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Complex Quantum Hydrodynamics in Momentum Space with Broken Time-Reversal Symmetry

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Abstract: Shortly after Schrödinger's wave mechanics in terms of complex wave functions was published, Madelung formulated this theory in terms of two real hydrodynamic-like equations. This version is also the formal basis of Bohmian mechanics, albeit with a different ontological interpretation. A point of criticism raised by Pauli against Bohmian mechanics is its missing symmetry between position and momentum that is present in classical phase space as well as in the quantum mechanical position and momentum representations. Both Madelung's quantum hydrodynamics formulation and Bohmian mechanics are usually expressed only in position space. Recently, with the use of complex quantities, we were able to provide a hydrodynamic formulation also in momentum space. In this paper, we extend this formalism to include dissipative systems with broken time-reversal symmetry. In classical Hamiltonian mechanics and conventional quantum mechanics, closed systems with reversible time-evolution are usually considered. Extending the discussion to include open systems with dissipation, another form of symmetry is broken, that under time-reversal. There are different ways of describing such systems; for instance, Langevin and Fokker–Planck-type equations are commonly used in classical physics. We now investigate how these aspects can be incorporated into our complex hydrodynamic description and what modifications occur in the corresponding equations, not only in position, but particularly in momentum space.

Keywords: complex quantum hydrodynamics; momentum space; broken time-reversal symmetry

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1. Introduction

In 1926, Schrödinger published his communications on wave mechanics [1,2] in terms of a linear differential equation for a complex wave function $\psi(\vec{r},t)$. Using a polar ansatz in terms of amplitude and phase of this complex function $\psi(\vec{r},t) = \sqrt{\rho(\vec{r},t)}\exp\left(\frac{\mathrm{i}}{\hbar}S(\vec{r},t)\right)$, already in the same year Madelung [3] found a reformulation in terms of two real equations that have close similarity with equations known from hydrodynamics. One is a continuity equation for the square of the amplitude, interpreted as a probability density $\rho(\vec{r},t) = \psi^*(\vec{r},t)\psi(\vec{r},t)$, containing a contribution from the phase via a velocity field in a convection current. The other is a modified Hamilton–Jacobi-type equation for the phase $S(\vec{r},t)$, containing a contribution from the amplitude via a so-called "quantum potential", V_{qu}, apparently representing a typical quantum mechanical contribution and providing another coupling between the two equations for amplitude and phase. Madelung considered V_{qu} as the origin of "internal forces" of the continuum. Around 25 years later, Bohm used the same hydrodynamic formulation for his approach to find a deterministic version of quantum mechanics in terms of hidden variables [4,5]. Although formally equivalent, the approaches are ontologically totally different. Bohm assumed V_{qu} to be a characteristic quantum mechanical potential that establishes a difference between the classical Hamilton–Jacobi equation and the quantum mechanical version of Madelung. (At first sight this

seems reasonable, as formally for $\hbar \to 0$ the expression for V_{qu} (see Section 2, Equation (5)) vanishes, but even for the harmonic oscillator this is not quite correct, as has been shown by Ling [6].) Concerning the hidden variable aspect, in Bohm's approach it is assumed that from the velocity field in the continuity equation via integration a so-called Bohmian trajectory can be obtained, that is interpreted as a real path in physical space taken by a really existing quantum particle. The results obtained from Bohm's version and those from the conventional formulation that is associated with the Copenhagen interpretation are both in agreement with experiment (although recently some experiments [7–9] claim to be able to exclude the existence of hidden variables), therefore, there seems to exist no clear experimental evidence to confirm which approach is the correct one.

However, there was already early criticism by Pauli [10], pointing out that Bohmian mechanics is only formulated in position space, therefore, the symmetry between position and momentum that exists in the classical phase space formulation is missing in Bohm's approach. In recent work [11], we were able to show that this symmetry can be achieved if the hydrodynamic formulation is generalized to complex quantities. On the other hand, it was also possible to show [12] that the Bohmian trajectories are not real trajectories in physical space, but have to be considered in a probabilistic context. Nevertheless, Bohmian mechanics can still provide interesting information for problems such as tunneling processes and particularly the numerical results can be quite useful.

In this paper, we do not want to enter any ontological discussions, but want to concentrate on the hydrodynamic form of the equations of motion, its extension to incorporate complex quantities, and its generalization to include open systems.

The symmetry between position and momentum, addressed by Pauli, is not the only relevant one in classical and quantum mechanics. Another important symmetry is the one under time-reversal (or, according to Wigner, motion-reversal). While the fundamental equations of classical and quantum mechanics are invariant under time-reversal, daily experience shows irreversible time-evolution and dissipation of energy (often combined, but not necessarily always). The question arises, how these phenomena can be incorporated into the classical and quantum mechanical description. In the conventional approach, the system of interest is coupled to an environment with a large number of degrees of freedom (in the limit, infinitely many) and the whole, system plus environment, is considered a closed Hamiltonian system [13]. This approach not only leads to extensive, cumbersome, and expensive calculations, but also faces additional problems. In one of the most commonly used methods, where the system is coupled to a bath of harmonic oscillators (often attributed to Caldeira and Legget [14,15], although other authors also followed this line of thinking), there can show up the problem that the corresponding density operator is no longer positive definite. This problem can be avoided using a mathematically motivated extension of the evolution operator that guarantees this positivity [16,17], but in this case, the physical meaning of the terms that are added is not uniquely determined, but "guided by intuition" [18]. An approach to minimize the number of external degrees of freedom has been used by Bateman [19], taking into account only one environmental degree of freedom that absorbs the energy dissipated by the system of interest. However, the physical meaning of this external degree of freedom is rather obscure.

Finally, there are the so-called effective methods, that do not take into account any environmental degrees of freedom but only consider the effect of the environment on the system of interest. Classical models such as the Langevin equation in the trajectory picture or, equivalently, Fokker–Planck-type equations in the picture of distribution functions serve as a basis for these modifications.

There are also approaches to this problem using non-canonical extensions of classical mechanics with subsequent canonical quantization [20,21], but in this paper we restrict our discussion to approaches where the canonical variables can be associated with the physical ones.

These effective approaches usually lead to nonlinear modifications of the time-dependent Schrödinger equation. In order to find the additional (nonlinear) terms describing the effect

of the environment, two different types of approaches are frequently taken. One is based on the Langevin equation, that contains a linear velocity dependent friction force. As criterion for the definition of the friction term in the Schrödinger equation, it is requested that the mean value of the negative gradient of the friction term W provides the friction force proportional to velocity or momentum, i.e., $\langle -\nabla W \rangle = -\gamma \langle \vec{p} \rangle = -m\gamma \langle \vec{v} \rangle$ (with $\langle \ldots \rangle = \int \mathrm{d}\vec{r}\psi^*(\vec{r},t) \ldots \psi(\vec{r},t)$). However, this is a very vague request that can be fulfilled by several different approaches; however, most of them lead to unphysical results [22,23]. A major problem of these attempts is caused by the fact that they only add a real contribution to the Hamiltonian, providing the requested friction force for the (then irreversible) Ehrenfest equation of motion, however, the imaginary part of the corresponding Schrödinger equation stays unchanged, thus still leading to the reversible continuity equation for the probability density. An additional imaginary part to the Hamiltonian was obtained by Gisin [24,25], using a modified derivation of a generalized master equation, but now for a pure state wave function instead of the density operator. Although the imaginary contribution leads to a non-Hermitian Hamiltonian, corresponding to a non-unitary time-evolution, normalizability of the corresponding wave functions can be achieved. However, this approach leads to a wrong expression for the energy dissipation.

Another alternative starts from the distribution function viewpoint, adding a time-symmetry-breaking diffusion term to the continuity equation, turning it into a Fokker–Planck-type equation, in position space called the Smoluchowski equation. This provides the irreversibility of the time-evolution, but the dissipative aspect is related to the phase of the wave function. To find the corresponding contribution, a separation of the Smoluchowski equation for $\rho(\vec{r},t) = \psi^*(\vec{r},t)\psi(\vec{r},t)$ into two complex conjugate equations for ψ and ψ^* is necessary. This cannot be achieved in general, but a specific separation condition does not only allow this separation, leading to a modified Schrödinger equation with complex logarithmic nonlinearity, but also provides the contribution from the phase that leads to the correct friction force. Moreover, unphysical results such as a wrong frequency for the damped harmonic oscillator or the violation of the uncertainty principle do not show up in this case. The inclusion of a diffusion term into the equation of motion for the probability density was later supported by Doebner and Goldin [26] using group theoretical arguments.

In the classical theory of Brownian motion, the interaction of the system with the environment is (artificially) divided into a dissipative friction force proportional to velocity and a stochastic force, vanishing on average. In the effective nonlinear quantum mechanical models the friction force is related to an additional real contribution to the Hamiltonian. The additional contribution from an imaginary term to the continuity equation, in our case the additional diffusion term in the Smoluchowski equation, should, therefore, be related to the stochastic force. Hence, our extension of the Madelung picture to include open systems could also be considered in the context of a recent stochastic quantum hydrodynamic model [27,28]. This model relates vacuum fluctuations to dark matter and, thus, provides even connections with cosmological models, thus extending substantially the domain of possible applicability of our approach.

Our nonlinear Schrödinger equation based on the separation of a Smoluchowski equation for the probability density and its relation to other nonlinear ones will be used to find modifications of our complex hydrodynamic approach to include effects due to the interaction with a dissipative environment in position as well as in momentum space.

To make the paper self-contained, in Section 2, the hydrodynamic version of quantum mechanics according to Madelung and our complex generalization are recapitulated in position space, as well as its specific form for Gaussian wave packets. An equivalent formulation in momentum space shows how the corresponding symmetry between position and momentum can be achieved.

The discussion of the irreversible, dissipative case starts in Section 3 in position space. The complex logarithmic effective interaction term provides contributions to the two hydrodynamic equations. These contributions are compared with those from two

similar approaches, one in terms of an anti-commutator of position and momentum and another in terms of a modified kinetic energy term. Both also possess exact Gaussian wave packet solutions and lead to the same physical results as the logarithmic nonlinear Schrödinger equation. The equivalence in terms of our complex hydrodynamic quantities will be explained in detail.

In Section 4, the discussion of irreversible, dissipative systems is extended to momentum space. In this case, the formulation in terms of the logarithm of the wave function does not provide the requested friction force in the expected way, however, the two other approaches present apparently reasonable alternatives. The consequences for the two hydrodynamic equations in momentum space are shown and discussed.

Finally, in the conclusions, open questions concerning the logarithmic approach and the two alternative approaches are addressed.

2. Complex Hydrodynamic Description in Position and Momentum Space

2.1. Position Space

We start our discussion with the time-dependent Schrödinger equation in position space, and restrict this discussion to one dimension (extension to higher dimensions is possible), which will be indicated by a subscript "x".

In position space, the physical position variable is represented by a c-number x, and the physical momentum variable p by a differential operator, $p_{\mathrm{op}} = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x}$. The corresponding Schrödinger equation is

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi_x(x,t) = \Big\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\Big\}\psi_x(x,t) = \mathrm{H_L}\psi_x(x,t) \tag{1}$$

where the potential $V(x)$, in principle, can also contain an explicit time-dependence (as in the case of the parametric oscillator with time-dependent frequency $\omega(t)$, leading to a Hamiltonian that is no longer a constant of motion).

With the polar ansatz,

$$\psi_x(x,t) = \sqrt{\rho_x(x,t)}\,\exp\big(\frac{\mathrm{i}}{\hbar}S_x(x,t)\big) \tag{2}$$

for the wave function, the complex Schrödinger Equation (1) can be rewritten in terms of two coupled real hydrodynamic-like equations, the continuity equation

$$\frac{\partial}{\partial t}\rho_x + \frac{\partial}{\partial x}\Big[\rho_x\frac{1}{m}\frac{\partial}{\partial x}S_x\Big] = 0 \tag{3}$$

and a modified Hamilton–Jacobi equation,

$$\frac{\partial}{\partial t}S_x + \frac{1}{2m}\Big(\frac{\partial}{\partial x}S_x\Big)^2 + V(x) + V_{\mathrm{qu,x}} = 0 \tag{4}$$

with the so-called quantum potential

$$V_{\mathrm{qu,x}} = -\frac{\hbar^2}{2m}\frac{\frac{\partial^2}{\partial x^2}\sqrt{\rho_x}}{\sqrt{\rho_x}}. \tag{5}$$

For the extension to our complex formulation, we define the complex quantities $\mathrm{F_c}$, corresponding to physical quantities F with the associated quantum mechanical operators $\mathrm{F_{op}}$ in the a-representation (in this paper, "a" is position "x" or momentum "p") according to

$$\mathrm{F_c} = \frac{\langle \mathrm{a}|\mathrm{F_{op}}|\psi(t)\rangle}{\langle \mathrm{a}|\psi(t)\rangle} = \mathrm{F_R} + \mathrm{iF_I}. \tag{6}$$

In position space, this means for the canonical variables

$$\mathrm{X_c} = x, \tag{7}$$

$$\mathrm{P_c} = \frac{\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x}\psi_x}{\psi_x} = \frac{\partial}{\partial x}S_x - \mathrm{i}\frac{\hbar}{2}\frac{\frac{\partial}{\partial x}\rho_x}{\rho_x} = \mathrm{P_R} + \mathrm{iP_I}, \tag{8}$$

where in general the mean value of the imaginary parts vanishes, $\langle \mathrm{F_I}\rangle = 0$, i.e., the mean value is completely determined by the real part, $\langle \mathrm{F_c}\rangle = \langle \mathrm{F_R}\rangle$. (It should be emphasized, that this is usually no longer the case if one considers powers of $\mathrm{F_c}$, so in general $\langle \mathrm{F_c^2}\rangle = \langle \mathrm{F_R^2} - \mathrm{F_I^2}\rangle + \mathrm{i}2\langle \mathrm{F_R F_I}\rangle$ also has contributions from the imaginary part $\mathrm{F_I}$ in the real part of $\langle \mathrm{F_c^2}\rangle$; an example is $V_{\mathrm{qu,x}}$, that entirely depends on $\mathrm{F_I}$).

In terms of the complex variables, the hydrodynamic equations can be rewritten as

$$\frac{\partial}{\partial t}\rho_x + \frac{\partial}{\partial x}\left[\rho_x \frac{1}{m}\mathrm{P_R}\right] = 0 \tag{9}$$

$$\frac{\partial}{\partial t}S_x + \frac{1}{2m}\mathrm{P_R^2} + V(x) + V_{\mathrm{qu,x}} = 0 \tag{10}$$

with the quantum potential in the form

$$V_{\mathrm{qu,\,x}} = -\frac{1}{2m}\left[\mathrm{P_I^2} - \hbar\frac{\partial}{\partial x}\mathrm{P_I}\right]. \tag{11}$$

Taking the spatial derivative of (10) leads to the Euler-type equation

$$\left(\frac{\partial}{\partial t} + \frac{1}{m}\mathrm{P_R}\frac{\partial}{\partial x}\right)\mathrm{P_R} = \frac{\mathrm{D}}{\mathrm{D}t}\mathrm{P_R} = -\frac{\partial}{\partial x}\left(V(x) + V_{\mathrm{qu,\,x}}\right), \tag{12}$$

i.e., a Newton-type equation for a co-moving observer with the substantial time-derivative $\frac{\mathrm{D}}{\mathrm{D}t}$.

The time-dependent Schrödinger Equation (1) possesses exact analytic solutions in the form of Gaussian wave packets if the potential is at most quadratic in the position variable, particularly for the harmonic oscillator, the parametric oscillator with time-dependent frequency $\omega = \omega(t)$ and, in the limit $\omega \longrightarrow 0$, the free motion.

In these cases, the wave packet can be written in the form

$$\psi_{\mathrm{WP,\,x}}(x,t) = N_x(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\left[\frac{m}{2}\mathcal{C}\tilde{x}^2 + \langle p\rangle\tilde{x} + K_x(t)\right]\right\} \tag{13}$$

with $\tilde{x} = x - \langle x\rangle = x - \eta(t)$, $\langle x\rangle = \int_{-\infty}^{+\infty}\mathrm{d}x\,\psi_x^* x\psi_x = \eta(t)$, $\langle p\rangle = m\dot{\eta}$.

Gaussian functions are completely determined by their maximum, here $\langle x\rangle = \eta(t)$, and their width $\sqrt{\langle\tilde{x}^2\rangle}$ with $\langle\tilde{x}^2\rangle = \langle x^2\rangle - \langle x\rangle^2$, and in our case by the time-evolution of these parameters. The width is connected with the complex coefficient $\mathcal{C}(t)$ of the quadratic term in the exponent of the Gaussian wave packet. Therefore, the dynamics of the wave packet is completely determined by the two equations of motion for $\eta(t)$ and $\mathcal{C}(t)$, that can be obtained by inserting the ansatz (13) into Equation (1), leading to

$$\ddot{\eta} + \omega^2\eta = 0, \tag{14}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C} + \mathcal{C}^2 + \omega^2 = 0. \tag{15}$$

Equation (15) for the complex quantity $\mathcal{C}(t)$ is a nonlinear Riccati equation. Riccati equations have the property that they can be linearized, in our case to a complex Newtonian equation formally equivalent to (14) (for further details, see, e.g., [23]). With $\mathcal{C} = \frac{\dot{\lambda}}{\lambda}$ this leads to

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\lambda + \omega^2\lambda = 0 \tag{16}$$

with complex λ.

Not only can the position uncertainty $\langle\widetilde{x}^2\rangle$ be expressed in terms of $\mathcal{C} = \mathcal{C}_\mathrm{R} + \mathrm{i}\mathcal{C}_\mathrm{I}$, but also the correlation of position and momentum uncertainties $\langle[\widetilde{x},\widetilde{p}]_+\rangle$ (with $[\quad,\quad]_+ =$ anti-commutator) and $\langle\widetilde{p}^2\rangle$.

For the following, we need

$$\mathcal{C}_\mathrm{R} = \frac{1}{2m}\frac{\langle[\widetilde{x},\widetilde{p}]_+\rangle}{\langle\widetilde{x}^2\rangle} = \frac{1}{2}\frac{\frac{\partial}{\partial t}\langle\widetilde{x}^2\rangle}{\langle\widetilde{x}^2\rangle}, \tag{17}$$

$$\mathcal{C}_\mathrm{I} = \frac{\hbar}{2m}\frac{1}{\langle\widetilde{x}^2\rangle}. \tag{18}$$

In terms of the complex Riccati variable $\mathcal{C}(t)$, the complex momentum can be expressed as

$$\mathrm{P_c} = m\mathcal{C}\widetilde{x} + \langle p\rangle = \widetilde{\mathrm{P}}_\mathrm{c} + \langle p\rangle = \mathrm{P_R} + \mathrm{iP_I} = m\mathcal{C}_\mathrm{R}\widetilde{x} + \langle p\rangle + \mathrm{i}m\mathcal{C}_\mathrm{I}\widetilde{x}. \tag{19}$$

This quantity will be useful for the description of open dissipative systems in position space, as will be discussed in Section 3

2.2. Momentum Space

In momentum space, the momentum operator is a c-number and the position operator is represented by the differential operator $x_\mathrm{op} = -\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial p}$. In order to allow comparison with position space, the oscillator potential proportional to x^2 is considered, equivalent to the kinetic energy, that is proportional to p^2. In addition, this choice also guarantees analytical solutions in the form of Gaussian wave packets in momentum space.

The corresponding Schrödinger equation in (again one-dimensional) momentum space is

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi_p(p,t) = \Big\{\frac{p^2}{2m} - \frac{m}{2}\omega^2\hbar^2\frac{\partial^2}{\partial p^2}\Big\}\psi_p(p,t), \tag{20}$$

where now the subscript "p" indicates quantities in momentum space. Using again a polar ansatz according to

$$\psi_p(p,t) = \sqrt{\rho_p(p,t)}\exp\big(\frac{\mathrm{i}}{\hbar}S_p(p,t)\big) \tag{21}$$

the two real hydrodynamic equations corresponding to the complex Equation (20) are

$$\frac{\partial}{\partial t}\rho_p + \frac{\partial}{\partial p}\Big[\rho_p\big(m\omega^2\frac{\partial}{\partial p}S_p\big)\Big] = 0, \tag{22}$$

$$\frac{\partial}{\partial t}S_p + \frac{p^2}{2m} + \frac{m}{2}\omega^2\Big[\big(\frac{\partial}{\partial p}S_p\big)^2 - \hbar^2\frac{\frac{\partial^2}{\partial p^2}\sqrt{\rho_p}}{\sqrt{\rho_p}}\Big] = 0. \tag{23}$$

In analogy to the quantum potential $V_{\mathrm{qu,\,x}}$ in position space, one could consider the term

$$V_\mathrm{qu,p} = -\frac{m}{2}\omega^2\hbar^2\frac{\frac{\partial^2}{\partial p^2}\sqrt{\rho_p}}{\sqrt{\rho_p}} \tag{24}$$

as a quantum potential $V_{\mathrm{qu,\,p}}$ in momentum space.

The complex canonical variables in momentum space are now given by

$$\mathrm{P_c} = p \tag{25}$$

$$\mathrm{X_c} = -\frac{\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial p}\psi_p}{\psi_p} = -\frac{\partial}{\partial p}S_p + \mathrm{i}\frac{\hbar}{2}\frac{\frac{\partial}{\partial p}\rho_p}{\rho_p} = \mathrm{X_R} + \mathrm{iX_I}, \tag{26}$$

again with $\langle\mathrm{X_I}\rangle = 0$, i.e., vanishing imaginary part, and $\langle\mathrm{X_c}\rangle = \langle\mathrm{X_R}\rangle$. In terms of the complex variable, Equations (22) and (23) can be rewritten as

$$\frac{\partial}{\partial t}\rho_p + \frac{\partial}{\partial p}\Big[\rho_p\big(-m\omega^2 \mathrm{X_R}\big)\Big] = 0, \tag{27}$$

$$\frac{\partial}{\partial t}S_p + \frac{p^2}{2m} + V(\mathrm{X_R}) + V_{\mathrm{qu,p}} = 0. \tag{28}$$

The potential term in Equation (28) can be written in terms of $\mathrm{X_R}$ as

$$V(\mathrm{X_R}) = \frac{m}{2}\omega^2 \mathrm{X_R^2}, \tag{29}$$

which allows us to express the convection term in momentum space in Equation (27) as a flux connected with a change in time of momentum according to

$$\dot{\mathrm{P}} = -m\omega^2 \mathrm{X_R} = -\frac{\partial}{\partial x}V(x)\big|_{x=\mathrm{X_R}}. \tag{30}$$

An attempt to obtain an Euler equation corresponding to Equation (12) does not make sense, as "x" is not an independent variable in momentum space, therefore, a derivative with respect to this variable always vanishes. However, we have seen in Equations (27) and (28), a replacement of $V(x)$ by $V(\mathrm{X_R})$ provides reasonable expressions. Therefore, looking for an Euler equation connected with Equation (28), one can at least consider the potential term, the force that can be derived from it in terms of $\mathrm{X_R}$, and the corresponding mean value.

In this sense, the negative derivative of $V(\mathrm{X_R})$ with respect to $\mathrm{X_R}$ provides, according to (30), a force $-\frac{\partial}{\partial \mathrm{X_R}}V(\mathrm{X_R}) = -m\omega^2\mathrm{X_R}$, and, since $\langle \mathrm{X_R}\rangle = \langle x\rangle$, considering the mean values this is identical with the classical force, i.e., $\langle -\frac{\partial}{\partial \mathrm{X_R}}V(\mathrm{X_R})\rangle = -m\omega^2\langle x\rangle$.

Furthermore, in momentum space, the time-dependent Schrödinger equation, here in the form of (20), has analytical solutions in the form of Gaussian wave packets. In this case, they can be formulated as

$$\psi_{\mathrm{WP,\,p}}(p,t) = N_p(t)\exp\Big\{-\frac{\mathrm{i}}{\hbar}\Big[\frac{m}{2}\Big(\frac{1}{\mathcal{C}}\Big)\frac{\widetilde{p}^2}{m^2} + \langle x\rangle\widetilde{p} + K_p(t)\Big]\Big\} \tag{31}$$

with $\widetilde{p} = p - \langle p\rangle = p - m\dot{\eta}$. The purely time-dependent terms $N_p(t)$ and $K_p(t)$ are, like their counterparts in position space, not relevant for the following discussion.

Inserting wave packet (31) into (20) provides the two equations that determine the time-evolution of the maximum and width of the Gaussian function (31),

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle p\rangle + m\omega^2\langle x\rangle = m\big(\ddot{\eta} + \omega^2\eta\big) = 0, \tag{32}$$

$$-\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{\mathcal{C}}\Big) + \omega^2\Big(\frac{1}{\mathcal{C}}\Big)^2 + 1 = 0. \tag{33}$$

The complex Riccati equation (33) for the quantity determining the width of the wave packet in momentum space can again be linearized via a logarithmic derivative, in this case $\big(\frac{1}{\mathcal{C}}\big) = \frac{\dot{\lambda}}{\lambda}$, leading to the same complex linear Newtonian equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\lambda + \omega^2\lambda = 0 \tag{34}$$

as in position space.

In momentum space, the real and imaginary parts of the inverse of $\mathcal{C}(t)$ are useful and can be expressed in terms of the uncertainties, in analogy to (17) and (18), as

$$\Big(\frac{1}{\mathcal{C}}\Big)_{\mathrm{R}} = \frac{m}{2}\frac{\langle[\widetilde{x},\widetilde{p}]_+\rangle}{\langle\widetilde{p}^2\rangle}, \tag{35}$$

$$\Big(\frac{1}{\mathcal{C}}\Big)_{\mathrm{I}} = -\frac{m\hbar}{2}\frac{1}{\langle\widetilde{p}^2\rangle}. \tag{36}$$

Note that $\left(\frac{1}{\mathcal{C}}\right)_{\mathrm{R}} \neq \frac{1}{\mathcal{C}_{\mathrm{R}}}$ and $\left(\frac{1}{\mathcal{C}}\right)_{\mathrm{I}} \neq \frac{1}{\mathcal{C}_{\mathrm{I}}}$!

In terms of the complex Riccati variable $\mathcal{C}(t)$ or its inverse, the complex position can be written as

$$\mathrm{X_c} = \frac{1}{\mathcal{C}}\frac{\widetilde{p}}{m} + \langle x\rangle = \widetilde{\mathrm{X}}_{\mathrm{c}} + \langle x\rangle = \mathrm{X_R} + \mathrm{i}\mathrm{X_I} = \left(\frac{1}{\mathcal{C}}\right)_{\mathrm{R}}\frac{\widetilde{p}}{m} + \langle x\rangle + \mathrm{i}\left(\frac{1}{\mathcal{C}}\right)_{\mathrm{I}}\frac{\widetilde{p}}{m}. \tag{37}$$

Next, it will be analyzed how this complex hydrodynamic description can be extended to also include environmental effects, leading to irreversible time-evolution and dissipation of energy in position as well as in momentum space.

3. Complex Quantum Hydrodynamics for Open Systems in Position Space

When talking about an open system, we mean a system of interest that interacts with some environment, also called a reservoir or heatbath. In principle, this is an isolated system if system plus reservoir and their corresponding degrees of freedom are all considered explicitly. Usually, the number of degrees of freedom is, due to the environmental ones, too large to be treated in detail (on the order of 10^{24}) and generally one is not even interested in the details of the environmental ones (as usually they are averaged out in the end). Therefore, it appears reasonable to look for alternative descriptions of the dynamics of the system of interest, where only the effect of the environment on the system enters, but no details of the interaction with the environmental degrees of freedom. This is already so in the classical case and leads in the trajectory picture to the Langevin equation that can be written in the form

$$\ddot{\eta} + \gamma\dot{\eta} + \omega^2\eta = f(t), \tag{38}$$

where here a harmonic force is considered, but any other conservative force is also possible. The term $-\gamma\dot{\eta}$, which does not occur in the Newtonian equation of motion (14), represents a damping friction force proportional to velocity $\dot{\eta}$ with friction coefficient γ and determines the long-term evolution of the system. Furthermore, $f(t)$ is proportional to a stochastic fluctuating force without analytic time-dependence, but with statistical properties, such as a vanishing mean value of $f(t)$. A physically equivalent description of this situation can also be given in terms of a one-body distribution function, leading to Fokker–Planck-type equations, in position space particularly called the Smoluchowski equation, where an additional diffusion term $-D\frac{\partial^2}{\partial x^2}\rho_x$ breaks the time-symmetry of the continuity Equation (3).

Corresponding quantum mechanical models use modifications of the (one-body) Schrödinger equation. There are different ways to reach this goal, but not all of them are successful. One group of approaches uses modifications of the classical Lagrange and Hamilton formalism that supply an equation of motion including the linear velocity dependent friction force contained in the Langevin equation. This goes beyond the usual classical formalism by involving non-canonical transformations of position and momentum variables. Subsequent canonical quantization leads to Hamiltonian operators that are still linear, but can be explicitly time-dependent. The most popular is the approach by Caldirola [20] and Kanai [21]. The classical non-canonical transformation, however, requires a quantum mechanical analogue, influencing the meaning of the wave function. If this aspect is not taken into account consistently, unphysical results such as the violation of the uncertainty principle arise. A proper treatment of these approaches is discussed in [23]. In the following, we want to avoid possible complications due to non-canonical transformations from the beginning and consider only approaches where physical variables correspond to canonical ones.

Approaches of this kind usually lead to modifications of the Schrödinger equation with additional nonlinear terms, $\mathrm{W}(x,t;\psi_x)$. Often these terms are chosen in a way that their mean value vanishes, which can be achievedf by changing to $\widetilde{\mathrm{W}} = \mathrm{W} - \langle \mathrm{W}\rangle$. This has two effects: (1) The mean value of the nonlinear Hamiltonian is still the mean value of the linear one, $\mathrm{H_L}$, i.e., the mean value of kinetic and potential energies, only now calculated with the solution ψ_{NL} of the nonlinear equation. (2) If W is complex or imaginary, the subtraction of the mean value $\langle \mathrm{W}\rangle$ still formally allows normalization of the solution ψ_{NL},

and, thus, the conventional probabilistic interpretation. Therefore, the general form of these modified Schrödinger equations is

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi_{\mathrm{NL}} = \left(\mathrm{H_L} + \widetilde{\mathrm{W}}\right)\psi_{\mathrm{NL}} \tag{39}$$

with $\langle\widetilde{\mathrm{W}}\rangle = 0$ and hence $\langle E\rangle_{\mathrm{NL}} = \langle \mathrm{H_L}\rangle_{\mathrm{NL}}$, where the subscript NL indicates that the mean values are calculated with ψ_{NL}, not ψ_{L}. The additional term $\widetilde{\mathrm{W}}$ in the Schrödinger equation leads to additional contributions to the hydrodynamic equations. The continuity Equation (3) turns into

$$\frac{\partial}{\partial t}\rho_x + \frac{\partial}{\partial x}\left[\rho_x\frac{1}{m}\frac{\partial}{\partial x}S_x\right] - \frac{2}{\hbar}\widetilde{\mathrm{W}}_{\mathrm{I}} = 0, \tag{40}$$

the modified Hamilton–Jacobi Equation (4) into

$$\frac{\partial}{\partial t}S_x + \frac{1}{2m}\left(\frac{\partial}{\partial x}S_x\right)^2 + V(x) + V_{\mathrm{qu,\,x}} + \widetilde{\mathrm{W}}_{\mathrm{R}} = 0, \tag{41}$$

where $\widetilde{\mathrm{W}} = \widetilde{\mathrm{W}}_{\mathrm{R}} + \mathrm{i}\widetilde{\mathrm{W}}_{\mathrm{I}}$.

There are different strategies to find a suitable "friction term" $\widetilde{\mathrm{W}}$. One of them is based on the trajectory picture and the corresponding Langevin equation, requesting, according to Ehrenfest, that the mean values fulfill the classical equation of motion including the friction force, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle p\rangle + m\omega^2\langle x\rangle + \gamma\langle p\rangle = 0 \tag{42}$$

with

$$\gamma\langle p\rangle = \langle\frac{\partial}{\partial x}\widetilde{\mathrm{W}}\rangle \tag{43}$$

(assuming that the fluctuating force also vanishes when the quantum mechanical mean value is taken).

However, the requirements (43) that have to be fulfilled by $\widetilde{\mathrm{W}}$ are so vague that many approaches can achieve this goal, but most of them show unphysical results in other respects. Amongst the most frequently used or cited ones are those of Kostin [29] with

$$\widetilde{\mathrm{W}}_{\mathrm{Kos}} = \frac{\gamma}{2}\frac{\hbar}{\mathrm{i}}\left(\ln\frac{\psi}{\psi^*} - \langle\ln\ln\frac{\psi}{\psi^*}\rangle\right) \tag{44}$$

and Albrecht [30] with

$$\widetilde{\mathrm{W}}_{\mathrm{Al}} = \gamma\langle p\rangle(x - \langle x\rangle), \tag{45}$$

but both provide the wrong frequency for the damped harmonic oscillator, even worse, since they are real terms, the hydrodynamic equation for the density of this irreversible process is still the reversible continuity equation.

An improvement concerning the latter criticism had been achieved by Süssmann [31] using

$$\widetilde{\mathrm{W}}_{\mathrm{Sü}} = \frac{\gamma}{2}\Big[(x - \langle x\rangle), p\Big]_+ \tag{46}$$

in terms of the anti-commutator $[\quad,\quad]_+$.

This term provides a symmetry-breaking contribution to the continuity equation but still has the wrong frequency for the damped oscillator.

This problem was solved by Hasse [22], combining the approaches of Albrecht and Süssmann to

$$\widetilde{\mathrm{W}}_{\mathrm{Has}} = \frac{\gamma}{4}\Big[(x - \langle x\rangle), (p + \langle p\rangle)\Big]_+ = \frac{\gamma}{4}[\widetilde{x}, \widetilde{p}]_+ + \frac{\gamma}{2}\langle p\rangle\widetilde{x} = \frac{1}{2}\left(\widetilde{\mathrm{W}}_{\mathrm{Al}} + \widetilde{\mathrm{W}}_{\mathrm{Sü}}\right). \tag{47}$$

The only remaining problem is that $\langle \mathrm{W_{Has}} \rangle \neq 0$ is possible, but since $\langle \mathrm{W_{Has}} \rangle$ does not affect the equations of motion for the maximum and width of the wave packet solutions, this problem can be resolved, as will be shown below.

A different kind of approach is based on the distribution function picture, here particularly on breaking the time-reversal symmetry of the continuity equation by adding a diffusion term. This guarantees irreversibility for this aspect, but does not yet provide the dissipative friction term in the Ehrenfest equation that is related to the phase of the wave function. In order to obtain this information, the equation for the density $\rho_x(x,t)$ has to be separated into two conjugate complex equations, one for $\psi_{\mathrm{NL}}(x,t)$, the other for $\psi^*_{\mathrm{NL}}(x,t)$. Without the diffusion term, this separation is possible, as has been shown by Madelung [32] and Mrowka [33] in their method to derive the Schrödinger equation without making use of the Lagrange/Hamilton formalism. An attempt to follow their method is not successful in general, as the diffusion term does not allow this separation. However, one can analyze if there are particular conditions, especially having Gaussian solutions in mind, where such a separation is still possible.

It was shown in [34] that such a separation condition for the Smoluchowski equation is given by

$$-D_x \frac{\frac{\partial^2}{\partial x^2}\rho_x}{\rho_x} = \gamma\big(\ln \rho_x - \langle \ln \rho_x \rangle\big), \tag{48}$$

leading to a nonlinear Schrödinger equation with complex logarithmic nonlinearity (the logarithmic expression on the right-hand side of Equation (48) shows similarities with definitions used for quantum entropy, a point that needs further investigation),

$$\mathrm{i}\hbar \frac{\partial}{\partial t}\psi_x(x,t) = \Big\{ -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) + \gamma\big(\frac{\hbar}{\mathrm{i}} \ln \psi_x - \langle \frac{\hbar}{\mathrm{i}} \ln \psi_x \rangle\big)\Big\}\psi_x(x,t). \tag{49}$$

The complex nonlinear term can be written in terms of real and imaginary parts as

$$\widetilde{\mathrm{W}}_{\mathrm{SCH}} = \frac{\gamma}{2}\frac{\hbar}{\mathrm{i}}\Big(\ln \frac{\psi}{\psi^*} - \big\langle \ln \frac{\psi}{\psi^*} \big\rangle\Big) + \frac{\gamma}{2}\frac{\hbar}{\mathrm{i}}\Big(\ln(\psi\psi^*) - \langle \ln(\psi\psi^*) \rangle\Big) = \widetilde{\mathrm{W}}_{\mathrm{SCH,\,R}} + \mathrm{i}\widetilde{\mathrm{W}}_{\mathrm{SCH,\,I}}, \tag{50}$$

where the real part $\widetilde{\mathrm{W}}_{\mathrm{SCH,\,R}}$ is identical with Kostin's term (44), the imaginary part $\mathrm{i}\widetilde{\mathrm{W}}_{\mathrm{SCH,\,I}}$, corresponding to the diffusion term according to (48), solves all the problems of Kostin's approach.

The connection between the imaginary part and the equation for the density is clear, the contribution to the modified Hamilton–Jacobi equation results from the real part, $\widetilde{\mathrm{W}}_{\mathrm{SCH,\,R}}$, according to (41). Using the polar form of the wave function, this leads to

$$\frac{\partial}{\partial t}S_x + \frac{1}{2m}\big(\frac{\partial}{\partial x}S_x\big)^2 + V(x) + V_{\mathrm{qu,\,x}} + \gamma\big(S_x - \langle S_x \rangle\big) = 0 \tag{51}$$

Taking the spatial derivative $\frac{\partial}{\partial x}$ of Equation (51) and using our complex hydrodynamic notation with $\frac{\partial}{\partial x}S_x = \mathrm{P_R}$, one obtains

$$\Big(\frac{\partial}{\partial t} + \frac{1}{m}\mathrm{P_R}\frac{\partial}{\partial x}\Big)\mathrm{P_R} = \frac{\mathrm{D}}{\mathrm{D}t}\mathrm{P_R} = -\frac{\partial}{\partial x}\big(V(x) + V_{\mathrm{qu,\,x}}\big) + \gamma \mathrm{P_R} \tag{52}$$

with $-\frac{\partial}{\partial x}\widetilde{\mathrm{W}}_{\mathrm{R}} = -\gamma \mathrm{P_R}$ and $\langle -\frac{\partial}{\partial x}\widetilde{\mathrm{W}}_{\mathrm{R}} \rangle = -\gamma\langle p \rangle$.

The requirement according to Ehrenfest is also fulfilled for the wave packet, which can be shown by inserting ansatz (13) into the nonlinear Schrödinger Equation (49), leading for the evolution of the maximum to a Newtonian equation including the friction term,

$$\ddot{\eta} + \gamma\dot{\eta} + \omega^2\eta = 0. \tag{53}$$

The complex Riccati equation for $\mathcal{C}(t)$, determining the evolution of the wave packet width, is modified by an additional linear term according to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C} + \gamma\mathcal{C} + \mathcal{C}^2 + \omega^2 = 0. \tag{54}$$

Using again a logarithmic derivative, i.e., $\mathcal{C} = \frac{\dot{\widetilde{\lambda}}}{\widetilde{\lambda}}$ with $\widetilde{\lambda} = \lambda \exp(\frac{\lambda}{2}t)$, (54) can be linearized to

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\widetilde{\lambda} + \gamma \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\lambda} + \omega^2 \widetilde{\lambda} = 0, \tag{55}$$

i.e., the same Newtonian equation as (53), but now for a complex quantity.

The connection between $\mathcal{C}_\mathrm{R}$, $\mathcal{C}_\mathrm{I}$, and the uncertainties (see Equations (17) and (18)) remains unchanged.

The only known analytical solutions of the time-dependent Schrödinger equation are Gaussian wave packets. These solutions exist for potentials that are at most quadratic in the position variable. (An exception is the motion in a magnetic field, where the appearing terms can also be restricted to fulfill this requirement.) The same also applies to the nonlinear modifications of the time-dependent Schrödinger equation. Although all the modifications discussed so far possess this type of solutions, not all of them provide physically sound results. Our approach, leading to a complex logarithmic nonlinearity starting from a Fokker–Planck-type equation for the density, shows none of the shortcomings of other approaches. It not only breaks the time-reversal symmetry of the density equation via introduction of a diffusion term, but it also leads via our separation condition to an averaged Langevin-type equation with a time-symmetry-breaking friction force and the correct reduced frequency $\Omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$ for the damped harmonic oscillator. In addition, the (complex) Riccati equation determining the evolution of the wave packet width is modified by a linear term which, after linearization of the Riccati equation, leads to a (complex) friction term.

Starting from the Langevin picture, the only approach leading to the same results is the one of Hasse, however, with the (small) disadvantage that its mean value does not always vanish. This shortcoming can be cured by subtracting the mean value of Hasse's friction term W_Has, which does not affect the other above-mentioned results, i.e., one obtains

$$\widetilde{\mathrm{W}}_\mathrm{Has} = \mathrm{W}_\mathrm{Has} - \langle \mathrm{W}_\mathrm{Has} \rangle. \tag{56}$$

For Gaussian wave packets, this leads to identical results with $\widetilde{\mathrm{W}}_\mathrm{SCH}$, i.e.,

$$\widetilde{\mathrm{W}}_\mathrm{Has}\psi_{\mathrm{WP,\,x}}(x,t) = \widetilde{\mathrm{W}}_\mathrm{SCH}\psi_{\mathrm{WP,\,x}}(x,t). \tag{57}$$

In the classical case, the rate of energy dissipation due to the friction force only depends on the kinetic energy T of the system, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}E = \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{T} + V) = -2\gamma \mathrm{T}, \tag{58}$$

therefore, it seems reasonable that the form of "the friction term should not explicitly depend on the potential of the problem but somehow be connected with the kinetic energy of the system" [23].

While "in the derivation of the friction term in position space" in our approach "the comparison of the diffusion term proportional to $\frac{\partial^2}{\partial x^2}\rho_x$ with the logarithm $\ln \rho_x$ was essential, now the procedure on the level of the complex function ψ_x can be reversed" [23]. That means, $\ln \psi_x(x,t)$ has to be expressed in terms of $\frac{\partial^2}{\partial x^2}\psi_x(x,t)$, "or, more precisely, $\widetilde{\mathrm{W}}_\mathrm{SCH}$ in terms of the kinetic energy term, which shall be written as $\widetilde{\mathrm{W}}_\mathrm{D}$. Requiring that the condition

$$\widetilde{\mathrm{W}}_\mathrm{SCH}\psi_{\mathrm{WP,\,x}} = \widetilde{\mathrm{W}}_\mathrm{D}\psi_{\mathrm{WP,\,x}} \tag{59}$$

must be fulfilled, it is possible to express $\widetilde{\mathrm{W}}_\mathrm{SCH}$", [23] with the help of $\mathcal{C}$, in terms of the kinetic energy (and its mean value) as

$$\widetilde{\mathrm{W}}_\mathrm{D} = \frac{\gamma}{\mathcal{C}}\frac{1}{2m}\left(p_\mathrm{op}^2 - \langle p_\mathrm{op}^2 \rangle\right) = \frac{\gamma}{\mathcal{C}}\left(\frac{\widetilde{p}_\mathrm{op}^2}{2m} - \left\langle \frac{\widetilde{p}_\mathrm{op}^2}{2m} \right\rangle\right) + \frac{\gamma}{\mathcal{C}}\langle p \rangle \frac{\widetilde{p}_\mathrm{op}}{m}, \tag{60}$$

$\widetilde{p}_{\mathrm{op}} = p_{\mathrm{op}} - \langle p \rangle$.

This form of the friction term turned out to be valid in position and momentum space [35] and will, therefore, also be useful in the next section. In position space it can be written in the form

$$\widetilde{\mathrm{W}}_{\mathrm{D}} = \frac{\gamma}{\mathcal{C}}\Big(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \Big\langle -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Big\rangle\Big). \tag{61}$$

Since all three versions of the friction term lead to the same results for the Gaussian wave packet solutions, we want to check if a unified formulation in terms of our complex hydrodynamic quantities is possible, i.e., is there a common way to formulate $\widetilde{\mathrm{W}}_{\mathrm{c,\,x}} = \frac{\widetilde{\mathrm{W}}_{\mathrm{x}}\psi_{\mathrm{WP,\,x}}}{\psi_{WP,x}}$ in terms of the relevant complex variable, in position space $\mathrm{P_c} = \widetilde{\mathrm{P}}_{\mathrm{c}} + \langle p \rangle = m\mathcal{C}\widetilde{x} + \langle p \rangle$ (the subscript "c" will be skipped in the following, only "x" remains to indicate that we consider position space). From the logarithm of the wave packet $\psi_{\mathrm{WP,\,x}}(x,t)$, Equation (13), follows

$$\widetilde{\mathrm{W}}_{\mathrm{SCH,\,x}} = \gamma\mathcal{C}\frac{m}{2}\big(\widetilde{x}^2 - \langle\widetilde{x}^2\rangle\big) + \gamma\langle p\rangle\widetilde{x} \tag{62}$$

which can be rewritten, using $\widetilde{x} = \frac{1}{m\mathcal{C}}\widetilde{\mathrm{P}}_{\mathrm{c}}$, in terms of $\widetilde{\mathrm{P}}_{\mathrm{c}}$ as

$$\widetilde{\mathrm{W}}_{\mathrm{SCH,\,x}} = \frac{\gamma}{\mathcal{C}}\Big[\frac{1}{2m}\big(\widetilde{\mathrm{P}}_{\mathrm{c}}^2 - \langle\widetilde{\mathrm{P}}_{\mathrm{c}}^2\rangle\big)\Big] + \gamma\langle p\rangle\widetilde{x}. \tag{63}$$

Looking at the modified Hasse term and replacing p by p_{op}, the complex formulation leads to

$$\begin{aligned}\widetilde{\mathrm{W}}_{\mathrm{Has,\,x}} &= \frac{\gamma}{4}\Big(\big[\widetilde{x},\widetilde{\mathrm{P}}_{\mathrm{c}}\big]_+ - \big\langle\big[\widetilde{x},\widetilde{\mathrm{P}}_{\mathrm{c}}\big]_+\big\rangle\Big) + \gamma\langle\mathrm{P_c}\rangle\widehat{x} \\ &= \frac{\gamma}{\mathcal{C}}\Big[\frac{1}{2m}\big(\widetilde{\mathrm{P}}_{\mathrm{c}}^2 - \langle\widetilde{\mathrm{P}}_{\mathrm{c}}^2\rangle\big)\Big] + \gamma\langle p\rangle\widetilde{x},\end{aligned} \tag{64}$$

where $\langle\mathrm{P_c}\rangle = \langle p\rangle$ was used.

Finally, using $p_{\mathrm{op}} = \widetilde{p}_{\mathrm{op}} + \langle p\rangle$, $\widetilde{\mathrm{W}}_{\mathrm{D,\,x}}$ attains the form

$$\widetilde{\mathrm{W}}_{\mathrm{D,\,x}} = \frac{\gamma}{\mathcal{C}}\frac{1}{2m}\Big[\big(\widetilde{\mathrm{P}}_{\mathrm{c}}^2 - \langle\widetilde{\mathrm{P}}_{\mathrm{c}}^2\rangle\big) + 2\langle p\rangle\widetilde{\mathrm{P}}_{\mathrm{c}}\Big] \tag{65}$$

which again turns into (63) for $\widetilde{\mathrm{P}}_{\mathrm{c}} = m\mathcal{C}\widetilde{x}$.

Therefore, all three formulations can be expressed in the same form (63), using our complex quantity $\mathrm{P_c}$, or $\widetilde{\mathrm{P}}_{\mathrm{c}} = \mathrm{P_c} - \langle p\rangle$, respectively.

The friction force, obtained by taking the negative spatial derivative of $\widetilde{\mathrm{W}}_{\mathrm{c}}$ then turns out to be

$$-\frac{\partial}{\partial x}\widetilde{\mathrm{W}}_{\mathrm{c}} = -\big(\gamma m\mathcal{C}\widetilde{x} + \gamma\langle p\rangle\big) = -\gamma\mathrm{P_c} \tag{66}$$

where the first term is related to the uncertainties via $\mathcal{C}$ and vanishes on average, just like the stochastic fluctuating force in the Langevin equation. The second term yields via

$$-\gamma\langle\mathrm{P_c}\rangle = -\gamma\langle p\rangle \tag{67}$$

the linear velocity or momentum dependent friction force.

In the next section, how far this description of dissipative systems also holds in momentum space will be analyzed.

4. Complex Quantum Hydrodynamics for Open Systems in Momentum Space

In position space, a conservative force can be derived from a potential $V(x)$ by taking its negative derivative with respect to space, $-\frac{\partial}{\partial x}V(x)$. In certain approaches it was assumed that the friction force proportional to velocity or momentum can be obtained in a similar way by taking the spatial derivative of a kind of "friction potential" $\widetilde{\mathrm{W}}$. In classical

Hamilton–Jacobi theory, the momentum is the spatial derivative of the action function S. In Schrödinger's wave mechanics, he introduced his wave function $\psi(x,t)$ via the action according to [1]

$$S = \frac{\hbar}{\mathrm{i}} \ln \psi(x,t) \tag{68}$$

(his first assumption, the coefficient of the logarithm would be $\hbar$, he later on corrected to $\frac{\hbar}{\mathrm{i}}$ when he had to accept that ψ is complex.). Therefore, our friction term in Equation (49), together with Schrödinger's definition (68), leads to the desired result for the friction force. In momentum space, the situation is different. Since now momentum is the independent variable, a derivative with respect to the position x of an action depending on p in the quantum mechanical case would make no sense (a derivative with respect to p would provide a positive "force" proportional to x).

One could still think about a derivative with respect to $\mathrm{X_R}$, as in the case of the conservative potential force in momentum space, but looking at the definition (31) of $\psi_{\mathrm{WP,\,p}}(p,t)$ already shows that a resulting force would have the wrong, positive sign. Therefore, definitions of the friction term using the action via $\ln \psi_p$ seem to be problematic in momentum space.

There still remain the two other formulations in terms of x and p and the corresponding operators and mean values, that are in position space equivalent to the formulation using the action, and to each other.

In momentum space, the position is represented by the differential operator $\mathrm{X_{op}} = -\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial p}$. Considering the modification (56) of Hasse's friction term, one replaces x by this differential operator and uses the definition of our complex hydrodynamic quantities, then $\widetilde{x}$ can be replaced by $\widetilde{\mathrm{X}}_\mathrm{c}$, leading to the modified Hasse friction term in the form

$$\begin{aligned} \widetilde{\mathrm{W}}_{\mathrm{Has,\,p}} &= \frac{\gamma}{2}\left(\widetilde{p}\widetilde{\mathrm{X}}_\mathrm{c} - \langle \widetilde{p}\widetilde{\mathrm{X}}_\mathrm{c}\rangle\right) + \gamma\langle p\rangle\widetilde{\mathrm{X}}_\mathrm{c} \\ &= \frac{\gamma}{2}\left(\widetilde{p}\widetilde{\mathrm{X}}_\mathrm{R} - \langle \widetilde{p}\widetilde{\mathrm{X}}_\mathrm{R}\rangle\right) + \gamma\langle p\rangle\widetilde{\mathrm{X}}_\mathrm{R} \\ &+ \mathrm{i}\frac{\gamma}{2}\left(\widetilde{p}\mathrm{X}_\mathrm{I} - \langle \widetilde{p}\mathrm{X}_\mathrm{I}\rangle\right) + \mathrm{i}\gamma\langle p\rangle\mathrm{X}_\mathrm{I} \end{aligned} \tag{69}$$

where $\mathrm{X_I} = \widetilde{\mathrm{X}}_\mathrm{I}$, as $\langle \mathrm{X_I}\rangle = 0$ due to its definition (see Equation (26)). Furthermore, in this form, a derivative $\frac{\partial}{\partial x}$ does not make sense, however, as in the case of the conservative potential, a derivative $\frac{\partial}{\partial \mathrm{X_R}} = \frac{\partial}{\partial \widetilde{\mathrm{X}}_\mathrm{R}}$ is possible, leading to

$$-\frac{\partial}{\partial \mathrm{X_R}}\widetilde{\mathrm{W}}_{\mathrm{Has,\,p}} = -\frac{\gamma}{2}\widetilde{p} - \gamma\langle p\rangle \quad \text{with} \quad \left\langle -\frac{\partial}{\partial \mathrm{X_R}}\widetilde{\mathrm{W}}_{\mathrm{Has,\,p}}\right\rangle = -\gamma\langle p\rangle. \tag{70}$$

The mean value of the first term of the force vanishes, like the stochastic fluctuating force in the Langevin equation, the second term is identical with the mean value of the force and represents the classical friction force linear in momentum.

The friction term (69) causes additional terms in the hydrodynamic equations. For the modified Hamilton–Jacobi equation, this leads to

$$\frac{\partial}{\partial t}S_p + \frac{p^2}{2m} + \frac{m}{2}\omega^2\mathrm{X_R^2} + V_{\mathrm{qu,\,p}} + \gamma\left(\frac{1}{2}\left[\widetilde{p}\widetilde{\mathrm{X}}_\mathrm{R} - \langle \widetilde{p}\widetilde{\mathrm{X}}_\mathrm{R}\rangle\right] + \langle p\rangle\widetilde{\mathrm{X}}_\mathrm{R}\right) \tag{71}$$

with a conservative potential term $V(\mathrm{X_R}) = \frac{m}{2}\omega^2\mathrm{X_R^2}$, depending on the real part $\mathrm{X_R}$ of $\mathrm{X_c}$. A negative derivative with respect to $\mathrm{X_R}$ provides

$$-\frac{\partial}{\partial \mathrm{X_R}}\left(V(\mathrm{X_R}) + \widetilde{\mathrm{W}}_{\mathrm{Has,\,p,\,R}}\right) = -m\omega^2\mathrm{X_R} - \gamma\left(\frac{1}{2}\widetilde{p} + \langle p\rangle\right) \tag{72}$$

which supplies on average the correct expressions for the classical conservative and dissipative forces,

$$\Big\langle -\frac{\partial}{\partial \mathrm{X_R}}\big(V(\mathrm{X_R})+\widetilde{\mathrm{W}}_{\mathrm{Has,\,p,\,R}}\big)\Big\rangle = -m\omega^2\langle x\rangle - \gamma\langle p\rangle. \tag{73}$$

The continuity equation for ρ_p gains, according to (40), an additional term $-\frac{2}{\hbar}\widetilde{\mathrm{W}}_{\mathrm{I}}\rho_p$, where $\widetilde{\mathrm{W}}_{\mathrm{I}}$ can be obtained from $\widetilde{\mathrm{W}}_{\mathrm{Has,\,p}}$ in our complex formulation as

$$\widetilde{\mathrm{W}}_{\mathrm{Has,\,p,\,I}} = \frac{\gamma}{2}\frac{\hbar}{2}\Big[\widetilde{p}\frac{\frac{\partial}{\partial p}\rho_p}{\rho_p} - \Big\langle\widetilde{p}\frac{\frac{\partial}{\partial p}\rho_p}{\rho_p}\Big\rangle\Big] + \frac{\hbar}{2}\frac{1}{\rho_p}\big(\gamma\langle p\rangle\frac{\partial}{\partial p}\rho_p\big) \tag{74}$$

leading to

$$-\frac{2}{\hbar}\widetilde{\mathrm{W}}_{\mathrm{Has,\,p,\,I}}\rho_p = -\gamma\langle p\rangle\frac{\partial}{\partial p}\rho_p - \frac{\gamma}{2}\rho_p\Big(\widetilde{p}\frac{\frac{\partial}{\partial p}\rho_p}{\rho_p} - \Big\langle\widetilde{p}\frac{\frac{\partial}{\partial p}\rho_p}{\rho_p}\Big\rangle\Big). \tag{75}$$

The first term on the right-hand side supplies an additional friction contribution to the drift term, i.e., it changes the conservative contribution $\dot{\mathrm{P}} = -m\omega^2\mathrm{X_R}$ (see Equation (30)) into

$$\dot{\mathrm{P}} = -m\omega^2\mathrm{X_R} - \gamma\langle p\rangle. \tag{76}$$

For the Gaussian wave packets that we consider, the second term in (75) can be related to a diffusion term, using $\widetilde{p}\frac{\frac{\partial}{\partial p}\rho_p}{\rho_p} = -\frac{\widetilde{p}^2}{\langle\widetilde{p}^2\rangle}$ and $\Big\langle\widetilde{p}\frac{\frac{\partial}{\partial p}\rho_p}{\rho_p}\Big\rangle = -1$, leading to the modified (Fokker–Planck-type) equation for the density $\rho_p(p,t)$,

$$\frac{\partial}{\partial t}\rho_p + \frac{\partial}{\partial p}\Big[\rho_p\big(-m\omega^2\mathrm{X_R} - \gamma\langle p\rangle\big)\Big] + \frac{\gamma}{2}\langle\widetilde{p}^2\rangle\frac{\partial^2}{\partial p^2}\rho_p = 0. \tag{77}$$

The drift term contains (at least on average) the correct contribution from the force, the diffusion term is analogous to the one in position space, only the diffusion coefficient $D_x = \frac{\gamma}{2}\langle\widetilde{x}^2\rangle$ is replaced by $D_p = \frac{\gamma}{2}\langle\widetilde{p}^2\rangle$; however, the sign of the diffusion term is different from the expected one. The reason for this needs further investigation.

In position space, for Gaussian wave packets the friction term using the modified kinetic energy operator provides the same results as the modified Hasse term $\widetilde{\mathrm{W}}_{\mathrm{Has}}$. The connection between these two approaches in momentum space can be shown, using again our complex hydrodynamic quantities.

In momentum space, p_{op} is just the c-number p and $\frac{1}{m\mathcal{C}}\widetilde{p}_{\mathrm{op}}$ turns into $\frac{1}{m\mathcal{C}}\widetilde{p} = \widetilde{\mathrm{X}}_{\mathrm{c}}$. Therefore, the friction term (60) can be written as

$$\begin{aligned}\widetilde{\mathrm{W}}_{\mathrm{D,\,p}} &= \frac{\gamma}{\mathcal{C}}\Big[\frac{1}{2m}\big(\widetilde{p}^2 - \langle\widetilde{p}^2\rangle\big)\Big] + \gamma\langle p\rangle\frac{1}{m\mathcal{C}}\widetilde{p} \\ &= \frac{\gamma}{2}\big(\widetilde{p}\widetilde{\mathrm{X}}_{\mathrm{c}} - \langle\widetilde{p}\widetilde{\mathrm{X}}_{\mathrm{c}}\rangle\big) + \gamma\langle p\rangle\widetilde{\mathrm{X}}_{\mathrm{c}} = \widetilde{\mathrm{W}}_{\mathrm{Has,\,p}} = \widetilde{\mathrm{W}}_{\mathrm{c,\,p}}.\end{aligned} \tag{78}$$

Therefore, these two formulations of the friction term are equivalent in momentum space, at least for Gaussian wave packets. Using these terms in the modified time-dependent Schrödinger equation in momentum space leads to the correct equations of motion for the maximum and width of the Gaussian wave packet. The resulting equations of motion for the maximum and width are then

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle p\rangle + \gamma\langle p\rangle + m\omega^2\langle x\rangle = m\big(\ddot{\eta} + \gamma\dot{\eta} + \omega^2\eta\big) = 0 \tag{79}$$

and

$$-\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{\mathcal{C}}\Big) + \gamma\Big(\frac{1}{\mathcal{C}}\Big) + \omega^2\Big(\frac{1}{\mathcal{C}}\Big)^2 + 1 = 0. \tag{80}$$

The equation for the maximum is obviously correct. Equation (80), determining the time-evolution of the uncertainties, can be linearized to the same equation as in position space using $\left(\frac{1}{\mathcal{C}}\right) = \frac{\widetilde{\lambda}}{\widetilde{\lambda}}$, i.e.,

$$\ddot{\widetilde{\lambda}} + \gamma\dot{\widetilde{\lambda}} + \omega^2\widetilde{\lambda} = 0. \tag{81}$$

As the time-evolution of the uncertainties is independent of the representation in which it is formulated, and both Riccati equations can be linearized to the same complex Newtonian equation (81), the time-evolution of the uncertainties in momentum space derived from $\widetilde{W}_{\mathrm{Has,\,p}}$ and $\widetilde{W}_{\mathrm{D,\,p}}$ should be correct.

5. Conclusions

In Madelung's hydrodynamic formulation of quantum mechanics, one complex equation, Schrödinger's wave equation, is replaced by two real ones with formal similarity to hydrodynamic equations, a continuity equation and a modified Hamilton–Jacobi equation with an additional term depending on the amplitude of the wave function. Taking the spatial derivative of this equation provides an Euler-type equation, i.e., a Newton-type equation with a substantial time-derivative, describing the evolution for a co-moving observer, including a contribution from the additional (quantum potential) term.

An extension to momentum space is possible in terms of complex quantities, obtained by application of the quantum operators on the complex wave function in momentum space. Furthermore, also in momentum space, the complex wave equation can be replaced by two hydrodynamic equations, again a continuity equation and a modified Hamilton–Jacobi equation.

The continuity equations in both spaces contain a drift term represented by the application of the derivative of a flux term with respect to the corresponding independent variable. The flux term is given by the product of the time-derivative of the relevant variable and the probability density in the respective space. In position space, this time-derivative is expressed in terms of P_R, the real part of the complex hydrodynamic momentum variable P_c in position space; in momentum space in terms of X_R, the corresponding real part of the complex variable X_c in momentum space. The mean values of the expressions representing the time-derivatives correspond to the classical expressions in the respective flux terms in classical phase space.

The interaction with an environment causes irreversibility of the dynamics, which can be taken into account via an additional diffusion term in the continuity equation, and dissipation of energy, which can be taken into account via a friction force proportional to the velocity or momentum, which can be taken into account by an additional term in the modified Hamilton–Jacobi equation that leads to this friction force in the corresponding Euler equation.

In position space, three possible formulations for (nonlinear) friction terms were discussed, all providing the same equations of motion for the mean values (including the friction force) and, thus, for the maximum of the analytic Gaussian wave packet solution, as well as a modified complex Riccati equation that determines the time-evolution of the wave packet width. These terms not only provide the same (correct) equations of motion for the maximum and width of the wave packet solutions, but even more generally, can be expressed in equivalent form in terms of the complex hydrodynamic quantities in position as well as (partially) in momentum space.

This can be shown in position space by replacing $\widetilde{x}$ with the complex conjugate variable via $\widetilde{x} = \frac{1}{m\mathcal{C}}\widetilde{P}_c$, leading to

$$\begin{aligned}\widetilde{W}_{c,\,x} &= \gamma\mathcal{C}\frac{m}{2}\left(\tilde{x}^2-\langle\tilde{x}^2\rangle\right)+\gamma\langle p\rangle\tilde{x} \\ &= \frac{\gamma}{4}\Big(\left[\tilde{x},\widetilde{P}_c\right]_+-\left\langle\left[\tilde{x},\widetilde{P}_c\right]_+\right\rangle\Big)+\gamma\langle P_c\rangle\frac{1}{\mathcal{C}}\frac{\widetilde{P}_c}{m} \\ &= \frac{\gamma}{\mathcal{C}}\Big(\frac{\widetilde{P}_c^2}{2m}-\langle\frac{\widetilde{P}_c^2}{2m}\rangle\Big)+\frac{\gamma}{\mathcal{C}}\langle P_c\rangle\frac{\widetilde{P}_c}{m} \\ &= \frac{\gamma}{\mathcal{C}}\Big(\frac{P_c^2}{2m}-\langle\frac{P_c^2}{2m}\rangle\Big)\end{aligned} \tag{82}$$

showing the connection between the three different approaches.

In momentum space, the logarithmic ansatz appears not to be sufficient, but the other two approaches again provide the desired equations of motion for the maximum and width of the wave packet solution in momentum space. That the equation of motion for the maximum is correct is obvious. That the Riccati equation for the wave packet width is correct follows from the fact that it can be linearized to the same complex Newtonian Equation (55) including the friction force as in position space. Since the time-evolution of the uncertainties should not depend on the representation in which it is described, this proves that the momentum space version of the Riccati equation is as correct as the one in position space.

From the friction term $\widetilde{W}_{c,\,p}$ in momentum space follow two additional contributions to the continuity equation. One is a friction force in the convection term, leading, in analogy to the classical case, to a term correctly representing the time-derivative of momentum including friction. The other contribution is an additional diffusion term with diffusion coefficient $D_p=\frac{\gamma}{2}\langle\tilde{p}^2\rangle$.

In the modified Hamilton–Jacobi equation, additional terms depending on X_R are provided by $\widetilde{W}_{c,p}$. Treating them in the same way as the contribution from the conservative potential in momentum space allows us to define a force via negative derivative with respect to X_R, $\frac{\partial}{\partial X_R}$, supplying a force that on average is identical with the sum of classical potential and friction forces.

Again, the two friction terms $\widetilde{W}_{Has,\,p}$ and $\widetilde{W}_{D,\,p}$ not only supply the correct equations of motion for the wave packet maximum and width, but also can be expressed in the same form in terms of our complex hydrodynamic quantities in momentum space. For this purpose, $\tilde{p}$ is replaced by $\tilde{p}=m\mathcal{C}\widetilde{X}_c$, showing that the two approaches are related via

$$\begin{aligned}\widetilde{W}_{c,\,p} &= \frac{\gamma}{\mathcal{C}}\Big(\frac{p^2}{2m}-\langle\frac{p^2}{2m}\rangle\Big) \\ &= \frac{\gamma}{\mathcal{C}}\Big(\frac{\tilde{p}^2}{2m}-\langle\frac{\tilde{p}^2}{2m}\rangle\Big)+\gamma\langle p\rangle\widetilde{X}_c \\ &= \frac{\gamma}{4}\Big(\left[\widetilde{X}_c,\tilde{p}\right]_+-\left\langle\left[\widetilde{X}_c,\tilde{p}\right]_+\right\rangle\Big)+\gamma\langle p\rangle\widetilde{X}_c \\ &= \gamma\mathcal{C}\frac{m}{2}\Big(\widetilde{X}_c^2-\langle\widetilde{X}_c^2\rangle\Big)+\gamma\langle p\rangle\widetilde{X}_c\end{aligned} \tag{83}$$

Therefore, to transition from position to momentum space means to replace $\tilde{x}$ or $\tilde{p}$ by the complex hydrodynamic variable in the canonical conjugate space, i.e., $\tilde{x}\longleftrightarrow\frac{\widetilde{P}_c}{m}$ in position space, $\frac{\tilde{p}}{m}\longleftrightarrow\widetilde{X}_c$ in momentum space.

There still remain a few open questions: (1) Why is the logarithmic ansatz connected with the action not providing the desired result in momentum space? (2) Why does the diffusion term in momentum space have the wrong sign? Answers to these questions are probably not trivial, but may supply further insight into the problem. Therefore, this will be a topic of our future investigations.

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