Title	Discrete Integrable System and Invariant Variety of Periodic Points
Author(s)	Yumibayashi, Tsukasa
Citation	
Issue Date	2015-03-25
URL	http://hdl.handle.net/10748/7142
DOI	
Rights	
Type	Thesis or Dissertation
Textversion	ETD

TOKYO METROPOLITAN UNIVERSITY

首都大学東京

http://www.tmu.ac.jp/

Discrete Integrable System

and

Invariant Variety of Periodic Points

Tsukasa YUMIBAYASHI

Department of Physics, Tokyo Metropolitan University,
Minamiohsawa 1-1, Hachiohji, Tokyo, 192-0397 Japan

A Dissertation in candidacy for

the degree of Doctor of Philosophy

2015

As far as the laws of mathematics refer to reality, they are not certain, and as far as they are certain, they do not refer to reality.

$\sim$ *Albert Einstein* $\sim$

Abstract

In this thesis, we discuss the nature of periodic points of discrete integrable systems. We consider, in particular, integrable rational maps and/or algebraic difference equations(ADE), whose behavior we can decide precisely for all initial conditions.

It was shown, in the last ten years, that periodic points of such a system form an algebraic variety different for each period if the system has a sufficient number of invariants. Since every variety is determined only by information of the invariants it is called an invariant variety of periodic points(IVPP).

It was also suggested that the existence of an IVPP might be sufficient to characterize integrability of rational maps. This is because we can prove that the coexistence of an IVPP and a discrete set of periodic points of any period is forbidden in one map. Thus an IVPP guarantees non existence of the Julia set, a fractal set of unstable periodic points which characterizes non-integrability of the system.

Having studied the properties of IVPPs in many discrete maps we encountered various interesting phenomena common in such systems. For example, the algebraic varieties associated with IVPPs of different periods <u>intersect</u> each other. The intersections form a variety which is singular because every point of the variety is occupied by points of different periods simultaneously. The main purpose of this thesis is to explore where and how the intersections of IVPPs can take place.

First by studying the periodicity conditions for the maps in detail we will arrive at a proposition that the intersections are possible only on the singularities of the maps. In the case of rational maps the zero set of denominators form a variety of singular points(SP), while the points satisfying 0/0 form a variety of indeterminate points(IDP). In the ADE case the IDPs can be determined from the implicit function

theorem. Many integrable maps are investigated by means of computer algebra to confirm our proposition.

Based on this observation we have found the following phenomena in this work.

- Let us consider a d-dimensional map with p invariants. After elimination of p variables by using all the invariants, we obtain ADE for $(d-p)$ variables whose coefficients are parameterized by the invariants. If we fix the parameters such that the invariants specify the IVPP of period n, the ADE becomes a recurrence equation of period n. Namely, all solutions of this ADE are n periodic for any initial point.

- On the other hand if we parameterize the SP of a rational map by the invariants, any point on the SP must be mapped, after n steps, to the IVPP of period n. In other words, the singular points of the map are the source of IVPPs. This fact enables us to derive IVPPs of all periods iteratively. Moreover we will show that this phenomenon can be associated with a "projective resolution" of "triangulated category".

- Finally we investigate how the transition takes place between an integrable map and a non-integrable map. To this end we introduce an arbitrary parameter a to an integrable map, such that the map becomes integrable at $a = 0$. When a is finite the repelling periodic points of the map form the Julia set. As a varies the value continuously every point of the Julia set moves along an algebraic curve. We see that some of them approach IVPPs in the $a = 0$ limit but a large part of them approach the singular points of the integrable map, so that the points become singular loci of the algebraic curves which are highly degenerate for each period.

Contents

Abbreviation words	Original words
SP	Singular points
IDP	Indeterminate points
ADE	Algebraic difference equation
IADE	Ideal form of the ADE
VPP	Variety of periodic points
UC-VPP	Uncorrelated-VPP
C-VPP	Correlated-VPP
FC-VPP	Full correlated-VPP
IVPP	Invariant variety of periodic points
RE	Recurrence equation
SC	Singularity confinement
LV	Lotka-Volterra
KdV	Korteweg-de Vries
HM eq.	Hirota-Miwa equation
KP	Kadomtsev-Petviashvili

Chapter 1

Introduction

As an introduction to this thesis, we would like to explain first the motivation to study discrete integrable systems. Then we review briefly the basic ideas on discrete dynamical systems. The summary of the results of this thesis is presented in the last part of this chapter.

1.1 Motivation

We want to understand physical phenomena in general. The most of phenomena appear in a sequence of transition of one physical state into another. We can say we understand the phenomena if we can describe it by means of mathematical formulas and if we can predict the behavior of the sequence for all initial conditions.

If we use the terms of the category theory[1] this process of understanding physical phenomena is nothing but a "natural transformation ($\mathscr{P}$) " of "a category of physical states ($\mathcal{PS}$)" with "endofunctor of transitions (End($\mathcal{PS}$))" to "a category of mathematical objects ($\mathcal{MO}$)" with "endofunctor of transformations (End($\mathcal{MO}$))".

$$\mathscr{P} \; : \; \mathcal{PS} \text{ with End}(\mathcal{PS}) \; \rightarrow \; \mathcal{MO} \text{ with End}(\mathcal{MO}).$$

A "transition" $\mathscr{T} \in \mathrm{End}(\mathcal{PS})$ induces a transition of physical states X to $X_{\mathscr{T}}[1] := \mathscr{T}(X)$ for any $X \in \mathrm{Ob}(\mathcal{PS})$. A "transformation" $\mathcal{P}(\mathscr{T}) \in \mathrm{End}(\mathcal{MO})$ that describes the transition $\mathscr{T}$ induces a transformation of mathematical objects $\mathscr{P}(X)$ to $\mathscr{P}(X)_{\mathscr{P}(\mathscr{T})}[1] := \mathscr{P}(\mathscr{T})(\mathscr{P}(X))$. Therefore,

$$\text{Transition} : \mathscr{T} : X \to X_{\mathscr{T}}[1] \quad \Rightarrow \quad \text{Transformation} : \mathscr{P}(\mathscr{T}) : \mathscr{P}(X) \to \mathscr{P}(X)_{\mathscr{P}(\mathscr{T})}[1].$$

In general, a transition $\mathscr{T} \in \mathrm{End}(\mathcal{PS})$ is iterated as

$$X_{\mathscr{T}}[2] := \mathscr{T}(X_{\mathscr{T}}[1]) = \mathscr{T}^{(2)}(X) := \mathscr{T} \circ \mathscr{T}(X),$$

because a transition is given by the low of physics. Therefore a transformation $\mathscr{P}(\mathscr{T}) \in \mathrm{End}(\mathcal{MO})$ is also iterated.

From this correspondence, we see that the pair $\left(\left\{ \mathscr{P}(\mathscr{T})^{(t)} \right\}_{t\in\mathbb{Z}}, \mathcal{MO} \right)$ is the "discrete dynamical system"[2], which describes physical phenomena. Therefore the understanding physical phenomena amounts to understanding the "discrete dynamical system" $\left(\left\{ \mathscr{P}(\mathscr{T})^{(t)} \right\}_{t\in\mathbb{Z}}, \mathcal{MO} \right)$. This explains the reason why we study discrete dynamical systems.

In order to understand physical phenomena it is important to know if we can predict behavior of the sequence of transition of the states. In the side of mathematics it amounts to discriminating integrable systems from non-integrable ones. For this purpose we study, in this thesis, discrete integrable systems, such as rational maps[3] and algebraic dynamical equations(ADE)[4][5] [1] .

In the recent works[7][8][9] it was found that integrable rational maps are characterized by the existence of algebraic varieties of periodic points different for each period, if there exist a sufficient number of invariants. Such a variety is called an invariant variety of periodic points(IVPP), since it is determined by only information

[1]An ADE is also called an algebraic relation in [6].

of the invariants. We are interested in finding necessary and sufficient conditions for a discrete system being integrable. For this purpose it will be worthwhile to clarify the role of the IVPPs in integrable systems.

Finally we would like to mention that, although we use many mathematical tools throughout this work, our main concern is on physical phenomena. Since we encounter many new phenomena, which have never been formulated before in mathematics, we are forced to introduce many conjectures or ansatzes to explain the phenomena. A large part of arguments owe to the observation by using computer algebra [2]. In fact the computer algebra was used not only to confirm all propositions and conjectures, but also to find new phenomena.

1.2   What is the Dynamical System and Integrability?

In this section, we are going to make our question more precise. We must specify what is meant by "understanding a discrete dynamical system". The mathematical physics has an answer to this question, *i.e.,* "It is to determine whether the system is integrable or non integrable".

We shall give, in the following subsections, "definitions" of discrete dynamical systems, discrete maps and discrete integrable systems, which we adopt in this work.

1.2.1   Dynamical System and Map

First we give a definition of a dynamical system over $\mathbb{C}$[2][10].

---

[2]We used Maple throughout this work.

A d dimensional dynamical system over $\mathbb{C}$ is a pair $\left(\left\{F^{(t)}\right\}_{t\in\mathbb{A}},\mathbb{C}^d\right)$ such that

$$F^{(t)}:\mathbb{C}^d\to\mathbb{C}^d,$$

$$F^{(0)}:=\mathrm{id}_{\mathbb{C}^d},\quad F^{(t')}\circ F^{(t)}=F^{(t+t')},\quad {}^\forall t,t'\in\mathbb{A}$$

and the "set of time" $\mathbb{A}$ is a commutative monoid.

If the set of time $\mathbb{A}$ is $\mathbb{R}_{\geq0}$, then the dynamical system $\left(\left\{F^{(t)}\right\}_{t\in\mathbb{R}_{\geq0}},\mathbb{C}^d\right)$ is called "continuum dynamical system", and if the set of time $\mathbb{A}$ is $\mathbb{Z}_{\geq0}$, then the dynamical system $\left(\left\{F^{(t)}\right\}_{t\in\mathbb{Z}_{\geq0}},\mathbb{C}^d\right)$ is called "discrete dynamical system". In addition, if a map F is invertible, then the time set of discrete dynamical system $\mathbb{Z}_{\geq0}$ can extend to $\mathbb{Z}$.

Hence the discrete dynamical system $\left(\left\{F^{(t)}\right\}_{t\in\mathbb{Z}_{\geq0}},\mathbb{C}^d\right)$ is given by iterations of the map F. In this thesis, we denote the process that carries a point $\boldsymbol{x}^t\in\mathbb{C}^d$ to $\boldsymbol{x}^{t+1}:=F(\boldsymbol{x}^t)\in\mathbb{C}^d$ by a map F,

$$F:\boldsymbol{x}^t\mapsto\boldsymbol{x}^{t+1}.$$
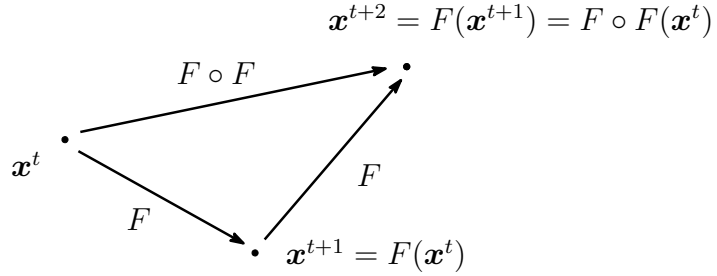
Figure 1.1: Time evolutions by iteration of the map F

1.2.2 Difference Equation

In addition, a physical phenomenon is described by means of "equations of motion" either differential or difference equations[2][11], dependent on a continuum or a discrete dynamical system, respectively. So we give a definition of the difference equation which describes a discrete dynamical system.

A d dimensional "first order" difference equation $\tilde{F}$ over $\mathbb{C}$ is defined as,

$$\tilde{F}_i(\boldsymbol{x}^t, \boldsymbol{x}^{t+1}) = 0, \quad \boldsymbol{x}^t, \boldsymbol{x}^{t+1} \in \mathbb{C}^d, \quad t \in \mathbb{Z}, \quad i = 1, \ldots, d.$$

We can get a map $F : \boldsymbol{x}^t \mapsto \boldsymbol{x}^{t+1}$ by solving[3] the difference equation $\tilde{F}$ for the variable $\boldsymbol{x}^{t+1}$.

We give a small remark about the reason that our object is a "first order" difference equation. It is because we can always rewrite a d dimensional N-th order difference equation to a $d + (N-1)d$ dimensional first order difference equation by adding $(N-1)d$ variables $x^t_{d+(i-1)(N-1)+T} := x^{t+T}_i, i = 1, \ldots, d, T = 1, \ldots, N-1$.

1.2.3 Time Evolution of Difference Equation

The time evolution of the difference equations $\tilde{F}$ is given by elimination of the variable $\boldsymbol{x}^{t+1}$ from the set of difference equations,

$$\tilde{F}_i(\boldsymbol{x}^t, \boldsymbol{x}^{t+1}) = 0, \quad \tilde{F}_i(\boldsymbol{x}^{t+1}, \boldsymbol{x}^{t+2}) = 0, \quad i = 1, \ldots, d,$$

from which we get a new set of difference equations $\tilde{F}^{(2)}$

$$\tilde{F}_i^{(2)}(\boldsymbol{x}^t, \boldsymbol{x}^{t+2}) = 0, \quad \boldsymbol{x}^t, \boldsymbol{x}^{t+2} \in \mathbb{C}^d, \quad i = 1, \ldots, d.$$

In this manner, we can decide the time evolution of the difference equation $\tilde{F}$ at any time.

---

[3]It does not mean to solve the difference equation.

1.3   Integrability of Dynamical System

In this section we discuss the notion of integrability.

1.3.1   Integrability of a Map and Difference Equation

If the map F and the difference equation $\tilde{F}$ can give "solutions" $\boldsymbol{x}\left[\boldsymbol{x}^0\right]$ for "any" initial point $\boldsymbol{x}^0 \in \mathbb{C}^d$,

$$\boldsymbol{x}\left[\boldsymbol{x}^0\right] : t \mapsto \boldsymbol{x}^t\left[\boldsymbol{x}^0\right], \quad \boldsymbol{x}^0, \boldsymbol{x}^t\left[\boldsymbol{x}^0\right] \in \mathbb{C}^d, \quad t \in \mathbb{Z},$$

then F is called an integrable map, and $\tilde{F}$ an integrable difference equation.

This definition is like "solvability". However, in general, integrability and solvability are not the same. Therefore we should decide our position about the meaning of integrability, among many possible candidates. In this thesis, we choose a position as follows. If a map has "soliton solutions" the map is called a soliton equation. There have been known many soliton equations which are integrable in the sense that all solutions can be obtained, by means of the inverse scattering method, for any initial conditions. In particular, according to the Sato theory[12] of the KP hierarchy[13], infinitely many soliton equations can be derived from a single discrete equation called the Hirota-Miwa equation(HM eq.)[14][15], whose solutions are associated with the points of the universal Grassmannian. Therefore, we consider, in this thesis, about maps which are given mainly by the HM eq., when we must consider them explicitly.

The meaning of this definition is that we can find a point $\boldsymbol{x}^t\left[\boldsymbol{x}^0\right]$ that is given for any t of the map F and difference equation $\tilde{F}$ starting from any initial point $\boldsymbol{x}^0$. An integrable system is understandable, because we know all behavior of it for an arbitrary initial point. On the other hand, a non integrable system is not

understandable, because there exist some orbits which we have no way to predict the behavior before we calculate it explicitly.

If a map is described by a rational function, it is called a rational map[3], while a difference equation described by a polynomial is called an algebraic difference equation(ADE)[4][5]. These are the main systems we study in this thesis. Therefore we do not distinguish words between map and difference equation.

1.3.2 Integrability v.s. Invariants

In this subsection we introduce the notion of invariants. An invariant $H(\boldsymbol{x}) \in \mathbb{C}$ of a map and a difference equation is defined when $H(\boldsymbol{x})$ satisfies

$$H(\boldsymbol{x}^t) = H(\boldsymbol{x}^{t+1}(\boldsymbol{x}^t)).$$

Here we should remark about the relation between our definition of integrability and the invariants. It is often said that maps and difference equations are integrable when they have enough number of "invariants". This is, however, incorrect, because we can give infinite number of exceptions of this definition.

The reason of this error might come from the comparison with the well known Liouville-Arnold theorem[16] for a continuous Hamiltonian dynamical system[17] which is integrable when it has enough number of independent invariants. When the time evolution is discrete, the theorem does not work anymore.

We would like to emphasize that integrability of a general dynamical system is not determined by the number of invariants. For example, the following two 2 dimensional rational maps

$$F_{\text{2dMöb}} : (x_1^t, x_2^t) \mapsto (x_1^{t+1}, x_2^{t+1}) = \left(x_1^t \frac{1 - x_2^t}{1 - x_1^t}, x_2^t \frac{1 - x_1^t}{1 - x_2^t} \right),$$

and

$$F_{\text{2dlog}} : (x_1^t, x_2^t) \mapsto (x_1^{t+1}, x_2^{t+1}) = \left((x_1^t)^2 x_2^t (1 - x_1^t), \frac{1}{x_1^t(1 - x_1^t)} \right),$$

have an invariant

$$H(\boldsymbol{x}^t) = x_1^t x_2^t.$$

Now we restrict the maps $F_{\text{2dMöb}}$ and F_{2dlog} on the level set $\{\, \boldsymbol{x} \in \mathbb{C}^2 \mid H(\boldsymbol{x}) = h \,\}$ that is a surface on a constant parameter $h \in \mathbb{C}$ of the invariant $H(\boldsymbol{x}^t)$. These maps become 1 dimensional maps on the level set, because the number of independent variables of d dimensional map with p invariants is $d - p$. After the elimination of x_2 using the invariant and $y := x_1$, we see that $F_{\text{2dMöb}}$ becomes the Möbius map[18]

$$(F_{\text{2dMöb}})_h : y^t \mapsto y^{t+1} = \frac{y^t - h}{1 - y^t},$$

that is known integrable, and F_{2dlog} becomes the logistic map[19]

$$(F_{\text{2dlog}})_h : y^t \mapsto y^{t+1} = h y^t (1 - y^t),$$

that is known non integrable. Because of this reason the number of invariants is not a proper quantity to define integrability.

1.3.3 Non Integrable System and Julia Set

In general, a non integrable discrete system F has a chaos. In particular, a non integrable system has (non trivial) Julia set $J(F)$[10].

In one dimensional case the Julia set $J(F)$ is a boundary of a set of convergent points $K(F)$ that is called filled Julia set. It has some properties as follows

- $J(F) = J(F^{(t)}), \quad t \in \mathbb{N},$

- $J(F) = F(J(F)) = F^{(-1)}(J(F)).$

Therefore the Julia set $J(F)$ is an invariant of the map F.

A $K((F_{2\text{dMöb}})_h)$ is $\{1\}$ for any $h \in \mathbb{C}$, therefore the Möbius map has no Julia set. And we present a filled Julia set of the $(F_{2\text{dlog}})_h$ at $h = 3.3$ in Figure 1.2.
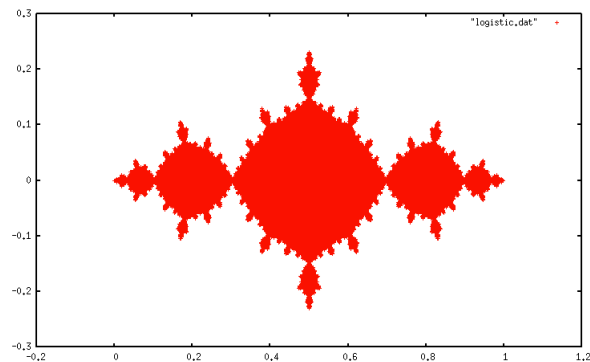
Figure 1.2: Filled Julia set of the $(F_{2\text{dlog}})_h$ $h = 3.3$

In this thesis, we discuss higher dimensional maps, in which the precise definition of the Julia set has not been known yet. Hence, we assume that a non integrable discrete system has the Julia set, given by the closure of unstable periodic points.

1.3.4 Integrability Test and Singularity Confinement(SC)

In this subsection, we discuss about the integrability test of a discrete dynamical system. The singularity confinement(SC)[20][21][22][23] was a candidate of the integrability test of a discrete dynamical system analogous to the Painlevé test[24] [4] for second order differential equations.

The SC for a rational map F is defined by the following conditions:

---

[4]It is not proved yet. Recently, it is found[25] that the Painlevé 6, for example has chaotic orbits.

- An initial point $\boldsymbol{x}^0$ is chosen at a zero of one of denominators $D_i, {}^\exists i \in \{1, \ldots, d\}$. Then $\boldsymbol{x}^0$ goes to infinity by the map F.

$$\boldsymbol{x}^0 \in \{ \, \boldsymbol{x} \in \mathbb{C}^d \mid D_i(\boldsymbol{x}) = 0, \ {}^\exists i \in \{1, \ldots, d\} \, \} \to F(\boldsymbol{x}^0) = \infty.$$

- The point $\boldsymbol{x}^0$ returns to a finite point after finite steps $N_{sc} \in \mathbb{N}$ of the map F.

$$\boldsymbol{x}^0 \to \cdots \to F^{(N_{sc})}(\boldsymbol{x}^0) < \infty.$$

Since the SC has counter examples[26], it is not a sufficient of integrability test. However it seems an important nature of discrete integrable systems.

Although there are many other useful methods to judge integrability, such as algebraic entropy[26], Liapnov numbers[27], and so on, we do not discuss them in this thesis, because we do not use numerical calculation.

1.3.5   Continuous and Ultradiscrete Limits

We have so far discussed discrete integrable systems described by a "discrete time" and "continuous dependent variables". But a large part of physical phenomena are dynamical systems of continuous time. As far as integrability is concerned, however, the gap between discrete and continuous time is not big. Almost every discrete integrable system can be transferred to a continuum integrable system. It will be achieved in the $\epsilon = 0$ limit of the following procedure

$$f(t+1) = e^{\frac{\mathrm{d}}{\mathrm{d}t}} f(t) \to f(t+\epsilon) = e^{\epsilon \frac{\mathrm{d}}{\mathrm{d}t}} f(t) = f(t) + \epsilon \frac{\mathrm{d}}{\mathrm{d}t} f(t) + \ldots,$$

without losing integrability[28].

Another possible limit of the discrete integrable system is called an ultradiscrete system that is described by "discrete time and discrete dependent variables"[2][28].

The procedure of ultradiscritization is defined by taking the $\epsilon = 0$ limit of the following scheme:

$$x + y = z \quad \Leftrightarrow \quad \max(X, Y) = Z \quad (\epsilon \to +0), \quad x = e^{X/\epsilon}, \ y = e^{Y/\epsilon}, z = e^{Z/\epsilon}, \quad \text{etc.}$$

Many discrete integrable maps have been shown to remain integrable after this procedure[2].

1.4 Behavior of Periodic Points

We are interested in the behavior of periodic points of a rational map or an ADE. In fact we are going to show that periodic points characterize an integrable system. On the other hand the Julia set, which was introduced in §1.2.7, can be also defined as "the closure of the set of repelling periodic points"[10]. In other words, an analysis of periodic points should provide a key to discriminate integrable systems from non integrable ones.

In this section, we assume that the map F is d dimensional rational map or ADE with p invariants $\boldsymbol{H}(\boldsymbol{x}^t) \in (\mathbb{C}[\boldsymbol{x}^t])^p$ [5] .

1.4.1 Fixed Points and Periodic Points

If a point $\boldsymbol{x} \in \mathbb{C}^d$ satisfies the condition,

$$\boldsymbol{x} = F(\boldsymbol{x}) \ \Rightarrow \ [F - \mathrm{id}]\,(\boldsymbol{x}) = 0,$$

the point $\boldsymbol{x} \in \mathbb{C}^d$ is called a fixed point of the map F. The fixed points of $F^{(n)}$ are called n periodic points of the map F, if they are not periodic points of divisors of n.

---

[5]Here $\mathbb{C}[\boldsymbol{x}^t]$ is a polynomial ring for the variable $\boldsymbol{x}^t$[29].
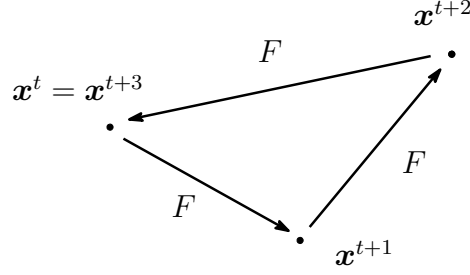
Figure 1.3: Points of period 3 of the map F

1.4.2 Invariant Variety of Periodic Points(IVPP) and IVPP Theorem

A "variety" of periodic points(VPP) of period n of the map F is defined by the conditions

$$\left[F^{(n)} - \mathrm{id}\right](\boldsymbol{x}) = 0 \quad \Rightarrow \quad \bar{\Gamma}^{(n)}_{j_n}(\boldsymbol{y}, \boldsymbol{h}) = 0, \quad \boldsymbol{y} \in \mathbb{C}^{d-p}, \quad \boldsymbol{h} \in \mathbb{C}^p, \quad j_n = 1, \ldots, J_n,$$

where $\boldsymbol{y} \in \mathbb{C}^{d-p}$ is the variable of the level set $\boldsymbol{H}(\boldsymbol{x}) = \boldsymbol{h} \in \mathbb{C}^p$, and $J_n \leq d - p$.
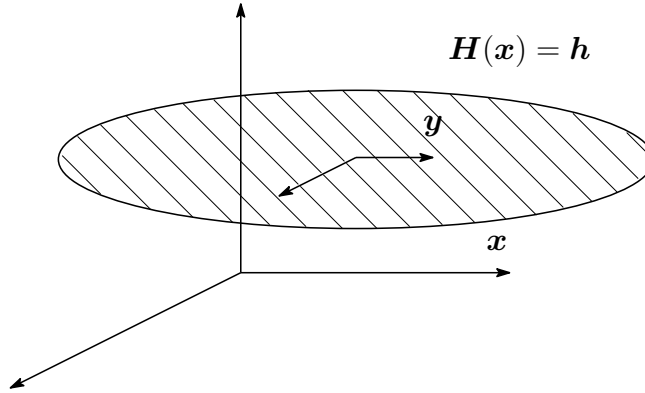
Figure 1.4: Variable $\boldsymbol{y}$ on level set $\boldsymbol{H}(\boldsymbol{x}) = \boldsymbol{h}$

A VPP of period n has some cases as follows[7][8][9]:

- Uncorrelated (UC-VPP):

$$\Gamma_{k_n}^{(n)}(\boldsymbol{y}, \boldsymbol{h}) = 0, \quad k_n = 1, \ldots, K_n, \quad K_n = J_n,$$

- Correlated (C-VPP):

$$\Gamma_{k_n}^{(n)}(\boldsymbol{y}, \boldsymbol{h}) = 0, \quad \gamma_{l_n}^{(n)}(\boldsymbol{h}) = 0, \quad k_n = 1, \ldots, K_n, \quad l_n = 1, \ldots, L_n, \quad K_n + L_n = J_n,$$

- Full Correlated (FC-VPP):

$$\gamma_{l_n}^{(n)}(\boldsymbol{h}) = 0, \quad l_n = 1, \ldots, L_n, \quad L_n = J_n.$$

where $\Gamma_{k_n}^{(n)}(\boldsymbol{y}, \boldsymbol{h})$ is a function of the variable $\boldsymbol{y}$ on the level set and the invariants $\boldsymbol{h}$, while $\gamma_{l_n}^{(n)}$ is a function of the invariants $\boldsymbol{h}$ only.

A FC-VPP is given by only information of the invariants, thus it is also called an invariant variety of periodic points(IVPP)[7][8][9].

We should give a small remark about the dimension of a VPP. In the case $K_n = d - p$, the dimension of the UC-VPP on a level set is 0, and in the case $L_n = d - p$, the dimension of the IVPP on a level set is p. Therefore, in the case of maximal, the UC-VPP is a set of discrete points and the IVPP is a set of continuous points.
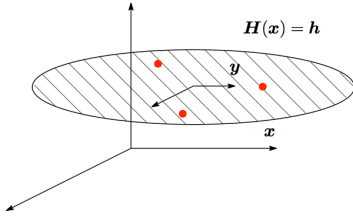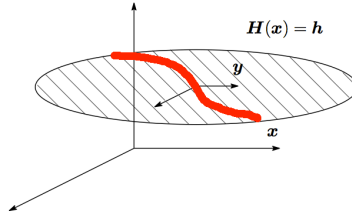
Figure 1.5: UC-VPP   Figure 1.6: C-VPP   Figure 1.7: FC-VPP
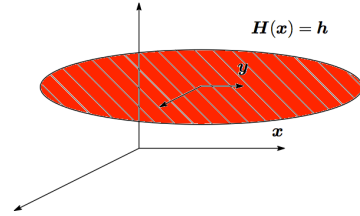
IVPP Theorem

An IVPP has an important property that is called IVPP theorem[7][8][9].

First, we introduce an "axiom" as follows,

> ⌐ Axiom ───
>
>   VPPs of different periodicity have no intersection on "generic points".
>
> └──

Where, "generic points" will be defined in Chap.2. Based on this axiom, we can prove IVPP theorem as follows,

> ⌐ IVPP Theorem ──
>
>   Let F be a d dimensional "rational map" with p invariants. If $p \geq d/2$, an IVPP
>
>   and a UC-VPP of any period do not exist in one map, simultaneously.
>
> └──

By IVPP theorem, a rational map that has the IVPP can be analyzed by the variables on the level set and the parameters correspondent to the invariants.

1.4.3 IVPP/Julia Set and Integrable/Non Integrable System

A Julia set has definitions in many ways [6] . In particular, the following definition is convenient for our argument,

$$J(F) := \overline{\bigcup_n \{\text{"repelling" periodic points of period } n\}} \subset \overline{\bigcup_n \text{Period}(F, n)}.$$

Therefore, the Julia set is characterized by periodic points.

───

[6]In general, some definitions are not equivalent for a higher dimensional map.

In general, an IVPP is a continuum set that has integer dimension, but the Julia set is a fractal set[30] that has fractal dimension of non integer [7].

Therefore, as a corollary of the IVPP theorem we conjecture as follows,

---
IVPP/Julia Set Conjecture

If a map F has an IVPP/Julia set then the map F can't have a Julia set/IVPP.

---

and

Integrable/Non Integrable Conjecture

If a map F has an IVPP/Julia set then the map F is Integrable/Non Integrable.

Thus the existence or nonexistence of the IVPP/Julia set of the map F is a test for the decision of Integrable/Non Integrable system. One of researches about this problem is the transition of Integrable/Non Integrable system[8].

1.4.4 Recurrence Equation(RE)

If a VPP of period n of a map F is $\mathbb{C}^d$, then the map F is called an n periodic recurrence equation(RE)[31][32]. Of course, a RE is an integrable system.

Now, we give some examples of the RE[31][32],

- 2 periodic RE:
$$x^t \mapsto x^{t+1} = \frac{a}{x^t}, \quad {}^\forall a \in \mathbb{C} \setminus \{0\},$$

- 5 periodic RE:
$$(x^{t-1}, x^t) \mapsto x^{t+1} = \frac{1 + x^t}{x^{t-1}},$$

---
[7]We need some attention for this statement, because, an IVPP is for each period, while a Julia set is for all.

- 8 periodic RE:
$$(x^{t-2}, x^{t-1}, x^t) \mapsto x^{t+1} = \frac{1 + x^{t-1} + x^t}{x^{t-2}}.$$

They are REs of period 2, period 5 and period 8, respectively, which have been given in [31]. Yahagi and Hirota have derived many other examples[32].

Furthermore in ref. [33] a method was developed to derive infinitely many such mappings associated with IVPP.

$$F_i(\boldsymbol{x}^t, \boldsymbol{x}^{t+1}) = 0, \quad \gamma_j^{(n)}(\boldsymbol{H}(\boldsymbol{x}^t)) = 0, \quad i = 1, \ldots, d, \quad j = 1, \ldots, d - p$$
$$\Rightarrow \quad \tilde{F}_{n,k}(\boldsymbol{z}^t, \boldsymbol{z}^{t+1}) = 0, \quad \boldsymbol{z} := (x_1, \ldots, x_p) \in \mathbb{C}^p, \quad k = 1, \ldots, p,$$

This RE is period n and p dimensional map with $2p - d$ invariants.

1.5 Summaries of This Thesis

We give summaries of our thesis in this section.

In Chap.2 we first give definitions of a rational map, an ADE and some notations. An ADE is a generalization of a rational map. Since an ADE is multivalued in general, the time evolutions of ADE becomes implicit. We will overcome this difficulty, however, by using the notion of elimination ideal[34].

Compared with a rational map, an ADE has good properties as follows

- Invariant/Parameter Duality: By IVPP theorem, it is a good idea to restrict the map to a level set. In general, a rational map restricted on the level set is not a rational map, but an ADE remains the same.

- IVPP/RE Duality: Similarly, a rational map on an IVPP is not a rational map, while an ADE remains the same.

In other words, we think that the category of rational map is not available, and the category of ADE is available. It is the reason that we investigate an ADE in addition to a rational map.

We extend, in this chapter, the IVPP theorem, derived for a rational map, to an ADE. In the derivation of the IVPP theorem[8], an axiom is introduced such that "VPPs of different periodicity have no intersection on generic points". As we extend the theorem to an ADE, we are to investigate relations of the intersections of IVPPs and the "indeterminate points(IDP)" of ADE.

In Chap.3 we study the Invariant/Parameter duality. Upon elimination of p independent variables from a rational map by using p invariants, we will find an ADE. This enables us to investigate explicitly the relations between a rational map and an ADE, which we have just discussed above.

In Chap.4 we discuss a method to derive IVPPs iteratively from a singular point of a rational map. This is possible when the singularity is confined and the point is specified only by the invariants of the map.

We deepen this idea in Chap 5. As we discussed in §1.1 from the view of category theory, an integrable system is a category in which objects are the states and the functor is the change of the states. Keeping this correspondence in mind we will show that the generation of IVPPs from the singularity can be associated with the "projective resolution" of the "triangulated category".

In Chap.6 we discuss about a transition of Integrable/Non Integrable systems. This problem has been studied intensively in the literature. The transition to a non integrable system was investigated perturbatively. An important result is well known as the fixed point theorem[35][36]. It tells us that the perturbation causes

a production of a number of unstable periodic points in the neighborhood of every periodic point of the original map.

In order to understand this phenomenon in our terminology, we deform one of integrable system by introducing a parameter. We can specify an algebraic curve along which a periodic point of any period moves as the parameter changes. From the fixed point theorem we expect that all such points approach some points on IVPPs in the integrable limit. In fact some part of the Julia set approach IVPPs. But a large part of them move to the singular points of the map, instead of IVPPs. We emphasize that this phenomenon is difficult to find by a traditional perturbative method.

Chapter 2

IVPP Theorem and Intersections of VPPs

In this chapter, we give our set up and extension of the IVPP theorem for a rational map to an algebraic difference equation(ADE). In addition, we consider carefully the conditions for the intersections of VPPs. We will find that they are on indeterminate points(IDP) of the map.

2.1 Rational Map

2.1.1 Rational Map

A d dimensional rational map $F : \boldsymbol{x}^t \mapsto \boldsymbol{x}^{t+1}$ is defined by

$$F : \boldsymbol{x}^t := (x_1^t, x_2^t, \ldots, x_d^t) \mapsto \boldsymbol{x}^{t+1} := (x_1^{t+1}, x_2^{t+1}, \ldots, x_d^{t+1}), \quad \boldsymbol{x}^t, \boldsymbol{x}^{t+1} \in \mathbb{C}^d,$$

$$x_i^{t+1} := \frac{N_i(\boldsymbol{x}^t)}{D_i(\boldsymbol{x}^t)}, \quad N_i(\boldsymbol{x}^t), D_i(\boldsymbol{x}^t) \in \mathbb{C}[\boldsymbol{x}^t], \quad i = 1, \ldots, d,$$

where each pair of the denominator $D_i(\boldsymbol{x}^t)$ and the numerator $N_i(\boldsymbol{x}^t)$ are coprime for all $i = 1, \ldots, d$. The time evolution of the rational map F is given by an iteration

of the map,

$$F^{(2)} : \boldsymbol{x}^t \mapsto \boldsymbol{x}^{t+2} = F^{(2)}(\boldsymbol{x}^t) := F \circ F(\boldsymbol{x}^t),$$

$$x_i^{t+2} := \frac{N_i^{(2)}(\boldsymbol{x}^t)}{D_i^{(2)}(\boldsymbol{x}^t)}, \quad N_i^{(2)}(\boldsymbol{x}^t), D_i^{(2)}(\boldsymbol{x}^t) \in \mathbb{C}[\boldsymbol{x}^t], \quad i = 1, \ldots, d.$$

In general, $T \in \mathbb{N}$ times evolution of the map F is given by T iterations of the map,

$$F^{(T)} : \boldsymbol{x}^t \mapsto \boldsymbol{x}^{t+T} = F^{(T)}(\boldsymbol{x}^t) := \overbrace{F \circ F \circ \cdots \circ F}^{T}(\boldsymbol{x}^t),$$

$$x_i^{t+T} := \frac{N_i^{(T)}(\boldsymbol{x}^t)}{D_i^{(T)}(\boldsymbol{x}^t)}, \quad N_i^{(T)}(\boldsymbol{x}^t), D_i^{(T)}(\boldsymbol{x}^t) \in \mathbb{C}[\boldsymbol{x}^t], \quad i = 1, \ldots, d,$$

A rational map F is called a bi-rational map, when the inverse map of F is also a rational map,

$$
\begin{aligned}
F^{(-1)} \quad : \quad & \boldsymbol{x}^t := (x_1^t, x_2^t, \ldots, x_d^t) \\
& \mapsto \boldsymbol{x}^{t-1} := (x_1^{t-1}, x_2^{t-1}, \ldots, x_d^{t-1}), \quad \boldsymbol{x}^t, \boldsymbol{x}^{t-1} \in \mathbb{C}^d,
\end{aligned}
$$

$$x_i^{t-1} := \frac{N_i^{(-1)}(\boldsymbol{x}^t)}{D_i^{(-1)}(\boldsymbol{x}^t)}, \quad N_i^{(-1)}(\boldsymbol{x}^t), D_i^{(-1)}(\boldsymbol{x}^t) \in \mathbb{C}[\boldsymbol{x}^t], \quad i = 1, \ldots, d.$$

All our examples in this work are bi-rational maps.

2.1.2 Singular Points(SP) and Indeterminate Points(IDP)

A rational map F has "singular points(SP)" that are zero points of the denominators,

$$\mathrm{SP}(F, i) := \left\{ \boldsymbol{x} \in \mathbb{C}^d \mid D_i(\boldsymbol{x}) = 0 \right\}, \quad i = 1, \ldots, d, \tag{2.1}$$

$$\mathrm{SP}(F) := \bigcup_{i=1}^{d} \mathrm{SP}(F, i). \tag{2.2}$$

In addition, if there exist, at the same time, zero points of the numerators $N_i(\boldsymbol{x})$ and the denominators $D_i(\boldsymbol{x})$, then they are called "indeterminate points(IDP)",

$$\mathrm{IDP}(F, i) := \left\{ \boldsymbol{x} \in \mathbb{C}^d \mid D_i(\boldsymbol{x}) = 0, N_i(\boldsymbol{x}) = 0 \right\}, \quad i = 1, \ldots, d, \tag{2.3}$$

$$\mathrm{IDP}(F) := \bigcup_{i=1}^{d} \mathrm{IDP}(F, i). \tag{2.4}$$

Of course, we notice immediately the relation,

$$\mathrm{IDP}(F) \subset \mathrm{SP}(F).$$

In this thesis, we also discuss about an algebraic difference equation(ADE). We might think the ADE

$$\tilde{F}_i(\boldsymbol{x}^t, \boldsymbol{x}^{t+1}) := x_i^{t+1} D_i(\boldsymbol{x}^t) - N_i(\boldsymbol{x}^t) = 0, \quad N_i(\boldsymbol{x}^t), D_i(\boldsymbol{x}^t) \in \mathbb{C}[\boldsymbol{x}^t], \quad i = 1, \ldots, d,$$

as an "implicitization"[34] of the rational map (2.1.1). Similarly the implicitization of the inverse map $F^{(-1)}$ is given by

$$\tilde{F}_i^{(-1)}(\boldsymbol{x}^t, \boldsymbol{x}^{t-1}) := x_i^{t-1} D_i^{(-1)}(\boldsymbol{x}^t) - N_i^{(-1)}(\boldsymbol{x}^t) = 0,$$

$$N_i^{(-1)}(\boldsymbol{x}^t), D_i^{(-1)}(\boldsymbol{x}^t) \in \mathbb{C}[\boldsymbol{x}^t], \quad i = 1, \ldots, d.$$

The implicitization of a rational map is often more convenient than the rational map, because we can apply some algebraic methods (*e.g.* Gröbner basis) to the former.

Finally we notice that the IDP of an implicitization of a rational map is determined by the implicit function theorem[37], as follows:

$$\mathrm{IDP}(\tilde{F}, i) := \left\{ \boldsymbol{x} \in \mathbb{C}^d \mid \partial_{x_i'} \tilde{F}_i(\boldsymbol{x}, \boldsymbol{x}') = D_i(\boldsymbol{x}) = 0 \right\} = \mathrm{IDP}(F, i).$$

Now we use the fact about the implicitization of a rational map such that if a denominator $D_i(\boldsymbol{x})$ is zero, then $N_i(\boldsymbol{x})$ is also zero.

2.1.3 Fixed Points and IDP

If a point $\boldsymbol{x} \in \mathbb{C}^d$ satisfies the condition,

$$\tilde{F}_i(\boldsymbol{x}, \boldsymbol{x}) = x_i D_i(\boldsymbol{x}) - N_i(\boldsymbol{x}) = 0, \quad i = 1, \ldots, d,$$

the point $\boldsymbol{x} \in \mathbb{C}^d$ is called a fixed point of the implicitization of the rational map F.

If $D_{i'}(\boldsymbol{x}) = 0, i' \in \{1, \ldots, d\}$ then $N_{i'}(\boldsymbol{x}) = 0$, while if $D_i(\boldsymbol{x}) \neq 0, i = 1, \ldots, d$, $\boldsymbol{x} \in \mathbb{C}^d$ satisfies the condition,

$$x_i = \frac{N_i(\boldsymbol{x})}{D_i(\boldsymbol{x})}, \quad i = 1, \ldots, d.$$

In this case the point $\boldsymbol{x} \in \mathbb{C}^d$ is called a fixed point of the rational map F. Therefore the set of fixed points

$$\mathrm{FP}(\tilde{F}) := \left\{ \boldsymbol{x} \in \mathbb{C}^d \ \middle| \ \tilde{F}_i(\boldsymbol{x}, \boldsymbol{x}) = x_i D_i(\boldsymbol{x}) - N_i(\boldsymbol{x}) = 0, \ i = 1, \ldots, d \right\}, \qquad (2.5)$$

and

$$\mathrm{FP}(F) := \left\{ \boldsymbol{x} \in \mathbb{C}^d \ \middle| \ x_i = \frac{N_i(\boldsymbol{x})}{D_i(\boldsymbol{x})}, \ i = 1, \ldots, d \right\},$$

satisfy the relation

> **FP/IDP Conjecture**
>
> $$\mathrm{FP}(\tilde{F}) = \mathrm{FP}(F) \cup \mathrm{IDP}(F). \qquad (2.6)$$

This is an important relation, but is not completely proved yet. Therefore, this relation is a conjecture, at this moment.

2.1.4 Example

Rational Map

In this thesis, we often use the 2 dimensional Möbius map, defined by

$$F_{\mathrm{2dM\ddot{o}b}} : (x_1^t, x_1^t) \mapsto (x_1^{t+1}, x_2^{t+1}) = \left(x_1^t \frac{1 - x_2^t}{1 - x_1^t}, x_2^t \frac{1 - x_1^t}{1 - x_2^t} \right). \qquad (2.7)$$

It has an inverse map

$$F_{\text{2dMöb}}^{(-1)} : (x_1^t, x_1^t) \mapsto (x_1^{t-1}, x_2^{t-1}) = \left(x_1^t \frac{1 + x_2^t}{1 + x_1^t}, x_2^t \frac{1 + x_1^t}{1 + x_2^t} \right),$$

that is also a rational map. Therefore the 2 dimensional Möbius map is a bi-rational map. This is one of the most useful and convenient examples, which enables us to check the validity of our argument.

Implicitization of Rational Map

The implicitization of the 2 dimensional Möbius map is

$$(\tilde{F}_{\text{2dMöb}})_i(\boldsymbol{x}^t, \boldsymbol{x}^{t+1}) = x_i^{t+1}(1 - x_i^t) - x_i^t(1 - x_{i+1}^t) = 0, \quad i \in \mathbb{Z}/2\mathbb{Z}.$$

Time Evolution

The time evolution of the 2 dimensional Möbius map proceeds as

$$
\begin{aligned}
(x_1^t, x_2^t) \;\to\;& \left(x_1^t \frac{1 - x_2^t}{1 - x_1^t}, x_2^t \frac{1 - x_1^t}{1 - x_2^t} \right) \\
\to\;& \left(x_1^t \frac{1 - 2x_2^t + x_1^t x_2^t}{1 - 2x_1^t + x_1^t x_2^t}, x_2^t \frac{1 - 2x_1^t + x_1^t x_2^t}{1 - 2x_2^t + x_1^t x_2^t} \right) \to \dots \text{etc.}
\end{aligned}
$$

SP and IDP

The SP and IDP of the 2 dimensional Möbius map are given by

$$
\begin{aligned}
\text{SP}(F_{\text{2dMöb}}, 1) &= \left\{ (1, x_2) \in \mathbb{C}^2 \;\middle|\; {}^{\forall} x_2 \in \mathbb{C} \right\}, \\
\text{SP}(F_{\text{2dMöb}}, 2) &= \left\{ (x_1, 1) \in \mathbb{C}^2 \;\middle|\; {}^{\forall} x_1 \in \mathbb{C} \right\},
\end{aligned}
$$

$$\text{IDP}(F_{\text{2dMöb}}, 1) = \text{IDP}(F_{\text{2dMöb}}, 2) = \{(1, 1)\}.$$

Fixed Point and IDP

The set of fixed points of the 2 dimensional Möbius map are given by

$$\text{FP}(F_{\text{2dMöb}}) = \left\{ \, (x, x) \in \mathbb{C}^2 \mid x \neq 1, \ x \in \mathbb{C} \, \right\},$$

$$\text{FP}(\tilde{F}_{\text{2dMöb}}) = \left\{ \, (x, x) \in \mathbb{C}^2 \mid x \in \mathbb{C} \, \right\} = \text{FP}(F_{\text{2dMöb}}) \cup \text{IDP}(F_{\text{2dMöb}}).$$

2.2 Algebraic Difference Equation(ADE)

2.2.1 ADE

A d dimensional algebraic difference equation(ADE) $F : \boldsymbol{x}^t \mapsto \boldsymbol{x}^{t+1}$ is defined by

$$\tilde{F}_i(\boldsymbol{x}^t, \boldsymbol{x}^{t+1}) = 0, \quad \tilde{F}_i(\boldsymbol{x}^t, \boldsymbol{x}^{t+1}) \in \mathbb{C}[\boldsymbol{x}^t, \boldsymbol{x}^{t+1}], \quad i = 1, \ldots, d. \tag{2.8}$$

2.2.2 Time Evolution of ADE

In contrast to the rational map case, the point $\boldsymbol{x}^{t+1}$ after the time evolution from the initial point $\boldsymbol{x}^t$ is difficult to determine in the ADE case. Therefore, we need a new way to study the time evolution of an ADE.

Now, we define an ideal [1] form of the ADE $\tilde{F}$(IADE) by

$$I_{\tilde{F}}^{t,t+1}(\boldsymbol{x}^t, \boldsymbol{x}^{t+1}) := \left(\tilde{F}_1(\boldsymbol{x}^t, \boldsymbol{x}^{t+1}), ..., \tilde{F}_d(\boldsymbol{x}^t, \boldsymbol{x}^{t+1}) \right). \tag{2.9}$$

Thus, we define an ADE $\tilde{F}^{(2)}$ or an IADE $I_{\tilde{F}}^{t,t+2}(\boldsymbol{x}^t, \boldsymbol{x}^{t+2})$ that is a time evolution of the ADE $\tilde{F}$, by the elimination ideal,

$$I_{\tilde{F}}^{t,t+2}(\boldsymbol{x}^t, \boldsymbol{x}^{t+2}) := \text{Gb}\left(I_{\tilde{F}}^{t,t+1}(\boldsymbol{x}^t, \boldsymbol{x}^{t+1}), I_{\tilde{F}}^{t+1,t+2}(\boldsymbol{x}^{t+1}, \boldsymbol{x}^{t+2}) \right) \cap \mathbb{C}[\boldsymbol{x}^t, \boldsymbol{x}^{t+2}],$$

---

[1] In Appendix A, we give a very short explanation of affine algebraic variety and this notation.

where Gb $\left(I_{\tilde{F}}^{t,t+1}(\boldsymbol{x}^t, \boldsymbol{x}^{t+1}), I_{\tilde{F}}^{t+1,t+2}(\boldsymbol{x}^{t+1}, \boldsymbol{x}^{t+2}) \right)$ represents the Gröbner basis [2] [34][39] of the ordering $\boldsymbol{x}^{t+2} > \boldsymbol{x}^t > \boldsymbol{x}^{t+1}$. In general, we can define the T-th time evolution of the IADE $I_{\tilde{F}}^{t,t+T}(\boldsymbol{x}^t, \boldsymbol{x}^{t+T})$ recursively.

2.2.3 Time Evolution and "Exact Sequence"

Proposition

The IADE $I_{\tilde{F}}^{t,t+1}(\boldsymbol{x}^t, \boldsymbol{x}^{t+1})$ *of a bi-rational map* F *and the* IADE $I_{\tilde{F}^{(-1)}}^{t,t+1}(\boldsymbol{x}^{t+1}, \boldsymbol{x}^t)$ *of the inverse map* $F^{(-1)}$ *are equivalent* [3] :

$$I_{\tilde{F}}^{t,t+1}(\boldsymbol{x}^t, \boldsymbol{x}^{t+1}) \sim I_{\tilde{F}^{(-1)}}^{t,t+1}(\boldsymbol{x}^{t+1}, \boldsymbol{x}^t). \tag{2.10}$$

Proof: The bi-rational map is bijective. Thus, a bi-rational map F gives the same points solved by both of the variables $\boldsymbol{x}^t$ and $\boldsymbol{x}^{t+1}$.

Lemma 1

In this thesis, we assume [4] that the number of bases of the IADE $I_{\tilde{F}}^{t,t+2}(\boldsymbol{x}^t, \boldsymbol{x}^{t+2})$ is d, and these bases are represented in the form $\tilde{F}_i^{(2)}(\boldsymbol{x}^t, \boldsymbol{x}^{t+2}), i = 1, \ldots, d$. Therefore, the relation,

$$I_{\tilde{F}}^{t,t+2}(\boldsymbol{x}^t, \boldsymbol{x}^{t+2}) \sim I_{\tilde{F}^{(2)}}^{t,t+1}(\boldsymbol{x}^t, \boldsymbol{x}^{t+2}), \tag{2.11}$$

is satisfied.

---

[2] The Gröbeer basis is a possible basis which generates the ideal given by some algorithm e.g. Buchberger's algorithm[39].

[3] This equivalence is defined as (A.1) in Appendix A.

[4] This assumption has some difficulty, in the sense that we need "minimal Gröbner basis"[40] for a map chosen by usual Gröbner basis.

Lemma 2

From the nature of the IADE, it has an important property:

$$I_{\tilde{F}}^{t,t'}(\boldsymbol{x}^t, \boldsymbol{x}^{t'}) + I_{\tilde{F}}^{t',t''}(\boldsymbol{x}^{t'}, \boldsymbol{x}^{t''}) = I_{\tilde{F}}^{t,t'}(\boldsymbol{x}^t, \boldsymbol{x}^{t'}) + I_{\tilde{F}}^{t,t''}(\boldsymbol{x}^t, \boldsymbol{x}^{t''}). \qquad (2.12)$$

Lemma 3

From the Lem. 2, we find an "exact sequence":

$$
\begin{array}{ccc}
I_{\tilde{F}}^{\bullet} := I_{\tilde{F}}^{0,1}(\boldsymbol{x}^0, \boldsymbol{x}^1) & \longrightarrow & I_{\tilde{F}}^{0,1}(\boldsymbol{x}^0, \boldsymbol{x}^1) + I_{\tilde{F}}^{1,2}(\boldsymbol{x}^1, \boldsymbol{x}^2) \\
\searrow & & \downarrow \\
& I_{\tilde{F}}^{0,1}(\boldsymbol{x}^0, \boldsymbol{x}^1) + I_{\tilde{F}}^{0,2}(\boldsymbol{x}^0, \boldsymbol{x}^2) & \longrightarrow \quad \cdots \\
& & \searrow \quad \cdots
\end{array}
$$

or, equivalently,

$$
\begin{aligned}
I_{\tilde{F}}^{\bullet} := I_{\tilde{F}}^{0,1}(\boldsymbol{x}^0, \boldsymbol{x}^1) \quad \to \quad & I_{\tilde{F}}^{0,1}(\boldsymbol{x}^0, \boldsymbol{x}^1) + I_{\tilde{F}}^{0,2}(\boldsymbol{x}^0, \boldsymbol{x}^2) \\
& \to I_{\tilde{F}}^{0,2}(\boldsymbol{x}^0, \boldsymbol{x}^2) + I_{\tilde{F}}^{0,3}(\boldsymbol{x}^0, \boldsymbol{x}^3) \to \cdots \qquad (2.13)
\end{aligned}
$$

From this argument we have the following result:

Proposition

An iteration of the ADE can be identified with an "exact sequence" (2.13) of the IADE up to "common factors".

Remark

We notice that $I_{\tilde{F}}^{\bullet}$ is not a completely "exact sequence", because these terms have common factors, *e.g.* fixed points. Thus we should remove some parts from $I_{\tilde{F}}^{\bullet}$ to make it exact.

This indeterminacy by common factors produces a delicate problem. In chapter 6, we discuss about this problem as null system of "triangulated category" in the case of Hirota-Miwa equation(HM eq.).

2.3   IVPP of ADE

The most important object of this thesis is an invariant variety of periodic points(IVPP). In this section, we define the IVPP of ADE.

Invariant

An invariant $H(\boldsymbol{x}^t) \in \mathbb{C}[\boldsymbol{x}^t]$ of an ADE $\tilde{F}$ is defined to satisfy the relation,

$$H(\boldsymbol{x}^t) = H(\boldsymbol{x}^{t+1}(\boldsymbol{x}^t)), \quad H(\boldsymbol{x}^t) \in \mathbb{C}[\boldsymbol{x}^t].$$

We assume that an ADE $\tilde{F}$ has p invariants $\boldsymbol{H}(\boldsymbol{x}^t) := (H_1(\boldsymbol{x}^t), \ldots, H_p(\boldsymbol{x}^t))$ [5] .

Level Set

A level set of an ADE $\tilde{F}$ constrained by the parameter set $\boldsymbol{h} \in \mathbb{C}^p$ is defined as,

$$\mathrm{LS}(\tilde{F}, \boldsymbol{h}) := \left\{\, \boldsymbol{x} \in \mathbb{C}^d \,\middle|\, \boldsymbol{H}(\boldsymbol{x}) - \boldsymbol{h} = 0 \,\right\}. \tag{2.14}$$

VPP and IVPP

A variety of periodic points(VPP) of period $n \geq 2$ of an ADE $\tilde{F}$ is defined by

$$\mathrm{Period}(\tilde{F}, n) := \left\{\, \boldsymbol{x} \in \mathbb{C}^d \,\middle|\, \tilde{F}_i^{(n)}(\boldsymbol{x}, \boldsymbol{x}) = 0, \tilde{F}_i^{(m)}(\boldsymbol{x}, \boldsymbol{x}) \neq 0, n|m, i = 1, \ldots, d \,\right\}.$$
$$\tag{2.15}$$

---

[5]In general, invariants may not be polynomials, but we assume this, in this work, for convenience.

We notice a relation

$$\text{Period}(F, n) = \text{FP}(F^{(n)}) \setminus \left(\bigcup_{n|m} \text{FP}(F^{(m)}) \right), \qquad (2.16)$$

where $n|m$ means that m is a divisor of n. Hence a $\text{Period}(\tilde{F}, n)$ of rational map F satisfies

---

Period/IDP Proposition

$$\text{Period}(\tilde{F}, n) = \text{Period}(F, n) \cup \text{IDP}(F^{(n)}) \qquad (2.17)$$

---

by FP/IDP conjecture. In general, the condition for VPP of period n on the level set $\text{LS}(F, \boldsymbol{h})$

$$\bar{\Gamma}^{(n)}_{j_n}(\boldsymbol{y}, \boldsymbol{h}) = 0, \quad j_n = 1, \dots, J_n, \qquad (2.18)$$

is given by the following "mixed conditions"

$$\Gamma^{(n)}_{k_n}(\boldsymbol{y}, \boldsymbol{h}) = 0, \quad k_n = 1, \dots, K_n, \qquad (2.19)$$

$$\gamma^{(n)}_{l_n}(\boldsymbol{h}) = 0, \quad l_n = 1, \dots, L_n, \qquad (2.20)$$

where $\boldsymbol{y} \in \mathbb{C}^{d-p}$ is a variable on the level set, and $J_n = K_n + L_n \leq d - p$.

- If $L_n = 0$ then a VPP of period n is called "uncorrelated(UC)-VPP".

- If $K_n \neq 0, L_n \neq 0$ then a VPP of period n is called "correlated(C)-VPP".

- If $K_n = 0$ then a VPP of period n is called "full correlated(FC)-VPP".

A FC-VPP is also called an IVPP. The IVPP also can be represented as,

$$\text{Period}(F, n) := \bigcup_{\boldsymbol{h} \in \left\{ \boldsymbol{h} \in \mathbb{C}^p \,\middle|\, \gamma^{(n)}_{l_n}(\boldsymbol{h}) = 0, \ l_n = 1, \dots, L_n \right\}} \text{LS}(F, \boldsymbol{h}).$$

In sum, any point on the level set $\text{LS}(F, \boldsymbol{h})$ for the parameter set $\left\{ \boldsymbol{h} \in \mathbb{C}^p \,\middle|\, \gamma^{(n)}_{l_n}(\boldsymbol{h}) = 0, \ l_n = 1, \dots, L_n \right\}$, is an n periodic point.

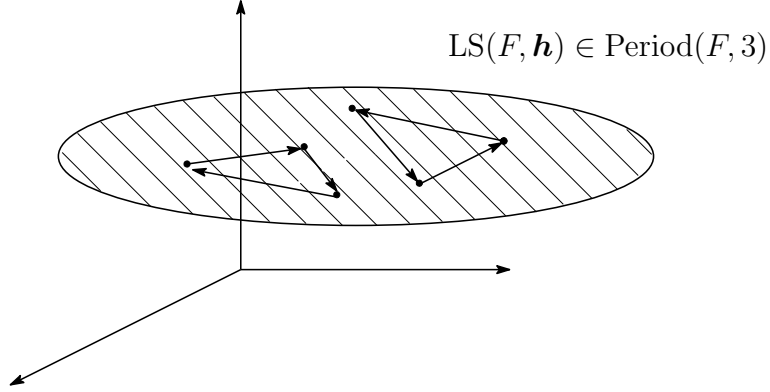$$\mathrm{LS}(F, \boldsymbol{h}) \in \mathrm{Period}(F, 3)$$

Figure 2.1: An IVPP of period 3 of the map F

2.3.1 Example

For an example, we use again the 2 dimensional Möbius map (2.7).

Invariant

The invariant $H(\boldsymbol{x}^t)$ of the 2 dimensional Möbius map is given

$$H(\boldsymbol{x}^{t+1}(\boldsymbol{x}^t)) = x_1^{t+1}(\boldsymbol{x}^t)x_2^{t+1}(\boldsymbol{x}^t) = \left(x_1^t \frac{1 - x_2^t}{1 - x_1^t}\right) \cdot \left(x_2^t \frac{1 - x_1^t}{1 - x_2^t}\right) = x_1^t x_2^t. \qquad (2.21)$$

VPPs

- $\mathrm{Period}(F_{\mathrm{2dMöb}}, 2) = \emptyset,$

- $\mathrm{Period}(F_{\mathrm{2dMöb}}, 3) = \{\, \boldsymbol{x} \in \mathbb{C}^2 \mid x_1 x_2 + 3 = H(\boldsymbol{x}) + 3 = 0 \,\},$

- $\mathrm{Period}(F_{\mathrm{2dMöb}}, 4) = \{\, \boldsymbol{x} \in \mathbb{C}^2 \mid x_1 x_2 + 1 = H(\boldsymbol{x}) + 1 = 0 \,\},$

- $\mathrm{Period}(F_{\mathrm{2dMöb}}, 5) = \{\, \boldsymbol{x} \in \mathbb{C}^2 \mid (x_1 x_2)^2 + 10 x_1 x_2 + 5 = (H(\boldsymbol{x}))^2 + 10 H(\boldsymbol{x}) + 5 = 0 \,\},$
  etc.

Therefore VPPs of the 2 dimensional Möbius map are IVPPs. We give a picture of IVPPs of 2 dimensional Möbius map.

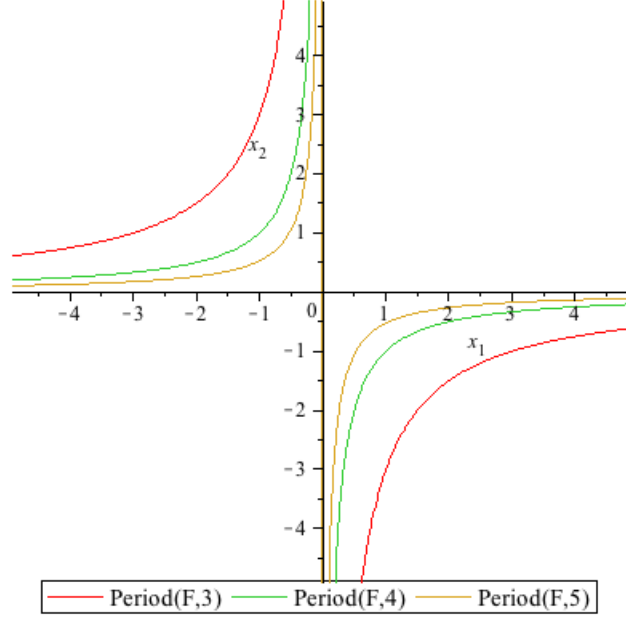Figure 2.2: IVPPs of 2 dimensional Möbius map

2.4   IVPP Theorem

The IVPP theorem for a rational map is also true for an ADE by a similar proof.

First, one more time, we state the "axiom" as follows,

> **Axiom**
>
> VPPs of different periodicity have no intersection on "generic points".

Where, "generic points" mean that are not on IDP. On this axiom, we can prove IVPP theorem as follows,

---
IVPP Theorem

Let $\tilde{F}$ be a d dimensional ADE with p invariants. If there exists $n \geq 2$ such that $p \geq J_n$, and a VPP of period n is an IVPP, then a VPP of period m is not an UC-VPP for any $m \geq 2$.

Proof:

By the assumption, the condition for the VPP being period n is given by only the parameter $\boldsymbol{h}$ of the invariants, and is written as

$$\gamma_{l_n}^{(n)}(\boldsymbol{h}) = 0, \quad l_n = 1, \ldots, L_n = J_n, \quad \boldsymbol{h} \in \mathbb{C}^p. \tag{2.22}$$

where $\boldsymbol{y}$ is a variable on $\mathrm{LS}(F, \boldsymbol{h})$. Notice that if we discuss on p dimensional invariant space, the condition (2.22) determines a $p - J_n$ dimensional variety in the p dimensional invariant space. In other words, there is no IVPP, in general, when $p \geq J_n$ is not satisfied.

Now, we assume that J_m conditions of period m are UC-VPP,

$$\Gamma_{k_m}^{(m)}(\boldsymbol{y}, \boldsymbol{h}) = 0, \quad k_m = 1, \ldots, K_m = J_m, \quad \boldsymbol{y} \in \mathbb{C}^{d-p}, \quad \boldsymbol{h} \in \mathbb{C}^p.$$

This condition is solvable for the parameter $\boldsymbol{h} \in \mathbb{C}^p$ of the IVPP of period n. It is impossible by the axiom. Therefore, we proved the statement.

$$(\mathrm{Q.E.D})$$

Remark

By this proof, we can find some propositions as follows.

- There is not FC-VPP of period n in general when $p \geq J_n$ is not satisfied.

- IVPP theorem is satisfied when the intersections of the VPPs are not on generic points.

In the next section, we discuss about the above propositions.

2.5   Conditions for Intersections of VPPs

In this section, we consider conditions for intersections of VPPs.

Let $\tilde{F}$ be a d dimensional ADE with p invariants. We assume that there exist $\alpha(=n, m) \geq 2$, such that the period α conditions on the level set,

$$\bar{\Gamma}_{j_\alpha}^{(\alpha)}(\boldsymbol{y}, \boldsymbol{h}) = 0, \quad j_\alpha = 1, \ldots, J_\alpha, \quad \boldsymbol{y} \in \mathbb{C}^{d-p}, \quad \boldsymbol{h} \in \mathbb{C}^p,$$

are written as

$$\Gamma_{k_\alpha}^{(\alpha)}(\boldsymbol{y}, \boldsymbol{h}) = 0, \quad k_\alpha = 1, \ldots, K_\alpha, \qquad \gamma_{l_\alpha}^{(\alpha)}(\boldsymbol{h}) = 0, \quad l_\alpha = 1, \ldots, L_\alpha,$$

$$\boldsymbol{y} \in \mathbb{C}^{d-p}, \quad \boldsymbol{h} \in \mathbb{C}^p,$$

where $\boldsymbol{y} \in \mathrm{C}^{d-p}$ are the variables on the level set and $J_\alpha = K_\alpha + L_\alpha \leq d - p$.

If $L_\alpha \leq p$ then we get periodic points of period α,

$$\boldsymbol{y}_\alpha(\boldsymbol{Y}_\alpha, \boldsymbol{h}_\alpha(\boldsymbol{H}_\alpha), \boldsymbol{H}_\alpha) \in \mathbb{C}^{K_\alpha}, \quad \boldsymbol{h}_\alpha(\boldsymbol{H}_\alpha) \in \mathbb{C}^{L_\alpha}, \quad \boldsymbol{Y}_\alpha \in \mathbb{C}^{d-p-K_\alpha}, \quad \boldsymbol{H}_\alpha \in \mathbb{C}^{p-L_\alpha},$$

where $\boldsymbol{Y}_\alpha \in \mathbb{C}^{d-p-K_\alpha}$ and $\boldsymbol{H}_\alpha \in \mathbb{C}^{p-L_\alpha}$ are free parameters.

In these assumptions, there can be some situations as follows,

- $K_n \geq K_m, L_n \geq L_m$, i.e. $J_n \geq J_m$.

- $K_n \geq K_m, L_n \leq L_m$.

- Alternative case of n and m.

In the case $K_n \geq K_m, L_n \geq L_m$:

$$(\hat{y}_m)_{k_m}(\boldsymbol{Y}_n, \boldsymbol{H}_n) = (\hat{y}_n)_{k_m}(\boldsymbol{Y}_n, \boldsymbol{H}_n), \quad k_m = 1, \ldots, K_m,$$

$$(\hat{h}_m)_{l_m}(\boldsymbol{H}_n) = (\hat{h}_n)_{l_m}(\boldsymbol{H}_n), \quad l_m = 1, \ldots, L_m,$$

$$\boldsymbol{Y}_n \in \mathbb{C}^{d-p-K_n}, \quad \boldsymbol{H}_n \in \mathbb{C}^{p-L_n}.$$

Hence we get

- $p \geq L_m + L_n \geq J_n + J_m - (d - p), \ (d \geq J_n + J_m)$: There exist intersections of the VPP of period n and the VPP of period m on the variables $\boldsymbol{y} \in \mathbb{C}^{d-p}$.

- $p \geq L_n + J_m$: There exist intersections of the VPP of period n and the VPP of period m on the parameters $\boldsymbol{h} \in \mathbb{C}^p$.

- $d \geq J_n + J_m$: There exist intersections of the VPP of period n and the VPP of period m.

In the case $K_n \geq K_m, L_n \leq L_m$:

$$(\hat{y}_m)_{k_m}(\boldsymbol{Y}_n, \boldsymbol{H}_m) = (\hat{y}_n)_{k_m}(\boldsymbol{Y}_n, \boldsymbol{H}_m), \quad k_m = 1, \ldots, K_m,$$

$$(\hat{h}_m)_{l_n}(\boldsymbol{H}_m) = (\hat{h}_n)_{l_n}(\boldsymbol{H}_m), \quad l_n = 1, \ldots, L_n,$$

$$\boldsymbol{Y}_n \in \mathbb{C}^{d-p-K_n}, \quad \boldsymbol{H}_m \in \mathbb{C}^{p-L_m}.$$

Hence we get

- $p \geq L_n + L_m \geq J_n + J_m - (d - p), \ (d \geq J_n + J_m)$: There exist intersections of the VPP of period n and the VPP of period m on the variables $\boldsymbol{y} \in \mathbb{C}^{d-p}$.

- $p \geq L_n + J_m$: There exist intersections of the VPP of period n and the VPP of period m on the parameters $\boldsymbol{h} \in \mathbb{C}^p$.

- $d \geq J_n + J_m$: There exist intersections of the VPP of period n and the VPP of period m.

Special case $J_n = J_m = d - p$

In fact, every our example satisfies $J_n = J_m = d - p$, and this IVPP satisfies $L_n = d - p$. Here we consider concrete cases, *i.e.*, the case $J_n = J_m = d - p$, and $L_n = L_m = d - p$.

If $J_n = J_m = d - p$

- $p \geq L_n + L_m \geq d - p$, $(p \geq d/2)$: There exist intersections of the VPP of period n and the VPP of period m on the variables $\boldsymbol{y} \in \mathbb{C}^{d-p}$.

- $2p - d \geq L_n$: There exist intersections of the VPP of period n and the VPP of period m on the parameters $\boldsymbol{h} \in \mathbb{C}^p$.

- $p \geq d/2$: There exist intersections of the VPP of period n and the VPP of period m.

If $L_n = L_m = d - p$

- $p \geq 2d/3$: There exist intersections of the VPP of period n and the VPP of period m on the parameters $\boldsymbol{h} \in \mathbb{C}^p$.
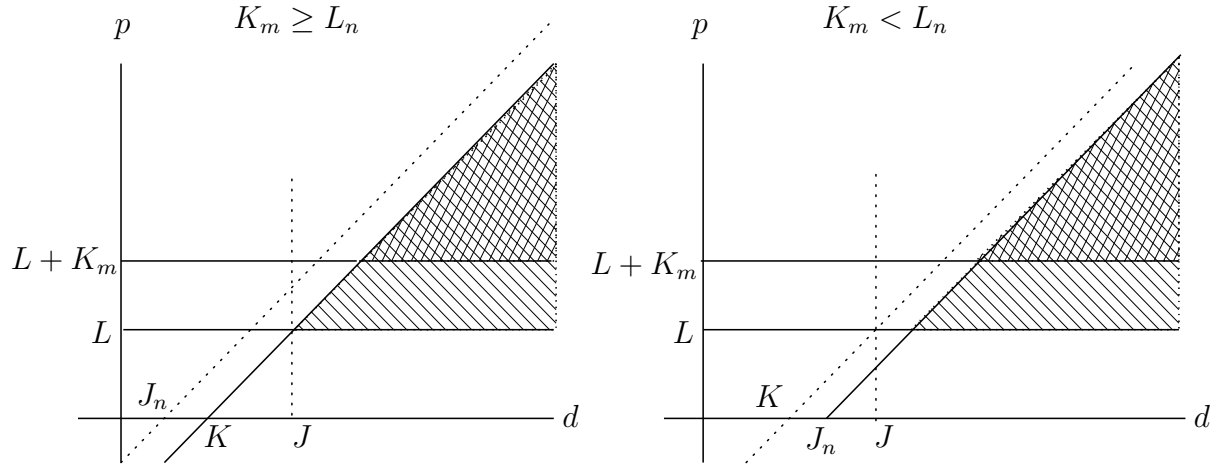

Figure 2.3: Existence Area of Intersections of VPPs

2.6 Origins of Intersection of VPPs

In the previous section, we considered about intersections of VPPs. Our discussion is based on counting the dimension and counting the number of conditions. Hence, it does not prohibit that VPPs have intersections. In this section, we consider about this possibility.

2.6.1 IDP

By the condition for the IVPP theorem that the VPPs have no intersection on generic points, intersections of the VPPs must be "singular points" of some kind.

In the case of rational map F, $\mathrm{Period}(F, n)$ and $\mathrm{Period}(\tilde{F}, n)$ satisfy,

$$\mathrm{Period}(\tilde{F}, n) = \mathrm{Period}(F, n) \cup \mathrm{IDP}(F^{(n)}),$$

by the Period/IDP conjecture. Therefore $\mathrm{Period}(F, n)$, $\mathrm{Period}(\tilde{F}, n)$, $\mathrm{Period}(F, m)$ and $\mathrm{Period}(\tilde{F}, m)$ satisfy,

$$\begin{aligned} \mathrm{Period}(\tilde{F}, n) \cap \mathrm{Period}(\tilde{F}, m) \;\; &= \;\; (\mathrm{Period}(F, n) \cap \mathrm{Period}(F, m)) \\ &\qquad \cup (\mathrm{IDP}(F^{(n)}) \cap \mathrm{IDP}(F^{(m)})). \quad (2.23) \end{aligned}$$

Here, we assume that the intersections of VPPs are not generic points. Hence we can get more conjectures as follows.

Conjecture for rational map

$$\mathrm{Period}(\tilde{F}, n) \cap \mathrm{Period}(\tilde{F}, m) = \mathrm{IDP}(F^{(n)}) \cap \mathrm{IDP}(F^{(m)}) \qquad (2.24)$$

and

2.6.2   Common Factors

We consider other source of intersections of VPPs that is given by common factors
of the conditions for a VPP. We give only the easiest case as follows:

- The conditions of the period $n \geq 2$,

$$\tilde{F}_i^{(n)}(\boldsymbol{x}, \boldsymbol{x}) = 0, \quad i = 1, \ldots, d, \quad \boldsymbol{x} \in \mathbb{C}^d,$$

  can be rewritten on the $\mathrm{LS}(F, \boldsymbol{h})$ as,

$$\bar{\Gamma}_{j_n}^{(n)}(\boldsymbol{y}, \boldsymbol{h}) = 0, \quad j_n = 1, \ldots, J_n, \quad \boldsymbol{y} \in \mathbb{C}^{d-p}, \quad \boldsymbol{h} \in \mathbb{C}^p.$$

  In general, the conditions of the VPP are not just the conditions of the IVPP,
  but become

$$K_{j_n}^{(n)}(\boldsymbol{y}, \boldsymbol{h}) \gamma_{j_n}^{(n)}(\boldsymbol{h}) = 0, \quad j_n = 1, \ldots, J_n, \quad \boldsymbol{y} \in \mathbb{C}^{d-p}, \quad \boldsymbol{h} \in \mathbb{C}^p.$$

  We now must take care of the extra factor $K_{j_n}^{(n)}(\boldsymbol{y}, \boldsymbol{h}), \ j_n = 1, \ldots, J_n$.

- The easiest case, we assume that the conditions $\gamma_{j_n}^{(n)}(\boldsymbol{h}) = 0, j_n = 1, \ldots, J_n$
  give a generic VPP($i.e.$ IVPP). Any other $m \geq 2$ periodic conditions must be
  written as $\gamma_{j_m}^{(m)}(\boldsymbol{h}) = 0, \ j_m = 1, \ldots, J_m$ by the IVPP theorem. In other words,
  the zero point conditions of the factors $K_{j_m}^{(m)}(\boldsymbol{y}, \boldsymbol{h}), \ j_m = 1, \ldots, J_m$ give the
  intersections of the VPP of period n and the VPP of period m. Therefore,
  the zero points of the conditions of the factors $K_{j_m}^{(m)}(\boldsymbol{y}, \boldsymbol{h}), \ j_m = 1, \ldots, J_m$ are
  some kinds of singular points.

2.6.3 Singular Points of Variety

Finally, we discuss about an origin of intersection of VPPs from the view of singular points of variety. We consider only the easiest case in the following.

Singular points of an IVPP of period n with respect to the invariants,

$$\partial_{h_k}(K_{j_n}^{(n)}(\boldsymbol{y}, \boldsymbol{h})\gamma_{j_n}^{(n)}(\boldsymbol{h})) = 0, \quad k = 1, \ldots, p, \quad j_n = 1, \ldots, d - p, \tag{2.25}$$

are calculated as,

$$\partial_{h_k}(K_{j_n}^{(n)}(\boldsymbol{y}, \boldsymbol{h})\gamma_{j_n}^{(n)}(\boldsymbol{h})) = 0$$
$$(\partial_{h_k}K_{j_n}^{(n)}(\boldsymbol{y}, \boldsymbol{h}))\gamma_{j_n}^{(n)}(\boldsymbol{h}) + K_{j_n}^{(n)}(\boldsymbol{y}, \boldsymbol{h})(\partial_{h_k}\gamma_{j_n}^{(n)}(\boldsymbol{h})) = 0.$$

Hence, the singular points of the variety, *i.e.*, the IVPP are given by the following conditions,

$$K_{j_n}^{(n)}(\boldsymbol{y}, \boldsymbol{h}) = 0, \quad j_n = 1, \ldots, d - p,$$

or

$$(\partial_{h_k}\gamma_{j_n}^{(n)}(\boldsymbol{h})) = 0, \quad j_n = 1, \ldots, d - p.$$

The zero points conditions of the common factors $K_{j_n}^{(n)}(\boldsymbol{y}, \boldsymbol{h}), j_n = 1, \ldots, d - p$ give the conditions of the singular points of the variety. Therefore we "push forward" the results of the previous subsection:

> **Conjecture of Singular Points of Variety**
>
> If there exist intersections of VPPs, then they are on singular points of the VPPs.

2.7 Examples

2.7.1 2 dimensional Möbius Map

The 2 dimensional Möbius map has 1 invariant, then it does not satisfy the intersection condition

$$p \geq \frac{2d}{3}. \tag{2.26}$$

of VPPs in the case of $L_n = d - p$. Hence, IVPPs of 2 dimensional Möbius map has no intersection.

2.7.2 3 dimensional Lotka-Volterra Map

Next, we discuss about the 3 dimensional Lotka-Volterra(3dLV) map[41][42]. It is a 3 dimensional map which has two invariants, then it satisfies the intersection condition of VPPs (2.26) in the case $L_n = d - p$. Hence, IVPPs of 3dLV map has intersections.

Rational Map

$$(F_{3\mathrm{dLV}})_i : \boldsymbol{x}^t \to x_i^{t+1} := x_i^t \frac{1 - x_{i+1}^t + x_{i+1}^t x_{i+2}^t}{1 - x_{i+1}^t + x_{i+2}^t x_i^t}, \quad i \in \mathbb{Z}/3\mathbb{Z}. \tag{2.27}$$

ADE

$$
\begin{aligned}
(\tilde{F}_{3\mathrm{dLV}})_i(\boldsymbol{x}^t, \boldsymbol{x}^{t+1}) \ := \ & x_i^{t+1}(1 - x_{i+1}^t + x_{i+2}^t x_i^t) \\
& - x_i^t(1 - x_{i+1}^t + x_{i+1}^t x_{i+2}^t) = 0, \quad i \in \mathbb{Z}/3\mathbb{Z}. \tag{2.28}
\end{aligned}
$$

Invariants

There are two invariants, which we denote by $f = H_1(\boldsymbol{x}^t), g = H_2(\boldsymbol{x}^t)$ for convenience,

$$f \quad := \quad x_1^t x_2^t x_3^t - (1 - x_1^t)(1 - x_2^t)(1 - x_3^t), \tag{2.29}$$

$$g \quad := \quad 1 + (1 - x_1^t)(1 - x_2^t)(1 - x_3^t). \tag{2.30}$$

IVPPs

The conditions of the IVPPs of this map have been known[7][8][9]：

$$\gamma^{(2)}(f, g) \quad = \quad g, \tag{2.31}$$

$$\gamma^{(3)}(f, g) \quad = \quad f^2 + fg + g^2, \tag{2.32}$$

$$\gamma^{(4)}(f, g) \quad = \quad f^3 + (1 - g)(f + g)^3, \tag{2.33}$$

$$\text{etc.}$$

We give Figure 2.4, Figure 2.5 of IVPPs of 3dLV map for period 2 and 4. The IVPP of period 3 is invisible in real number space, since that is in complex number space.
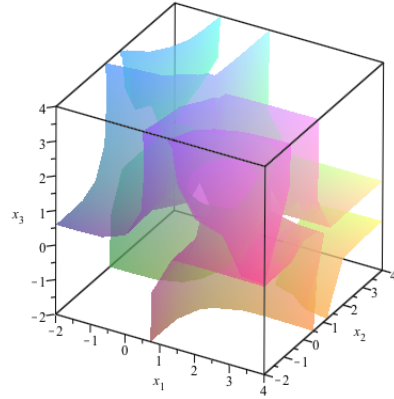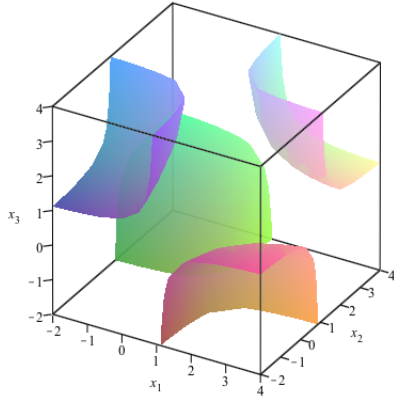

Figure 2.4: IVPP of period 2 of 3dLV map Figure 2.5: IVPP of period 4 of 3dLV map

IDP and Intersections of IVPPs

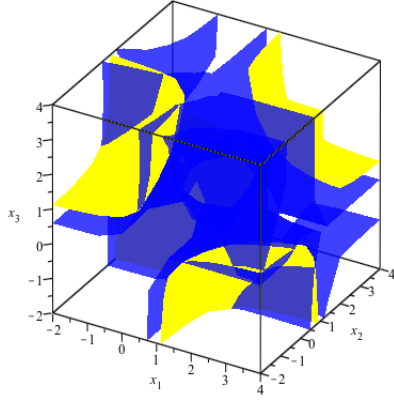There are some intersections of IVPPs which we can see in Figure 2.6, Figure 2.7.

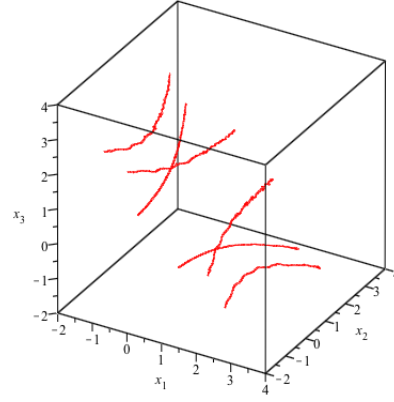Figure 2.6: Intersections of IVPPs of period 2,4 of 3dLV map

Figure 2.7: Intersections of IVPPs of period 2,4 of 3dLV map

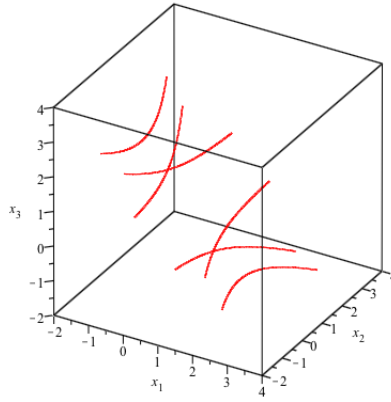In particular, IDP of the 3dLV map is given as follows.

Figure 2.8: IDP of 3dLV map

Thus we can see that these give the same points.

Common Factors and Intersections of IVPPs

Next we consider the common factors of the periodic conditions, which we discussed in §2.6.2. Here, because the formulas become too large, we present only the 2 period case as follows.

$$
\begin{aligned}
K_1^{(2)}(\boldsymbol{x}) &= x_3 x_1^2 x_2^2 - 2 x_3 x_1^2 x_2 + x_3 x_1^2 + x_1 x_2^2 x_3 - x_3^2 x_1 x_2^2 + x_1 x_2 x_3 \\
&\quad -2 x_3 x_1 + x_2^2 x_3^2 - 2 x_2^2 x_3 + x_2^2 - x_2 + x_2 x_3 + x_3, \\
K_2^{(2)}(\boldsymbol{x}) &= -x_3^2 x_1^2 + x_2 x_3^2 x_1^2 - x_3^2 x_1 x_2^2 + 2 x_1 x_2^2 x_3 - x_1 x_2^2 - x_3^2 x_1 x_2 \\
&\quad -x_1 x_2 x_3 + 2 x_1 x_2 + 2 x_3^2 x_1 - x_3 x_1 - x_1 + x_3 - x_3^2, \\
K_3^{(2)}(\boldsymbol{x}) &= -x_2^2 x_1^2 + x_3 x_1^2 x_2^2 - x_2 x_3^2 x_1^2 - x_3 x_1^2 x_2 + 2 x_2 x_1^2 - x_1^2 \\
&\quad +2 x_3^2 x_1 x_2 - x_1 x_2 x_3 - x_1 x_2 + x_1 - x_2 x_3^2 + 2 x_2 x_3 - x_2.
\end{aligned}
$$

We can easily check that these give the set of $\mathrm{IDP}(F_{\mathrm{3dLV}})$.

Singular Points of Variety and Intersections of IVPPs

We show the singular points of IVPP. Because, the IVPP of period 2 has no singular point and the case of other periods are too difficult to write down, we consider only the period 3 case here. However, we can check that any periods have the same situations as the case of 3 period.

$$
\begin{aligned}
\partial_f \gamma^{(3)}(f, g) &= 2f + g, \\
\partial_g \gamma^{(3)}(f, g) &= f + 2g,
\end{aligned}
$$

therefore we get singular points of the IVPP of period 3 at

$$
f = 0, \quad g = 0.
$$

We notice that the singular points of the variety include the $\mathrm{IDP}(F_{\mathrm{3dLV}})$.

2.7.3   3 dimensional Korteweg-de Vries Map

Since we must know IVPPs of the map to check the validity of our propositions we next study the 3 dimensional Korteweg-de Vries(3dKdV) map[41][42], which has been well studied.

Rational Map

$$(F_{\text{3dKdV}})_i : \boldsymbol{x}^t \to x_i^{t+1} := x_i^t \frac{1 - x_i^t x_{i+2}^t + x_i^t x_{i+1}^t (x_{i+2}^t)^2}{1 - x_i^t x_{i+1}^t + (x_i^t)^2 x_{i+1}^t x_{i+2}^t}, \quad i \in \mathbb{Z}/3\mathbb{Z}. \tag{2.34}$$

ADE

$$\begin{aligned}(F_{\text{3dKdV}})_i(\boldsymbol{x}^t, \boldsymbol{x}^{t+1}) &:= x_i^{t+1}(1 - x_i^t x_{i+1}^t + (x_i^t)^2 x_{i+1}^t x_{i+2}^t) \\ &\quad - x_i^t(1 - x_i^t x_{i+2}^t + x_i^t x_{i+1}^t (x_{i+2}^t)^2) = 0, \; i \in \mathbb{Z}/3\mathbb{Z}. \end{aligned} \tag{2.35}$$

Invariants

There are two invariants, which we denote by $f = h_1(\boldsymbol{x}^t), g = h_2(\boldsymbol{x}^t)$ for convenience,

$$\begin{aligned} f &= 1 + x_1^t x_2^t x_3^t, & (2.36) \\ g &= 1 + (1 + x_1^t x_2^t)(1 + x_2^t x_3^t)(1 + x_3^t x_1^t). & (2.37) \end{aligned}$$

IVPPs

We can find the IVPPs in [7],

$$\begin{aligned} \gamma^{(2)}(f, g) &= g, & (2.38) \\ \gamma^{(3)}(f, g) &= (f^2 - 2f + g)^2 - gf(f - 2), & (2.39) \\ \gamma^{(4)}(f, g) &= (f^2 - 2f + g)^3 - (g - 1)f^3(f - 2)^3, & (2.40) \\ & \text{etc.} \end{aligned}$$

We give Figure 2.9, Figure 2.10 of IVPPs of 3dKdV map for period 2 and 4. The IVPP of period 3 is again invisible in real number space.
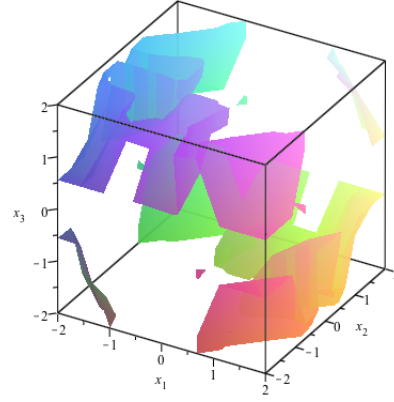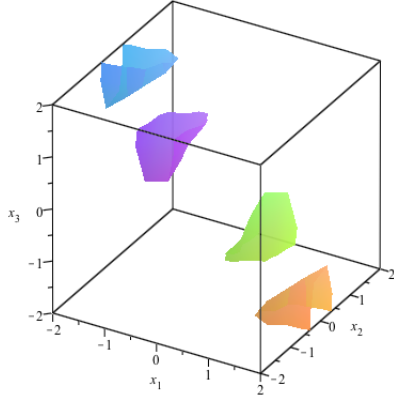

Figure 2.9: IVPP of period 2 of 3dKdV map

Figure 2.10: IVPP of period 4 of 3dKdV map

Intersection of IVPPs

There are some intersections of IVPPs which we can see in Figure 2.11, Figure 2.12.
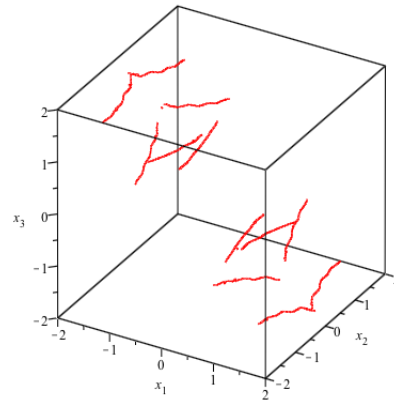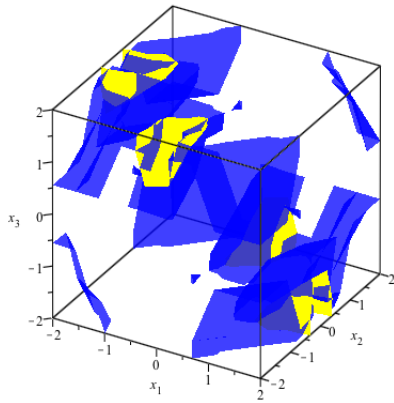

Figure 2.11: Intersections of IVPPs of period 2,4 of 3dKdV map

Figure 2.12: Intersections of IVPPs of period 2,4 of 3dKdV map

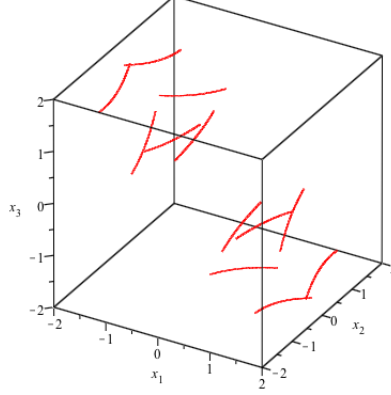In particular, IDP of the 3dKdV map is given as follows.

Figure 2.13: IDP of 3dKdV map

Thus we can see that these give the same points.

Common Factors and Intersections of IVPPs

Next we also show the common factors of the periodic conditions about only the 2 period as follows.

$$
\begin{aligned}
K_1^{(2)}(\boldsymbol{x}) &= x_2^3 x_1^3 x_3^3 + x_2^2 x_1^3 x_3^2 - x_2^4 x_1^2 x_3^3 + x_2^3 x_1^2 x_3^2 + 2x_2^2 x_1^2 x_3 - 2x_3^3 x_2^3 x_1 \\
&\quad + x_2^2 x_1 - x_2^2 x_1 x_3^2 + x_1 x_3 x_2 - x_3^3 x_2^2 - 2x_3^2 x_2 + x_2 - x_3, \\
K_2^{(2)}(\boldsymbol{x}) &= x_2^3 x_1^3 x_3^3 - x_1^3 x_3^4 x_2^2 - 2x_1^3 x_3^3 x_2 - x_1^3 x_3^3 + x_2^3 x_1^2 x_3^2 + x_3^3 x_2^2 x_1^2 \\
&\quad - x_3^2 x_2 x_1^2 - 2x_1^2 x_3 + 2x_2^2 x_1 x_3^2 + x_1 x_3 x_2 - x_1 + x_3 + x_3^2 x_2, \\
K_3^{(2)}(\boldsymbol{x}) &= x_2^3 x_1^4 x_3^2 - x_2^3 x_1^3 x_3^3 + 2x_2^3 x_1^3 x_3 - x_2^2 x_1^3 x_3^2 + x_2^3 x_1^2 - x_3^3 x_2^2 x_1^2 \\
&\quad + x_2^2 x_1^2 x_3 - 2x_3^2 x_2 x_1^2 - x_1^2 x_3 + 2x_2^2 x_1 - x_1 x_3 x_2 - x_1 + x_2.
\end{aligned}
$$

We can easily check that these give the set of IDP(F_{3dKdV}). In addition, the common factors of the other periods are the same situations as the 3dLV map case.

Singular Points of Variety and Intersections of IVPPs

We show the singular points of IVPP of only 3 period. Because, IVPP of period 2 has no singular point, and other periods are difficult to draw here. However, we can check that any periods have the same situations as the case of 3 period.

$$
\begin{aligned}
\partial_f \gamma^{(3)}(f, g) &= 4f^3 - 12f^2 + 2fg + 8f - 2g, \\
\partial_g \gamma^{(3)}(f, g) &= f^2 - 2f + 2g,
\end{aligned}
$$

therefore we get a singular points of variety of IVPP of period 3 is

$$
\begin{aligned}
f = 0, \quad & g = 0, \\
f = 1, \quad & g = \frac{1}{2}, \\
f = 2, \quad & g = 0.
\end{aligned}
$$

We notice that the singular points of the variety include the $\mathrm{IDP}(F_{3\mathrm{dKdV}})$.

2.7.4 Discussion

In this section, we showed two examples, 3dLV map and 3dKdV, about intersections of IVPPs. These examples have intersections of IVPPs in many forms, IDP, common factors and singular points of IVPP. However, all intersections of IVPPs are on IDP of ADE, therefore, we checked a correctness of the axiom of IVPP theorem in these examples.

Chapter 3

Invariant/Parameter Duality

In this chapter, we discuss an ADE on a level set such that it is also an ADE. In particular, an ADE constrained on an IVPP provides a recurrence equation(RE). In addition, the ADE on the level set can be interpreted as an IVPP on a "parameter space", hence we can consider higher dimensional IVPPs on the parameter space.

3.1   Invariant/Parameter Duality

In the discussion thus far, we explained that the level set of the ADE and the VPP of ADE are important for the IVPP theorem. However, counting of dimension of each ADE and setting variables and parameters are tedious.

Hence, we propose a new ADE $\tilde{F}_h$ that is restricted the ADE $\tilde{F}$ on the LS$(\tilde{F}, \boldsymbol{h})$. The IADE of the ADE $\tilde{F}_h$ is defined as,

$$
\begin{aligned}
I_{\tilde{F}_h}^{t,t+1}(\boldsymbol{y}^t, \boldsymbol{y}^{t+1}) \quad &:= \quad \mathrm{Gb}\left(I_{\tilde{F}}^{t,t+1}(\boldsymbol{x}^t, \boldsymbol{x}^{t+1}), (\boldsymbol{h} - \boldsymbol{h}(\boldsymbol{x}^t))\right) \cap \mathbb{C}[\boldsymbol{y}^t, \boldsymbol{y}^{t+1}], \\
&\qquad \boldsymbol{y}^t, \boldsymbol{y}^{t+1} \in \mathbb{C}^{d-p}, \ \boldsymbol{h} \in \mathbb{C}^p,
\end{aligned}
\tag{3.1}
$$

where $\boldsymbol{y} := (x_1, \ldots, x_{d-p})$. The ADE $\tilde{F}_h$ has no invariant, but it has p parameters.

From the definition we find a relation as follows:

Invariant/Parameter Duality

A d dimensional ADE $\tilde{F}$ with p invariants $\Leftrightarrow$ A $d - p$ dimensional ADE $\tilde{F}_h$ with p parameters.

This relation is called Invariant/Parameter duality. By this duality, we can discuss in the world in which the variables are separated by the variables on the level set and the parameters of the invariants.

3.2   Invariant/Parameter Duality and IVPP/RE Duality

We can rewrite IVPP theorem by using the Invariant/Parameter duality.

Remark

An ADE $\tilde{F}_h$ with p parameters satisfies,

- If the ADE $\tilde{F}_h$ with p parameters becomes an RE of period n on a parameter $\boldsymbol{h}_n \in \mathbb{C}^p$ space, there exist parameters $\boldsymbol{h}_n \in \mathbb{C}^p$ that satisfy the n periodic condition of the ADE $\tilde{F}_h$.

Therefore, by the Invariant/Parameter duality, we can find a relation between an IVPP and an RE.

IVPP/RE Duality Theorem

If an ADE $\tilde{F}$ satisfies the same conditions of IVPP theorem and the ADE $\tilde{F}$ has an IVPP of period $n \geq 2$, then an ADE $\tilde{F}_h$ that is restricted on the LS$(\tilde{F}, \boldsymbol{h})$, becomes an RE of period n when we take the parameters $\boldsymbol{h} \in \mathbb{C}^p$ on the IVPP.

We can find a new theorem that is given by IVPP theorem and IVPP/RE duality theorem as,

RE Theorem

Let $\tilde{F}_h$ be a d dimensional ADE with p parameters. If $p \geq d$ and there exists $n \geq 2$ such that a VPP of period n is given by only information of the parameters, then VPP of period m for any $m \geq 2$ are given by only information of the parameters. In other words, if the ADE $\tilde{F}_h$ has parameters such that the ADE $\tilde{F}_h$ becomes an RE of period n, then the ADE $\tilde{F}_h$ has parameters that the ADE $\tilde{F}_h$ becomes an RE of period m for any $m \geq 2$.

Proof:

- abbreviation

In addition, we define that IVPP theorem and RE theorem are called together the IVPP/RE theorem. The IVPP/RE theorem suggests that there exist a series of IVPP/RE from an ADE.

3.3 Examples

3.3.1 3 dimensional Lotka-Volterra Map

ADE on Level Set

The ADE $(\tilde{F}_{3\text{dLV}})_h$ restricted on the level set $\text{LS}(\tilde{F}_{3\text{dLV}}, (r, s))$, $r, s \in \mathbb{C}$ of $\tilde{F}_{3\text{dLV}}$ (2.28) is given by the following "bi-quadratic equation"[7][8][9],

$$
\begin{aligned}
(\tilde{F}_{3\text{dLV}})_h(y^t, y^{t+1}) \;=\; & a_1(r, s)(y^t)^2(y^{t+1})^2 + b_1(r, s)(y^{t+1}(y^t)^2 + (y^{t+1})^2 y^t) \\
& +c_1(r, s)((y^{t+1})^2 + (y^t)^2) + d_1(r, s)y^t y^{t+1} \\
& +e_1(r, s)(y^t + y^{t+1}) + f_1(r, s),
\end{aligned} \tag{3.2}
$$

where $y = x_1$, and new invariants,

$$
r \;:=\; x_1 x_2 x_3, \tag{3.3}
$$

$$
s \;:=\; (1 - x_1)(1 - x_2)(1 - x_3), \tag{3.4}
$$

and the parameters,

$$
a_1(r, s) = r + 1, \quad b_1(r, s) = s - 1 - 2r, \quad c_1(r, s) = r - s, \tag{3.5}
$$

$$
d_1(r, s) = 1 + 3r + rs + s^2, \quad e_1(r, s) = -r(1 + s), \quad f_1(r, s) = 0. \tag{3.6}
$$

It is important to notice that an iteration of (3.2) yields the same bi-quadratic form every time,

$$
\begin{aligned}
(\tilde{F}_{3\text{dLV}})_h^{(T)}(y^t, y^{t+T}) \;=\; & a_T(r, s)(y^t)^2(y^{t+T})^2 + b_T(r, s)(y^{t+T}(y^t)^2 + (y^{t+T})^2 y^t) \\
& +c_T(r, s)((y^{t+T})^2 + (y^t)^2) + d_T(r, s)y^t y^{t+T} \\
& +e_T(r, s)(y^t + y^{t+T}) + f_T(r, s).
\end{aligned} \tag{3.7}
$$

Hence, if there exist parameters (r, s) that satisfy,

$$
a_{n+1}(r, s) = a_1(r, s), \ldots, f_{n+1}(r, s) = f_1(r, s), \tag{3.8}
$$

then these parameters give the n period IVPP/RE.

Therefore it is apparent that periodic points of different periods are specified only by the different sets of parameters.

It was shown in [7][8][9] that many integrable maps, including the symmetric QRT map, can be reduced to the bi-quadratic equation (3.2) when $p = d - 1$, and generate IVPPs.

REs

We obtain REs from the IVPPs of the ADE $\tilde{F}_{3\mathrm{dLV}}$ upon elimination of s and putting $r = 0$ for convenience:

- 2 periodic RE
$$y^t y^{t+1} - (y^t + y^{t+1}) = 0.$$

- 3 periodic RE
$$(y^t)^2 (y^{t+1})^2 + ((y^t)^2 + (y^{t+1})^2) - (y^t)^2 y^{t+1} - 2y^t (y^{t+1})^2 + y^t y^{t+1} = 0.$$

- 4 periodic RE
$$(y^t)^2 (y^{t+1})^2 + ((y^t)^2 + (y^{t+1})^2) - 2y^t (y^{t+1}) = 0.$$

etc.

It will be worthwhile to see how they look like. Although the invariants are complex numbers, we can draw a curve associated with $\gamma^{(n)}(r, s) = 0$, if we constrain r and s to real values. Figure 3.1 shows the result. IVPPs of different periods from 2 to 10 are drawn by different colors.
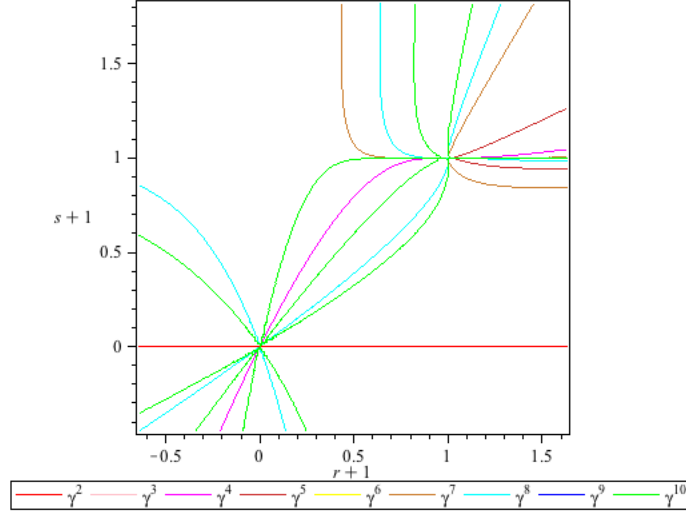
Figure 3.1: IVPPs of 3dLV map

3.3.2 3 dimensional Korteweg-de Vries Map

ADE on Level Set

Restricted on the level set $\mathrm{LS}(\tilde{F}_{\mathrm{3dKdV}}, (r, s))$, $r, s \in \mathbb{C}$ of 3dKdV map(2.35) is also given by the bi-quadratic equation,

$$
\begin{aligned}
(\tilde{F}_{\mathrm{3dKdV}})_h(y^t, y^{t+1}) \;=\; & a_1(r,s)(y^t)^2(y^{t+1})^2 + b_1(r,s)(y^{t+1}(y^t)^2 + (y^{t+1})^2 y^t) \\
& + c_1(r,s)((y^{t+1})^2 + (y^t)^2) + d_1(r,s)y^t y^{t+1} \\
& + e_1(r,s)(y^t + y^{t+1}) + f_1(r,s), \quad\quad\quad (3.9)
\end{aligned}
$$

where $y = x_1$, and new invariants,

$$
\begin{aligned}
r \;&:=\; x_1 x_2 x_3, &\quad (3.10) \\
s \;&:=\; (1 + x_1 x_2)(1 - x_2 x_3)(1 - x_3 x_1), &\quad (3.11)
\end{aligned}
$$

and the parameters,

$$
a_1(r,s) = 0, \quad b_1(r,s) = r(1+s), \quad c_1(r,s) = r^2 + s, \quad\quad (3.12)
$$

– 52 –

$$d_1(r,s) = r^2s + 3r^2 - 1, \quad e_1(r,s) = r(-1 + 2r^2 + s), \quad f_1(r,s) = r^2(r-1)(r+1). \quad (3.13)$$

REs

We obtain REs from the IVPPs of the ADE $\tilde{F}_{\text{3dKdV}}$ upon elimination of s and putting $r = 0$ for convenience:

- 2 periodic RE

$$y^t + y^{t+1} = 0.$$

- 3 periodic RE

$$((y^t)^2 + (y^{t+1})^2) + y^t y^{t+1} = 0.$$

- 4 periodic RE

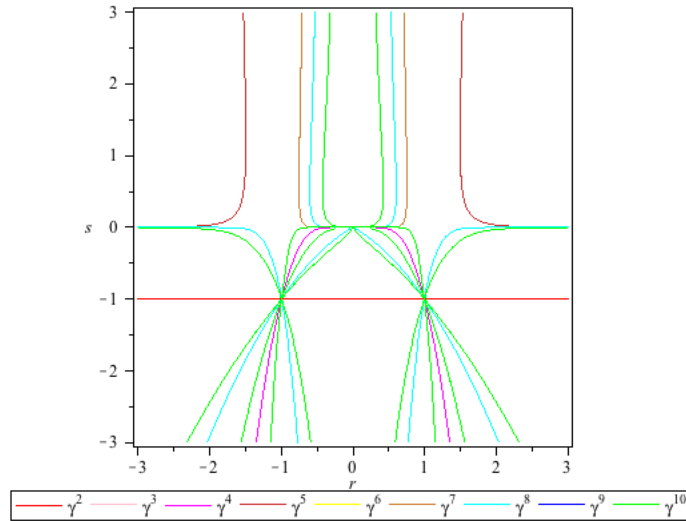$$(y^t)^2 + (y^{t+1})^2 = 0.$$

etc.

Figure 3.2: IVPPs of 3dKdv map

Chapter 4

Derivation of IVPPs from SC

In this chapter, we give a method of derivation of IVPPs of a rational map that is given by singularity confinement. In addition, we want to propose that this phenomena determines an integrability test of the map, in other words, this phenomenon is a candidate of the test of a judgement of integrability of the map.

4.1 SC and Derivation of IVPPs

4.1.1 SC

The singularity confinement(SC) is a phenomenon of a rational map such that a point, which is mapped to infinity, returns to finite region after finite steps. Although this phenomenon had been expected to play the role of discrete version of the Painlevé test for the integrability of second order difference equations, some counter examples have been found.

In [8], by studying 3dLV map, the authors found the fact that an IVPP appears as a zero set of the denominator at every step of iteration of the map after it is recovered from a singularity. We are going to develop an algorithm in this section

which enables us to generate IVPPs from singularities in general case.

Before the discussion of the algorithm it is useful to explain the mechanism of SC. Let us first rewrite the notion of zero set of denominators. Namely we denoted by $\mathrm{SP}(F)$ the set of zero points of the denominators of the rational map F in (2.2).

$$\mathrm{SP}(F) := \bigcup_{i=1}^{d} \mathrm{SP}(F, i), \quad \mathrm{SP}(F, i) := \left\{ \, \boldsymbol{x} \in \mathbb{C}^d \mid D_i(\boldsymbol{x}) = 0 \, \right\}.$$

All points on $\mathrm{SP}(F)$ are mapped to ∞, where ∞ means

$$\infty \sim \frac{1}{0}. \tag{4.1}$$

When F has its inverse $F^{(-1)}$, we denote by $\mathrm{SP}(F^{(-1)})$ the zero set of the denominators of $F^{(-1)}$. A point $\boldsymbol{x} \in F^{(-1)}\big(\mathrm{SP}(F^{(-1)})\big)$, which is at ∞, is mapped to a finite point $F(\boldsymbol{x}) \in \mathrm{SP}(F^{(-1)})$. The SC will take place if there exists a finite number N_{sc} such that

$$F^{(N_{sc})}(\mathrm{SP}(F)) \subset \mathrm{SP}(F^{(-1)}).$$

We call the minimum of N_{sc} the SC step number of F.

4.1.2 Example

For example in the 3dLV map(4.2) case,

$$
\begin{aligned}
F_{\mathrm{3dLV}} \quad &: \boldsymbol{x}^t \\
&\mapsto \quad \boldsymbol{x}^{t+1} = \left(x_1^t \frac{1 - x_2^t + x_2^t x_3^t}{1 - x_3^t + x_3^t x_1^t}, \ x_2^t \frac{1 - x_3^t + x_3^t x_1^t}{1 - x_1^t + x_1^t x_2^t}, \ x_3^t \frac{1 - x_1^t + x_1^t x_2^t}{1 - x_2^t + x_2^t x_3^t} \right), \tag{4.2}
\end{aligned}
$$

a point

$$\boldsymbol{x}^0 = \left(x_1^0, x_2^0, \frac{1}{1 - x_1^0} \right) \in \mathrm{SP}(F_{\mathrm{3dLV}}, 1) \subset \mathrm{SP}(F_{\mathrm{3dLV}}), \tag{4.3}$$

which satisfies

$$D_1(\boldsymbol{x}^0) = 0, \tag{4.4}$$

is mapped iteratively according to

$$\boldsymbol{x}^0 \rightarrow (\infty, 0, 1) \rightarrow (1, 0, \infty) \rightarrow \left(\frac{1}{1 - x_1^0}, x_2^0, x_1^0 \right) \in \mathrm{SP}(F_{\mathrm{3dLV}}^{(-1)}, 3) \subset \mathrm{SP}(F_{\mathrm{3dLV}}^{(-1)}), \quad (4.5)$$

hence $N_{sc} = 3$.

4.1.3   Derivation of IVPPs

Now we notice that if $\boldsymbol{x}^0 \in \mathrm{FP}(F)$ is a period n point with $n \geq N_{sc} - 1$, its image $F^{(n+1)}(\boldsymbol{x}^0)$ must be divergent, *i.e.,*

$$\boldsymbol{x}^0 \in \mathrm{SP}(F) \cap \mathrm{Period}(F, n) \quad \Longrightarrow \quad F^{(n+1)}(\boldsymbol{x}^0) \text{ is divergent}, \quad (4.6)$$

hence we get

$$\mathrm{SP}(F^{(n+1)}|_{\mathrm{SP}(F)}) \subset \mathrm{Period}(F, n) \quad (4.7)$$

by $F^{(n+1)}(\boldsymbol{x}^0) = F(\boldsymbol{x}^0)$. Generally speaking the converse of (4.6) is not true. An important observation in [8], however, is that, in the 3dLV case, the divergence of $F^{(n+1)}(\boldsymbol{x}^0)$ is sufficient to determine the set of period n points.

Let us see how this happens. Since the map (4.2) has two invariants

$$\begin{aligned}
f &= x_1 x_2 x_3 - (1 - x_1)(1 - x_2)(1 - x_3), \\
g &= 1 + (1 - x_1)(1 - x_2)(1 - x_3),
\end{aligned} \quad (4.8)$$

we can solve (4.4) to express $\boldsymbol{x}^0$ of (4.3) in terms of the invariants. In fact we obtain

$$\boldsymbol{x}^0 = \left(\frac{f}{f + g}, r\frac{g}{f}, \frac{f + g}{g} \right), \qquad r := f + g - 1 = x_1 x_2 x_3. \quad (4.9)$$

Hence all images of $\boldsymbol{x}^0$ are expressed only by the invariants. On the other hand, the condition (4.4) also implies that $D_1^{(n+1)}(\boldsymbol{x}^0)$ must vanish at period n points. Now

we recall that, in the 3dLV case, all periodic points of period n are on the IVPP uniquely determined by the single function $\gamma^{(n)}(f, g)$ of the invariants.

$$
\begin{aligned}
\gamma^{(2)}(f, g) &= g, \\
\gamma^{(3)}(f, g) &= f^2 + fg + g^2, \\
\gamma^{(4)}(f, g) &= f^3 + (1 - g)(f + g)^3,
\end{aligned}
$$

etc.

Therefore $D_1^{(n+1)}(\boldsymbol{x}^0)$ must be proportional to $\gamma^{(n)}(f, g)$. In other words, we can generate IVPPs of all periods in the denominators of $x_1^{n+1}(\boldsymbol{x}^0)$, $n = 2, 3, 4, ...$ iteratively. This is the phenomenon observed in [8], which takes place irrespective whether $\boldsymbol{x}^0$ belongs to Period(F, n) or not.

4.1.4   Generalization and Algorithm

We can use the same procedure to derive all IVPPs if the map has $d - 1$ invariants and at least one IVPP. In this case the IVPP of period n is determined by a single polynomial $\gamma^{(n)}(\boldsymbol{h})$ of the invariants. But when $p \leq d - 2$, the condition $\boldsymbol{x}^0 \in \mathrm{SP}(F)$ is not sufficient to determine $\boldsymbol{x}^0$ by the invariants. Therefore it is certainly not clear, in general cases, whether IVPPs of all periods are generated by the map recovered from the singularities.

We now ask if there exists a way to derive IVPPs by the SC when $d/2 \leq p \leq d-2$. To answer this question we notice that, in addition to $\boldsymbol{x}^0 \in \mathrm{SP}(F)$, we need $d - p - 1$ extra conditions to write down $\boldsymbol{x}^0$ by the invariants. We can obtain such conditions by forcing $d - p$ independent denominators of the images $F^{(m)}(\boldsymbol{x}^0)$ of $\boldsymbol{x}^0$ to vanish when $m < N_{sc}$. If they are chosen properly IVPPs must be derived from the periodicity conditions iteratively.

Without loss of generality we assume $\boldsymbol{x}^0 \in \text{SP}(F, 1)$, or equivalently $D_1(\boldsymbol{x}^0) = 0$. Now suppose this condition also implies $D_1^{(2)}(\boldsymbol{x}^0) = 0$. Then it is clear that $D_1^{(n)}(\boldsymbol{x}^0) = 0$ for all n. Therefore the SC can take place only if $D_1^{(2)}(\boldsymbol{x}^0)$ does not vanish identically when $D_1(\boldsymbol{x}^0)$ vanishes, unless we enforce it to vanish. In other words we can adopt the conditions

$$D_1^{(j)}(\boldsymbol{x}^0) = 0, \qquad j = 1, 2, ..., d - p,$$

to express $\boldsymbol{x}^0$ in terms of the invariants.

Summarizing these arguments we are ready to propose the algorithm, which enables us to generate IVPPs from the SC. We present it here in more precise form as follows.

We consider a rational map which has p invariants such that $p \geq d/2$.

1. Solve the d equations

$$\begin{aligned} \boldsymbol{H}(\boldsymbol{x}) &= \boldsymbol{h}, \quad \boldsymbol{h} \in \mathbb{C}^p, \\ D_1^{(j)}(\boldsymbol{x}) &= 0, \quad j = 1, 2, ..., d - p \end{aligned} \qquad (4.10)$$

   for $\boldsymbol{x}$ to write the initial point $\boldsymbol{x}^0$ by the invariants.

2. Compute $F^{(n)}(\boldsymbol{x}^0)$, $n \geq N_{sc}$ iteratively to find $D_1^{(n)}(\boldsymbol{x}^0)$, which is a polynomial functions of the invariants.

3. Let $P^{(N_{sc}-1)}(\boldsymbol{h})$ be a set of $d - p$ irreducible polynomials, one from each of the set

$$\left\{ D_1^{(N_{sc})}(\boldsymbol{x}^0), \ D_1^{(N_{sc}+1)}(\boldsymbol{x}^0), \ ..., \ D_1^{(N_{sc}+d-p-1)}(\boldsymbol{x}^0) \right\}.$$

   If the polynomials in $P^{(N_{sc}-1)}(\boldsymbol{h})$ are all independent, the intersection of the elements is a set of periodic points of period $N_{sc} - 1$.

4. By the same reason the intersection of the set of polynomials $P^{(n)}(\boldsymbol{h})$ chosen from

$$\left\{ D_1^{(n+1)}(\boldsymbol{x}^0),\ D_1^{(n+2)}(\boldsymbol{x}^0),\ ...,\ D_1^{(n+d-p)}(\boldsymbol{x}^0) \right\}, \qquad n \geq N_{sc} - 1,$$

is a set of periodic points of period n.

$$\begin{aligned}
\boldsymbol{x}^0 \ \in\ & \left(\bigcap_{j=1}^{d-p} \mathrm{SP}(F^{(j)}, 1) \right) \cap \mathrm{Period}(F, n) \\
& \rightarrow F^{(n+j)}(\boldsymbol{x}^0), \quad j = 1, \ldots, d-p \quad \text{are divergent}, \qquad (4.11)
\end{aligned}$$

Our algorithm is given by an ansatz as

Derivation of IVPPs Ansatz

$$\bigcap_{j=1}^{d-p} \mathrm{SP} \left(F^{(n+j)}|_{\bigcap_{j'=1}^{d-p} \mathrm{SP}(F^{(j)},1)}, 1 \right) \subset \mathrm{Period}(F, n) \qquad (4.12)$$

It is clear that if there exists $P^{(N_{sc}-1)}(\boldsymbol{h})$, it must be $\mathrm{Period}(F, N_{sc}-1)$. Therefore, when N_{sc} is finite, hence the SC takes place, and if we can derive $P^{(n)}(\boldsymbol{h})$ by our algorithm, the IVPP theorem guarantees integrability of the map.

4.2 Examples

This section is devoted to application of the algorithm we developed in the previous section. In order to examine our integrability test, we would like to recall that only maps which have sufficient number of invariants, satisfying $p \geq d/2$, can pass the test. This is not a sufficient condition for the integrability but a necessary condition for an IVPP to exist.

Although there have been known many integrable maps, their invariants have not been given explicitly. Exceptions are the Lotka-Volterra maps of all dimensions. We will study mainly this series in this thesis, because we can test our algorithm in various dimensions, (see Appendix B).

4.2.1 2 dimensional Möbius Map

Before discussing the LV series, we study the case of 2 dimensional Möbius map(2.7) that has the invariant

$$h := x_1 x_2.$$

Following to the first step of our algorithm we find the initial point of the map which is parameterized by the invariant r:

$$\boldsymbol{x}^0 = (1, h) \in \mathrm{SP}(F_{2d\mathrm{Möb}}, 1).$$

Then, by the second step, we see that the SC undergoes as

$$\boldsymbol{x}^0 \to (\infty, 0) \to (-1, -h) \to \left(-\frac{1+h}{2}, \ -\frac{2r}{1+h} \right)$$

$$\to \left(-\frac{1+3h}{3+h}, \ -\frac{h(3+h)}{1+3h} \right) \to \left(-\frac{1+6h+h^2}{4(1+h)}, \ -\frac{4h(1+h)}{1+6h+h^2} \right) \to \cdots$$

The SC step number is 2, in this map.

We can derive analytic expression of $\boldsymbol{x}^{n+1}$ for all n,

$$\boldsymbol{x}^{n+1} = \left(h^{1/2} \frac{(1-h^{1/2})^n + (1+h^{1/2})^n}{(1-h^{1/2})^n - (1+h^{1/2})^n}, \ h^{1/2} \frac{(1-h^{1/2})^n - (1+h^{1/2})^n}{(1-h^{1/2})^n + (1+h^{1/2})^n} \right),$$

from which we find all $\gamma^{(n)}(h)$'s as follows,

$$
\begin{aligned}
\gamma^{(3)}(h) &= 3 + h, \\
\gamma^{(4)}(h) &= 1 + h, \\
\gamma^{(5)}(h) &= 5 + 10h + h^2, \\
&\quad \text{etc.}
\end{aligned}
$$

4.2.2 3 dimensional Korteweg-de Vries Map

In the case of 3d KdV map(2.34):

Parametrization of $\mathrm{SP}(F_{3d\mathrm{KdV}}, 1)$ **and SC**

$$
\boldsymbol{x}^0 = \left(-\frac{f^2 - 2f + g}{g(f-1)}, \ -\frac{f(f-1)(f-2)}{f^2 - 2f + g}, \ \frac{g(f-1)}{f(f-2)} \right) \in \mathrm{SP}(F_{3d\mathrm{KdV}}, 1),
$$

$$
\boldsymbol{x}^0 \rightarrow \ (\infty, 0, 1-f) \rightarrow (\infty, 1-f, 0),
$$
$$
\rightarrow \left(-\frac{f^2 - 2f + g}{g(f-1)}, \ \frac{g(f-1)}{f(f-2)}, \ -\frac{f(f-1)(f-2)}{f^2 - 2f + g} \right) \rightarrow \cdots
$$

The steps of SC is 3.

IVPP

$$
\begin{aligned}
\gamma^{(2)}(f,g) &= g, \\
\gamma^{(3)}(f,g) &= (f^2 - 2f + g)^2 - gf(f-2), \\
\gamma^{(4)}(f,g) &= (f^2 - 2f + g)^3 - (g-1)f^3(f-2)^3, \\
\gamma^{(5)}(f,g) &= (f^2 - 2f + g)^6 - gf^2(f-2)^2(f^2 - 2f + 3)(f^2 - 2f + g)^3 \\
&\quad -3g^3 f(f-2)(f-1)^2(f^2 - 2f + g)^2 - g^3 f^3(f-2)^3(g-1)^2,
\end{aligned}
$$

etc.

4.2.3 4 dimensional Lotka-Volterra Map

The Lotka-Volterra(LV) map of arbitrary dimension d was introduced in [41][42], which we review briefly in Appendix B. Because we have already discussed the 3 dimensional case, we will study the cases $d = 4, 5$ in this and the following subsections.

The d dimensional LV map and invariants are given in (B.1) and (B.2). Solving (B.1) for (X_1, X_2, X_3, X_4) in the $d = 4$ case we obtain

$$F_{\text{4dLV}} : \boldsymbol{x}^t \mapsto \boldsymbol{x}^{t+1} = \left(x_1^t \frac{1 - x_2^t - x_3^t + x_2^t x_3^t + x_3^t x_4^t}{1 - x_3^t - x_4^t + x_3^t x_4^t + x_4^t x_1^t}, x_2^t \frac{1 - x_3^t - x_4^t + x_3^t x_4^t + x_4^t x_1^t}{1 - x_4^t - x_1^t + x_4^t x_1^t + x_1^t x_2^t}, \right.$$

$$\left. x_3^t \frac{1 - x_4^t - x_1^t + x_4^t x_1^t + x_1^t x_2^t}{1 - x_1^t - x_2^t + x_1^t x_2^t + x_2^t x_3^t}, x_4^t \frac{1 - x_1^t - x_2^t + x_1^t x_2^t + x_2^t x_3^t}{1 - x_2^t - x_3^t + x_2^t x_3^t + x_3^t x_4^t} \right).$$

There are three invariants H_1, H_2 and R, which are given explicitly in (B.2). If we define $r := R$, $f = H_1 - H_2 - 1$, $g := 1 - H_1$ they are

$$\begin{cases} r &= x_1 x_2 x_3 x_4, \\ f &= x_1 x_2 x_3 x_4 - (1 - x_1)(1 - x_2)(1 - x_3)(1 - x_4), \\ g &= 2 - q_1 - q_2 - q_3 - q_4, \end{cases}$$

where we use the notation $q_j = x_j (1 - x_{j-1})$.

Since $d - p = 1$, the single condition $D_1(\boldsymbol{x}) = 0$ is sufficient to determine the initial point $\boldsymbol{x}^0$ on $\text{SP}(F_{\text{4dLV}}, 1)$. We find

$$\boldsymbol{x}^0 = \left(\frac{(-g^2 + g - gf + 2f + gN_4)f}{2(gf - g^2 r + f^2)}, -\frac{gf - 2gf - f + f^2 + fN_4}{2f}, \right.$$

$$\left. -\frac{(-gf + 2gr + f - f^2 + fN_4)g}{2(-gf - g^2 r + f^2 + g^2 f)}, \frac{-g - 2f + g^2 + gf + gN_4}{2(g - 1)g} \right)$$

where

$$N_4 := \sqrt{1 - 2g - 2f + g^2 + 2gf + f^2 + 4r - 4gr}.$$

$\boldsymbol{x}^0$ is then mapped to

$$\boldsymbol{x}^0 \rightarrow \left(\infty, 0, \frac{f + g - 1 - N_4}{2(g - 1)}, 1 \right)$$

$$\rightarrow \left(1, \frac{f + g - 1 + N_4}{2(g - 1)}, 0, \infty \right) \rightarrow \boldsymbol{x}^3 \rightarrow \cdots$$

where

$$\boldsymbol{x}^3 := \left(\frac{-g - 2f + g^2 + gf + gN_4}{2(g - 1)g}, -\frac{(-gf + 2gr + f - f^2 + fN_4)g}{2(-gf - g^2 r + f^2 + g^2 f)}, \right.$$

$$\left. -\frac{gf - 2gr - f + f^2 + fN_4}{2f}, \frac{(-g^2 + g - gf + 2f + gN_4)f}{2(gf - g^2 r + f^2)} \right)$$

hence the SC step number is 3. If we continue the map further the IVPPs of this map are generated from the denominators of x_1^n, $n = 3, 4, 5, ...$ as

$$
\begin{aligned}
\gamma^{(2)}(f, g, r) &= g, \\
\gamma^{(3)}(f, g, r) &= f^2 - g^2 r + g^2 f, \\
\gamma^{(4)}(f, g, r) &= -2g^2 r + g^2 f + 2f^2, \\
\gamma^{(5)}(f, g, r) &= r^2 f g^6 + 3f^5 g^2 + 3g^4 r^2 f^2 - 3r f^4 g^2 - 4g^4 r f^3 + f^6 - r^3 g^6 + g^4 f^4, \\
&\text{etc.}
\end{aligned}
\tag{4.13}
$$

In this way integrability of the 4dLV map is shown by our test.

4.2.4   5 dimensional Lotka-Volterra Map

The maps we studied so far are the case of $d - p = 1$. When $d - p \geq 2$, we must consider, according to our algorithm in §4.1.4, more condition (4.10) in addition to $D_1(\boldsymbol{x}) = 0$. Since the number p of the invariants of the d dimensional LV map is $[\frac{d+2}{2}]$, the 5dLV map is an example of $d - p = 2$ case.

By solving (B.1) in the Appendix B for $(X_1, X_2, X_3, X_4, X_5)$ the 5dLV map is given by

$$
F_{5\text{dLV}} : \boldsymbol{x}^t \mapsto \boldsymbol{x}^{t+1} = \left(x_1^t \frac{D_5(\boldsymbol{x}^t)}{D_1(\boldsymbol{x}^t)}, \ x_2^t \frac{D_1(\boldsymbol{x}^t)}{D_2(\boldsymbol{x}^t)}, \ x_3^t \frac{D_2(\boldsymbol{x}^t)}{D_3(\boldsymbol{x}^t)}, \ x_4^t \frac{D_3(\boldsymbol{x}^t)}{D_4(\boldsymbol{x}^t)}, \ x_5^t \frac{D_4(\boldsymbol{x}^t)}{D_5(\boldsymbol{x}^t)} \right)
$$

where we used the notation

$$
D_i(\boldsymbol{x}) := (1 - x_{i+2})\Big(x_i x_{i+4} + (1 - x_{i+3})(1 - x_{i+4}) \Big) + x_{i+2} x_{i+3} x_{i+4} x_i, \quad i \in \mathbb{Z}/5\mathbb{Z}.
$$

This map has three invariants,

$$
\begin{cases}
r &= x_1 x_2 x_3 x_4 x_5, \\
f &= x_1 x_2 x_3 x_4 x_5 - (1 - x_1)(1 - x_2)(1 - x_3)(1 - x_4)(1 - x_5), \\
g &= 2 - q_1 - q_2 - q_3 - q_4 - q_5 + x_1 x_2 x_3 x_4 x_5.
\end{cases}
\tag{4.14}
$$

Therefore we impose two conditions $D_1(\boldsymbol{x}) = 0$ and $D_1^{(2)}(\boldsymbol{x}) = 0$ to obtain the initial point $\boldsymbol{x}^0$ on $\mathrm{SP}(F_{5\mathrm{dLV}}, 1) \cap \mathrm{SP}(F_{5\mathrm{dLV}}^{(2)}, 1)$,

$$\boldsymbol{x}^0 = \left(\frac{f^2 g}{B}, r\frac{B - fA}{-f^3}, \frac{B - fg^2}{B - fA}, \frac{(B - gf^2)f}{(B - fg^2)(f - g)}, \frac{B(g - f)}{(B - gf^2)g} \right),$$

where

$$A := f^2 - gf + g^2, \quad B := rA - f^2.$$

The SC mapping undergoes as

$$\boldsymbol{x}^0 \to \left(\infty, 0, \frac{g}{g - f}, \frac{f}{g}, 1 \right) \to \left(\frac{0}{0}, 0, 1, \frac{0}{0}, \frac{0}{0} \right) \to \left(1, 0, \frac{0}{0}, \frac{0}{0}, \frac{0}{0} \right)$$

$$\to \left(\frac{g}{g - f}, 0, \infty, 1, \frac{f}{g} \right) \to \boldsymbol{x}^5 \to \boldsymbol{x}^6 \to \cdots \quad (4.15)$$

where

$$\boldsymbol{x}^5 = \left(\frac{B - fg^2}{B - fA}, r\frac{B - fA}{-f^3}, \frac{f^2 g}{B}, \frac{B(g - f)}{(B - gf^2)g}, \frac{(B - gf^2)f}{(B - fg^2)(f - g)} \right).$$

The SC step number is 5. $0/0$ in (4.15) means that the denominator and the numerator of the component become zero separately, so that we can not determine its value. In other words the point is indeterminate.

The IVPPs derived by our algorithm are as follows,

$$
\begin{aligned}
\gamma_1^{(3)}(f, g, r) &= g - f, \quad \gamma_2^{(3)}(f, g, r) = r - f - 1, \\
\gamma_1^{(4)}(f, g, r) &= 2f^2 - 2fg + g^2, \quad \gamma_2^{(4)}(f, g, r) = f^2 + f - g - rf, \\
\gamma_1^{(5)}(f, g, r) &= f^4 - 3gf^3 + 4f^2 g^2 - 2fg^3 + g^4, \\
\gamma_2^{(5)}(f, g, r) &= -2f^3 + 2f^2 - rf^2 + (6f^2 - f)g - (3f - 1)g^2 + 2g^3, \\
&\quad \text{etc.}
\end{aligned}
$$

Thus our algorithm has been proved to work when $d - p \neq 1$ as well.

Chapter 5

SC and "Projective Resolution" of "Triangulated Category"

In this chapter we propose a method to characterize the Hirota-Miwa equation(HM eq.) by means of the theory of "triangulated category". In particular we show in detail how the SC, a phenomenon which was proposed to characterize integrable maps, can be associated with the "projective resolution" of the "triangulated category".

5.1   Geometrical Feature of the Hirota-Miwa Equation(HM eq.)

Before starting our argument we review briefly some features of the HM eq.[14][15].

$$a_{14}a_{23}\tau_{14}(p)\tau_{23}(p) - a_{24}a_{13}\tau_{24}(p)\tau_{13}(p) + a_{34}a_{12}\tau_{34}(p)\tau_{12}(p) = 0, \qquad (5.1)$$

$$\tau(p) \in \mathbb{C}, \quad p = (p_1, p_2, p_3, p_4) \in \mathbb{C}^4, \quad a_{ij} = -a_{ji} \in \mathbb{C}.$$

In above and hereafter we use the abbreviations, such as

$$\tau_j(p) := \mathrm{D}_j\tau(p) = \tau(p + \delta_j), \quad \tau_{ij}(p) := \mathrm{D}_i\mathrm{D}_j\tau(p) = \tau(p + \delta_i + \delta_j), \qquad (5.2)$$

$$\delta_j := (\delta_{1j}, \delta_{2j}, \delta_{3j}, \delta_{4j}),$$

with the Kronecker symbol δ_{ij}.

1) If we substitute

$$\tau(p) = \prod_{i,j} \left(\frac{E(z_i, z_j)}{z_i - z_j} \right)^{p_i p_j} \theta\left(\zeta + \sum_j p_j w(z_j) \right), \quad a_{ij} = z_i - z_j, \tag{5.3}$$

into (5.1), we obtain an identity, called Fay's trisecant formula, for the hyperelliptic function θ and the prime form $E(z_i, z_j)$ defined on a Riemann surface of arbitrary genus[43][44].

2) From this single equation all soliton equations in the Kadomtsev—Petviashvili(KP) hierarchy can be derived corresponding to various continuous limits of independent variables[14][15].

3) The solutions of (5.1) were identified with the points of universal Grassmannian by Sato and known as the τ functions[12].

5.1.1 Nature of τ Functions

It was shown in [45] that the τ functions can be represented by means of tachyon correlation functions of the string theory. Since it provides the most convenient formulation in our argument we will use the notion of string theory in what follows. The 4 point string (tachyon) correlation function is given by

$$\Phi(p, z; G) = \langle 0 | V(p_1, z_1) V(p_2, z_2) V(p_3, z_3) V(p_4, z_4) | G \rangle, \tag{5.4}$$

where $z = (z_1, z_2, z_3, z_4) \in \mathbb{Z}^4$ is a set of parameters determined by a_{ij}'s of the equation (5.1). Here

$$V(p_j, z_j) =: \exp(i p_j X(z_j)) :,$$

is the vertex operator of momentum p_j of a string attached at z_j of the string world sheet specified by the state vector $|G\rangle$. The string coordinate $X(z)$ is an operator which acts on the state $|\cdot\rangle$, while the symbol :: means the normal order product. It was proved in [45] that the substitution of the ratio

$$\tau(p) = \frac{\Phi(p, z; G)}{\Phi(p, z; 0)},\qquad(5.5)$$

into (5.1) yields exactly Fay's formula associated with the Riemann surface Figure 5.1 corresponding to the world sheet $|G\rangle$. Hence the point z_j on the world sheet is a puncture of the Riemann surface. Notice that, since we do not integrate over z_j's, thus has no problem of divergence, we define the vertex operator $V(p, z)$ with no ghost field c.
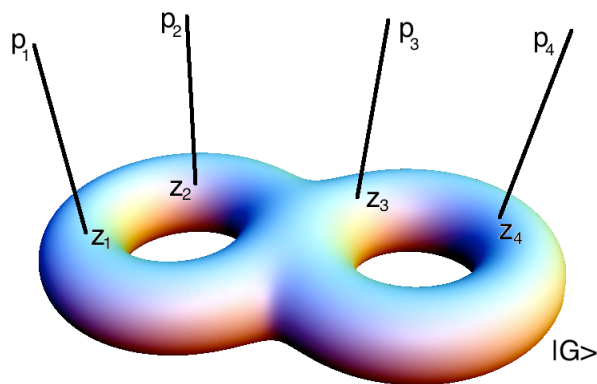

Figure 5.1: Riemann surface

Although every solution of (5.1) is obtained by specifying the state $|G\rangle$ of the general solution (5.5), we do not discuss explicit forms of $|G\rangle$ in this thesis. Therefore we simply write $\Phi(p, z; G)$ as $\Phi(p, z)$ unless it is necessary. On the other hand the main property of τ functions is determined by the nature of the vertex operators as we will see now. Since they satisfy

$$V(p, z)V(p', z') = (-1)^{pp'}V(p', z')V(p, z),\qquad(5.6)$$

we see immediately that the fields $\psi_\pm(z) := V(\pm 1, z)$ have the properties

$$\psi_\pm(z)\psi_\pm(z') = -\psi_\pm(z')\psi_\pm(z), \quad \psi_\pm(z)\psi_\mp(z') = -\psi_\mp(z')\psi_\pm(z), \tag{5.7}$$

and, in particular, the following holds:

$$\psi_+(z)\psi_+(z) = 0, \quad \psi_-(z)\psi_-(z) = 0. \tag{5.8}$$

Hence $\psi_\pm(z)$ are Grassmann fields. These relations are checked in Appendix C.

By taking this property into account we define operators $\hat{D}_s^{\pm 1}$ by

$$\hat{D}_s^{\pm 1}\Phi(p, z) := \langle 0|V(p_1, z_1)V(p_2, z_2)V(p_3, z_3)V(p_4, z_4)\psi_\pm(z_s)|G\rangle, \tag{5.9}$$

to describe the insertion of $\psi_\pm(z_s)$ into $\Phi(p, z)$ of (5.4). When z_s of $\psi_\pm(z_s)$ coincides with one of $z = (z_1, z_2, z_3, z_4)$ in (5.9), say $s = j$, the insertion of $\psi_\pm(z_s)$ is equivalent to change p_j to $p_j \pm 1$, up to a phase factor which comes from the exchange of order of $\psi_\pm(z_j)$ with vertex operators. If we use the notations such as

$$\Phi_j(p, z) = \hat{D}_j\Phi(p, z), \quad \Phi_{ij}(p, z) = \hat{D}_i\hat{D}_j\Phi(p, z), \qquad i, j = 1, 2, 3, 4,$$

in these particular cases, we obtain

$$\Phi_{ij}(p, z) = -\Phi_{ji}(p, z), \qquad i, j = 1, 2, 3, 4,$$

hence

$$\Phi_{ii}(p, z) = 0, \qquad i = 1, 2, 3, 4. \tag{5.10}$$

This is an expression of (5.8). Thus we have found that the zero of $\Phi(p, z)$ is associated with a coincidence of two punctures on the Riemann surface.

In the theory of KP hierarchy[46] [47] [48] the operators $\mathrm{e}^{\psi_\pm(z_j)} = 1 + \psi_\pm(z_j)$ are known as elements of the symmetry group $GL(\infty)$ which act on the state $|G\rangle$. Correspondingly we call

$$\hat{D}_j = \mathrm{e}^{\hat{D}_j} - 1,$$

a 'difference' operator, in analogy with the differential operator ∂_j.

Despite of this odd behavior of the correlation function Φ under the operation of $\hat{D}_j^{\pm}$, the solutions (5.5) of the HM eq. behave regularly. It owes to the fact that both $\Phi(p, z; G)$ and $\Phi(p, z; 0)$ are shifted by $\hat{D}_j$ simultaneously in $\tau(p)$. The extra phase factors arising from the exchange of order of vertex operators and $\psi_{\pm}(z_j)$ in (5.9) cancel exactly. As a result we find

$$\hat{D}_j \tau(p) = D_j \tau(p) = \tau(p + \delta_j) = \tau_j(p),$$

in these particular cases, we obtain

$$\tau_{ij}(p) = \tau_{ji}(p), \qquad i, j = 1, 2, 3, 4,$$

in agreement with our previous notation (5.2). Nevertheless it is important to notice that the zero point of Φ function (5.10) is an indeterminate point of $\tau(p)$. This fact will play a central role in our analysis of the SC in §5.4.

$$\tau_{ii}(p) = \frac{\Phi_{ii}(p, z; G)}{\Phi_{ii}(p, z; 0)} = \frac{0}{0} = ?. \tag{5.11}$$

5.1.2 Difference Geometry on Lattice Spaces

Although the variable p of the τ function is on $\mathbb{C}^4$, the solutions of the HM eq. are on a lattice space $\mathbb{Z}^4$ embedded in $\mathbb{C}^4$, which is fixed once an 'initial point' $p_0 \in \mathbb{C}^4$ of τ is fixed. Let us call this lattice space

$$\Xi_4(p_0) := \left\{ p \in \mathbb{C}^4 \mid p - p_0 \in \mathbb{Z}^4 \right\}.$$

Since, however, the 'initial point' p_0 does not appear explicitly in our discussion, we simply write $\Xi_4(p_0)$ as Ξ_4. Moreover we often write $p - p_0$ as $p \in \Xi_4$ unless there is a confusion.

In order to study the HM eq. within the framework of the theory of category, it will be useful to study its geometrical feature on the lattice space Ξ_4. For this purpose we introduce a notion of 'difference form' on the lattice space in this subsection.

4 dimensional 'Difference' Forms

Let us define an exterior 'difference' operator d_B by

$$\mathrm{d}_B \omega := \sum_{j=1}^{4} \mathrm{D}_j \omega \wedge \mathrm{d}p_j, \qquad {}^{\forall}\omega \in \Omega\left[\Xi_4\right] := \left\{\, \omega \mid \omega : \Xi_4 \to \mathbb{C} \,\right\}. \tag{5.12}$$

We must emphasize that the form $\mathrm{D}_j \omega(p)$ is on Ξ_4 if $p \in \Xi_4$, but, in contrast to the differential form, it is not at the same point p but is at $p + \delta_j$. In particular, the operation of d_B to $\omega(p)$ increases the value of the sum $\sum_{i=1}^{4} p_i$ of the components of p by 1. To describe the situation more precisely we define a subspace of Ξ_4 by

$$\Xi_4^{(n)} := \left\{\, p \in \Xi_4 \;\middle|\; \sum_{i=1}^{4} p_i = n \in \mathbb{Z} \,\right\},$$

so that

$$\bigcup_{n \in \mathbb{Z}} \Xi_4^{(n)} = \Xi_4.$$

We notice that $\Xi_4^{(n)}$ is a lattice hyperplane in Ξ_4. In particular $\Xi_4^{(1)}$ is the hyperplane which includes the four points $\{\delta_1,\ \delta_2,\ \delta_3,\ \delta_4\}$. All other hyperplanes are parallel to $\Xi_4^{(1)}$.

Each hyperplane is embedded in a three dimensional lattice space $\mathbb{Z}^3$. In fact the points of $\Xi_4^{(n)}$ occupy all corners of octahedra which fill $\mathbb{Z}^3$ together with tetrahedra, as it is illustrated in Figure 5.2.
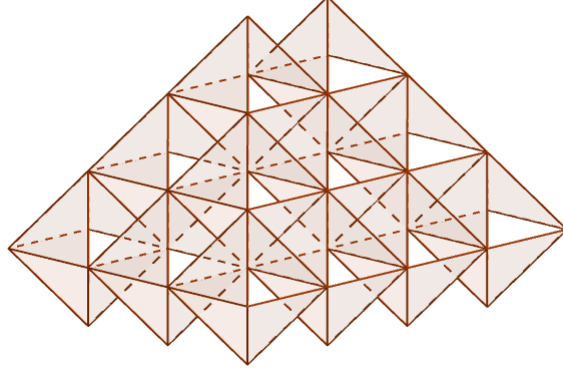
Figure 5.2: $\Xi_4^{(n)}$

If $p \in \Xi_4^{(n)}$, the forms $\mathrm{D}_j\omega(p) = \omega_j(p)$ are on $\Xi_4^{(n+1)}$ for all j, hence

$$\omega^{-1}\mathrm{D}\omega : \Xi_4^{(n)} \to \Xi_4^{(n+1)},$$

with $\mathrm{D} = (\mathrm{D}_1, \mathrm{D}_2, \mathrm{D}_3, \mathrm{D}_4)$. Since all functions in the HM eq.(5.1) are of the form $\tau_{ij}(p) = \mathrm{D}_i\mathrm{D}_j\tau(p)$, they are on $\Xi_4^{(n+2)}$ if $p \in \Xi_4^{(n)}$. Moreover the six functions $\tau_{ij}(p)$ in (5.1) are at the six corners of the octahedron whose center is at p. Hence the HM eq. determines relations among functions on $\Xi_4^{(n+2)}$. In other words solutions of the HM eq. are different if they are on different hyperplanes. We mention that this is a result of the fact that the HM eq. is a Plücker relation.

Let us define $\Omega^{(n)}\left[\Xi_4^{(0)}\right]$ by

$$\Omega^{(n)}\left[\Xi_4^{(0)}\right] := \left\{ \sum_{i_1 \ldots i_n} \omega_{i_1 \ldots i_n} \mathrm{d}p_{i_1} \wedge \cdots \wedge \mathrm{d}p_{i_n} \;\middle|\; \omega_{i_1 \ldots i_n} : \Xi_4^{(n)} \to \mathbb{C} \right\}, \quad n \in \mathbb{Z},$$

when $p \in \Xi_4^{(n)}$. We then naturally obtain a graded algebra

$$\bigoplus_{n \in \mathbb{Z}} \mathbb{C}\Omega^{(n)}\left[\Xi_4^{(0)}\right], \tag{5.13}$$

on which d_B acts by

$$d_B^{(n)} : \Omega^{(n)} \left[\Xi_4^{(0)} \right] \to \Omega^{(n+1)} \left[\Xi_4^{(0)} \right]. \tag{5.14}$$

We give a remark that the maximal of n of $\Omega^{(n)} \left[\Xi_4^{(0)} \right]$ seems 4, but we impose a connection condition later, so that we do not have to limit n.

3 dimensional 'Difference' Forms

From Figure 5.2 we can see that the lattice space $\Xi_4^{(n+2)}$ consists of parallel planes, each filled by triangles of octahedra, as illustrated in Figure 5.3. If we fix the direction of the planes parallel to the direction of $-p_4$, we can specify them by the values of $t := \sum_{a=1}^{3} p_a$. Such a plane is defined by

$$\Xi_3^{(t,n+2)} := \left\{ (p_1, p_2, p_3) \in \mathbb{Z}^3 \,\middle|\, \sum_{a=1}^{3} p_a = t \in \mathbb{Z}, \ (p_1, p_2, p_3, p_4) \in \Xi_4^{(n+2)} \right\},$$

$$\bigcup_{t \in \mathbb{Z}} \Xi_3^{(t,n+2)} = \Xi_4^{(n+2)}.$$

Since $n+2 \ (= \sum_{i=1}^{4} p_i) = t + p_4$ is fixed, the planes are perpendicular to the direction of $-p_4$.
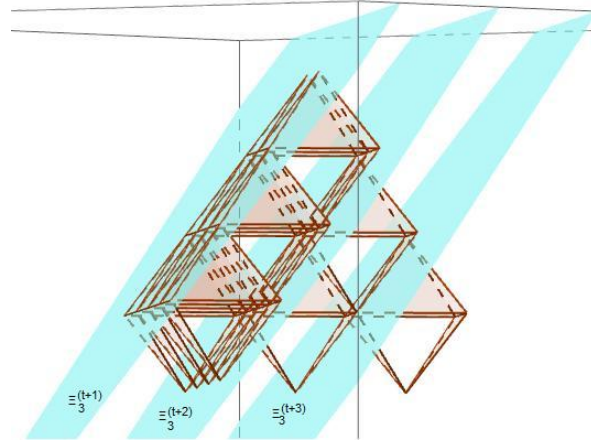

Figure 5.3: $\Xi_3^{(t,n+2)}$

We denote a point on $\Xi_3^{(t,n+2)}$ by $\boldsymbol{p} = (p_1, p_2, p_3) \in \mathbb{Z}^3$. Corresponding to d_B of (5.14) we define the 3 dimensional exterior 'difference' operator d_T by

$$\mathrm{d}_T\omega[t] = \sum_{a=1}^{3} \mathrm{D}_a\omega[t] \wedge \mathrm{d}p_a, \quad {}^\forall\omega[t] \in \Omega\left[\Xi_3^{(t,n+2)}\right] := \left\{\, \omega[t] \,\middle|\, \omega[t] : \Xi_3^{(t,n+2)} \to \mathbb{C} \,\right\},$$

which we call the shift operator. Since, similar to the 4 dimensional case, $\omega(\boldsymbol{p})$ and $\mathrm{D}_a\omega(\boldsymbol{p})$ are on different planes, we define graded forms of degree t by $\omega[t]$ when $\boldsymbol{p} \in \Xi_3^{(t,n+2)}$. The graded algebra

$$\bigoplus_{t\in\mathbb{Z}} \mathbb{C}\Omega^{(t,n+2)}\left[\Xi_3^{(0,n+2)}\right],$$

$$\Omega^{(t,n+2)}\left[\Xi_3^{(0,n+2)}\right] := \left\{\, \sum_{a_1\dots a_t} \omega_{a_1\dots a_t}\mathrm{d}p_{a_1} \wedge \cdots \wedge \mathrm{d}p_{a_t} \,\middle|\, \omega_{a_1\dots a_t} : \Xi_4^{(t,n+2)} \to \mathbb{C} \,\right\}, \quad t \in \mathbb{Z},$$

is generated by

$$\mathrm{d}_T : \Omega^{(t,n+2)}\left[\Xi_3^{(0,n+2)}\right] \to \Omega^{(t+1,n+2)}\left[\Xi_3^{(0,n+2)}\right]. \tag{5.15}$$

5.2 Dynamical Feature of the HM eq.

Based on the geometrical structure of the HM eq. we studied in the previous section, we discuss dynamical feature of the HM eq. in this section.

5.2.1 Dynamical Evolution

First, we give a pair of 4 dimensional 'difference' 2-forms

$$F := \sum_{i,j=1}^{4} F_{ij}\mathrm{d}p_i \wedge \mathrm{d}p_j, \qquad F_{ij} := a_{ij}\tau_{ij}, \tag{5.16}$$

$$\tilde{F} := \sum_{i,j=1}^{4} \tilde{F}_{ij}\mathrm{d}p_i \wedge \mathrm{d}p_j, \qquad \tilde{F}_{ij} := {}^*a_{ij}\tau_{ij}, \tag{5.17}$$

with $^*a_{ij} = \sum_{kl} \epsilon_{ijkl} a_{kl}$, where ϵ_{ijkl} is the Levi-Civita symbol. We can easily check that each of the followings[49]

$$\det F_{ij} = 0, \qquad \det \tilde{F}_{ij} = 0, \qquad (5.18)$$

is equivalent to the HM eq. (5.1). By this reason we call F of (5.16) the HM 2-form in this section.

In this subsection we fix the hyperplane $\Xi_4^{(n+2)}$, hence we simply denote $\Xi_3^{(t,n+2)}$ by $\Xi_3^{[t]}$, and study behavior of a particular solution of the HM eq. For this purpose we rewrite the HM 2-form by using the operator d_T of (5.15) as

$$F = \sum_{i,j=1}^{4} F_{ij} dp_i \wedge dp_j = d_T \left(F_4 dp_4 + \sum_{b=1}^{3} F_b dp_b \right),$$

and see that the six components F_{ij} split into two parts

$$d_T F_4 \;\; = \;\; \sum_{a=1}^{3} F_{a4} dp_a, \qquad (5.19)$$

$$d_T \sum_{b=1}^{3} F_b dp_b \;\; = \;\; \sum_{b,c=1}^{3} F_{bc} dp_b \wedge dp_c, \qquad (5.20)$$

corresponding to 3 dimensional 1-form and 2-form, respectively.

Let us denote by $\tilde{S}_j[t+1]$ and $S_j[t+2]$ the triangles in an octahedron which are perpendicular to p_j and parallel each other. From Figure 5.3 we see that every octahedron is put between two nearest planes, such that two parallel triangles are on each plane. In fact the triangle $\tilde{S}_4[t+1]$ is on $\Xi_3^{[t+1]}$ and $S_4[t+2]$ is on $\Xi_3^{[t+2]}$ if $\tau(\boldsymbol{p})$ is on $\Xi_3^{[t]}$. Define $\mathcal{O}[t] = (\tilde{S}_4[t+1], S_4[t+2])$ with

$$\tilde{S}_4[t+1] \;\; := \;\; \Big(\tau_{14}[t+1], \tau_{24}[t+1], \tau_{34}[t+1] \Big),$$
$$S_4[t+2] \;\; := \;\; \Big(\tau_{23}[t+2], \tau_{31}[t+2], \tau_{12}[t+2] \Big).$$

Then the procedure to solve initial value problem is to determine the following sequence

$$\mathcal{O} \xrightarrow{\mathrm{d}_T} \mathcal{O}[1] \xrightarrow{\mathrm{d}_T} \mathcal{O}[2] \xrightarrow{\mathrm{d}_T} \mathcal{O}[3] \xrightarrow{\mathrm{d}_T} \cdots \tag{5.21}$$
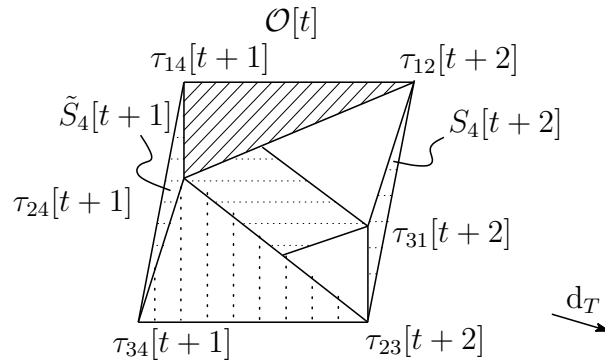
when information $\mathcal{O}$ at initial time $t = 0$ is given.

Figure 5.4:

5.2.2 Deterministic Rule for the Flow of Information

We want to know how information on $\mathcal{O}[t]$ transfers to other octahedra as t increases. The HM eq. determines a relation between $\tilde{S}_4[t+1]$ and $S_4[t+2]$. But it does not tell us how the information of $\tilde{S}_4[t+1]$ transfers to $S_4[t+2]$. In order to solve an initial value problem we must know some deterministic rules which decide uniquely the local flow of information. In this subsection we set up a rule for the flow of information in an octahedron.

Let us consider two lattice points p and p' on Ξ_4 which are separated by

$$p' - p = m \in \mathbb{Z}^4.$$

If they are on the same hyperplane $\Xi_4^{(n)}$, the separation m must satisfy

$$m_1 + m_2 + m_3 + m_4 = 0. \tag{5.22}$$

If p and p' are neighbor of an octahedron,

$$m = \delta_i - \delta_j, \qquad i \neq j, \tag{5.23}$$

corresponding to the edge parallel to the vector $p_i - p_j$. There will be many possible routs along which information can transfer between two points p and p' fixed arbitrary. Since an addition of a path of the type (5.23) does not change the condition (5.22), any routs can connect p and p' as long as they are connected by edges of octahedra.

When we decide the rule of transfer of information we must keep in mind the following items:

- Information at $p + \delta_i + \delta_j$ is transfered properly to $p' + \delta_k + \delta_l$ if all operators corresponding to all possible routs $\{r\}$ change $\tau_{ij}(p)$ to $\tau_{kl}(p')$ uniquely.

$$\mathrm{D}_r(ij, kl)_{p,p'} : \tau_{ij}(p) \xrightarrow{r} \tau_{kl}(p'), \quad {}^\forall r \in \{\text{possible routs}\}.$$

- Our system is deterministic if a rule of transfer of information is fixed along edges of an octahedron, and is the same for all octahedra.

- The system is integrable if this rule is sufficient to predict the values of τ on $\Xi_3^{[t]}$ for all t when the values on $\Xi_3^{[0]}$ are given arbitrary.

There are 12 edges in an octahedron $\mathcal{O}$. Every object τ_{ij} at a corner has connections to its four neighbors but no direct connection to its diagonal one. Since we are interested in a flow of information from one corner to another of $\mathcal{O}$, we must decide direction (hence a rule) of the flow. In other words we fix the order of points in $\mathcal{O}$. A natural way is the cyclic ordering of suffixes, $i.e.$,

$$1 < 2 < 3 < 4 < 1 < 2 < 3. \tag{5.24}$$

We notice in Figure 5.4 that every pair of corners connected by an edge have a common suffix, like (τ_{14}, τ_{12}). Therefore we define the direction of transfer by

$$D(ij, ik)_{p,p} : \tau_{ij}(p) \longrightarrow \tau_{ik}(p), \quad iff \quad k < i < j. \tag{5.25}$$

The action of $D(kl, ij)_{p,p}$ is to remove punctures at z_i and z_j from $|G\rangle$ and insert other punctures at z_k and z_l. It will be convenient to represent this action explicitly by means of the operators:

$$D(ij, kl)_{p.p} = D_k D_l D_j^{-1} D_i^{-1}. \tag{5.26}$$

We can easily check the rule of product,

$$D(ij, kl)_{p,p} \circ D(kl, mn)_{p,p} = D(ij, mn)_{p,p}.$$

From

$$D(ij, kl)_{p,p} \circ D(kl, ij)_{p,p} = D(ij, ij)_{p,p} = \mathrm{id}_{ij_{p,p}},$$

we see that $D(ij, ij)_{p,p}$ is an isomorphism.

Since corners connected by an edge have a common suffix, (5.26) simplifies

$$D(ij, ik)_{p,p} = D_k D_j^{-1},$$

whereas the morphism connecting the diagonals of an octahedron can be obtained by a product of morphisms.

5.2.3 Transfer of Information along a Chain of Octahedra

In this subsection we want to see how information flows along a chain of octahedra. Let us denote by $\mathcal{O}[0] = (\tilde{S}_4(p)[1], S_4(p)[2])$ the octahedron whose center is at $p \in \Xi_4^{(n)}$. There are three octahedra which share three edges of the triangle $S_4(p)[2]$. Since these three neighbors are on the same hyperplane $\Xi_4^{(n+2)}$ their centers must be

at $p + \delta_1 - \delta_4$, $p + \delta_2 - \delta_4$, $p + \delta_3 - \delta_4$, respectively. Let $\mathcal{O}[1]$ be one of them, say $\mathcal{O}[0]$. Then these octahedra are connected as illustrated in Figure 5.5, where we use the abbreviations:

$$\tilde{S}_4[1] = (X, Y, Z) = \Big(\tau_{14}[1], \tau_{24}[1], \tau_{34}[1]\Big), \quad S_4[2] = (X', Y', Z') = \Big(\tau_{23}[2], \tau_{31}[2], \tau_{12}[2]\Big).$$
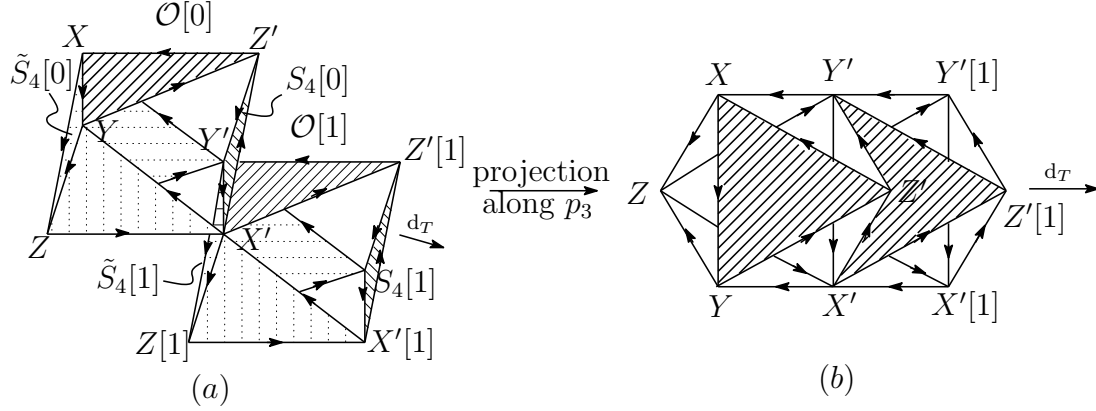

Figure 5.5:

For the information of $\mathcal{O}[0]$ to transfer to its neighbor $\mathcal{O}[1]$ properly, we must impose the following conditions :

$$\tau_{14}[2] = \tau_{31}[2], \quad \tau_{24}[2] = \tau_{23}[2]. \tag{5.27}$$

This means the connecting condition in the lattice space, the following:

$$\Omega^{(1,n)}\left[\Xi_3^{(1,n)}\right] \sim \Omega^{(2,n)}\left[\Xi_3^{(0,n)}\right]. \tag{5.28}$$

Repeating this procedure we can define a chain of octahedra.

5.3   View from the Category Theory

We are now ready to summarize our result in the previous section in terms of the category theory.

Comparing our arguments in §5.2.2 and §5.2.3 with the set of axioms Tr1 $\sim$ Tr4 of the triangulated category in Appendix D, we naturally find the following correspondence:

$$
\begin{array}{lcl}
\text{objects} & \longleftrightarrow & \tau_{ij} \in \bigcup_{a=1,2} \bigcup_{t \in \mathbb{Z}} \Omega^{(a,n+2)} \left[\Xi_3^{(t,n+2)} \right], \\[2mm]
\text{morphism} & \longleftrightarrow & \left\{ \, \mathrm{D}_r(ij,kl)_{p,p'}, \; ^{\forall}r \in \{\text{possible routs}\}, \; p,p' \in \Xi^{(n+2)}. \, \right\}, \\[2mm]
\text{shift functor} & \longleftrightarrow & \mathrm{d}_T : \mathcal{O}[t] \to \mathcal{O}[t+1], \\[2mm]
\text{octahedron axiom} & \longleftrightarrow & \left\{ \, \mathrm{D}(ij,ik)_{p,p}, \; \text{iff} \; k < i < j, \; p \in \Xi^{(n+2)} \, \right\}
\end{array}
$$

$$(5.29)$$

We have thus found various type of categories which are related from each other. Among others our interpretation of the information flow of the HM eq. by means of the triangulated category is the most fundamental. However the correspondence of the flow with the triangulated category seems not quite right by two reasons.

1. An addition of objects is no longer an object in general, because our objects are solutions of a nonlinear equation. Although most of studies of triangulated category have been based on additive categories in mathematics, the above axioms themselves do not use the notion of additivity. Therefore nonadditive nature of our theory will not cause any problem.

2. Second reason is that we have not yet defined null object 0, which appears in the first axiom Tr1. This is the main subject which we are going to discuss in the following section.

5.4   Localization and SC

It is well known that a localization of a triangulated category is also a triangulated category. We show in this section that the SC of a rational map obtained from

the τ function of the HM eq. can be described in terms of the localization of the triangulated category.

5.4.1 Reduction of the Lattice Space

We have studied difference geometry of 4 and 3 dimensions in §5.2. Our concern in this section is a 2 dimensional lattice space.

If we fix $n = \sum_{j=1}^{4} p_j - 2$ and $t = \sum_{a=1}^{3} p_a$ we are left with 2 dimensional lattice space, which we parameterize by

$$q = p_1 + p_2, \quad j := p_2 - p_1.$$

Like the higher dimensional lattice cases we define 2 dimensional lattice space

$$\Xi_2^{(q,t,n+2)} := \left\{ (p_1, p_2) \in \mathbb{Z}^2 \,\middle|\, p_1 + p_2 = q \in \mathbb{Z}, \ (p_1, p_2, p_3) \in \Xi_3^{(t,n+2)} \right\},$$

$$\bigcup_{q \in \mathbb{Z}} \Xi_2^{(q,t,n+2)} = \Xi_3^{(t,n+2)},$$

and the exterior difference operator d_Q by

$$\mathrm{d}_Q \tau(\boldsymbol{p}) = \mathrm{D}_1 \tau(\boldsymbol{p}) \mathrm{d}p_1 + \mathrm{D}_2 \tau(\boldsymbol{p}) \mathrm{d}p_2,$$

which displaces the lattice space

$$\omega^{-1} \mathrm{D}\omega : \Xi_2^{(q,t,n+2)} \to \Xi_2^{(q+1,t,n+2)}.$$

Because $t = q + p_3$, we can fix p_3 instead of t. In this frame of coordinate, the change of t is exactly the same with the change of q. We recall that the diagram Figure 5.5 (b) was the projection Figure 5.5 (a) along p_3. Since $j = p_2 - p_1 \in \mathbb{Z}$ is still free we denote $\tau_j^t := \tau(\boldsymbol{p})$ and consider the lattice space

$$\left\{ (j, t) \in \mathbb{Z}^2 \,\middle|\, \Delta t = \Delta q, \ p_2 - p_1 = j \in \mathbb{Z}, \ (p_1, p_2) \in \Xi_2^{(q,t,n+2)} \right\}.$$

In our previous section we discussed the transfer of information along the chain of octahedra. We now extend the study to consider a transfer of information of many octahedra linked along a line in the j direction.

We have to mention that the connection condition (5.27) is already taken into account in this expression. Moreover the projection along p_3 enforces degeneration of $Z'[t]$ in $\mathcal{O}[t]$ and $Z[t+2]$ in $\mathcal{O}[t+2]$. Therefore the shift operation d_T brings $\tilde{S}_4[t+1]$ directly to $\tilde{S}_4[t+2]$, so that τ_j^t is determined uniquely for all t and j as we can see in the Figure 5.6.
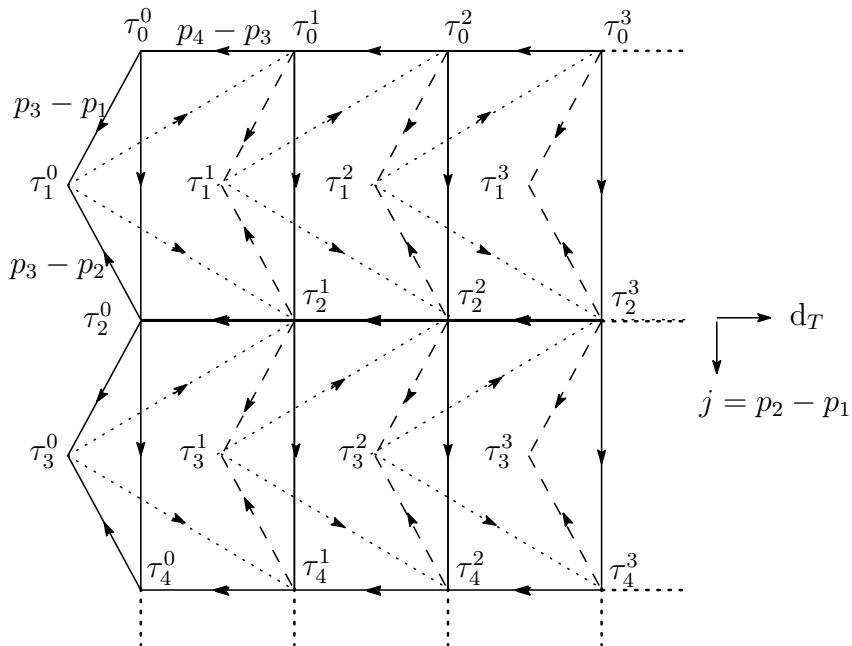
Figure 5.6:

In the theory of KP hierarchy it is known that we can either truncate the function τ_j^t, or impose periodicity in the direction of j at any value, with no violation of integrability. For example we can impose

$$\tau_{j+d}^t = \tau_j^t, \tag{5.30}$$

to obtain a reduced map of d dimension. In this case it is more convenient to consider

$$\boldsymbol{\tau}^t = (\tau_0^t, \tau_1^t, \tau_2^t, \tau_3^t, ..., \tau_{d-1}^t),$$

instead of a triangle $\tilde{S}_4[t+1]$ of each octahedron separately. When $d = 3$, the chain of Figure 5.6 becomes a chain of triangles.

5.4.2 Localization of a Triangulated Category

The theory of triangulated category tells us that, if there is a null system, the theory can be localized such that the localized theory again satisfies the axioms of triangulated category[50][51]. A null system $\mathcal{N} \subset \mathcal{T}$ of the triangulated category $\mathcal{T}$ is a set of objects defined by

1. $0 \in \mathcal{N}$, where 0 is null object that is object as both of initial object and terminal object.

2. $Z \in \mathcal{N}$ if $X, Y \in \mathcal{N}$ and $X \to Y \to Z \to X[1]$ is a distinguished triangle.

3. $X[1] \in \mathcal{N} \quad iff \quad X \in \mathcal{N}$.

For any triangulated category $\mathcal{T}$ and a null system $\mathcal{N} \subset \mathcal{T}$, we define a multiplicative system by

$$S(\mathcal{N}) := \left\{ \mathcal{G} \;\middle|\; X \xrightarrow{\mathcal{G}} Y \to Z \to X[1], \; X, Y \in \mathcal{T}, \; Z \in \mathcal{N} \right\}.$$

Then the localization is defined by the functor $\mathcal{T} \to \mathcal{T}/S(\mathcal{N})$. The following theorem is known in the theory of triangulated category:

Theorem

  $\mathcal{T}/S(\mathcal{N})$ *is again a triangulated category whose null object is 0 itself.*

Therefore, in order to discuss the localization of our system we must know a null system of our "triangulated category". Our objects are solutions $\{\tau_{ij}\}$ of the HM eq. which are assigned at corners of each octahedron. They are generically finite, because the τ functions are defined by the ratios (5.5) of correlation functions such that the zeros of correlation functions cancel from each other.

As we explained in §5.1.1 the correlation functions $\{\Phi_{ij}\}$ vanish by themselves when two punctures encounter on the Riemann surface. It is, however, important to notice that, when the correlation functions $\Phi(p, z; G)$ and $\Phi(p, z; 0)$ vanish simultaneously, the cancellation of their zeros does not mean the value of their ratio being definite. Let $\lambda(p)$ be the ratio of the correlation functions at the point where they vanish, $i.e.$

$$\lambda(p) := \left\{ \left. \frac{\Phi(p, z; G)}{\Phi(p, z; 0)} \; \right| \; \Phi(p, z; G) = \Phi(p, z; 0) = 0 \right\}, \qquad (5.31)$$

then $\lambda(p)$ is set of indeterminate points at $p \in \Xi_4^{(n+2)}$ in general, hence can take any value. Since zero is not excluded in (5.31) we call the zero of $\lambda(p)$ the null object and denote by 0, $i.e.,$

$$0 \in \lambda(p).$$

We now focus our attention to this subtle object in the following discussion and show how the localization of triangulated category resolves the subtlety.

The localization of our system will be introduced by considering rational maps of the τ functions. To be specific we consider some reduced flow diagrams of Figure 5.6 which satisfy the condition (5.30). In particular we study in detail rational maps defined by the following variables:

$$x_j^t = \begin{cases} \dfrac{\tau_{j+\epsilon+1}^t \tau_{j-\epsilon}^{t+1}}{\tau_{j+1}^t \tau_j^{t+1}}, & j, \epsilon = 1, 2, 3, ..., d, \quad \text{LV}, \\[4mm] \dfrac{\tau_{j+1}^t \tau_{j-\epsilon}^{t+1}}{\tau_j^t \tau_{j+1-\epsilon}^{t+1}}, & j, \epsilon = 1, 2, 3, ..., d, \quad \text{KdV}. \end{cases} \qquad (5.32)$$

The maps which are obtained by the new variables LV and KdV of (5.32) are nothing but the LV map and the KdV map[41][42].

An important feature of the variables (5.32) is that they are invariant under the local gauge transformation of τ_j^t

$$\mathcal{G}[\mu, \nu] : \tau_j^t \to \exp\left(\int^t \mu(t', j)\mathrm{d}t' + \int^j \nu(t, j')\mathrm{d}j'\right) \tau_j^t, \tag{5.33}$$

where $\mu(t, j)$ and $\nu(t, j)$ are arbitrary functions. For example, we can write the right hand side of (5.32) as

$$x_j^t = \begin{cases} \dfrac{\Phi_{j+\epsilon+1}^t(p, z)\Phi_{j-\epsilon}^{t+1}(p, z)}{\Phi_{j+1}^t(p, z)\Phi_j^{t+1}(p, z)}, & j, \epsilon = 1, 2, 3, ..., d, \quad \text{LV}, \\[2em] \dfrac{\Phi_{j+1}^t(p, z)\Phi_{j-\epsilon}^{t+1}(p, z)}{\Phi_j^t(p, z)\Phi_{j+1-\epsilon}^{t+1}(p, z)}, & j, \epsilon = 1, 2, 3, ..., d, \quad \text{KdV}. \end{cases} \tag{5.34}$$

This follows from the fact that the denominator of $\tau_{j+k}^{t+u}(p)$ is given by

$$\Phi_{j+k}^{t+u}(p, z, 0) = (z_1 - z_2)^{(p_1-k)(p_2+k)}(z_3 - z_4)^{(p_3+u)(p_4-u)},$$

so that all denominators of τ functions in (5.32) are eliminated exactly from the expression.

We notice that we can not distinguish a change of $\lambda(p)$ in (5.31) with the gauge transformation

$$\mathcal{G}[\mu, \nu] : \lambda(p) \to \lambda'(p). \tag{5.35}$$

This means that, if $\Lambda(\infty)$ is the set of all possible $\lambda(p)$, *i.e.*,

$$\Lambda(\infty) := \left\{ \lambda(p) \mid p \in \Xi_4^{(n+2)} \right\}. \tag{5.36}$$

$\Lambda(\infty)$ is invariant under the gauge transformation.

Now suppose that τ_j^{t+1} (or $\tau_{j+1-\epsilon}^{t+1}$ in the KdV case) in the denominator of x_j^t in (5.32) is the null object, hence takes the value zero. Then x_j^t and also x_{j-1}^{t+1} diverge while all components of $\boldsymbol{x}^{t+u}$ with $u \geq 2$ are finite, as far as other τ functions are finite. This owes to the fact that the same τ function does not propagate beyond two steps. There is no way to determine the values of τ^{t+1}'s because the null object is invariant under the gauge (5.35), *i.e.*,

$$\mathcal{G}[\mu, \nu] : 0 \to \mathcal{G}[\mu, \nu]0 = 0.$$

In other words the null object is transferred to an indeterminate object,

$$\mathrm{d}_T : \{0\} \to \lambda(p). \tag{5.37}$$

which is an element of $\Lambda(\infty)$. It should be emphasized that $\Lambda(\infty)$ does not appear in the localized theory, because the localized variables x_j^t's are gauge invariant. Thus we are strongly suggested to identify $\Lambda(\infty)$ with the null system $\mathcal{N}$ of our map

$$\mathcal{N} \sim \Lambda(\infty). \tag{5.38}$$

Our argument in the rest of this paper will be devoted to support this conjecture.

5.4.3   SC of 3 dimensional Maps

To proceed our argument further we consider the case $d = 3$ and $\epsilon = 1$ in (5.32) for simplicity. Then the HM eq. becomes the following rational maps, 3dLV map(4.2) and 3dKdV map(2.34). These maps have two invariants (2.30) and (2.37), respectively.

We can solve the initial value problem of the HM eq. following to the algorithm:

A1 Fix initial values $\boldsymbol{\tau}^0 = (\tau_0^0, \tau_1^0, \tau_2^0)$ and $\boldsymbol{\tau}^1 = (\tau_0^1, \tau_1^1, \tau_2^1)$ by hand to determine $\boldsymbol{x} = (x_1, x_1, x_3)$.

A2 $\boldsymbol{x}^t := (x_1^t, x_2^t, x_3^t), \quad t \geq 1$ are obtained as functions of $\boldsymbol{x}$ iteratively by the maps 3dLV map or 3dKdV map.

A3 $\boldsymbol{\tau}^{t+1} = (\tau_0^{t+1}, \tau_1^{t+1}, \tau_2^{t+1}), \quad t \geq 1$ are determined by (5.32) from $\boldsymbol{\tau}^t$ and $\boldsymbol{x}^t$.

Needless to say this procedure of solving the HM eq. (5.1) is compatible with the flow of information through the chain of octahedra, since the rational maps 3dLV map and 3dKdV map are derived from the HM eq. by the transformation of dependent variables (5.32). The algorithm is certainly deterministic, since values of $\boldsymbol{\tau}^t$ for all $t \geq 2$ are determined if the initial values $\boldsymbol{\tau}^0$ and $\boldsymbol{\tau}^1$ are fixed. As we will show in the following, however, it becomes not clear how the null object appears during the procedure.

The SC of the 3dLV and 3dKdV map have been studied in detail in the previous chapter. To see what happens we review this problem from the view point of the theory of category.

Since we are interested in studying the SC we fix the initial conditions such that $\boldsymbol{x}^1$ is divergent. Without loss of generality this condition is satisfied by requiring for the denominator of x_1^1 to vanish. Let us solve the 3dLV map case following to our algorithm.

A1 We fix the initial values $\boldsymbol{\tau}^1$ at

$$\boldsymbol{\tau}^1 = (\lambda_0, \lambda_1, \lambda_2),$$

and, instead of fixing $\boldsymbol{\tau}^0$ by hand, we require

(a) denominator of x_1^1 vanishes:

$$1 - x_3^0 + x_3^0 x_1^0 = 0.$$

(b) invariants f, g are fixed by

$$f = x_1^0 x_2^0 x_3^0 - (1 - x_1^0)(1 - x_2^0)(1 - x_3^0), \quad g = 1 + (1 - x_1^0)(1 - x_2^0)(1 - x_3^0),$$

from which we obtain

$$\boldsymbol{x}^0 = \left(\frac{f}{f+g}, r\frac{g}{f}, \frac{f+g}{g} \right), \quad r := f + g - 1 = x_1^0 x_2^0 x_3^0, \tag{5.39}$$

and

$$\boldsymbol{\tau}^0 = \left(f\lambda_1, \ g\lambda_2, \ (f+g)\lambda_0 \right).$$

A2 Iteration of the 3dLV yields the sequence of SC,

$$\boldsymbol{x} \to (\infty, 0, 1) \to (1, 0, \infty) \to \boldsymbol{x}^3 \to \boldsymbol{x}^4 \to \cdots \tag{5.40}$$

A3 (a) From $\boldsymbol{x}^1 = (\infty, 0, 1)$, we find $\tau_1^2 = 0$ and $\tau_2^2 \lambda_2 = \tau_0^2 \lambda_1$. Since an overall factor is irrelevant we obtain

$$\boldsymbol{\tau}^2 = (\lambda_2, 0, \lambda_1).$$

(b) From $\boldsymbol{x}^2 = (1, 0, \infty)$ we find $\tau_0^3 \lambda_2 = \tau_1^3 \lambda_1$, but τ_2^3 is undetermined.

(c) Since $\boldsymbol{x}^t$'s are finite for all $t \geq 3$, the rest of $\boldsymbol{\tau}^{t+1}$ are determined for all $t \geq 3$, thus we obtain, up to overall factors,

$$\boldsymbol{\tau}^0 \to (\lambda_0, \lambda_1, \lambda_2) \to (\lambda_2, 0, \lambda_1) \to (\lambda_0', \lambda_1', \lambda_2')$$

$$\to \left(\lambda_2' \alpha^{(2)}, \lambda_0' \gamma^{(2)}, \lambda_1' \beta^{(2)} \right) \to \left(\lambda_1' \alpha^{(3)}, \lambda_2' \gamma^{(3)}, \lambda_0' \beta^{(3)} \right) \to \cdots \tag{5.41}$$

Here we defined new functions

$$(\lambda_0', \lambda_1', \lambda_2') := (\tau_0^3, \tau_1^3, \tau_2^3),$$

which are free as far as

$$\lambda_0' \lambda_2 = \lambda_1' \lambda_1, \tag{5.42}$$

is satisfied.

From this result it is clear how the SC undergoes. The singularities of $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ in (5.40) come from $\tau_1^2 = 0$. This null object is the source of the singularities. This information, however, can transfer only to its neighbor since τ_1^2 does not appear beyond $\boldsymbol{x}^2$. Hence it does not transfer directly to remote objects.

We can extend the sequence of (5.41) to the left, if we apply the inverse map of 3dLV to $\boldsymbol{x}$. We find, with $\lambda_{t+3} := \lambda_t,\ \lambda'_{t+3} := \lambda'_t$,

$$
\boldsymbol{\tau}^{t+2} = \begin{cases}
\left(\lambda'_{1-t}\alpha^{(t)},\ \ \lambda'_{2-t}\gamma^{[t]},\ \ \lambda'_{-t}\beta^{(t)} \right), & t \geq 2, \\
(\lambda'_0, \lambda'_1, \lambda'_2), & t = 1, \\
(\lambda_2,\ 0,\ \lambda_1), & t = 0, \\
(\lambda_0, \lambda_1, \lambda_2), & t = -1, \\
\left(\lambda_{2-t}\beta^{(-t)}, \lambda_{-t}\gamma^{[-t]}, \lambda_{1-t}\alpha^{(-t)} \right), & t \leq -2.
\end{cases}
\tag{5.43}
$$

Here we denote by $\gamma^{[t]}$ the product

$$
\gamma^{[t]} := \prod_{t|t'} \gamma^{(t')}.
$$

We can summarize our result of this subsection by the diagram in Figure 5.7.
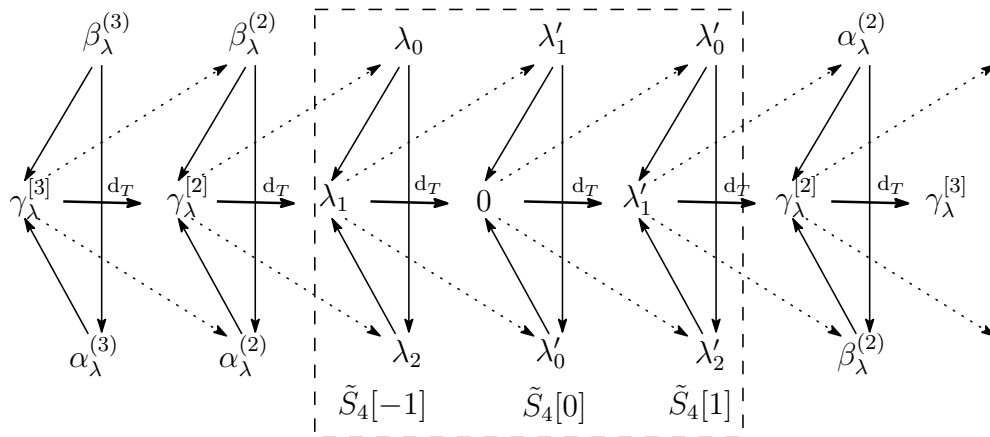
Figure 5.7: $\boldsymbol{\tau}^{[t]},\ t \in \mathbb{Z}$

Here we used the notations $\alpha_\lambda^{(t)}, \beta_\lambda^{(t)}$ and $\gamma_\lambda^{[t]}$ to represent the elements of $\boldsymbol{\tau}$ in (5.43).

From this analysis we learned that the existence of a null object introduces free functions $(\lambda'_0, \lambda'_1, \lambda'_2)$ in addition to the initial values $(\lambda_0, \lambda_1, \lambda_2)$. Since it is constrained by (5.42) and the overall factor is irrelevant there is only one degree of freedom. As we have seen it comes from the gauge invariance of the null object $\mathcal{G}[\mu, \nu]0 \sim 0$.

We are now ready to apply the localization theorem of the category theory discussed in §5.4.2. The diagram of Figure 5.7 shows how the effect of this gauge freedom propagates as t increases. It is important to notice that all objects in three triangles $\tilde{S}_4[-1], \tilde{S}_4[0], \tilde{S}_4[1]$ are indeterminate. In other words

$$\tilde{S}_4[-1], \tilde{S}_4[0], \tilde{S}_4[1] \subset \Lambda(\infty).$$

If we accept the conjecture (5.38) to identify $\mathcal{N} \sim \Lambda(\infty)$, we naturally define our multiplicative system by the set of gauge transformations

$$S(\mathcal{N}) = \{\, \mathcal{G}[\mu, \nu] \mid \mathcal{G}[\mu, \nu] : \lambda(p) \to \lambda'(p), \ 0 \to \mathcal{G}[\mu, \nu]0 = 0 \,\},$$

so that our local theory is obtained simply by gauge fixing all λ_j's and λ'_j's at 1.
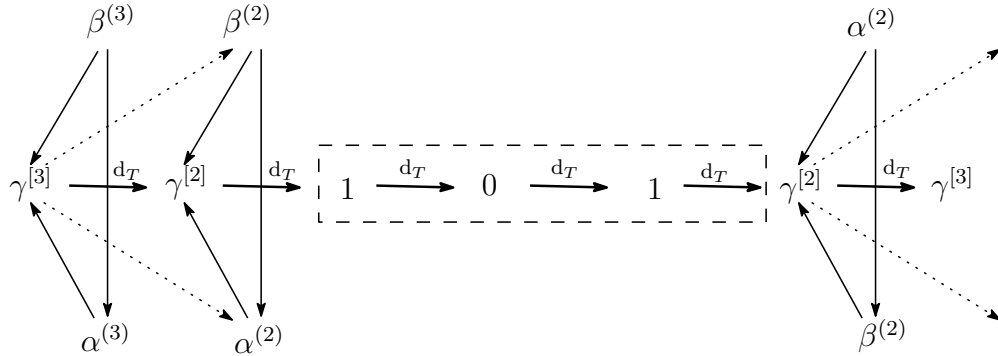
Figure 5.8:

5.4.4 "Projective Resolution"

Let us consider a subchain of Figure 5.7,
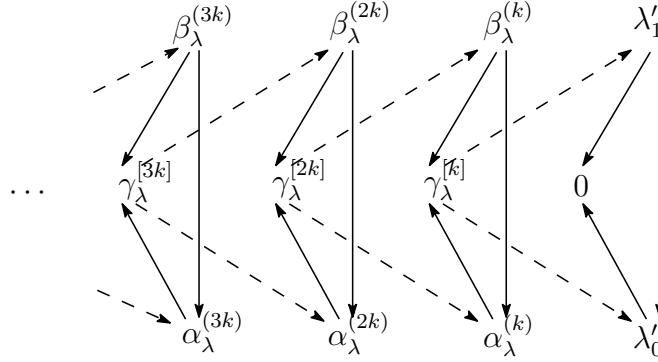
Figure 5.9:

which can be obtained by iteration of the map $\boldsymbol{x}^t \to \boldsymbol{x}^{t+k}$. Especially in the diagram
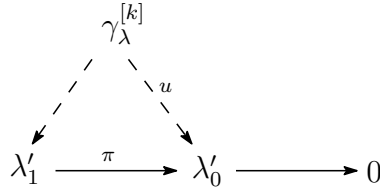

Figure 5.10:

the morphism $u : \gamma_\lambda^{[k]} \to \lambda_0'$ passes the epimorphism $\pi : \lambda_1' \to \lambda_0'$. Hence the object $\gamma_\lambda^{[k]}$ is a "projective object" for all k, and the "exact sequence"

$$P := \cdots \to \gamma_\lambda^{[4]} \to \gamma_\lambda^{[3]} \to \gamma_\lambda^{[2]} \to \lambda_1 \to \lambda_0' \to 0, \tag{5.44}$$

is a "projective resolution" of λ_0'.

This is a result of the theory of triangulated category in mathematics. It tells us that infinitely many projections by $\gamma_\lambda^{[t]}$'s constitute the object λ_0'.

5.5 Discussion

5.5.1 Localization and 3 dimensional Maps

While $\mathcal{HM}$ has a null set $\Lambda(\infty)$, however, $\Lambda(\infty)$ does not appear in the maps of the variables defined by (5.32), for example. Moreover we discussed the localization of $\mathcal{HM}$ by means of the gauge transformations $\mathcal{S}(\Lambda(\infty))$. Therefore we notice, in the case of 3dLV map, a relation as follows:

$$3 \text{ dimensional maps} \sim \mathcal{HM}/\mathcal{S}(\Lambda(\infty)).$$

5.5.2 Derivation of IVPPs

Since $\tau_1^2 = 0$, a point satisfying $\tau_1^{t+2} = 0$ must include periodic points of period t. On the other hand $\tau_1^{t+2} \propto \gamma^{[t]}$ holds as we can see from (5.43). If we repeat the procedure explained in the previous subsections we derive a sequence of polynomial functions of the invariants. Thus we have found an important fact:

⎧ Derivation of IVPPs by SC and "Projective Resolution" ⎤

  P of (5.44) is a chain of polynomial functions whose zero sets are IVPPs.

5.5.3 IDP of IVPPs

Now we must call attention to another remarkable feature of IVPPs. In previous chapter, we have found that IVPPs of all periods intersect on a IDP. It is a set of "singular points" by two reasons.

1. Every point on IDP is singular because it is a point which is occupied by periodic points of all periods simultaneously.

2. It is an indeterminate point of the map. Namely IDP is a set of points on which the denominators and the numerators of the rational map vanish simultaneously.

Now we can translate information of IDP into the language of τ functions using the formula (5.32). In the case of LV map the points $\boldsymbol{x}^1 = (\infty, 0, 1)$ and $\boldsymbol{x}^2 = (1, 0, \infty)$ are on SP. From the algorithm of §5.4.3 they correspond exactly to the three triangles $\left(\tilde{S}_4[-1], \tilde{S}_4[0], \tilde{S}_4[1] \right)$ of the chain in Figure 5.7. The latter belongs to $\Lambda(\infty)$, which is a set of indeterminate points again, but of the τ functions instead of the map functions.

The correspondence of IDP with $\Lambda(\infty)$ is clear because the indeterminacy of both functions comes from the same source, *i.e.*, the zero set of correlation functions $\Phi(p, z; G)$. Moreover from this correspondence we see that λ_0' and λ_1 in P of (5.44) must be objects in which all IVPPs are degenerate. In other words the null set is a source from which IVPPs are generated. This will provide us an interpretation of the "projective resolution".

Chapter 6

Transition of Integrable/Non Integrable System

In this thesis we have so far studied mainly properties of integrable systems. In order to understand integrable systems, however, it is quite important to study them in comparison with non integrable systems. We show in this chapter that, when a non-integrable rational map changes to an integrable one continuously, a large part of the periodic points approach indeterminate points(IDP) of the map along algebraic curves. Not only this type of work has not been discussed in other literatures, but our study is still at the beginning[52, 53]. Therefore we consider here deformations of only two maps, the 2 dimensional Möbius map and the 3dLV map, to clarify our argument.

6.1 2 dimensional Möbius Map

We consider one of the deformations of integrable map. First, we show the case of 2 dimensional Möbius map (2.7).

$$F_{\mathrm{2dM\ddot{o}b}}(x,y) \to (F_{\mathrm{2dM\ddot{o}b}})_a(x,y) = \left(x\frac{1-y}{1-x-a}, \ y\frac{1-x}{1-y} \right), \qquad (6.1)$$

where $x, y \in \mathbb{C}$ are variables of the map and $a \in \mathbb{C}$ is a continuous parameter.

6.1.1 Fixed points

There are two fixed points of $(F_{\mathrm{2dM\ddot{o}b}})_a$ at

$$(x,y) = (0,0) \text{ and } (-a,0),$$

Although the fixed points are not considered being periodic in general, they form the line $y = x$ in the integrable limit in this particular map (6.1).

6.1.2 Period 2 points

When $a \neq 0$, we must solve the periodicity condition to find the period 2 points. They are at $(x,y) = (x_\pm^2, y_\pm^2)$, where

$$(x_\pm^2, y_\pm^2) = \left(1 - \frac{a}{2} \pm \sqrt{\frac{a(a^2-4)}{a-4}}, \ 1 + \frac{a(2-a)}{4} \mp \frac{a}{2}\sqrt{\frac{a(a^2-4)}{a-4}} \right), \qquad (6.2)$$

Now we recall that, in the $a = 0$ case, there is no IVPP of period 2, while the period 2 points of (6.2) exist at $a = 0$. But the point $(x,y) = (1,1)$, where the period 2 points of (6.2) approach, is exactly the IDP of the map $F_{\mathrm{2dM\ddot{o}b}}$. Therefore all period 2 points approach the IDP in this case, and none of them move to IVPP, in the integrable limit.

The explicit expression of the points like (6.2) is not easy to find as the period n becomes large. It will be more convenient to present the polynomial function $K_a^{(n)}(x)$

from which we can derive $x_j^{(n)}$ by solving $K_a^{(n)}(x) = 0$, and another polynomial function $L_a^{(n)}(x)$ so that $y_j^{(n)}$ is given by $y_j^{(n)} = L_a^{(n)}(x_j^{(n)})$. In the period 2 case we obtain

$$\begin{aligned}
K_a^{(2)}(x) &= (a-4)x^2 + (a-2)(a-4)x - 2(a-2)(a-1), \\
L_a^{(2)}(x) &= \left(1 - \frac{a}{2}\right)(x+a),
\end{aligned} \tag{6.3}$$

from which we find (6.2) immediately.

Now we want to know the paths of the periodic points along which they move as a changes and approaches in the limit $a = 0$. We can do it if we eliminate the parameter a from $K_a^{(2)}(x) = 0$ and $y - L_a^{(2)}(x) = 0$ of (6.3). The result we obtain is the algebraic curve $G^{(2)}(x,y) = 0$, with

$$G^{(2)}(x,y) = (2-x)^2(1-y)^2 - 3(1-x)(1-y)(2-x) + (1-x)^2(2-x+xy). \tag{6.4}$$

This curve certainly passes the $\mathrm{IDP}(F_{2\mathrm{dMöb}}) = (1,1)$, hence $G^{(2)}(1,1) = 0$, as we can check easily. From (6.3) we see that it corresponds the integrable limit $a = 0$. To see the paths of period 2 points, we can draw the curve on the (x,y) real plane. We find a curve in Figure 6.1.
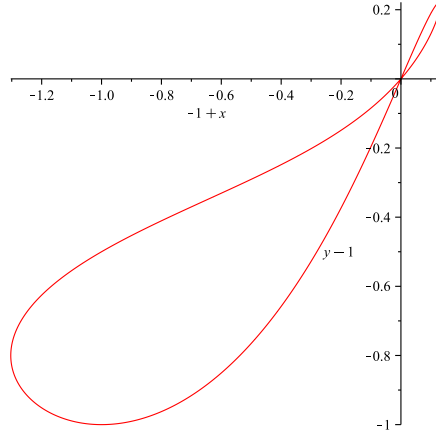
Figure 6.1: Path of period 2 points.

Notice that the $\mathrm{IDP}(F_{2\mathrm{dM\ddot{o}b}}) = (1, 1)$ is shifted to the origin in this graph. From this picture it is apparent that the IDP is the singular locus of the curve (6.4). In fact we can convince ourselves that the multiplicity of the curve at the point $(1, 1)$ is two. Since $K_a^{(2)}(x)$ and $L_a^{(2)}(x)$ are smooth functions of a, two periodic points must approach IDP along this curve at the same time, continuously as a becomes small.

Similarly, we can give curves of some other periods, as shown in Figure 6.2. In the picture the red, blue and green curves correspond to period 2, 3, and 4, respectively: In Figure 6.3 the details of these curves near $(x, y) = (1, 1)$ are shown.
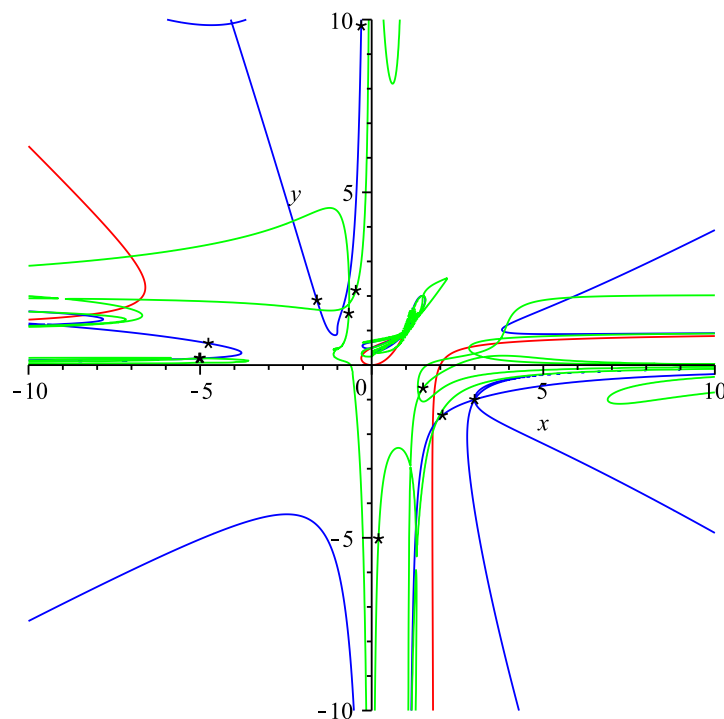
Figure 6.2: Paths of points of period two, three and four.
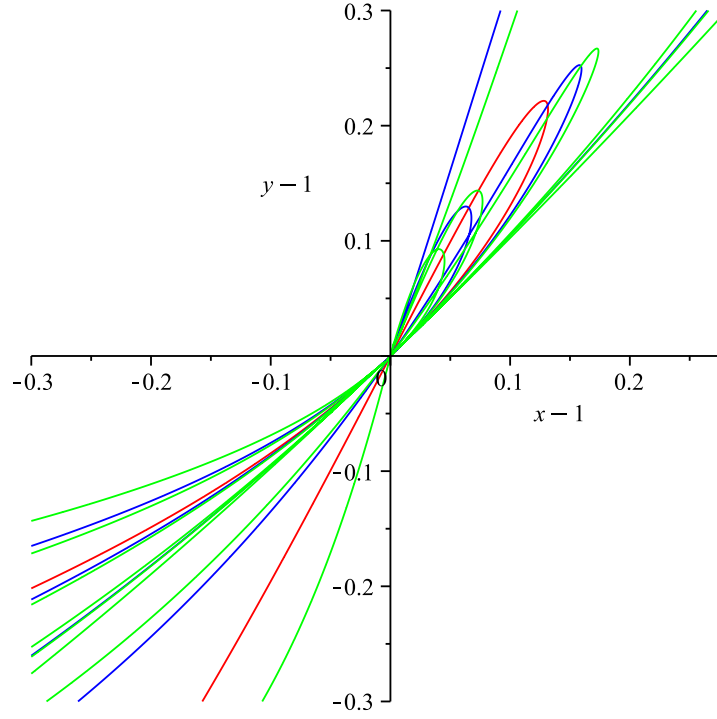
Figure 6.3: Details of the paths of period 2,3 and 4.

6.2 3 dimensional Lotka-Volterra Map

We can apply the same method to see how the periodic points of the deformed 3dLV map

$$
\begin{aligned}
F_{\text{3dLV}}(x, y, z) \quad &\rightarrow \quad (F_{\text{3dLV}})_{a,b}(x, y, z) \\
&= \left(x \frac{1 - y + yz}{1 + a - z + zx}, y \frac{1 + b - z + zx}{1 - x + xy}, z \frac{1 - x + xy}{1 - y + yz} \right), (6.5)
\end{aligned}
$$

move. In this case we have two parameters a, b. As they change every periodic point must move on a surface instead of a curve. In fact, after elimination of a, b from the

period 2 conditions, we obtain a formula

$$K_{a,b}^{(2)}(x, y, z) = 0, \tag{6.6}$$

which represents a complex surface in $\mathbb{C}^3$. The surface is shown in Figure 6.4, and we show a detail of one of the intersection curves in Figure 6.5.
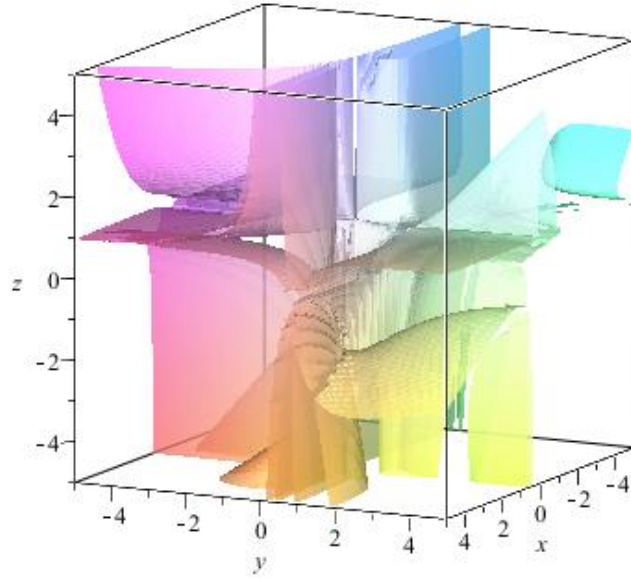
Figure 6.4: period 2 surface $K^{(2)} = 0$
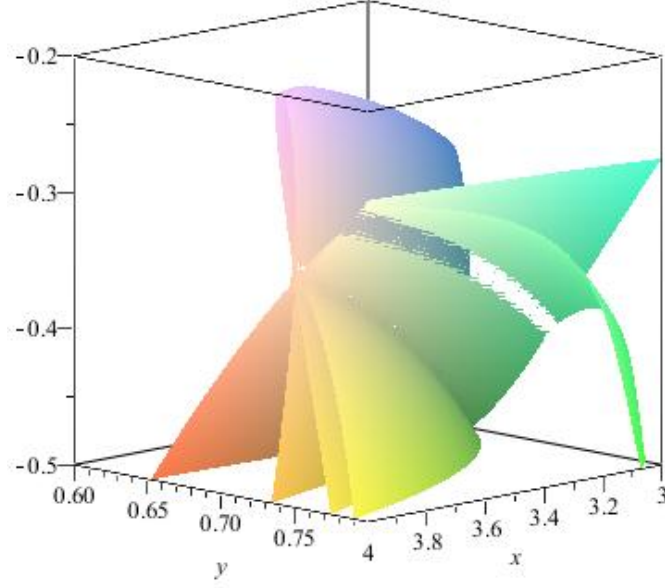
Figure 6.5: details near IDP(F_{3dLV})

The curves along which the surface (6.6) intersect with itself is not difficult to find. They consist of two independent curves

$$
\begin{aligned}
\Lambda^+ &:= \left\{ \left(1 - \frac{1}{t}, \frac{1}{1-t}, t \right) \in \mathbb{C}^3 \,\middle|\, t \in \mathbb{C} \right\}, \\
\Lambda^- &:= \left\{ \left(\frac{1}{1-t}, 1 - \frac{1}{t}, t \right) \in \mathbb{C}^3 \,\middle|\, t \in \mathbb{C} \right\}.
\end{aligned}
\tag{6.7}
$$

The curve of Figure 6.5 corresponds to Λ^+. From our experience in the deformed 2dMöbius map, all surfaces of different periods must intersect along the same curves $\Lambda^\pm$. It is, however, very difficult to draw surfaces of large period numbers.

A remarkable fact is that the curves we observed in Figure 6.5 coincide exactly

with $\Lambda^{\pm}$. This means that almost all periodic points approach $\Lambda^{\pm}$ in the integrable limit.

6.3 Discussion

Some remarks are in order: We have presented the behavior of periodic points here only those of small number of period. Nevertheless we are certain that all other periodic points of higher periods behave similar, because of the IVPP theorem. In particular the periodic points will approach the IDP as deformation parameters become zero, if they do not move to the IVPPs. Since this is true for all periods, we can say that a large part of the Julia set approach the IDP.

Chapter 7

Conclusion

In this chapter, we give conclusion about this thesis.

7.1 ADE Dynamical System

First, we extended the dynamical system of a mapping to an ADE, in which time evolution is given by elimination ideal. However an ADE has IDP with "gauge transformation", then the sequence of iteration of the mapping does not become an "exact sequence". For one of the solutions of this problem, in the case of HM eq., we considered IDP with gauge transformation as null system. Therefore, within the frame work of localization of triangulated category, the sequence of iteration of the mapping becomes an "exact sequence". In other words, an ADE dynamical system is identified with an "exact sequence" of iteration of the mapping.

7.2 IVPP Theorem

Next, we proved IVPP theorem that is the main theorem of this thesis. Because IVPP theorem gave Integrable/Non Integrable conjecture that is a judgemental con-

jecture of integrability. In Chap.4, we discussed a method to derive IVPPs by SC. Therefore we can determinate the Integrable/Non Integrable system by this method. Furthermore, we considered the structure of this method as "projective resolution" of "triangulated category", in the case of the HM eq.. Hence we understand the sufficient condition of integrability has "projective resolution" structure. In addition, by the Invariant/Parameter duality, this structure provides an ADE resolution by REs of all periods like a Fourier expansion.

7.3   Intersections of IVPPs

In addition, we considered about intersections of IVPPs that were seemed a problem of IVPP theorem. We discussed about the conditions for the existence of intersections of VPPs, which are the origins of IDP, common factors and singular points of IVPP, and we analyzed some examples. The reasons of these origins, we considered in Chap.6. We deformed the Integrable/Non Integrable systems in the case of 2 dimensional Möbius map and the 3dLV map. We studied the fate of Julia set to IDP and IVPPs, and thus found that infinitely many periodic points go to IDP.

Acknowledgements

Appendix A

Affine Algebraic Variety

Affine Algebraic Variety

Let $f := \{f_1, \ldots, f_n\}$ be elements of $\mathbb{C}[\boldsymbol{x}]$. We denote the (affine algebraic) variety $\mathcal{V}(f)$ that is generated by f as follows,

$$\mathcal{V}(f) := \left\{ \boldsymbol{x} \in \mathbb{C}^d \mid f_i(\boldsymbol{x}) = 0, \ i = 1, \ldots, d \right\}.$$

Ideal and Variety

We denote the ideal I_f that is generated by f as follows,

$$I_f := (f_1, \ldots, f_N) := \left\{ g(\boldsymbol{x}) \in \mathbb{C}[\boldsymbol{x}] \ \middle| \ g(\boldsymbol{x}) = \sum_{i=1}^{N} h_i(\boldsymbol{x}) f_i(\boldsymbol{x}), \ h_i(\boldsymbol{x}) \in \mathbb{C}[\boldsymbol{x}], \ i = 1, \ldots, N \right\}.$$

Moreover, we denote the variety $\mathcal{V}(I_f)$ that is generated by I_f as follows,

$$\mathcal{V}(I_f) := \left\{ \boldsymbol{x} \in \mathbb{C}^d \mid g(\boldsymbol{x}) = 0, \ {}^{\forall}g(\boldsymbol{x}) \in I_f \right\}.$$

In addition, we define an equivalence between ideal I_1 and I_2 when the ideals I_1

and I_2 generate the same variety [1] ,

$$I_1 \sim I_2 \quad \Leftrightarrow \quad \mathcal{V}(I_1) = \mathcal{V}(I_2). \tag{A.1}$$

Ideal Calculations and Variety

Ideals I_f and $I_{f'}$ are defined addition $I_f + I_{f'}$ as follows,

$$I_f + I_{f'} = (f, f')$$

This addition has important property as follows,

$$\mathcal{V}(I_f + I_{f'}) = \mathcal{V}(I_f) \cap \mathcal{V}(I_{f'}).$$

Similarly, Ideals I_f and $I_{f'}$ are product $I_f \cdot I_{f'}$ as follows,

$$I_f \cdot I_{f'} = \left\{ g''(\boldsymbol{x}) \in \mathbb{C}[\boldsymbol{x}] \;\middle|\; g''(\boldsymbol{x}) = \sum_{j=1}^{n} g_j(\boldsymbol{x}) g_j'(\boldsymbol{x}), \; g_j(\boldsymbol{x}) \in I_f, \; g_j'(\boldsymbol{x}) \in I_{f'}, \; n \in \mathbb{N} \right\}$$

This product has important property as follows,

$$\mathcal{V}(I_f \cdot I_{f'}) = \mathcal{V}(I_f) \cup \mathcal{V}(I_{f'}).$$

---

[1]In formal mathematical definition, a variety is given by a radical of an ideal[29][38]. Therefore

$$I_1 = I_2 \Leftrightarrow \mathcal{V}(I_1) = \mathcal{V}(I_2).$$

is satisfied. However, we do not use a radical of an ideal for our convenience.

Appendix B

d dimensional Lotka-Volterra Map

The original Lotka-Volterra map of d dimension was defined [41][42] by

$$X_j^{t+\delta}\left(1 - \delta X_{j-1}^{t+\delta}\right) = X_j^t\left(1 - \delta X_{j+1}^{t+\delta}\right), \quad j = 1, \ldots, d, \tag{B.1}$$

$$X_j^t = X_{j+d}^t, \quad \text{(periodic condition)},$$

which becomes, by taking the zero limit of the the minimal step of time δ, integrable continuous time Lotka-Volterra equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}X_j(t) = X_j(t)\left[X_{j-1}(t) - X_{j+1}(t)\right], \quad j = 1, \ldots, d.$$

Hereafter we fix $\delta = 1$ and denote $t = n \in \mathbb{N}$.

Invariants of the map (B.1) were also derived in [41][42], but in implicit form. We present here their explicit form cited from [7] [8][9],

$$H_k := \begin{cases} 1 - (-1)^d q_1 q_2 \cdots q_d, & k = 0 \\ \sum'_{j_1, j_2, \ldots, j_k} q_{j_1} q_{j_2} \cdots q_{j_k}, & k = 1, 2, \ldots, [d/2] \\ 0, & k = [d/2] + 1, \ldots, d - 1 \\ -1, & k = d, \end{cases} \tag{B.2}$$

$$q_j := X_j(1 - X_{j-1}),$$

here $[d/2] = d/2$ if d is even and $[d/2] = (d-1)/2$ if d is odd. The prime in the summation $\sum'$ means that the summation must be taken over all possible combinations of $j_1, j_2, ..., j_k$ but excluding direct neighbors. Since H_0 can be represented by other H_k's and

$$R := X_1 X_2 \cdots X_d$$

it is convenient to use R instead of H_0. The total number of the invariants is

$$p = \left[\frac{d+2}{2} \right].$$

Appendix C

String/Soliton Correspondence

String Theory

Classical Theory

First, we give notations of (closed) string theory[54]. The equation of motion(EOM) of a string about complex coordinate $z, \bar{z}$ is given by

$$\partial_z \partial_{\bar{z}} X(z, \bar{z}) = 0.$$

Therefore, the solution of this EOM, is the following form

$$X(z, \bar{z}) = X(z) + X(\bar{z}),$$

where

$$X(z) = \frac{1}{2} x + \frac{i}{4} \alpha_0 \log z + \sum_{n \neq 0} \frac{\alpha_n}{n} z^n, \quad X(\bar{z}) = \frac{1}{2} x + \frac{i}{4} \alpha_0 \log \bar{z} + \sum_{n \neq 0} \frac{\tilde{\alpha}_n}{n} \bar{z}^n.$$

In this appendix, we consider $X(z)$, a part of $X(z, \bar{z})$.

Quantum Theory

In addition, the first (canonical) quantization of the string is given by canonical commutative relation(CCR) as follows,

$$\left[i\frac{x}{2}, -\frac{\alpha_0}{4}\right] = -1, \quad \left[\frac{i}{n}\alpha_n, i\alpha_{-m}\right] = -\delta_{nm}, \quad n, m \in \mathbb{N}. \tag{C.1}$$

Therefore, we can take a differential representation of CCR(C.1) by new variables $t_n, n = 0, 1, 2, \ldots$

$$t_0 = i\frac{x}{2}, \quad t_n := \frac{i}{n}\alpha_n, \quad \frac{\partial}{\partial t_0} := -\frac{1}{4}\alpha_0, \quad \frac{\partial}{\partial t_n} := i\alpha_{-n}, \quad n \in \mathbb{N}.$$

Vertex Operator

The tachyon vertex operator, that is given by the state/operator correspondence[54], is defined as

$$|p, z\rangle \sim V(p, z) := \mathrm{e}^{ipX(z)} = \mathrm{e}^{ipX^-(z)}\mathrm{e}^{ipX^+(z)},$$

where

$$X^-(z) := -i\sum_{n=0}^{\infty} z^n t_n, \quad X^+(z) := -i\left(\log z\frac{\partial}{\partial t_0} - \sum_{n=1}^{\infty}\frac{z^{-n}}{n}\frac{\partial}{\partial t_n}\right).$$

Product of Vertex Operators

Lemma

We need Baker-Campbell-Hausdorff's formula for our calculation. In particular, if $[A, B]$ is a scalar then it is written as follows,

$$\mathrm{e}^A\mathrm{e}^B = \mathrm{e}^{[A,B]}\mathrm{e}^B\mathrm{e}^A.$$

Normal Ordering

Normal ordering is defined as follows,

$$: V(p, z)V(p', z') : := V^-(p, z)V^-(p', z')V^+(p, z)V^+(p', z'),$$

where $V^\pm(p, z) := e^{ipX^\pm(z)}$.

Vertex Operator Algebra

We want to check the relation(5.6)

$$V(p, z)V(p', z') = (-1)^{pp'}V(p', z')V(p, z).$$

Check

$$
\begin{aligned}
V(p, z)V(p', z') &= e^{ipX^-(z)}e^{ipX^+(z)}e^{ip'X^-(z')}e^{ip'X^+(z')} \\
&= e^{ipX^-(z)}\left(e^{-pp'[X^+(z),X^-(z')]}e^{ip'X^-(z')}e^{ipX^+(z)}\right)e^{ip'X^-(z')} \\
&= e^{-pp'[X^+(z),X^-(z')]} : V(p, z)V(p', z') :,
\end{aligned}
$$

where,

$$
\begin{aligned}
[X^+(z), X^-(z')] &= -\left[\left(\log z \frac{\partial}{\partial t_0} - \sum_{n=1}^\infty \frac{z^{-n}}{n}\frac{\partial}{\partial t_n}\right), \sum_{m=0}^\infty z'^m t_m\right] \\
&= -\left(\log z \left[\frac{\partial}{\partial t_0}, t_0\right] - \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{1}{n}\frac{z'^m}{z^n}\left[\frac{\partial}{\partial t_n}, t_m\right]\right) \\
&= -\left(\log z - \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{1}{n}\frac{z'^m}{z^n}\delta_m^n\right) \\
&= -\left(\log z - \sum_{n=1}^\infty \frac{1}{n}\left(\frac{z'}{z}\right)^n\right) \\
&= -\left(\log z + \log\left[1 - \left(\frac{z'}{z}\right)\right]\right) \\
&= -\log(z - z').
\end{aligned}
$$

Hence we get

$$
\begin{aligned}
V(p,z)V(p',z') &= (z-z')^{pp'} : V(p,z)V(p',z') : \\
&= (-1)^{pp'} V(p',z')V(p,z)
\end{aligned}
$$

String/Soliton Correspondence

The Miwa transformation[15] is given as,

$$
t_0 = \sum_{j=1}^{\infty} p_j \log z_j, \quad t_n = -\frac{1}{n} \sum_{j=1}^{\infty} p_j z_j^{-n}, \quad n \in \mathbb{N}.
$$

Therefore we can get

$$
-i\frac{\partial}{\partial p_j} = -i\sum_{n=0}^{\infty} \frac{\partial t_n}{\partial p_j}\frac{\partial}{\partial t_n} = -i\left(\log z_j \frac{\partial}{\partial t_0} - \sum_{n=1}^{\infty} \frac{z_j^{-n}}{n}\frac{\partial}{\partial t_n} \right), \quad j = 1,\ldots,4,
$$

i.e.

$$
e^{ip_j X^+(z_j)} = e^{p_j \frac{\partial}{\partial p_j}}.
$$

Moreover, we can get

$$
\begin{aligned}
e^{ip_j X^-(z_j)} &= \exp\left[-p_j \sum_{n=1}^{\infty} z_j^n t_n \right] \\
&= \exp\left[-p_j \sum_{\substack{i=1 \\ i\neq j}}^{\infty} p_i \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{z_j}{z_i}\right)^n \right] \\
&= \exp\left[-p_j \sum_{\substack{i=1 \\ i\neq j}}^{\infty} p_i \log\left[1 - \left(\frac{z_j}{z_i}\right) \right] \right] \\
&= \prod_{\substack{i=1 \\ i\neq j}}^{\infty} \left[1 - \left(\frac{z_j}{z_i}\right) \right]^{-p_j p_i}.
\end{aligned}
$$

Hence we can get a relation between the vertex operator and the shift operator as follows,

$$V(p_j, z_j) = \prod_{\substack{i=1 \\ i \neq j}}^{\infty} \left[1 - \left(\frac{z_j}{z_i} \right) \right]^{-p_j p_i} e^{p_j \frac{\partial}{\partial p_j}}.$$

Appendix D

Triangulated Category

In order to see this correspondence more in detail let us first recall the axioms of triangulated category[50][56].

Definition

Let $\mathcal{D}$ be an additive category, X, Y, Z, X', Y', Z' be objects and u, v, w be morphisms of $\mathcal{D}$. The structure of a *triangulated category* on $\mathcal{D}$ is defined by the *shift functor T* and the class of *distinguished triangles* satisfying the following axioms:

Tr1 (1)Any triangle of the form

$$X \xrightarrow{\text{id}} X \longrightarrow 0 \longrightarrow T(X)$$

  is in the class of distinguished triangles.

  (2) Any triangle isomorphic to a distinguished triangle is distinguished.

  (3) Any morphism $u : X \to Y$ can be completed to a distinguished triangle

$$X \xrightarrow{u} Y \longrightarrow C(u) \longrightarrow T(X)$$

  by the object $C(u)$ obtained by morphism u.

Tr2 The triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$$

is a distinguished triangle if and only if

$$Y \xrightarrow{v} Z \xrightarrow{w} T(X) \xrightarrow{-T(u)} T(Y)$$

is a distinguished triangle.

Tr3 Suppose there exists a commutative diagram of distinguished triangles,

$$
\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & T(X) \\
u\downarrow & & v\downarrow & & & & \downarrow T(u) \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & T(X').
\end{array}
$$

Then this diagram can be completed to a commutative diagram by a (not necessarily unique) morphism $w : Z \to Z'$.

Tr4 (the octahedron axiom) Let $X \xrightarrow{u} Y \xrightarrow{v} Z$ be a triangle. Then the following commutative diagram holds:

$$(D.1)$$

Bibliography

[1] S. MacLane, *"Categories for the working mathematician"*, Graduate texts in mathematics, vol. 5, Springer-Verlag, 1998.

[2] R. Hirota and D. Takahashi, *"Discrete and Ultradiscrete Systems"* (In Japanese), Kyoritsu Pub., 2003.

[3] A. F. Beardon, *"Iteration of rational functions"*, Graduate Texts in Mathematics, vol. 132, Springer-Verlag, New York, 1991.

[4] R. M. Cohn, *"Difference algebra"* , Interscience Publishers John Wiley and Sons, New York-London-Sydeny, 1965.

[5] W. Michael, *"Algebraic difference equations"*, Lecture notes, `http://www.algebra.rwth-aachen.de/en/Mitarbeiter/Wibmer/AlgebraicDifferenceEquations.pdf`, 2013.

[6] A.P. Veselov, *Uspekhi mat. nauk*, **46**:5(281), 3-45, 1991.

[7] S. Saito and N. Saitoh, *J. Phys. Soc. Jpn.*, **76** No.2 p.024006, 2007.

[8] S. Saito and N. Saitoh *J. Math. Phys*, **51** 063501, 2010.

[9] S. Saito and N. Saitoh, *"Invariant varieties of periodic points"* in Mathematical Physics Research Developments, 2008 Nova Science Publishers, Inc., Capt.3 pp 85-139, 2008.

[10] R. L. Devaney, *"An Introduction to Chaotic Dynamical Systems"*, Westview Press, 2003.

[11] E. A. Coddington and N. Levinson, *"Theory of Ordinary Differential Equations"*, McGraw-Hill, 1955.

[12] M. Sato and Y. Sato, Research Institute of Mathematical Studies Report, **388**, 183; *ibd* **414**, 181 (19981), 1980.

[13] K. Takasaki, *"The World of Integrable Systems"* (In Japanese), Kyoritsu Pub., 2001.

[14] R. Hirota, J. *Phys. Soc. Jpn.*, **50** 3787, 1981.

[15] T. Miwa, *Proc. Japan Acad.*, **58A** 9, 1982.

[16] V. I. Arnold, "*Mathematical Methods of Classical Mechanics*", Springer-Verlag, 1978.

[17] R. S. MacKay and J. D. Meiss, *"Hamiltonian Dynamical Systems"*, a reprint selection, Adam-Hilgar Press, 1987.

[18] F. Klein, *"Lectures on the ikosahedron and the solution of equations of the fifth degree"*, Dover edition, 1888.

[19] M. Robert, *"Simple mathematical models with very complicated dynamics"*, Nature 261(5560):459-467, 1976.

[20] B. Grammaticos, A. Ramani, and V. Papageorgiou, *Phys. Rev. Lett.* **67**, 1825, 1991.

[21] A. Ramani, B. Grammaticos, J.Satsuma, *Phys. Lett.* A 169, 323-328, 1992.

[22] B. Grammaticos, A. Ramani and K. M. Tamizhmani, T. Tamizhmani, and A. S. Carstea, *Advances in Difference Equations*, Volume 2008, Article ID 317520, 2008.

[23] S. Lafortune and A. Goriely, *J. Math. Phys.* **45**, 1191-1208, 2004.

[24] R. Conte and M. Musette, *"The Painlevé handbook"*, Springer, Dordrecht, 2008.

[25] K. Iwasaki and T. Uehara, *Math. Ann.*, **338**, 295-345, 2007.

[26] J. Hietarinta and C. Viallet, *Phys. Rev. Lett.*, **81**, 325, 1998.

[27] J.-P. Eckmann, and D. Ruelle, *Rev. Mod. Phys.*, **57**, 616, 1985.

[28] Y. Nakamura, et. al., *"The Applications of Mathematical science of The Integrable systems"* (In Japanese), Shokabo, 2000.

[29] M. F. Atiyah and I. G. MacDonald, *"Introduction to commutative algebra"*, Addison-Wesley Publish- ing Co., Reading Mass.-London-Don Mills, 1969.

[30] B. B. Mandelbrot, *"The fractal geometry of nature"*, New York: W. H. Freeman and Company, 1982.

[31] R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete Mathematics* (Addison-Wesley), 1994.

[32] R. Hirota and H. Yahagi, *J. Phys. Soc. Jpn.*, **71**, 2867, 2002.

[33] S. Saito and N. Saitoh, *J. Phys. A: Math. Theor.*, **40**, 12775-12787, 2007.

[34] D. Cox, J. Little and D. O'Shea. *"Ideals, Varieties, and Algorithms"*. Undergradu-. ate Texts in Mathematics. Springer-Verlag, New York, second edition, 1997.

[35] G. D. Birkhoff, *Trans. Amer. Math. Soc..* **14**, 14, 1913.

[36] G. D. Birkhoff, *Acta Math.*, **47** 297, 1925.

[37] W. Rudin, Principles of Mathematical Analysis, Mcgraw-Hill,1953-1976.

[38] R. Hartshorne, *"Algebraic geometry"*, Graduate Texts in Mathematics, No. 52. Springer-Verlag, New. York-Heidelberg, 1977.

[39] B. Buchberger, *An Algorithm for Finding a Basis for the Residue Class Ring of a Zero-Dimensional Polynomial Ideal*, Ph.D. thesis, University of Innsbruck, Austria, (1965).

[40] M. G. Marinari, T. Mora and H. M. Moeller, *J. AAECC.*, **4**, 103-145, 1993.

[41] R. Hirota, S. Tsujimoto and T. Imai, *Difference Scheme of Soliton Equations,* in *Future Directions of Nonlinear Dynamics in Physical and Biological Systems*, ed. by P.L.Christiansen at al., (Plenum Press, New York, 1993) p.7.

[42] R. Hirota, and S. Tsujimoto, *J.Phys.Soc.Jpn.*, **64** 3125-3127, 1995.

[43] J. D. Fay, *"Theta Functions on Riemann Surfaces"*, edited by A. Dold and B.Eckmann, Lecture Notes in Mathematics Vol.352, Springer-Verlag, Berlin, 1973.

[44] I. M. Krichever, *Russ. Math. Surv.*, **32**, 185 (1977).

[45] S. Saito, *Phys. Rev. Lett.*, **59** 1798, 1987.

[46] M. Kashiwara and T. Miwa, *Proc. Jpn. Acad.*, **57A** 342, 1981.

[47] E. Date, M. Kashiwara and T. Miwa, *Proc. Jpn. Acad.*, **57A** 387, 1981.

[48] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, *J. Phys. Soc. Jpn.*, **50** 3806, 3813, 1981.

[49] N. Shinzawa and S. Saito, *J. Phys. A: Math. Gen.*, **31** 4533, 1998.

[50] M. Kashiwara and P. Schapira, *"Sheaves on manifolds"*, Berlin : Springer, 1990.

[51] H. Krause, *"Derived categories, resolutions, and Brown representability"*, arXiv: math.KT/0511047.

[52] N. Saitoh and S. Saito, *J.Phys. Soc. Jpn.*, **77** 024001, 2008.

[53] S. Saito, N. Saitoh, H. Harada, T. Yumibayashi and Y. Wakimoto, AIP Advances, AIP ID: 003306ADV, 2013.

[54] J. Polchinski, *"String Theory"*, Cambridge University Press, 1998.

[55] Y. Yamada, *"An Introduction to Conformal Field Theory"*(In Japanese), Baifukan, 2006.

[56] S. Gelfand and Yu. Manin, *"Methods of Homological Algebra"*, Springer, 1996.

[57] S. Saito and N. Saitoh, *J.Math. Phys.*, **28** 1052, 1987.

[58] R. Sato and S. Saito, *J.High Energy Phys.*, JHEP 11, p047, 2004.

[59] S. Saito, *J. Nonlinear Math. Phys.*, **19** 1250032, 2012.

[60] T. Yumibayashi, S. Saito, and Y. Wakimoto, *Phys. Lett. A.*, **378** 2014.

[61] S. Saito, T. Yumibayashi, and Y. Wakimoto, *Prog. Theor. Exp. Phys.*, 023A08, 2014.

[62] B. Grammaticos, A. Ramani and V. Papageogiou, *Phys. Rev. Lett.*, **67** 1825, 1991.

[63] A. Ramani, B. Grammaticos and J. Hietarinta, *Phys. Rev. Lett.*, **67** 1829, 1991.

[64] M. Mulase, *J.Differential Geom.*, **19**, 403, 1984.

[65] T. Shiota, *Invent. Math.*, **83** 333, 1986.

[66] L. E. Reichl, *"Transition to Chaos"*, Springer-Verlag, 1992.

[67] J. W. Milnor, *"Singular Points of Complex Hypersurfaces"*, Princeton University Press, 1968.