Appendix B 5. The mixing with A^0 's, however, is not resonant, which follows from the structure of the mixing operator, $\partial_\mu a \cdot A_\mu^0$ [see Eq. (B40) in Appendix B], and their effect may be neglected.

Reference [55] considered a generic expansion in terms of light pseudoscalar and heavy mesons matrices preserving the $SU_L(3) \otimes SU_R(3)$ symmetry, then dropped some of the operators for the sake of simplicity, and then fixed the expansion coefficients by requiring to recover the observable parameters such as masses and decay widths. An important point is that the ALP- P_h^0 mixing is very sensitive to the presence of the dropped operators; we will come back to this later.

IV. ALP PRODUCTION

We consider the following ALP production modes: the proton bremsstrahlung (approximating the initial state radiation production), quark fragmentation, two- and three-body decays of light mesons ($\eta, \eta', K_S, K^+, \rho^0, \omega$), the Drell-Yan process, and decays of B mesons. The diagrams of these production processes are summarized in Fig. 3.

Decays of ρ^0 and ω within the three-flavor case are considered for the first time; we discuss their description in more detail in Appendix C 1.

For the description of the bremsstrahlung, we follow the quasireal approximation from Refs. [60–62]. We adopt the ALP-nucleon form factor, needed for accounting for the non-point-like structure of the proton, from [33]. We also incorporate the intrinsic theoretical uncertainty of the quasireal approximation—the hard scale defining the allowed ranges of the proton virtuality; see Refs. [42,62,63]. For the Drell-Yan production, we follow Ref. [14]; in particular, we accommodate the theoretical uncertainties by varying factorization and renormalization scales of the hard process $GG \rightarrow a$. They are sizable because the production vertex scales with α_s .

Qualitatively, the production in the fragmentation chain occurs when, because of the mixing, each P^0 appearing at the end of the fragmentation is “replaced” by the ALP with a tiny probability [42]. The probability is given in terms of “generalized” mixing angles, including the contributions from both the ALP-meson mixing and the multifield operators to account for the cancellation of the unphysical dependence on the chiral rotation (2); see Appendix C 2. This process is attractive because of a relatively small uncertainty compared to the proton bremsstrahlung for

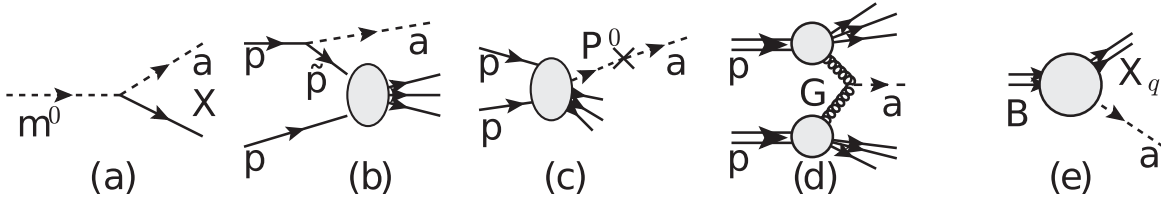


FIG. 3. Diagrams of the production of ALPs at proton accelerator experiments: decays of light mesons (a), initial state radiation known as proton bremsstrahlung (b), final radiation processes in quark fragmentation (c), Drell-Yan process (d), and decays of B mesons into an ALP and a hadronic state including an s or d quark, $X_{s/d}$ (e).

masses $m_a \lesssim 1$ GeV. We incorporate the production in the quark fragmentation via the mixing with light P^0 's in PYTHIA8 [64]; see Ref. [43].

The mixing with heavier resonances contributes to bremsstrahlung and fragmentation. However, including the contribution is a nontrivial task. In the bremsstrahlung case, it requires knowing the axial-vector proton elastic form factor in the timelike region, to which the heavy mesons contribute. It may be parametrized in terms of the Breit-Wigner contributions of various resonances (as is done, e.g., for dark photons in Ref. [63]). The coefficients in front of these contributions cannot be all fixed without accounting for the experimental data, which is very limited in the axial-vector case. As for the production in fragmentation, it requires knowing the fragmentation function into heavy hadrons. The latter must be tuned on the experimental data on the fluxes, which is lacking for heavy resonances. Therefore, our results on these modes are conservative; we leave studying these interesting questions to future work.

The summed contribution of decays of light mesons, the proton bremsstrahlung, and quark fragmentation replaces the “flux-times-mixing-angle” approach of Ref. [13], widely used in the literature [26]; see Appendix C.

The probabilities of all the production processes for the beam and target setup corresponding to the SHiP experiment for the gluonic ALPs are shown in Fig. 4 (top panel). The uncertainty in the proton bremsstrahlung (a common feature for the other LLPs; see Ref. [42]) and the Drell-Yan process may reach 1–2 orders in magnitude. The production in the fragmentation chain is one of the dominant modes for gluonic ALPs, whereas decays of B mesons become important for the fermionic ALPs.

V. ALP DECAY MODES

Similar to the previous studies [12,14,19,49], we consider exclusive ALP decay modes $a \rightarrow 2\gamma, 3\pi, 2\pi\gamma, KK\pi, \eta^{(\prime)}\pi\pi, 4\pi/2\rho^0, 2\omega, 3\eta, 2\eta\pi$ and perturbative decays $a \rightarrow 2G, 2s$. The heavy pseudoscalar excitations have the same decay modes, so the ALP mixing with them resonantly enhances the corresponding decay widths.

The total ALP hadronic decay widths for the models of fermionic and gluonic ALPs are shown in Fig. 4 (bottom panel). We include there two calculations: using our results and utilizing the state-of-the-art approach similar to [12,14], obtained in the κ_q -dependent way (for the coupling solely to gluons) and dropping the mixing with heavy pseudoscalar mesons. To fix the κ_q parameters, we adopt the common choice $\kappa_q = m_q^{-1}/\text{Tr}[m_q^{-1}]$, where m_q is the quark mass matrix.

In the mass range $m_a \lesssim 2$ GeV, the exclusive widths differ by up to a few orders of magnitude. For the gluonic ALPs, our prediction is much smaller than the state-of-the-art results in the domain $m_a \gtrsim 1$ GeV. It is caused by the partial cancellation between the contributions coming from the diagonalization (3) and the rotation (4), which reflects

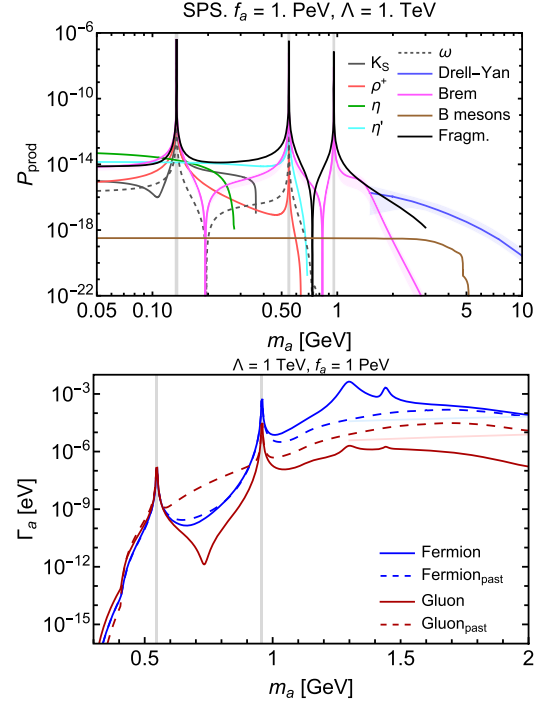


FIG. 4. Phenomenology of ALPs. Top panel: probabilities of various processes producing the gluonic ALPs (the yield of ALPs per proton on target) assuming the setup of a 400 GeV proton beam hitting the molybdenum target, corresponding to the SHiP experiment [31]. The bands denote theoretical uncertainties. Bottom panel: the total hadronic decay width of the gluonic (the blue lines) and fermionic ALPs (the red lines). The dark solid lines are obtained using the approach of our work, whereas the dashed lines closely resemble the results of Refs. [12,14]. The light color lines show the perturbative QCD hadronic decay widths. The vertical bands denote the vicinities of π^0 , η , and η' masses where the description of the ALP phenomenology based on the mixing with mesons breaks down.

the genuine property of the gluonic interaction in Eq. (1). Because of the same cancellation, the mixing with heavy pseudoscalar mesons does not contribute significantly. These factors prevent making a matching of the exclusive width with the perturbative width into gluons. The situation is different for the ALPs universally coupled to fermions, for which the heavy pseudoscalars increase the width by up to 2 orders of magnitude.

The predictive power of our approach (and, in principle, any ALP phenomenology study) is tied to the maturity of the effective description of the Standard Model’s heavy pseudoscalar sector encapsulated in Refs. [54,55]. First, experimental information on many resonances is scarce, resulting in sizable uncertainties on their partial widths [59]. Second, the internal quark composition of some of the heavy pseudoscalars remains unsettled [56]. Finally, for simplicity, Ref. [55] (studying P_h resonances) omitted several operators whose inclusion could materially alter ALP observables; see Appendix B 7. However, including them would require a full reanalysis and refit of the

operator expansion coefficients, a task that lies beyond the scope of the present work. Nevertheless, our entire framework is implemented in a public *Mathematica* notebook, so additional operators can be incorporated straightforwardly. Continued progress in light-meson spectroscopy—through both new data and theoretical tools such as the ELSM and the unitarization approach [59]—is therefore essential for sharpening ALP phenomenology predictions.

VI. CONCLUSIONS

We have systematically explored the phenomenology of the ALPs in the GeV mass range at accelerator experiments. Our study has two main outcomes. First, the approach we developed (Sec. III) is simultaneously free from unphysical chiral rotation, Eq. (2), and includes the ALP mixing with heavy meson resonances such as heavy pseudoscalar mesons—the points that were missing in the previous studies; see Sec. II. The heavy mesons have been incorporated using the framework of the extended linear sigma model; see Sec. III A. Incorporating these two features changes the production probabilities of ALPs by up to an order of magnitude and their decay widths by up to 2 orders of magnitude (see Fig. 5), especially in the mass range $1 \text{ GeV} \lesssim m_a \lesssim 2 \text{ GeV}$.

However, the properties of the heavy resonances are not well understood; see discussion in Secs. IV and V. This ignorance translates into the unavoidable limitations of understanding the ALP interactions. In light of this, our framework may serve as the new state-of-the-art study of the ALP phenomenology, with the possibility of easily incorporating new interactions once the maturity of the meson spectroscopy improves.

The revised ALP phenomenology has a significant impact on the parameter space of ALPs, affecting the scientific program of currently running and future experiments (see Fig. 1).

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are openly available [65].

APPENDIX A: APPENDIXES

In this appendix, we describe the ALP interactions with various mesons and nucleons following the self-consistent approach presented in the paper. It is organized as follows.

Appendix B is devoted to the addition of ALPs to the ChPT, describing the light pseudoscalar meson nonet and incorporating interactions with various other mesons. There, Appendix B 1 describes ChPT. In Appendix B 2, we add ALPs, define the procedure of the diagonalization of the quadratic part of the ALP-meson Lagrangian, calculate the mixing angles, and outline our strategy to incorporate other mesons in the interaction Lagrangian. In Appendix B 3, we discuss the addition of light scalar, vector, and tensor mesons. Appendix B 4 describes the ALP-nucleon interaction. Appendix B 5 is devoted to the description of the ALP interaction with axial-vector and heavy pseudoscalar mesons. Appendix B 6 is devoted to cross-checks of our approach. Finally, Appendix B 7 highlights the limitations of our approach, relating its maturity to the incomplete knowledge of meson spectroscopy in the mass range $M < 2 \text{ GeV}$.

Appendix C uses the results of the previous section to describe different ALP production mechanisms, including decays of light mesons and the production in the quark fragmentation. There, we also outline the problem of the widely adopted description [13], approximating the flux of ALPs by the flux of mesons P^0 times the squared mixing angle $|\theta_{P^0 a}|^2$.

Most of the results of these sections are obtained (unless stated otherwise) using our *Mathematica* notebook where we carefully implement the ALP phenomenology.⁴

APPENDIX B: MESON SPECTROSCOPY AND ALP INTERACTIONS

1. Pure ChPT

The pure ChPT Lagrangian we will use in our studies is

$$\begin{aligned} \mathcal{L}_{\text{ChPT, min}} = & \frac{f_\pi^2}{2} B_0 \text{Tr}[\Sigma \hat{m}_q^\dagger + \hat{m}_q \Sigma^\dagger] \\ & + \mathcal{L}_{\text{anomaly}} + \frac{f_\pi^2}{4} \text{Tr}[D_\mu \Sigma D^\mu \Sigma^\dagger]. \end{aligned} \quad (\text{B1})$$

Here,

- (i) $B_0 = m_\pi^2 / (m_u + m_d)$ and $f_\pi = 93 \text{ MeV}$ is the pion decay constant.
- (ii) Σ is the matrix of the pseudoscalar mesons:

$$\begin{aligned} \Sigma = \exp \left[\frac{2i\mathcal{P}}{f_\pi} \right], \\ \mathcal{P} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{3}} + \frac{\eta'}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{3}} + \frac{\eta'}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{\eta}{\sqrt{3}} + 2\frac{\eta'}{\sqrt{6}} \end{pmatrix}, \end{aligned} \quad (\text{B2})$$

⁴Available online [50].

while $D_\mu \Sigma = \partial_\mu \Sigma + ieA_\mu \Sigma$ is the covariant derivative. Here, we have fixed the $\eta - \eta'$ mixing angle as $\theta_{\eta\eta'} = \arcsin(-1/3)$, which provides a reasonable agreement with the experimental data while allowing us to provide the results in a simple analytic form [12].

- (iii) $\hat{m}_q = \text{diag}(m_u, m_d, m_s)$ is the matrix of quark masses.
- (iv) $\mathcal{L}_{\text{anomaly}}$ is the QCD anomaly-breaking term of the $U_A(1)$ symmetry:

$$\mathcal{L}_{\text{anomaly}} = -m_0^2 \eta_0^2 / 2, \quad (\text{B3})$$

with $\eta_0 = \cos(\theta_{\eta\eta'})\eta' - \sin(\theta_{\eta\eta'})\eta$ being its Goldstone. The coefficient m_0 is fixed in a way such that after summing the ChPT mass term [coming from the first summand in Eq. (B1)] and the m_0 term, there is no $\eta - \eta'$ mixing for the given $\theta_{\eta\eta'}$:

$$m_0^2 = \frac{3m_\pi^2(2m_s - m_u - m_d)}{2(m_u + m_d)}. \quad (\text{B4})$$

The inclusion of η' as in (B2) and the anomalous mass term (B3) are consistent with large- N_c arguments and $SU(3)_L \otimes SU(3)_R$ symmetry [66]. In particular, keeping N_c arbitrary and expressing η_0 in terms of Σ , the anomalous mass term is proportional to $-1/N_c \ln(\det[\Sigma])^2 \sim i/N_c \text{Tr}[\mathcal{P}]^2$, which is demanded by the $1/N_c$ counting rules.

Calculating the masses of η and η' , π^0 , and using the explicit form the isospin breaking parameter $\delta = (m_d - m_u)/(m_d + m_u)$, one may get the following consistency relations between quark and meson masses which we will use below:

$$\begin{aligned} m_u &= m_d \frac{1 - \delta}{1 + \delta}, & m_d &= (1 + \delta) \frac{m_{\pi^0}^2}{2m_\eta^2 - m_{\pi^0}^2} m_s, \\ m_{\eta'}^2 &= 4m_\eta^2 - 3m_{\pi^0}^2, & m_{K^+}^2 &= m_\eta^2 - \frac{\delta m_{\pi^0}^2}{2}. \end{aligned} \quad (\text{B5})$$

Here and below, we will keep only the terms up to quartic in meson fields. For these purposes, for example, in $\text{Tr}[D_\mu \Sigma D^\mu \Sigma]$, we have to expand $\Sigma = \sum_{n=0}^4 \frac{1}{n!} (\frac{2i\mathcal{P}}{f_\pi})^n$ and then drop all the terms with the dimensionality higher than 4.

2. Adding ALPs to ChPT and strategy of adding other excitations

Let us start with the interaction Lagrangian

$$\mathcal{L}_a = c_G \frac{a}{f_a} \frac{\alpha_s}{4\pi} G_{\mu\nu}^a \tilde{G}^{\mu\nu, a} + \frac{\partial_\mu a}{f_a} \sum_q c_q \bar{q} \gamma^\mu \gamma_5 q. \quad (\text{B6})$$

Similarly to the studies [18,19], we assume that this Lagrangian is defined at some scale $\Lambda \gg \Lambda_{\text{electroweak}}$. The scales of interest are $\Lambda_Q \simeq m_a$, and the Lagrangian (B6) nontrivially evolves down to this scale: The flow induces the couplings to quarks $c_q(\Lambda_Q) \neq c_q(\Lambda)$. In particular, even if $c_q(\Lambda) = 0$, we end up with nonzero $c_q(\Lambda_Q)$; for the choice $\Lambda = 1$ TeV commonly used in the literature, $c_q \simeq 10^{-2}$ [14,19].

The analog of the Lagrangian (B6) at the scale Λ_Q can be translated to the effective Lagrangian in terms of mesonic degrees of freedom after performing the rotation

$$q \rightarrow \exp \left[-\frac{ic_G a}{f_a} \hat{\kappa}_q \gamma_5 \right] q, \quad \text{Tr}[\kappa_q] = 1, \quad (\text{B7})$$

eliminating the direct coupling to gluons. We parametrize the κ_q matrix by

$$\hat{\kappa}_q = \text{diag}(\kappa_u, \kappa_d, 1 - \kappa_u - \kappa_d). \quad (\text{B8})$$

The corresponding Lagrangian in terms of the mesons is [12,18,33]

$$\begin{aligned} \mathcal{L}_{\text{ChPT, min}} &= \frac{1}{2} (\partial_\mu a)^2 - \frac{m_a^2}{2} a^2 + \frac{f_\pi^2}{2} B_0 \text{Tr}[\Sigma \hat{m}_q^\dagger + \hat{m}_q \Sigma^\dagger] \\ &+ \mathcal{L}_{\text{anomaly}} + \frac{f_\pi^2}{4} \text{Tr}[D_\mu \Sigma D^\mu \Sigma^\dagger] \\ &+ \frac{f_\pi^2}{2} \frac{\partial_\mu a}{f} \text{Tr}[(\hat{c}_q + c_G \kappa_q)(\Sigma D^\mu \Sigma^\dagger - \Sigma^\dagger D^\mu \Sigma)]. \end{aligned} \quad (\text{B9})$$

Here,

- (i) $\hat{c}_q = \text{diag}(c_u, c_d, c_s)$ is the direct ALP coupling to quarks. To shorten the expressions, below, we set them to zero, although keeping in all numerical calculations.
- (ii) \hat{m}_q becomes modified by the chiral transformation:

$$\hat{m}_q = \exp \left[-ic_G \frac{a}{f} \kappa_q \right] m_q \exp \left[-ic_G \frac{a}{f} \kappa_q \right]. \quad (\text{B10})$$

- (iii) $\mathcal{L}_{\text{anomaly}}$ is still given by Eq. (B3), given that the Lagrangian (B9) is written already after performing the chiral rotation (B7). Before the rotation, it would have been given in terms of $(2c_G a/f_a - \text{Tr}[2\mathcal{P}/f_\pi])^2$ [Eq. (11) from [66]], given that the ALP serves as a spurion $\theta(x)$ of the $U_A(1)$ transformation.

The part of the Lagrangian (B9) quadratic in the ALP field a and flavorless pseudoscalar mesons $P^0 = \eta, \eta', \pi^0$ has the form

$$\mathcal{L}_{P^0 a, \text{quad}} = \frac{1}{2} K_{ij} \partial_\mu X_i \partial^\mu X_j - \frac{1}{2} M_{ij} X_i X_j, \quad (\text{B11})$$

where $X = (a, \pi^0, \eta, \eta')$. The mass and kinetic matrices are, respectively,

$$\hat{K} = \begin{pmatrix} 1 & \epsilon c_G (\kappa_u - \kappa_d) & \sqrt{\frac{2}{3}} \epsilon c_G (2\kappa_d + 2\kappa_u - 1) & -\frac{\epsilon c_G (\kappa_d + \kappa_u - 2)}{\sqrt{3}} \\ \epsilon c_G (\kappa_u - \kappa_d) & 1 & 0 & 0 \\ \sqrt{\frac{2}{3}} \epsilon c_G (2\kappa_d + 2\kappa_u - 1) & 0 & 1 & 0 \\ -\frac{\epsilon c_G (\kappa_d + \kappa_u - 2)}{\sqrt{3}} & 0 & 0 & 1 \end{pmatrix} \quad (\text{B12})$$

and

$$\hat{M} = \begin{pmatrix} m_a^2 & M_{a\pi^0} & M_{a\eta} & M_{a\eta'} \\ M_{a\pi^0} & m_{\pi^0}^2 & -\sqrt{\frac{2}{3}} \delta m_{\pi^0}^2 & -\frac{\delta m_{\pi^0}^2}{\sqrt{3}} \\ M_{a\eta} & -\sqrt{\frac{2}{3}} \delta m_{\pi^0}^2 & m_\eta^2 & 0 \\ M_{a\eta'} & -\frac{\delta m_{\pi^0}^2}{\sqrt{3}} & 0 & m_{\eta'}^2 \end{pmatrix}, \quad (\text{B13})$$

where

$$M_{a\pi^0} = -\epsilon c_G m_{\pi^0}^2 ((\delta + 1)\kappa_d + (\delta - 1)\kappa_u), \quad (\text{B14})$$

$$M_{a\eta} = \sqrt{\frac{2}{3}} \epsilon c_G (m_{\pi^0}^2 (\delta \kappa_d - \delta \kappa_u + 1) + 2m_\eta^2 (\kappa_d + \kappa_u - 1)), \quad (\text{B15})$$

$$M_{a\eta'} = \frac{\epsilon c_G (m_{\pi^0}^2 ((\delta + 3)\kappa_d - (\delta - 3)\kappa_u - 2) - 4m_\eta^2 (\kappa_d + \kappa_u - 1))}{\sqrt{3}}. \quad (\text{B16})$$

For brevity, we have set here $c_q = 0$. However, including the quark couplings would be straightforward. In Eq. (B13), we used the relation for the u and d quark masses in terms of δ and m_s [see Eq. (B5)], *a priori* assuming that the shift in the meson masses due to ALPs is negligibly small (which is true as far as $f_\pi/f_a \ll 1$).

A generic transformation simultaneously diagonalizing these matrices in the $\mathcal{O}(\delta)$ order is [12]

$$a = a_{\text{phys}} - \sum_{P^0 = \pi^0, \eta, \eta'} h(P^0, m_{P^0}) P^0_{\text{phys}}, \quad (\text{B17})$$

$$P^0 = P^0_{\text{phys}} - \sum_{P^{0'} \neq P^0} \frac{M_{P^0 P^{0'}}}{m_{P^0}^2 - m_{P^{0'}}^2} + h(P^0, m_a) a_{\text{phys}}, \quad (\text{B18})$$

where

$$h(P^0, m_X) = \frac{1}{m_a^2 - m_{P^0}^2} \left[M_{aP^0} - m_X^2 K_{aP^0} + \sum_{P^{0'} \neq P^0} M_{P^0 P^{0'}} \frac{M_{aP^{0'}} - m_X^2 K_{aP^{0'}}}{m_X^2 - m_{P^{0'}}^2} \right]. \quad (\text{B19})$$

Below, we will drop the index ‘‘phys.’’

Introducing the parameter $\epsilon = f_\pi/f_a$, the mixing angles (the coefficients in the expansion of P^0 in front of a) become

$$\theta_{\pi^0 a} = \frac{\frac{1}{3} \epsilon c_G \delta m_{\pi^0}^2 \left(\frac{(2m_a^2 - m_{\eta'}^2 - m_{\pi^0}^2)}{(m_a^2 - m_{\eta'}^2)} - \frac{2(m_a^2 - 2m_\eta^2 + m_{\pi^0}^2)}{(m_a^2 - m_\eta^2)} \right)}{m_a^2 - m_{\pi^0}^2} + \epsilon c_G (\kappa_d - \kappa_u), \quad (\text{B20})$$

$$\theta_{\eta a} = \frac{\sqrt{\frac{2}{3}}\epsilon c_G(m_a^2 + m_{\pi^0}^2 - 2m_\eta^2)}{m_a^2 - m_\eta^2} - 2\sqrt{\frac{2}{3}}\epsilon c_G(\kappa_d + \kappa_u), \quad (\text{B21})$$

$$\theta_{\eta' a} = -\frac{\epsilon c_G(2m_a^2 - m_{\eta'}^2 - m_{\pi^0}^2)}{\sqrt{3}(m_a^2 - m_{\eta'}^2)} + \frac{\epsilon c_G(\kappa_d + \kappa_u)}{\sqrt{3}}. \quad (\text{B22})$$

If utilizing the relations (B5), the quadratic part of the Lagrangian becomes diagonalized up to terms $\mathcal{O}(\delta)$ inclusive. We note the presence of the pole terms from η, η' in the $\pi^0 a$ mixing and, vice versa, $\theta_{\eta a/\eta' a}$ include pole from π^0 . It is caused by the $\pi^0 - \eta/\eta'$ mixing induced by the mass matrix (B13).

The resonant enhancement is present independently of the order of the chiral expansion, as it originates from a phenomenological quadratic Lagrangian between ALPs and P . The case $m_a \rightarrow m_P$ corresponds to the maximal mixing between the ALP and the P states. Besides the fact that such a large mixing would require a fine-tuning of the mass parameters that is rather implausible, it would modify the properties of the P in a way that is incompatible with experimental findings (e.g., with the measured rate of the $\pi^0 \rightarrow \gamma\gamma$). In practice, we exclude the vicinity of the masses of light excitations P^0 from our analysis (see the vertical bands in Figs. 1 and 4).

Let us now outline our strategy to add other mesons to the ALP interaction. For this, we will utilize the transformations of the Σ and \mathcal{P} matrices under the quark chiral rotation (B7) driven by κ_q :

$$\Sigma \rightarrow \exp\left[-ic_G\kappa_q \frac{a}{f_a}\right]\Sigma \exp\left[-ic_G\kappa_q \frac{a}{f_a}\right], \quad (\text{B23})$$

$$\mathcal{P} \rightarrow \mathcal{P} + \epsilon c_G\kappa_q a. \quad (\text{B24})$$

κ_q -invariant interactions of ALPs with the other mesons can be obtained with the help of the Lagrangian written in terms

$$\mathcal{S} = \begin{pmatrix} a_0^0 - \sigma \sin(\theta_s) + f_0(980) \cos(\theta_s) & \sqrt{2}a_0^+ & \sqrt{2}\kappa^+ \\ \sqrt{2}a_0^- & -a_0^0 - \sigma \sin(\theta_s) + f_0(980) \cos(\theta_s) & \sqrt{2}\kappa \\ \sqrt{2}\kappa^- & \sqrt{2}\bar{\kappa} & \sqrt{2}\sigma \cos(\theta_s) + \sqrt{2}f_0(980) \sin(\theta_s) \end{pmatrix}, \quad (\text{B25})$$

with $\theta_s = -21^\circ$ being the scalar mixing angle and κ is also known as $K_0^*(700)$. The interaction coefficients have been fixed to describe the $\pi\pi$ and πK scattering data and also to reproduce the decay $\eta' \rightarrow \eta\pi\pi$.

of the objects (B23) and (B24), subsequently inserted linear transformation (B18), and keeping only $\mathcal{O}(\epsilon, \delta, \epsilon \cdot \delta)$ terms.

Having such a Lagrangian, we are ready to calculate the ALP decay widths. To do this, we first compute the matrix element of the given process and ensure that it is κ_q independent. Then, following Ref. [12], we adopt the phenomenological suppression factors $F(m)$ for the whole matrix element, accounting for the QCD sum rules. We consider the limit $\mathcal{O}(\delta)$ everywhere except for the process $a \rightarrow 3\pi$, for which the squared matrix element is already $\propto \delta^2$.

Below, we describe how we add various excitations: lightest scalar, vector, tensor, axial-vector, and heavy pseudoscalar.

Regarding the sector of scalar, vector, and tensor mesons (see Appendix B 3 for the detailed description), we mainly follow Refs. [14,49]. They slightly refine the treatment of these interactions in Ref. [12] but study only the ALPs without the coupling to gluons. The refinements include imposing unitarity constraints restricting the contribution of certain excitations to various kinematic phase space of ALP decays; including proper polarization sum of vector and tensor mesons when treating them as intermediate states; and the coupling of the tensor mesons to the stress-energy tensor in terms of pseudoscalar mesons rather than just the minimal coupling via the metrics. However, there is an important difference between our present analysis and Refs. [14,49] concerning the sector of scalar mesons in our present analysis; it is discussed in Appendix B 3 a.

The addition of axial-vector and heavy pseudoscalar mesons is the new result that we discuss in Appendix B 5. It is based on the so-called extended linear sigma model from Refs. [54–56].

3. Adding scalar, vector, and tensor resonances a. Scalar mesons

We add the interactions with the lightest scalar mesons following Refs. [53,67], which introduced $SU(3)$ invariant description in terms of the matrices Σ and the scalar nonet matrix

These studies defined η and η' in terms of the isoscalar and singlet particles η_8 and η_0 in a different way compared to the ALP studies. Namely, the mixing angle in [53] differs from the angle $\theta_{\eta\eta'} = \arcsin(-1/3)$, commonly used when

describing the mixing of ALPs with these mesons. Thus, incorporating the interactions from [53,67] into the ALP phenomenology “out of the box” would lead to inconsistency.

To fix the problem, we start with Eq. (B.4) of [67]. It is written in terms of phenomenological constants A , B , C , and D and the matrix of the pseudoscalar nonet Σ (denoted as ξ there). As usual, we replace it with Eq. (B23) and perform the expansion in terms of \mathcal{P} . The parameters A , B , C , and D are fitted to the observational data— π and K scattering for A and B and η' decays for C and D . Because of the different definition of η and η' we use in our expansion, the η, η' couplings depend differently on C and D than in the expansion (A.1) of Ref. [53], so we cannot just use their reported values—it would destroy the predictions on $\eta' \rightarrow \eta\pi\pi$ decay. Instead, we recompute the couplings of scalar mesons to η, η' in our expansion and set their values to recover the width of the process $\eta' \rightarrow \eta 2\pi$, consistent with the experimental value [59].

We also utilize these couplings when adding the S -wave amplitude to the ALP decays into $KK\pi$ using the $BABAR$ fit from [49].

b. Vector mesons

The hidden local symmetry approach of including the interaction of vector mesons adds the following Lagrangian [51,52]:

$$\begin{aligned} \mathcal{L}_{\text{vec+an}} = & -\frac{3g^2}{8\pi^2 f_\pi} \epsilon^{\mu\nu\alpha\beta} \text{Tr}[\mathcal{P}(x) \partial_\mu V_\nu(x) \partial_\alpha V_\beta(x)] \\ & + \frac{7}{60\pi^2 f_\pi^5} \epsilon^{\mu\nu\alpha\beta} \text{Tr}[\mathcal{P}(x) \partial_\mu \mathcal{P} \partial_\nu \mathcal{P} \partial_\alpha \mathcal{P} \partial_\beta \mathcal{P}] \quad (\text{B26}) \end{aligned}$$

$$+ 2f_\pi^2 \text{Tr} \left| gV_\mu - eA_\mu Q - \frac{i}{2f_\pi^2} [\mathcal{P}, \partial_\mu \mathcal{P}] \right|^2. \quad (\text{B27})$$

Here, $g \approx m_\rho / \sqrt{2} f_\pi$, $Q = \text{diag}(2/3, -1/3, -1/3)$ is the quark charge matrix, V_μ is the matrix of vector mesons,

$$V_\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\rho^0 + \omega}{\sqrt{2}} & \rho^+ & K^{*+} \\ \rho^- & \frac{-\rho^0 + \omega}{\sqrt{2}} & K^{*0} \\ K^{*-} & \bar{K}^{*0} & \phi \end{pmatrix} \quad (\text{B28})$$

with K^* being associated with $K^*(892)$, and A_μ is the EM field. As discussed, to define the ALP interactions, we start with the corresponding Lagrangian, replace the \mathcal{P} matrix with Eq. (6), and perform the diagonalization (3).

The Lagrangian induces the mixing between A_μ and ρ^0 , ω , and ϕ , which effectively generates, e.g., electromagnetic decays of P^0 . This way, it is responsible for all the vertices of interactions, in particular, for the κ_q cancellation. In particular, it includes the effect of the chiral rotation (B7)

coming from the noninvariance of the fermion path integration measure, i.e., reproduces the chiral anomaly at the language of the quark bound states rather than quarks themselves.

Let us show how the κ_q dependence vanishes by calculating the coupling of the ALPs to two photons. Expanding Eq. (B27), for the $a\gamma\gamma$ piece, we get

$$\mathcal{L}_{a\gamma\gamma} = \frac{\alpha_{\text{EM}} \epsilon}{\pi f_\pi} c_{\gamma\gamma}^{\text{eff}} F_{\mu\nu} \tilde{F}^{\mu\nu}, \quad (\text{B29})$$

where

$$c_{\gamma\gamma}^{\text{eff}} = \frac{1}{9} (6c_G(3\kappa_u + 1) - 4\sqrt{6}\theta_{\eta a} - 7\sqrt{3}\theta_{\eta' a} - 9\theta_{\pi^0 a}). \quad (\text{B30})$$

The κ_q dependence vanishes after inserting the expressions (B20)–(B22). The first term has the same structure as the term $\propto \text{Tr}[\kappa_q Q^2]$, which is explicitly added in the literature [20] because of the path integral measure noninvariance under the chiral rotation.

We perform two cross-checks. First, in the limit of the “ P^0 -like” ALP (i.e., setting $\theta_{\rho^0 a} = \delta_{\rho^0 a}$ and $c_G = 0$), the Lagrangian (B29) predicts the decay widths of π^0 , η , and η' mesons matching the observed data with $\simeq 10\%$ accuracy, typical for the model of hidden local symmetry [46,51].

Second, let us consider the two-flavor limit. This is achieved by setting $\theta_{\eta/\eta' a} \rightarrow 0$, dropping the π^0 - η/η' mixing terms in Eq. (B21), and assuming $\kappa_u = 1 - \kappa_d$ (this is the relation between κ_u and κ_d in the case of the two-flavor setup). Keeping now explicitly the ALP derivative coupling to quarks $\mathcal{L} \ni \partial_\mu a / f_a \sum_q c_q \bar{q} \gamma^\mu \gamma_5 q$, for the expression (B30) we obtain

$$c_{\gamma\gamma}^{\text{eff}} = c_G \left(-\delta \frac{m_{\pi^0}^2}{m_{\pi^0}^2 - m_a^2} - \frac{5}{3} \right) - \frac{m_a^2 (c_u - c_d)}{m_{\pi^0}^2 - m_a^2}, \quad (\text{B31})$$

which exactly matches Eq. (92) in Ref. [18] (after taking into account the relation $c_{qq} \equiv 2c_q$ there).

c. $s \rightarrow d$ transition operator

Next, let us proceed to the process $K \rightarrow a\pi$, generated by the $s \rightarrow d$ transition. The Standard Model analog of this process, $K \rightarrow \pi\pi$, receives contributions from two operators [68] classified by the transformation properties of the chiral operators— $SU(3)$ octet and 27-plet. The coupling constant in front of the latter is severely suppressed, $G_{27}/G_8 \approx 0.05$. Despite this, the Standard Model decay is driven by G_{27} , as the G_8 contribution is proportional to the tiny factor $(m_{\pi^+}^2 - m_{\pi^0}^2)/m_K^2$. However, this is no longer the case for ALPs, and the octet typically makes the main contribution (see, e.g., [21]).

We implement the octet operator; the implementation of the 27-plet may be, in principle, done analogously. The matrix element of the process has the form

$$\begin{aligned} \mathcal{M}_{K^+ \rightarrow \pi^+ a} = & \frac{1}{3} i f_\pi G_8 [6\epsilon c_G (-m_a^2 (2\kappa_d + \kappa_u) + m_{K^+}^2 (\kappa_d + \kappa_u) + m_{\pi^+}^2 \kappa_d + m_a^2 - m_{\pi^+}^2) \\ & + \sqrt{3} \theta_{a\eta'} (m_a^2 + 2m_{K^+}^2 - 3m_{\pi^+}^2) + 2\sqrt{6} \theta_{\eta a} (m_{K^+}^2 - m_a^2) + 3\theta_{\pi^0 a} (m_a^2 - m_{\pi^+}^2)]. \end{aligned} \quad (\text{B32})$$

Note that, unlike the work [20], we do not decouple the η' meson, which leads to the qualitative difference in the scaling of the matrix element with the ALP mass. Namely, after inserting the explicit form of the mixing angles (which identically cancels the κ_q dependence) and working in the limit $\delta = 0$, we get

$$\mathcal{M}_{K^+ \rightarrow \pi^+ a} \approx 8i\epsilon c_G f_\pi G_8 \frac{(m_{K^+}^2 - m_{\pi^+}^2)^2}{m_a^2 - m_{\eta'}^2}. \quad (\text{B33})$$

d. Tensor mesons

We add the following Lagrangian of the interaction of P_s with the tensor meson f_2 [52]:

$$\mathcal{L}_T = -g_T \frac{f_\pi^2}{4} \text{Tr} \left[\left(\partial^\mu \Sigma^\dagger \partial^\nu \Sigma - \frac{1}{2} g^{\mu\nu} \partial^\alpha \Sigma^\dagger \partial_\alpha \Sigma \right) \mathbf{f}_2 \right] f_{2\mu\nu}, \quad (\text{B34})$$

where $\mathbf{f}_2 = \text{diag}(1/2, 1/2, 0)$ is the $SU(3)$ generator of the tensor meson. The coupling $g_T = 13.1 \text{ GeV}^{-1}$ recovers the decay widths of the f_2 meson [49].

Here, we use the replacement (B23) to achieve the κ_q independence. As an illustration, consider the vertex $f_2 a \eta'$:

$$\begin{aligned} V_{f_2 a \eta'}^{\mu\nu} = & \frac{1}{6} \epsilon g_T (-g^{\mu\nu} (p_a \cdot p_\eta) + p_a^\mu p_\eta^\nu + p_a^\nu p_\eta^\mu) (\sqrt{2} \theta_{a\eta'} \\ & + \sqrt{6} c_G (\kappa_d + \kappa_u) + 2\theta_{\eta a}) \end{aligned} \quad (\text{B35})$$

Inserting the explicit expressions (B21) and (B22) for the c_G -dependent piece of the coefficient in front of the tensor structure, we get

$$c_G \frac{(m_\eta^2 - m_{\pi^0}^2)(m_a^2 - 2m_\eta^2 + m_{\pi^0}^2)}{(m_a^2 - m_\eta^2)(m_a^2 - 4m_\eta^2 + 3m_{\pi^0}^2)}. \quad (\text{B36})$$

4. Interaction with nucleons

We follow Ref. [33] and consider the low-momentum transfer ALP-nucleon interaction in the form

$$\mathcal{L}_{app} = g_{app} \partial_\mu a \bar{p} \gamma^\mu \gamma_5 p. \quad (\text{B37})$$

Here,

$$\begin{aligned} g_{app} = & \frac{(4D_s + 3D - F)\theta_{a\eta'}}{2\sqrt{3}} + \epsilon c_G (\kappa_d (F - D) + 2F\kappa_u) \\ & + \frac{\theta_{\eta a} (D_s + 2F)}{\sqrt{6}} + \frac{1}{2} (D + F) \theta_{\pi^0 a} \end{aligned} \quad (\text{B38})$$

is the ALP-nucleon coupling, with D_s , F , and D being phenomenological constants. Using Eqs. (B20)–(B22) and also the relation (B5), the explicit form of g_{app} becomes

$$\begin{aligned} g_{app} = & \frac{\epsilon c_G (m_{\pi^0}^2 - m_\eta^2)(m_a^2 (3D_s + 2D) - 2m_\eta^2 (2D_s + D + F) + m_{\pi^0}^2 (D_s + 2F))}{f_\pi (m_a^2 - m_\eta^2)(m_a^2 - m_{\eta'}^2)} \\ & + \delta \frac{\epsilon c_G m_{\pi^0}^2 (D + F)(m_{\pi^0}^2 - m_\eta^2)(m_a^2 - 2m_\eta^2 + m_{\pi^0}^2)}{f_\pi (m_{\pi^0}^2 - m_a^2)(m_a^2 - m_\eta^2)(m_a^2 - m_{\eta'}^2)}. \end{aligned} \quad (\text{B39})$$

5. Adding axial-vector and heavy pseudoscalar excitations

In the mass range $m_a \gtrsim 1 \text{ GeV}$, axial-vector mesons A_μ and heavy pseudoscalar excitations P_h are essential to describe the ALP interactions. The reason is that ALPs mix

with them. For excited pseudoscalar mesons, the mixing structure is similar to the light pseudoscalar case discussed in Appendix B 2: The chiral rotation modifies the meson kinetic and mass terms, whereas the derivative coupling translates to the analog of the last term in Eq. (B9). The axial-vector sector generically adds the mixing terms of the type $L \ni M_{AP} A_\mu \partial^\mu P$, where M_{AP} is the mixing coefficient.

It turns out that the $A - a$ mixing is not as important as the mixing with pseudoscalar mesons: Unlike the latter, it does not get resonantly enhanced at the vicinity of the A mass. To understand this feature, let us treat the $A_\mu \partial^\mu a$ term as the interaction term and use it as an internal vertex of some matrix element \mathcal{M} of the ALP production or decay. It has the form

$$\begin{aligned} \mathcal{M} &\propto M_{A^0 a} p_a^\mu D_{\mu\nu}^{A^0} \tilde{\mathcal{M}}^\nu \\ &= M_{A^0 a} \frac{(1 - \frac{m_a^2}{m_{A^0}^2})}{m_a^2 - m_{A^0}^2 - i\Gamma_{A^0} m_{A^0}} p^\nu \tilde{\mathcal{M}}_\nu \\ &\equiv \theta_{aA^0}^{\text{eff}} p^\nu \tilde{\mathcal{M}}_\nu, \end{aligned} \quad (\text{B40})$$

where $\theta_{aA^0}^{\text{eff}}$ is the mixing angle. We see that it vanishes at $m_a = m_{A^0}$ and, hence, lacks the enhancement. Therefore, compared to the mixing with the heavy pseudoscalar mesons, it is expected to be subdominant. In addition, incorporating them in a consistent way with the other mesons that are already included in our approach is a nontrivial task. For these two reasons, we will drop their interactions with ALPs. We return to this question in Appendix B 7.

Nevertheless, as we will see, the mixing between axial vectors and light pseudoscalar mesons are important to

describe the interactions of pseudoscalar excitations with vector and light pseudoscalar mesons. Therefore, we briefly describe their phenomenology in Appendix B 5 a.

Our treatment of their interactions follows Refs. [54–56]. It utilizes the description of different vector, axial-vector, heavy scalar, and pseudoscalar excitations using the $SU(3)$ symmetry, assuming that masses originate from a condensate of the heavy scalar nonet, and fixing the interaction couplings by matching the theoretical prediction of various decay widths with experimentally observed data. This model is called the extended linear sigma model. The crucial point is that we turn off the contributions from axial-vector and heavy pseudoscalar sectors for the scale $m \lesssim 1$ GeV, which is explained by the incompleteness of the ELSM in the meson spectrum mass range $M < 1$ GeV (see Appendix B 7).

The mixing of ALPs with P_h contributes to all “ P_h -like” decay modes, including 3π , $KK\pi$, $\eta 2\pi$, and many others.

a. Mixing of axial-vector and pseudoscalar mesons

In this section, we follow the approach of including the axial-vector mesons presented in Ref. [54].

We start with the Lagrangian describing the sector of axial-vector A , vector V , and pseudoscalar mesons P :

$$\begin{aligned} \mathcal{L}_0 &= \text{Tr}[(D_\mu \Phi)^\dagger (D_\mu \Phi)] - m_0^2 \left(\frac{G}{G_0}\right)^2 \text{Tr}(\Phi^\dagger \Phi) - \lambda_1 [\text{Tr}(\Phi^\dagger \Phi)]^2 - \lambda_2 \text{Tr}(\Phi^\dagger \Phi)^2 \\ &\quad - \frac{1}{4} \text{Tr}(L_{\mu\nu}^2 + R_{\mu\nu}^2) + \text{Tr} \left[\left(\left(\frac{G}{G_0}\right)^2 \frac{m_1^2}{2} + \Delta \right) (L_\mu^2 + R_\mu^2) \right] + \text{Tr}[H(\Phi + \Phi^\dagger)] + \text{Tr}(\Phi^\dagger \Phi E_0 + \Phi \Phi^\dagger E_0) \\ &\quad + c_1 (\det \Phi - \det \Phi^\dagger)^2 + i \frac{g_2}{2} (\text{Tr}\{L_{\mu\nu}[L^\mu, L^\nu]\} + \text{Tr}\{R_{\mu\nu}[R^\mu, R^\nu]\}) \\ &\quad + \frac{h_1}{2} \text{Tr}(\Phi^\dagger \Phi) \text{Tr}(L_\mu^2 + R_\mu^2) + h_2 \text{Tr}[|L_\mu \Phi|^2 + |\Phi R_\mu|^2] + 2h_3 \text{Tr}(L_\mu \Phi R^\mu \Phi^\dagger) \\ &\quad + g_3 [\text{Tr}(L_\mu L_\nu L^\mu L^\nu) + \text{Tr}(R_\mu R_\nu R^\mu R^\nu)] + g_4 [\text{Tr}(L_\mu L^\mu L_\nu L^\nu) + \text{Tr}(R_\mu R^\mu R_\nu R^\nu)] \\ &\quad + g_5 \text{Tr}(L_\mu L^\mu) \text{Tr}(R_\nu R^\nu) + g_6 [\text{Tr}(L_\mu L^\mu) \text{Tr}(L_\nu L^\nu) + \text{Tr}(R_\mu R^\mu) \text{Tr}(R_\nu R^\nu)]. \end{aligned} \quad (\text{B41})$$

Here,

$$\Phi(x) = \langle \Phi \rangle + S_h + i\mathcal{P}(x) \quad (\text{B42})$$

is the (heavy scalar-)pseudoscalar nonet, with S_h being the heavy scalar nonet, and

$$\langle \Phi \rangle = \frac{1}{\sqrt{2}} \text{diag} \left(\frac{\phi_N}{\sqrt{2}}, \frac{\phi_N}{\sqrt{2}}, \phi_S \right) \quad (\text{B43})$$

being the scalar condensate giving mass to the fields.

$$L_\mu = V_\mu + A_\mu \quad \text{and} \quad R_\mu = V_\mu - A_\mu \quad (\text{B44})$$

are, correspondingly, left and right axial and vector nonets, with V_μ defined in Eq. (B28), and A_μ the axial-vector nonet:

$$A_\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{f_{1N} + a_1^0}{\sqrt{2}} & a_1^+ & K_1^+ \\ a_1^- & \frac{f_{1N} - a_1^0}{\sqrt{2}} & K_1^0 \\ K_1^- & \bar{K}_1^0 & f_{1S} \end{pmatrix}. \quad (\text{B45})$$

f_{1N} is associated to $f_1(1285)$, f_{1S} to $f_1(1415)$, and K_1 to $K_1(1270)$. $L_{\mu\nu} = \partial_\mu L_\nu - \partial_\nu R_\mu$ (and similarly $R_{\mu\nu}$) is the field strength. Finally, $D_\mu \Phi = \partial_\mu \Phi - ig_1(L_\mu \Phi - \Phi R_\mu)$ is

the covariant derivative, with g_1 being an interaction coupling.

The relevant values of the couplings entering Eq. (B41) are summarized in Table III in Ref. [54]; the rest are set to zero. Finally, the glueball G is set to its vacuum expectation value G_0 , whereas $E_0 = \text{diag}(0, 0, \epsilon_S)$.

We are interested in the mixing terms between A and P particles. Let us, therefore, turn off vector and heavy scalar fields S_h .⁵ Thus, the matrix $\Phi(x)$ has the form $\Phi(x) = \langle \Phi \rangle + iP(x)$. The quadratic part of the resulting interaction is

$$\mathcal{L}_{VA}^{(2)} = \sum_{A,X=\{P,a\}} \frac{M_{AX}}{2} A_\mu \partial^\mu X. \quad (\text{B46})$$

With this Lagrangian, we reproduce the Lagrangian of the pure A - P mixing from Ref. [54].

It is possible to eliminate the mixing terms by performing the shift

$$A^\mu \rightarrow A^\mu + M_{A^0P} Z_P \partial^\mu P, \quad P \rightarrow Z_P P, \quad (\text{B47})$$

where Z_P is the renormalization constant, which ensures that the kinetic term for P has the canonical form after the shift. In the ELSM, Z_P enters the expressions of the P masses in terms of the parameters of the model (B41), as well as the P 's interactions with other mesons.

Our approach to treating this mixing is to assume that the Lagrangian (B41) and the shift (B47) generate the ChPT

mass term (B1). In other words, in the sector of pseudo-scalar mesons only, the ELSM predictions exactly match the ChPT description. However, we will utilize it in the next section, where we describe the interactions of heavy pseudoscalar nonet.

For completeness, however, we provide the mixing coefficients between the neutral A^0 and ALPs which can be obtained if including them in the same fashion as in the other interactions:

$$M_{a^0_a} = g_1 \phi_N (c_G (\kappa_d - \kappa_u) - \theta_{\pi^0_a}), \quad (\text{B48})$$

$$M_{f_1(1285)_a} = -\frac{1}{3} g_1 \phi_N (\sqrt{3} \theta_{\eta'_a} + 3c_G (\kappa_d + \kappa_u) + \sqrt{6} \theta_{\eta_a}), \quad (\text{B49})$$

$$M_{f_1(1415)_a} = \frac{1}{3} g_1 \phi_S (-2\sqrt{3} \theta_{\eta'_a} + 6c_G (\kappa_d + \kappa_u - 1) + \sqrt{6} \theta_{\eta_a}). \quad (\text{B50})$$

As it should be, it is κ_q independent; it also does not vanish in the limit $c_{u,d,s} \rightarrow 0$.

b. Heavy pseudoscalar sector

To describe the interactions of ALPs with heavy pseudoscalar mesons, we follow Ref. [55], which incorporated them in the ELSM. The starting Lagrangian is

$$\begin{aligned} \mathcal{L}_{P_h} = & \text{Tr}[(D_\mu \Phi_h)^\dagger (D_\mu \Phi_h)] + \alpha \text{Tr}[(D_\mu \Phi_h)^\dagger (D_\mu \Phi) + (D_\mu \Phi)^\dagger (D_\mu \Phi_h)] - (m_0^*)^2 \left(\frac{G}{G_0}\right)^2 \text{Tr}(\Phi_h^\dagger \Phi_h) \\ & - \lambda_0 \left(\frac{G}{G_0}\right)^2 \text{Tr}(\Phi_h^\dagger \Phi + \Phi^\dagger \Phi_h) - \lambda_1^* \text{Tr}(\Phi_h^\dagger \Phi_h) \text{Tr}(\Phi^\dagger \Phi) - \lambda_2^* \text{Tr}(\Phi_h^\dagger \Phi_h \Phi^\dagger \Phi + \Phi_h \Phi_h^\dagger \Phi \Phi^\dagger) \\ & - \kappa_1 \text{Tr}(\Phi_h^\dagger \Phi + \Phi^\dagger \Phi_h) \text{Tr}(\Phi^\dagger \Phi) - \kappa_2 [\text{Tr}(\Phi_h^\dagger \Phi + \Phi^\dagger \Phi_h)]^2 - \kappa_3 \text{Tr}(\Phi_h^\dagger \Phi + \Phi^\dagger \Phi_h) \text{Tr}(\Phi_h^\dagger \Phi_h) - \kappa_4 [\text{Tr}(\Phi_h^\dagger \Phi_h)]^2 \\ & - \xi_1 \text{Tr}(\Phi_h^\dagger \Phi \Phi^\dagger \Phi + \Phi_h \Phi^\dagger \Phi \Phi^\dagger) - \xi_2 \text{Tr}(\Phi_h^\dagger \Phi \Phi_h^\dagger \Phi + \Phi^\dagger \Phi_h \Phi^\dagger \Phi_h) - \xi_3 \text{Tr}(\Phi^\dagger \Phi_h \Phi_h^\dagger \Phi_h + \Phi \Phi_h^\dagger \Phi_h \Phi_h^\dagger) - \xi_4 \text{Tr}(\Phi_h^\dagger \Phi_h)^2 \\ & + \text{Tr}(\Phi_h^\dagger \Phi_h E_1 + \Phi_h \Phi_h^\dagger E_1) + c_1^* [(\det \Phi - \det \Phi_h^\dagger)^2 + (\det \Phi^\dagger - \det \Phi_h)^2] + c_{1E}^* (\det \Phi_h - \det \Phi_h^\dagger)^2 \\ & + \frac{h_1^*}{2} \text{Tr}(\Phi_h^\dagger \Phi + \Phi^\dagger \Phi_h) \text{Tr}(L_\mu^2 + R_\mu^2) + \frac{h_{1E}^*}{2} \text{Tr}(\Phi_h^\dagger \Phi_h) \text{Tr}(L_\mu^2 + R_\mu^2) \\ & + h_2^* \text{Tr}(\Phi_h^\dagger L_\mu L^\mu \Phi + \Phi^\dagger L_\mu L^\mu \Phi_h + R_\mu \Phi_h^\dagger \Phi R^\mu + R_\mu \Phi^\dagger \Phi_h R^\mu) + h_{2E}^* \text{Tr}[|L_\mu \Phi_h|^2 + |\Phi_h R_\mu|^2] \\ & + 2h_3^* \text{Tr}(L_\mu \Phi_h R^\mu \Phi^\dagger + L_\mu \Phi R^\mu \Phi_h^\dagger) + 2h_{3E}^* \text{Tr}(L_\mu \Phi_h R^\mu \Phi_h^\dagger). \end{aligned} \quad (\text{B51})$$

Here,

$$\Phi_h = S_{hh} + \frac{i}{\sqrt{2}} \begin{pmatrix} \frac{\eta_N + \pi_E^0}{\sqrt{2}} & \pi_E^+ & K_E^+ \\ \pi_E^- & \frac{\eta_N - \pi_E^0}{\sqrt{2}} & K_E^0 \\ K_E^- & \bar{K}_E^0 & \eta_N \end{pmatrix} \quad (\text{B52})$$

⁵We have *a posteriori* checked that their large mass, absence of mixing with ALPs, and presence of the lighter scalar nonet makes their contribution to the processes with ALPs subdominant.

is the excited scalar-pseudoscalar nonet, with S_{hh} denoting the second heavy scalar fields. Here, π_E is associated to $\pi(1300)$, K_E to $K(1460)$, η_N to $\eta(1295)$, and η_S to $\eta(1440)$. The association assumes that the mentioned states are two-quark bound states, although there is ambiguity. As in the axial-vector sector, the covariant derivative has the form $D_\mu \Phi_h = \partial_\mu \Phi_h - g_{1E}(L_\mu \Phi_h - \Phi_h R_\mu)$, and $G = G_0$.

Following Ref. [55], we drop the terms proportional to α , λ_0^* , λ_1^* , ξ_1 , h_1^* , and κ_{1-4} . It implies, in particular, the absence of the mixing between the heavy and light pseudoscalar mesons. We comment on its consequences in Appendix B 7. We also set $S_{hh} = 0$, as, similarly to the nonet S_h , they are not expected to contribute sizably to the interactions.

The interactions of Φ_h with vector and light pseudoscalar mesons are obtained with the help of the transformation (B47); it induces the vertices $\Phi_h \Phi V$ and $\Phi_h \Phi \Phi \Phi$, dominating the decays of Φ_h and, hence, ALPs, due to the mixing with heavy pseudoscalars.

After inserting the explicit form of Φ matrix, Eq. (B42), one arrives at the following mass and kinetic matrices between the ALP and the neutral mesons $P_h^0 = \pi^0(1300)$, $\eta(1295)$, $\eta(1440)$, similar to Eqs. (B12) and (B13):

$$\hat{M} = \begin{pmatrix} m_a^2 & M_{a\pi^0(1300)} & M_{a\eta(1295)} & M_{a\eta(1440)} \\ M_{a\pi^0(1300)} & m_{\pi^0(1300)}^2 & 0 & 0 \\ M_{a\eta(1295)} & 0 & m_{\eta(1295)}^2 & 0 \\ M_{a\eta(1440)} & 0 & 0 & m_{\eta(1440)}^2 \end{pmatrix} \quad (\text{B53})$$

with

$$M_{a\pi^0(1300)} = -c_G \frac{f_\pi}{2f_a} (\kappa_d - \kappa_u) (2m_0^{*2} + (\lambda_2^* - \xi_2) \phi_N^2), \quad (\text{B54})$$

$$M_{a\eta(1295)} = c_G \frac{f_\pi}{2f_a} (\kappa_d + \kappa_u) (2m_0^{*2} + (\lambda_2^* - \xi_2) \phi_N^2), \quad (\text{B55})$$

$$M_{a\eta(1440)} = -c_G \frac{f_\pi}{f_a} \sqrt{2} (\kappa_d + \kappa_u - 1) \times (m_0^{*2} + (\lambda_2^* - \xi_2) \phi_S^2 - 2\epsilon_S^E), \quad (\text{B56})$$

and

$$\hat{K} = \begin{pmatrix} 1 & \frac{f_\pi(c_G(\kappa_u - \kappa_d) - c_d + c_u)}{f_a} & \frac{f_\pi(c_G(\kappa_d + \kappa_u) + c_d + c_u)}{f_a} & \frac{\sqrt{2}f_\pi(c_s - c_G(\kappa_d + \kappa_u - 1))}{f_a} \\ \frac{f_\pi(c_G(\kappa_u - \kappa_d) - c_d + c_u)}{f_a} & 1 & 0 & 0 \\ \frac{f_\pi(c_G(\kappa_d + \kappa_u) + c_d + c_u)}{f_a} & 0 & 1 & 0 \\ \frac{\sqrt{2}f_\pi(c_s - c_G(\kappa_d + \kappa_u - 1))}{f_a} & 0 & 0 & 1 \end{pmatrix}. \quad (\text{B57})$$

Unlike the case of the light pseudoscalar sector, there is no mixing between various mesons from the beginning—the mixing is solely between P_h^0 and a . Utilizing the relation between the parameters of the Lagrangian and the masses of the heavy pseudoscalars [with the help of our *Mathematica* notebook, we recovered Eqs. (21)–(27) in Ref. [55]], for the resulting mixing angles, we get

$$\theta_{\pi^0(1300)a} = -\frac{f_\pi}{f_a} \left(c_G(\kappa_u - \kappa_d) + \frac{m_a^2(c_u - c_d)}{m_a^2 - m_{\pi^0(1300)}^2} \right), \quad (\text{B58})$$

$$\theta_{\eta(1295)a} = -\frac{f_\pi}{f_a} \left(c_G(\kappa_d + \kappa_u) + \frac{m_a^2(c_d + c_u)}{m_a^2 - m_{\eta(1295)}^2} \right), \quad (\text{B59})$$

$$\theta_{\eta(1440)a} = -\sqrt{2} \frac{f_\pi}{f_a} \left(c_G(1 - \kappa_d - \kappa_u) + \frac{m_a^2 c_s}{m_a^2 - m_{\eta(1440)}^2} \right). \quad (\text{B60})$$

Unlike the light pseudoscalars, heavy pseudoscalar mesons have a non-negligible decay width, which smears the resonant enhancement. To include this effect in practice, we ensure that the κ_q dependence vanishes in the zero-width limit, explicitly set $\kappa_q = 0$, and replace the denominators of the mixing angles: $m_a^2 - m_{P_h^0}^2 \rightarrow m_a^2 - m_{P_h^0}^2 - i\Gamma_{P_h^0} m_{P_h^0}$.

c. How the c_G -mediated contribution vanishes

Now, let us consider the contribution of the Lagrangian (B51) to some processes. Choosing, for instance, the

process $a(\rightarrow \eta^*(1440)) \rightarrow KK\pi$, we get for the corresponding Lagrangian

$$\begin{aligned} \mathcal{L}_a = & \frac{if_\pi}{2f_a} a(x)\partial^\mu K^-(x)K_\mu^{*+}(x)h_3^*\theta_{K_1^-(1270)K^-}[\sqrt{2}(\phi_S(\theta_{\eta(1295)a} + \theta_{\pi^0(1300)a}) - \phi_N\theta_{\eta(1440)a}) \\ & + 2c_G(\phi_N(\kappa_d + \kappa_u - 1) + \sqrt{2}\phi_S\kappa_u)]. \end{aligned} \quad (\text{B61})$$

After using Eqs. (B58)–(B60), for the expression in the brackets, we get

$$\frac{2c_s\phi_N}{m_a^2 - m_{\eta(1440)}^2} - \frac{\sqrt{2}\phi_S(c_d(m_{\eta(1295)}^2 - m_{\pi^0(1300)}^2) + c_u(2m_a^2 - m_{\eta(1295)}^2 - m_{\pi^0(1300)}^2))}{(m_a^2 - m_{\eta(1295)}^2)(m_a^2 - m_{\pi^0(1300)}^2)}; \quad (\text{B62})$$

i.e., whereas the $c_{u,d,s}$ terms obviously survive, the dependence on c_G vanishes identically. It happens because of the simple structure of the interaction operators and the trivial structure of heavy pseudoscalars utilized in Ref. [55]: Unlike the case of light pseudoscalars, $\eta(1295)$ and $\eta(1440)$ exactly match the isoscalar and singlet components of the nonet; also, there is no analog of the isospin breaking.

The same cancellation happens also for any other interaction induced by the Lagrangian (B51). Therefore, if dropping the terms proportional to α , λ_0^* , λ_1^* , ξ_1 , h_1^* , and κ_{1-4} (as is done in [55]), the ALPs coupled to gluons can experience these interactions only via tiny couplings $c_{u,d,s}$ induced because of the renormalization flow of the ALP gluonic operator from the scale Λ .

6. Cross-checks of our approach

We have validated our combined approach in various independent ways:

- (i) Comparing symbolic expressions for various quantities with the literature. For instance, assuming the two-flavor limit, we recovered the mass matrix (B13), the ALP-photon coupling, and the ALP decay width into three pions with the works [15,18] (see also Appendix B 3 b). Next, we have compared the expressions for the pure ChPT contribution to the matrix elements of ALP decays to Ref. [12] and mostly found the exact agreement. The only exception concerns the κ_q -dependent piece, which is absent for the $a \rightarrow \eta\pi\pi$ matrix element in [12]. For the added interactions with scalar, vector, and tensor mesons (where we followed Ref. [49]), we have found the agreement with [49] except for the sector of the scalar mesons, modified by our treatment (discussed in Appendix B 3 a).
- (ii) Ensuring κ_q independence of various vertices and matrix elements. When computing different quantities, we insert an explicit form of the mixing angles and calculate the κ_q -dependent pieces. They always vanish.

(iii) Comparing the widths of the Standard Model processes with their measured values. We consider the processes $\eta \rightarrow \pi^+\pi^-\gamma$, $\eta' \rightarrow \eta 2\pi$, $\eta' \rightarrow 4\pi$, and $\pi^0/\eta/\eta' \rightarrow 2\gamma$, which in the model we use goes via the mixing of vector mesons with photons [51] and intermediate vector and scalar excitations [52,53]. We find good agreement within 10%. This deviation is subdominant compared to the other uncertainty sources (highlighted in Appendix B 7) and also uncertainties coming from the experimental setup [which may include a cascade enhancement of the ALP production in the thick target—already an $\mathcal{O}(1)$ effect] and lack of knowledge about the meson's spectra. Regarding the newly added axial-vector and heavy pseudoscalar sectors, we have recovered the results of Refs. [54,55].

(iv) Reproducing the results of Ref. [14], which studied the ALPs universally coupled to fermions, after turning off the contributions from axial-vector and heavy pseudoscalar mesons. For this setup, the hadronic ALP widths differ within a factor of 1.5. The discrepancy is caused by the improvement of the sector of the interactions with the scalar mesons made in the given paper (see Appendix B 3 a).

Let us finalize the discussion by comparing our calculation of the ALP decay width with the results of Ref. [12]. In Fig. 5, we show the result of our approach, the κ -dependent approach without the inclusion of the mixing with heavy excitations, and the total width from Fig. S1 of [12]. For masses $m_a \lesssim 0.5$ GeV, the width from [12] closely matches our results; at masses $m_\eta \lesssim m_a \lesssim m_{\eta'}$, it deviates from both our curves (still being closer to the κ -independent approach); whereas at higher masses, it is much closer to the κ -dependent calculations. This behavior is explained by the fact that, at low masses, the ALP width is dominated by the decay into two photons $a \rightarrow \gamma\gamma$, for which Ref. [12] used the κ_q -invariant description, manually including the contribution from the chiral rotation (recall the discussion in Appendix B 3). For higher masses, hadronic decay modes (for which the κ -dependent

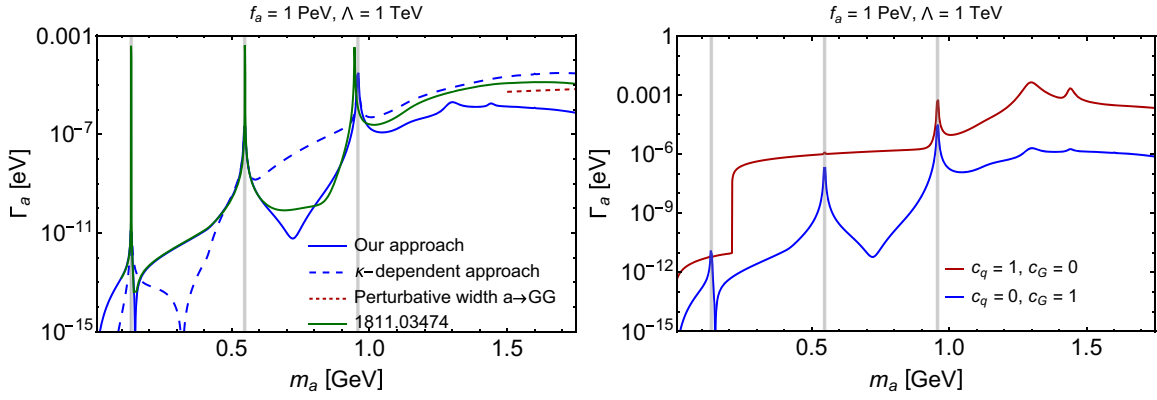


FIG. 5. Left panel: the comparison of the total ALP decay widths obtained in this work and in Ref. [12]. The meaning of the blue lines and the dashed red line is the same as in Fig. 4 in the main text. Right panel: the comparison of the total decay widths of the fermionic and gluonic ALPs. For the fermionic coupling case, the jump at $m_a = 2m_\mu$ is caused by the turning on of the dimuon decay, which dominates until the mass $m_a \simeq m_{\eta'}$.

approach has been used in [12]) become more and more important, and the discrepancy quickly accumulates, reaching 1–2 orders of magnitude.

The result of [12] is similar to our calculations obtained within the κ_q -dependent approach [which does not utilize the rotations (B24) and (B23)]. The discrepancy arises mainly because of the differences in the treatment of light resonances, which we described in Appendix B 2. In the same figure (right panel), we compare the total decay widths of ALPs that are universally coupled to fermions and those coupling solely to gluons. In the fermionic case, the effects of the heavy pseudoscalar resonances are much stronger [see Eq. (B60)], leading to a significant enhancement of the decay width.

7. Limitations of our approach

Our approach to ALP phenomenology, summarized in Appendix B, has limitations intimately related to the maturity of the frameworks describing effective interactions of mesons. There are two major problems.

The first problem arises because there is no unique comprehensive framework describing all the meson excitations with masses below 2 GeV [56]. For instance, ELSM is fitted to describe the main decays of some of the pseudoscalar, scalar, vector, and tensor mesons. On the other hand, it does not consistently incorporate the sectors

of light scalar nonet and pseudovector fields, as well as anomalous interactions with vector mesons. The latter may be essential for decays of light pseudoscalar mesons [52,53]. Partially because of this, it does not fit the experimental data on various subdominant decays of mesons which are essential for describing the ALP decay pattern, such as $\eta \rightarrow \pi^+ \pi^- \gamma$ or $\eta' \rightarrow 4\pi$. Because of this, we use the hybrid framework, utilizing the ELSM to describe the sector of heavy pseudoscalars.

Next, some heavy pseudoscalar excitations, such as $\pi(1800)$, are not implemented as well. Because of the resonantly enhanced mixing of such mesons with the ALPs, it would severely influence the ALP phenomenology.

One of the main reasons for this is the lack of robust experimental data on these resonances [59], leading to huge uncertainties in decay widths and masses of some resonances. It leads to a complexity of interpreting various meson states assigned in Appendix B 5 as two-quark bound states may be four-quark bound states or an admixture between two-quark and four-quark bound states [56]; the latter would directly change the interaction operators of mesons and, hence, the ALP phenomenology.

An independent problem concerns the ambiguity of the phenomenological Lagrangians. For example, consider again the interaction Lagrangian of heavy pseudoscalar mesons P_h as defined in Ref. [55]:

$$\begin{aligned}
 \mathcal{L}_{P_h} = & \text{Tr}[(D_\mu \Phi_h)^\dagger (D_\mu \Phi_h)] + \alpha \text{Tr}[(D_\mu \Phi_h)^\dagger (D_\mu \Phi) + (D_\mu \Phi)^\dagger (D_\mu \Phi_h)] - (m_0^*)^2 \left(\frac{G}{G_0}\right)^2 \text{Tr}(\Phi_h^\dagger \Phi_h) \\
 & - \lambda_0 \left(\frac{G}{G_0}\right)^2 \text{Tr}(\Phi_h^\dagger \Phi + \Phi^\dagger \Phi_h) - \lambda_1^* \text{Tr}(\Phi_h^\dagger \Phi_h) \text{Tr}(\Phi^\dagger \Phi) - \lambda_2^* \text{Tr}(\Phi_h^\dagger \Phi_h \Phi^\dagger \Phi + \Phi_h \Phi_h^\dagger \Phi \Phi^\dagger) \\
 & - \kappa_1 \text{Tr}(\Phi_h^\dagger \Phi + \Phi^\dagger \Phi_h) \text{Tr}(\Phi^\dagger \Phi) - \kappa_2 [\text{Tr}(\Phi_h^\dagger \Phi + \Phi^\dagger \Phi_h)]^2 - \kappa_3 \text{Tr}(\Phi_h^\dagger \Phi + \Phi^\dagger \Phi_h) \text{Tr}(\Phi_h^\dagger \Phi_h) - \kappa_4 [\text{Tr}(\Phi_h^\dagger \Phi_h)]^2 \\
 & - \xi_1 \text{Tr}(\Phi_h^\dagger \Phi \Phi^\dagger \Phi + \Phi_h \Phi^\dagger \Phi \Phi^\dagger) - \xi_2 \text{Tr}(\Phi_h^\dagger \Phi \Phi_h^\dagger \Phi + \Phi^\dagger \Phi_h \Phi^\dagger \Phi_h) - \xi_3 \text{Tr}(\Phi^\dagger \Phi_h \Phi_h^\dagger \Phi_h + \Phi \Phi_h^\dagger \Phi_h \Phi^\dagger) - \xi_4 \text{Tr}(\Phi_h^\dagger \Phi_h)^2 \\
 & + \text{Tr}(\Phi_h^\dagger \Phi_h E_1 + \Phi_h \Phi_h^\dagger E_1) + c_1^* [(\det \Phi - \det \Phi_h^\dagger)^2 + (\det \Phi^\dagger - \det \Phi_h)^2] + c_{1E}^* (\det \Phi_h - \det \Phi_h^\dagger)^2 \\
 & + \frac{h_1^*}{2} \text{Tr}(\Phi_h^\dagger \Phi + \Phi^\dagger \Phi_h) \text{Tr}(L_\mu^2 + R_\mu^2) + \frac{h_{1E}^*}{2} \text{Tr}(\Phi_h^\dagger \Phi_h) \text{Tr}(L_\mu^2 + R_\mu^2) \\
 & + h_2^* \text{Tr}(\Phi_h^\dagger L_\mu L^\mu \Phi + \Phi^\dagger L_\mu L^\mu \Phi_h + R_\mu \Phi_h^\dagger \Phi R^\mu + R_\mu \Phi^\dagger \Phi_h R^\mu) + h_{2E}^* \text{Tr}[|L_\mu \Phi_h|^2 + |\Phi_h R_\mu|^2] \\
 & + 2h_3^* \text{Tr}(L_\mu \Phi_h R^\mu \Phi^\dagger + L_\mu \Phi R^\mu \Phi_h^\dagger) + 2h_{3E}^* \text{Tr}(L_\mu \Phi_h R^\mu \Phi_h^\dagger). \tag{B63}
 \end{aligned}$$

When fitting the coefficients of this Lagrangian to the data, Ref. [55] dropped for simplicity the terms inducing the mixing between light and heavy pseudoscalars. As we have discussed in Appendix B 5 b, in this case, κ_q invariance also leads to the cancellation of the κ_q -independent pieces in the

interaction between ALPs and P_h . On the other hand, the dropped terms may be essential for the ALPs.

Indeed, let us consider the second term, which induces such mixing, and perform the usual diagonalization (B18). Among the other terms, it includes the ALP-meson mixing which does not vanish in the limit $c_{u,d,s} \rightarrow 0$:

$$\begin{aligned}
 \mathcal{L}_\alpha \subset & \alpha \frac{f_\pi}{f_a} c_G a(x) \left[\eta_{1440}(x) \frac{\sqrt{2}(m_a^2 - m_{\pi^0}^2)(m_\eta^2 - m_{\eta'}^2)}{3(m_a^2 - m_\eta^2)(m_a^2 - m_{\eta'}^2)} \right. \\
 & \left. - \eta_{1295}(x) \frac{(m_a^2(m_\eta^2 + 2m_\eta^2 - 3m_{\pi^0}^2) + 2m_{\pi^0}^2 m_\eta^2 + m_\eta^2(m_{\pi^0}^2 - 3m_{\eta'}^2))}{3(m_a^2 - m_\eta^2)(m_a^2 - m_{\eta'}^2)} \right]. \tag{B64}
 \end{aligned}$$

Treating it for the moment as the additional mixing, we see that the c_G -dependent piece involving the ALP- P_h^0 mixing no longer vanishes [there is no corresponding ‘‘counter-term’’ from the rotation (B24) and (B23)] and may sizably contribute to the ALP decay widths. However, including them would require a complete reanalysis of the whole ELSM fit (as additional mixing between P_h and P may be non-negligible), which goes beyond the scope of our study.

Therefore, we conclude that the robustness of the description of the ALP phenomenology is significantly limited. Nevertheless, incorporating the interactions in *Mathematica* notebook, we provide a state-of-the-art

analysis that can be easily improved in the future once the meson spectroscopy becomes more mature.

APPENDIX C: PRODUCTION OF ALPS

Below, we briefly discuss our method to describe the production modes of ALPs (more details may be found in Ref. [43]).

The commonly adopted approach to describe ALP production [13,26] was to approximate the flux of ALPs by the formula

$$\frac{d^2 N_a}{d\phi_a dE_a} = \left(\frac{d^2 N_a}{d\phi_a dE_a} \right)_{\text{direct+light decays}} + \sum_{B, X_{s/d}} N_B \cdot \text{Br}(B \rightarrow X_{s/d} a) \frac{df_a^{(B)}}{d\phi_a dE_a}. \tag{C1}$$

Here, ϕ and E stay for the polar angle and energy correspondingly.

$$\left(\frac{d^2 N_a}{d\phi_a dE_a} \right)_{\text{direct+light decays}} = \sum_{P^0} |\theta_{P^0 a}|^2 \frac{d^2 N_{P^0}}{d\phi_{P^0} dE_{P^0}} \Big|_{\phi_{P^0}, E_{P^0} \rightarrow \phi_a, E_a} \tag{C2}$$

describes the contribution of the ALPs produced by deep inelastic scatterings and decays of light mesons, with $\phi_{P^0}, E_{P^0} \rightarrow \phi_a, E_a$ being some transformation relating meson's kinematic to the ALP kinematics. The second summand describes the production of the ALPs by decays of B mesons, with N_B being the total number of B mesons of the given type produced in collisions, $\text{Br}(B \rightarrow X_{s/d} a)$ the branching ratio of the decay into an ALP and a hadronic state containing an s/d quark, and $f_a^{(B)}$ the distribution in polar angle and energy of the ALPs produced in this decay (it is normalized by one).

The term (C2) literally means “the flux of ALPs is the summed fluxes of mother mesons times the squared mixing angle.”

This approximation has two problems. First, it explicitly depends on unphysical κ_q parameters entering $\theta_{P^0 a}$. Second, it wrongly describes the mass dependence of the ALP flux, in terms of both the integrated value and the kinematics (the proposed ϕ, E transformation is ambiguous and leads to unphysical bumps in the angular distribution of ALPs at low center-of-mass frame collision energy). A clear illustration is to consider the ALPs from the mixing with pions; while $\simeq 60\%–70\%$ of pions are produced by decays of heavier mesons, this is not the case of the ALPs with $m_a \gg m_{\pi^0}$.

As we have discussed in the main paper, we explicitly decompose this production into separate modes: the ALP production in the proton bremsstrahlung (the initial state radiation process), quark fragmentation (mostly the final state radiation process relevant for masses $m_a \lesssim 2$ GeV), gluon fusion or Drell-Yan process (the final state radiation relevant for heavier ALPs), and decays of light mesons (see also Ref. [42] for the definition of the processes). The descriptions of the proton bremsstrahlung and gluon fusion processes may be found in Refs. [14,33,42], whereas below we discuss the production in decays of mesons and the quark fragmentation.

1. Decays of mesons

Let us now present the method to calculate the branching ratios of various mesons, denoted by h , into ALPs. First, using the Lagrangian with the ALP interactions, we compute the matrix element of the process, \mathcal{M} . Then, we calculate the decay widths using the standard formulas [59]:

$$\Gamma_{h \rightarrow a}^{\text{2-body}} = \frac{|\mathcal{M}|^2 |\mathbf{p}|}{8\pi m_h^2} \quad (\text{C3})$$

for two-body decays, where $|\mathbf{p}|$ is the momentum of the decay product at the rest frame of the decaying h , and

$$\Gamma_{h \rightarrow a}^{\text{3-body}} = F_s \frac{1}{(2\pi)^3 8m_h} \int dE_1 dE_3 |\mathcal{M}|^2, \quad (\text{C4})$$

for three-body decays, where $E_{1,3}$ are the energies of the particles 1 and 3 in the process $h \rightarrow 1 + 2 + 3$ at h 's rest frame and F_s is the symmetry factor, being $1/n!$ if there are n identical particles in the final state and 1 otherwise. We will consider the following decay processes:

$$\begin{aligned} \rho^\pm &\rightarrow \pi^\pm a, & \eta &\rightarrow \pi^+ \pi^- a, & \eta &\rightarrow 2\pi^0 a, \\ K_S &\rightarrow \pi^0 a, & \omega &\rightarrow \pi^+ \pi^- a, & \omega &\rightarrow a\gamma. \end{aligned} \quad (\text{C5})$$

As a cross-check, considering the “pionlike” ALP, we have checked that, together with the production via fragmentation, decays of mesons accumulate $\simeq 90\%$ of the Standard Model neutral pion production flux. The processes that we drop are, e.g., $\Lambda \rightarrow pa$ and $K^\pm \rightarrow a\pi^\pm$. The first process has too limited phase space and produces very soft ALPs (as most of the momentum is carried away by the outgoing proton). In their turn, K^\pm are long-lived and typically do not have time to decay before interacting with the infrastructure surrounding the collision point (or target for beam dump experiments). Given the complexity of describing its decays, and also the suppression of the corresponding event yield, it can be safely neglected.

2. Fragmentation

Qualitatively, the production in the quark fragmentation effectively replaces a meson P^0 occurring at the last stage of the fragmentation chain with the ALP a .

To account for the proper ALP mass dependence of the flux, we incorporate the production of ALPs via mixing with $P^0 = \pi^0, \eta, \eta'$ in PYTHIA8 [64]. The production of the ALPs from the mixing with the given meson P^0 is described by P^0 's fragmentation function, carefully tuned to the data on the meson fluxes available at different facilities and having at most $\mathcal{O}(1)$ uncertainty. More details can be found in Ref. [43].

The mixing with heavier mesons (axial-vector mesons A and heavy pseudoscalars P_h^0) cannot be properly accommodated, as PYTHIA8 misses these particles in the spectrum. Even if including them, it would be unrealistic to properly accommodate the contribution of these mesons because of the complexity of properly tuning PYTHIA8 setup to describe. Therefore, our estimate of the production in this channel is conservative.

The rate of the replacement $P^0 \rightarrow a$ cannot be given by the mixing angle $\theta_{P^0 a}$, as it is clearly κ_q dependent and misses contributions from the direct operator (see Fig. 2). Instead, we replace it with the “effective” rate $\Theta_{P^0 a}$, which we define from some typical processes involving the $P^0 a$ mixing and then extrapolate on the whole fragmentation process. Schematically, this rate has the form

$$\Theta_{P^0 a} = \theta_{P^0 a} + \dots, \quad (\text{C6})$$

where \dots stays for the direct operator contribution and other mixings.

Specifically, to define Θ for the mixing with $P^0 = \pi^0, \eta, \eta'$, we consider, correspondingly, the following processes:

$$\pi^0 \pi^0 \rightarrow \pi^0 a, \quad \pi^0 \eta \rightarrow \pi^0 a, \quad \pi^0 \eta' \rightarrow \pi^0 a. \quad (\text{C7})$$

The specific processes have been chosen because they are the simplest scattering processes and because the $P^0 a$

mixing directly contributes to them. Namely, their matrix elements have the form

$$\mathcal{M}_{\pi^0 P^0 \rightarrow \pi^0 a} = \Theta_{\pi^0 a} \cdot \tilde{\mathcal{M}}. \quad (\text{C8})$$

Explicitly, we get

$$\begin{aligned} \Theta_{\pi^0 a} &= \theta_{\pi^0 a} + c_G [\kappa_u - \kappa_d - \delta(\kappa_d + \kappa_u)(\sqrt{3}\theta_{\pi^0 \eta'} + \sqrt{6}\theta_{\pi^0 \eta} + 1)] \\ &\quad - \frac{1}{3} \delta(3\theta_{\pi^0 \eta'} + 3\sqrt{2}\theta_{\pi^0 \eta} + \sqrt{3})\theta_{\eta' a} - \frac{1}{3} \delta\theta_{\eta a}(3\sqrt{2}\theta_{\pi^0 \eta'} + 6\theta_{\pi^0 \eta} + \sqrt{6}), \end{aligned} \quad (\text{C9})$$

$$\begin{aligned} \Theta_{\eta a} &= \theta_{\eta a} + \frac{1}{2} c_G [\kappa_d(\delta(2\sqrt{2}\theta_{\pi \eta'} + \theta_{\pi \eta} + \sqrt{6}) + \sqrt{6}) + \kappa_u(\sqrt{6} - \delta(2\sqrt{2}\theta_{\pi \eta'} + \theta_{\pi \eta} + \sqrt{6}))] \\ &\quad - \frac{1}{2} \delta\theta_{\pi^0 a}(2\sqrt{2}\theta_{\pi \eta'} + \theta_{\pi \eta} + \sqrt{6}) + \frac{\theta_{\eta' a}}{\sqrt{2}}, \end{aligned} \quad (\text{C10})$$

$$\begin{aligned} \Theta_{\eta' a} &= \theta_{\eta' a} + c_G [\kappa_d(\delta(-\theta_{\pi^0 \eta'} + 2\sqrt{2}\theta_{\pi^0 \eta} + \sqrt{3}) + \sqrt{3}) + \kappa_u(\delta(\theta_{\pi^0 \eta'} - 2\sqrt{2}\theta_{\pi^0 \eta} - \sqrt{3}) + \sqrt{3})] \\ &\quad + \delta \cdot \theta_{\pi^0 a}(\theta_{\pi^0 \eta'} - 2\sqrt{2}\theta_{\pi^0 \eta} - \sqrt{3}) + \sqrt{2}\theta_{\eta a}. \end{aligned} \quad (\text{C11})$$

Here,

$$\theta_{\pi^0 \eta} = -\frac{\sqrt{\frac{2}{3}}m_{\pi^0}^2}{m_{\eta}^2 - m_{\pi^0}^2}, \quad \theta_{\pi^0 \eta'} = -\frac{m_{\pi^0}^2}{\sqrt{3}(m_{\eta'}^2 - m_{\pi^0}^2)} \quad (\text{C12})$$

are the mixing angles between π^0 and η/η' (modulus δ) originating from the pure ChPT [see the mass matrix (B13)].

Inserting them and the ALP- P^0 mixing angles (B20)–(B22) in Eqs. (C9)–(C11), it can be seen that the κ_q dependence drops out.

Note that using $\Theta_{P^0 a}$ instead of $\theta_{P^0 a}$ is also needed in the case of ALPs with the dominant coupling to quarks.

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