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PAPER

Variation of Planck's quantum of action in Entangled Relativity

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Abstract

Entangled Relativity is a recent non-linear reformulation of General Relativity that does not include Planck's constant $\hbar$ or Newton's gravitational constant G in its fundamental structure. One of its key predictions is that $\hbar$ emerges as a dynamical field, potentially varying across space and time. In this study, we estimate the magnitude of such variations in three different astrophysical environments: the weak gravitational fields of the Sun and Earth, the intermediate regime of white dwarfs, and the strong fields found in neutron stars. In the Solar System, the relative change in $\hbar$ is minimal, reaching at most $\sim 2.5 \times 10^{-12}$. In white dwarfs, depending on central density, variations range from $\sim 7 \times 10^{-10}$ to $\sim 10^{-6}$. For neutron stars, the variation can be as high as 1.5% at the surface relative to a remote observer, and up to 5.7% at the center. These results suggest that, if Entangled Relativity accurately describes gravity, spatial variations of Planck's constant could become an observable signature, particularly in the context of dense stellar objects.

1. Introduction

Entangled Relativity is a general theory of relativity that is a non-linear reformulation of General Relativity, which requires the existence of matter in order (even) to be defined, therefore realizing Einstein's original idea that a satisfying theory of relativity cannot allow for the existence of spacetimes in a vacuum [1–7]. Indeed, vacuum solutions imply that inertia—which is defined from the metric tensor in a relativistic theory—could be defined in the total absence of matter, which would *de facto* violate the *principle of relativity of inertia* [1–7] that Einstein named *Mach's principle* in [2]. Despite its very unusual action—see equation (1) below—Entangled Relativity has been shown to possess General Relativity as a limit in fairly generic (classical) situations [8–14], which indicates that, at least up to further scrutiny, the theory might be viable from an observational point of view.

Entangled Relativity has the conceptual advantage over General Relativity that it requires fewer dimensionful constants in its definition, both at the classical and at the quantum level [15, 16]. But a consequence of this is that both G and $\hbar$ can vary in time and space. The purpose of this paper is to numerically evaluate these variations in the two opposite situations that are the weak field limit of the Solar System on the one hand, and the strong field limit of white dwarfs and neutron stars on the other hand. As we will see, while the variations of G and $\hbar$ is completely negligible in the Solar System, it may reach a few percent of variation between the surface of the densest possible neutron stars and a remote observer.

2. Theory

Entangled Relativity can be defined by its path integral as follows [8, 15, 16]

$$Z = \int [\mathcal{D}g] \prod_j [\mathcal{D}f_j] \exp\left[-\frac{\mathrm{i}}{2\epsilon^2}\int \mathrm{d}^4_g x \frac{\mathcal{L}_m^2(f,g)}{R(g)}\right], \tag{1}$$

where $\int[\mathcal{D}]$ relates to the sum over all possible (non-redundant) field configurations, R is the usual Ricci scalar that is constructed upon the metric tensor g, $\mathrm{d}_g^4x := \sqrt{-|g|}\mathrm{d}^4x$ is the spacetime volume element, with $|g|$ the metric g determinant, and $\mathcal{L}_m$ is the Lagrangian density of matter fields f—which could be the current *standard model of particle physics* Lagrangian density, but most likely a completion of it. Just as General Relativity, Entangled Relativity does not specify the material content of the Universe. $\mathcal{L}_m$ also depends on the metric tensor, *a priori* through the usual *comma-goes-to-semicolon rule* [17]—although, strictly speaking, this is required only in the limit where Entangled Relativity becomes equivalent to General Relativity—in order to recover the Equivalence Principle in that limit. Entangled Relativity was presented for the first time in [8], and the theory was named for the first time in [9]. The name follows from the fact that matter and curvature cannot be treated separately at the level of the action in this framework—such that they are *entangled*, in the etymological sense. Hence, the name has nothing to do with *quantum entanglement*, *a priori*.

One can check that the integral in the quantum phase in equation (1) has the dimension of an energy squared. Thus, the only parameter of the theory is a quantum of energy squared ϵ^2. This parameter and the causal structure constant c—hidden in the spacetime volume element—are the only two universal constants of the theory[4]. The value of ϵ does not affect the classical phenomenology of the theory. But more importantly, neither G nor $\hbar$ appear in the formulation of the theory. As a consequence, from the onset, one knows that neither G nor $\hbar$ are fundamental constants in Entangled Relativity. As a result, one should expect them to vary—as we will explicitly see in sections 3 and 4.3 respectively.

Indeed, this should be compared with the path integral formulation of the Core theory[5], which is expressed as

$$Z_{\mathrm{C}} = \int[\mathcal{D}g]\prod_j[\mathcal{D}f_j]\exp\left[\frac{\mathrm{i}}{\hbar c}\int \mathrm{d}_g^4x\left(\frac{R(g)}{2\kappa_{\mathrm{GR}}}+\mathcal{L}_m^{\mathrm{SM}}(f,g)\right)\right], \tag{2}$$

where f_j are the matter fields of the standard model of particles—such as fermions and gauge bosons, and the Higgs. There are three universal constants in this formulation: the quantum constant $\hbar$ (Planck's), the causal structure constant c [6] and the constant of gravity $G = c^4\kappa_{\mathrm{GR}}/(8\pi)$ (Newton's). From these constants, one can construct an energy scale, a mass scale, a time scale and a length scale, known as the Planck energy (E_{P}), mass (m_{P}), time (t_{P}) and length (l_{P}) respectively:

$$E_{\mathrm{P}} = \sqrt{\frac{\hbar c^5}{G}}, m_{\mathrm{P}} = \frac{E_{\mathrm{P}}}{c^2}, t_{\mathrm{P}} = \sqrt{\frac{\hbar G}{c^5}}, l_{\mathrm{P}} = ct_{\mathrm{P}}. \tag{3}$$

One can directly see the fundamental role of the Planck length for pure quantum General Relativity from its path integral formulation

$$Z_{\mathrm{GR}} = \int[\mathcal{D}g]\exp\left[\frac{\mathrm{i}}{16\pi l_{\mathrm{P}}^2}\int \mathrm{d}_g^4x R(g)\right], \tag{4}$$

for which the Planck length squared l_{P}^2 plays the role of a quantum of area, the same way Planck's constant $\hbar$ plays the role of a quantum of action in standard quantum field theory and quantum mechanics. From various theoretical arguments [19–24], the Planck length and time are thought to be the elementary units of spacetime. These elementary units therefore play a crucial role in attempts to construct a quantum theory of General Relativity [21, 25–27].

Standard quantum field theory on the other hand arises from the assumption that gravity can entirely be neglected. One typical example is the path integral of the standard model of particle physics, which reads

$$Z_{\mathrm{SM}} = \int\prod_i[\mathcal{D}f_i]\exp\left[\frac{\mathrm{i}}{\hbar c}\int \mathrm{d}^4x\mathcal{L}_m^{\mathrm{SM}}(f)\right]. \tag{5}$$

It is noteworthy that the linear nature of the coupling between gravity and matter in the quantum phase of the Core theory implies that it is possible to consider one sector while entirely forgetting about the other, as exemplified in equations (4) and (5). While this separation is obviously artificial within the framework of the Core theory of physics—since we know that both gravity and matter exist—it is

[4] Besides potential additional dimensionful constants within $\mathcal{L}_m$—see discussion in [16].

[5] That is, the current standard model of physics, as named by [18].

[6] Which turns out to be equal to the speed of light in vacuum on scales for which spacetime can be approximated to be flat.

nevertheless often used in practice, either in order to simplify the physics of the quantum matter sector, such as in the standard model of particle physics, or to simplify the physics of the quantum gravitational sector, such as in causal dynamical triangulation for instance [26, 27]. Interestingly, one can no longer make such a separation in Entangled Relativity, and this echoes Einstein's original wish that a satisfactory theory of relativity should not allow for the existence of spacetimes without matter [1–7].

Even more interestingly, it is not possible to define a Planck length and a Planck time using the universal constants of Entangled Relativity in equation (1) [16]. Even though the non-perturbative behavior of the theory defined in equation (1) is currently unknown, the absence of Planck's units of space and time seems to indicate that the quantum gravity behavior of the theory should be qualitatively different from that of quantum General Relativity.

3. Classical field equations

3.1. Original form

The same way that the field equations of classical physics in the Core theory follow from the extremization of the quantum phase in (2), the field equations of classical physics in Entangled Relativity follow from the extremization of the quantum phase in equation (1), that is

$$\Theta = -\frac{1}{2\epsilon^2}\int d_g^4x \frac{\mathcal{L}_m^2(f,g)}{R(g)}. \tag{6}$$

In particular, the metric field equation reads [8]

$$\frac{\mathcal{L}_m^2}{R^2}\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right) = -\frac{\mathcal{L}_m}{R}T_{\mu\nu} + \left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\right)\frac{\mathcal{L}_m^2}{R^2}, \tag{7}$$

with

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_m\right)}{\delta g^{\mu\nu}}. \tag{8}$$

Let us note that the stress-energy tensor is not conserved in general, as one has

$$\nabla_\sigma\left(\frac{\mathcal{L}_m}{R}T^{\alpha\sigma}\right) = \mathcal{L}_m\nabla^\alpha\left(\frac{\mathcal{L}_m}{R}\right). \tag{9}$$

The matter field equation, for any tensorial matter field χ, gets modified due to the non-linear coupling between matter and curvature as follows

$$\frac{\partial\mathcal{L}_m}{\partial\chi} - \frac{1}{\sqrt{-|g|}}\partial_\sigma\left(\frac{\partial\sqrt{-|g|}\mathcal{L}_m}{\partial\left(\partial_\sigma\chi\right)}\right) = \frac{\partial\mathcal{L}_m}{\partial\left(\partial_\sigma\chi\right)}\frac{R}{\mathcal{L}_m}\partial_\sigma\left(\frac{\mathcal{L}_m}{R}\right). \tag{10}$$

As usual in $f(R)$ theories [28], the differential equation on the extra-degree of freedom is given by the trace of the metric field equation (7), it reads

$$3\Box\frac{\mathcal{L}_m^2}{R^2} = -\frac{\mathcal{L}_m}{R}\left(T-\mathcal{L}_m\right). \tag{11}$$

This means that while R/T is a constant in General Relativity, $R/\mathcal{L}_m$ is a field in Entangled Relativity.

It is worth noting that if $\mathcal{L}_m = T$, then the equation for $R/\mathcal{L}_m$ remains unsourced, allowing $R/\mathcal{L}_m = R/T$ to be constant, akin to General Relativity. The precise cancellation on the right-hand side of equation (11) when $\mathcal{L}_m = T$ has been termed an *intrinsic decoupling* in [29]—in the context of scalar–tensor theories.

3.2. A classically equivalent scalar–tensor formulation—provided that $\mathcal{L}_m \neq \varnothing$

Quite intriguingly, these equations can also be derived from the following (pretty standard) Einstein–dilaton phase [8, 16]:

$$\theta_{\text{Ed}} = \frac{1}{\epsilon^2}\int d_g^4x\frac{1}{\kappa}\left(\frac{R(g)}{2\kappa}+\mathcal{L}_m(f,g)\right), \tag{12}$$

provided that $\mathcal{L}_m \neq \varnothing$[7], and where κ is a dimensionful scalar-field. Obviously, the value of ϵ does not impact the classical limit of the theory. The classical equivalence between the original $f(R, \mathcal{L}_m)$ theory in equation (6) and the *Einstein–dilaton* theory in equation (12), stems from a very well known fact: non-linear algebraic functions of the Ricci scalar in the action are equivalent to having an additional scalar degree-of-freedom with gravitational strength [28, 30, 31]. As a consequence, it indicates that the theory defined in equation (6) should be immune to the Ostrogradskian instability and to the non-well-posedness of the Cauchy problem despite not being of second order—just as $f(R)$ theories [30–32].

Indeed, the Euler–Lagrange equation for the scalar-field implies

$$\kappa = -\frac{R}{\mathcal{L}_m}, \tag{13}$$

instead of $\kappa = -R/T$ in General Relativity, whereas the metric field equation is

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu} + \kappa^2 \left(\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box\right) \kappa^{-2}, \tag{14}$$

with

$$\nabla_\sigma \left(\kappa^{-1} T^{\alpha\sigma}\right) = \mathcal{L}_m \nabla^\alpha \kappa^{-1}. \tag{15}$$

Using equation (13), the trace of the metric field equation can be rewritten as follows

$$3\kappa^2 \Box \kappa^{-2} = \kappa \left(T - \mathcal{L}_m\right). \tag{16}$$

The matter field equation, for any tensorial matter field χ, reads

$$\frac{\partial \mathcal{L}_m}{\partial \chi} - \frac{1}{\sqrt{-|g|}} \partial_\sigma \left(\frac{\partial \sqrt{-|g|} \mathcal{L}_m}{\partial \left(\partial_\sigma \chi\right)}\right) = \frac{\partial \mathcal{L}_m}{\partial \left(\partial_\sigma \chi\right)} \kappa \partial_\sigma \kappa^{-1}. \tag{17}$$

Let us also note that, defining $\vartheta := \bar{\kappa}/\kappa$, where $\bar{\kappa}$ is a normalization constant whose value depends on the considered initial boundary or background conditions of the field ϑ, one has

$$\Theta_{\rm Ed} = \frac{1}{\bar{\kappa}\epsilon^2} \int \mathrm{d}_g^4 x \left(\frac{\vartheta^2 R(g)}{2\bar{\kappa}} + \vartheta \mathcal{L}_m(f, g)\right), \tag{18}$$

which looks much more common. One could also use the field redefinition $\phi = \vartheta^2$, such that Entangled Relativity corresponds to a special case of the class of theories considered in [29]

$$\Theta_{\rm Ed} = \frac{1}{\bar{\kappa}\epsilon^2} \int \mathrm{d}_g^4 x \left(\frac{\phi R(g)}{2\bar{\kappa}} + \sqrt{\phi} \mathcal{L}_m(f, g)\right), \tag{19}$$

which all possess an *intrinsic decoupling* that will be further discussed in section 4. However, such a choice limits κ to be positive, whereas the original form of the theory does not have this restriction *a priori*. Note that this action differs slightly from that of standard Einstein–dilaton theories (see equation (87)), and that the resulting phenomenologies of the two classes of theories also differ, as explained in detail in appendix.

3.3. Propagation of light

In the following, the electromagnetic field is not considered as a significant source of curvature. Applying equation (17) on the specific case of electromagnetism $\mathcal{L}_m = F^2/(2\mu_0)$—where $F^2 = F_{\alpha\beta} F^{\alpha\beta}$, with $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ the Faraday tensor and A^μ the electromagnetic field—one gets

$$\nabla_\sigma \left(\kappa^{-1} \, F^{\mu\sigma}\right) = 0. \tag{20}$$

Using the Lorenz gauge ($\nabla_\sigma A^\sigma = 0$), it reduces to:

$$-\Box A^\mu + g^{\mu\epsilon} R_{\gamma\epsilon} A^\gamma - \left(\nabla^\mu A^\sigma - \nabla^\sigma A^\mu\right) \partial_\sigma \ln \kappa = 0. \tag{21}$$

[7] If $\mathcal{L}_m = \varnothing$ in equation (12), then it does not correspond to Entangled Relativity, as defined in equation (6).

Following the analysis made in [17], we expand the 4-vector potential in order to derive the geometric optic approximation as follows:

$$A^\mu = \Re\left\{\left(a^\mu + \varepsilon b^\mu + O\left(\varepsilon^2\right)\right)\exp^{\mathrm{i}\theta/\varepsilon}\right\}, \tag{22}$$

where ε is the ratio between a typical wavelength λ of the electromagnetic radiation and the minimum between the inverse of the square-root of the typical component of the Riemann tensor $\mathfrak{R}$ and a typical length $\mathfrak{L}$ over which the amplitude, polarization and wavelength of the waves vary [17]. The two first leading orders of equation (21) respectively give:

$$k_\sigma k^\sigma = 0, \tag{23}$$

where $k_\sigma \equiv \partial_\sigma \theta$ is the wave vector, and

$$a^\mu \nabla_\sigma k^\sigma + 2k^\sigma \nabla_\sigma a^\mu = (k^\sigma a^\mu - k^\mu a^\sigma)\,\partial_\sigma \ln\kappa. \tag{24}$$

Remembering that the Lorenz gauge condition gives $k_\sigma a^\sigma = 0$ at the leading order, one gets:

$$k^\sigma \nabla_\sigma k^\mu = 0. \tag{25}$$

This equation is the usual null-geodesic equation, showing that the specific coupling of Entangled Relativity do not affect light ray trajectories at the geometric optic approximation. However, defining $a^\mu = af^\mu$, the propagation equation for the scalar amplitude (a) as well as the propagation equation for the polarization vector (f^μ) are modified according to:

$$k^\sigma \nabla_\sigma a = -\frac{a}{2}\nabla_\sigma k^\sigma + \frac{1}{2}ak^\sigma \partial_\sigma \ln\kappa, \tag{26}$$

$$k^\sigma \nabla_\sigma f^\mu = -\frac{1}{2}k^\mu f^\sigma \partial_\sigma \ln\kappa. \tag{27}$$

From there follows that the conservation law of the intensity (or, 'photon number') is modified [33, 34]:

$$\nabla_\sigma\left(k^\sigma a^2\right) = -a^2 k^\sigma \partial_\sigma \ln\kappa. \tag{28}$$

This notably implies that there is a transfer of energy between the electromagnetic field and the gravitational scalar degree-of-freedom when $\partial\kappa \neq 0$. This also notably means that the distance luminosity is modified with respect to General Relativity [33, 34].

4. The slowly varying κ limit: variation of G and $\hbar$

As we will further see in section 5, the scalar-field's perturbation κ varies generically much less than the metric's perturbations. This is due to what has been named an *intrinsic decoupling* in [29]—that is, the cancellation in the right hand side of (16) when $\mathcal{L}_m = T$, in addition to the fact that one has $\mathcal{L}_m \approx T$ quite generically in the observable Universe. For example, $\mathcal{L}_m = T$ holds for a Universe solely made of dust and electromagnetic radiation—which is a reasonable approximation of our own Universe. This provides a qualitative explanation for why one can anticipate General Relativity to be a generic limit of Entangled Relativity.

4.1. General relativistic limit

One can check from equations (13)–(16) that for any matter field that is such that $\mathcal{L}_m = T$ on-shell—such as for a dust field for instance, for which one has $\mathcal{L}_m = -c^2\sum_i m_i\delta^{(3)}(\vec{x}-\vec{x}_i(t))/(\sqrt{-g}u^0) = T$, where m_i represents the conserved mass of dust particles along their geodesics [35]—all the solutions of General Relativity are also solutions of Entangled Relativity, where $\kappa = \text{constant} = 8\pi G/c^4 \equiv \kappa_{\mathrm{GR}}$. In particular, equation (13) simply corresponds to the trace of Einstein's equation, $R = -\kappa_{\mathrm{GR}}T$, in that situation.

This notably means that all the solutions of General Relativity with both a dust field and electromagnetic radiation are also solutions of Entangled Relativity, since both are such that $\mathcal{L}_m = T$.

Hence, because as far as we know our present Universe is well approximated as being composed of dust particles and electromagnetic radiation, one should not expect strong deviations of Entangled Relativity from General Relativity in our present Universe.

Recovering General Relativity:
Let us note that it is obvious that Entangled Relativity is no different from General Relativity for a Universe that would only be made of a dust field, as then one would have

$$\Theta_{\mathrm{Ed}} = \frac{1}{\epsilon^2} \int \mathrm{d}_g^4 x \frac{R(g)}{2\kappa^2} - \sum_i \int \frac{m_i c^2}{\kappa \epsilon^2} \, c\mathrm{d}\tau, \tag{29}$$

such that, performing the conformal transformation $\tilde{g}_{\alpha\beta} = (\bar{\kappa}/\kappa)^2 g_{\alpha\beta}$, where $\bar{\kappa}$ simply is a normalization constant, it would reduce to

$$\Theta_{\mathrm{Ed}} = \frac{1}{\bar{\kappa}\epsilon^2} \int \mathrm{d}^4 x \sqrt{-\tilde{g}} \frac{1}{2\bar{\kappa}} \left(\tilde{R} - 2\tilde{g}^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi\right) - \sum_i \frac{m_i c^2}{\bar{\kappa}\epsilon^2} \int c\mathrm{d}\tilde{\tau}, \tag{30}$$

with φ being an uncoupled scalar-field defined as $\varphi := \sqrt{3} \, \ln \kappa/\bar{\kappa}$+constant [9–11, 13, 14], and $\mathrm{d}\tilde{\tau} = \bar{\kappa}/\kappa \, \mathrm{d}\tau$. Therefore, in that situation, Entangled Relativity would be nothing more than General Relativity with an additional uncoupled—and therefore irrelevant—scalar-field [36]. In other words, one recovers General Relativity in that situation when the theory is written in the Einstein conformal frame[8]. In particular, one can see that particles follow geodesics of the Einstein frame metric.

4.2. Value of the quantum of energy

More generally, one recovers standard physics at leading order when the variation of κ can be neglected—such as in the Solar System for instance, see section 5.2. Indeed, assuming that $\kappa = \kappa_0$ is constant at the background level equation (12) reduces to

$$\Theta_{\mathrm{Ed}} = \frac{1}{\kappa_0 \epsilon^2} \int \mathrm{d}_g^4 x \left(\frac{R(g)}{2\kappa_0} + \mathcal{L}_m(f, g)\right). \tag{31}$$

However, as we saw, the quantum phase of the Core theory of Physics [18] reads

$$\Theta_{\mathrm{C}} = \frac{1}{\hbar c} \int \mathrm{d}_g^4 x \left(\frac{R(g)}{2\kappa_{\mathrm{GR}}} + \mathcal{L}_m^{\mathrm{SM}}(f, g)\right). \tag{32}$$

Therefore, the same way that one identifies κ_{GR} to be $8\pi G/c^4$ in General Relativity from the (perturbative) Newtonian limit of the theory—which has to reproduce the trajectories of massive bodies predicted by the theory of Newton at leading order—one can deduce from the $\kappa \approx$ constant limit of Entangled Relativity that

$$\kappa_0 = \kappa_{\mathrm{GR}} = \frac{8\pi G}{c^4}, \tag{33}$$

where G is the constant of Newton[9], as well as

$$\kappa_0 \epsilon^2 = \hbar c, \tag{34}$$

such that the only free parameter of the theory ϵ turns out to be the reduced Planck energy

$$\epsilon = \sqrt{\frac{\hbar c}{\kappa_{\mathrm{GR}}}} = \frac{E_{\mathrm{P}}}{\sqrt{8\pi}}. \tag{35}$$

[8] The original conformal frame in which the theory takes the form of equation (6) has be named the *entangled frame* in [37].
[9] We will make that more precise in section 5.2.1.

4.3. $\hbar$ is not a constant

Equation (34) indicates that $\hbar$ was proportional to κ in the previous example. But while this value can be set to be a constant at some scale—say, the scale of the solar system dynamics for instance, see section 5.2, or the scale of a table-top experiment—it cannot be set to be a constant in general—say, at the cosmological scale, see for instance [33]. Therefore, equation (34) actually indicates that $\hbar$ varies proportionally to κ, which is a scalar field—that is

$$\hbar(x) = \frac{\epsilon^2 \kappa(x)}{c}. \tag{36}$$

So even if at the scale of particle physics experiments, the variation of $\hbar$ can be neglected—because it varies even less than the spacetime metric that is entirely neglected in particle physics experiments—it may vary over larger scale in either space or time as we will further see in section 5. Typically, at the scale of a quantum phenomenon—such as an atomic transition—the spacetime metric can safely be assumed to be flat, and consequently, κ can also be treated as constant. Therefore, quantum field theory on flat spacetime still applies locally. However, the local value of κ may vary from one region to another over scales much larger than those of quantum phenomena. As a result, $\hbar := \kappa\epsilon^2/c$ will effectively take on different values at different locations across such large scales[10].

In particular, one has $\hbar \propto \kappa$, such that

$$\frac{\delta\vartheta}{\vartheta} = \frac{1}{2}\frac{\delta\phi}{\phi} = -\frac{\delta\kappa}{\kappa} = -\frac{\delta\hbar}{\hbar}, \tag{37}$$

for the various parameterization considered in section 3.

The variation of $\hbar$ is a particular signature of Entangled Relativity, which is rooted in the non-linearity of its action, see equation (1) and from the fact that Planck's quantum of action is not a fundamental parameter of the theory. This situation is analogous to string theory where $\hbar$ is also not a fundamental parameter (the fundamental parameters being c and the string length). On the other hand, in standard tensor–scalar theories (generalized Brans–Dicke or Einstein–dilaton), $\hbar$ is a fundamental parameter and is therefore constant. We explore and compare these 4 cases (string theory, generalized Brans–Dicke, Einstein–dilaton and Entangled Relativity) in details in the appendix, with a special care on clarifying what are the varying constants in each class of theory.

5. Estimations of the magnitude of the variation of $\hbar$ and observational consequences

In this section, we will study some astrophysical consequences of this theory. We will first derive an analytical modeling of some observational consequences induced by Entangled Relativity in the Solar System and show that Entangled Relativity is currently indistinguishable from General Relativity with current observations. We will then turn into some strong field considerations.

5.1. On-shell matter Lagrangian

In certain simple situations, it is feasible to derive the on-shell matter Lagrangian from first principles. For example, in the case of a charged black hole, the on-shell matter Lagrangian is given by $\mathcal{L}_m = E^2/2$ in natural units, where E denotes the magnitude of the electric field of the black hole [11, 13].

However, this direct derivation is not typically applicable in general, and in particular not for complex astronomical bodies. As a result, one must rely on a set of assumptions that might or might not hold true in specific contexts. Therefore, while Entangled Relativity does not have free parameters in its definition, its phenomenological outcomes can still depend on the underlying assumptions made to determine the on-shell value of the matter Lagrangian density.

In what follows, we shall use the assumption that we are dealing with a perfect fluid, and that the rest mass energy density ρ_r is conserved

$$\nabla_\sigma(\rho_r u^\sigma) = 0, \tag{38}$$

such that the on-shell Lagrangian is $\mathcal{L}_m = -\rho$, where ρ is the total energy density given by [9, 38]

$$\rho = \rho_r\left(c^2 - \frac{P}{\rho_r} + \int \frac{\mathrm{d}P}{\rho_r}\right). \tag{39}$$

[10] Moreover, let us note that because κ generally varies much less than the spacetime metric in Entangled Relativity—see, e.g. section 5—its variation can still be neglected at leading order in quantum field theory on curved spacetime. Thus, the effect of κ-variation should be treated as a (small) correction to standard quantum field theory in curved spacetime.

In that situation, the scalar-field has been named *a pressuron* in [33], because it is only sourced by pressure—as we will notably see in equation (41).

Let us note that it has been argued that the on-shell matter Lagrangian could be exactly equal to the trace of the stress-energy tensor in some situations as well [39–41], which is a different assumption than the one we will use later on. However, under this assumption ($\mathcal{L}_m = T$) and assuming the usual definition of the total energy density equation (39) still holds, one can show that the rest mass energy density is no-longer conserved since [9]

$$\nabla_\sigma \left(\rho_r u^\sigma\right) = 3P \frac{\rho_r}{\rho + P} u^\sigma \partial_\sigma \ln \kappa \qquad \text{if } \mathcal{L}_m = T, \tag{40}$$

Nevertheless, if $\mathcal{L}_m = T$ for the celestial bodies considered in what follows, one would not have any deviation from General Relativity, because of the exact cancellation in the right hand side of equation (16). It notably also induces that the rest-mass energy density would also still be conserved, simply because $\partial_\sigma \kappa = 0$ in that situation in equation (40). Let us stress that the fact that $\nabla_\sigma(\rho_r u^\sigma) = 0$ for both $\mathcal{L}_m = -\rho$ and $\mathcal{L}_m = T$ is unique to theories with *intrinsic decoupling* [29] in the landscape of theories with non-minimal scalar-matter coupling [42].

Because one recovers General Relativity exactly in the Solar System, for white dwarfs and neutron stars if $\mathcal{L}_m = T$, we will only consider the case $\mathcal{L}_m = -\rho$ in what follows.

Let us stress however that knowing which on-shell matter Lagrangian is more appropriate for various celestial bodies—if not a mixture of different type of on-shell matter Lagrangians depending on the various fields in various states—is still an open question.

5.2. In the Solar System

In the Solar-System regime, experimental constraints on relativistic gravity are usually interpreted within the post-Newtonian framework, which expands the equations of motion in powers of $v/c \approx \sqrt{GM/(rc^2)}$. The Newtonian order corresponds to the leading order, while higher post-Newtonian corrections describe relativistic effects of increasing order—see e.g. [43].

In the following, we explicitly show that the equations of motion in Entangled Relativity coincide with those of General Relativity up to this 1.5 post-Newtonian order—for both massive bodies and light. Consequently, ER reproduces all the weak-field predictions of GR that are currently tested in the Solar System—including the dynamics of massive bodies and the propagation of light—so that present observations cannot distinguish between the two theories.

5.2.1. The post-Newtonian metric

Assuming $\mathcal{L}_m = -\rho$, equation (16) becomes

$$\kappa^2 \Box \kappa^{-2} = \kappa P, \tag{41}$$

where P is the pressure of the considered fluid that source the gravitational field equations. In either the harmonic or standard post-Newtonian gauges, the first order post-Newtonian metric can be parameterized by a potential w, a vector potential w^i and the post-Newtonian parameters γ and β [29]

$$g_{00} = -1 + 2\frac{w}{c^2} - 2\beta \frac{w^2}{c^4} + \mathcal{O}\left(1/c^6\right), \tag{42a}$$

$$g_{0i} = -2\left(1+\gamma\right)\frac{w^i}{c^3} + \mathcal{O}\left(1/c^5\right), \tag{42b}$$

$$g_{ij} = \delta_{ij}\left(1 + 2\gamma \frac{w}{c^2}\right) + \mathcal{O}\left(1/c^4\right). \tag{42c}$$

Injecting this metric in the field equations (14) and (41) results in $\gamma = \beta = 1$ and

$$w = w_{\rm GR} - \frac{1}{c^2} G \int \frac{P(\mathbf{x}')\,\mathrm{d}^3x'}{|\mathbf{x} - \mathbf{x}'|} + \mathcal{O}\left(1/c^4\right),$$

$$=: w_{\rm GR} + \frac{1}{c^2}\delta w + \mathcal{O}\left(1/c^4\right), \tag{43a}$$

$$w^i = w^i_{\rm GR} + \mathcal{O}\left(1/c^2\right) \tag{43b}$$

where $w_{\rm GR}$ and $w^i_{\rm GR}$ are the expressions of the potentials predicted by General Relativity. For instance, in the Einstein–Infeld–Hoffmann–Droste–Lorentz approximation[11], they read as follows

$$w_{\rm GR} = w_0 - \frac{1}{c^2}\Delta + O\left(c^{-4}\right), \tag{44}$$

$$w^i_{\rm GR} = \sum_A \frac{Gm_A}{r_A} v^i_A + O\left(c^{-2}\right), \tag{45}$$

where

$$w_0 = \sum_A \frac{Gm_A}{r_A}, \tag{46}$$

with $\boldsymbol{r}_A = \boldsymbol{x} - \boldsymbol{x}_A(t)$, $r_A = |\boldsymbol{r}_A|$, and

$$\Delta = \sum_A \frac{Gm_A}{r_A}\left\{-2v_a^2 + \sum_{B\neq A}\frac{Gm_B}{r_{BA}} + \frac{1}{2}\left[\frac{(\boldsymbol{r}_A\boldsymbol{v_A})^2}{r_A^2} + \boldsymbol{r}_A\boldsymbol{a}_A\right]\right\}, \tag{47}$$

with $r_{AB} = |\boldsymbol{x}_B - \boldsymbol{x}_A|$ [48], and $8\pi G = c^4\kappa_0$, where κ_0 is the background value of κ. From (43a) and (42a), one can see that the metric characterizing the considered theory differs from the General Relativity metric at the first post-Newtonian level by a $1/c^4$ term in the temporal component of the metric. The solution of the scalar-field equation (41) can be written as

$$\frac{\delta\kappa}{\kappa_0} = \frac{\delta\hbar}{\hbar_0} = -\delta w + \mathcal{O}\left(1/c^2\right), \tag{48}$$

where $\delta\kappa := c^4(\kappa - \kappa_0)$, or equivalently $\delta\hbar := c^4(\hbar - \hbar_0)$[12].

5.2.2. Trajectory of test particles

We have seen that the weak-field perturbative metric that derives from the phase equation (12) differs from General Relativity at the first post-Newtonian level. However, massive particles do not follow the geodesics of the metric $g_{\alpha\beta}$ because of the coupling with the scalar field. The invariance of the phase equation (12) under diffeomorphisms leads to the conservation equation in equation (15) that can be rewritten as follows

$$\nabla_\sigma T^{\mu\sigma} = -\left(\mathcal{L}_m g^{\mu\sigma} - T^{\mu\sigma}\right)\frac{\partial_\sigma\kappa}{\kappa}. \tag{49}$$

The stress-energy tensor of non-interactive test particles is $T^{\alpha\beta} = \rho\, U^\alpha U^\beta$, where $U^\alpha = \mathrm{d}x^\alpha/(c\mathrm{d}\tau)$ with τ the proper time of the fluid's elements. Then, noting that the material Lagrangian reduces to $\mathcal{L}_m = -\rho = -\rho_r$ and the matter fluid current is conserved $\nabla_\sigma(\rho U^\sigma) = 0$ [29, 38], one gets

$$U^\sigma\nabla_\sigma U^\mu = \left(g^{\mu\sigma} + U^\mu U^\sigma\right)\frac{\partial_\sigma\kappa}{\kappa}. \tag{50}$$

The left-hand side of this equation represents the standard geodesic equation, whereas the right-hand side denotes a non-inertial acceleration arising from the scalar coupling. The change in the acceleration of massive test particles compared to General Relativity stems from both the altered metric, as indicated by equation (43a), and the non-inertial acceleration introduced by equation (50). When parameterized by the time-coordinate, the free-fall equation is expressed as:

$$\begin{aligned}\frac{\mathrm{d}^2x^i}{\mathrm{d}t} &= a^i_{\rm GR} + c^{-2}\left\{\partial_i\delta w + \frac{\partial_i\delta\kappa}{\kappa_0}\right\} + \mathcal{O}\left(1/c^4\right)\\ &= a^i_{\rm GR} + \mathcal{O}\left(1/c^4\right),\end{aligned} \tag{51}$$

where $a^i_{\rm GR}$ is the standard relativistic acceleration. The relation (48) shows there is an exact cancellation between the non-inertial acceleration $\vec{a}_{\rm NI}$ ($a^i_{\rm NI} \equiv c^{-2}\partial_i\delta\kappa/\kappa_0$) and the part of the inertial acceleration

[11] These equations of motion have been first written by Lorentz & Droste in 1917 ([44, 45], for an English translation: [46]), then by Einstein, Infeld & Hoffmann in 1939 [47].

[12] Here, the 'c^4' are just meant to keep track of the post-Newtonian orders. It simply means that the corrections to General Relativity are of order $\mathcal{O}(c^{-4})$, whereas the leading post-Newtonian terms in General Relativity are of order $\mathcal{O}(c^{-2})$ [29].

coming from the modification of the metric $\vec{a}_{\delta w}$ ($a^i_{\delta w} \equiv c^{-2}\partial_i \delta w$). Consequently, even though the trajectories of massive test particles no longer align with the geodesics of the metric $g_{\alpha\beta}$, they remain consistent with those from General Relativity up to the 1.5 post-Newtonian level.

Another way to verify this is by looking at the Einstein frame metric—that is, by doing a conformal transformation of the metric $\tilde{g}_{\alpha\beta} = (\bar{\kappa}/\kappa)^2 g_{\alpha\beta}$. Indeed, we have seen in section 4.1 that particles follow geodesics of the Einstein frame metric. Therefore, in order to show that particles follow the same trajectories as in General Relativity at the 1.5 post-Newtonian level, one just has to check that the Einstein frame metric is precisely the one of General Relativity at the 1.5 post-Newtonian level. After doing the conformal transformation, one checks that the Einstein frame metric indeed reads

$$\tilde{g}_{00} = -1 + 2\frac{w_{\mathrm{GR}}}{c^2} - 2\frac{w_{\mathrm{GR}}^2}{c^4} + \mathcal{O}\left(1/c^6\right), \tag{52a}$$

$$\tilde{g}_{0i} = -4\frac{w^i_{\mathrm{GR}}}{c^3} + \mathcal{O}\left(1/c^5\right), \tag{52b}$$

$$\tilde{g}_{ij} = \delta_{ij}\left(1 + 2\frac{w_{\mathrm{GR}}}{c^2}\right) + \mathcal{O}\left(1/c^4\right). \tag{52c}$$

5.2.3. Shapiro delay and the deflection of light

However, one expects the full c^{-4} metric to deviate from the one of General Relativity. Therefore, because that light rays still follow null-geodesics in Entangled Relativity—see section 3.3—one expects a modification of the Shapiro delay and the deflection of light with respect to General Relativity at the c^{-4} level. However, current solar system experiments are unable to measure the effect of the Shapiro delay and deflection of light to such precision [43, 49, 50].

5.2.4. Estimation of $\delta\omega$ for the Earth and the Sun

One can define a mass M_A^P of the body A that is related to the pressure of the body only, which reads as follows

$$M_A^P := 4\pi \int_A \frac{r^2\ P(r)}{c^2} \mathrm{d}r, \tag{53}$$

such that

$$c^2\frac{\kappa(r) - \kappa_0}{\kappa_0} = c^2\frac{\hbar(r) - \hbar_0}{\hbar_0} = \frac{\delta w}{c^2} = \frac{GM_A^P}{r} + \mathcal{O}\left(c^{-4}\right), \tag{54}$$

outside the body A, for spherically symmetric objects. On the other hand, the Newtonian potential of the body A at leading order in equation (43a) simply is $w_A = GM_A/r_A + \mathcal{O}(c^{-2})$, with $M_A := 4\pi \int_A r^2\ \rho(r)\mathrm{d}r$. Let us now provide some numerical estimates for the Earth and the Sun:

- $A =$ Earth: Using the preliminary reference Earth model (PREM) that infers the internal structure of the Earth from seismic waves [51], see figure 1, one finds $M_\oplus^P = 8.0 \times 10^{14}$ kg, and $M_\oplus = 6.3 \times 10^{24}$ kg. Hence, the magnitude of the deviation from General Relativity would be of the order 10^{-10} for the Earth if $\mathcal{L}_m = -\rho$ on shell.
- $A =$ Sun: Using the density profile of the model S for the Sun [52], see figure 2, one finds $M_\odot^P = 2.3 \times 10^{24}$ kg and $M_\odot = 2.0 \times 10^{30}$ kg. Hence, the magnitude of the deviation from General Relativity is of the order 10^{-6} for the Sun if $\mathcal{L}_m = -\rho$ on shell.

5.2.5. Variation of $\hbar$ between the surface of the Sun and an observer on Earth

From equation (37), one can translate the estimation of the relative variation of κ into a relative variation of $\hbar$. The variation of $\hbar$ in the potential of the Sun between the surface and an observer located on the Earth reads

$$\frac{\Delta\hbar}{\hbar} = \frac{GM_\odot^P}{c^2}\left(\frac{1}{r_\odot} - \frac{1}{r_\oplus}\right) \approx \frac{GM_\odot^P}{r_\odot c^2} \sim 2.5 \times 10^{-12}, \tag{55}$$

where $r_\odot$ and $r_\oplus$ are the position of the surface of the Sun and of the Earth respectively, in heliocentric coordinates.

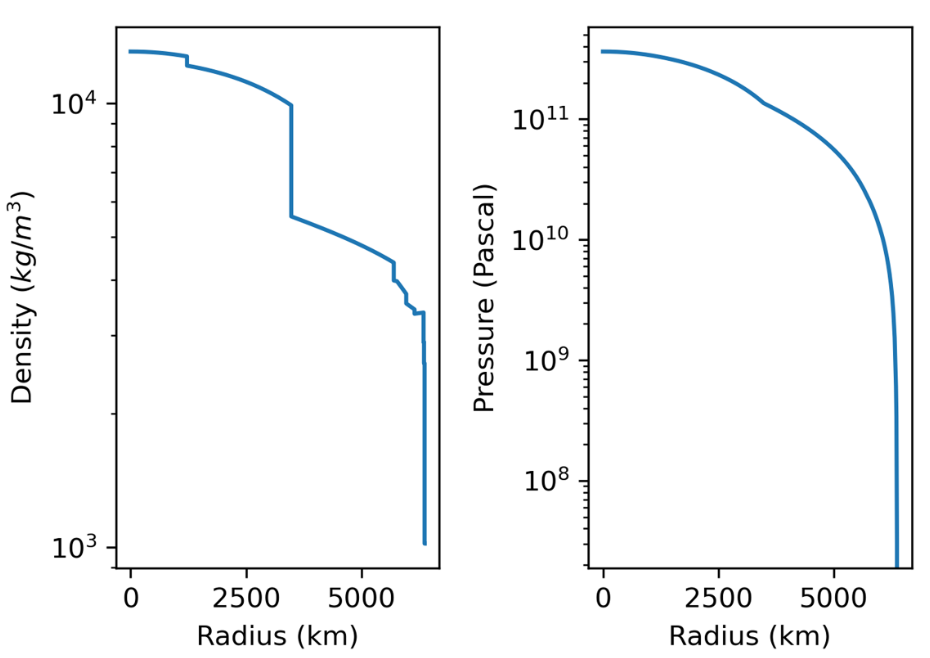

Figure 1. PREM model of the Earth density and pressure profiles [51]. Data kindly provided by Yanick Ricard from the ENS Lyon.

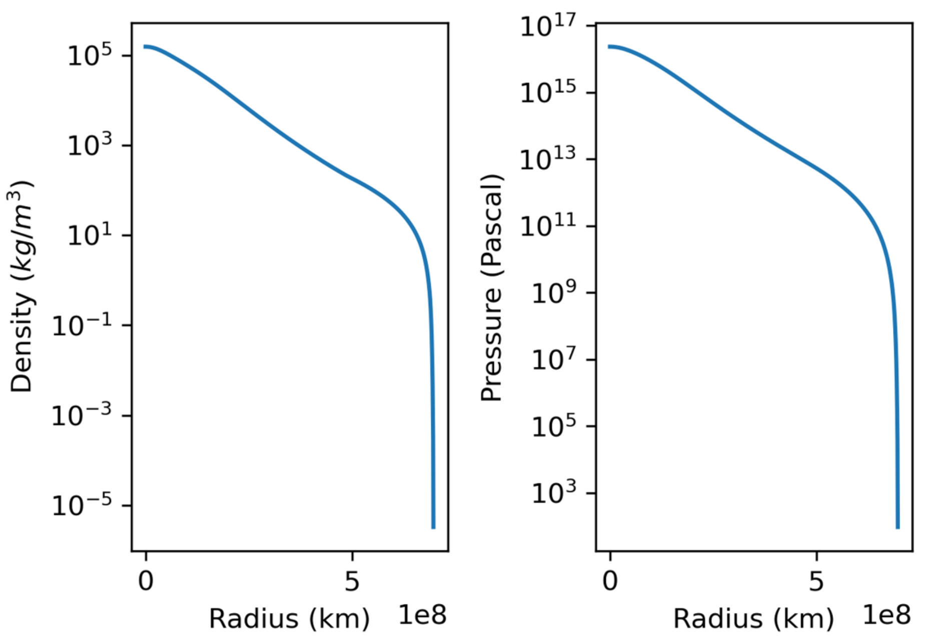

Figure 2. Model S of the Sun density and pressure profiles [52].

5.3. For white dwarfs

White dwarfs exhibit strong different behavior depending on their densities. To model them we simply use the Chandrasekhar equation of state for stellar matter, which can be written as follow [53, 54] :

$$P = A\left(2x^3\sqrt{x^2+1} - 3x\sqrt{x^2+1} + 3\operatorname{arcsinh}(x)\right), \tag{56}$$

and

$$\rho = Bx^3, \tag{57}$$

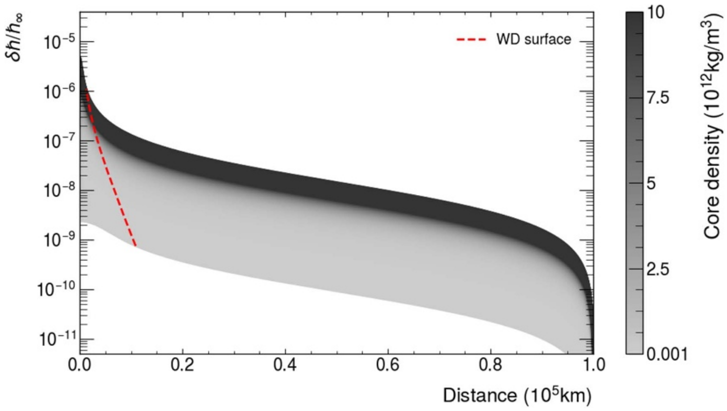

Figure 3. Plot of the relative variation of $\hbar$ depending on a white dwarf's central density versus the distance. The plot shows the behavior of $\delta\hbar/\hbar_\infty$ for different central densities, ranging from 10^9 to 10^{13} kg m^{-3} (from non-relativistic to ultra-relativistic). The vertical red curve represents the radii of the stars. Here $\delta\hbar = \hbar - \hbar_\infty$. For the highest central density the expected relative variation between the surface of the star and a remote observer is : 10^{-6} and for the lowest ones 7×10^{-10}.

with

$$A = \frac{\pi m_e^4 c^5}{3\hbar^3 (2\pi)^3} = 6.02 \times 10^{21}\,\text{Pa}, \tag{58}$$

and

$$B = \frac{8\pi m_e^3 c^3 m_P \mu}{3\hbar^3 (2\pi)^3} = 9.82\mu \times 10^8\,\text{kg}\cdot\text{m}^{-3}, \tag{59}$$

and with $x = \frac{p_0}{mc}$, where p_0 is the maximum momentum of the particle, m_e is the electron mass, m_P is the proton mass, and $\mu = \frac{A}{Z}$ is the molecular weight. Here, we considered an Oxygen gas so we have $\mu = \frac{15.999}{8}$.

This equation of state assumes that the white dwarf is well modeled by a Oxygen degenerate Fermi gas [54, 55]. If $\hbar$ varies, it affects the equation of state—through A and B—and must therefore be accounted for from the outset. However, in the case of white dwarfs, the difference is entirely negligible—assuming a constant or varying $\hbar$ in the equation of state yields essentially identical results. This is because the variation in $\hbar$ is extremely small, resulting in a perturbative effect that can be neglected at leading order. This issue will be discussed in more detail for neutron stars in section 5.4.2, where the effect will also be shown to remain small, despite their much higher densities. The simulations in figure 3 presents the variation of $\hbar$ versus the distance for white dwarfs with a central density that goes from 10^9 to 10^{13} kg·m^{-3} as the maximal value[13]. These simulations were obtained by solving the Tolman–Oppenheimer–Volkoff modified equations [9]. The Tolman–Oppenheimer–Volkoff equations describe hydrostatic equilibrium in spherically symmetric, self-gravitating objects within General Relativity, by balancing the pressure gradient against the gravitational pull of the enclosed mass. They therefore determine the internal structure and mass–radius relation of relativistic stars. In Entangled Relativity, the Tolman–Oppenheimer–Volkoff system is modified through the modified field equations, which lead to the additional dependence of the effective coupling constants—in particular $\hbar$ and G—on the scalar field. This involves solving the field equations given in equation (14), (15) and (16) in spherical symmetry, using the equation of state from equations (56)–(59) with the assumption that $\hbar$ is constant.

Solving the Tolman–Oppenheimer–Volkoff modified equations is a Boundary Value Problem, because one aims to find the initial condition at the center of the star ϕ_c that would lead to the wanted normalized value at infinity $\phi_\infty = 1$, for every considered value of the central density. This boundary value problem can be solved using a *shooting method* combined with a *dichotomic search*. The process begins by setting an initial guess for the central value of the scalar field. A numerical integration is performed, and the resulting boundary value of the scalar field is compared to the desired value. If the boundary value does not match the target, the central value is adjusted iteratively using a *dichotomic search*, which systematically narrows the range of possible central values. The loop continues until the difference between

[13] Maximum value of a white dwarf central density is still under debate, we used the extreme result presented in [56].

the calculated boundary value and the desired boundary value falls below a specified tolerance, defined as a small parameter times the desired value. In our simulations, the precision has been set to achieve $\phi_\infty = 1 \pm 10^{-5}$.

Let us also note that for the two extreme cases (low and high densities), they can be modeled by a simple polytropic equation of state of the form $P \propto \rho^\gamma$. These are obtained taking the case where $x \to 0$ for the low densities and $x \to \infty$ for the high ones in equations (56)–(57), such that one has $P \propto \rho^{4/3}$ or $P \propto \rho^{5/3}$ for the non-relativistic or ultra-relativistic cases, respectively. In particular, in the ultra-relativistic case, one has

$$P = 2Ax^4, \text{ and } \rho = Bx^3, \tag{60}$$

such that

$$P = 2\frac{A}{B^{4/3}}\rho^{4/3}. \tag{61}$$

From the definition of A and B, one obtains that $A/B^{4/3} \propto \hbar$. For the non-relativistic case, on the contrary, one has

$$P = \frac{8}{5}Ax^5, \text{ and } \rho = Bx^3, \tag{62}$$

such that

$$P = \frac{8}{5}\frac{A}{B^{5/3}}\rho^{5/3}, \tag{63}$$

and one verifies that $A/B^{5/3} \propto \hbar^2$, such that one effectively has

$$P = \frac{8}{5}\left(\frac{\hbar}{\hbar_0}\right)^2 \left.\frac{A}{B^{5/3}}\right|_0 \rho^{5/3}, \tag{64}$$

where $X|_0 \equiv X(\hbar_0)$.

Solving for the modified Tolman–Oppenheimer–Volkoff equations using the Chandrasekhar equation of state provide a solution for the radial profile of the scalar field which can be translated into an evolution of $\hbar$ as a function of the radial coordinate. The evolution of $\hbar$ as a function of the distance from the center of the star for various white dwarfs density is presented in figure 3. The maximal variation of $\hbar$ between the surface of a white dwarf and a remote observer is at the 10^{-6} level for the densest ultra-relativistic white dwarfs, but could be as low as 7×10^{-10} level for the less dense of the non-relativistic white dwarfs. The codes that generates all the figures presented in this manuscript, are freely available on github [57].

5.4. For neutron stars

We will now focus on celestial bodies inducing stronger gravitational field and we will estimate the variation of $\hbar$ around such bodies and assess its detectability.

5.4.1. Neglecting the magnetic field

In this study, we adopt a simple polytropic equation of state, as our primary aim is to get an idea of the orders of magnitude one can expect for the variation of $\hbar$.

The equation of state that we shall consider reads

$$P = K\rho^\gamma, \tag{65}$$

with $\gamma = 5/3$ and $K = 1.475 \times 10^{-3}\,(\text{fm}^3\,\text{MeV}^{-1})^{2/3}$ [9, 59].

As before, we simulate the behavior of neutron stars by solving the modified Tolman–Oppenheimer–Volkoff equations [9]. The simulations have been made of central densities in the range between 100 and 2000 MeV fm^{-3}. However, we only keep the simulations that are such that the speed of sound inside the neutron star is always less than the conservative limit usually considered of $c/\sqrt{3}$ [58], which corresponds to a central density of 1578 MeV fm^{-3} for the considered equation of state.

As we can see in figure 4, the relative variation of $\hbar$ between the surface of the neutron star and a remote observer range from 1.5% for most compact case to 0.1% for the least compact case.

The relative variation of $\hbar$ between the center of the neutron star and a remote observer can be a bit more significant, as one can see in figure 5. Indeed, the relative variation of $\hbar$ between the center of

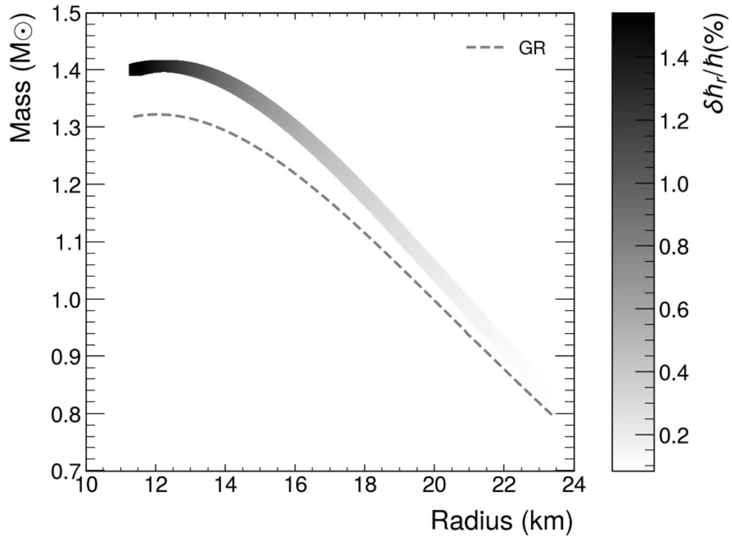

Figure 4. Simulations of the relative variation of the quantum of action between the **surface** of a neutron star and a remote observer $\delta\hbar_r/\hbar$, where $\delta\hbar_r = \hbar|_{\text{NS radius}} - \hbar|_\infty$ for neutron stars with the equation of state given in equation (65) with various radii and masses. The (absolute) maximal relative difference is 1.5% for a neutron star of 1.4 solar mass and 11.4 km. The (absolute) minimal relative difference is 0.1% for a neutron star of 0.8 solar mass and 23.5 km. The simulations have been made for central densities between 100 and 2000 MeV fm^{-3}, but the plot only shows the simulations for which the speed of sound is less than the conservative limit usually considered of $c/\sqrt{3}$ [58] in the whole neutron star, which corresponds to a central density of 1578 MeV fm^{-3} for the considered equation of state. The masses and radii for the simulations in General Relativity with the same equation of state are given in the dashed-line for comparison.

Table 1. Table summarizing the relative variation of $\hbar$ between the center (or the surface) of a neutron star and a remote observer and the associated mass and radius.

	Least compact	Most compact
$\frac{\delta\hbar}{\hbar}$% center	0.3	5.7
$\frac{\delta\hbar}{\hbar}$% surface	0.1	1.5
Mass ($M_\odot$)	0.8	1.4
Radius (km)	23.5	11.4

the neutron star and a remote observer range from 5.7% for most compact case to 0.3% for the least compact case. These results are summarized in table 1.

As mentioned above, this begs the question about the modification of nuclear physics in the neutron star and the respective expected modifications of the equations of state. It should not be excluded that this could eventually be used to constrain the plausible equations of states within the framework of Entangled Relativity. The codes that generates all the figures presented in this manuscript, are freely available on github [57].

5.4.2. Impact of the variation of $\hbar$ *on the equation of state*

In the previous section, and in section 5.3, we assumed that the equation of state was independent of the value of $\hbar$. However, we know that this is only valid as a first order approximation because the equation of state depends on nuclear physics, and nuclear physics depends on the value of $\hbar$. Nevertheless, given the very small amplitude of the variation of $\hbar$ found in the simulation, it seems likely that assuming that $\hbar$ is constant in the equation of state is a good approximation, and that a variation of $\hbar$ in the equation of state can be taken into account perturbatively.

In order to check this assumption, we assumed that the equation of state that we previously used had the same dependence on $\hbar$ as the degenerate Fermi gas [53, 54], that is, $P \propto \hbar^2 \rho^{5/3}$—see equation (64). Given that $\hbar^2 \propto \phi^{-1}$, the polytropic equation (65) is thus modified as follows :

$$P = K\left(\frac{\hbar}{\hbar_0}\right)^2 \rho^\gamma = \frac{K}{\phi}\rho^\gamma, \tag{66}$$

where $\hbar_0$ is the background value of $\hbar$. While this assumption does not correspond to the actual dependence of a neutron star equation of state on $\hbar$ *a priori*, because the nuclear material of neutron stars is not a degenerated Fermi gas, it is a useful model that allows to assess the impact of retroaction of the

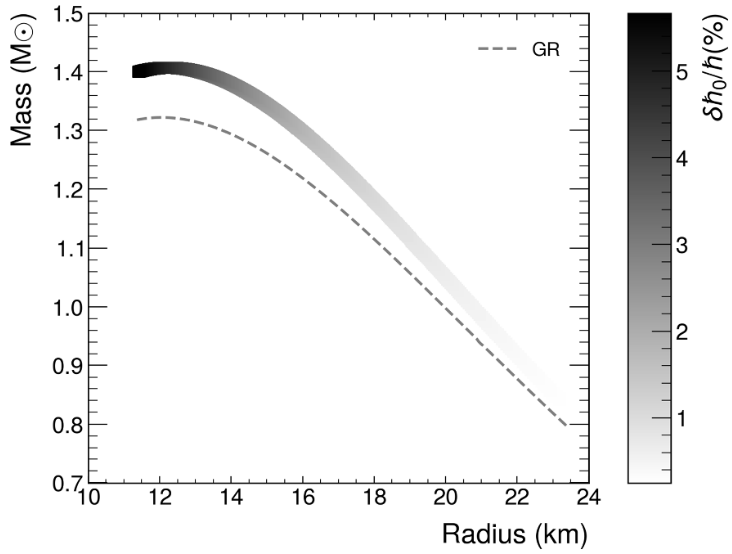

Figure 5. Simulations of the relative variation of the quantum of action between the **center** of the neutron star and a remote observer $\delta\hbar_0/\hbar$, where $\delta\hbar_0 = \hbar|_0 - \hbar|_\infty$ for neutron stars with the equation of state given in equation (65) with various radii and masses. The (absolute) maximal relative difference is 5.7% for a neutron star of 1.4 solar mass and 11.4 km. The (absolute) minimal relative difference is 0.3% for a neutron star of 0.8 solar mass and 23.5 km. The simulations have been made for central densities between 100 and 2000 MeV fm^{-3}, but the plot only shows the simulations for which the speed of sound is less than the conservative limit usually considered of $c/\sqrt{3}$ [58] in the whole neutron star, which corresponds to a central density of 1578 MeV fm^{-3} for the considered equation of state. The masses and radii for the simulations in General Relativity with the same equation of state are given in the dashed-line for comparison.

variation of $\hbar$ on the equation of state for the typical matter densities of neutron stars[14]. Nevertheless, we also test other hypothetical dependence below, notably in order to check the perturbative nature of the effect.

Results:
What we find is that, as expected, the variation of $\hbar$ is already so small in the neutron star that it does not impact the equation of state significantly, such that the results are very similar whether or not the variation of $\hbar$ is considered in the equation of state. This is exemplified in figure 6 where is represented the variation of $\hbar$ versus the radius of a neutron star with a central density of 1000 MeV fm^{-3} for two cases : (i) one where we neglect the dependency of the equation of state to the scalar field (orange line) and (ii) one where we include the back reaction of the variation of $\hbar$ into the equation of state (blue line).

As shown in figure 6, neglecting the variation of $\hbar$ in the equation of state introduces an error of only 11% at the center of the star and at its surface, where the error at a location $r = \mathfrak{R}$ is defined as follows

$$\text{Error } \hbar(\mathfrak{R}) = \frac{\hbar(\mathfrak{R})\,|_{\mathrm{w}} - \hbar(\mathfrak{R})\,|_{\mathrm{wo}}}{\Delta\hbar(\mathfrak{R})\,|_{\mathrm{w}}}, \tag{67}$$

where 'w' and 'wo' stand for 'with retroaction' and 'without retroaction' respectively and $\Delta\hbar(\mathfrak{R}) = \hbar(\mathfrak{R}) - \hbar_\infty$.

This shows that the variation of $\hbar$ in the equation of state can be considered as a perturbative effect that can be neglected at leading order.

Test with other dependencies:
In the previous paragraphs, we considered a quadratic dependency of the pressure to $\hbar$ and show that the back reaction from the variation of the equation of state can safely be neglected in neutron stars. We will now assess if this conclusion holds also for other dependency and consider that $P \propto \hbar^n$. If the effect is perturbative, it should then scale linearly with the power of the dependence as

[14] We note that deriving the exact dependence of a neutron star's equation of state on $\hbar$ from first principles is challenging, as it involves complex nuclear physics, which itself depends on the value of $\hbar$. Therefore, we leave the detailed derivation of this dependence for future studies.

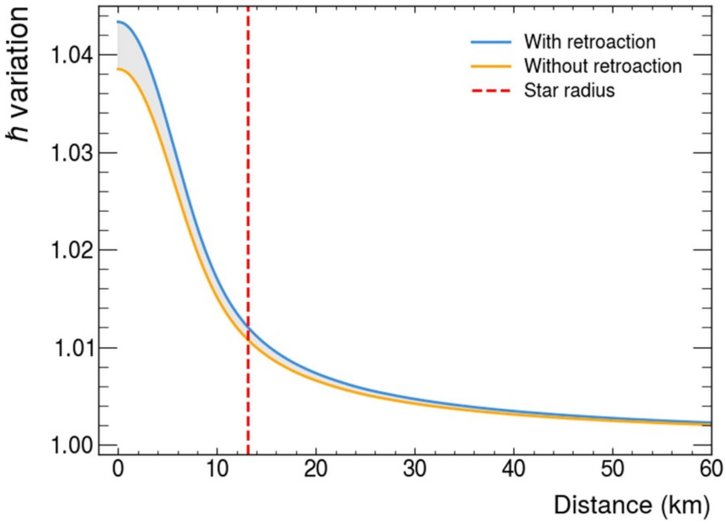

Figure 6. Plot of the variation of $\hbar$ for a neutron star with a central density of 1000 MeV fm^{-3}. The blue curve shows the variation of $\hbar$ when taking into account its variation into the equation of state, its '*retroaction*', whereas the orange curve is obtained by neglecting the variation of $\hbar$ in the equation of state. Neglecting the retroaction of $\hbar$ in the equation of state results in only a 11% error in the evaluation of the variation of $\hbar$ at the center and at the surface.

Table 2. table summarizing the relative variation of $\hbar$ at the surface of the neutron star after the numerical integration, by computing the relative difference of the values of $\hbar$ with and without considering the retroaction that comes from the variation of $\hbar$ in the equation of state equation (66), for the same central density of 1000 MeV fm^{-3}.

n	-3	-2	-1	1	2	3
Error$_*$ $\hbar$%	-15	-10	-5	5	11	16

$$\hbar = \hbar_0 + \delta\hbar \Rightarrow \hbar^n \approx \hbar_0^n \left(1 + n\frac{\delta\hbar}{\hbar_0}\right), \tag{68}$$

$$\Rightarrow P \approx P|_0 \left(1 + n\frac{\delta\hbar}{\hbar_0}\right), \tag{69}$$

where $X|_0 \equiv X(\hbar_0)$ and therefore everything that follows should keep the linear dependence in n.

We perform several integrations of the Tolman–Oppenheimer–Volkoff equations, considering various n, and compare the obtained values of $\hbar$ at the radius of the star, to the values we obtained when this retroaction is neglected.

The results for various dependencies are given in table 2 and one can indeed check the roughly linear behavior of the effect.

The effect of the retroaction of $\hbar^n$ has been tested for multiple central densities from 100 MeV fm^{-3} to 1500 MeV fm^{-3}. With higher central densities, and with higher values of n, it progressively loses its linearity as the central density increases, since the effect becomes less and less perturbative.

As a consequence, despite the fact that one does not know precisely what is the equation of state dependence on $\hbar$ for the nuclear material of neutron stars, its effect on the overall numerical integration should remain small anyway.

On the speed of sound limit:
An important aspect is that the maximal central density depends on the equation of state. Specifically, the limit of our simulation is determined by constraining the sound velocity not to exceed $c/\sqrt{3}$ in the neutron star, following [58]. When an initial central density is chosen, it corresponds to a specific pressure defined by the equation of state, which is subsequently influenced by the retroaction effects. Consequently, the maximum central density differs depending on whether or not the retroaction of $\hbar$ on the equation of state is taken into account. Indeed we previously saw that the maximal central density of neutron star is 1578 MeV fm^{-3}. With the retroaction this maximal central density is modified to

1323 MeV fm^{-3}. The maximum central densities show a relative difference of 16% between the cases with and without retroaction for a dependency of the form $P \propto \hbar^2$.

Comparison of maximum mass between Entangled Relativity with and without retroaction and General Relativity:
A previous study shown that the maximum mass of Entangled Relativity neutron stars is predicted to be always larger than that given in General Relativity [9]. In this result, the equation of state was assumed to be independent of $\hbar$. But if one takes into account the backreaction of $\hbar$ onto the equation of state, one can expect bigger variation with respect to General Relativity. Indeed, in terms of ADM mass the result shows that a General Relativity neutron star has a maximum mass of 1.32 $M_\odot$ with an associated density of 1270 MeV fm^{-3}, while the neutron stars predicted by Entangled Relativity have a maximum mass of 1.41 $M_\odot$ with an associated density of 1266 MeV fm^{-3}. Considering the retroaction : $P \propto \hbar^2$, the maximum mass predicted is 1.49 $M_\odot$ with a density of 1323 MeV fm^{-3} which corresponds to the maximum density reachable in Entangled Relativity with the dependency $P \propto \hbar^2$, and the simple equation of state equation (66). This result aligns with the previous one, confirming that Entangled Relativity neutron stars are always more massive than the General Relativity ones.

Realistic tabulated equation of state.
Incorporating a varying $\hbar$ into realistic tabulated equations of state (e.g. CompOSE [60, 61]) would require a careful reformulation of the underlying microphysics and numerical tables, which is beyond the scope of this paper and left for future dedicated investigations.

5.4.3. Magnetars

The simulations discussed earlier presuppose $\mathcal{L}_m = -\rho$ on-shell for nuclear material of white dwarf and neutron stars. However, unless the derivation is explicitly made from first principles, we cannot rule out the possibility of $\mathcal{L}_m = T$ instead [39–41]. If this were the case, there would be no deviation for white dwarfs and neutron stars–assuming their magnetic field is neglected. This is obvious from equation (16), but it has also been explicitly confirmed in the code that simulates white dwarfs and neutron stars without magnetic fields.

On the other hand, when the substantial magnetic field of certain neutron stars is taken into account,[15] the situation is slightly different. Indeed, for a magnetic field, the scalar field is sourced as follows

$$3\kappa^2 \Box \kappa^{-2} = \kappa \frac{B^2}{2\mu_0}, \tag{70}$$

as exemplified for instance with the charged black-hole solutions [11, 13] or with Schwarzschild black-holes immersed in a magnetic background field [14]. Consequently, one could anticipate a deviation from General Relativity for magnetars within the framework of Entangled Relativity, irrespective of the on-shell value of the Lagrangian for the star's nuclear material. However, one would have to go beyond the spherical symmetry to simulate magnetars as the magnetic field would be axi-symmetric at leading order, and the generation of the magnetic field within the magnetar may be difficult to model. Nevertheless, the magnitude of the source term ($B^2/2\mu_0$) from the magnetic field (around 10^{11} Tesla [64]) in equation (70) is typically 10^{-7} relatively smaller than the typical energy densities—or pressure, since $\rho \sim P/c^2$ for the densest neutrons stars—which source the scalar field κ, see equation (16). This shows that the magnetic field is expected to produce a small perturbation to the calculations presented in the previous section. However, in ultra-strong magnetic fields beyond the Schwinger limit, quantum and nonlinear electrodynamics effects become relevant [65], potentially enhancing the influence of a varying $\hbar$.

5.5. Black holes

Vacuum solutions are ill-defined in Entangled Relativity, because $\mathcal{L}_m/R$ is ill-defined in equation (7) when both $\mathcal{L}_m$ and R are null form the outset. As a consequence, none of the vacuum solutions of General Relativity are, strictly speaking, solutions of Entangled Relativity. Nevertheless, its has been shown on multiple occasions already [11, 13, 14] that vacuum solutions of General Relativity are accurate approximations of solutions of Entangled Relativity when the matter fields density goes to zero—which we shall name the 'vacuum limit' hereafter. More recently [37], it has even been argued that all

[15] Such as ultra-magnetized objects, e.g. perhaps, GLEAM-X J162759.5-523 504.3 [62, 63].

vacuum solutions of General Relativity are limits of solutions of Entangled Relativity in the vacuum limit. In order to illustrate this, let us have a look at a spherical black-hole immersed in a matter field. Because an electromagnetic field is arguably the easiest field to consider outside the black-hole, a spherical black-hole immersed in a magnetic, or an electric, field has been derived in [14] and reads as follows

$$ds^2 = \Lambda^{-\frac{28}{13}} r^2 \sin^2\theta \mathrm{d}\varphi^2 + \Lambda^{\frac{20}{13}} \left[\left(1 - \frac{r_s}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 - \left(1 - \frac{r_s}{r}\right) c^2 \mathrm{d}t^2 \right], \tag{71}$$

where r_s is the Schwarzschild radius and with

$$\Lambda = 1 + \frac{13}{48} B^2 r^2 \sin^2\theta, \tag{72}$$

using units such that

$$\lim_{A^\alpha \to 0} \kappa = 1, \tag{73}$$

where A^α is the electromagnetic 4-vector. The solution for the electromagnetic 4-vector is

$$A = \frac{B}{2\Lambda} r^2 \sin^2\theta \ \mathrm{d}\varphi. \tag{74}$$

It corresponds to a magnetic field pointing along the z-direction. Let us note that the solution is such that [14]

$$\mathcal{L}_m = -B^2 \Lambda^{-44/13} \left(1 - \frac{r_s \ \sin^2\theta}{r}\right), \tag{75}$$

and

$$R = B^2 \Lambda^{-46/13} \left(1 - \frac{r_s \ \sin^2\theta}{r}\right), \tag{76}$$

such that one has

$$\kappa = -\frac{R}{\mathcal{L}_m} = \Lambda^{-2/13}. \tag{77}$$

Therefore, in the vacuum limit $B \to 0$, $\Lambda \to 1$ and the black-hole is accurately approximated by a Schwarzschild black-hole and the scalar degree-of-freedom κ becomes constant—as one generally expects from the no-hair theorem in scalar–tensor theories [66].

Although the interstellar medium is not generally expected to correspond to a magnetic field oriented in a specific direction, such an example provides a simple exact solution for a spherical black hole immersed in an interstellar field. Therefore, given the typically low density of interstellar matter, one should not expect any significant variation of $\hbar$ around astrophysical black holes.

6. Discussion and conclusion

Entangled Relativity can be viewed as a non-linear reformulation of General Relativity in terms of its field content, although it leads to a slightly distinct phenomenology. This comes at the price that neither Newton's constant G nor Planck's quantum of action $\hbar$ are fundamental constants in this framework. While it is quite usual to have theories that predict that G varies, such as the usual Brans–Dicke theories (see appendix), the predicted variation of $\hbar$ is much more surprising. One can compute the amplitude of the variation of $\hbar$ predicted by the theory. Interestingly, while it relies on some modeling assumptions in each considered case, as always in astrophysics, it does not depend on any theoretical parameter that could be fine tuned *a posteriori*.

In this paper, we estimated the level of variation of $\hbar$ in three distinct cases: in the Solar System, where we found that the variation is at the 10^{-12} level, for white dwarfs, for which the relative variation seems to be less than 10^{-6} between their surface and a remote observer, and neutron stars, for which the relative variation can be as much as 1.5%. While we expect these values to marginally change if one uses different modeling assumptions for white dwarfs and neutron stars—that is, different equations of state—the values provided in this paper provide a good order of magnitude, provided that the main assumption $\mathcal{L}_m = -\rho$ discussed in section 5.1 is accurate for the bodies considered in this study[16].

6.1. Possible observational signatures of a variation of $\hbar$

The implications, in terms of observables, of $\hbar$ taking different values for an observed phenomenon and its observer remain to be derived from first principles. In particular, this question should be tackled in order to test this prediction of the theory.

One possible expectation could be that it would be akin to an apparent variation of the fine structure constant α, since the definition of the fine structure constant depends on $\hbar$ as $\alpha = e^2/(4\pi\epsilon_0\hbar c)$, where ϵ_0 is the vacuum permittivity, and e is the elementary charge of an electron, both constant in Entangled Relativity[17]. Hence, $\delta\alpha/\alpha = -\delta\hbar/\hbar$. Therefore, one might argue that we are not too far from being able to start testing this prediction. Indeed, on the one hand, our investigations suggest that such types of variation could be up to a few 10^{-6} level for white dwarfs in Entangled Relativity—see figure (3)—provided that a variation of $\hbar$ indeed leads to similar observational signatures as a variation of α. On the other hand, current observational constraints on variation of α are at the 10^{-5} level [67]. It should be noted, however, that the phenomenology of an $\hbar$-varying theory and that of an α-varying theory are expected to differ, as one parameter is universal whereas the other is not.

Our study shows that no significant variation of κ and $\hbar$ is expected around astrophysical black holes, as they are predicted to be identical to those from General Relativity under standard astrophysical conditions [11, 13, 14], see section 5.5. This implies that the scalar degree of freedom remains constant around astrophysical black holes, leading to no variation in κ and $\hbar$.

Beyond the aforementioned possibility, another observational consequence of a variation in $\hbar$ would be a modification of the black-body radiation law. Indeed, black-body radiation is a universal effect—in the sense that it does not depend on the composition of the elements constituting the black body—and it does not depend on any of the standard model parameters nor on Newton's constant. The black-body radiation law depends explicitly on the value of $\hbar$, and a change in $\hbar$ cannot be entirely reabsorbed by adjusting $k_B T$, where k_B is the Boltzmann constant and T the temperature of the black-body. In other words, one can use the spectrum of a black-body in order to fit both $\hbar$ and $k_B T$ independently.

A variation in the black-body radiation law could, in principle, be constrained at the cosmological level using the cosmic microwave background radiation, provided that one also takes into account the non-conservation of light intensity along geodesics, as follows from equation (28), which induces an additional chemical potential in the black-body spectrum [34]. Nevertheless, although a dedicated study has yet to be carried out, it does not seem very likely that the theory could be constrained using the black-body radiation of the cosmic microwave background, given that the scalar degree of freedom freezes at the cosmological level for matter sources composed of dust and/or null radiation. Indeed, $\mathcal{L}_m = T$ for dust and null radiation, such that the scalar degree of freedom is not sourced, while the expansion of the Universe introduces a friction term in its dynamics (i.e. $\Box\phi = \ddot{\phi} + 3H\dot{\phi} = 0$, where H is the Hubble parameter [12, 15]). Therefore, one should not expect a strong variation in the shape of the cosmic microwave background's black-body spectrum in Entangled Relativity. In any case, given that $\hbar \propto G$ in Entangled Relativity, a cosmological variation in $\hbar$ would have to be consistent with the constraints obtained on the cosmological variation of Newton's constant [68, 69], as well as on local temporal constraints [43, 70].

Perhaps probing the black-body radiation law in compact objects—where one expects small but quantitative variations in $\hbar$[18]; see section 5—could be more promising.

6.2. Conclusion

In summary, the predicted variation of $\hbar$ opens an entirely new window of potentialities to explore in fundamental physics. All of this comes from a very conservative theory, as Entangled Relativity merely replaces the coupling between matter and curvature at the formulation level of the theory. While Entangled Relativity requires fewer fundamental constants than standard physics, it recovers standard physics up to yet non-observable differences, and it seems to make an explicit connection between the

[16] If $\mathcal{L}_m = T$ instead, as argued notably in [39–41] in other contexts, then only a strong magnetic field could lead to a variation of $\hbar$, but it would be several orders of magnitude smaller than the values derived in this paper—see, section 5.4.3.

[17] According to [16], α is defined as $\alpha := e^2/(4\pi\epsilon^2\kappa)$ in Entangled Relativity, in the units that are such that $e = 5.38 \times 10^{-14}\,\mathrm{kg}^{1/2}\mathrm{m}^{3/2}\mathrm{s}^{-1}$ and $\epsilon_0 := 1$.

[18] Provided that the assumption $\mathcal{L}_m = -\rho$ used in this paper for perfect fluids is correct—see section 5.1.

quantum and gravitational worlds through the relation $\kappa \propto \hbar$ provided in equation (34). More importantly, the prediction that $\hbar$ varies akin to a gravitational scalar field may become testable in the foreseeable future, notably potentially with the accurate measurement of white dwarfs' spectra. However, several important aspects have to be tackled before the predicted variation of $\hbar$ can actually be tested.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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Appendix. Differences between theories

The goal of this appendix is to provide a detailed comparison of various theories and to explicitly show what is constant and what varies in each of them. The methodology is to write the path integral of the theory and to identify its various 'constants'. We will first briefly consider the case of string theory, which is an example of a theory where $\hbar$ is not constant, and varies inversely proportionally to G. We will then review one of the main result from this paper by showing that $\hbar$ and G vary proportionally in Entangled Relativity and show that this feature is different from minimally coupled tensor–scalar theory (generalized Brans–Dicke or Bergmann-Wagoner-Nordvert theory) or from Einstein–dilaton theories.

Let us first sketch the general methodology that will be used in this appendix. We start from a general theory, whose path integral reads as follows

$$Z = \int [\mathcal{D}g] \prod_j [\mathcal{D}f_j] \exp\left[\frac{\mathrm{i}}{\xi_F}\int \mathrm{d}_g^4 x F\right], \tag{78}$$

where F is a function that defines the theory and ξ_F is a quantum parameter whose dimension is $[\xi_F] = [F] \times L^4$, such that the quantum phase $\Theta = \int \mathrm{d}_g^4 x F/\xi_F$ is dimensionless. F is assumed to be a differentiable function of the fields involved in the definition of the Lagrangian density.

If F is affine in the matter Lagrangian density $\mathcal{L}_m$—i.e. if $F = \mathcal{L}_m + g(R, \Phi, \partial_\alpha \Phi \partial^\alpha \Phi, \text{etc.})$—then ξ_F has the dimension of an energy times a length, and its value must be $\xi_F = \hbar c$ in order to recover standard quantum physics in the limit where the effects of gravity can be neglected. Indeed, assuming for instance that

$$F = g(R) + \mathcal{L}_m. \tag{79}$$

Then, at any scale at which one can neglect gravity, the path integral of particle physics reduces to

$$Z \approx \int \prod_j [\mathcal{D}f_j] \exp\left[\frac{\mathrm{i}}{\xi_F}\int \mathrm{d}_g^4 x \mathcal{L}_m\right]. \tag{80}$$

Comparing this partition function with the one from the Core theory, see equation (2), one can directly identify ξ_F to $\hbar c$, which means that, particle physics experiments are sensitive to $\xi_F = \hbar c$. In particular, as explained in section 6.1, a measurement of the black-body spectrum will be sensitive only to this parameter.

However, when the function F is not necessarily linear with respect to the Lagrangian of the standard model of particle physics, the dimension of the quantum parameter ξ_F appearing in the path integral will depend on the dimension of the function F. For this reason, in general, the quantum parameter ξ_F does not have the dimension of an action times a velocity. In such a case, the fundamental quantum parameter of the theory cannot be a *quantum of action* like in the Core theory. A typical example where it is not the case is string theory, which is discussed in the next subsection. Furthermore, as discussed in the main part of this paper and in section A.4 below, the non-linear nature of the Entangled Relativity action also implies that the quantum parameter of this theory cannot be identified to $c\hbar$.

In summary, whenever the function F does not have the dimension of an energy density, the quantum parameter ξ_F cannot be identified to $c\hbar$ in the limit at which gravity is neglected

A.1. String theory

Here, we will briefly remind some known results from string theory regarding the fact that the quantum parameter of the theory is not the Planck constant—but the *string length*—and, as a consequence, the fact that $\hbar$ is not constant in string theory.

String theory, being purely geometrical, only contains two fundamental constant, c for relativistic invariance and the string length λ_s for quantization. Planck's and Newton's parameter appear only through Planck's length, a calculable fraction of λ_s [71]. Indeed, the string path integral reads as follows [71]

$$Z_{\text{string}} = \int \mathrm{d}X^\mu \exp\left(-\frac{\mathrm{i}}{\pi\,\lambda_s^2} A_{\text{rea}}(X^\mu)\right), \tag{81}$$

where the Nambu–Gotto so-called *action* actually has the dimension of an *area*[19] and reads

$$A_{\text{rea}}(X^\mu) = \int \mathrm{d}\sigma^i\,\mathrm{d}\sigma^j \sqrt{-\det(h_{ij})}, \tag{82}$$

in which $\mathrm{d}\sigma^i\,\mathrm{d}\sigma^j$ is the surface element and h_{ij} is the induced metric.

The important point for our discussion is the fact that the only fundamental dimensionful constant is a length λ_s, also known as the string length. In other words, one can equivalently say that λ_s^2 is a *quantum of area* in string theory[20]. From there, one can define a space-time varying *quantum of action* as follows

$$\hbar(X^\mu) = \frac{\lambda_s^2 c}{2\alpha'(X)}, \tag{83}$$

where α' is the Regge slope, a classical quantity with dimension of length/mass proportional to Newton's constant [71], which depends on the expectation value of the dilaton field.

In conclusion, string theory is a known example of a theory whose quantum parameter ξ_F cannot be identified as the Planck constant and where, as a consequence, the Planck constant becomes a quantity that can vary in space and time.

A.2. Minimally coupled tensor–scalar theories

In this section, we will consider the case of minimally coupled tensor–scalar theories, i.e. theories where there exists a conformal frame where the scalar field does not explicitly coupled to the fields of the standard model of particles. These are known as generalized Brans–Dicke theory or sometimes known as Bergmann–Wagoner–Nordtvedt theory, see [72–74]. The action of tensor–scalar theories reads as equation (78) with

$$F_{\text{TS}} = \mathcal{L}_m + \frac{1}{2\kappa}\left(\Phi R - \frac{\omega(\Phi)}{\Phi}(\partial_\sigma\Phi)^2 - V(\Phi)\right), \tag{84}$$

where κ is a constant and ω and V some functions of the scalar field Φ. In the general relativistic limit ($\Phi \approx$constant), the path integral of the theory from equation (78) becomes

$$Z_{\text{TS}} = \int [\mathcal{D}g] \prod_j [\mathcal{D}f_j] \exp\left[\frac{\mathrm{i}}{\xi_F}\int \mathrm{d}_g^4 x \left(\frac{\Phi R}{2\kappa} - \frac{V(\Phi)}{2\kappa} + \mathcal{L}_m\right)\right]. \tag{85}$$

Comparing this equation with the corresponding one from the Core theory, see equation (2), one can clearly identify[21].

$$\frac{\kappa}{\Phi(x)} = \frac{8\pi G(x)}{c^4}, \quad \text{and} \quad \xi_F = \hbar c. \tag{86}$$

[19] Let us stress that the Polyakov so-called *action* also actually has the dimension of an *area*, $A_{\text{rea}} = \frac{1}{2}\int \mathrm{d}^2\sigma \sqrt{-h}\,h^{ab} g_{\mu\nu}(X)\,\partial_a X^\mu(\sigma)\,\partial_b X^\nu(\sigma)$, such that the following discussion does not depend on which *area* is being considered.

[20] Provided that one considers equation (82) as the fundamental object in String theory.

[21] More accurately, one has to derive the post-Newtonian metric to identify Newton's constant as measured with experiments on Earth, see e.g. [75].

For this reason, a measurement of the black-body radiation spectrum would be sensitive to $\hbar$ as explained in section 6.1 and in the beginning of this appendix. Given that both ξ_F and c are set to be constant by definition in this theory, $\hbar$ is a constant as well.

In conclusion, in minimally coupled tensor–scalar theories, Newton's constant $G(x)$ varies in space and time, whereas $\hbar$ is a constant. Let us stress that this would be the case for any theory where F is $F = \mathcal{L}_m + g(R, \Phi, \partial_\sigma\Phi\,\partial^\sigma\Phi, \text{etc})$.

A.3. Einstein–dilaton theories

Let us now consider the case of a non-minimally coupled tensor–scalar theory, i.e. in a theory where there is an explicit coupling between the scalar field and the standard model of particles. Such theories are sometimes known as Einstein–dilaton theories [76, 77]. The path integral of Einstein–dilaton theories reads as equation (78) with

$$F_{\text{Ed}} = \frac{1}{2\kappa}\left(f(\varphi)R - \frac{\omega(\varphi)}{\varphi}(\partial_\mu\varphi)^2 - V(\varphi)\right) + \mathcal{L}_m[g, \Psi_i] + \mathcal{L}_{\text{int}}[g, \Psi_i, \varphi]\,, \tag{87}$$

where κ is a constant, f and ω are some functions of the scalar field φ, Ψ_i the various fields of the standard model, $\mathcal{L}_m$ the standard model Lagrangian and $\mathcal{L}_{\text{int}}$ characterizes the interaction between the scalar field and matter. A widely studied phenomenological interaction Lagrangian [76] is provided by

$$\begin{aligned}\mathcal{L}_{\text{int}} = &\frac{D_e(\varphi)}{4e^2}F_{\mu\nu}F^{\mu\nu} - \frac{D_g(\varphi)\,\beta_3(g_3)}{2g_3}G^a_{\mu\nu}G_a^{\mu\nu}\\ &- \sum_{i=e,u,d}\left(D_{m_i}(\varphi) + \gamma_{m_i}D_g(\varphi)\right)m_i\bar{\psi}_i\psi_i,\end{aligned} \tag{88}$$

where $F_{\mu\nu}$ is the Faraday tensor, $G^a_{\mu\nu}$ the gluons tensor, g_3 the strong force coupling constant, $\beta_3(g_3) = \mu\,\partial\ln g_3/\partial\mu$ its beta function relative to the quantum scale invariance violation, μ the energy scale of the relevant physical processes, m_i the fermions mass, ψ_i their spinor, and $\gamma_{m_i} = -\mu\,\partial\ln m/\partial\mu$ the beta function relative to the dimensional anomaly of the fermions masses coupled to the gluons. The $D_i(\varphi)$ functions describe the different couplings between the matter fields and the dilaton. D_e characterizes the φ dependency of the fine structure constant, D_g the φ dependency of the QCD mass scales Λ_3, and D_{m_i} the quarks masses[22]. Note that the action (87) differs from the tensor–scalar formulation of Entangled gravity presented in equation (18).

This interaction between the scalar field and matters induces a spatio-temporal dependence of the constant of nature as follows [76][23]:

$$\alpha(x) = [1 + D_e(\varphi(x))]\,\bar{\alpha}, \tag{89a}$$

$$\Lambda_3(x) = \left[1 + D_g(\varphi(x))\right]\bar{\Lambda}_3, \tag{89b}$$

$$m_e(x) = [1 + D_{m_e}(\varphi(x))]\,\bar{m}_e, \tag{89c}$$

$$m_q(x) = \left[1 + D_q(\varphi(x))\right]\bar{m}_q, \quad q = u, d\,, \tag{89d}$$

where barred quantities are background quantities that are independent of φ.

In the general relativistic limit ($\varphi \approx$ constant), the partition function of the theory from equation (78) becomes

$$Z_{\text{Ed}} = \int [\mathcal{D}g]\prod_j[\mathcal{D}f_j]\exp\left[\frac{\mathrm{i}}{\xi_F}\int \mathrm{d}^4_g x\left(\frac{f(\varphi)R}{2\kappa} - \frac{V(\varphi)}{2\kappa} + \mathcal{L}_m + \mathcal{L}_{\text{int}}\right)\right]. \tag{90}$$

Similarly to what is done in the previous section, comparing this equation with the corresponding one from the Core theory, see equation (2), one can clearly identify

$$\frac{\kappa}{f(\varphi(x))} = \frac{8\pi G(x)}{c^4}, \quad \text{and} \quad \xi_F = \hbar c\,. \tag{91}$$

For this reason, a measurement of the black-body radiation spectrum would be sensitive to $\hbar$ as explained in section 6.1 and in the beginning of this appendix. Let us insist that Einstein–dilaton theories have a richer phenomenology compared to generalized Brans–Dicke because some constants of Nature are now effectively varying, as shown in equations (89a).

[22] Let us note that these couplings are just non-linear generalizations of the couplings considered in [76].
[23] A review on varying constants is provided in [69].

In conclusion, in Einstein–dilaton theories, the Newton parameter and some non-universal parameters of the standard model of particles—such as the electromagnetic fine structure constant or the fermions' mass—can vary in space and time, while the Planck constant remains a fundamental constant of the theory.

A.4. Entangled Relativity

For completeness, we will now consider the case of Entangled Relativity (already presented in details in section 3), following the same methodology as the one used in the previous section.

The path integral of Entangled Relativity reads (see equation (6))

$$Z_{\mathrm{ER}} = \int [\mathcal{D}g] \prod_j [\mathcal{D}f_j] \exp\left[\frac{\mathrm{i}}{\xi_F}\int \mathrm{d}^4_g x \left(-\frac{1}{2}\frac{\mathcal{L}_m^2}{R}\right)\right]. \tag{92}$$

Since the integral in the quantum phase has the dimension of an energy squared, the quantum parameter ξ_F has the dimension of an energy squared. Similarly to the case of string theory discussed in section A.1, this quantum parameter cannot be identified to the Planck constant. As a consequence, the Planck constant is not a fundamental constant of Entangled Relativity.

As explained in section 3.2, when the Matter Lagrangian is non vanishing, the previous path integral equation (92) is equivalent to

$$Z_{\mathrm{ER}} = \int [\mathcal{D}g] \prod_j [\mathcal{D}f_j] \exp\left[\frac{\mathrm{i}}{\xi_F}\int \mathrm{d}^4_g x \frac{1}{\kappa}\left(\frac{R}{2\kappa} + \mathcal{L}_m\right)\right], \tag{93}$$

at least at the perturbative level, where κ is here a scalar field (which can be identified to $\kappa(x) = -\mathcal{L}_m/R$).

Similarly to what is done in the previous sections, comparing this equation with the corresponding one from the Core theory, see equation (2), one can clearly identify

$$\kappa(x) = \frac{8\pi G(x)}{c^4}, \quad \text{and} \quad \xi_F \kappa(x) = c\hbar(x), \tag{94}$$

which shows that both Newton's constant and the effective Planck constant depend directly on the local value of the scalar field $\kappa(x)$, and therefore can vary in space and/or time. For this reason, a measurement of the black-body radiation spectrum would be sensitive to $\hbar(x) = \xi_F/(c\kappa(x))$, and therefore could be used to detect a a variation of Planck constant.

In conclusion, in Entangled Relativity, Newton's constant and Planck's quantum of action are not fundamental constants of nature, but emerge in the low energy regime, similarly to the example with string theory described in section A.1.

A.5. Summary

In this appendix, we have explicitly derived in several theories which constants are varying in space and time, and which ones are fundamental constants, starting from first principle and from the expression of the theories in terms of their path integrals. To summarize:

- in the Core theory of physics, the fundamental constants are $\hbar$, c and G.
- in minimally coupled tensor–scalar theory (generalized Brans–Dicke theory), $\hbar$ and c are both fundamental constants of the theory, while Newton's parameter G can vary in space and time.
- in tensor–scalar theory with a non-minimal coupling between the scalar field and matter defined in equation (87), $\hbar$ and c are fundamental constants as well, but Newton's parameter G and other non-universal parameters of the standard model of particle physics like the fine-structure constant or the fermions' mass can vary in space and time.
- in string theory, the string length λ_s and c are the fundamental parameters of the theory. As a consequence, $\hbar$ and G can vary in space and time.
- the non-linearity of the action of Entangled Relativity—see equation (92)—implies directly that the quantum parameter ξ_F of the theory cannot be the usual quantum of action of Planck $\hbar$. As a result, ξ_F—which can also be written as ϵ^2 since ξ_F has the dimension of energy squared, and can be identified the reduced Planck energy, see section 4.2—and c are the fundamental parameters of the theory. As a consequence, both Newton's and Planck's parameters can vary in space and time.

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