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# Non-relativistic limits and 3D massive gravity

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**Abstract.** The non-relativistic version of General Relativity, called Newton-Cartan gravity, has received renewed attention as a tool to explore (non-perturbative) properties of non-relativistic quantum field theories. The first part of this talk is a non-specialist introduction to Newton-Cartan geometry; in particular, it is shown how Newton-Cartan gravity arises as a special limit of General Relativity. The second part reviews a novel non-relativistic limit of the 3D massive spin-2 Fierz-Pauli theory to a spin-2 planar Schrödinger equation that has been proposed to describe the so-called GMP gapped spin-2 mode of fractional Quantum Hall states. The talk concludes with speculations on an extension of this non-relativistic limit to interacting massive 3D gravity theories.

## 1. Introduction

Einstein's General Relativity is a cornerstone of modern physics; it provides an accurate description of many gravitational phenomena not explained by Newton's theory, such as the bending of light by the sun and the recently observed merging of two black holes. As the name suggests, it is an extension of Special Relativity to allow for arbitrary frames of reference, but a similar extension is also possible for Galilean Relativity and this leads to novel reformulations of Newtonian gravity, notably Newton-Cartan (NC) theory; this is a frame-independent geometrical formulation consistent with the Galilean principle of equivalence. In contrast to General Relativity, it is based on a non-Riemannian geometry in which spacetime is foliated by spaces that are specified by an absolute time on which all observers agree.

Recently the realization has dawned that non-relativistic geometry is applicable to much more than Newton's law of gravity, which can only be used to describe gravity with weak gravitational fields and at velocities that are low with respect to the speed of light. Different models of non-relativistic "gravity" have emerged in the course of investigations into *non-gravitational* phenomena, and this has now evolved into a tool kit that can be used to investigate the non-perturbative properties of non-relativistic quantum field theories (QFTs) via the holographic principle, which states that a classical gravitational theory defined in a volume with a certain background asymptotic geometry can be used to describe the strong coupling limit of a QFT defined on the asymptotic boundary, even though this QFT has no obvious relation to gravity.

The holographic principle first arose as a general idea in the context of black hole physics in General Relativity, but later found a concrete realization within string theory [1], where it goes under the name of the AdS/CFT correspondence (sometimes called gauge/gravity duality); in this realization one considers string theory (usually in a limit in which it reduces to supergravity) on Anti-de Sitter (AdS) spacetime, which is a maximally-symmetric spacetime with constant negative curvature, and the boundary QFT is a Conformal Field Theory (CFT), which implies self-similarity at all distance scales. The isometries of the AdS background become the conformal symmetries of the QFT and, roughly speaking, the extra dimension of the gravitational theory becomes the energy scale of the CFT.

About ten years after the introduction of the AdS/CFT correspondence, it was realized that it can be extended to *non-relativistic* holography by replacing the background AdS spacetime with one whose isometry group is non-relativistic, like the Schrödinger or Lifshitz symmetry groups that are typical of a wide class of field theories describing non-relativistic phenomena. Non-relativistic holography has been used to study phenomena, such as cold atoms, quantum-critical points, strange metals and high-temperature superconductivity [2]. Recently, NC geometry has emerged within this type of non-relativistic holography as a background geometry of the non-relativistic boundary QFT [3].

There exists another approach to non-relativistic holography in which not only is the boundary QFT non-relativistic, but so also is the string theory. The non-relativistic strings that underlie this type of holography came into the picture about fifteen years ago as a possibly solvable special corner of string theory [4]. This alternative approach to non-relativistic holography is not yet well explored because not much is known at present about non-relativistic string theory. Nevertheless, it is now known that it leads to a non-relativistic gravity in the bulk that differs from NC gravity; this *String NC Gravity* plays the same role for non-relativistic string theory as General Relativity plays for relativistic strings [4, 5, 6].

Independent of developments in holography, it was realized in the condensed matter community that reparametrization invariance is not only relevant to gravity but also plays a crucial role in Effective Field Theories (EFTs), where one writes hydrodynamic equations or effective Lagrangians in an expansion organized according to the number of derivatives. Coupling an EFT to an arbitrary gravitational background enables one to study generic features of this EFT in an arbitrary frame. This has led to significant benefits, such as providing strong constraints on effective Lagrangians, enabling the definition of conserved currents, such as a non-relativistic version of the energy-momentum-stress tensor, and simplifying the study of linear response.

The application of non-relativistic gravity in this new context was pioneered by Son and coworkers [7] but the detailed understanding of how to couple a non-relativistic EFT to a general NC gravitational background was achieved only a few years ago. These results can be obtained systematically by starting from a relativistic EFT in a general gravitational background that includes an auxiliary zero-flux vector field [8]. By taking a special limit of this EFT in a frame-independent way, one automatically obtains a non-relativistic EFT in a NC background.

This talk is organized as follows. In the first part we recall how the kinematics of NC gravity follows from gauging a specific non-relativistic algebra, and we show how the same results can be derived by taking a non-relativistic limit of General Relativity augmented with a zero-flux gauge field. Since this limit converts General Relativity into NC gravity, we will call it a 'force limit'. In the second part of the talk, we will discuss a new application of massive gravity in the context of the Fractional Quantum Hall (FQH) effect. To be precise, we will show that in the case of a free Fierz-Pauli theory in 2+1 dimensions, i.e. linearized massive 3D gravity, there exists a novel 'particle limit' that yields a 3D spin-2 planar Schrödinger equation of the type that occurs in the spectrum of Fractional Quantum Hall states, where they are called GMP modes. In the conclusions we discuss issues arising in attempts to extend this result to include interactions.

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## 2. Newton-Cartan gravity from gauging the Bargmann algebra

Our starting point is the following set of Galilei symmetries connecting non-relativistic inertial frames:

- time translations: $\delta t = \xi^0\,,$
- space translations: $\delta x^i = \xi^i\,, \qquad i = 1, 2, 3\,,$
- spatial rotations: $\delta x^i = \lambda^i{}_j\, x^j\,,$
- Galilean boosts: $\delta x^i = \lambda^i t\,.$

Here $\xi^0, \xi^i, \lambda^i{}_j$ and $\lambda^i$ are the constant parameters of the different symmetries. These symmetries are identical to the Poincaré symmetries except for the Galilean boosts which differ from the Lorentzian boosts in the sense that under Galilean boosts the spatial coordinates $x^i$ transform to the absolute time coordinate $t$, but $t$ does not transform back into the spatial coordinates $x^i$.

It turns out that gauging the Galilei algebra associated with the above symmetries is not sufficient to obtain NC gravity. One way to see this is to realize that the spin-connection fields associated with the spatial rotations and boost symmetries are dependent fields in General Relativity and one expects this to be the same in NC gravity. In General Relativity this is obtained by setting the torsion equal to zero which give the 24 so-called conventional constraints needed to solve the 24 spin-connection fields associated with the Lorentz transformations. The problem with the Galilei algebra is that the curvature of time translations does not contain a spin-connection field and hence cannot be used for obtaining a conventional constraint. Therefore, there are not enough conventional constraints to solve for all spin-connections. One way to circumvent this issue is to work with a centrally extended Galilei algebra, called the Bargmann algebra. It turns out that the curvature corresponding to the central charge transformations does contain the boost spin-connection fields. By setting this curvature to zero one acquires the 6 missing conventional constraints. Together with the 18 conventional constraints that follow from setting the curvature corresponding to the spatial translations equal to zero, one is now able to solve for the 12 spatial rotation connection fields $\Omega_\mu{}^{AB}$ ($\mu = 0, 1, 2, 3; A = 1, 2, 3$) and the 12 Galilean boost connection fields $\Omega_\mu{}^A$.

We now consider the gauging of the Bargmann algebra [9] which is based on a similar gauging procedure developed in the supergravity community many years ago [10, 11]. Our starting point is the set of commutation relations defining the Bargmann algebra

$$[J_{AB}, P_C] = -2\delta_{C[A}P_{B]}\,, \qquad [J_{AB}, G_C] = -2\delta_{C[A}G_{B]}\,,$$
$$[G_A, H] = -P_A\,, \qquad [G_A, P_B] = -\delta_{AB}Z\,, \tag{1}$$

where $\{H, P_A, J_{AB}, G_A, Z\}$ are the generators of time translations, space translations, spatial rotations, Galilean boosts and central charge transformations, respectively. In this gauging procedure we associate to every generator/symmetry a gauge field plus gauge parameters that are arbitrary functions of spacetime together with the covariant curvatures, see Table 1 below. Note that we have left out the parameters corresponding to the time and space translations. Instead, we assume that all gauge fields transform as covariant vectors under general coordinate transformations with parameters $\xi^\mu(x)$.

One can now show that by imposing the 24 conventional constraints

$$R_{\mu\nu}{}^A(P) = R_{\mu\nu}(Z) = 0 \tag{2}$$

one can solve for the 24 connection fields $\Omega_\mu{}^{AB}$ and $\Omega_\mu{}^A$. Furthermore, we impose the 6 geometric constraints

$$R_{\mu\nu}(H) = 2\partial_{[\mu}\tau_{\mu]} = 0 \qquad \to \qquad \tau_\mu = \partial_\mu\tau\,, \tag{3}$$

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**Table 1.** This table indicates for every symmetry of the Bargmann algebra the corresponding generators, gauge fields, local gauge parameters and covariant curvatures.

| symmetry | generators | gauge field | parameters | curvatures |
|---|---|---|---|---|
| time translations | $H$ | $\tau_\mu$ | – | $R_{\mu\nu}(H)$ |
| space translations | $P_A$ | ${E_\mu}^A$ | – | ${R_{\mu\nu}}^A(P)$ |
| Galilean boosts | $G_A$ | ${\Omega_\mu}^A$ | $\lambda^A(x^\nu)$ | ${R_{\mu\nu}}^A(G)$ |
| spatial rotations | $J_{AB}$ | ${\Omega_\mu}^{AB}$ | $\lambda^{AB}(x^\nu)$ | ${R_{\mu\nu}}^{AB}(J)$ |
| central charge transf. | $Z$ | $M_\mu$ | $\sigma(x^\nu)$ | $R_{\mu\nu}(Z)$ |

defining absolute time. Due to this constraint the time difference $\Delta T$ between two events becomes independent of the path that the observer has traveled between these two events:

$$\Delta T = \int_{\mathcal{C}} \mathrm{d}x^\mu \tau_\mu = \int_{\mathcal{C}} \mathrm{d}\tau \,. \tag{4}$$

One thus ends up with three independent gauge-fields $\{\tau_\mu, {E_\mu}^A, M_\mu\}$ that transform under general coordinate transformations, with parameters $\xi^\mu$, as covariant vectors and under spatial rotations, Galilean boosts and central charge transformations as follows:

$$\begin{aligned} \delta\tau_\mu &= \xi^\lambda\partial_\lambda\tau_\mu + \partial_\mu\xi^\lambda\tau_\lambda \,, \\ \delta {E_\mu}^A &= \xi^\lambda\partial_\lambda {E_\mu}^A + \partial_\mu\xi^\lambda {E_\lambda}^A + {\lambda^A}_B\, {E_\mu}^B + \lambda^A\tau_\mu \,, \\ \delta M_\mu &= \xi^\lambda\partial_\lambda M_\mu + \partial_\mu\xi^\lambda M_\lambda + \partial_\mu\sigma + \lambda_A\, {E_\mu}^A \,. \end{aligned} \tag{5}$$

One may define the following Bargmann-invariant metrics

$$\tau_{\mu\nu} = \tau_\mu\tau_\nu \,, \qquad\qquad h^{\mu\nu} = {E^\mu}_A {E^\nu}_B\, \delta^{AB} \,,$$

one in the time direction and a separate one in the spatial directions. One cannot define a boost-invariant metric with upper indices in the time direction or lower indices in the spatial directions. A Galilean boost-invariant metric can be obtained by adding terms proportional to the central charge gauge field $M_\mu$ but in that case one ends up with a metric that is not invariant under the central charge gauge transformations.

## 3. Newton-Cartan gravity from taking a force limit of General Relativity

Before discussing how to take the non-relativistic limit of General Relativity, it is instructive to consider first the Inönü-Wigner contraction of the Poincaré algebra to the Bargmann algebra. Our starting point is the Poincaré algebra plus an additional U(1) generator $\mathcal{Z}$ that commutes with all the Poincaré generators:

$$\big[\hat{P}_{\hat{A}}, \hat{M}_{\hat{B}\hat{C}}\big] = 2\,\eta_{\hat{A}[\hat{B}}\,\hat{P}_{\hat{C}]} \,, \quad \big[\hat{M}_{\hat{A}\hat{B}}, \hat{M}_{\hat{C}\hat{D}}\big] = 4\,\eta_{[\hat{A}[\hat{C}}\,\hat{M}_{\hat{D}]\hat{B}]} \quad \text{plus} \quad \hat{\mathcal{Z}} \,. \tag{6}$$

Here $\{\hat{P}_{\hat{A}}, \hat{M}_{\hat{A}\hat{B}}\}$ are the generators of spacetime translations and Lorentz generators, respectively. We have indicated all relativistic generators and indices with a hat to distinguish them from the non-relativistic case. The extra U(1) generator $\hat{\mathcal{Z}}$ is needed because the Bargmann algebra contains one generator more than the Poincaré algebra. Next, we decompose $\hat{A} = (0, A)$ and relate the Poincaré $\otimes$ U(1) generators $\{\hat{P}_0, \hat{P}_A\, \hat{M}_{A0}, \hat{M}_{AB}\}$ and $\hat{\mathcal{Z}}$ to the non-relativistic Bargmann generators $\{H, P_A, G_A, J_{AB}, Z\}$ as follows

$$\hat{P}_0 = \frac{1}{2\omega}\, H + \omega\, Z\,, \qquad \hat{\mathcal{Z}} = \frac{1}{2\omega}\, H - \omega\, Z\,, \tag{7}$$

$$\hat{P}_A = P_A\,, \qquad \hat{M}_{AB} = J_{AB}\,, \qquad \hat{M}_{A0} = \omega\, G_A\,, \tag{8}$$

where we have introduced a contraction parameter $\omega$. In a second step, taking the limit $\omega \to \infty$, we obtain the Bargmann algebra including the following commutator containing the central charge generator $Z$:

$$\big[P_A, G_B\big] = \delta_{AB}\, Z\,. \tag{9}$$

**Figure 1.** This figure compares the Inönü-Wigner contraction of the Poincaré algebra with the non-relativistic limit of General Relativity discussed in the text.

| | | |
|---|---|---|
| Poincaré $\otimes$ U(1) | $\overset{\text{'gauging'}}{\Longrightarrow}$ | GR plus $\partial_\mu \hat{M}_\nu - \partial_\nu \hat{M}_\mu = 0$ |
| contraction $\Downarrow$ | | $\Downarrow$ the NC limit |
| Bargmann | $\overset{\text{'gauging'}}{\Longrightarrow}$ | Newton-Cartan gravity |

Inspired by the above Inönü-Wigner contraction we now define the non-relativistic limit of General Relativity as follows, see Figure 1. Using a second-order formulation, we first introduce, on top of the Vierbein field, a vector field $\hat{M}_\mu$ with $\partial_{[\mu}\hat{M}_{\nu]} = 0$ [8]. Next, we relate the relativistic gauge fields $\{\hat{E}_\mu{}^{\hat{A}}, \hat{M}_\mu\}$ to the non-relativistic gauge fields $\{\tau_\mu, E_\mu{}^A, M_\mu\}$ of NC gravity as follows:

$$\hat{E}_\mu{}^0 = \omega\, \tau_\mu + \frac{1}{2\omega}\, M_\mu\,, \qquad \hat{M}_\mu = \omega\, \tau_\mu - \frac{1}{2\omega}\, M_\mu\,, \qquad \hat{E}_\mu{}^A = E_\mu{}^A\,. \tag{10}$$

In a second step we take the limit $\omega \to \infty$ in the relativistic transformation rules. In this way we obtain the correct non-relativistic transformation rules (5). The same limit can be used to derive the NC gravity equations of motion from the Einstein equations. Note that the standard textbooks on General Relativity usually go straight from General Relativity to Newtonian gravity skipping the general frame formulation of NC gravity.

A characteristic feature of the force limit is that the graviton, the carrier of the relativistic gravitational force, has disappeared and we are left with the Newtonian gravitational force that works instantaneously without being carried by a particle that moves at a finite velocity. The same force limit can also be applied to relativistic matter such as a real Klein-Gordon scalar field $\Phi$ with wave equation

$$\frac{1}{c^2}\ddot{\Phi} - \nabla^2\Phi + \Big(\frac{mc}{\hbar}\Big)^2\, \Phi = 0\,, \tag{11}$$

where $c$ is the velocity of light. Clearly, taking the limit $c \to \infty$ the mass term blows up. This can be circumvented by taking the limit such that the (reduced) Compton wavelength $\lambda = \hbar/mc$ is fixed. This leads to the Yukawa equation

$$\nabla^2 \Phi = \frac{1}{\lambda^2} \Phi \,, \tag{12}$$

where the massless spin-0 particle described by the Klein-Gordon equation has disappeared. Instead one is left with a scalar force satisfying the Yukawa equation (12).

## 4. A new particle limit of 3D Fierz-Pauli

Recently, there has been a lot of interest in constructions of *massive gravity*, which is a deformation of General Relativity, where one assigns the graviton a certain mass. The mass parameter can be made extremely small such that it affects only the workings of General Relativity at cosmological scales, where we are facing unsolved mysteries such as dark matter and dark energy. Sometime ago we have constructed new models of massive gravity in *three* spacetime dimensions, not by adding a mass term for the graviton but instead by adding higher-derivative terms to the usual Einstein-Hilbert kinetic term [12]. Our model is called *New Massive Gravity (NMG)* in the literature. Linearization about a Minkowski vacuum yields a free field theory that is equivalent to the three-dimensional version of the massive spin-2 field theory of Fierz and Pauli. There are various so-called *bi-metric* three-dimensional massive gravity theories that have the same linearized limit, and NMG may itself be viewed as the simplest example, with an auxiliary tensor field as the second metric [12]. The introduction of a second metric enables one to describe a massive graviton propagating in a general curved background different from the Minkowski vacuum.

As it happens, the states that occur in the FQH Effect have a so-called Girvin-MacDonald-Platzman (GMP) gapped spin-2 mode that can be considered as a non-relativistic massive spin-2 particle whose dynamics is described by a spin-2 planar Schrödinger equation [13]. Following suggestions of a geometrical interpretation of these GMP states [14], Gromov and Son recently showed that this spin-2 planar Schrödinger equation emerges upon linearization about flat space of a particular non-relativistic bi-metric EFT that shares features with some of the bi-metric massive gravity theories proposed in the gravity literature [15]. The second metric of the EFT proposed in [15] can be used to describe the FQH effect in a curved ambient space. However, the metrics used in [15] are *space* metrics only and not spacetime metrics. This has the disadvantage that the Galilean boost symmetry is not realized. Nevertheless, the EFT of [15] has been the starting point for detailed investigations of the GMP mode in the FQH Effect.

It is natural to ask oneself the question: *Is the bi-metric EFT proposed by Gromov and Son the non-relativistic limit of some relativistic bi-metric massive gravitational theory, perhaps New Massive Gravity (NMG)?* Although we do not answer this question in this talk, we will show that the Schrödinger equation proposed to describe the GMP mode can be obtained by taking a novel non-relativistic limit of the three-dimensional Fierz-Pauli theory. We will call this limit a particle limit to distinguish it from the force limit in which the relativistic particle disappears.

A stumbling block for deriving the EFT for a non-relativistic massive spin-2 particle is that the corresponding Schrödinger equation is complex, whereas the gravitational fields one starts with are real. Indeed, to obtain a Schrödinger equation for a massive spin-0 particle one must consider a *complex* scalar field $\Phi$ with wave equation

$$\frac{1}{c^2}\ddot{\Phi} - \nabla^2\Phi + \left(\frac{mc}{\hbar}\right)^2 \Phi = 0 \,. \tag{13}$$

To avoid infinities we redefine

$$\Phi = e^{-\frac{i}{\hbar}(mc^2)t}\Psi \tag{14}$$

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so that the Klein-Gordon equation becomes

$$-\frac{1}{2mc^2}\Big(i\hbar\frac{d}{dt}\Big)^2\Psi - i\hbar\dot{\Psi} - \frac{\hbar^2}{2m}\nabla^2\Psi = 0\,. \tag{15}$$

After this redefinition we take the limit $c \to \infty$ and obtain the desired spin-0 Schrödinger equation

$$i\hbar\dot{\Psi} + \frac{\hbar^2}{2m}\nabla^2\Psi = 0\,. \tag{16}$$

Extending this complex limit procedure to higher spins we find the generic feature that each complexified helicity mode yields a separate Schrödinger equation. At the linearized level, this procedure is easy to perform for any spin. However, the complexification is non-trivial if one wishes to include interactions like in the spin-2 case we have in mind. Another issue is that a parity even 3D massive spin-2 theory contains two helicities and therefore leads to *two* Schrödinger equations instead of the single one that occurs in the EFT proposed by Gromov and Son. To circumvent this issue it would be desirable if we could take a non-relativistic limit of a *real* massive Fierz-Pauli (FP) theory without doubling the degrees of freedom by complexification. We will now show that precisely in the 3D case discussed by Gromov and Son, the 3D FP equations have a novel non-relativistic limit to the single planar spin-2 Schrödinger equation that occurs in the proposal of [15]. For simplicity, we first explain how this novel limit works for the massive Proca theory and then discuss its extension to the massive FP theory.

The novel limit in the case of a real Proca theory works as follows. [1] Our starting point is the 3D free Proca Lagrangian for a real vector field $A_\mu = (A_0, \mathbf{A})$. Elimination of the auxiliary field $A_0$ yields a Lagrangian that can be simplified further by redefining the 2-vector $\mathbf{A}$ into a new vector field $\mathbf{B}$ [16]. The key trick is now that the 2 real components $B_1$ and $B_2$ of the 2-vector $\mathbf{B}$ can be used to define a complex scalar $B$ as follows:

$$B = \frac{1}{\sqrt{2}}\big(B_1 + iB_2\big)\,. \tag{17}$$

One can show that in terms of this complex scalar, the Lagrangian reads [16]

$$\mathcal{L} = \frac{1}{c^2}\dot{B}^\star\dot{B} + B^\star\nabla^2 B - \Big(\frac{m}{\hbar}\Big)^2 B^\star B\,. \tag{18}$$

If we now set

$$B = e^{-imc^2t}\Psi[1]\,, \tag{19}$$

for a new complex variable $\Psi[1]$, we may take the $c \to \infty$ limit to arrive at the Galilean invariant Lagrangian density

$$\mathcal{L}_{\rm NR} = 2im\bar{\Psi}[1]\dot{\Psi}[1] + \bar{\Psi}[1]\nabla^2\Psi[1]\,. \tag{20}$$

The field equation is

$$-\frac{1}{2m}\nabla^2\Psi[1] = i\dot{\Psi}[1]\,, \tag{21}$$

which is the planar spin-1 Schrödinger equation. Note that the redefinition (19) breaks parity in the limit $c \to \infty$ which explains why we end up with a single parity non-invariant Schrödinger equation.

It is relatively straightforward to extend the above limit to the spin-2 case. For more details, see [16]. Our starting point in the spin-2 case is the 3D real FP Lagrangian. The basic field is now a symmetric traceless tensor field $f_{\mu\nu}$:

$$f_{\mu\nu} = f_{\nu\mu}\,, \qquad \eta^{\mu\nu}f_{\mu\nu} = 0\,. \tag{22}$$

[1] For a more detailed description of this limit, see [16].

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In the same way that the real Proca vector decomposes under the spatial SO(2) as $3 \to 1 + 2$ or

$$A_\mu \quad \to \quad A_0 \quad + \quad A_1 + iA_2\,, \tag{23}$$

the symmetric traceless FP tensor decomposes under the spatial SO(2) as $5 \to 1 + 2 + 2$ or

$$f_{\mu\nu} \quad \to \quad f_{11} + f_{22} \quad + \quad f_{01} + if_{02} \quad + \quad \frac{1}{2}(f_{11} - f_{22}) + if_{12}\,. \tag{24}$$

After integrating out the auxiliary singlet and doublet one ends up with a Lagrangian for the complex scalar $\frac{1}{2}(f_{11} - f_{22}) + if_{12}$ that is identical to the Lagrangian that we encountered in the spin-1 case except that under spatial rotations in the spin-2 case the complex scalar rotates twice as fast as in the spin-1 case. We therefore precisely end up with the 3D planar spin-2 Schrödinger equation that occurs in the proposal of [15].

So far, we derived the planar Schrödinger equation only for a *free* spin-2 particle, or spin-2 GMP mode in the condensed matter context. It is a major challenge to extend the analysis including general interactions. As a first step in this direction, we will show how to obtain a Schrödinger potential from a so-called generalized null-reduction [17]. [2]

Our starting point is some solution of the full 4D Einstein field equations: $G_{mn} = 8\pi G_N T_{mn}$, where $G_{mn}$ is the Einstein tensor, $G_N$ is Newton's constant, and $T_{mn}$ is some specified source tensor (we set $c = 1$). Given a 4-metric that solves these equations, we linearize about it to find the following equations for the metric perturbation tensor:

$$D^2 h_{mn} - 2D_{(m}h_{n)} + D_m\partial_n h + \; 2\left[R_{p(m}h_{n)}{}^p + R_{pmnq}h^{pq}\right] = 0\,, \tag{25}$$

where $D$ is the covariant derivative with respect to the affine connection for which $R^p{}_{mnq}$ is the Riemann tensor and $R_{mn}$ the Ricci tensor. Furthermore, we have $h_m = g^{pq}D_p h_{qm}$ and $h = g^{mn}h_{mn}$. Choosing lightcone coordinates $x^m = \{x^+, x^-, x^i\}$ where $i = 1, 2$ we now take as our background the Brinkmann-wave metric

$$ds^2 = 2dx^+dx^- + 2v(x^+, \mathbf{x})(dx^+)^2 + d\mathbf{x}\cdot d\mathbf{x}\,. \tag{26}$$

The function $v$ is independent of $x^-$, which ensures that $\partial_-$ is a null Killing vector field; in particular, constant $v$ yields Minkowski spacetime. One may verify that the background Einstein equations are satisfied if

$$\nabla^2 v = 8\pi G_N T_{++}\,, \tag{27}$$

where $T_{++}$ must be the *only* non-zero component of $T_{mn}$ [17].

We next impose the gauge condition $h_{m-} = 0$ after which the dynamical equations (25) reduce to [17]

$$D^2 h_{mn} + 2R_{mpqn}h^{pq} = 0\,. \tag{28}$$

We need consider only the equation for $h_{ij}$, which is

$$2\partial_-\partial_+ h_{ij} - 2v\partial_-^2 h_{ij} + \nabla^2 h_{ij} = 0\,. \tag{29}$$

Only the traceless part of $h_{ij}$ is non-zero, and we can trade this for the traceless transverse metric perturbation

$$\Psi[2] = h_{11} + ih_{12}\,. \tag{30}$$

Imposing the generalized null-reduction condition [17]

$$\partial_-\Psi[2] = i(m/\hbar)\Psi[2]\,, \tag{31}$$

[2] A more detailed version of the derivation given below can be found in [17].

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where $m$ is a positive mass parameter, we find that the equation for $\Psi[2]$ becomes a spin-2 planar Schrödinger equation with a Hamiltonian that includes a Schrödinger potential [17]

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V(t, \mathbf{x}) , \qquad V = mv , \tag{32}$$

where $t = x^+$. This concludes our derivation of the Schrödinger potential.

## 5. Discussion

In this talk of two parts we first showed how Newton-Cartan gravity arises from a non-relativistic limit of General Relativity (GR). We then explained how the Schrödinger equation in two space dimensions can be found from a novel non-relativistic limit of the 3D Fierz-Pauli theory for a free relativistic spin-2 particle. This was a Schrödinger equation for a free particle that was originally proposed in the context of a geometric description of "gapped spin-2 excitations" observed in Fractional Quantum Hall states but in this context, a potential term, implying some kind of interaction, also appears.

We reviewed briefly a method of arriving at this more general Schrödinger equation by a "generalized null reduction" from (4D) GR; one starts by considering the linearized Einstein field equations in a particular "Brinkmann wave" background solution of the full Einstein equations. The planar spin-2 Schrödinger equation is then found by the null reduction method, the potential term being induced from the starting background solution. An open problem at the moment is to obtain this result as a non-relativistic limit of some interacting version of the 3D Fierz-Pauli theory.

The problem of introducing interactions into the FP theory in a way consistent with the principles of GR is a difficult one, but a simple solution to this problem in the 3D case was found ten years ago; this is the generalization of 3D GR known as "New Massive Gravity" (NMG) The 'null-reduction-from-4D' route suggests that one should consider linearization about some background solution of NMG. The only solution with the required symmetries would appear to be the Minkowski vacuum, which does not seem to be a promising starting point. However, an interesting feature of NMG is that there may be a Minkowski vacuum even when the (3D) cosmological constant is non-zero, so there remain possibilities to be explored here.

It is clearly a major challenge to go beyond the Schrödinger potential and construct a boost-invariant extension of the interacting EFT proposed in [15]. To achieve this, we have two general strategies in mind. One strategy is to extend the limit technique that works so well at the linearized level to the non-linear case by making use of the fact that General Relativity in three spacetime dimensions allows a simplified description in terms of a so-called Chern-Simons action for the relativistic Poincaré algebra. The simple form of the Chern-Simons action leaves open the possibility to not only consider a real metric and take the limit using the trick we discussed in this talk but also to temporarily extend the theory and start from a *complex* metric, take the conventional non-relativistic limit and, after the limit has been taken, truncate the theory by imposing a judiciously chosen reality condition.

An alternative strategy we have in mind is to construct the desired EFT directly by making use of Chern-Simons actions that are based on *non-relativistic* algebras. This is non-trivial, since a requirement for the construction of a Chern-Simons action is that the algebra in question allows a so-called *non-degenerate bi-linear form.* This requirement is satisfied by the Poincaré algebra but not by the Galilei or Bargmann algebra. However, precisely in three spacetime dimensions, the Galilei algebra allows for *two* central extensions corresponding to the so-called *extended Bargmann algebra* that does allow a non-degenerate bilinear form. We already constructed the Chern-Simons action for this extended Bargmann algebra in [18]. The challenge now is to construct a bi-metric EFT that is based upon *two* of these algebras introducing a mass parameter that does not describe the massive deformation of a non-relativistic gravity theory but, instead, corresponds to the physical mass of the GMP mode. At the time of writing both strategies are under investigation.

A fascinating feature of this work is that it relates recent results obtained in the gravity community to new developments taking place in the condensed matter community. In both communities, a lot of effort has been put into constructing extensions of the gravity and EFT models in question by adding higher-derivative terms and/or by considering spins higher than two. Higher-derivative terms are natural in any model of quantum gravity, while EFTs are based upon an expansion in terms of derivatives. Similarly, higher spins are believed to play a crucial role in solving the problem of quantum gravity, whereas there exist higher-spin versions of the GMP mode, called 'magneto-rotons' [19]. In both cases, even an infinite number of higher spins based on the area-preserving symmetries have been considered. In view of this, we foresee exciting and unexpected interactions between the gravity and condensed matter communities.

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