INTERMEDIATE PARTICLES IN S-MATRIX THEORY AND CALCULATION OF HIGHER ORDER EFFECTS IN THE PRODUCTION OF INTERMEDIATE VECTOR BOSONS

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Thesis-1963-Veltman

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PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE WISKUNDE EN NATUURWETENSCHAPPEN AAN DE RIJKSUNIVERSITEIT TE UTRECHT, OP GEZAG VAN DE RECTOR MAGNIFICUS DR. W. C. VAN UNNIK, HOOGLERAAR IN DE FACULTEIT DER GODGELEERDHEID, VOLGENS BESLUIT VAN DE SENAAT DER UNIVERSITEIT TEGEN DE BEDENKINGEN VAN DE FACULTEIT DER WISKUNDE EN NATUURWETENSCHAPPEN TE VERDEDIGEN OP MAANDAG 22 APRIL 1963, DES NAMIDDAGS TE VIER UUR PRECIES

DOOR

MARTINUS JUSTINUS GODEFRIDUS VELTMAN

GEBOREN TE WAALWIJK

Promotor: Prof. Dr. L. Ch. P. Van Hove

CONTENTS

Chapter I

INTRODUCTION AND SUMMARY

Ever since the introduction of the Fermi theory[1]) describing β-decay the theory of weak interactions, despite spectacular successes, has been in a rather unsatisfactory state, at least from a theoretical point of view. Many new processes behaving in many respects like β-decay have been discovered, but no systematic scheme giving a unified description of all processes of this kind has been found despite many years of thorough investigation. Also the field theoretical description of these processes is very poor; the Fermi theory of weak interactions is non-renormalizable, which means that one does not know any method to extract results beyond the lowest order of perturbation theory.

Nevertheless some progress has been made. The fact that the theory is non-renormalizable indicates that the structure of the particles involved plays an important role. This idea has received further amplification since the discovery of parity non-conservation[2]) and the subsequent proposal of the specific form of the interaction known as the V–A theory[3]). This latter theory suggests very strongly a structure whereby the weak interactions are mediated by a new particle of spin one, i.e., a vector boson, in much the same way as electromagnetic interactions are mediated by the photon. To account for the existing experimental data, this boson, which is usually called the intermediary boson, must be charged and have a mass that exceeds the mass of the K meson. As far as its other quantum numbers are concerned great difficulties appear. Thus, to account for the observed processes, one apparently needs at least 6 different types of intermediary bosons. On the other hand, however, the problem of the radiative decay of the muon, experimentally unobserved, but theoretically possible if there is only one kind of neutrino and if an intermediary boson exists, has been resolved by the experimental discovery of the existence of two different neutrinos[4]).

From the field theoretical point of view, such an intermediary boson introduces into the weak interactions a structure which leads to a less divergent, although still non-renormalizable, theory. However, in at least two respects the theoretical situation is not satisfactorily clarified: (i) no systematic study has been made of renormalizable theories containing unstable particles, and (ii) theories with derivative couplings, to which the theory with intermediary vector boson can be reduced, are not yet studied very extensively.

Our Chapter II presents a contribution to the first of these problems. We investigate with the help of diagram techniques perturbation theory for a model containing an unstable particle, limiting ourselves for simplicity to scalar particles. It turns out that properties of renormalizability and causality are the same as those in theories containing only stable particles. With respect to unitarity the situation, however, is deeply modified and more complicated. In fact, we must make a rearrangement of the perturbation series to get the theory into a satisfactory form. The method followed in attacking this problem is the following. First, a mathematical identity relating diagrams from different orders in the perturbation series is proved. Next, the main difficulty in the usual perturbation expansion is exhibited and remedied by a rearrangement of the perturbation series. The rest of the chapter is devoted to an investigation of the properties of this rearranged series; it is shown that the resulting theory is unitary, causal, and renormalizable, and further that it is still a solution of the original field equations.

Chapter III discusses a feature of the theory of intermediary vector bosons of more direct experimental interest. In the near future the question of the existence of an intermediate vector boson will be experimentally investigated. This will be done with the help of highly energetic neutrinos used to create vector bosons through the reactions $\nu_\mu \to \mu^- + W^+$ or $\nu_e \to e^- + W^+$ (e = electron, μ = muon, ν_e and ν_μ the corresponding neutrinos, W = intermediary boson). These processes are kinematically impossible in free space, and in practice they will occur most conveniently in the Coulomb field of a nucleus. To lowest order in both the electromagnetic and weak coupling constants, the first process has been calculated by several authors [5]. However, if the materials used have a high nuclear charge Z the perturbation parameter $Ze^2/\hbar c$ related to the nuclear Coulomb field can be so large (for lead one has $Ze^2/\hbar c \sim 0.6$) that lowest order perturbation theory becomes questionable. In view of the future interpretation of experimental facts, we study in Chapter III the higher order corrections in $Ze^2/\hbar c$ which are of main interest at high energies. A calculation is made in a high energy approximation with the help of techniques introduced earlier by Bethe-Maximon [6]) in connection with creation of an electron-positron pair by a photon in a Coulomb field. In two respects, however, we go beyond this treatment: firstly, a term in the equations of motion that cannot be treated with the Bethe-Maximon techniques is necessary for our purpose and is included in our approximation, and secondly, in order to improve our results in the region of low muon (or electron) energy, we introduce additional terms in such a way that the high energy behaviour is not affected, while the matrix element remains correct up to second order in the coupling constant $Ze^2/\hbar c$ at all lepton energies. The difficulties arising from the charge distribution of the nucleus have been dealt with in an approximative way to improve the validity of our results at neutrino energies near the threshold for intermediate boson production.

Acknowledgements. The author is greatly indebted to Professor L. Van Hove for his most valuable advice and criticism throughout this work. He wishes to express his gratitude to Dr. S. Berman and Dr. K. Symanzik for a number of illuminating discussions.

In addition, the author wants to thank the "Stichting voor Fundamenteel Onderzoek der Materie" for the opportunity to perform, at the University of Utrecht, a part of the present investigation in the framework of its research programme.

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CHAPTER II

UNITARITY AND CAUSALITY IN A RENORMALIZABLE FIELD THEORY WITH UNSTABLE PARTICLES

Synopsis

The problems of unitarity, causality and renormalizability are treated in a field theory containing an unstable particle. Perturbation theory is suitably modified and leads to an implicit equation for complete propagators. The S-matrix constructed with these propagators and connecting stable particle states only is shown to be unitary, renormalizable and causal. It is also shown to give rise to interpolating Heisenberg fields which verify the original field equations.

1. *Introduction.* In recent years many authors discussed unstable particles in the framework of quantum field theory. In particular, Matthews and Salam[1]) gave suitable definitions of mass and lifetime of an unstable particle in terms of its field theoretical propagator. As shown by Jacob and Sachs[2]) these definitions are in agreement with the experimental situation.

In the present paper we study another aspect of a field theory with unstable particles, namely the questions of unitarity, causality and renormalization in perturbation theory. To be more explicit suppose that we have a situation where an unstable scalar particle, say A-particle, can decay into two identical stable scalar particles, say φ-particles. In setting up perturbation theory for such a model one starts by introducing the "bare" fields A and φ, obeying the Klein-Gordon equation and coupled to each other in some way specified by an interaction Lagrangian. Ordinary perturbation theory leads then, however, to a very undesirable feature, namely the unstable A-particles appear at infinite times in incoming and outgoing states. A realistic theory cannot have this feature, but if one just removes the unstable particle states from the in- and out-states one is faced with the problem of unitarity of the resulting truncated S-matrix. The problem can now be stated as follows: consider the Hilbert space of stable particle states. Is it then possible to construct by suitable modification of perturbation theory an S-matrix which is unitary in this Hilbert space. The answer is

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yes, and the present paper gives the necessary modification of perturbation theory. Of course, such an S-matrix still has to meet in addition the requirements of renormalizability and causality (where one can use for the latter for instance the definitions of Lehmann-Symanzik-Zimmermann[3]) or Bogoliubov-Shirkov-Medvedev-Polivanov[4])), but it will be shown that in this respect no particular difficulties arise in the present case as compared with a theory describing stable particles only.

There is still a further point of interest in connection with the above mentioned S-matrix. From the fact that no unstable particles appear in ingoing or outgoing states one expects that it should be possible to reformulate the theory in terms of the stable field operators only, and indeed this will turn out to be the case. Such a treatment, however, leads to a Lagrangian density involving fields at different space time points (non-local interaction) and again in setting up perturbation theory in this new framework one encounters difficulties with respect to unitarity[5]). Clearly, our treatment of the unstable particle case may then be of help in understanding non-local theories, in the sense that we have a concrete model of a theory with non-local interaction, which, in a non-trivial way, satisfies the requirements of unitarity, causality and renormalizability.

Our method will now be as follows: for a simple model of unstable A-particles and stable φ-particles we construct an S-matrix which can be expected to satisfy the requirements formulated above. This S-matrix will be given in terms of diagrams, with propagators which satisfy the Lehmann-representation[6]) and a certain implicit equation derived in section 3. It will be shown then that this S-matrix is unitary in the Hilbert space of stable particle states, that it is a solution of the field equations of the model, and finally that it is causal in the sense of Bogoliubov. Throughout the work renormalization is taken care of.

In section 2 we derive a general identity valid for diagrams with propagators obeying the Lehmann representation. In section 3 the model mentioned above is introduced explicitly, and the difficulties resulting in a perturbation expansion from the unstability of the A-particle are exhibited. As a result perturbation theory is reformulated, leading to an implicit equation for the propagators involved. Next an S-matrix involving stable particles only is constructed in section 4 and its unitarity properties are established with the help of the identity derived in section 2. Section 5 gives the connection of this S-matrix with the field equations of the model. Finally, in section 6, using the techniques of section 2 and 5, it is shown that causality holds in the sense of Bogoliubov.

2. *The cutting formula.* This formula describes certain properties of diagrams arising from the properties of field propagators. The complete propagator associated with a field $\varphi(x)$ (which for simplicity we suppose to

be a real scalar field) is defined by:

$$\bar{\Delta}_F(x) = \Theta(x_0)\,\bar{\Delta}^+(x) + \Theta(-x_0)\,\bar{\Delta}^-(x)$$

$$\bar{\Delta}^+(x) = i\langle 0\,|\varphi(x)\,\varphi(0)\,|0\rangle \qquad \bar{\Delta}^-(x) = i\langle 0\,|\varphi(0)\,\varphi(x)|\,0\rangle$$

$$\bar{\Delta}^-(x) = -(\bar{\Delta}^+(x))^* \qquad \Theta(u) = \begin{cases} 1 & u > 0 \\ \frac{1}{2} & u = 0 \\ 0 & u < 0 \end{cases} \tag{2.1}$$

Inserting intermediate states in the expressions for $\bar{\Delta}^+$ one observes that $\bar{\Delta}^+$ is a positive frequency function with respect to time and consequently $\bar{\Delta}^-$ a negative frequency function. Following Lehmann[6]) we write (we take the metric so that $kx = \boldsymbol{kx} - k_0x_0$):

$$\bar{\Delta}^\pm(x) = \frac{i}{(2\pi)^3}\int \mathrm{d}_4k\, \mathrm{e}^{ikx}\Theta(\pm k_0)\int_0^\infty \mathrm{d}\kappa^2\delta(k^2+\kappa^2)\,\rho(-\kappa^2) \tag{2.2}$$

with $\rho(-\kappa^2) \geqslant 0$.

Using (2.1), (2.2) and some structure properties of diagrams we will be able to prove the cutting formula. In its simplest form, for a diagram consisting of two points x and x' and one propagator, it reduces to

$$\bar{\Delta}_F(x-x') - \bar{\Delta}^+(x-x') - \bar{\Delta}^-(x-x') - \bar{\Delta}_F^*(x-x') = 0$$

or:

$$2iIm\bar{\Delta}_F(x-x') = \bar{\Delta}^+(x-x') + \bar{\Delta}^-(x-x')$$

which clearly is an immediate consequence of (2.1). In much the same way for a general diagram the cutting formula gives an expression for the imaginary part of the diagram. It will be used later on to prove unitarity. Further it will give some information concerning the space-time behaviour of diagrams which we will show to be related to causality.

We will start now by proving the cutting formula without explicit reference to diagrams. In order to simplify matters we leave the space variables aside and in the rest of this paragraph x will stand for time only.

Let there be given a number of functions $f_i^+(x)$, $f_i^-(x)$ so that $(f_i^+(x))^* = -f_i^-(x)$, and so that $f_i^+(x)$ contains only positive frequencies while $f_i^-(x)$ contains only negative ones, i.e. they can be written in the form:

$$f_i^\pm(x) = \int \mathrm{e}^{-ipx}\,\Theta(\pm p)\,f_i^\pm(p)\,\mathrm{d}p \tag{2.3}$$

This is clearly compatible with the reality requirement above. We define functions $f_i(x)$ by:

$$f_i(x) = \Theta(x)\,f_i^+(x) + \Theta(-x)\,f_i^-(x)$$

so that

$$f_i^*(x) = -\Theta(x)\,f_i^-(x) - \Theta(-x)\,f_i^+(x) \tag{2.4}$$

The analogy of these equations with (2.1), (2.2) is obvious. For $x = 0$ we have $f_l(0) = -f_l^*(0)$. The inversion formulae of (2.4) are

$$\begin{aligned} f_l^+(x) &= \Theta(x)\, f_l(x) - \Theta(-x)\, f_l^*(x) \\ f_l^-(x) &= -\Theta(x)\, f_l^*(x) + \Theta(-x)\, f_l(x) \end{aligned} \tag{2.5}$$

We now consider a function F of n variables $x_1 \dots x_n$, which can be written as a product of $f_l(x)$ functions, with as arguments of these functions only differences $x_i - x_j$:

$$F(x_1 \dots x_n) = \prod_r f_{l_r}(x_{\alpha_{1r}} - x_{\alpha_{2r}}) \tag{2.6}$$

Any function f_l can appear several times. As for the variables $x_1 \dots x_n$ we assume that each appears at least once and that $\alpha_{1r} \neq \alpha_{2r}$ for each r.

We will now replace in a systematic way in $F(x_1 \dots x_n)$ functions f by f^+, f^- or f^*. By definition the function

$$F_m^+(x_1 \dots x_{i_1-1} x_{i_1+1} \dots x_{i_m-1}\, x_{i_m+1} \dots x_n \,|\, x_{i_1} \dots x_{i_m}) \tag{2.7}$$

is constructed from $F(x_1 \dots x_n)$ by replacing a function $f_{l_r}(x_{\alpha_{1r}} - x_{\alpha_{2r}})$ by

$f_{l_r}^+(x_{\alpha_{1r}} - x_{\alpha_{2r}})$ if $x_{\alpha_{1r}}$ appears on the left and $x_{\alpha_{2r}}$ on the ight of the bar
$f_{l_r}^-(x_{\alpha_{1r}} - {}_{\alpha_{2r}}x)$ if $x_{\alpha_{1r}}$ appears on the right and $x_{\alpha_{2r}}$ on the left of the bar
$-f_{l_r}^*(x_{\alpha_{1r}} - x_{\alpha_{2r}})$ if $x_{\alpha_{1r}}$ and $x_{\alpha_{2r}}$ appear on the right of the bar.

If both $x_{\alpha_{1r}}$ and $x_{\alpha_{2r}}$ appear on the left hand side of the bar we leave the function $f_{l_r}(x_{\alpha_{1r}} - x_{\alpha_{2r}})$ unchanged. Suppose now that x_a is smaller than or equal to any of the other x. One sees very simply with the help of (2.4) or (2.5) that

$$\begin{aligned} &F_m^+(x_1 \dots x_{i_1-1} x_{i_1+1} \dots x_a \dots x_n \,|\, x_{i_1} \dots x_{i_m}) = \\ &\quad F_{m+1}^+(x_1 \dots x_{i_1-1} x_{i_1+1} \dots x_n \,|\, x_a x_{i_1} \dots x_{i_m}) \qquad x_a \leqslant x_1 \dots x_n \end{aligned} \tag{2.8}$$

In words: the functions F_m^+ do not change if the smallest x_a of all x is moved from the right to the left of the bar and vice versa. Take now a subset $x_{j_1} \dots x_{j_p}$ containing x_a. We write for brevity

$$x_{j_q} = y_q$$

From (2.8) follows

$$\begin{aligned} \sum_{q=0}^{p} \sum_{\{q\}} (-1)^q F_{m+q}^+(x_1 \dots y_1 \dots x_{i_1-1} x_{i_1+1} \dots y_{s_1-1} y_{s_1+1} \dots \\ \dots y_p \dots x_n \,|\, x_{i_1} \dots x_{i_m} y_{s_1} \dots y_{s_q}) = 0 \end{aligned} \tag{2.9}$$

$\{q\}$ means summation over all choices of q numbers $s_1 \dots s_q$ out of $1 \dots p$ with $s_1 < s_2 \dots < s_q$. The functions F_i^+ in this equation differ only in the position of the y with respect to the bar. The result follows by observing that each function F_i^+ with the smallest variable to the left of the bar cancels a function F_{i+1}^+ with the smallest variable to the right of the bar.

Suppose now that (2.9) is integrated over a number of variables from

— to + infinity. (2.9) can be true in the whole region of integration only if the subset $y_1 = x_{j_1} \dots y_p = x_{j_p}$ contains all integration variables as well as the smallest (or one of the smallest) of the variables which are not integrated over. In this way we arrive at the *cutting formulae*:

$$\int \mathrm{d}y_{\alpha_1} \dots \mathrm{d}y_{\alpha_\nu} \sum_{q=0}^{p} \sum_{\{q\}} (-1)^q F^+_{m+q}(x_1 \dots y_1 \dots x_{i_1-1} x_{i_1+1} \dots \\ \dots y_{s_1-1} y_{s_1+1} \dots y_p \dots x_n | x_{i_1} \dots x_{i_m} y_{s_1} \dots y_{s_q}) = 0 \tag{2.10}$$

It holds if there exists an i with $i \neq \alpha_1 \dots \alpha_\nu$ such that

$$y_i \leqslant x_i \text{ for all } i \neq j_1, \dots j_p$$

One easily proves a similar formula for a function F^-, defined from F^+ by interchange of f^+ and f^- functions, if one replaces $\leqslant$ by $\geqslant$ in the inequalities just given for the y_i.

Up to now we have not made use of the fact that f^+ and f^- contain only positive or negative frequencies (see (2.3)). We note that due to (2.3) many terms in (2.10) actually vanish. This can be understood by introducing (2.3) into (2.10) and performing the integrations over the integration variables y_{α_i} (of course also for the f and f^* functions Fourier representations must be inserted). Each integration gives rise to a δ-function of an (algebraic) sum of variables p_β. Every f^+ or f^- function containing a y_{α_i} gives rise to a $\Theta(p_\beta)$ or $\Theta(-p_\beta)$, and it may well happen that the product of these Θ functions and the δ function is always zero. This happens for instance for the product $\delta(p_1 + p_2)\, \Theta(p_1)\, \Theta(p_2)$. As can easily be established in general with the definition given above for the F^+ functions, a term in (2.10) will be zero if from the set of integration variables $y_{\alpha_1} \dots y_{\alpha_\nu}$ a subset $y_{\beta_1} \dots y_{\beta_\mu}$ can be found which is entirely situated on one side of the bar and such that every function depending on the difference of one of the y_β and one of the other variables (y as well as x) is a f_l^+ or f_l^-.

We will now picture (2.10) with the help of diagrams (consisting of lines and vertices). The functions F can be pictured in the normal way: any variable x corresponds with a vertex (integration variables correspond to what will be called "internal vertices", whereas any other vertices will eventually be end-point of an external line, reason why we will call them "external vertices") and any function $f_l(x_j - x_i)$ with a line connecting x_i and x_j. For simplicity we assume that the functions $f_l(x)$ are symmetrical with respect to the change $x \to -x$, otherwise an arrow would have to be attached to all lines. This symmetry implies $f_l^+(-x) = f_l^-(x)$. It holds for the propagators (2.1), (2.2). For the functions F_m^+ we use the same diagrams with the difference that the vertices that appear in F_m^+ to the right of the bar are marked with a small circle. We henceforth will speak accordingly of marked and unmarked vertices. A line between a marked and an unmarked vertex corresponds to an f^+ or f^- function, whereas a line between two marked vertices corresponds to a $-f^*$ function. Lines between unmarked vertices are f functions. We now remark that the functions f^+ and f^- can always be written in the form $f^+(+x) = f^-(-x)$ with $x = z - y$ where z is marked and y unmarked. One sees now that if, as is usually done, we define energy as the variable associated with x in Fourier space, *viz.* p in (2.3), it will always be positive.

Equation (2.10) now gives a relation between a set of diagrams which differ from each other by some of the unmarked vertices being replaced by marked ones and vice versa. As a result of energy conservation (expressed by the aforementioned appearance of δ-functions) all diagrams having a set of marked internal vertices that is connected with the rest of the diagram through lines representing f^+ or f^- functions only are zero. This leads to a new method of picturing the terms F^+: any term F^+ is represented by a diagram in which a number of lines are cut by a line (called the "cut") that is shaded on one side. All vertices on the shaded side of the cut are to be considered as marked vertices, all vertices on the unshaded side as unmarked vertices. Or: lines intersected by the cut represent f^+ or f^- functions, lines on the shaded side $-f^*$ functions, lines on the unshaded side f functions. In going from unshaded to shaded region energy has to be positive. A simple example of identity (2.10) is represented in fig. 1. In fig. 2 we see how this can be pictured in a simple way for a diagram with 3 external vertices and an unspecified number of internal vertices.

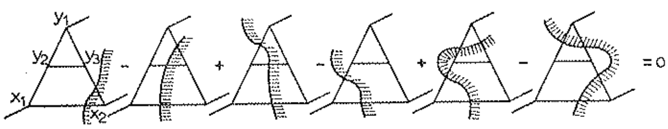

Fig. 1. y_2 and y_3 are internal, x_1, x_2 and y_1 external vertices. $y_1 \leqslant x_1$ and x_2. Note that x_1 is unmarked and x_2 marked in each term of the identity; the marking of the y's changes from term to term.

Let us now consider (2.10) for some special cases. A very important case is that where the subset $x_{j_1} \ldots x_{j_p} (= y_1 \ldots y_p)$ contains all x. In (2.10) the term with $q = 0$ is exactly the original function F from (2.7), while the term with $q = n$ is up to a sign the complex conjugate of F. One finds then that the real or imaginary part of a diagram equals the sum of all cut diagrams (with the appropriate signs). This will be the form in which the cutting formula is best suited to prove unitarity. Another important case is provided when the subset $x_{j_1} \ldots x_{j_p}$ contains all x except one external vertex x_b, and when one of the $x_{j_1} \ldots x_{j_p}$ is an external vertex x_a with $x_a \leqslant x_b$.

One has then an equation stating that the sum of all cut diagrams with x_a and x_b unmarked equals minus the sum of all cut diagrams with x_b unmarked and x_a marked. In this form the cutting formula will be used in connection with causality.

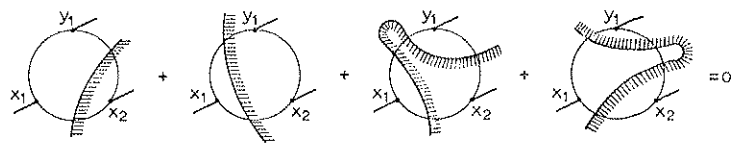

Fig. 2. $y_1 \leqslant x_1, x_2$. The circle stands for a general diagram, and we think all vertices and lines located inside the circle. Only the external vertices are indicated. The first diagram stands for all diagrams with x_1 and y_1 unmarked, x_2 marked, and such that one can go from x_1 to y_1 along lines and vertices without crossing the cut. The second diagram has y_1 and x_2 marked with a path from x_2 to y_1 that does not cross the cut. The third and fourth diagrams have y_1 unmarked and marked respectively and each path connecting any two vertices out of y_1, x_1, x_2 crosses the cut.

3. *Implicit equation for propagators.* We will now discuss the propagators of a field theory with unstable particles in somewhat more detail. To this purpose we introduce a model, namely a field theory with two kinds of particles, uncharged scalar bosons for simplicity, with an interaction Lagrangian density:

$$L_I = \frac{g}{2} (\varphi^2(x)\, A(x) + A(x)\, \varphi^2(x)) \tag{3.1}$$

The physical masses M of the A-field and m of the φ-field shall be such that the A-field be unstable, i.e.:

$$M > 2m \tag{3.2}$$

We now note that for $g = 0$ the A-particle can be expected to exist at all times, whereas for $g \neq 0$ it is unstable, which means that no A-particle will be found at infinite times, i.e. in asymptotic states. We expect therefore difficulties in the application of perturbation theory to this model, for instance with respect to unitarity of the S-matrix. Furthermore, as will be seen later in greater detail, the A-field can be eliminated from the theory whereby the Lagrangian density becomes non-local (involves fields at different space-time points) and it is well known that perturbation theory for a non-local interaction Lagrangian gives rise to difficulties with respect to unitarity [5].

In the following we will see how these difficulties arise from an unjustified handling of perturbation theory. The Feynman rules for the present model are: any diagram consists of two kinds of lines, namely φ-lines and A-lines which we shall respectively represent by solid and dotted lines. An internal line connects two vertices, an external one is connected to one vertex only. A vertex is end-point of two φ-lines and one A-line. To a φ or A-line between the vertices x_i and x_j belongs respectively a factor (propagator):

$$i\Delta_F(x_i - x_j) = i(2\pi)^{-4} \int d_4k \, e^{ik(x_i - x_j)} \, (k^2 + m^2 - i0)^{-1}$$

or a factor (3.3)

$$i\Delta'_F(x_i - x_j) = i(2\pi)^{-4} \int d_4k \, e^{ik(x_i - x_j)} \, (k^2 + M^2 - i0)^{-1}$$

A diagram has a factor

$$\frac{g^n i^n}{n!}$$

where n is the number of vertices in the diagram. Furthermore there are simple combinatorial factors due to the undistinguishability of particles. For instance the second order diagram of fig. 3 gets a factor 2.

Fig. 3. Second order self energy diagram.

We now investigate the complete propagator for the A-field. To begin with we consider the lowest order self-energy-diagram of fig. 3. Neglecting terms that vanish after renormalization one finds:

$$F(x_1 - x_2) = (2\pi)^{-4} \int d_4k \, e^{ik(x_1 - x_2)} \{(k^2 + M^2)^2 \, R_2(k^2) + \\ + i\Theta(-k^2 - 4m^2) \, I_2(k^2)\} \qquad (3.4)$$

In this $R_2(k^2)$ and $I_2(k^2)$ are real functions of k^2, of order g^2. This lowest order diagram can be inserted 0, 1, 2 ... times in a A-line. The Fourier transform of the sum of the resulting expressions is a geometrical series with argument:

$$\frac{(k^2 + M^2)^2 \, R_2(k^2) + i\Theta(-k^2 - 4m^2) \, I_2(k^2)}{k^2 + M^2 - i0} \qquad (3.5)$$

Because of (3.2) there will be a neighbourhood of $k^2 = -M^2$ where (3.5) in absolute value is larger then unity, for $I_2(-M^2) \neq 0$ for arbitrary $g \neq 0$. Therefore this series will not converge and perturbation theory is not valid in this region. It is exactly this divergence of perturbation theory which is responsible for the difficulties with unstable particles and non-local field-theories.

The way whereby this difficulty can be circumvented is well-known [7]).

One uses perturbation theory in the region where the series converges and uses then the analytical continuation of the result in the difficult regions. We will adopt the same technique here. The underlying idea is that single self-energy diagrams alone are meaningless quantities, but that repeated insertion, summation and analytical continuation leads to a reasonable result. We will do this for the A and φ-particles. Consider for a certain particle all selfenergy diagrams that do not contain any further self-energy parts in the internal lines. We will call such self-energy diagrams simple proper diagrams (s.p.d.) for the given particle. Their Fourier transforms depend on the square of the four-momentum and on the propagators Δ_F, Δ'_F of φ and A-particle prespectively, see (3.3). The sum of the contributions for all s.p.d. of the φ or A-particle respectively will be denoted by

$$E(\Delta_F, \Delta'_F, k^2), \qquad E'(\Delta_F, \Delta'_F, k^2)$$

E, E' are thus functionals of the functions $\Delta_F(x)$ and $\Delta'_F(x)$, or of their Fourier transforms $\Delta_F(k^2)$ and $\Delta'_F(k^2)$. We will always adopt the convention

$$\Delta(x) = (2\pi)^{-4} \int e^{ikx} \Delta(k) \, d_4 k$$

Let us for the moment neglect renormalization terms. In the region where perturbation theory converges we define functions $\Delta_F^{(n)}(k^2)$, $\Delta_F'^{(n)}(k^2)$ by iteration.

$$\begin{aligned} i\Delta_F^{(n)}(k^2) &= i\Delta_F(k^2) + \sum_{m=1}^{\infty} (i\Delta_F(k^2))^{m+1} E^m(\Delta_F^{(n-1)}, \Delta_F'^{(n-1)}, k^2) = \\ &= i\Delta_F(k^2) \left[1 - i\Delta_F(k^2) E(\Delta_F^{(n-1)}, \Delta_F'^{(n-1)}, k^2)\right]^{-1} \end{aligned} \tag{3.6}$$

$$i\Delta_F'^{(n)}(k^2) = i\Delta'_F(k^2)[1 - i\Delta'_F(k^2) E'(\Delta_F^{(n-1)}, \Delta_F'^{(n-1)}, k^2)]^{-1}$$

Here $\Delta_F^{(0)}(k^2) = \Delta_F(k^2)$, $\Delta_F'^{(0)}(k^2) = \Delta'_F(k^2)$, see (3.3). Outside the region of convergence the functions $\Delta_F^{(n)}(k^2)$, $\Delta_F'^{(n)}(k^2)$ are defined by analytical continuation of (3.6). Assuming that the limits exist we have for the complete propagators

$$\begin{aligned} \bar{\Delta}_F(k^2) &= \lim_{n\to\infty} \Delta_F^{(n)}(k^2) \\ \bar{\Delta}'_F(k^2) &= \lim_{n\to\infty} \Delta_F'^{(n)}(k^2) \end{aligned} \tag{3.7}$$

From (3.6) and (3.7) follows the implicit equations for complete propagators:

$$\begin{aligned} \bar{\Delta}_F(k^2) &= \Delta_F(k^2)[1 - i\Delta_F(k^2) E(\bar{\Delta}_F, \bar{\Delta}'_F, k^2)]^{-1} \\ \bar{\Delta}'_F(k^2) &= \Delta'_F(k^2)[1 - i\Delta'_F(k^2) E'(\bar{\Delta}_F, \bar{\Delta}'_F, k^2)]^{-1} \end{aligned} \tag{3.8}$$

Taking renormalization into account we subtract purely imaginary terms $iZ_2\delta m^2$, $iZ'_2\delta M^2$, $-i(Z_2 - 1)(k^2 + m^2)$ and $-i(Z'_2 - 1)(k^2 + M^2)$ in such a way that a development of the imaginary parts of E and E' in powers of $(k^2 + m^2)$ and $(k^2 + M^2)$ respectively contains only second or higher order powers in these quantities. In this Z_2, Z'_2, δm^2 and δM^2 are constants and the notation used is the standard one.

In our case the $\bar{\Delta}_F(k^2)$ will take the form:

$$\begin{aligned} \bar{\Delta}_F(k^2) &= [k^2 + m^2 - (k^2 + m^2)^2 R(k^2) - i\Theta(-k^2 - 9m^2) I(k^2)]^{-1} \\ \bar{\Delta}'_F(k^2) &= [k^2 + M^2 - (k^2 + M^2)^2 R'(k^2) - i\Theta(-k^2 - 4m^2) I'(k^2)]^{-1} \end{aligned} \tag{3.9}$$

M and m are the physical masses. R, I, R', and I' are real functions of second and higher order in g, analogous to R_2 and I_2 in (3.4), (3.5). If the A-particle is unstable, i.e. $M > 2m$, the propagator $\bar{\Delta}'_F(k^2)$ will have no pole at the point $k^2 + M^2 = 0$. In section 4 where unitarity is considered we will see that it is exactly the absence of this pole which will ensure unitarity of the theory without A-particles in the asymptotic in- or out-states.

The equations (3.8) are quite similar to equations obtained by Symanzik[8]) for the cases of retarded and time ordered functions. By means of them one can separate renormalization effects from the S-matrix. One uses then an S-matrix without self-energy parts, but with complete propagators that have to satisfy the implicit equation. If no vertex divergencies occur (as in the super-renormalisable model considered here all renormalization problems are separated from the actual S-matrix calculation. In case of vertex-divergencies one has to go further than the two-point function and also the three point function has to be treated. The situation is then more complicated due to overlapping divergencies but can be handled following the rules of renormalization theory as developed for stable particles. For further details we refer to Symanzik's work. In the following we will use (3.8) as a starting point without reference to its derivation. We will suppose that (3.8) has a solution that obeys Lehmann's representation

$$\begin{aligned} \bar{\Delta}_F(k^2) &= \int_0^\infty \rho(-\kappa^2)\, \Delta_F(k^2, \kappa^2)\, \mathrm{d}\kappa^2 \\ \bar{\Delta}'_F(k^2) &= \int_0^\infty \rho'(-\kappa^2)\, \Delta_F(k^2, \kappa^2)\, \mathrm{d}\kappa^2 \\ \Delta_F(k^2, \kappa^2) &= (k^2 + \kappa^2)_P^{-1} + i\pi\delta(k^2 + \kappa^2) = (k^2 + \kappa^2 - i0)^{-1} \end{aligned} \tag{3.10}$$

The weight functions ρ, ρ' are non-negative and vanish for positive argument. They are related to the functions I, I' in (3.9) as indicated in (3.11).

Our task will be to prove that such a solution gives rise to a unitary S-matrix connecting φ-particle states only. We close this section by writing down for the complete propagators the formulae needed in order to apply the cutting formula in the following section. From (3.10)

$$\begin{aligned} \bar{\Delta}_F(x) &= \Theta(x_0)\, \bar{\Delta}^+(x) + \Theta(-x_0)\, \bar{\Delta}^-(x) \\ \bar{\Delta}'_F(x) &= \Theta(x_0)\, \bar{\Delta}'^+(x) + \Theta(-x_0)\, \bar{\Delta}'^-(x) \end{aligned}$$

with:

$$\begin{aligned} \bar{\Delta}^\pm(x) &= i(2\pi)^{-3} \int \mathrm{d}_4k\, e^{ikx}\Theta(\pm k_0)\, \rho(k^2) \\ \bar{\Delta}'^\pm(x) &= i(2\pi)^{-3} \int \mathrm{d}_4k\, e^{ikx}\Theta(\pm k_0)\, \rho'(k^2) \end{aligned}$$

Clearly these equations coincide with (2.1), (2.2).

We note also

$$\begin{aligned} \bar{\Delta}^\pm(k^2) &= 2\pi i\Theta(\pm k_0)\, \rho(k^2) = 2i\Theta(\pm k_0)\, \mathrm{Im}\, \bar{\Delta}_F(k^2) \\ \bar{\Delta}'^\pm(k^2) &= 2\pi i\Theta(\pm k_0)\, \rho'(k^2) = 2i\Theta(\pm k_0)\, \mathrm{Im}\, \bar{\Delta}'_F(k^2) \end{aligned} \tag{3.11}$$

4. *Unitarity of the S-matrix.* We start now by defining the S-matrix of our model in terms of diagrams. It shall consist of all possible diagrams constructed according to the rules given in section 3, with the exclusion of diagrams that contain self-energy parts and/or outgoing A-lines. The correspondence between the diagrams and the analytical expressions associated with them will be as before, with the exception that internal A and φ lines now correspond to the complete propagators $\bar{\Delta}'_F$ and $\bar{\Delta}_F$ that are solutions of the implicit equation (3.8) and obey the Lehmann representation. The complete analytical expression for the S-matrix can be written in the form

$$S = 1 + \sum_{n=3}^{\infty} \int \ldots \int \mathrm{d}x_1 \ldots \mathrm{d}x_n f_n(x_1 \ldots x_n) : \varphi_1 \ldots \varphi_n : \tag{4.1}$$

The term $n = 3$ is of course zero because of energy momentum conservation. Nevertheless we retain it explicitly for future use. In this $\varphi_i \equiv \varphi_{in}(x_i)$ is the free in-field corresponding to the φ-particle. The commutator of φ_i with itself is just the free-field commutator Δ for a particle with mass m. The functions $f_n(x_1 \ldots x_n)$ are the sums of the contributions arising from all diagrams with n vertices $x_1 \ldots x_n$ as external vertices. In the vertices $x_1 \ldots x_n$ one internal A-line, one internal φ-line and one external φ-line come together. The n external φ-lines are not included in the evaluation of $f_n(x_1 \ldots x_n)$, but correspond to the operators $\varphi_1 \ldots \varphi_n$ in (4.1), where: $:\varphi_1 \ldots \varphi_n:$ is the usual ordered product.

Equation (4.1) gives us as well defined expression for the S-matrix, for which we must prove unitarity. If in (4.1) we separate the unit term from the rest by writing $S = 1 + T$ unitarity takes the form

$$T + T^\dagger = -TT^\dagger \tag{4.2}$$

T is given by the sum over n in the right hand side of (4.1). Let us write down $TT^\dagger$:

$$TT^\dagger = \sum_{n=3}^{\infty} \sum_{n'=3}^{\infty} \int \mathrm{d}x_1 \ldots \mathrm{d}x_n \, \mathrm{d}x'_1 \ldots \mathrm{d}x'_{n'} f_n(x_1 \ldots x_n) \, f^*_{n'}(x'_1 \ldots x'_{n'}) \cdot \\ :\varphi_1 \ldots \varphi_n: \; :\varphi_{1'} \ldots \varphi_{n'}:$$

By standard techniques [9]) one can reduce the product of the two ordered products to a sum of ordered products by introducing the contraction of two field operators:

$$\overline{\varphi_i \varphi_{i'}} \equiv \langle 0 \, | \varphi_{in}(x_i) \, \varphi_{in}(x_{i'}) | \, 0 \rangle = -i\Delta^+(x_i - x_{i'}) \\ = \frac{1}{(2\pi)^3} \int \mathrm{d}_4 k \; \mathrm{e}^{ik(x_i - x_{i'})} \Theta(k_0) \, \delta(k^2 + m^2) \tag{4.3}$$

Clearly $TT^\dagger$ can be associated with diagrams obtained by connecting, through lines carrying Δ^+ functions, the various diagrams of T (propagators $\bar{\Delta}_F$, $\bar{\Delta}'_F$) with the various diagrams of $T^\dagger$ (propagators $\bar{\Delta}^*_F$, $\bar{\Delta}'_F{}^*$).

Next we inspect $T + T^\dagger$. Consider a diagram of T with m vertices $y_1 \dots y_m$, among which ν internal vertices $y_{\alpha_1} \dots y_{\alpha_\nu}$ and $n = m - \nu$ external vertices which we call $x_1 \dots x_n$. Its contribution to the function $f_n(x_1 \dots x_n)$ can be written in the form:

$$(i)^{m+a+b} \int d_4y_{\alpha_1} \dots d_4y_{\alpha_\nu} F(y_1 \dots y_m) \tag{4.4}$$

In this a is the number of internal φ-lines and b the number of internal A-lines. We separated out the factor i attached to each internal line (see (3.3)) in order to be able to apply the cutting formula to $F(y_1 \dots y_m)$, which is seen to be of the form (2.6). We apply now (2.10) for the case that the subset $x_{j_1} = y_1 \dots x_{j_p} = y_p$ consists of all vertices ($p = m$) and get

$$(i)^{m+a+b} \int d_4y_{\alpha_1} \dots d_4y_{\alpha_\nu} \sum_{q=0}^{m} \sum_{\{q\}} (-1)^q F_q(y_1 \dots y_{i_1-1}\, y_{i_1+1} \dots \dots y_m | y_{i_1} \dots y_{i_q}) = 0 \tag{4.5}$$

The term with $q = 0$ equals (4.4) while the term with $q = m$ equals

$$(-i)^{m+a+b} \int d_4y_{\alpha_1} \dots d_4y_{\alpha_\nu} F^*(y_1 \dots y_m) \tag{4.6}$$

where we used

$$F(|y_1 \dots y_m) = (-1)^{a+b} F^*(y_1 \dots y_m)$$

as can be inferred from the rules for the construction of the modified F functions (2.7). Obviously the term with $q = m$ can be interpreted as a contribution to $T^\dagger$. As described in section 2 the terms with $1 < q < m$ can be pictured as cut diagrams, where to the cut lines correspond $\bar{\Delta}^\pm$ or $\bar{\Delta}'^\pm$ functions, while all internal lines on the shaded side give rise to $-\bar{\Delta}_F^*$ or $-\bar{\Delta}_F'^*$ functions. If now in the cutting formula (4.5) we move all terms with $1 < q < m$ to the right-hand side, we see that the left-hand side, containing only (4.4) and (4.6), exactly corresponds to the left-hand side of the unitarity equation (4.2). The right-hand side will reduce to the right-hand side of (4.2) if we are able to achieve two aims: eliminate all diagrams with cuts which cross A-lines, and replace in the other diagrams $\bar{\Delta}^\pm$ by Δ^+ for all cut φ-lines. This is done as follows. We first convert all $\bar{\Delta}^-$ and $\bar{\Delta}'^-$ functions into $\bar{\Delta}^+$ and $\bar{\Delta}'^+$ functions with the help of the identities valid in the case of symmetrical propagators:

$$\bar{\Delta}^+(x) = \bar{\Delta}^-(-x), \quad \bar{\Delta}'^+(x) = \bar{\Delta}'^-(-x)$$

(there is no loss of generality here, one can if necessary circumvent this step). We next use the identities (3.8), (3.9) and (3.11); going over to the Fourier representation we have:

$$\begin{aligned} \bar{\Delta}^+(k^2) &= \Theta(k_0)[|\bar{\Delta}_F(k^2)|^2\, 2i \operatorname{Re} E(\bar{\Delta}_F, \bar{\Delta}'_F, k^2) + 2\pi i \delta(k^2 + m^2)] \\ \Delta'^+(k^2) &= \Theta(k_0)\, |\bar{\Delta}'_F(k^2)|^2\, 2i \operatorname{Re} E'(\bar{\Delta}_F, \bar{\Delta}'_F, k^2) \end{aligned} \tag{4.7}$$

The crucial point in these equations is that $\bar{\Delta}^+(k^2)$, belonging to the stable φ-particle, contains $\Delta^+(k^2) = 2\pi i \Theta(k_0)\, \delta(k^2 + m^2)$ as a separate term whereas $\bar{\Delta}'^{+}(k^2)$ contains no such contribution. The following step is to express $\operatorname{Re} E$ and $\operatorname{Re} E'$ in terms of cut self-energy diagrams. Consider E and E'. They are the sums of all s.p.d. with $\bar{\Delta}_F$ and $\bar{\Delta}'_F$ propagators, are thus themselves sums of contributions corresponding with diagrams, and accordingly we can apply the cutting formula to them, always taking the case where the subset $y_1 \dots y_p$ consists of all vertices. One finds that $|\bar{\Delta}_F|^2\, 2 \operatorname{Re} E$ (and $|\bar{\Delta}'_F|^2\, 2 \operatorname{Re} E'$) can be regarded as corresponding to the sum of all cuttings of all s.p.d. with $\bar{\Delta}_F$, $\bar{\Delta}'_F (\bar{\Delta}^*_F, \bar{\Delta}'^*_F)$ on the unshaded (shaded) side. Returning then to (4.5), and the original diagrams we can now picture the contribution of the terms in $\operatorname{Re} E$ and $\operatorname{Re} E'$ to a cut line of the original diagrams as obtained by substituting to it a cut self-energy

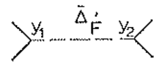

Fig. 4*a*. Example of a diagram as represented by a formula of the type (4.4). We added external lines.

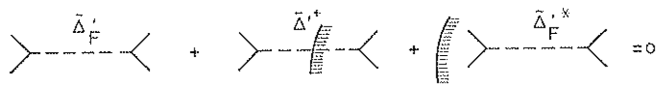

Fig. 4*b*. Equation (4.5) for the diagram of figure 4*a*.

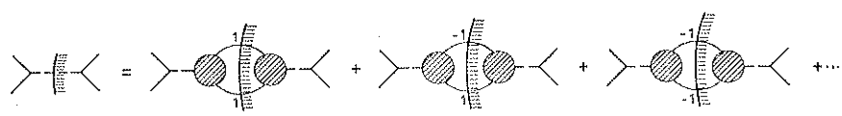

Fig. 4*c*. First step in the reduction of $\bar{\Delta}'^{+}$. A $\Theta(k_0)\,\delta(k^2+m^2)$ factor is indicated by a cut line with a number 1. $\Theta(k_0) \operatorname{Re} E$ or $\Theta(k_0) \operatorname{Re} E'$ factors are indicated by cut lines with the number -1. The shaded circles stand for the sum of all diagrams with three lines outgoing as indicated.

diagram, more precisely a cut s.p.d.*). This can be repeated ad infinitum, and the only contributions from cut lines which are eventually left are the $2\pi i\Theta(k_0)\,\delta(k^2+m^2) = \Delta^{+}(k^2)$. They originate from cut φ-lines, and cut A-lines are left with no contribution at all. The last step in the proof of (4.2) consists in showing that one gets all diagrams of $TT^\dagger$ by treating in the manner indicated above all diagrams of T. This is a simple combinatorial problem which we do not consider further here.

As an illustration figure 4 gives an example of the reduction procedure for a simple diagram. Factors are not indicated for simplicity.

5. *S-matrix and field equations.* In this section we will show that the S-matrix defined in section 4 is a solution of the field equations belonging

*) One easily verifies that the shaded area of the original diagram and the shaded of the s.p.d. substituted into it are both on the same side. This follows from the sign of the energy specified by $\Theta(k_0)$ in (4.5).

to a Lagrangian density

$$L = -\tfrac{1}{2}\left\{\left(\frac{\partial\varphi}{\partial x_\mu}\right)^2 + m^2\varphi^2 + \left(\frac{\partial A}{\partial x_\mu}\right)^2 + M^2A^2\right\} + L_I \tag{5.1}$$

where L_I is as given in (3.1) and all renormalization terms are suppressed. In order to prove field equations we define from the S-matrix an interpolating field $\varphi(x)$ with the property that for infinite times its expectation values equal those of the in- or out-fields. This interpolating field shall be defined so that the vacuum expectation value of the product of the field with itself equals $\bar{\Delta}^+$ (see (2.1)), and we will show then that it satisfies the integral form of the field equations derived from L, see (5.11) below. In this form of the equations, A is explicitly given in terms of φ.

Let us start by recalling the definition of the outfield[10])

$$\varphi_{\text{out}}(x_i) \equiv \varphi_{\underline{i}} \equiv S^\dagger\varphi_{\text{in}}(x_i)\, S = S^\dagger\varphi_i S = \varphi_i + S^\dagger[\varphi_i, S] \tag{5.2}$$

From the commutation rule

$$[\varphi_i, \varphi_j] = -i\Delta(x_i - x_j) = (2\pi)^{-3}\int \mathrm{d}_4k\, \mathrm{e}^{ikx}\,\delta(k^2+m^2)\,\varepsilon(k_0)$$

$$\varepsilon(k_0) = -1 + 2\Theta(k_0)$$

one finds by introducing (4.1) in (5.2):

$$\varphi_{\underline{i}} = \varphi_i - \int \mathrm{d}_4x\Delta(x_i - x)\, j(x)$$

where $j(x)$ is given by

$$j(x) = iS^\dagger[\sum_{n=3}^{\infty}\ \sum_{j=1}^{n} \int \mathrm{d}x_1 \ldots \mathrm{d}\check{x}_j \ldots \mathrm{d}x_n.$$

$$f_n(x_1 \ldots x_{j-1}, x, x_{j+1} \ldots x_n) :\varphi_1 \ldots \check{\varphi}_j \ldots \varphi_n: \tag{5.3}$$

In this we used the notation

$$a_1 \ldots a_{i-1}a_{i+1} \ldots a_n = a_1 \ldots \check{a}_i \ldots a_n$$

Because of the fact that the Fourier transform of $\Delta(x_i - x)$ contains a $\delta(k^2 + m^2)$ function we need up to now the function $f_n(x_1 \ldots x_n)$ only on the mass shell (i.e. of the Fourier transform $f_n(k_1 \ldots k_n)$ *of* $f_n(x_1 \ldots x_n)$ we need only the values for $k_i^2 = -m^2$, $i = 1, \ldots, n$). Of course the f_n given by diagrams and Feynman rules have well-defined values off the mass shell also, but in the S-matrix only the mass shell values are required. We now define $j(x)$ also off the mass shell (i.e. for $k^2 \neq -m^2$, where k is the 4-momentum associated with x):

$$j(x) = iS^\dagger[\sum_{n=3}^{\infty}\ \sum_{j=1}^{n} \int \mathrm{d}x_1 \ldots \mathrm{d}\check{x}_j \ldots \mathrm{d}x_n.$$

$$f'_n(x_1 \ldots x_{j-1}, x, x_{j+1} \ldots x_n) :\varphi_1 \ldots \check{\varphi}_j \ldots \varphi_n:$$

where the Fourier transformed $f'_n(k_1 \ldots k_n)$ of $f'_n(x_1 \ldots x_n)$ is related to

$f_n(k_1 \dots k_n)$ as calculated by the ordinary Feynman rules through

$$f'_n(k_1 \dots k_n) = N(k_1^2) \dots N(k_n^2)\, f_n(k_1 \dots k_n)$$

$$N(k_i^2) = \frac{\bar{\Delta}_F(k_i^2)}{\Delta_F(k_i^2)} = 1 + iE(\bar{\Delta}_F, \bar{\Delta}'_F, k_i^2)\, \bar{\Delta}_F(k_i^2) \tag{5.4}$$

This definition coincides on the mass shell with (4.1) and (5.3) because of the fact that $N(-m^2) = 1$. In the following we will use the notation:

$$\sum_{n=3}^{\infty} \sum_{j=1}^{n} \int \mathrm{d}x_1 \dots \mathrm{d}\check{x}_j \dots \mathrm{d}x_n \cdot f'_n(x_1 \dots x_{j-1}, x, x_{j+1}, \dots x_n).$$

$$:\varphi_1 \dots \check{\varphi}_j \dots \varphi_n: = \frac{\delta S}{\delta \varphi_{\mathrm{in}}(x)} = \frac{\delta S}{\delta x} \tag{5.5}$$

We stress the fact that the δ/δ symbols are introduced as a notation rather than indicating functional differentiation with respect to a field. However, one can establish that well-known properties of functional derivatives also hold in this case. For instance defining $\delta S^\dagger/\delta x$ as the hermitian conjugate of (5.5) one has

$$iS^\dagger \frac{\delta S}{\delta x} = -i \frac{\delta S^\dagger}{\delta x} S (= j(x)) \tag{5.6}$$

or

$$\frac{\delta S^\dagger S}{\delta x} = 0$$

The proof of this equation, obviously related to unitarity, is not trivial and needs an interpretation of the factor $N(k^2)$ in terms of diagrams. From (5.4) we see that the factors $N(k_i^2)$ in f'_n can be interpreted as unity plus all (renormalized) s.p.d. connected to the original f_n diagrams by a $\bar{\Delta}_F$ propagator (fig. 5*a*). The proof of (5.6) is now entirely analogous to the proof of unitarity in section 4: apply the cutting formula to all diagrams contributing to the $f'_n(x_1 \dots x \dots x_n)$; reduce all $\bar{\Delta}^{\pm}$, $\bar{\Delta}'^{\pm}$ functions in the manner used in the unitarity proof. The terms corresponding to diagrams with x unmarked give the left-hand side of (5.6), the others give the right-hand side. The necessity of the additional diagrams introduced in f'_n as compared to f_n (see an example in fig. 5*a*) is illustrated in fig. 5*b* and 5*c*. The occurrence of

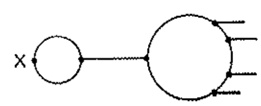

Fig. 5*a*. Diagrams appearing in $\delta S/\delta x$ in addition to those that appear also in S. As indicated in fig. 2 the circles stand for general diagrams with external vertices indicated by dots. Their necessity is illustrated in fig. 5*b* and *c*.

diagrams of the type shown in fig. 5*b* implies diagrams as in fig. 5*c*, which themselves require that the definition of $\delta S/\delta x$ contains diagrams as in fig. 5*a*.

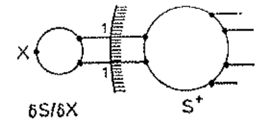

Fig. 5*b*. Diagram giving non zero contribution to $S^\dagger(\delta S/\delta x)$ if the Fourier variable associated with x satisfies $k^2 < -4m^2$. As in fig. 4 the 1 indicates a Δ^+ function.

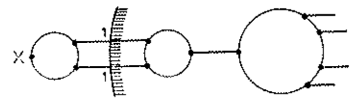

Fig. 5*c*. Type of diagram appearing in $S^\dagger(\delta S/\delta x)$ which requires the consideration in $\delta S/\delta x$ of the extra diagrams as given in fig. 5*a*.

We now define the interpolating field $\varphi(x)$:

$$\varphi(x) = \varphi_{\text{in}}(x) - \int \mathrm{d}_4x' \Delta_R(x - x')\, j(x') \tag{5.7}$$

and we will be concerned with the properties of this field in the remaining part of the present section. As a first point we observe that $j(x')$ is used also off the mass shell because $\Delta_R(k^2) \neq 0$ for $k^2 \neq -m^2$:

$$\Delta_R(x) = \Theta(x_0)\, \Delta(x) = (2\pi)^{-4} \int \mathrm{d}_4k\, \mathrm{e}^{ikx}[k^2 + m^2 - i\varepsilon(k_0)0]^{-1}$$

One verifies further that for arbitrary states $|\alpha\rangle$ and $|\beta\rangle$:

$$\lim_{t\to\pm\infty} \langle\alpha\, |\varphi(x)|\, \beta\rangle = \langle\alpha\, |\varphi_{\substack{\text{out}\\ \text{in}}}(x)|\, \beta\rangle$$

Also from (5.7) together with $S|1\rangle = |1\rangle$ where $|1\rangle$ is the one particle state one derives

$$\langle 0\, |\varphi(x)|\, 1\rangle = \langle 0\, |\varphi_{\text{in}}(x)|\, 1\rangle$$

We prove now

$$i\langle 0\, |\varphi(x)\, \varphi(x')|\, 0\rangle = \bar{\Delta}^+(x - x') \tag{5.8}$$

Inserting (5.7) we have:

$$\langle 0\, |\varphi(x)\, \varphi(x')|\, 0\rangle = \langle 0\, |\varphi_{\text{in}}(x)\, \varphi_{\text{in}}(x')|\, 0\rangle + \\ + \int \mathrm{d}_4y\, \mathrm{d}_4y' \Delta_R(x - y)\, \Delta_R(x' - y')\, \langle 0\, |j(y)\, j(jy')|\, 0\rangle$$

Inserting intermediate states into $\langle 0\, |j(y)\, j(y')|\, 0\rangle$ we see that this function contains positive frequencies only, i.e. it can be written in the form:

$$\int \mathrm{d}_4k\, \mathrm{e}^{ik(y-y')}\, \Theta(k_0)\, g(k^2)$$

This enables us to write

$$i\langle 0\, |\varphi(x)\, \varphi(x')|\, 0\rangle = i\langle 0\, |\varphi_{\text{in}}(x)\, \varphi_{\text{in}}(x')|\, 0\rangle + \\ + i \int \mathrm{d}_4y\, \mathrm{d}_4y' \Delta_F(x - y)\, \langle 0\, |j(y)\, j(y')|\, 0\rangle\, \Delta_F^*(y' - x') \tag{5.9}$$

where we used

$$\Delta_R(x) = \Delta_F(x) - \Delta^-(x) = \Delta_F^*(x) + \Delta^+(x) \tag{5.10}$$

We now study $\langle 0 | j(y)\, j(y') | 0 \rangle$. On substituting

$$j(y) = iS^+ \frac{\delta S}{\delta y} \qquad j(y') = -i \frac{\delta S^\dagger}{\delta y'} S$$

one finds with $S|0\rangle = |0\rangle$:

$$\langle 0 | j(y)\, j(y') | 0 \rangle = \sum_{n=6}^{\infty} \sum_{m=3}^{n-3} \sum_{j=m+1}^{n} \sum_{i=1}^{m} \int \mathrm{d}x_1 \ldots \mathrm{d}\check{x}_i \ldots \mathrm{d}\check{x}_j \ldots \mathrm{d}x_n \cdot$$
$$\cdot f_m'(x_1 \ldots x_{i-1}, y, x_{i+1} \ldots x_m)\, f_{n-m}'^{*}(x_{m+1} \ldots x_{j-1}, y', x_{j+1} \ldots x_n) \cdot$$
$$\cdot \langle 0 | :\varphi_1 \ldots \check{\varphi}_i \ldots \varphi_m: \, :\varphi_{m+1} \ldots \check{\varphi}_j \ldots \varphi_n: | 0 \rangle$$

The vacuum expectation value can be reduced to a sum of products of Δ^+ functions, and again with the help of cutting formula and reduction technique we find:

$$\langle 0 | j(y)\, j(y') | 0 \rangle = (2\pi)^{-4} \int \mathrm{d}_4 k\, e^{ik(y-y')} \Theta(k_0)\, N(k^2)\, N^*(k^2) \cdot 2\, \mathrm{Re}\, E(\bar{\Delta}_F, \bar{\Delta}_F', k^2).$$

This result inserted in (5.9) gives together with (5.4), (4.7) and (4.3) the desired result (5.8).

Let us now try to prove that $\varphi(x)$ is a solution of the field equations. These equations, derived from the Lagrangian (5.1) and put in integral equation form, are (Δ_R' is Δ_R witd M instead of m)

$$\begin{aligned} A(x) &= -g \int \mathrm{d}_4 x' \Delta_R'(x - x') \{\varphi^2(x') - \langle 0 | \varphi^2(x') | 0 \rangle - \text{ren. terms.}\} \\ \varphi(x) &= \varphi_{\text{in}}(x) - g \int \mathrm{d}_4 x' \Delta_R(x - x') \{\varphi(x')\, A(x') + A(x')\, \varphi(x') - \text{ren. terms}\} \end{aligned} \tag{5.11}$$

The absence of A_{in} field is due to the instability of the A-particle. Indeed the matrix elements of the A-field must vanish for infinite times. Because of the absence of A_{in} field one can eliminate the A-field from the equations (5.11). One has then in the φ-field a non-local theory of the type investigated by Bloch, ref. 5. The exact form of the renormalization terms will be given later.

Let us consider the operator $\varphi^2(x)$. With the help of (5.7), (5.6) and the relations

$$\varphi_{\text{in}}^2(x) = :\varphi_{\text{in}}^2(x): + \langle 0 | \varphi_{\text{in}}^2(x) | 0 \rangle$$
$$\frac{\delta S^\dagger}{\delta x_j} \frac{\delta S}{\delta x_i} + \frac{\delta S^\dagger}{\delta x_i} \frac{\delta S}{\delta x_j} = -S^\dagger \frac{\delta^2 S}{\delta x_j \delta x_i} - \frac{\delta^2 S^\dagger}{\delta x_j \delta x_i} S$$

(the latter can be proved by the methods described before) one finds:

$$\begin{aligned} \varphi^2(x) = {} & \tfrac{1}{2} :\varphi_{\text{in}}^2(x): + \tfrac{1}{2} \langle 0 | \varphi_{\text{in}}^2(x) | 0 \rangle + i \int \Delta_R(x - x')\, \varphi_{\text{in}}(x) \frac{\delta S^\dagger}{\delta x'} S\, \mathrm{d}_4 x' + \\ & - \tfrac{1}{2} \int \Delta_R(x - x')\, \Delta_R(x - x'') \frac{\delta^2 S^\dagger}{\delta x' \delta x''} S\, \mathrm{d}_4 x'\, \mathrm{d}_4 x'' + \\ & + \text{herm. conjugate} \end{aligned} \tag{5.12}$$

We introduce some notations. Let the operator B be a linear combination of the ordered products $:\varphi_1 \dots \varphi_n:$. A product of $\varphi_{\text{in}}(x)$ with B can be split up in a part where $\varphi_{\text{in}}(x)$ is contracted with a φ_{in} in B, and the rest, and we will adopt a notation exhibiting this splitting explicitly

$$\varphi_{\text{in}}(x)\, B = \overline{\varphi_{\text{in}}(x)\, B} + :\varphi_{\text{in}}(x)\, B:$$

For instance with $B = :\varphi_1 \dots \varphi_n:$ we have

$$\overline{\varphi_{\text{in}}(x)\, B} = \sum_{j=1}^{n} \langle 0\, |\varphi_{\text{in}}(x)\, \varphi_j|\, 0\rangle : \varphi_1 \dots \check{\varphi}_j \dots \varphi_n:$$

$$:\varphi_{\text{in}}(x)\, B: = :\varphi_{\text{in}}(x)\, \varphi_1 \dots \varphi_n:$$

We write further

$$\frac{\delta S}{\delta x} = \int \mathrm{d}_4 x' N(x - x')\, \frac{\delta S}{\delta \underline{x}'}$$

$$N(x) = (2\pi)^{-4} \int \mathrm{d}_4 k\; \mathrm{e}^{ikx} N(k^2)$$

$\delta S/\delta \underline{x}$ is $\delta S/\delta x$ without the factor $N(k^2)$. With these notations and (5.10) we can write

$$\int \mathrm{d}_4 x' \Delta_R(x - x')\, \varphi_{\text{in}}(x)\, \frac{\delta S^\dagger}{\delta x'}\, S = \int \mathrm{d}_4 x'\, \mathrm{d}_4 x'' \cdot$$

$$\cdot \{\bar{\Delta}_F^*(x - x') + \Delta^\dagger(x - x')\} \left\{ -i\Delta^+(x - x'')\, \frac{\delta^2 S^\dagger}{\delta \underline{x}' \delta \underline{x}''}\, S + :\varphi_{\text{in}}(x)\, \frac{\delta S^\dagger}{\delta \underline{x}'} : S \right\}$$

Further we have:

$$:\varphi_{\text{in}}^2(x): = :\varphi_{\text{in}}^2(x): S^\dagger S = -2i \int \mathrm{d}_4 x' \Delta^+(x - x')\; :\varphi_{\text{in}}(x)\, \frac{\delta S^\dagger}{\delta x'} : S +$$

$$- \int \mathrm{d}_4 x'\, \mathrm{d}_4 x'' \Delta^+(x - x')\, \Delta^+(x - x'')\, \frac{\delta^2 S^\dagger}{\delta x' \delta x''}\, S + :\varphi_{\text{in}}^2(x)\, S^\dagger : S$$

Insertion of the last two formulae into (5.12) gives after some algebra:

$$\varphi^2(x) = \tfrac{1}{2} \langle 0\, |\varphi_{\text{in}}^2(x)|\, 0\rangle - \frac{i}{2}\, BS + \text{herm. conj.}$$

with

$$B = -2 \int \mathrm{d}_4 x' \bar{\Delta}_F^*(x - x')\; :\varphi_{\text{in}}(x)\, \frac{\delta S^\dagger}{\delta \underline{x}'} : +$$

$$-i \int \mathrm{d}_4 x'\, \mathrm{d}_4 x'' \bar{\Delta}_F^*(x - x')\, \bar{\Delta}_F^*(x - x'')\, \frac{\delta^2 S^\dagger}{\delta \underline{x}' \delta \underline{x}''} + i\; :\varphi_{\text{in}}^2(x)\, S^\dagger : \qquad (5.13)$$

The operator B is very similar to the operator $\delta S^\dagger/\delta x$. The first term corresponds to all diagrams of the $S^\dagger$-matrix where one external line ending in the vertex x, where in addition one external φ-line but no A-line is attached (fig. 6*a*). The second term contains all diagrams of $S^\dagger$ with two outgoing lines replaced by two internal lines ending in the vertex x to which no A-line is attached (fig. 6*b*). The third term contains all diagrams of $S^\dagger$ supplemented by an external vertex with two outgoing φ-lines and without A-line (fig. 6*c*).

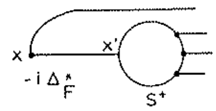

Fig. 6*a*. Diagrams contributing to the first term of the right-hand side of (5.13).

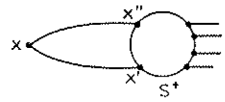

Fig. 6*b*. Diagrams contributing to the second term of the right-hand side of (5.13).

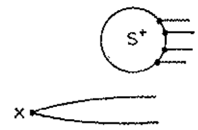

Fig. 6*c*. Diagrams contributing to the last term of the right-hand side of (5.13).

There is also a factor i for the new vertex as required by the Feynman rules given before. Now the second term contains also diagrams from which a part without external φ-line can be separated by cutting one internal line (fig. 7). We recognize this part, together with the internal A-line involved,

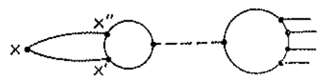

Fig. 7. Diagram contained in fig. 6*b* having the property that a part without external lines can be separated by cutting only one A-line.

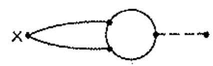

Fig. 8. The factor $N'(k^2) - 1$. The circle stands for all possible diagrams with three outgoing lines as indicated. Together with the two φ-lines it gives exactly all s.p.d. for the A-field, i.e. a contribution $E'(\bar{\Delta}_F, \bar{\Delta}_F', k^2)\; i\bar{\Delta}_F'(k^2)$.

as the analogon of $N(k^2) - 1$ for the A-field, and shall call it $N'(k^2) - 1$ (fig. 8). Taking all terms together we can write:

$$gB = N'^*(k^2)\; C$$

$$N'(k^2) = \frac{\bar{\Delta}_F'(k^2)}{\Delta_F'(k^2)}$$

where C contains all diagrams having one external vertex without A-line and from which no part without external lines can be separated by cutting one A-line.

We observe now that because

$$\bar{\Delta}'_F(-M^2) \neq \infty \quad \text{and} \quad \Delta'_F(-M^2) = \infty$$
$$N'(-M^2) = 0$$

This makes it possible to add formally diagrams having outgoing A-lines with the factor $N'(k^2)$ to our S-matrix without influencing the physical results (since their contribution vanishes on the mass shell). The operator $g(-iBS - i\langle 0 |BS| 0\rangle)$ can be seen then as

$$-i \frac{\delta S^\dagger}{\delta A_{\text{in}}(x)} S - i\langle 0 | \frac{\delta S^\dagger}{\delta A_{\text{in}}(x)} S| 0\rangle = j'(x)$$

i.e. as a current $j'(x)$ from which an A-field can be generated

$$A(x) = -\int d_4x' \Delta'_R(x - x')\, j'(x') \tag{5.14}$$

Again, for this A one can prove

$$i \langle 0 |A(x)\, A(x')| 0\rangle = \bar{\Delta}'^{+}(x - x')$$

Furthermore, because of the important relation

$$\int d_4x' \Delta'(x - x')\, j'(x') = 0$$

characteristic for unstable particles and due to the vanishing of $N'(k^2)$ on the mass shell, we find as expected

$$\lim_{t=\pm\infty} \langle \alpha |A(x)| \beta\rangle = 0$$

We remark that from the diagrams in fig. 7 the infinite renormalization terms still have to be subtracted. This is done as indicated after (3.8) and the outcome is that we have proved the first of the field equations (5.11), namely (5.14) with the following current

$$j'(x) = g\{\varphi^2(x) - \langle 0 |\varphi^2(x)| 0\rangle\} - \delta M^2 Z'_2 A(x) - (Z'_2 - 1)(\Box^2 - M^2)\, A(x)$$

The derivation of the other field equation is somewhat simpler due to the absence of the A_{in} field. The essential reasoning being the same, however, we do not give the proof here.

6. *Causality.* If we eliminate the A-field we can regard our theory as an example of a non-local theory in the field φ. Despite its non-local character it must fulfil the requirements of causality, and it is of interest to verify

this explicitly. To this end we use Bogoliubov's form of causality (ref. 4);

$$\frac{\delta}{\delta\varphi_{\text{in}}(x_1)} \left\{ \frac{\delta S^\dagger}{\delta\varphi_{\text{in}}(x_2)} S \right\} = 0 \qquad x_1 \lesssim x_2$$

($x_1 \lesssim x_2$ means spacelike separated x_1 and x_2 or $x_{01} < x_{02}$) or

$$\frac{\delta^2 S^\dagger}{\delta\varphi_{\text{in}}(x_1)\,\delta\varphi_{\text{in}}(x_2)} S = -\frac{\delta S^\dagger}{\delta\varphi_{\text{in}}(x_2)} \frac{\delta S}{\delta\varphi_{\text{in}}(x_1)} \tag{6.1}$$

Fig. 9 expresses this causality condition for a general diagram and one immediately recognizes it as deducible from a special case of the cutting

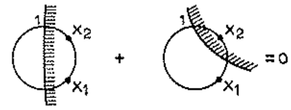

Fig. 9. Diagrams contributing to (6.1). The number 1 added to the cut means that all cut lines give Δ^+ functions. Only the relevant external vertices x_1 and x_2 are explicity indicated.

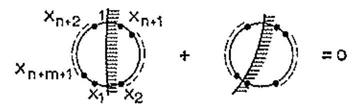

Fig. 10. Diagram interpretation of (6.2).

formula. The validity of the latter therefore establishes causality. In its most general form the cutting formula gives rise to equations generalizing (6.1) with more derivatives, i.e. more vertices x_i. For an arbitrary number of vertices x_i one has (we write again $\delta/\delta x_i$ for $\delta/\delta\varphi_{\text{in}}(x_i)$):

$$\frac{\delta}{\delta x_1} \left\{ \frac{\delta^{(n)} S^\dagger}{\delta x_2 \ldots \delta x_{n+1}} \cdot \frac{\delta^{(m)} S}{\delta x_{n+2} \ldots \delta x_{n+m+1}} \right\} = 0 \tag{6.2}$$

if $x_1 \lesssim x_2, \ldots x_{n+m+1}$

Fig. 10 expresses this in terms of diagrams. It should be remarked, however, that formula (6.2) can be derived from (6.1) and consequently contains no additional information.

Acknowledgements. The author is greatly indebted to Prof. L. Van Hove, who suggested the present investigation, for guidance in all stages of the work. He also profited greatly from discussions with Dr. K. Symanzik. Finally he gratefully acknowledges stimulating discussions with the staff members of the Institute for Theoretical Physics of the University, Utrecht.

This work is part of the research program of the "Stichting voor Fundamenteel Onderzoek der Materie" (F.O.M.), which is financially supported by the Nederlandse Organisatie voor zuiver Wetenschappelijk Onderzoek (Z.W.O.).

Received 5-11-62

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Chapter III

HIGHER ORDER CORRECTIONS TO THE COHERENT PRODUCTION OF VECTOR BOSONS IN THE COULOMB FIELD OF A NUCLEUS

Synopsis

The wave function for a vector boson in a Coulomb field is obtained in a high energy approximation. The Furry wave function for a lepton in a Coulomb field is reconsidered and extended by taking into account a term hitherto neglected. Both wave functions are then applied to the coherent production of vector bosons and muons by neutrinos in the Coulomb field of a nucleus. For low-energy muons the matrix element is improved by taking into account all second order effects for the muon. The effect of nuclear structure is briefly discussed.

1. *Introduction*. Recently, Lee, Markstein and Yang[1]) calculated the coherent production cross section of intermediate vector bosons by neutrinos in lowest order in the e.m. coupling constant. Their calculations indicate a rather strong dependence of the cross section on the mass and the magnetic moment of the vector boson and it will thus be probable that mass and magnetic moment are determined from experimental cross sections. Clearly any uncertainty in the theoretical cross section will come out then as an uncertainty in these parameters and it seems desirable to investigate higher order corrections. The process in question is very similar to electron-positron pair creation by a photon, which has been studied extensively. For the case of large Z where higher order effects in Ze^2 are important, the pair production has been calculated in a high energy approximation by Bethe and Maximon[2]), and it is the purpose of the present paper to do the same type of calculation for production of vector bosons (hereafter called W). This problem can be split into two parts, i) the calculation of the wave function of a vector boson in a Coulomb field, and ii) the application of this wave function to a special problem, for instance neutrino induced production or muon induced production. Clearly, in attacking the problem in this way, we have the advantage of finding a wave function which is of use independently of the way in which other particles (muon or electron) enter the problem. For instance, such a wave function can be used also in high energy neutrino induced processes, where the muon (or electron) comes out with

low velocity, a situation which we shall see to be of practical interest.

In section 2, kinematical considerations will reveal that coherent production necessarily involves a relativistic vector boson and a momentum transfer whose magnitude is small with respect to the energy of the W. In section 3, we solve the equation of motion of a vector boson in a Coulomb field for high energies and small momentum transfer. Section 4 treats the Dirac equation in an analogous way. In section 5, the wave functions of sections 3 and 4 are applied to the problem of neutrino induced W production, both outgoing particles being relativistic. To improve the matrix element in the case of a slow outgoing muon certain second order terms are added as discussed in section 6. In section 7 the important nuclear structure effects are discussed. Section 8, finally, summarizes the results of sections 5 to 7.

2. *Kinematics.* For definiteness we consider a specific process:

$$\nu + Z \to \mu + W + Z$$

where ν = neutrino, Z = nucleus of charge Ze, μ = muon, W = vector boson. Here and in what follows we will work in the laboratory system of reference and neglect the recoil energy of the nucleus. Putting $\hbar = c = 1$ and denoting the momentum and energy of the neutrino, μ-meson and vector boson by $(\boldsymbol{q}, E)$, $(\boldsymbol{p}, \varepsilon)$ and $(\boldsymbol{k}, \omega)$ respectively, we have the relations:

$$\boldsymbol{Q} = \boldsymbol{q} - \boldsymbol{k} - \boldsymbol{p} \quad q = |\boldsymbol{q}| = E \quad p = |\boldsymbol{p}| = \sqrt{\varepsilon^2 - m^2}$$
$$k = |\boldsymbol{k}| = \sqrt{\omega^2 - M^2} \quad E = \varepsilon + \omega \quad Q = |\boldsymbol{Q}|$$

where $\boldsymbol{Q}$ is the momentum absorbed by the nucleus and m and M are the masses of muon and vector boson. For given initial energy E the minimum amount of momentum to be absorbed is given by

$$Q_m = E - \sqrt{E^2 - (M + m)^2}$$

This is when $\boldsymbol{q}$, $\boldsymbol{k}$ and $\boldsymbol{p}$ are all in the same direction, while ε and ω are given by

$$\varepsilon = m \frac{E}{M + m} \qquad \omega = M \frac{E}{M + m}$$

The restriction to the coherent process means $Q \lesssim [\text{nucl. dim.}]^{-1}$, which gives for instance for lead $Q \lesssim 40$ MeV/c. However, larger momentum transfers will also occur, although at a much reduced rate depending on the details of the nuclear form factor. Taking for definiteness $Q \lesssim 100$ MeV/c we get with $M = 600$ MeV and $m = 100$ MeV that $E > 2500$ MeV. This gives to ε and ω relativistic values which can be considered large with respect to Q.

Suppose now that we want to observe muons coming out in directions orthogonal to the incident ν. The region of special interest is then the low muon energy region where $\varepsilon \sim m$ and $Q \sim Q_m + m$. Of course, this Q will be above the upper limit [nucl. dim.]$^{-1}$, but, depending on the special form of the nuclear form factor, the excess will not be so strong as to suppress all processes of this form. The vector boson can never be slow, however, as its mass is much higher so that $Q_m + M$ is always far above threshold.

3. *Vector boson in Coulomb field.* The equation of motion of a W particle in the Coulomb field of a point charge Ze is (in addition to $\hbar = c = 1$ we set $Ze^2 = a$):

$$\frac{\partial^2}{\partial x_\mu^2}\varphi_\nu - \frac{\partial^2}{\partial x_\nu \partial x_\mu}\varphi_\mu - M^2\varphi_\nu = \frac{2a}{r}\frac{\partial \varphi_\nu}{\partial x_4} - \frac{a}{r}\frac{\partial \varphi_4}{\partial x_\nu} - \frac{a}{r}\delta_{\nu 4}\frac{\partial \varphi_\mu}{\partial x_\mu}$$
$$-\lambda\delta_{\nu 4}\left(\frac{\partial(a/r)}{\partial x_i}\right)\varphi_i - (1-\lambda)\left(\frac{\partial(a/r)}{\partial x_\nu}\right)\varphi_4 - \frac{a^2}{r^2}\varphi_\nu + \frac{a^2}{r^2}\delta_{\nu 4}\varphi_4. \tag{3.1}$$

Latin indices will always run from 1 to 3, while Greek ones take the values 1 to 4. λ is the parameter for the magnetic moment μ_W, defined by

$$\mu_W = \frac{e\hbar}{2Mc}\lambda$$

Further $r = |\boldsymbol{x}|$. In the following we shall often use $\boldsymbol{r} = \boldsymbol{x}$. The vector product of two four vectors A and B will be given by $(\boldsymbol{AB}) + A_4B_4$.

There is a subsidiary condition (given in (3.3) below) that can be found by applying the operator $\partial/\partial x_\nu$ to eq. (3.1) and summing over ν. This condition can be used to eliminate the second term in the left-hand side of (3.1). After some algebraic manipulations one finds:

$$(\Box - M^2)\,\varphi_\nu = \frac{2a}{r}\frac{\partial\varphi_\nu}{\partial x_4} - \frac{(2-\lambda)}{M^2}\left[\frac{\partial^2}{\partial x_\nu\partial x_i}, \frac{a}{r}\right]\frac{\partial\varphi_i}{\partial x_4} - \lambda\delta_{\nu 4}\left[\frac{\partial}{\partial x_i}, \frac{a}{r}\right]\varphi_i$$
$$+\frac{(1-\lambda)}{M^2}\left[\frac{\partial^3}{\partial x_\nu\partial x_i^2}, \frac{a}{r}\right]\varphi_4 - \frac{(1-\lambda)}{M^2}\left[\frac{\partial}{\partial x_\nu}, \frac{a}{r}\right]\left(\frac{\partial^2\varphi_4}{\partial x_4^2} + M^2\varphi_4\right)$$
$$+\frac{\lambda}{M^2}\left[\frac{\partial^2}{\partial x_\nu\partial x_i}, \frac{a}{r}\right]\frac{\partial\varphi_4}{\partial x_i} - \frac{a^2}{r^2}\varphi_\nu + \frac{(1-\frac{1}{2}\lambda)}{M^2}\left[\frac{\partial^2}{\partial x_\nu\partial x_i}, \frac{a^2}{r^2}\right]\varphi_i \tag{3.2}$$
$$+\frac{1}{M^2}\left[\frac{\partial}{\partial x_\nu}, \frac{a}{r}\right]\left\{\frac{a^2}{r^2}\varphi_4 - \lambda\frac{a}{r}\frac{\partial\varphi_4}{\partial x_4} + \lambda\left[\frac{\partial}{\partial x_i}, \frac{a}{r}\right]\varphi_i + \left((1-\lambda)\frac{\partial}{\partial x_4} + \frac{a}{r}\right)[A]\right\}$$
$$-\frac{(1-\frac{1}{2}\lambda)}{M^2}\left[\frac{\partial}{\partial x_\nu}, \frac{a^2}{r^2}\right]\frac{\partial\varphi_i}{\partial x_i} - \frac{a}{r}\delta_{\nu 4}[A]$$

with

$$[A] = -\frac{(2-\lambda)}{M^2}\frac{\partial(a/r)}{\partial x_i}\frac{\partial\varphi_i}{\partial x_4} + \frac{(1-\lambda)}{M^2}\left[\frac{\partial^2}{\partial x_i^2}, \frac{a}{r}\right]\varphi_4 + \frac{(1-\frac{1}{2}\lambda)}{M^2}\frac{\partial(a^2/r^2)}{\partial x_i}\varphi_i$$
$$+\frac{\lambda}{M^2}\frac{\partial(a/r)}{\partial x_i}\frac{\partial\varphi_4}{\partial x_i}.$$

As far as is possible the right-hand side has been written in terms of commutators of ar^{-1} with differential operators in order to facilitate the discussion below. The subsidiary condition is

$$\frac{\partial \varphi_\mu}{\partial x_\mu} = \frac{a}{r} \varphi_4 + [A] \tag{3.3}$$

One notes that at $r = \pm \infty$ this goes over into the subsidiary condition for a free particle.

We must now find a wave function that obeys this wave equation in the low momentum transfer limit. This corresponds to the limit of large r in (3.2) and, neglecting all terms of order r^{-2}, one is left with the equation:

$$(\Box - M^2)\,\varphi_\nu - \frac{2a}{r}\frac{\partial \varphi_\nu}{\partial x_4} = 0 \tag{3.4}$$

The monochromatic solutions of this equation are well known and can be written in terms of a confluent hypergeometric function[3]). They differ from each other in the asymptotic behaviour for $r \to \infty$. We restrict ourselves to the particular solution needed when the W occurs in the final state of a collision. This means that we have to take the solution with ingoing spherical waves[4]):

$$\varphi_\nu^{(0)}(x) = e^{i\omega t}\varphi_\nu^{(0)}(\boldsymbol{x}) = M(\omega)\, e_\nu(k)\, e^{-i\boldsymbol{kr}+i\omega t}\, {}_1F_1(-ia_1;\, 1;\, i\boldsymbol{kr} + ikr) \tag{3.5}$$

$$M(\omega) = e^{-(\pi a_1/2)}\Gamma(1 - ia_1) \quad \omega^2 = k^2 + M^2 \quad a_1 = a\frac{\omega}{k}$$

$e_\nu(k)$ is a polarisation vector depending on $\boldsymbol{k}$ and ω (see (3.7)). Clearly ${}_1F_1(-ia_1;\, 1;\, i\boldsymbol{kr} + ikr)$ fulfils:

$$\left(\nabla^2 - 2ik\nabla - \frac{2a_1k}{r}\right) {}_1F_1(-ia_1;\, 1;\, i\boldsymbol{kr} + ikr) = 0 \tag{3.6}$$

One way of seeing that this is the solution with the correct asymptotic behaviour, is by taking the first order term in a of its Fourier transform. This Fourier transform can only be defined if we include a damping term $\exp(-\lambda' r)$ with very small λ':

$$\int d_3 r\, e^{i(\boldsymbol{q}-\boldsymbol{k})\boldsymbol{r}-\lambda' r}\, {}_1F_1(-ia_1;\, 1;\, i\boldsymbol{kr} + ikr) =$$

$$= \left[\frac{8\pi\lambda'(1 + ia_1)}{((\boldsymbol{q} - \boldsymbol{k})^2 + \lambda'^2)} - \frac{1}{(q^2 - k^2 - 2ik\lambda' + \lambda'^2)}\, \frac{8\pi a_1(k + i\lambda')}{(\boldsymbol{q} - \boldsymbol{k})^2 + \lambda'^2}\right] \cdot$$

$$\cdot \left(\frac{(\boldsymbol{q} - \boldsymbol{k})^2 + 2(\boldsymbol{k}, \boldsymbol{q} - \boldsymbol{k}) + \lambda'^2 - 2ik\lambda'}{(\boldsymbol{q} - \boldsymbol{k})^2 + \lambda'^2}\right)^{ia_1}.$$

For $\boldsymbol{q} \neq \boldsymbol{k}$ and $\lambda' \to 0$ only the second term in square brackets is non-zero, and one sees that in first order in a it coincides with the first Born approximation for the potential $2a\omega r^{-1}$ and an ingoing spherical wave. The subsidiary condition gives a restriction on the polarization vector $e_\nu(k)$:

$$k_\nu e_\nu(k) = 0 \tag{3.7}$$

Clearly one can find 3 independent unit vectors that fulfil (3.7). This corresponds to the 3 polarization possibilities.

The wave function (3.5) is often not accurate enough for direct application. Therefore, we try to improve it by calculating the next higher order contribution. We do this by treating the r^{-2} terms in Born approximation, where (3.4) is considered as the unperturbed equation. Dropping all terms of order $a^2 r^{-3}$ in (3.2), inserting $\varphi_\nu^{(0)}$ in all r^{-2} terms and substituting $\varphi_\nu(x) = (\exp i\omega t)\, \varphi_\nu(\boldsymbol{x})$ we are left with:

$$\left(\nabla^2 + k^2 - \frac{2a_1 k}{r}\right)\varphi_j(\boldsymbol{x}) = -\frac{(2-\lambda)}{2M^2}\left[\frac{\partial^2}{\partial x_j \partial x_i}, \frac{2a_1 k}{r}\right]\varphi_i^{(0)}(\boldsymbol{x}) +$$
$$+\frac{(1-\lambda)}{2\omega M^2}\left[\frac{\partial^3}{\partial x_j \partial x_i^2}, \frac{2a_1 k}{r}\right]\varphi_4^{(0)}(\boldsymbol{x}) - \frac{(1-\lambda)}{2\omega M^2}\left[\frac{\partial}{\partial x_j}, \frac{2a_1 k}{r}\right](\omega^2 + M^2)\,\varphi_4^{(0)}(\boldsymbol{x}) +$$
$$-\frac{a^2}{r^2}\varphi_j^{(0)}(\boldsymbol{x}) - \frac{i\lambda k_i}{2\omega M^2}\left[\frac{\partial^2}{\partial x_j \partial x_i}, \frac{2a_1 k}{r}\right]\varphi_4^{(0)}(\boldsymbol{x}) \tag{3.8}$$

$$\left(\nabla^2 + k^2 - \frac{2a_1 k}{r}\right)\varphi_4(\boldsymbol{x}) = -\frac{(2-\lambda)\,\omega}{2M^2}\left[\frac{\partial}{\partial x_i}, \frac{2a_1 k}{r}\right]\varphi_i^{(0)}(\boldsymbol{x}) +$$
$$-\frac{\lambda}{2\omega}\left[\frac{\partial}{\partial x_i}, \frac{2a_1 k}{r}\right]\varphi_i^{(0)}(\boldsymbol{x}) + \frac{(1-\lambda)}{2M^2}\left[\frac{\partial^2}{\partial x_i^2}, \frac{2a_1 k}{r}\right]\varphi_4^{(0)}(\boldsymbol{x}) +$$
$$-\frac{a^2}{r^2}\varphi_4^{(0)}(\boldsymbol{x}) - \frac{i\lambda k_i}{2M^2}\left[\frac{\partial}{\partial x_i}, \frac{2a_1 k}{r}\right]\varphi_4^{(0)}(\boldsymbol{x}) \tag{3.9}$$

where we have used $\partial/\partial x_i\, \varphi_4^{(0)} = -ik_i\, \varphi_4^{(0)}$ + terms of order $a/r\ \varphi_4^{(0)}$. One notes that the right-hand side contains an r^{-2} term and terms which have a commutator of ∇ and r^{-1} as a factor. We will treat them separately mainly because the book-keeping becomes somewhat simpler. Consider the commutator terms first. They require the solution of an equation of the form

$$\left(\nabla^2 + k^2 - \frac{2a_1 k}{r}\right)\varphi_j(\boldsymbol{x}) = \left[f_{ji}(\nabla), \frac{2a_1 k}{r}\right]\varphi_i^{(0)}(\boldsymbol{x}) =$$
$$= M(\omega)\, e^{-ikr}\left[f_{ji}(\nabla - i\boldsymbol{k}), \frac{2a_1 k}{r}\right] e_i(k)\ {}_1F_1(-ia_1; 1; i\boldsymbol{kr} + ikr) \tag{3.10}$$

where f is some function that can be given in terms of a power series. A solution of this equation is:

$$\varphi_j(\boldsymbol{x}) = M(\omega)\, e^{-ikr}\{f_{ji}(\nabla - i\boldsymbol{k}) - f_{ji}(-i\boldsymbol{k})\}\cdot e_i(k)\ {}_1F_1(-ia_1; 1; i\boldsymbol{kr} + ikr)$$

as can be seen immediately by insertion in (3.10). Since the derivative of ${}_1F_1$ behaves asymptotically as r^{-1} this $\varphi_j(\boldsymbol{x})$ has the asymptotic behaviour required for our problem. By this method we have now the solution of (3.8) and (3.9) as far as the commutator terms are concerned. Consider next

$$\left(\nabla^2 + k^2 - \frac{2a_1k}{r}\right)\varphi_j(\boldsymbol{x}) = \frac{\alpha}{r^2} M(\omega)\, e_j(k)\, \mathrm{e}^{ikr}\, {}_1F_1(-ia_1; 1; i\boldsymbol{kr} + ikr) \quad (3.11)$$

α being some constant. We make the ansatz[5])

$$\varphi_j(\boldsymbol{x}) = \frac{\alpha M(\omega)}{4\pi} e_j(k) \int \mathrm{d}_3k' \frac{\mathrm{e}^{-ik'r}}{(k^2 - k'^2 + i0)} \cdot$$

$$\cdot \frac{1}{|\boldsymbol{k} - \boldsymbol{k}'|}\, {}_1F_1(-ia_1; 1; i\boldsymbol{k}'\boldsymbol{r} + ikr) \quad (3.12)$$

We insert this ansatz into the left-hand side of (3.11). In order to permit differentiation under the integral sign we make the $\boldsymbol{k}'$ integration uniformly convergent by adding a term $-\lambda' |\boldsymbol{k} - \boldsymbol{k}'|$ in the exponent. By using the equation for a confluent hypergeometric function

$$\left(x \frac{\mathrm{d}^2}{\mathrm{d}x^2} + (b - x)\frac{\mathrm{d}}{\mathrm{d}x} - a\right) {}_1F_1(a; b; x) = 0$$

we get without difficulty:

$$\left(\nabla^2 + k^2 - \frac{2a_1k}{r}\right)\varphi_j(\boldsymbol{x}) =$$

$$= \frac{\alpha M(\omega)}{4\pi} e_j(k) \int \mathrm{d}_3k' \frac{\mathrm{e}^{-ik'r - \lambda'|\boldsymbol{k}' - \boldsymbol{k}|}}{|\boldsymbol{k} - \boldsymbol{k}'|} \{{}_1F_1'' - 2\,{}_1F_1' + {}_1F_1\}$$

To calculate this we use the well-known representation:

$${}_1F_1(-ia_1; 1; i\boldsymbol{k}'\boldsymbol{r} + ikr) = \frac{1}{2\pi i} \int^{0^+, 1^+} \mathrm{d}t\, t^{-ia_1 - 1} (t - 1)^{ia_1}\, \mathrm{e}^{t(i\boldsymbol{k}'\boldsymbol{r} + ikr)} \quad (3.13)$$

The integrand has a cut from $t = 0$ to $t = 1$ and we are in the sheet where the arguments of t and $t - 1$ are zero on the real axis to the right of 1. The integration contour encircles the points 0 and 1 anti-clockwise. Taking the contour sufficiently close around the cut we can, for a given $\lambda' \neq 0$, exchange the t and $\boldsymbol{k}'$ integration. Performing the $\boldsymbol{k}'$ integration we get:

$$\left(\nabla^2 + k^2 - \frac{2a_1k}{r}\right)\varphi_j(\boldsymbol{x}) = \frac{\alpha M(\omega)}{8\pi i} e_j(k)\, \mathrm{e}^{-ikr} \cdot$$

$$\int^{0^+, 1^+} \mathrm{d}t\, t^{-ia_1 - 1} (t - 1)^{ia_1 + 2}\, \mathrm{e}^{ikrt + ikrt} \frac{4\pi}{r^2(t - 1)^2 + \lambda'^2} .$$

The t integrand has now acquired two poles which approach the point $t = 1$ as $\lambda' \to 0$ so that the contour is pinched from two sides. Therefore, we expand first the contour so that the poles come inside by adding the residues of the poles. These residues are seen to go to zero if $\lambda' \to 0$, and performing this limit we get the desired result (3.11). The solution (3.12) goes over into the normal plane wave Born approximation if we let $a_1 \to 0$. One establishes without difficulty that (3.12) has the proper asymptotic behaviour.

We can now write down the complete solution of (3.8) and (3.9):

$$\varphi_\nu(x) = \varphi_c + \varphi_d + \varphi_a \tag{3.14}$$

with

$$\varphi_c = \frac{M(\omega)}{\sqrt{2\omega V}}\, e^{-ikr+i\omega t}\, e_\nu(k)\ {}_1F_1(-ia_1;\,1;\,i\boldsymbol{kr} + ikr)$$

$$\varphi_d = \frac{M(\omega)}{\sqrt{2\omega V}}\, e^{-ikr+i\omega t}\, \Omega_{\nu\mu} e_\mu(k)\ {}_1F_1(-ia_1;\,1;\,i\boldsymbol{kr} + ikr)$$

$$\varphi_a = \frac{a^2}{4\pi}\,\frac{M(\omega)}{\sqrt{2\omega V}}\, e_\nu(k) \int d_3k'\ \frac{e^{-i\boldsymbol{k'r}+i\omega t}}{(k'^2 - k^2 - i0)}\ \frac{1}{|\boldsymbol{k} - \boldsymbol{k'}|}\ \cdot$$

$$\cdot\, {}_1F_1(-ia_1;\,1;\,i\boldsymbol{k'r} + ikr)$$

where we have added the usual normalization factor $(2\omega V)^{-\frac{1}{2}}$ [6]. Ω is given by:

$$\Omega_{ji} = -\frac{(2-\lambda)}{2M^2}\left\{\left(\frac{\partial}{\partial x_j} - ik_j\right)\left(\frac{\partial}{\partial x_i} - ik_i\right) + k_j k_i\right\}$$

$$\Omega_{j4} = \frac{(1-\lambda)}{2\omega M^2}\left\{\left(\frac{\partial}{\partial x_j} - ik_j\right)\left(\nabla^2 - 2ik_i\frac{\partial}{\partial x_i} - k^2\right) - ik_j k^2 + \right.$$
$$\left. - (\omega^2 + M^2)\frac{\partial}{\partial x_j}\right\} - \frac{\lambda}{2\omega M^2}\left\{\left(\frac{\partial}{\partial x_j} - ik_j\right)\left(\frac{\partial}{\partial x_i} - ik_i\right) ik_i + ik_j k^2\right\}$$

$$\Omega_{4i} = -\frac{(2-\lambda)\,\omega}{2M^2}\,\frac{\partial}{\partial x_i} - \frac{\lambda}{2\omega}\,\frac{\partial}{\partial x_i}$$

$$\Omega_{44} = \frac{(1-\lambda)}{2M^2}\left\{\nabla^2 - 2ik_i\frac{\partial}{\partial x_i}\right\} - \frac{\lambda}{2M^2}\, ik_i\,\frac{\partial}{\partial x_i}\,.$$

With the help of eq. (3.6) for ${}_1F_1$ and the condition (3.7) we finally reduce Ω to:

$$\Omega_{ji} = -\frac{(2-\lambda)}{2M^2}\left\{\frac{\partial^2}{\partial x_j \partial x_i} - ik_j\frac{\partial}{\partial x_i}\right\}$$

$$\Omega_{j4} = \frac{(1-\lambda)}{M^2}\left(\frac{\partial}{\partial x_j} - ik_j\right)\frac{a_1 k}{r\omega} +$$
$$+ \frac{\lambda}{2\omega M^2}\left\{-ik_i\frac{\partial^2}{\partial x_i \partial x_j} - k_j k_i\frac{\partial}{\partial x_i} + M^2\frac{\partial}{\partial x_j}\right\} \tag{3.15}$$

$$\Omega_{4i} = -\frac{2\omega^2 - \lambda k^2}{2M^2\omega} \frac{\partial}{\partial x_i}$$

$$\Omega_{44} = \frac{(1-\lambda)}{M^2} \frac{a_1 k}{r} - \frac{i\lambda}{2M^2} k_i \frac{\partial}{\partial x_i}$$

4. *Dirac particle in Coulomb field.* The Dirac equation for a particle in a Coulomb field is:

$$\left(\gamma^\mu \frac{\partial}{\partial x_\mu} - \gamma^4 \frac{a}{r} - m\right) \psi(x) = 0 \tag{4.1}$$

Multiplying with $(-\gamma^\mu \partial/\partial x_\mu + \gamma^4 a r^{-1} - m)$ we have:

$$(\Box - m^2)\, \psi(x) = \left(\frac{2a}{r} \frac{\partial}{\partial x_4} + a\gamma^4\gamma^i \frac{x_i}{r^3} - \frac{a^2}{r^2}\right) \psi(x). \tag{4.2}$$

There seems to be some confusion in the literature[7]) concerning the use of (4.2) instead of (4.1). In appendix A we show that in second order the diagrams of perturbation theory generated by (4.2) are the same as those generated by (4.1). For the complete proof of the equivalence of (4.2) and (4.1) (for asymptotically equal solutions) we refer to the work of B. Nagel[8]).

In a way completely analogous to the calculation in section 3, we find the solution $\psi(x)$ which takes into account exactly the first term in the right-hand side of (4.2) and in first approximation the terms of order r^{-2}. Again specializing to the case of an outgoing particle we have:

$$\begin{aligned}
\bar\psi(x) &= \bar\psi_c + \bar\psi_d + \bar\psi_a \\
\bar\psi_c &= \frac{N(\varepsilon)}{\sqrt{V}} \mathrm{e}^{-i\boldsymbol{pr} + i\varepsilon t}\, \bar u(\mathfrak{p})\; {}_1F_1(ia_2;\, 1;\, i\boldsymbol{pr} + ipr) \\
\bar\psi_d &= \frac{N(\varepsilon)}{\sqrt{V}} \mathrm{e}^{-i\boldsymbol{pr} + i\varepsilon t}\, \bar u(\mathfrak{p}) \frac{1}{2\varepsilon} \gamma^4\gamma^i \frac{\partial}{\partial x_i}\; {}_1F_1(ia_2;\, 1;\, i\boldsymbol{pr} + ipr) \\
\bar\psi_a &= \frac{a^2 N(\varepsilon)}{4\pi\sqrt{V}} \bar u(\mathfrak{p}) \int \mathrm{d}_3 p' \frac{\mathrm{e}^{-i\boldsymbol{p}'\boldsymbol{r} + i\varepsilon t}}{(p'^2 - p^2)\, |\boldsymbol{p} - \boldsymbol{p}'|}\; {}_1F_1(ia_2;\, 1;\, i\boldsymbol{p}'\boldsymbol{r} + ipr) \\
a_2 &= \frac{a\varepsilon}{p} \qquad N(\varepsilon) = \mathrm{e}^{(\pi/2)a_2}\, \Gamma(1 + ia_2) \quad \varepsilon^2 = p^2 + m^2
\end{aligned} \tag{4.3}$$

$\bar u(\mathfrak{p})$ is the Dirac spinor for a free muon of four impuls $\mathfrak{p}$, cf. ref. 6).

5. *The matrix element for neutrino induced W production.* In the foregoing we discussed the wave functions of vector boson and lepton in a Coulomb field of arbitrary strength in the low momentum transfer limit. We shall use now these functions for the particular case of W and muon production by a neutrino in the Coulomb field of a nucleus whereby the nucleus is assumed to remain in the same state (coherent process). The weak inter-

action will be treated in lowest order. The matrix element for this process corresponding to the diagram of fig. 1, will depend on the four-momenta q, p and k of the neutrino, muon and vector boson, and on the polarisation of these particles.

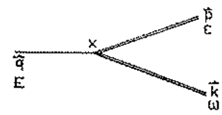

Fig. 1. Diagram for production of a W with momentum-energy $\boldsymbol{k}$, ω by a neutrino with momentum-energy $\boldsymbol{q}$, E.

Taking the neutrino to be in a definite state of helicity we need two indices t and s to indicate the polarisation of muon and vector boson respectively. Denoting the matrix element by $M(q, p, k, t, s)$ we have:

$$M(q, p, k, t, s) = ig \int d_4x \, \langle p, t \, |\bar{\psi}^m(x)| \, 0\rangle \, \gamma^\mu(1 + \gamma^5) \, \langle 0 \, |\psi^n(x)| \, q\rangle \cdot \cdot \langle k, s \, |\varphi_\mu(x)| \, 0\rangle. \qquad (5.1)$$

In here $\bar{\psi}^m$, ψ^n, φ_μ stand for muon, neutrino and vector boson field. We must now insert the wave functions calculated above together with a plane wave for the neutrino. The wave functions from sections 3 and 4 have the form:

$$\langle k, s \, |\varphi_\mu(x)| \, 0\rangle = \varphi_c + \varphi_d + \varphi_a$$
$$\langle p, t \, |\bar{\psi}^m(x)| \, 0\rangle = \bar{\psi}_c + \bar{\psi}_d + \bar{\psi}_a$$

where all φ and $\bar{\psi}$ are as indicated in (3.14) and (4.3), i.e., φ_c and $\bar{\psi}_c$ stand for the solution of the equations of motion with the Coulomb field r^{-1} alone (c.f. (3.4)), φ_d and $\bar{\psi}_d$ for the corrections arising from that part of the equation which could be written in terms of commutators of r^{-1} with V, and φ_a and $\bar{\psi}_a$ finally for the correction arising from the ar^{-2} term (c.f. (3.11)). Clearly, φ_d, φ_a are small compared with φ_c, and $\bar{\psi}_d$, $\bar{\psi}_a$ are small in comparison to $\bar{\psi}_c$. Inserting these expressions into (5.1) we have:

$$M(q, p, k, t, s) = \frac{ig}{\sqrt{V}} \int d_4x \{\bar{\psi}_c + \bar{\psi}_d + \bar{\psi}_a\} \, \gamma^\mu(1 + \gamma^5) \, \{u^n(q) \, e^{iqx}\} \cdot \{\varphi_c + \varphi_d + \varphi_a\}.$$

$u^n(q)$ is the Dirac spinor for a free neutrino of four impuls q.

Evaluating this expression we will neglect the term involving

$$(\bar{\psi}_d + \bar{\psi}_a)(\varphi_d + \varphi_a)$$

and we see that the matrix element is built up from five parts:

$$M = \frac{ig}{\sqrt{V}} \{I_1 + I_2 + I_3 + I_4 + I_5\}$$

with

$$
\begin{aligned}
I_1 &= \int d_4x \bar{\psi}_c \gamma^\mu (1 + \gamma^5)\, u^n(q)\, e^{iqx}\, \varphi_c \\
I_2 &= \int d_4x \bar{\psi}_c \gamma^\mu (1 + \gamma^5)\, u^n(q)\, e^{iqx}\, \varphi_d \\
I_3 &= \int d_4x \bar{\psi}_d \gamma^\mu (1 + \gamma^5)\, u^n(q)\, e^{iqx}\, \varphi_c \\
I_4 &= \int d_4x \bar{\psi}_c \gamma^\mu (1 + \gamma^5)\, u^n(q)\, e^{iqx}\, \varphi_a \\
I_5 &= \int d_4x \bar{\psi}_a \gamma^\mu (1 + \gamma^5)\, u^n(q)\, e^{iqx}\, \varphi_c
\end{aligned}
\tag{5.2}
$$

The integrals $I_1 - I_3$ are of the same type as those calculated by Nordsieck[9]) and Bethe-Maximon. Essentially they can all be derived from Nordsieck's result:

$$
\begin{aligned}
I(\lambda') &= \int d_3r \frac{e^{i\boldsymbol{Qr} - \lambda' r}}{r}\, {}_1F_1(ia_2; 1; i\boldsymbol{pr} + ipr)\, {}_1F_1(-ia_1; 1; i\boldsymbol{kr} + ikr) = \\
&= \frac{4\pi}{Q^2 + \lambda'^2} \left(\frac{2\boldsymbol{kQ} - 2ik\lambda' + Q^2 + \lambda'^2}{Q^2 + \lambda'^2} \right)^{ia_1} \left(\frac{2\boldsymbol{pQ} - 2ip\lambda' + Q^2 + \lambda'^2}{Q^2 + \lambda'^2} \right)^{-ia_2} \cdot \\
&\qquad \cdot F(-ia_1, ia_2; 1; 1 - x(\boldsymbol{Q}, \lambda'))
\end{aligned}
\tag{5.3}
$$

$$
x(\boldsymbol{Q}, \lambda') = \frac{(Q^2 + \lambda'^2)(Q^2 + 2\boldsymbol{kQ} + 2\boldsymbol{pQ} + 2\boldsymbol{pk} - 2pk + \lambda'^2 - 2i\lambda' k - 2i\lambda' p)}{(Q^2 + 2\boldsymbol{kQ} + \lambda'^2 - 2i\lambda' k)(Q^2 + 2\boldsymbol{pQ} + \lambda'^2 - 2i\lambda' p)}
$$

F is a normal hypergeometric function, see (5.10) below. We have put $\boldsymbol{Q} = \boldsymbol{q} - \boldsymbol{p} - \boldsymbol{k}$. The integral I_1 can be obtained by differentiation of $I(\lambda')$ with respect to λ', while the integrals I_2 and I_3 can be reduced to derivatives of $I(0)$ with respect to $\boldsymbol{p}$ and $\boldsymbol{k}$ for fixed $\boldsymbol{Q}$ with the help of identities of the type

$$
\frac{\partial}{\partial x_i}\, {}_1F_1(ia_2; 1; i\boldsymbol{pr} + ipr) = \frac{p}{r} \frac{\partial}{\partial p_i}\, {}_1F_1(ia_2; 1; i\boldsymbol{pr} + ipr)
$$

For completeness we write down explicitly the three types of integrals involved in (5.2):

$$
\begin{aligned}
&\int d_3r\, e^{i\boldsymbol{Qr}}\, {}_1F_1(ia_2; 1; i\boldsymbol{pr} + ipr)\, {}_1F_1(-ia_1; 1; i\boldsymbol{kr} + ikr) = \\
&\qquad = \left(- \frac{d}{d\lambda'} I(\lambda') \right)_{\lambda' = 0} \\
&= \frac{4\pi}{Q^2} \left(\frac{2\boldsymbol{kQ} + Q^2}{Q^2} \right)^{ia_1} \left(\frac{2\boldsymbol{pQ} + Q^2}{Q^2} \right)^{-ia_2} \left[\left\{ \frac{2a\varepsilon}{D_2} - \frac{2a\omega}{D_1} \right\} F(-ia_1, ia_2; 1; 1 - x) \right. \\
&\qquad \left. - 2i \left\{ \frac{Q^2(k + p)}{D_1 D_2} - \frac{(kD_2 + pD_1)\, x}{D_1 D_2} \right\} F'(-ia_1, ia_2; 1; 1 - x) \right]
\end{aligned}
\tag{5.4}
$$

$$
\begin{aligned}
&\int d_3r\, e^{i\boldsymbol{Qr}} \left\{ \frac{\partial}{\partial x_i}\, {}_1F_1(ia_2; 1; i\boldsymbol{pr} + ipr) \right\} {}_1F_1(-ia_1; 1; i\boldsymbol{kr} + ikr) \\
&\qquad = p \left(\frac{\partial}{\partial p_i} I(0) \right)_{\boldsymbol{Q} = \text{const.}}
\end{aligned}
$$

$$= \frac{4\pi}{Q^2}\left(\frac{2\boldsymbol{kQ}+Q^2}{Q^2}\right)^{i\alpha_1}\left(\frac{2\boldsymbol{pQ}+Q^2}{Q^2}\right)^{-i\alpha_2}\left[\frac{-2i\alpha\varepsilon}{D_2}Q_iF(-i\alpha_1, i\alpha_2; 1; 1-x)\right.$$

$$\left. -2p\left\{\frac{Q^2(Q_i+k_i-(k/p)\,p_i)}{D_1D_2}-\frac{xQ_i}{D_2}\right\}F'(-i\alpha_1, i\alpha_2; 1; 1-x)\right]$$

$$\int \mathrm{d}_3r\, \mathrm{e}^{i\boldsymbol{Qr}}\, {}_1F_1(i\alpha_2; 1; i\boldsymbol{pr}+ipr)\left\{\frac{\partial}{\partial x_i}\, {}_1F_1(-i\alpha_1; 1; i\boldsymbol{kr}+ikr)\right\} =$$

$$= k\left(\frac{\partial}{\partial k_i}\, I(0)\right)_{Q=\mathrm{const.}}$$

$$= \frac{4\pi}{Q^2}\left(\frac{2\boldsymbol{kQ}+Q^2}{Q^2}\right)^{i\alpha_1}\left(\frac{2\boldsymbol{pQ}+Q^2}{Q^2}\right)^{-i\alpha_2}\left[\frac{2i\alpha\omega}{D_1}Q_iF(-i\alpha_1, i\alpha_2; 1; 1-x)\right.$$

$$\left. -2k\left\{\frac{Q^2(Q_i+p_i-(p/k)\,k_i)}{D_1D_2}-\frac{xQ_i}{D_1}\right\}F'(-i\alpha_1, i\alpha_2; 1; 1-x)\right]$$

where now

$$x = x(\boldsymbol{Q}, 0) = \frac{Q^2(Q^2+2\boldsymbol{kQ}+2\boldsymbol{pQ}+2\boldsymbol{pk}-2pk)}{D_1D_2}$$

$$D_1 = Q^2 + 2\boldsymbol{Qk}$$
$$D_2 = Q^2 + 2\boldsymbol{Qp}$$

One observes that $x \to 0$ for $Q \to 0$. (Remember that the angle between $\boldsymbol{p}$ and $\boldsymbol{k}$ is of the order Q/k or Q/p).

We consider next I_4. The integral to be calculated is:

$$\int \mathrm{d}_3r \int \mathrm{d}_3k' \frac{\mathrm{e}^{i\boldsymbol{\alpha r}}}{(k'^2-k^2)\,|\boldsymbol{k}-\boldsymbol{k}'|}\, {}_1F_1(-i\alpha_1; 1; i\boldsymbol{k'r}+ikr)\, {}_1F_1(i\alpha_2; 1; i\boldsymbol{pr}+ipr) \quad (5.5)$$

with $\boldsymbol{\alpha} = \boldsymbol{Q} + \boldsymbol{k} - \boldsymbol{k}'$. This integral cannot be evaluated exactly and we must make some further approximation. Let us consider first the $\boldsymbol{r}$ integration:

$$\int \mathrm{d}_3r\, \mathrm{e}^{i\boldsymbol{\alpha r}}\, {}_1F_1(-i\alpha_1; 1; i\boldsymbol{k'r}+ikr)\, {}_1F_1(i\alpha_2; 1; i\boldsymbol{pr}+ipr) \quad (5.6)$$

In our approximation we need only take the leading part of this for small values of α. Now for small α (5.6) has a leading term of the $\delta(\boldsymbol{\alpha})$ type[10]. This allows us to change the factor $|\boldsymbol{k}-\boldsymbol{k}'|^{-1}$ in (5.5) so that the value of the $\boldsymbol{k}'$ integrand is not changed in the point $\boldsymbol{k}' = \boldsymbol{Q}+\boldsymbol{k}$ (or $\boldsymbol{\alpha} = 0$). A change that fulfils this condition is:

$$\frac{1}{|\boldsymbol{k}-\boldsymbol{k}'|} \to \frac{(\boldsymbol{Q}, \boldsymbol{k}'-\boldsymbol{k})}{Q\,|\boldsymbol{k}'-\boldsymbol{k}|^2}$$

This change has the further advantage that the new integrand has the same derivative with respect to $\boldsymbol{k}'$ in the point $\boldsymbol{k}' = \boldsymbol{Q}+\boldsymbol{k}$ as the old one. This ensures that this term gets the same phase factor as all other terms in the matrix element, which simplifies our final expression. The change just described makes the integral (5.5) of the type $\partial/\partial k_i\, I(\lambda')$. In the further treatment of (5.5) one can either do the $\boldsymbol{r}$ integration and the subsequent convolution over $\boldsymbol{k}'$, or one can prove that:

$$\int \mathrm{d}_3 k' \, \frac{\mathrm{e}^{-ik'r}}{(k'^2 - k^2 - i0)} \, \frac{(\boldsymbol{k}' - \boldsymbol{k})_i}{|\boldsymbol{k}' - \boldsymbol{k}|^2} \, {}_1F_1(-ia_1;\, 1;\, i\boldsymbol{k}'\boldsymbol{r} + ikr) = \\ = \frac{\pi^2}{ia_1 k} \, \mathrm{e}^{-i\boldsymbol{kr}} \, \frac{\partial}{\partial x_i} \, {}_1F_1(-ia_1;\, 1;\, i\boldsymbol{kr} + ikr)$$

We chose the latter way, the proof is given in appendix B.

At this place we want to clarify a little the above procedure. Adopting the same reasoning as given above one would expect a $\delta(\boldsymbol{Q})$ function for I_1. Indeed, on performing the differentiation with respect to λ' of $I(\lambda')$ one also gets a term of the form $\lambda'/(\lambda'^2 + Q^2)^2$ which is of the δ-type (see appendix C, eq. (C.1)). However, we consider only cases where $Q \neq 0$, so that this term does not contribute. Thus, in the case of I_1, the terms here neglected play a major role.

We can now write down the matrix element in terms of $I(\lambda')$ and

$$l^t_\mu = \tfrac{1}{2}(\bar{u}^m_t(p)\, \gamma^\mu\, (1 + \gamma^5)\, u^n(q)):$$

$$M(q, p, k, t, s) = 4\pi i g \delta(E - \varepsilon - \omega) \frac{M(\omega)\, N(\varepsilon)}{V^{\frac{3}{2}} \sqrt{2\omega}} \Big[- l^t_\mu e^s_\mu(k) \left(\frac{\mathrm{d}I(\lambda')}{\mathrm{d}\lambda'} \right)_{\lambda'=0} +$$

$$+ l^t_j e^s_i(k) \frac{(2-\lambda)}{2M^2} \left\{ ik_j k \frac{\partial}{\partial k_i} I(0) + ikQ_j \frac{\partial}{\partial k_i} I(0) + kp \frac{\partial^2}{\partial p_j \partial k_i} I_2 \right.$$

$$\left. + l^t_j e^s_4(k) \frac{(1-\lambda)\, a_1 k}{M^2 \omega} \left\{ -ik_j I(0) - iQ_j I(0) - p \frac{\partial}{\partial p_j} I_2 \right\} \right.$$

$$+ l^t_j e^s_4(k) \frac{\lambda}{2\omega M^2} \left\{ -(\boldsymbol{k}, \boldsymbol{Q})\, k \frac{\partial}{\partial k_j} I(0) - kpk_i \frac{\partial^2}{\partial k_j \partial p_i} I_2 - kk_j k_i \frac{\partial}{\partial k_i} I(0) \right.$$

$$\left. + M^2 k \frac{\partial}{\partial k_j} I(0) \right\}$$

$$+ l^t_4 e^s_i(k) \left(-\frac{2\omega^2 - \lambda k^2}{2M^2 \omega} \right) k \frac{\partial}{\partial k_i} I(0) +$$

$$+ l^t_4 e^s_4(k) \left\{ \frac{(1-\lambda)\, a_1 k}{M^2} I(0) - \frac{i\lambda k}{2M^2} k_i \frac{\partial}{\partial k_i} I(0) \right\}$$

$$+ \{\delta_{4i} l^t_\mu + \delta_{\mu i} l^t_4 - \delta_{\mu 4} l^t_i + \varepsilon_{\kappa 4 i \mu} l^t_\kappa\}\, e^s_\mu(k) \frac{p}{2\varepsilon} \frac{\partial}{\partial p_i} I(0)$$

$$+ l^t_\mu e^s_\mu(k) \frac{a^2}{4\pi} \left(\frac{\pi^2}{ia_1} \frac{Q_i}{Q} \frac{\partial}{\partial k_i} - \frac{\pi^2}{ia_2} \frac{Q_i}{Q} \frac{\partial}{\partial p_i} \right) I(0) \Big] \tag{5.7}$$

where we have used the following relations:

$$\gamma^{\mu}\gamma^{\alpha}\gamma^{\nu} = \delta_{\alpha\mu}\gamma^{\nu} + \delta_{\alpha\nu}\gamma^{\mu} - \delta_{\mu\nu}\gamma^{\alpha} + \varepsilon_{\kappa\mu\alpha\nu}\gamma^{\kappa}\gamma^{5}$$

$\varepsilon_{\lambda\mu\alpha\nu}$ is the completely antisymmetry tensor, $\varepsilon_{1234} = 1$.

$$\int \mathrm{d}_3 r\, e^{iQr}\, {}_1F_1(ia_2; 1; i\boldsymbol{pr} + ipr) \frac{\partial}{\partial x_i} \frac{k}{r} \frac{\partial}{\partial k_j}\, {}_1F_1(-ia_1; 1; i\boldsymbol{kr} + ikr) =$$

$$= -ik\, Q_i \frac{\partial}{\partial k_j} I(0) - kp \frac{\partial^2}{\partial p_i \partial k_j} I_2 \qquad (5.8)$$

$$I_2 = \int \mathrm{d}_3 r \frac{e^{iQr}}{r^2}\, {}_1F_1(ia_2; 1; i\boldsymbol{pr} + ipr)\, {}_1F_1(-ia_1; 1; i\boldsymbol{kr} + ikr)$$

We look now into questions of order of magnitude. I_1 (Eq. (5.2)) contains the two leading parts $\bar{\psi}_e$ and φ_e of muon and W wave functions. However, if $\boldsymbol{k}$, $\boldsymbol{p}$ and $\boldsymbol{q}$ point in the same direction and if $k/\omega \simeq p/\varepsilon$, we have

$$\frac{\varepsilon}{D_2} \sim \frac{\omega}{D_1}$$

which, as can be seen in (5.4), gives rise to a strong cancellation in I_1. Because of this cancellation, and because of the fact that the part from (5.4) containing a F' function is of lower order in the momentum transfer the integral I_1 becomes of the same order as I_2 to I_4. It is this fact*) which makes necessary the consideration of quantities of lower order, i.e., of quantities of the order of magnitude $I(0)$. It will further be clear from this that the term $(\mathrm{d}I(\lambda')/\mathrm{d}\lambda')_{\lambda'=0}$ must be manipulated without further approximations. We drop all terms containing I_2 because they are one order of magnitude smaller in Q than the terms $I(0)$. One may well ask why we did not drop directly the terms $\partial_i\partial_j$ and $r^{-1}\partial_j$ from $\Omega_{\mu\nu}$. The reason is that we want to be correct also up to first order in the coupling constant $\boldsymbol{a}$ and these terms contain first order contributions, that are extracted by the procedure (5.8). We insert now $I(\lambda')$ from (5.3) into (5.7) to get:

$$M(q, p, k, t, s) = \underline{\frac{8\sqrt{2}\,\pi^2 iga}{V^{\frac{3}{2}}\sqrt{\omega}}} \cdot M(\omega)\, N(\varepsilon)\, \underline{\delta(E - \varepsilon - \omega)}.$$

$$\cdot \left(\frac{2\boldsymbol{kQ} + Q^2}{Q^2}\right)^{ia_1} \left(\frac{2\boldsymbol{pQ} + Q^2}{Q^2}\right)^{-ia_2} \underline{\frac{1}{Q^2}} \left[V(x) \left\{ \underline{l^t_\mu e^s_\mu \left(\frac{2\varepsilon}{D_2} - \frac{2\omega}{D^1} + \right.} \right.\right.$$

$$\left. + \frac{\pi a}{2} \frac{Q}{D_1} + \frac{\pi a}{2} \frac{Q}{D_2} \right) - \underline{l^t_\mu e^s_\nu \frac{A_{\mu\nu}}{D_1}} +$$

$$\left. - \underline{(\delta_{4\nu} l^t_\mu + \delta_{\mu\nu} l^t_4 - \delta_{\mu 4} l^t_\nu + \varepsilon_{\kappa 4\nu\mu}\, l^t_\kappa)\, e^s_\mu \frac{iQ_\nu}{D_2}} \right\} +$$

*) A further enhancement of this effect comes about through the spinor factors in all these integrals.

$$+\frac{ia_1a_2}{a}W(x)\left\{l^t_\mu e^s_\mu\left(2x\left(\frac{k}{D_1}+\frac{p}{D_2}\right)-2Q^2\left(\frac{k+p}{D_1D_2}\right)+\right.\right.$$

$$+\frac{\pi ak}{\omega}\left(Q\,\frac{Q^2+\boldsymbol{pQ}-(p/k)(\boldsymbol{kQ})}{D_1D_2}-Q\,\frac{x}{D_1}\right)$$

$$+\frac{\pi ap}{\varepsilon}\left(Q\,\frac{Q^2+\boldsymbol{kQ}-(k/p)(\boldsymbol{pQ})}{D_1D_2}-Q\,\frac{x}{D_2}\right)\Bigg)+l^t_\mu e^s_\mu B_{\mu\nu}/D_1 \tag{5.9}$$

$$+(\delta_{4i}l^t_\mu+\delta_{\mu i}l^t_4-\delta_{\mu 4}l^t_i+\varepsilon_{\kappa 4i\mu}l^t_\kappa)\,e^s_\mu\left(\frac{-i}{2\varepsilon D_2}\right)\left(2xpQ_i-\frac{2Q^2p}{D_1}\left(Q_i+k_i-\frac{k}{p}\,p_i\right)\right\}\Bigg]$$

with $Q_\mu=(\boldsymbol{Q}\,;0)$, $e^s_\nu=e^s_\nu(k)$ and

$$A_{\mu\nu}=(k+Q)_\mu\left\{\frac{(2-\lambda)\,\omega}{M^2}\,Q_\nu+\frac{i(2-\lambda)(\boldsymbol{k},\,\boldsymbol{Q})+i(1-\lambda)\,Q^2}{M^2}\,\delta_{\nu 4}\right\}+$$

$$-\,i\lambda Q_\mu\delta_{\nu 4}+i\lambda\delta_{\mu 4}Q_\nu$$

$$B_{\mu\nu}=(k_\mu+Q_\mu)\left[\frac{(2-\lambda)\,k}{M^2}\,\beta_\nu-\frac{(2-\lambda)\,k}{M^2}\,\delta_{\nu 4}\beta_4+\right.$$

$$\left.+\frac{2i\lambda k}{\omega M^2}\,\delta_{\nu 4}\,\frac{Q^2}{D_1D_2}\left\{(\boldsymbol{Q},\boldsymbol{p})(\boldsymbol{Q},\boldsymbol{k})-\frac{Q^2}{2}\,(\boldsymbol{pk}-pk)\right\}\right]$$

$$-\frac{i\lambda k}{\omega}\,\delta_{\nu 4}\beta_\mu+\frac{i\lambda k}{\omega}\,\delta_{\mu 4}\beta_\nu+\frac{i\lambda k}{\omega M^2}\,\delta_{\nu 4}Q_\mu\,\frac{Q^2}{D_2}\,(\boldsymbol{pk}-pk)$$

$$-\frac{i\lambda k}{\omega M^2}\,\frac{Q^2}{D_2}\,(\boldsymbol{k},\,\boldsymbol{Q})\,\delta_{\nu 4}\left(p_\mu-\frac{p}{k}\,k_\mu\right)+\frac{i\lambda k}{\omega M^2}\,\frac{Q^2}{D_2}\,(\boldsymbol{k},\boldsymbol{Q})\,\delta_{4\nu}\delta_{4\mu}\left(p_4-\frac{p}{k}\,k_4\right)$$

$$\beta_\mu=xQ_\mu-\frac{Q^2}{D_2}\left(Q_\mu+p_\mu-\frac{p}{k}\,k_\mu\right)$$

Because $q_\mu l_\mu=0$ one may eventually write $-p_\mu$ for $(k+Q)_\mu$. Further:

$$V(x)=F(-ia_1;\,ia_2;\,1;\,1-x)$$

$$F(a,\,b;\,c;\,z)=1+\frac{ab}{1c}\,z+\frac{a(a+1)\,b(b+1)}{1\cdot 2\cdot c(c+1)}\,z^2+\ldots \tag{5.10}$$

$$a_1a_2W(x)=F'(-ia_1,\,ia_2;\,1;\,1-x)$$

$$x=\frac{Q^2\{q^2-(p+k)^2\}}{D_1D_2}$$

$$D_1=Q^2+2\boldsymbol{Qk}=(\boldsymbol{k}+\boldsymbol{Q})^2-k^2 \qquad D_2=Q^2+2\boldsymbol{Qp}=(\boldsymbol{p}+\boldsymbol{Q})^2-p^2.$$

In first order in a this formula agrees with the formula given in ref. 1 up to the cut-off function in Q describing nuclear effects (see section 7). To simplify the comparison we underlined the parts which contribute in first order in a. If $a_1 = a_2$, $V(x)$ and $W(x)$ are real functions of x. Davies, Bethe and Maximon[11]) investigated the behaviour of these functions in the neighbourhood of $x = 0$ with the result

$$V(0) = \frac{1}{\Gamma(1 - ia)\,\Gamma(1 + ia)} \qquad W(x) = -V(0) \log x \qquad 0 < x \ll 1 \tag{5.11}$$

Formula (5.9) is correct up to terms of order $aQ\omega^{-1}$ or $aQ\varepsilon^{-1}$, i.e., it contains all terms in first order in a irrespective of their order in Q/ω or Q/ε, and all terms in first order in Q/ω or Q/ε irrespective of their order in a.

6. *Corrections for low energy muons.* The present treatment breaks down for low energy muons, because of the fact that the second and third terms of the right-hand side of eq. (4.2), which we treated in Born approximation, cannot be considered as small with respect to the Coulomb term if $\varepsilon \simeq Q$. For the energies of interest with the present accelerators, as one can see from the energy distribution of the outgoing muons given in ref. 1, a considerable fraction of muons come out with relatively low energy. It is, therefore, highly desirable that we extend the region where our formula is valid into the low muon-energy direction. A very lucky circumstance is that the matrix element (5.9) deviates in structure really very little from what one would get by treating in (3.2) and (4.2) all right-hand side terms in Born approximation with respect to plane waves instead of Coulomb wave functions. This makes it possible to change the matrix element so that in the region of small p it is still correct up to second order in a. We want to stress the fact that no change has to be made in the normalization factor $N(\varepsilon)$ given in section 4, because it is related to the asymptotic behaviour of the Coulomb wave function which is determined by the Coulomb term alone.

Let us now try to find the necessary second order correction. Consider (4.2). Because of the fact that the Coulomb term is treated exactly, and because of the fact that the a^2/r^2 term has already been treated we need to solve (4.2) where in the right-hand side only the term x_i/r^3 is retained and ψ is replaced by ψ_a from (4.3):

$$(\nabla^2 + p^2)\,\psi(\boldsymbol{x}) = a\gamma^4\gamma^j \frac{x_j}{r^3} \frac{N^*(\varepsilon)}{2} e^{i\boldsymbol{pr}} \gamma^4\gamma^i u(\boldsymbol{p}) \frac{\partial}{\partial x_i} \cdot {}_1F_1(-ia_2;\, 1;\, -i\boldsymbol{pr} - ipr) \tag{6.1}$$

where we take from ${}_1F_1$ in the right-hand side only the lowest order part in a. Using a representation of the form (3.13) for ${}_1F_1$ we get:

$$\frac{\partial}{\partial x_i}\, {}_1F_1(-ia_2;\, 1;\, -i\boldsymbol{pr} - ipr) = \left(-ip_i - ip\frac{x_i}{r}\right) \frac{1}{2\pi i} \int^{0^+,1^+} \mathrm{d}t\; t^{-ia_2}\,(t-1)^{ia_2}\, e^{(-i\boldsymbol{pr} - ipr)t}$$

After partial integration we get in lowest order in a_2:

$$\frac{\partial}{\partial x_i}\, {}_1F_1(-ia_2;\, 1;\, -i\boldsymbol{pr} - ipr) \simeq$$

$$\simeq ia_2 \frac{(-ip_i - ip(x_i/r))}{(-i\boldsymbol{pr} - ipr)} (1 - e^{-i\boldsymbol{pr}-ipr})$$

We use the identity

$$\gamma^j\gamma^i = \delta_{ij} - \varepsilon_{jiv4}\gamma^v\gamma^4\gamma^5 = \delta_{ji} + f_{ji}$$

so that (6.1) transforms into:

$$(\nabla^2 + p^2)\,\psi(\boldsymbol{x}) = -\frac{ia^2}{2} N^*(\varepsilon)\, e^{i\boldsymbol{pr}} \left\{ \frac{1}{pr^3} + f_{ji}\frac{x_j p_i}{pr^3} \cdot \right.$$

$$\left. \cdot \frac{1}{\boldsymbol{pr} + pr} \right\} u(p)\,(1 - e^{-i\boldsymbol{pr}-ipr}). \qquad (6.2)$$

In using the solution of (6.2) in the matrix element no attempt will be made to incorporate further Coulomb effects of either muon or vector boson and this means that we need to calculate only the Fourier transform of the right-hand side of (6.2). This Fourier transform is given in appendix C, where also some other Fourier transforms pertinent to our problem are listed. The result (eqs. (C.9) and (C.10)) is exactly the same as what one would get by treating a diagram with two vertices of the type of the second term in the right-hand side of (4.2). We could have taken Dalitz'[12]) results for the second order Born approximation for electron scattering were it not for the fact that our initial muon is a virtual one, not being on the mass shell.

7. *The cut-off function for high momentum transfer.* The process under consideration requires high energy incident neutrinos and with the present accelerators most neutrinos will be at best in the threshold region for coherent W production. This means that the structure of the nucleus plays a very important if not decisive role. Of course, we are not able to treat the Coulomb potential of any realistic nuclear charge distribution consistently in the same way as we treated above the point charge Coulomb potential, and the only thing we can do is to make some reasonable guess on how to improve our formula for the higher momentum transfer where the nuclear structure is important.

Again, as in section 6, we will try to correct our matrix element so that in the momentum transfer region just mentioned it is still correct in first order in a and as far as possible also in second order in a. Let us start by writing down the potential as used in ref. 1, together with its Fourier transform (μ = constant related to the nuclear dimensions):

$$V(r) = \frac{a}{r}\left\{(1 - e^{-\mu r}) - \frac{\mu r}{2} e^{-\mu r}\right\}$$
$$V(Q) = \int d_3 r\, e^{iQr}\, V(r) = \frac{4\pi a}{Q^2(1 + (Q^2/\mu^2))^2} \tag{7.1}$$

The effect of the extra terms as compared with the point Coulomb field is, of course, to suppress the higher momentum transfer part. As a consequence, certain asymptotic properties, i.e., normalization constants and low momentum transfer behaviour, are not affected by the change from a point charge potential to the potential (7.1).

In order now to see how the potential appears in our matrix element we try to understand (5.9) in terms of a calculation with Feynman diagrams. Schematically we write:

$$M(q, p, k, t, s) = CV(x)(F_1 + F_2) + iF_3$$

with:

$$F_1 = \frac{a}{Q^2}\left(\frac{2\varepsilon}{D_2} - \frac{2\omega}{D_1}\right) l_\mu e_\mu - \frac{a}{Q^2}\frac{1}{D_1} A_{\mu\nu} l_\mu e_\nu$$
$$- \frac{1}{Q^2}\frac{iQ_\nu}{D_2}(\delta_{4\nu} l_\mu + \delta_{\mu\nu} l_4 - \delta_{\mu 4} l_\nu + \varepsilon_{\kappa 4\nu\mu} l_\kappa)\, e_\mu$$

$$F_2 = \frac{\pi a^2}{2Q}\left(\frac{1}{D_1} + \frac{1}{D_2}\right) l_\mu e_\mu - \frac{\pi a^2}{4pD_2}\left\{\Theta(p - |\boldsymbol{Q} + \boldsymbol{p}|) + \right.$$
$$\left. - \frac{p}{|\boldsymbol{Q} + \boldsymbol{p}|}\,\Theta(|\boldsymbol{Q} + \boldsymbol{p}| - p)\right\} l_\mu e_\mu$$

F_3 stands for all other terms, i.e., all terms containing $W(x)$ and the imaginary part of the second order muon correction. F_1 is precisely what one would get in a lowest order plane wave calculation (see fig. 2) and we see the usual

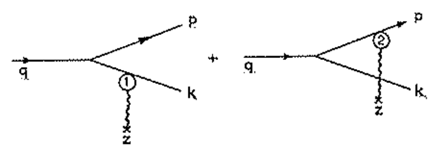

Fig. 2. The diagrams contributing to F_1

1 = Interaction given by right-hand side of eq. (3.2) in first order in a.
2 = Interaction given by right-hand side of eq. (4.2) in first order in a.

effect that the higher order diagrams involving the Coulomb potential have a relatively small effect, through the function $V(x)$ (whose lowest order contribution contains a^2 and is real). F_2 can be seen as arising from a second order calculation (see fig. 3), leaving aside second order effects of the Coulomb potential $2a\omega/r^{-1}$.

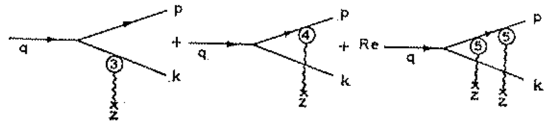

Fig. 3. The diagrams contributing to F_2

3 = a^2/r^2 interaction arising from the seventh term on the right-hand side of eq. (3.2).

4 = a^2/r^2 interaction arising from the third term in round brackets on the right-hand side of eq. (4.2).

Re = Real part of diagram whereby phase is fixed so that the first two diagrams are real.

5 = Interaction arising from the second term in round brackets on the right-hand side of eq. (4.2).

Clearly, in order to make the matrix element correct in first order in a we must make in F_1 the change

$$\frac{1}{Q^2} \to \frac{1}{Q^2(1 + (Q^2/\mu^2))^2}$$

In second order only the first term of F_2 (first and second diagrams of fig. 3) can be calculated. For this part we need to change Q^{-1}, being the Fourier transform of r^{-2}, into the Fourier transform $V'(Q)$ of $V(r)^2$ which can be calculated:

$$V'(Q) = \frac{1}{Q} - \frac{4}{\pi Q} \operatorname{arctg} \frac{Q}{\mu} + \frac{2}{\pi Q} \operatorname{arctg} \frac{Q}{2\mu} + \frac{2\mu^3}{\pi(Q^2 + 4\mu^2)^2} - \frac{2\mu}{\pi(Q^2 + \mu^2)} + \frac{2\mu}{\pi(Q^2 + 4\mu^2)}. \tag{7.2}$$

This is a rather clumsy function and we replace it by:

$$\frac{1}{Q} g(Q) = \frac{1}{Q}\left(\frac{1}{1 + (Q^2/\mu^2)}\right)^3 \left(A + BQ + \frac{C}{(1 + (Q^2/\mu^2))^6}\right) \tag{7.3}$$

with $A = 0.244$, $B = 0.790$ and $C = 0.529$. A comparison of the two functions in the range of interest is given in table I.

We are not able to calculate the changes involved in the other second order terms. One can only say that for higher Q some cut-off should appear, and one can establish the limiting behaviour in Q for $Q \to \infty$. It seems reasonable to take (7.2) as a model for the other terms, and we take therefore as cut-off function for the second and higher order terms the same function which multiplies Q^{-1} in (7.2) or (7.3).

At this point we want to remark that the extra terms in the equations of motion due to the difference between $V(r)$ and ar^{-1} can be treated in

TABLE I

Percentage deviation between (7.2) and (7.3) with $Q' = Q/\mu$	
Q'	$100\left(V'(Q') \Big/ \frac{1}{Q'}\, g(Q') - 1\right)$
0.1	8.09
0.2	−0.00
0.3	−1.37
0.4	−0.06
0.5	1.34
0.6	1.77
0.7	1.35
0.8	0.61
0.9	−0.07
1.0	−0.49
1.1	−0.64
1.2	−0.56
1.3	−0.35
1.4	−0.06
1.5	0.25
1.6	0.53
1.7	0.75
1.8	0.91
1.9	0.99
2.0	0.98
2.1	0.90
2.2	0.75
2.3	0.54
2.4	0.27
2.5	−0.05
2.6	−0.41
2.7	−0.80
2.8	−1.22
2.9	−1.66
3.0	−2.11
3.1	−2.58
3.2	−3.05
3.3	−3.52
3.4	−3.99
3.5	−4.46
3.6	−4.92
3.7	−5.37
3.8	−5.82
3.9	−6.26
4.0	−6.69

Born approximation in the way we treated the r^{-2} term, with an analogous ansatz (the eq. (3.12) above), which is seen to be correct if $\mu r \ll \boldsymbol{kr} + kr$ or $\boldsymbol{pr} + pr$, or $\mu \ll k$ or p and, of course, $kr \gg 1$, $pr \gg 1$.

However, at best one arrives at the same results as above and one is still left with the objection that such an approach is essentially inconsistent because one should treat the changes in the Coulomb term r^{-1} exactly. We think it therefore more realistic to introduce the cut-off as suggested by perturbation theory with respect to the coupling constant a.

8. *Final result.* For completeness we write down the whole matrix-element with the corrections indicated in sections 6 and 7. We remark that

$$|\mathbf{p} + \mathbf{Q}| - p > 0, \text{ as } 2\mathbf{Q}\mathbf{p} + Q^2 > 0$$

$$M(q, p, k, t, s) = \frac{8\sqrt{2}\pi^2 iga}{V^{\frac{3}{2}}\sqrt{\omega}} M(\omega)\, N(\varepsilon)\, \delta(E - \varepsilon - \omega) \left(\frac{D_1}{Q^2}\right)^{ia_1} \left(\frac{D_2}{Q^2}\right)^{-ia_2} \cdot$$

$$\cdot \frac{1}{Q^2(1+(Q^2/\mu^2))^2} \left[V(y) \left\{ l^t_\mu e^s_\mu \left(\frac{2\varepsilon}{D_2} - \frac{2\omega}{D_1} + \frac{\pi a}{2} \frac{Q}{D_1} h(Q) + \frac{\pi a}{2} \frac{Q}{D_2} h(Q) - \right.\right.\right.$$

$$\left. - \frac{\pi a}{4 D_2} \frac{Q^2}{\sqrt{Q^2 + 2\mathbf{Q}\mathbf{p} + p^2}} h(Q) \right) - l^t_\mu e^s_\nu \frac{A_{\mu\nu}}{D_1} +$$

$$- (\delta_{4\nu} l^t_\mu + \delta_{\mu\nu} l^t_4 - \delta_{\mu 4} l^t_\nu + \varepsilon_{\kappa 4\nu\mu} l^t_\kappa)\, e^s_\mu \frac{iQ_\nu}{D_2} +$$

$$+ (\delta_{4\mu} l^t_\nu - \delta_{\nu\mu} l^t_4 + \varepsilon_{\kappa\nu 4\mu} l^t_\kappa) e^s_\mu \varepsilon_{ji\nu 4} \frac{p_j Q_i}{D_2} \frac{a\pi}{4} \cdot$$

$$\cdot \frac{Q^2}{Q^2 p^2 - (\mathbf{Q}\mathbf{p})^2} \left(Q - \frac{Q^2 + \mathbf{p}\mathbf{Q}}{\sqrt{p^2 + 2\mathbf{p}\mathbf{Q} + Q^2}} \right) h(Q) \Big\} +$$

$$+ \frac{i a_1 a_2}{a} W(x)\, h(Q) \left\{ l^t_\mu e^s_\mu \left[2x \left(\frac{k}{D_1} + \frac{p}{D_2} \right) - 2Q^2 \left(\frac{k + p}{D_1 D_2} \right) + \right.\right.$$

$$+ \frac{\pi a k}{\omega} \left(Q \frac{Q^2 + \mathbf{p}\mathbf{Q} - (p/k)(\mathbf{k}\mathbf{Q})}{D_1 D_2} - Q \frac{x}{D_1} \right) +$$

$$\left. + \frac{\pi a p}{\varepsilon} \left(Q \frac{Q^2 + \mathbf{k}\mathbf{Q} - (k/p)(\mathbf{p}\mathbf{Q})}{D_1 D_2} - Q \frac{x}{D_2} \right) \right] + l_\mu e_\nu \frac{B_{\mu\nu}}{D_1} +$$

$$+ (\delta_{4i} l^t_\mu + \delta_{\mu i} l^t_4 - \delta_{\mu 4} l^t_i + \varepsilon_{\kappa 4 i\mu} l^t_\kappa)\, e^s_\mu \left(\frac{-i}{2\varepsilon D_2} \right) \cdot$$

$$\left. \cdot \left(2xpQ_i - \frac{2Q^2 p}{D_1} \left(Q_i + k_i - \frac{k}{p} p_i \right) \right) \right\}$$

$$+ \frac{ia}{4} (\delta_{4\mu} l^t_\nu - \delta_{\nu\mu} l^t_4 + \varepsilon_{\kappa\nu 4\mu} l^t_\kappa)\, e^s_\mu\, \varepsilon_{ji\nu 4} \frac{p_j Q_i}{D_2} \cdot h(Q) \cdot$$

$$\cdot \frac{Q^2}{Q^2 p^2 - (\mathbf{Q}\mathbf{p})^2} \left\{ \frac{Q^2 + \mathbf{p}\mathbf{Q}}{|\mathbf{Q} + \mathbf{p}|} \log \frac{\sqrt{Q^2 + 2\mathbf{Q}\mathbf{p} + p^2} - p}{\sqrt{Q^2 + 2\mathbf{Q}\mathbf{p} + p^2} + p} + \right.$$

$$\left. + \frac{\mathbf{Q}\mathbf{p}}{2p} \log \left(\frac{2\mathbf{p}\mathbf{Q} + Q^2}{Q^2} \right) \right\} +$$

$$- \frac{ia}{4} h(Q)\, l^t_\mu e^s_\mu \frac{Q^2}{D_2} \left\{ \frac{1}{p} \log \left(\frac{Q^2 + 2\mathbf{Qp}}{Q^2 + 2\mathbf{Qp} + p^2} \right) + \right.$$

$$\left. + \frac{1}{\sqrt{Q^2 + 2\mathbf{Qp} + p^2}} \log \frac{\sqrt{Q^2 + 2\mathbf{Qp} + p^2} + p}{\sqrt{Q^2 + 2\mathbf{Qp} + p^2} - p} \right\} \Bigg] . \qquad (8.1)$$

All notations are as in (5.9) with the additions:

$$h(Q) = \frac{1}{(1 + (Q^2/\mu^2))} \left\{ A + BQ + \frac{C}{(1 + (Q^2/\mu^2))^6} \right\}$$

$$A = 0.244, \qquad B = 0.790, \qquad C = 0.529$$

Finally, the relation between μ and nuclear dimension as given in ref. 1:

$$\mu = 3.440 \cdot A^{-\frac{1}{3}}\, 10^{13}\ \mathrm{cm}^{-1}$$
$$= 679 \cdot A^{-\frac{1}{3}}\ \mathrm{MeV}$$

where A is the mass number of the nucleus.

If we now consider (8.1) we see the following corrections to the lowest order calculation of ref. 1:

i) the Sommerfeld factor $M(\omega)\, N(\varepsilon)$,
ii) the four a^2 terms multiplied by $V(y)$,
iii) a factor $V(y)$,
iv) a group of terms multiplied by $W(x)$.
v) second order terms containing logarithms.

We remark that the imaginary parts of $V(y)$ and $W(x)$ are mostly very small (see limiting formulas (5.11)). However, due to the appearance of γ^5 in the l_μ this does not imply that there is no interference of the terms multiplied by $V(y)$ and the other terms.

Acknowledgements. The author is greatly indebted to Dr. S. M. Berman, who suggested the present problem, for much valuable guidance in the early stages of the work. He also wishes to thank Professor L. Van Hove and Dr. J. S. Bell for valuable discussions and helpful criticism throughout the work. Stimulating discussions with Professors E. H. Wichmann and C. N. Yang are gratefully acknowledged.

We show in this appendix that the second order diagrams belonging to (4.1) can be reformed into second order diagrams belonging to (4.2). The

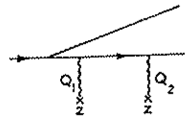

Fig. 4. Second order diagram belonging to (4.1).

lepton part of this second order diagram (fig. 4) is:

$$a^2\bar{u}(p)\,\frac{\gamma^4}{Q_2^2}\,\frac{i\gamma p' - m}{p'^2 + m^2}\,\frac{\gamma^4}{Q_1^2}\,\frac{i\gamma p'' - m}{p''^2 + m^2}\,\gamma^\mu(1+\gamma^5)\,u(q) \tag{A.1}$$

$$p' = p + Q_2, \quad p'' = p' + Q_1 = p + Q_1 + Q_2 \quad Q = Q_1 + Q_2$$

There is one integration over a closed loop.

In this appendix p, p', p'', Q_1, Q_2, Q are four vectors. Of course the fourth component of the Q's is zero. The three-vector part of a vector p will be denoted by $\boldsymbol{p}$. $\bar{u}(p)$ satisfies:

$$\bar{u}(p)(i\gamma^\nu p_\nu + m) = 0. \tag{A.2}$$

We take the $\boldsymbol{\gamma p}$ part of $\gamma p'$ in (A.1), commutate it with γ^4 to the left and apply (A.2). We get:

$$a^2\bar{u}(p)\,\frac{\gamma^4}{Q_1^2}\,\frac{i\boldsymbol{\gamma Q}_2 + 2i\gamma^4 p_4}{Q_2^2 + 2\boldsymbol{Q}_2\boldsymbol{p}}\,\frac{\gamma^4}{Q_1^2}\cdot\frac{i\gamma p'' - m}{Q^2 + 2\boldsymbol{Qp}}\,\gamma^\mu(1+\gamma^5)\,u(q) =$$

$$= a^2\bar{u}(p)\left(\frac{i\gamma^4(\boldsymbol{\gamma Q}_2)}{Q_2^2} - \frac{2\varepsilon}{Q_2^2}\right)\frac{1}{Q_2^2 + 2\boldsymbol{Q}_2\boldsymbol{p}}\,\frac{\gamma^4}{Q_1^2}\,\frac{(i\gamma p'' - m)}{Q^2 + 2\boldsymbol{Qp}}\,\gamma^\mu(1+\gamma^5)\,u(q)$$

We repeat this manipulation with $\boldsymbol{\gamma p}$ embodied in $\gamma p''$:

$$= a^2\bar{u}(p)\left(\frac{i\gamma^4(\boldsymbol{\gamma Q}_2)}{Q_2^2} - \frac{2\varepsilon}{Q_2^2}\right)\frac{1}{Q_2^2 + 2\boldsymbol{Q}_2\boldsymbol{p}}\,\frac{\gamma^3}{Q_1^2}\,\frac{(i\boldsymbol{\gamma Q}_1 + i\boldsymbol{\gamma Q}_2) + 2i\gamma^4 p_4}{Q^2 + 2\boldsymbol{Qp}}\,\cdot$$

$$\cdot\gamma^\mu(1+\gamma^5)\,u(q) +$$

$$- a^2\bar{u}(p)\left(\frac{[i\gamma^4(\boldsymbol{\gamma Q}_2),\, i\boldsymbol{\gamma p}]}{Q_2^2}\right)\frac{1}{Q_2^2 + 2\boldsymbol{Q}_2\boldsymbol{p}}\,\frac{\gamma^4}{Q_1^2}\,\frac{1}{Q^2 + 2\boldsymbol{Qp}}\,\gamma^\mu(1+\gamma^5)\,u(q) +$$

$$+ a^2\bar{u}(p)\left(\frac{[i\gamma^4 p_4,\, i\gamma^4(\boldsymbol{\gamma Q}_2)]}{Q_2^2}\right)\frac{1}{Q_2^2 + 2\boldsymbol{Q}_2\boldsymbol{p}}\,\frac{\gamma^4}{Q_1^2}\,\frac{1}{Q^2 + 2\boldsymbol{Qp}}\,\gamma^\mu(1+\gamma^5)\,u(q) =$$

$$= a^2\bar{u}(p)\left(\frac{i\gamma^4(\boldsymbol{\gamma Q}_2)}{Q_2^2} - \frac{2\varepsilon}{Q_2^2}\right)\frac{1}{Q_2^2 + 2\boldsymbol{Q}_2\boldsymbol{p}}\left(\frac{i\gamma^4\boldsymbol{\gamma Q}_1}{Q_1^2} - \frac{2\varepsilon}{Q_1^2}\right)\frac{\gamma^\mu(1+\gamma^5)}{Q^2 + 2\boldsymbol{Qp}}\,u(q) +$$

$$+ a^2\bar{u}(p)\left(\frac{i\gamma^4(\gamma \boldsymbol{Q}_2)}{Q_2^2} - \frac{2\varepsilon}{Q_2^2}\right)\frac{1}{Q_2^2 + 2\boldsymbol{Q}_2\boldsymbol{p}}\frac{\gamma^4}{Q_1^2}\frac{i\gamma\boldsymbol{Q}_2}{Q^2 + 2\boldsymbol{Q}\boldsymbol{p}}\gamma^\mu(1+\gamma^5)\,u(q) +$$

$$+ a^2\bar{u}(p)\left(\frac{+2\gamma^4\boldsymbol{Q}_2\boldsymbol{p} - 2\gamma\boldsymbol{Q}_2 p_4}{Q_2^2}\right)\frac{1}{Q_2^2 + 2\boldsymbol{Q}_2\boldsymbol{p}}\frac{\gamma^4}{Q_1^2}\frac{\gamma^\mu(1+\gamma^5)}{Q^2 + 2\boldsymbol{Q}\boldsymbol{p}}u(q) =$$

$$= a^2\bar{u}(p)\left(\frac{i\gamma^4(\gamma\boldsymbol{Q}_2)}{Q_2^2} - \frac{2\varepsilon}{Q_2^2}\right)\frac{1}{Q_2^2+2\boldsymbol{Q}_2\boldsymbol{p}}\left(\frac{i\gamma^4\gamma\boldsymbol{Q}_1}{Q_1^2} - \frac{2\varepsilon}{Q_1^2}\right)\frac{\gamma^\mu(1+\gamma^5)}{Q^2+2\boldsymbol{Q}\boldsymbol{p}}u(q)+$$

$$+a^2\bar{u}(p)\frac{1}{Q_1^2Q_2^2}\frac{1}{Q^2 + 2\boldsymbol{Q}\boldsymbol{p}}\gamma^\mu(1+\gamma^5)\,u(q).$$

After performing the integration over the closed loop the factor $Q_1^{-2}Q_2^{-2}$ goes into $0.5\pi Q^{-1}$, i.e. the Fourier transform of r^{-2} and we have the desired result.

APPENDIX B

Equivalence of ansatz and derivative formula.

$$I = \int \mathrm{d}_3k'\,\frac{\mathrm{e}^{-ik'r-\lambda|\boldsymbol{k}'-\boldsymbol{k}|}}{(k'^2 - k^2 - i0)^2}\frac{(\boldsymbol{k}'-\boldsymbol{k})_i}{(\boldsymbol{k}'-\boldsymbol{k})^2}\,{}_1F_1(-ia_1;1;i\boldsymbol{k}'\boldsymbol{r} + ikr) =$$

$$= \frac{\mathrm{e}^{-ikr}}{2\pi i}\int\limits^{0^+,1^+}\mathrm{d}t\,t^{-ia_1-1}\,(t-1)^{ia_1}\int\mathrm{d}_3q\,q_i\,\frac{\mathrm{e}^{iqr(t-1)-\lambda q}}{(q^2 - 2\boldsymbol{q}\boldsymbol{k} - i0)\,q^2}\,\mathrm{e}^{(ikr+ikr)t} \qquad \text{(B.1)}$$

We will set $\lambda = 0$.

Consider first the q-integration

$$I(q) = \int d_3q\,q_i\,\frac{\mathrm{e}^{iqr(t-1)}}{(q^2 - 2\boldsymbol{q}\boldsymbol{k} - \mathrm{i}0)\,q^2} = \frac{1}{i(t-1)}\frac{\partial}{\partial x_i}\int\mathrm{d}_3q\,\frac{\mathrm{e}^{iqr(t-1)}}{(q^2 - 2\boldsymbol{q}\boldsymbol{k} - i0)\,q^2}$$

We neglect difficulties around $q = 0$ as they are unreal in this case. Using Feynman's trick we get:

$$I(q) = \frac{1}{i(t-1)}\frac{\partial}{\partial x_i}\int\limits_0^1\mathrm{d}z\int\mathrm{d}_3q\,\frac{\mathrm{e}^{iqr(t-1)}}{\{(\boldsymbol{q} - \boldsymbol{k}z)^2 + (-ikz + 0)^2\}^2}\,.$$

Using (C.1) we get:

$$I(q) = \frac{\pi^2}{i(t-1)}\frac{\partial}{\partial x_i}\int\limits_0^1\mathrm{d}z\,\frac{\mathrm{e}^{(ikr+ikr)z(t-1)-0r(t-1)}}{-ikz+0}$$

$$= \frac{\pi^2}{ik}\int\limits_0^1\mathrm{d}z\left\{-\left(k_i + k\frac{x_i}{r}\right)\mathrm{e}^{(ikr+ikr)z(t-1)}\right\}$$

$$= \frac{-\pi^2}{ik}\,\frac{k_i + k(x_i/r)}{(i\boldsymbol{k}\boldsymbol{r} + ikr)(t-1)}\,\{\mathrm{e}^{(ikr+ikr)(t-1)} - 1\}.$$

There is no pole at $t = 1$, so $\lambda = 0$ is justified. We insert now $I(q)$ in (B.1), use the relation

$$t^{-ia_1-1}(t-1)^{ia_1-1} = \frac{1}{ia_1}\frac{\mathrm{d}}{\mathrm{d}t}t^{-ia_1}(t-1)^{ia_1}$$

and perform a partial integration (no "boundary" terms arise as the contour never goes through a cut):

$$I = \frac{\mathrm{e}^{-ikr}}{2\pi i}\cdot\frac{\pi^2}{ik\cdot ia_1}\left(k_i + k\frac{x_i}{r}\right)\int^{0^+,1^+}\mathrm{d}t\, t^{-ia_1}(t-1)^{ia_1}\cdot \mathrm{e}^{(i\boldsymbol{kr}+ikr)t}$$

$$= \frac{\mathrm{e}^{-ikr}}{a_1 k}\cdot\frac{\pi^2}{i}\frac{\partial}{\partial x_i}\,{}_1F_1(-ia_1;1;i\boldsymbol{kr}+ikr)$$

APPENDIX C

We list some 3 dim. Fourier transforms of interest to us. Everywhere $\beta = |\boldsymbol{\beta}|$, $\lambda > 0$.

$$\int \mathrm{d}_3r\, \mathrm{e}^{i\alpha r-\lambda r} = \frac{8\pi}{(\alpha^2+\lambda^2)^2},\ \lim_{\lambda\to 0} = \frac{2\pi^2}{\alpha^2}\delta(\alpha) \tag{C.1}$$

$$\int \mathrm{d}_3r\, \frac{1}{r}\mathrm{e}^{i\alpha r-\lambda r} = \frac{4\pi}{\alpha^2+\lambda^2} \tag{C.2}$$

$$\int \mathrm{d}_3r\, \frac{1}{r^2}\mathrm{e}^{i\alpha r-\lambda r} = \frac{4\pi}{\alpha}\,\mathrm{arctg}\,\frac{\alpha}{\lambda} \tag{C.3}$$

$$\int \mathrm{d}_3r\, \frac{x_i x_j}{r^4}\mathrm{e}^{i\alpha r-\lambda r} = -\delta_{ij}\left\{\frac{2\pi\lambda}{\alpha^2} - 2\left(\frac{\pi}{\alpha}+\frac{\pi\lambda^2}{\alpha^3}\right)\mathrm{arctg}\,\frac{\alpha}{\lambda}\right\}$$
$$+\frac{\alpha_i\alpha_j}{\alpha^2}\left\{\frac{6\pi\lambda}{\alpha^2} - 2\left(\frac{\pi}{\alpha}+\frac{3\pi\lambda^2}{\alpha^3}\right)\mathrm{arctg}\,\frac{\alpha}{\lambda}\right\} \tag{C.4}$$

$$\int \mathrm{d}_3r\, \frac{x_i}{r^3}\mathrm{e}^{i\alpha r-\lambda r} = \frac{4\pi i\alpha_i}{\alpha^2} - \frac{4\pi i\lambda\alpha_i}{\alpha^3}\mathrm{arctg}\,\frac{\alpha}{\lambda} \tag{C.5}$$

$$\int \mathrm{d}_3r\, \frac{x_i}{r^2}\mathrm{e}^{i\alpha r-\lambda r} = -\frac{4\pi i\alpha_i\lambda}{\alpha^2(\lambda^2+\alpha^2)} + \frac{4\pi i\alpha_i}{\alpha^3}\mathrm{arctg}\,\frac{\alpha}{\lambda} \tag{C.6}$$

$$\int \mathrm{d}_3r\, \frac{x_i}{r}\mathrm{e}^{i\alpha r-\lambda r} = \frac{8\pi i\alpha_i}{(\lambda^2+\alpha^2)^2} \tag{C.7}$$

$$\int \mathrm{d}_3r\, \frac{x_i x_j}{r^3}\mathrm{e}^{i\alpha r-\lambda r} = \delta_{ij}\left\{\frac{4\pi}{\alpha^2} - \frac{4\pi\lambda}{\alpha^3}\mathrm{arctg}\,\frac{\alpha}{\lambda}\right\}$$
$$-\frac{\alpha_i\alpha_j}{\alpha^2}\left\{\frac{8\pi}{\alpha^2} + \frac{4\pi\lambda^2}{\alpha^2(\alpha^2+\lambda^2)} + \frac{12\pi\lambda}{\alpha^3}\mathrm{arctg}\,\frac{\alpha}{\lambda}\right\}. \tag{C.8}$$

Finally the Fourier transforms needed in section 6.

$$\int \mathrm{d}_3 r \, \frac{1}{r^3} \left(1 - \mathrm{e}^{-i\boldsymbol{p}\boldsymbol{r} - ipr}\right) \mathrm{e}^{i\boldsymbol{\alpha}\boldsymbol{r}} = 2\pi \left\{ \log |1 - \beta^2| + \beta \log \left| \frac{1+\beta}{1-\beta} \right| + \right.$$

$$\left. + i\pi\Theta(\beta - 1) + i\pi\beta\Theta\,(1 - \beta) \right\} \tag{C.9}$$

with $\beta = \dfrac{p}{|\boldsymbol{\alpha} - \boldsymbol{p}|}$.

$$t_{ij} p_j \int \mathrm{d}_3 r \, \frac{x_i}{r^3(\boldsymbol{p}\boldsymbol{r} + pr)} \left(1 - \mathrm{e}^{-i\boldsymbol{p}\boldsymbol{r} - ipr}\right) \mathrm{e}^{i\boldsymbol{\alpha}\boldsymbol{r}} =$$

$$= 2\pi t_{ij} \alpha_i p_j \frac{1}{p^2\alpha^2 - (\boldsymbol{p}\boldsymbol{\alpha})^2} \left\{ \frac{p(\boldsymbol{\alpha}\boldsymbol{p} - \alpha^2)}{|\boldsymbol{p} - \boldsymbol{\alpha}|} \log \frac{p + |\boldsymbol{p} - \boldsymbol{\alpha}|}{p - |\boldsymbol{p} - \boldsymbol{\alpha}| - i0} - \right.$$

$$\left. - \frac{(\boldsymbol{p}\boldsymbol{\alpha})}{2} \log \left(\frac{\alpha^2 - 2\boldsymbol{\alpha}\boldsymbol{p}}{\alpha^2} \right)^2 + i\pi\alpha p \right\}. \tag{C.10}$$

To evaluate the last integral we reject terms of the form $t_{ij} p_i p_j$ and make use of what is essentially Feynman's rule:

$$\frac{1}{\boldsymbol{p}\boldsymbol{r} + pr} \left(1 - \mathrm{e}^{-i\boldsymbol{p}\boldsymbol{r} - ipr}\right) = i \int_0^1 \mathrm{d}z \, \mathrm{e}^{(-i\boldsymbol{p}\boldsymbol{r} - ipr)z} .$$

We also added a factor $e^{-\lambda r}$, $\lambda \to 0$ which is necessary to make the result unambigous in the region $p < |\boldsymbol{p} - \boldsymbol{\alpha}|$.

Received 24-10-62

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$$\int \mathrm{d}_3 r \, \mathrm{e}^{i\boldsymbol{\alpha}\boldsymbol{r} + ia\ln\left(\frac{\boldsymbol{p}\boldsymbol{r} + pr}{\boldsymbol{k}'\boldsymbol{r} + kr}\right)}$$

The logarithm has only angular dependence.
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SAMENVATTING

Ondanks spectaculaire successen is de theorie van zwakke wisselwerkingen, die zijn ontstaan heeft in het fundamentele werk van Fermi [1]) voor β-verval, nog steeds in een zeer onbevredigende toestand. Vele reacties die een sterke gelijkenis vertonen met β-verval zijn ontdekt, echter een universeel schema van selectieregels voor deze reacties is niet bekend. Ook de velden-theoretische beschrijving laat vanwege het niet-renormaliseerbare karakter van de Fermi-interactie nog veel te wensen over.

Toch is er de laatste jaren enige vooruitgang geboekt. Na de ontdekking van niet-pariteit behoud [2]) is tenminste de vorm van de Fermi-wisselwerking (de zogenaamde V–A theorie [3])) vrij definitief vastgesteld. Deze laatste theorie nu suggereert dat de zwakke wisselwerkingen tot stand komen via een intermediair deeltje van spin één op een manier die sterk overeenkomt met de wijze waarop via het foton de e.m.wisselwerking tussen geladen deeltjes tot stand komt. Deze intermediaire boson theorie is consistent met de experimenteel bekende feiten als het boson geladen is en een massa groter dan de massa van het K-meson heeft. Het probleem van selectieregels komt door de introductie van zo'n boson niet dichter bij een oplossing: minstens 6 boson typen zijn nodig om de bestaande interacties te beschrijven. Het experimenteel niet geobserveerde stralings-verval van het muon, mogelijk als slechts één soort neutrino's en een intermediair vector-boson bestaat is opgelost door de ontdekking van het bestaan van 2 soorten neutrino's [4]).

Van velden-theoretisch standpunt bezien is de introductie van een vector-boson een verbetering. Dat een theorie niet renormaliseerbaar is duidt op een diepere structuur van de wisselwerking, en een vector-boson theorie is precies een model voor zo'n structuur. Echter zelfs met vector-boson is de theorie niet-renormaliseerbaar, maar gezien de onvolledige wijze waarop theoriën met instabiele deeltjes en interacties met afgeleiden (waarop de vector-boson theorie kan worden gereduceerd) zijn bestudeerd, heeft deze uitspraak een voorlopig karakter.

Hoofdstuk II geeft een bijdrage aan het bovengenoemde probleem. Met behulp van diagram techniek wordt de storingstheorie voor een simpel model met instabiel deeltje onderzocht. Het blijkt dat eigenschappen betreffende renormaliseerbaarheid en causaliteit precies zo zijn als voor een theorie met louter stabiele deeltjes. Voor wat betreft unitariteit echter is de situatie gecompliceerder: om een bevredigende theorie te krijgen is een partiële sommatie in de storingsontwikkeling noodzakelijk.

De methode van Hoofdstuk II is nu als volgt: eerst wordt een formule bewezen die een verband legt tussen diagrammen van verschillende orde in de storingsreeks. Vervolgens wordt de moeilijkheid betreffende unitariteit gelocaliseerd en verholpen door een partiële sommatie. De rest van het hoofdstuk wordt dan besteed aan een onderzoek van de nieuwe storingsreeks en aangetoond wordt dat de resulterende theorie unitair, causaal en renormaliseerbaar is. Tenslotte wordt bewezen dat deze nieuwe storingsreeks een oplossing van de oorspronkelijke veldvergelijkingen is.

In hoofdstuk III wordt een probleem van meer practische aard bestudeerd. In de naaste toekomst zal de kwestie van existentie van een vectorboson experimenteel onderzocht worden. Dit zal worden gedaan met behulp van hoog energetische neutrino's die dan aanleiding geven tot vectorbosonen middels de reactie $\nu_\mu \rightarrow \mu^- + W^+$ of $\nu_e \rightarrow e^- + W^+$ (e = electron; μ = muon; ν_μ en ν_e zijn de overeenkomstige neutrino's; W = intermediair boson). Vanwege energie-impulsbehoud zijn deze processen in vacuum verboden, en practisch de meest aantrekkelijke methode is productie in het Coulomb-veld van een atoomkern. In laagste orde in zowel electronmagnetische als zwakke wisselwerkings koppelingsconstante is het eerstgenoemde proces berekend door diverse auteurs [5]). Voor kernen met een hoge lading Z echter kan de storings-parameter $Ze^2/\hbar c$ zodanig groot zijn (voor lood b.v. $Ze^2/\hbar c \sim 0.6$) dat laagste orde storingstheorie niet meer voldoende is. In hoofdstuk III bestuderen we de hogere orde correcties in $Ze^2/\hbar c$ in een hoge energiebenadering met behulp van technieken door Bethe en Maximon [6]) geïntroduceerd in verband met creatie van een electron-positron paar door een gamma quant in het Coulomb-veld van een kern. In twee opzichten echter gaat de behandeling verder: ten eerste, een term in de bewegingsvergelijking die niet met de Bethe Maximon techniek behandeld kan worden is nodig in ons geval en wordt dan ook in rekening gebracht, en ten tweede introduceren we extra termen in het matrix-element zodanig dat het hoge energie-gedrag ongewijzigd blijft terwijl het matrixelement correct blijft tot in tweede orde in $Ze^2/\hbar c$ voor alle lepton energieën. De moeilijkheden samenhangende met de afwijking van de ladingsverdeling van de kern van een puntlading worden behandeld op benaderende wijze, teneinde het matrixelement te verbeteren voor neutrino-energieën dicht bij de drempelwaarde voor het proces.

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