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# Gauge Field Theories, Quantum Space-Time and Some Applications

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ACADEMIC DISSERTATION

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## Abstract

In this thesis, the possibility of extending the Quantization Condition of Dirac for Magnetic Monopoles to noncommutative space-time is investigated. The three publications that this thesis is based on are all in direct link to this investigation. Noncommutative solitons have been found within certain noncommutative field theories, but it is not known whether they possess only topological charge or also magnetic charge. This is a consequence of that the noncommutative topological charge need not coincide with the noncommutative magnetic charge, although they are equivalent in the commutative context. The aim of this work is to begin to fill this gap of knowledge. The method of investigation is perturbative and leaves open the question of whether a nonperturbative source for the magnetic monopole can be constructed, although some aspects of such a generalization are indicated. The main result is that while the noncommutative Aharonov-Bohm effect can be formulated in a gauge invariant way, the quantization condition of Dirac is not satisfied in the case of a perturbative source for the point-like magnetic monopole.

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*Miklos Långvik*

# Contents

<b>List of publications</b>	<b>vii</b>
<b>1 Introduction</b>	<b>1</b>
<b>2 Quantum Field Theory on Noncommutative Space-Time</b>	<b>5</b>
2.1 Moyal star-product . . . . .	6
2.2 Broken Lorentz symmetry of the commutator . . . . .	9
2.3 The problems with $\theta^{i0} = 0$ . . . . .	14
2.4 Non-locality of the Quantum Field Theory . . . . .	16
2.4.1 UV/IR mixing and singularities . . . . .	17
2.4.2 Infinite non-locality vs. finite noncommutativity . . . . .	20
2.5 Noncommutative gauge field theories . . . . .	23
2.5.1 Charge quantization . . . . .	25
2.5.2 Seiberg-Witten-map . . . . .	27
<b>3 The Aharonov-Bohm Effect</b>	<b>30</b>
3.1 Wilson loops and the AB-effect . . . . .	30
3.2 The noncommutative AB-effect . . . . .	34

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<b>4</b>	<b>Magnetic Monopoles</b>	<b>37</b>
4.1	A source for the magnetic monopole . . . . .	39
4.2	The DQC from a noncommutative quantum mechanical model . . .	42
4.2.1	Problems in the second order of $\theta_k$ . . . . .	44
4.3	Magnetic monopoles and the Wu & Yang method . . . . .	48
4.4	Noncommutative Maxwell equations with a monopole . . . . .	51
4.4.1	The solution in the first order of $\theta_k$ . . . . .	54
4.4.2	The solution in the second order of $\theta_k$ . . . . .	56
<b>5</b>	<b>Conclusions</b>	<b>63</b>

# List of publications

- I. "*Gauge Covariance of the Aharonov-Bohm Phase in Noncommutative Quantum Mechanics*",  
M. Chaichian, M. Långvik, S. Sasaki and A. Tureanu,  
Phys. Lett. B **666** (2008) 199 [arXiv:0804.3565 [hep-th]].
- II. "*Dirac Quantization Condition for Monopole in Noncommutative Space-Time*",  
M. Chaichian, S. Ghosh, M. Långvik and A. Tureanu,  
Phys. Rev. D **79** (2009) 125029 [arXiv:0902.2453 [hep-th]].
- III. "*Wu-Yang Singularity-Free Gauge Transformations for Magnetic Monopoles in Noncommutative Space-Time*",  
M. Långvik, T. Salminen and A. Tureanu,  
Phys. Rev. D **83** (2011) 085006 [arXiv:1101.4540 [hep-th]].

# Chapter 1

## Introduction

The success of the standard model of particle physics, despite the ongoing search for the Higgs boson, has been striking. Especially when one takes into account that the standard model of particle physics does not encompass gravity, that up to present day still is best treated as a non-quantum theory. It is indeed interesting that one can remove the presence of gravity in spite of its obvious appearance to us in everyday life, and yet achieve a quantum theory with such rigor. The most trivial explanation for this success is due to scale. Gravity is so weak as a force between quantum particles that it may safely be neglected in most quantum treatments. However, when distances grow very small between particles, it can no longer be neglected and at this scale the predictions of the standard model of particle physics can no longer be trusted.

The description of gravity as a quantum theory has been a longstanding problem for theoretical physics. The problem of time and the question of background independence are possibly the most widely known obstacles for the construction of a theory of quantum gravity. But they are by far not an exhaustive description of the problems facing the construction of a theory of quantum gravity [1]. As the old bottom-up approach, in our case the construction of a quantum field-theory of gravity, has remained ever elusive, developments into completely novel models of physics have taken place. Perhaps the most widely spread is string theory [2], but quantum loop gravity [3] has also become a serious candidate in the search for a quantum theory of gravity. These two theories are today perhaps the most

rigorous and ambitious theories of quantum gravity. However, they are not without trouble. It seems that both approaches, apart from other difficulties that in this context can be viewed as minor, lack a clear connection to experimentally verifiable predictions<sup>1</sup>. These difficulties have resulted in different low-energy approaches to quantum gravity, some of which are not relying on low-energy limits of string theory or quantum loop gravity. Perhaps there is a scale that is not yet quantum gravity in its full rigor, but nevertheless contains effects resulting from quantum gravity. This is the arena where possibly noncommutative quantum field theory could play a role.

The subject of noncommutative spaces is not new and is familiar to anyone with some understanding of quantum mechanics. In quantum mechanics the position and the momentum operators do not commute  $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$  and phase-space becomes smeared out as a consequence. The study of these noncommutative algebraic spaces was pioneered by von Neumann (for a recent account see [4]), who referred to them as "pointless", in view of their lack of conventionally defined points. If we reverse the train of thought, we can envisage that a smeared out space-time could result in a noncommutative space-time with a smallest length scale. Given the problems with the UltraViolet (UV) divergences of quantum field theories in the early 20th century, this was one way thought to lead out of the problem. A smallest length-scale would simply imply a high momentum cutoff in the infinite integrals. The first paper on the subject of noncommutative space-time was written by H.S. Snyder [5] although W. Heisenberg is known to have had the original idea [6]. In Snyder's formulation space-time is noncommutative in the following sense:

$$[\hat{x}^\mu, \hat{x}^\nu] = \frac{ia^2}{\hbar}L^{\mu\nu}, \quad (1.1)$$

where  $a$  is a basic unit of length and  $L^{\mu\nu}$  are the generators of the Lorentz group.  $\hat{x}^\mu$  are the usual space-time coordinates that are now promoted to the status of operators. While the commutator (1.1) is Lorentz covariant, it does not preserve translational invariance. This problem was noted by Snyder in his original article [5]. C.N. Yang tried to salvage the situation [7] but had to introduce a five dimensional de Sitter space in order to do so. Today it is known that even ignoring the non-

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<sup>1</sup>Quantum Loop Gravity has one exception, and that is that area and volume are quantized within the theory. This is in principle, a verifiable experimental prediction.

translational invariance of (1.1), it will not lead to a UV-finite quantum field-theory [8].

Due to the big success of the renormalization program, the idea of the noncommutativity of space-time was abandoned for quite some time until it was revived in the 1980's when the notion of differential structure was generalized to include also noncommutative spaces [9]. This development led, amongst other developments, to the applications of Yang-Mills on a noncommutative torus [10] and the Connes-Lott model [11]. A further push in the noncommutative direction was given by the discovery of the UV-finite noncommutative fuzzy sphere [12], but it was not until the works of S. Doplicher, K. Fredenhagen and J.E. Roberts [13] and N. Seiberg and E. Witten [15], that noncommutativity became a popular scientific field within theoretical physics.

In the works [13], which are a revival of the idea in the work [14], it is argued that the existence of small black holes has an impact on space-time measurements. Indeed, if we try to measure i.e. the size of a very small particle in space-time, we need very high energy in order to localize the energy into a sufficiently small region of space-time. This localization will ultimately create a small black hole from which nothing can return and we have reached an upper limit on how accurately it is possible to do measurements in space-time. In [13], this argument leads to the commutator

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}, \tag{1.2}$$

where  $\theta_{\mu\nu}$  is a tensor, transforming covariantly under Lorentz transformations. The commutator (1.2) is formulated on a noncommutative algebra which is required to respect the Poincaré group symmetry. In this sense (1.2) is translationally invariant and differs from the algebra considered by Snyder (1.1). This approach to noncommutativity is termed the DFR (Doplicher-Fredenhagen-Roberts) approach.

The commutator (1.2) appears also as a low-energy limit of open string theory in a constant background field [15]. However, within this context  $\theta_{\mu\nu}$  is a constant antisymmetric *matrix*. The idea behind the noncommutativity of [15] is very similar to the noncommutativity arising in lowest level of the Landau problem and the Peierls substitution [16]. In the Peierls substitution, the energy levels of an originally 3-dimensional space are projected onto a noncommutative 2-dimensional

space in the vicinity of a strong magnetic field perpendicular to the noncommutative plane. Similarly in open string theory in a constant background field, the constant background field projects the open strings ending on a D-brane onto it, becoming a noncommutative field theory on the brane.

In this thesis we first review the most important aspects of noncommutative field theories up to date, giving special attention to the differences between noncommutative and commutative field theories. We then turn to describe in detail what has been found in the three articles that are the core of this thesis and place them in their proper context within noncommutative field theory.

## Chapter 2

# Quantum Field Theory on Noncommutative Space-Time

In this work we shall be most interested in noncommutativity of the type

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}, \quad (2.1)$$

where  $\theta_{\mu\nu}$  is a constant antisymmetric matrix and  $\hat{x}_\mu$  is an operator of the space-time position. At times we will make reference and comparison to other types of noncommutativity, but if nothing else is mentioned, this type of noncommutativity will be assumed. Noncommutativity of the type (2.1) is compatible with the noncommutativity that arises in string theory [15], but it is not equivalent to the DFR approach [13] where  $\theta_{\mu\nu}$  is assumed to be a Lorentz covariant tensor. Although the approach we use is equivalent to the Seiberg-Witten noncommutativity, we shall not be so concerned in this work with mapping the noncommutative gauge theories that we find to the commutative theory defined by the Seiberg-Witten map for gauge theories.

In this chapter, we begin by presenting the implementation of the commutator (2.1) as a Moyal star-product [17] in section 2.1. In section 2.2 we then discuss the symmetry of the noncommutative space-time, especially in connection with the broken Lorentz symmetry and how the light-cone structure of the theory changes as a consequence. We then move to the topic of infinite non-locality in section 2.4, which is present due to the constancy of  $\theta_{\mu\nu}$  and discuss the UV/IR mixing effect

and the difficulty of constructing a noncommutative field theory with a constant  $\theta_{\mu\nu}$  with a finite range of non-locality. We finish off the chapter with section 2.5, where a review of the construction of noncommutative gauge theories which we shall be needing in the following chapters is given.

## 2.1 Moyal star-product

In this section we shall present the construction of the Moyal star-product [17], which will allow us to implement the commutator (2.1) in a fairly simple way.

Having decided to take into account the possible shortest observable length-scale due to black hole formation at high energy as a commutator of space-time positions (2.1), we face the question of how to use the new operators in our theory. Fortunately there exists a very efficient method developed initially for the phase space of quantum mechanics that creates a one-to-one correspondence between functions (in this correspondence they are called symbols) and operators [18]. Within this method we define a Weyl operator by the map

$$W(a(x)) \quad \hat{a}_W(\hat{x}) = \int \tilde{a}(\tau) e^{i\hat{x}^m \tau_m} d^D \tau, \quad (2.2)$$

that takes the commutative function  $a(x)$  and turns it into an operator  $\hat{a}_W(\hat{x})$ . Here  $\tilde{a}(\tau) = \frac{1}{(2\pi)^n} \int e^{-ix^m \tau_m} d^n x$  and the map  $W(a(x))$  is called the Weyl map. It is clear that this definition of an operator is very similar to the Fourier transform of a function. This is the reason we require the function  $a(x)$  to satisfy the Schwartz condition

$$\sup_{x \in \mathbb{R}^n} x^\alpha \partial^\beta a(x) < \infty, \quad (2.3)$$

where  $\alpha$  and  $\beta$  are multi-indices of size  $n$ , which guarantees a sufficiently rapid decrease at infinity. This in turn leads to that we may use the Fourier transform of a function as a well-defined concept in the definition of the Weyl operators and it also requires that the functions we deform into operators be smooth.

The exponential in (2.2) sees to that the constructed operators  $\hat{a}_W(\hat{x})$  are symmetrically ordered w.r.t. the operators  $\hat{x}$ . Replacing coordinates  $\hat{x}^m$  by annihilation and creation operators one may also order operators in a different way e.g. normal

or Wick ordering. However, the observables of the theory should not be affected by operator ordering issues and as the Weyl symmetric ordering results in a star product that is manifestly Hermitian, we will be content with considering the Weyl symmetric ordering in this work.

We then define the multiplication of the operators (2.2) by demanding the property

$$W(a(x) \star b(x)) = W(a(x))W(b(x)), \quad (2.4)$$

where the  $\star$  symbolizes the multiplication of the commutative functions. This results in that the commutator (2.1) may be implemented as a deformed product of the commutative functions called the star product, provided we can find a representation for the star-product in (2.4). This representation can be given to us by e.g.:

$$a(x) \star b(x) = e^{\frac{i}{2}\theta^{kl}\partial_k^x\partial_l^y} a(x)b(y)\Big|_{x=y}. \quad (2.5)$$

This definition can be used also for polynomials, which do not belong to Schwartz functions but can be interpreted as Schwartz distributions.

If we insert the definition of the star-product (2.5) into  $a(x) \star b(x)$  where we write the functions  $a(x)$  and  $b(x)$  as Fourier expansions, i.e.

$$\begin{aligned} a(x) \star b(x) &= d^D\tau d^D\sigma \left( e^{\frac{i}{2}\theta^{kl}\partial_k^x\partial_l^y} \tilde{a}(\tau) e^{ix^m\tau_m} \tilde{b}(\sigma) e^{iy^n\sigma_n} \right) \Big|_{x=y} \\ &= d^D\tau d^D\sigma \tilde{a}(\tau) \tilde{b}(\sigma) \left( \sum_{j=0}^{\infty} \frac{1}{j!} \left( \frac{i}{2} \theta^{kl} \partial_k^x \partial_l^y \right)^j e^{ix^m\tau_m} e^{iy^n\sigma_n} \right) \Big|_{x=y} \\ &= d^D\tau d^D\sigma \tilde{a}(\tau) \tilde{b}(\sigma) \left( \sum_{j=0}^{\infty} \frac{1}{j!} \left( \frac{i}{2} \theta^{mn} (i\tau_m) (i\sigma_n) \right)^j e^{ix^m\tau_m} e^{iy^n\sigma_n} \right) \Big|_{x=y} \\ &= d^D\tau d^D\sigma \tilde{a}(\tau) \tilde{b}(\sigma) e^{-\frac{i}{2}\theta^{mn}\tau_m\sigma_n} e^{ix^m(\tau_m+\sigma_m)}. \end{aligned}$$

When we then turn the  $x$ :s into operators, we have

$$\begin{aligned} W(a(x) \star b(x)) &= d^D\tau d^D\sigma \tilde{a}(\tau) \tilde{b}(\sigma) e^{-\frac{i}{2}\theta^{mn}\tau_m\sigma_n} e^{i\hat{x}^m(\tau_m+\sigma_m)}, \\ &= d^D\tau d^D\sigma \tilde{a}(\tau) \tilde{b}(\sigma) e^{i\hat{x}^m\tau_m} e^{i\hat{x}^n\sigma_n} \\ &= W(a(x))W(b(x)), \end{aligned}$$

where in the second line we have used the Baker-Campbell-Hausdorff lemma in the form

$$e^{i\hat{x}^m \tau_m} e^{i\hat{x}^n \sigma_n} = e^{i\hat{x}^m (\tau_m + \sigma_m)} e^{-\frac{i}{2} \theta^{mn} \tau_m \sigma_n}.$$

Therefore the representation (2.5) satisfies the requirement (2.4), and we have found a good way to implement the commutator (2.4). One simply takes the usual products of fields in field theory and replaces them with star-products. It should be noted that the star product (2.5) is a special case of the more general star products appearing in deformation quantization of Poisson manifolds [19]. Therefore the noncommutative field theories that we shall work with in this thesis can be regarded as a special deformation of the commutative field theories.

The representation (2.5) is not the only representation of the star-product and depending on the problem at hand it may become convenient to use another one. We therefore give also two integral representations of the star-product which will become relevant to us when we discuss the infinite non-locality inherent in a theory with a noncommutativity of the type (2.1)

$$a(x) \star b(x) = \frac{1}{\pi^D \det \theta} \int d^D y \int d^D z a(y) b(z) e^{-2i\theta_{ij}^{-1}(x-y)^i (x-z)^j}, \quad (2.6)$$

$$= \frac{1}{(2\pi)^D} \int d^D y \int d^D z a(x_n - \frac{1}{2}y_n) b(x_m - \theta_{mp} z^p) e^{-iy \cdot z}, \quad (2.7)$$

where  $\theta_{ij}^{-1}$  is the inverse matrix of  $\theta_{ij}$ . If one space-time coordinate is chosen to be commutative w.r.t. all the other coordinates, the  $\theta$ -matrix does no longer poses an inverse, the most common choice being the time coordinate. However, although we shall consider time to be commutative, we will not be working with the inverse of  $\theta_{ij}$  and may disregard of this aspect.

As an instructive example of a noncommutative field theory we may consider a  $\phi^4$ -theory action. It becomes in the noncommutative case

$$S_{NC} = \int d^4 x \left( \frac{1}{2} \partial_\mu \phi \star \partial^\mu \phi - \frac{1}{2} m^2 \phi \star \phi - \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \right). \quad (2.8)$$

However, in this case, due to the integral  $\int d^4 x$ , we may do a partial integration of each derivative in the star-product of  $\phi$  in the terms with two fields to obtain

$$S_{NC} = \int d^4 x \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \right), \quad (2.9)$$

where the surface terms have been dropped and the other derivatives are equal to zero due to the antisymmetry of  $\theta_{\mu\nu}$ . One can then see that in noncommutative field theory the free action remains the same as in the commutative case, but the interaction terms change or are deformed.

We also note that the commutator (2.1) can now be given with aid of the Moyal star-product as

$$[x^\mu, x^\nu]_\star = i\theta^{\mu\nu}, \quad (2.10)$$

where we have introduced the Moyal star-commutator or Moyal bracket  $[x^\mu, x^\nu]_\star = x^\mu \star x^\nu - x^\nu \star x^\mu$ . From this point of view we may continue to refer to  $x^\mu$  as space-time coordinates in the commutative sense and do not need to make special reference to the operators that they correspond to. This will be helpful in the following section.

Before turning to the next section, we should also briefly note that we shall mostly be interested in the space-space noncommutativity  $\theta^{i0} = 0$  in this work. This choice is motivated as a result of the problems associated with the causality [20] and unitarity [21] of the theories with a time-like noncommutativity  $\theta^{i0} \neq 0$  (see section 2.3 for a more thorough discussion). However, it should be mentioned that light-like noncommutative theories  $\theta^{i0} = \theta^{i3}$ , with  $i = 1, 2$ , can also be obtained as limits of string theories [22] and quantized in the light-front formalism [23].

## 2.2 Broken Lorentz symmetry of the commutator

The commutation relation  $[x^\mu, x^\nu]_\star = i\theta^{\mu\nu}$ , where  $\theta^{\mu\nu}$  is constant and antisymmetric, is clearly not preserved under the Lorentz group  $O(1, 3)$ , although it remains intact under translations. Indeed, the largest subgroup of the Lorentz group under which the commutation relation remains intact is  $SO(1, 1) \times SO(2)$  [24], where the factor  $SO(1, 1)$  acts on the coordinates  $x_e = (x_0, x_1)$  and the factor  $SO(2)$  acts on the coordinates  $x_m = (x_2, x_3)$ <sup>1</sup>. When time commutes with all the other coordinates the largest preserved subgroup of the Lorentz group is  $O(1, 1) \times SO(2)$  and therefore

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<sup>1</sup>This is after a change of reference frame to a form where  $\theta^{\mu\nu}$  is block-diagonal. This can always be done for an antisymmetric matrix in even dimensions.

the total space-time symmetry group for the theory is given by  $[O(1, 1) \times SO(2)] \ltimes \mathcal{T}$ , where  $\mathcal{T}$  is the group of four dimensional translations.

Due to the breaking of Lorentz invariance of the commutator (2.1), we cannot adopt the usual formulation of light-cone causality in noncommutative field theories, but must redefine the concept of causality within these theories, if it exists at all. In the commutative case, microcausality is defined by demanding that the fields commute or anticommute outside of their light-cone. One can then envisage that in the noncommutative case the same can be done, although the fields must now commute or anticommute outside some other structure than the light-cone, as the maximal symmetry of space-time is now  $O(1, 1) \times SO(2)$  and not the Lorentz group  $O(1, 3)$ . A structure that is preserved under the demanded symmetry is the "light-wedge"  $V_+ = \{x \in \mathbb{R}^{1,3} \mid x_e^2 = 0\}$  [24]. Therefore we define microcausality with respect to the light-wedge. Fields have to commute or anticommute outside of it (see Figure 2.1<sup>2</sup>). From figure 2.1 we can also see that the maximal signal speed parallel to the coordinates  $x_m$  is infinity.

Although (2.1) is not Lorentz invariant, within field theory, it does preserve another symmetry called the twisted Poincaré symmetry [25]. It is a special example of the Drinfeld twist [26] defined within the context of quantum groups (for a review, see e.g. the books [27]). The twist deforms the universal enveloping algebra of the Poincaré algebra  $\mathcal{U}(\mathcal{P})$ . The result is that the commutation relations of the Poincaré algebra remain unchanged, i.e.

$$\begin{aligned} [P_\mu, P_\nu] &= 0 \\ [M_{\mu\nu}, M_{\alpha\beta}] &= -i(\eta_{\mu\alpha}M_{\nu\beta} - \eta_{\mu\beta}M_{\nu\alpha} - \eta_{\nu\alpha}M_{\mu\beta} + \eta_{\nu\beta}M_{\mu\alpha}) \\ [M_{\mu\nu}, P_\alpha] &= -i(\eta_{\mu\alpha}P_\nu - \eta_{\nu\alpha}P_\mu), \end{aligned} \quad (2.11)$$

but the coproduct  $\Delta_0(Y)$

$$\begin{aligned} \Delta_0 : \mathcal{U}(\mathcal{P}) &\rightarrow \mathcal{U}(\mathcal{P}) \otimes \mathcal{U}(\mathcal{P}) \\ \Delta_0(Y) &= Y \otimes 1 + 1 \otimes Y, \end{aligned}$$

of  $\mathcal{U}(\mathcal{P})$  changes to become

$$\Delta_t(Y) = \mathcal{F}\Delta_0(Y)\mathcal{F}^{-1}, \quad (2.12)$$

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<sup>2</sup>The figure is taken from [24].

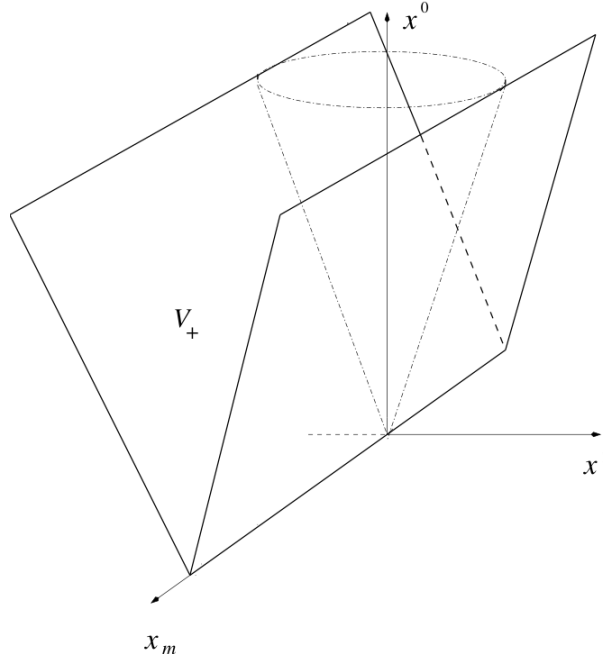


Figure 2.1: The light-cone and the "light-wedge".

where  $\mathcal{F} = \exp(\frac{i}{2}\theta^{\mu\nu}P_\mu \otimes P_\nu)$  is the twist element. The upshot of this is that we may freely use the usual Poincaré representations of the algebra in a field theory with a constant antisymmetric  $\theta^{\mu\nu}$ , since the Casimir invariants  $P^2$  and  $W^2$  of the commutative field theory, with  $W_\alpha = -\frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}M^{\beta\gamma}P^\delta$ , remain invariants under the twist. Simply put, the classification of the representations according to the eigenvalues of the operators  $W^2$  and  $P^2$  stays valid in the noncommutative theory. What is most interesting is that, although the Lorentz symmetry is broken, one manages to find some new kind of symmetry for the noncommutative theories with a constant antisymmetric  $\theta^{\mu\nu}$ . The twisted Poincaré symmetry and its role as a symmetry in a quantum field theory will be discussed some more within the context of noncommutative gauge theories in section 2.5.

The commutation relation (2.1) is not by far the only way to break Lorentz invariance. Many other attempts at Lorentz non-invariance exist that among other things would change the propagation speed of light. This has been desirable due to that experiments can detect Gamma Ray Bursts (GRB) including photons and

protons coming from Active Galactic Nuclei (AGN) with very high energy (see e.g. [28]), energies that exceed the Greisen-Zatsepin-Kuzmin (GZK) cutoff [29] which is due to considerations of Lorentz invariance. There are other explanations of the violation of the GZK cutoff, e.g. the decay of very heavy particles, but at present none of them are any more valid than the other and a possible explanation of its violation by Lorentz non-invariance remains a good candidate. Therefore, there does seem to be also some experimental motivation for considering Lorentz non-invariance as a good candidate for physics beyond the currently accepted models of physics. Since noncommutativity of space-time, as formulated in this work, is a Lorentz non-invariant theory, we shall dwell upon this issue a little further.

One of the first attempts aimed at just finding Lorentz violating terms is formulated in [30] and is today sometimes referred to as the Standard Model Extension (SME). This departure from the minimal standard model of particle physics  $U(1) \times SU(2) \times SU(3)$  has gauge invariance, energy-momentum conservation and Lorentz covariance under observer rotations and boosts, i.e. rotations and boosts of the observer's inertial frame. However, Lorentz covariance is violated under particle rotations and boosts, i.e. rotations and boosts of a localized particle or field that do not change the background expectation values. This peculiar kind of Lorentz violation is the result of a spontaneously broken Lorentz symmetry. This form of Lorentz violation is relevant to String Theories where it is expected that the higher dimensional Lorentz symmetry is spontaneously broken. If the breaking extends to our four macroscopic space-time dimensions, it could occur at the level of the standard model. This form of Lorentz violation is however not related to the Lorentz non-invariance of noncommutative quantum field theories, as these theories e.g. are power counting renormalizable and the noncommutative theories, due to UV/IR mixing [31] (see subsection 2.4.1 for a more thorough discussion of UV/IR mixing), certainly are not.

Another form of Lorentz violation is studied in Doubly Special Relativity (DSR) [32] models. These models have not only a highest signal speed, but also a highest energy/momentum, hence the name "doubly special". They were initially postulated to be related to loop quantum gravity, but it has been shown that loop quantum gravity does not presuppose the existence of a smallest length and hence highest energy or momentum [33]. The DSR models can thus far only be con-

structed in momentum space and typically they face problems such as, one does not know to which types of space-time they are related (if any) and how the observer in these theories is described. Therefore it is as of yet difficult to determine if they can be relevant to the Lorentz violation occurring in noncommutative theories with a constant  $\theta^{\mu\nu}$ , but at present it seems that they cannot, as there is no reason to consider a highest momentum/energy scale in noncommutative theories.

Although the SME construction is not related to noncommutative field theories and the relation to DSR theories is unclear, there is a third type of Lorentz violation in models that go by the name of Very Special Relativity (VSR) [34] which have been shown to be related to noncommutative theories [35]. In VSR models space-time symmetries are described by certain proper subgroups of the Poincaré group. These proper subgroups contain space-time translations and at least a 2-parameter subgroup of the Lorentz group isomorphic to that generated by  $K_x + J_y$  and  $K_y - J_x$ , where  $\mathbf{J}$  and  $\mathbf{K}$  are the generators of rotations and boosts respectively. The group generated by  $K_x + J_y$  and  $K_y - J_x$  is called  $T(2)$  and any space-time symmetry that consists of translations along with the Lorentz subgroup  $T(2)$  or three other Lorentz subgroups groups that may be formed by adjoining the generators  $J_z$  or  $K_z$  or both to  $T(2)$ , is referred to as VSR. The interest in VSR arose due to that the incorporation of either  $P$ ,  $T$  or  $CP$  enlarges these four subgroups to the full Lorentz group. Therefore, Lorentz violating effects are absent for any VSR theory containing any one of the aforementioned discrete symmetries. In [35], it is shown that the VSR with subgroup  $T(2)$  is equivalent to a theory with light-like noncommutativity i.e.  $\theta^{i0} = \theta^{i3}$ , with  $i = 1, 2$ . This implies that  $T(2)$  VSR invariant theories may be constructed as noncommutative theories with a constant light-like  $\theta^{\mu\nu}$ .

Another aspect of the broken Lorentz symmetry of (2.1) is that one has had reason to suspect that interacting noncommutative field theories are CPT-violating due to the result [36]. This result is however invalidated in [37], where specific counter examples are given. In fact, in noncommutative field theory the CPT-theorem holds [24, 38], with the exception of time-space noncommutative theories. An axiomatic formulation of a noncommutative CPT-theorem has also been put forth that supports this conclusion [24, 39].

## 2.3 The problems with $\theta^{i0} = 0$

If we accept that  $\theta^{\mu\nu}$  can have time/space noncommutativity, that is  $\theta^{i0} = 0$ , we encounter some interesting issues in noncommutative field theory. This might be expected because when time is noncommutative, it is difficult to construct any sensible Hamiltonian formalism for the theory. Indeed, if time is noncommutative, it is hard to say that one has a Hamiltonian at one instant of time, as every instant of time is related to another instant of time due to the infinite nonlocality of the star product (2.5). Therefore one should give special attention to e.g. the notion of causality in a theory of this kind. Here we shall concentrate on the issues of unitarity and causality but a more comprehensive review of the situation is given in [40].

In [20] it was found that noncommutative scalar  $\phi^4$  theory with time/space noncommutativity is acausal. This can be shown by calculating the wave functions of in and out states of a two to two particle scattering to lowest nontrivial order of the S-matrix. When one chooses the in state to be

$$\phi_{in}(p) \sim E_p \left( e^{-\frac{(p-p_0)^2}{\lambda}} + e^{-\frac{(p+p_0)^2}{\lambda}} \right), \quad (2.13)$$

the out state can be calculated [20] to be

$$\Phi_{out}(x) \sim g F(x; -\theta, \lambda, p_0) + 4 \bar{\lambda} e^{-\lambda \frac{x^2}{4}} e^{ip_0 x} + F(x; \theta, \lambda, p_0) + (p_0 \quad -p_0), \quad (2.14)$$

where

$$\frac{1}{-4i\theta} e^{-\frac{(x+8p_0\theta)^2}{64\theta^2\lambda}} e^{-i\frac{(x-\frac{p_0}{2\lambda^2\theta})^2}{16\theta}} e^{i\frac{p_0^2}{4\lambda^2\theta}} \equiv F(x; \theta, \lambda, p_0). \quad (2.15)$$

One can note that the outgoing wave packet splits into three parts concentrated at  $x = 8p_0\theta$ ,  $x = 0$  and  $x = -8p_0\theta$  respectively, which is perhaps counterintuitive, but not yet any cause for alarm. However, if we look at the last packet of the outgoing wave, which is delayed compared to the two other ones, we can see that it appears to originate before the ingoing wave hits the wall. What more, the advance of the wave packet is proportional to the energy, i.e.  $x = -8p_0\theta$  and the higher the energy, the bigger the advance of the wave packet compared to the incoming one. This does certainly suggest acausal behavior of the theory.

In [20] the calculation is done in the center of mass frame, so one can say that the back scattering considered, is the same as bouncing of a wall. This analogy can be used to make explicit the strange behavior of this process. One might think of the back scattering as a rigid rod of length  $L$  within a nonrelativistic theory. If one assumes that the rod reflects when its leading end strikes the wall, its center of mass would appear to reflect before it strikes the wall. However, this notion of rigid bodies is in serious conflict with Lorentz invariance and causality. In this theory these "rigid rods", the wave packets, would appear to expand in length as the energy grows. As this is contrary to the expectation of the Lorentz-Fitzgerald contraction one is lead to believe that time/space noncommutativity really leads to acausal field theories, as was suggested earlier.

Another suspicion of the pathology of the time/space noncommutative theories arises when one considers the unitarity of such theories. Due to the infinite number of derivatives in the Moyal star-product (2.5) one might be lead to doubt the unitarity of the S-matrix of these theories. However, due to the unitarity of string theories and because the space/space noncommutativity is a consequence of string theory in a low energy regime [15], one might expect that these theories would remain unitary. However, as there is no limit of string theory that in the low energy regime leads to time/space noncommutativity [41], these theories should be examined with extra care. Indeed, it has been shown that time/space noncommutativity leads to non-unitarity of the 1-loop diagrams of scalar  $\phi^3$  and  $\phi^4$  noncommutative field theories [21]. What is shown is that the cutting rules, which are a consequence of unitarity of field theory, are not satisfied in the case of time/space noncommutative theories. They are however satisfied for space/space noncommutative theories. If we have time/space noncommutativity, the quantity  $\theta^{\mu\nu} p_\mu p_\nu$  is not necessarily positive definite. For a noncommutative  $\phi^3$  theory this leads to a different result for the nonplanar<sup>3</sup> part of the one-loop integral

$$\text{Im } M = \frac{\lambda^2}{64\pi} \int_0^1 dx J_0\left(\frac{\theta^{\mu\nu}\theta^\gamma p_\mu p_\nu (m^2 + p^2 x(1-x))}{\phantom{\theta^{\mu\nu}\theta^\gamma p_\mu p_\nu (m^2 + p^2 x(1-x))}}\right), \quad (2.16)$$

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<sup>3</sup>By nonplanar is here meant a diagram that cannot be drawn on a plane without intersecting lines.

compared to that obtained from its cutting rules:

$$M^2 = \frac{\lambda^2}{2} \frac{1}{(2\pi)^2} \frac{d^3k}{2k_0} \frac{d^3q}{2q_0} \delta^4(p - k - q) \frac{1 + \cos(\theta^{\mu\nu} p_\mu k_\nu)}{2}. \quad (2.17)$$

The integral (2.17) is zero for  $\theta^{\mu\nu} p_\mu p_\nu < 0$  because energy-momentum conservation forbids a particle with space-like momenta to decay to two massive on-shell particles. A similar discrepancy is obtained for  $\phi^4$ -theory with time/space non-commutativity [21] and one therefore concludes that the cutting rules and hence perturbative unitarity is only satisfied for space/space noncommutative theories.

It should be mentioned that the problem of unitarity violation does not arise in the case of light-like noncommutativity  $\theta^{i0} = \theta^{i3}, i = 1, 2$  [22]. However, as they do violate the microcausality condition for the light-wedge [38], they are acausal macroscopically. Similarly, it is possible to construct noncommutative theories in the DFR approach where the problem of unitarity with time/space noncommutativity does not appear, although one does not resolve the problem of causality. In these theories the modified Feynman rules as used in [21] do not apply, but the Yang-Feldman approach can be used leading to a unitary field theory [42]. One may also use an interaction point time ordering procedure which is applied before integrations of the momenta are taken. In this case one finds a noncommutative quantum field theory which is mutually exclusive between the properties of unitarity and causality [43].

## 2.4 Non-locality of the Quantum Field Theory

After we have discussed the Lorentz invariance breaking of the space-time commutator (2.1) we turn to the issue of nonlocality that arises due to this commutator. Indeed, the constancy of  $\theta^{\mu\nu}$  does imply that all space-time points are inter-related in this theory. That is, any interaction taking place in this type of a space-time depends on all other space-time points and the interaction cannot be said to be happening at one space-time point. This type of nonlocal interaction that needs *all* the other space-time points for it to happen, is called infinite nonlocality. Another way to see this is to note that the Moyal star-product (2.5) contains an infinite number of derivatives that contribute to the Lagrangian of a noncommutative field

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theory only in the interaction terms, but to these terms, with an *infinite* number of derivatives. This infinite nonlocality leads to the probably most severe problem that noncommutative field theories face, the mixing of the UV and the IR [31]. The consequence of this mixing is that these theories are not renormalizable, at least not by any hitherto known mechanism.

### 2.4.1 UV/IR mixing and singularities

In [31] the one-loop behavior of scalar field theories is studied. It is concluded that the ordinary UV divergences are mixed with new IR divergences that appear due to the noncommutativity of space-time. In the following we shall explore this mechanism in more detail.

If we start from a scalar  $\phi^n$  field theory with the Euclidean action

$$S = \int d^4x \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 + \sum_{n=3}^M a_n g^{n-2} \phi_\star^n \right), \quad (2.18)$$

where  $\phi_\star^n$  denotes  $n$  star-products of the field  $\phi$ ,  $a_n$  are arbitrary constants and  $g$  are the coupling constants, we must find its Feynman rules to discuss its one-loop behavior. We can see from (2.18) that the free part of the field theory is the same as its commutative counterpart resulting in that the propagator is the same as in the commutative case. The difference to the commutative Feynman rules appears in the rules for the vertices. In momentum space every vertex has an extra phase factor

$$V(k_1, k_2, \dots, k_n) = e^{-\frac{i}{2} \sum_{i < j} \theta^{mn} k_m^i k_j^n}, \quad (2.19)$$

where  $k_m^i$  denotes the momentum of the  $i$ :th  $\phi$  flowing into the vertex and  $m$  is the index related to the noncommutative space-time coordinates through  $\theta^{mn}$ . This is in fact the only modification to the Feynman rules in momentum space compared to the same commutative theory [44]. The modification to the vertices serves to divide the Feynman diagrams into two distinct types: Planar diagrams and nonplanar diagrams. The planar diagrams are drawn as the commutative diagrams<sup>4</sup>, but

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<sup>4</sup>The difference to the commutative diagrams is a phase-factor (2.19) at each external line. Therefore the Feynman integrals do not change in the planar diagrams compared to the commutative Feynman integrals of the same diagram and these terms may therefore be renormalized by the introduction of counterterms as ordinarily in commutative theory.

the nonplanar diagrams are such that one cannot draw them on a plane without intersecting lines. This makes a substantial difference in how the Feynman integrals of these diagrams behave.

The complete vertex for both diagrams is [44]

$$e^{-\frac{i}{2} \sum_{i < j} \theta^{mn} p_m^i p_n^j} e^{-i \left( \frac{\theta^{mn}}{2} \sum_{i,j} C_{ij} k_m^i k_n^j \right)}, \quad (2.20)$$

where the  $p_m^i$ :s are external momenta, the  $k_m^i$  can be both internal and external momenta and the matrix  $C_{ij}$  counts the number of times the  $i$ :th momentum line crosses over the  $j$ :th momentum line. One can immediately see that planar graphs do not have the second exponent of (2.20) and consequently, they are not sensitive to the inner structure of the graph.

Next, we take scalar  $\phi^4$  theory as an example and calculate its Feynman integrals associated with the planar and nonplanar one loop diagrams (see figure 2.2<sup>5</sup>). These



Figure 2.2: The two one loop diagrams for noncommutative scalar  $\phi^4$  theory.

diagrams are terms of the 1 particle irreducible two point function which at lowest order is given by the inverse propagator  $\Gamma_0^{(2)} = p^2 + m^2$ . They have the form

$$\Gamma_{1\text{ planar}}^{(2)} = \frac{g^2}{3(2\pi)^4} \frac{d^4 k}{k^2 + m^2}, \quad (2.21)$$

$$\Gamma_{1\text{ nonplanar}}^{(2)} = \frac{g^2}{6(2\pi)^4} \frac{d^4 k}{k^2 + m^2} e^{i\theta^{mn} k_m p_n}. \quad (2.22)$$

After rewriting both integrals with a Schwinger parameter, then evaluating the  $k$  integrals and multiplying them by the regulating factor  $\exp(-1/(\Lambda^2 \alpha))$ , where  $\alpha$  is the Schwinger parameter, and integrating them over the Schwinger parameter, we

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<sup>5</sup>The figure is taken from [31].

end up with

$$\Gamma_{1\text{ planar}}^{(2)} = \frac{g^2}{48\pi^2} \left( \Lambda^2 - m^2 \ln\left(\frac{\Lambda^2}{m^2}\right) + O(1) \right), \quad (2.23)$$

$$\Gamma_{1\text{ nonplanar}}^{(2)} = \frac{g^2}{96\pi^2} \left( \Lambda_{eff}^2 - m^2 \ln\left(\frac{\Lambda_{eff}^2}{m^2}\right) + O(1) \right), \quad (2.24)$$

where  $\Lambda_{eff}^2 = \frac{1}{1/\Lambda^2 + \theta^{\mu\gamma}\theta_\gamma^\nu p_\mu p_\nu}$ . In the limit  $\Lambda \rightarrow \infty$  the planar one-loop contribution diverges, but the nonplanar contribution is clearly regulated by the noncommutativity of space-time. However, the nonplanar diagram diverges for  $p \rightarrow 0$  suggesting an IR singularity. This can be better seen if we write the total effective action to this one-loop order as

$$\begin{aligned} S_{1PI}^{(2)} = & \int d^4p \frac{1}{2} \left( p^2 + M^2 + \frac{g^2}{96\pi^2 (\theta^{\mu\gamma}\theta_\gamma^\nu p_\mu p_\nu + \frac{1}{\Lambda^2})} - \frac{g^2 M^2}{96\pi^2} \ln \frac{1}{M^2 (\theta^{\mu\gamma}\theta_\gamma^\nu p_\mu p_\nu + \frac{1}{\Lambda^2})} \right. \\ & \left. + \dots + \mathcal{O}(g^4) \right) \phi(p)\phi(-p), \end{aligned} \quad (2.25)$$

where  $M^2 = m^2 + \frac{g^2 \Lambda^2}{48\pi^2} - \frac{g^2 m^2}{48\pi^2} \ln\left(\frac{\Lambda^2}{m^2}\right) \dots$  is the renormalized mass. From (2.25) we can then find two cases:

1. In the zero momentum limit when  $\Lambda_{eff} \approx \Lambda$  we have the action

$$S_{1PI}^{(2)} = \int d^4p \frac{1}{2} (p^2 + M'^2) \phi(p)\phi(-p), \quad (2.26)$$

where  $M'^2 = M^2 + 3\frac{g^2 \Lambda^2}{96\pi^2} - \frac{3g^2 m^2}{96\pi^2} \ln\left(\frac{\Lambda^2}{m^2}\right) \dots$ . Here, the effective action diverges when one takes  $\Lambda \rightarrow \infty$ .

2. In the limit  $\Lambda \rightarrow \infty$  with  $\Lambda_{eff} \approx \frac{1}{\theta^{\mu\gamma}\theta_\gamma^\nu p_\mu p_\nu}$  we recover the action

$$\begin{aligned} S_{1PI}^{(2)'} = & \int d^4p \frac{1}{2} \left( p^2 + M^2 + \frac{g^2}{96\pi^2 \theta^{\mu\gamma}\theta_\gamma^\nu p_\mu p_\nu} - \frac{g^2 M^2}{96\pi^2} \ln \left( \frac{1}{m^2 \theta^{\mu\gamma}\theta_\gamma^\nu p_\mu p_\nu} \right) \right. \\ & \left. + \dots + \mathcal{O}(g^4) \right) \phi(p)\phi(-p), \end{aligned} \quad (2.27)$$

which diverges in the zero momentum limit.

However, depending on which limit one takes first, one ends up with a UV,  $\Lambda$  divergence in (2.26) or an IR,  $p \rightarrow 0$  divergence in (2.27). The noncommutativity of these two limits demonstrates the mixing of the UV and the IR.

The UV/IR mixing spoils the renormalizability of these quantum field theories. In the context of string theory, it can be related to the duality of open string theory at high energy and closed string theory at low energy [45]. Although it is not expectable that this phenomenon is absent in a gauge theory, it is still interesting to see how UV/IR mixing behaves in such a theory [46]. Especially the effect on the low energy regime, where the noncommutativity is not supposed to be detectable, is interesting due to the IR poles in the effective action. For instance, the dispersion relation for the transverse modes of a  $U_\star(1)$ <sup>6</sup> gauge boson at one-loop order can be found to have the form [46]

$$p_0^2 = p_3^2 + P^2, \quad (2.28)$$

where  $P$  is the spatial momentum chosen to be along the 1-direction. This is just as the ordinary commutative dispersion relation. However, when one chooses  $P$  to be along the 2-direction, one finds

$$p_0^2 = p_3^2 + P^2 + cg^2 \frac{1}{\theta^2 P^2}. \quad (2.29)$$

In both relations (2.28) and (2.29), the noncommutativity has been chosen as  $\theta^{12} = -\theta^{21} = \theta$  and the other directions are commutative. Thus, the dispersion relation becomes modified due to UV/IR mixing in the IR regime and we receive a contribution that we would not expect at low momenta, especially if we wish to consider the  $U_\star(1)$ -particle to be related to the photon at this energy.

## 2.4.2 Infinite non-locality vs. finite noncommutativity

If one could find a noncommutative quantum field theory without the UV/IR mixing, it would be a self consistent<sup>7</sup> quantum field theory with a minimal area. An interesting result in relation to this is obtained in [47], where it is concluded that the noncommutative  $\phi^4$  scalar field theory is renormalizable when one adds a harmonic term of the form  $\Omega(\theta_{\mu\nu}^{-1}x^\nu) \star (\theta^{-1\mu}_\sigma x^\sigma)$ , with  $\Omega$  a constant, to the Lagrangian. However, a Lagrangian with an explicit  $x$  dependence breaks translational invariance, but the progress in [47] has led to another model where the non-renormalizability

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<sup>6</sup>See section 2.5 for a definition of this gauge group.

<sup>7</sup>The theory would be self consistent in the same sense as e.g. supersymmetric quantum field theories are self consistent.

of the UV/IR mixing is resolved and the translational invariance of the scalar quantum field theory is preserved [48]. It is however not yet clear how this mechanism can be deployed in a gauge field theory.

We may also view the UV/IR mixing problem from a different angle. Due to the constancy of  $\theta^{\mu\nu}$  we have an infinite domain of validity of the noncommutativity. This is a clear signal of the UV/IR mixing. Because the noncommutativity is infinite, phenomena at short distance become linked with phenomena at long distance, the UV/IR mixing. That is why one might think that making  $\theta^{\mu\nu}$  a parameter that depends on the space-time position would help render the noncommutativity finite. The problem with this is that making  $\theta^{\mu\nu}$   $x$ -dependent spoils energy-momentum conservation in the quantum field theoretic sense. If this is the way we choose, then we cannot deploy quantum field theory in its usual sense, but must construct a new framework to work within. This seems to be a far too tedious approach and therefore other approaches should be tested at first.

In [49] it was attempted to reconcile the long and short distances in noncommutative quantum field theory by the introduction a support for the noncommutativity parameter inside a specific range<sup>8</sup>. However, it is difficult to construct an interaction that would remain nonlocal inside a finite range in this approach. Another difficulty in [49] is the choice of the observables that respect a new kind of microcausality due to the support of  $\theta^{\mu\nu}$  that reduces to the commutative microcausality outside the support of  $\theta^{\mu\nu}$ . In addition, one must deform the states in order to achieve finite noncommutativity and this deformation is highly nonunique. Since these problems together are rather severe, it was attempted in [50] to change the star product into something that would make it a theory of finite noncommutativity. In this approach it was required that the new product satisfies the commutator (2.1) and that it remains associative. One may for instance consider a Gaussian damping of the star product:

$$f(x) \star' g(x) := \int d^2z \int d^2y \frac{1}{\pi^2 \det \theta} \exp\left[\frac{2i}{\theta}(x-y+z-x)\right] \exp\left[-\frac{1}{\theta}((x-y)^2 + (x-z)^2)\right] f(y) g(z), \quad (2.30)$$

where the second exponential is the modification to the ordinary star product (2.6)

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<sup>8</sup>We shall from here on refer to constructions of noncommutativity of this kind as finite noncommutativity.

and we work in two dimensions for simplicity. This product is however not associative as can be seen by multiplication of plane waves, i.e.  $e^{ix \cdot p} \star' (e^{ix \cdot k} \star' e^{ix \cdot q}) = (e^{ix \cdot p} \star' e^{ix \cdot k}) \star' e^{ix \cdot q}$ .

Although the product (2.30) is nonassociative, we may still use it to calculate the equal time commutation relation of fields to see the effect of the Gaussian damping on microcausality in a quantum field theory with this product. From this calculation [50] one obtains a result that does not vanish but at infinity. This was to be expected since the infinite tails of the Gaussian distribution contribute to the product everywhere, but at infinity. This suggests that the product must be cutoff at some range, since an ordinary analytic function does not vanish but at infinity. A product of this type with a step-function cutoff is proposed [50], but one cannot even check its associativity due to the difficulty of analytical calculation with such a product.

In another approach [51], it was also concluded that the UV/IR mixing remains with a modified product of the type

$$f \star'' g = \frac{1}{(2\pi)^{\frac{d}{2}}} \int d^d p d^d q e^{ip \cdot x} \tilde{f}(q) \tilde{g}(p - q) e^{\alpha(p, q)}, \quad (2.31)$$

where  $\alpha(p, q)$  is an arbitrary function of  $p$  and  $q$ . This product was, amongst other things, required to remain associative and satisfy the commutator (2.1). It appears that the requirement of associativity is a rather strong restriction for a star-product. Therefore, it seems that a way out of the UV/IR mixing problem is not provided by modifying the star-product.

Another way out of this problem could be given by star-products on compact spaces, called fuzzy spaces (see e.g. [52] for a review). The fuzzy sphere  $S_N^2$  [12] is the most studied example, but the simplest four dimensional spaces are given by  $S_N^2 \times S_N^2$ , the noncommutative torus, and  $\mathbb{C}P_N^2$ , which has the symmetry group  $SU(3)$  [53]. These four dimensional compact noncommutative spaces are not free of trouble and it is argued that they have not yet been satisfactorily constructed [54] but they are very interesting mainly due to their UV finiteness. This can be easily understood by taking the fuzzy sphere as an example. It has a finite dimensional Lie algebra

$$[J_k, J_l] = \epsilon_{klm} J_m, \quad (2.32)$$

where the  $J_i$  are three  $j$ -dimensional matrices that form a basis for a  $j$  dimensional irreducible representation of the group  $SU(2)$ . The Casimir operator for this algebra is then given by

$$J_1^2 + J_2^2 + J_3^2 = \frac{1}{4}(j^2 - 1)I, \quad (2.33)$$

where  $I$  is the  $j$ -dimensional identity matrix. If one then defines the coordinates as  $x_a = kr^{-1}J_a$ , where  $k$  is a parameter defined to satisfy  $3r^4 = k^2(j^2 - 1)$  and  $r^2 = x^2 + y^2 + z^2$ , space has become truncated due to the relation to the finite dimensional Lie algebra. Thus, there are no UV divergences in a theory of this kind.

As a curiosity one may note that nonlocal field theories that are claimed to be unitary, causal, gauge invariant and even Lorentz invariant, in a manner of speaking, have been constructed (for a comprehensive review see [55]). However, in these theories one speaks of a quantum field theory on a stochastic space-time of extended objects that outside of their extent, obey Lorentz invariance and that inside their domain of nonlocality (related to the size of the object) act in a non-specified way. That is, nothing is said about how the nonlocality should manifest itself in these theories. One may implement it by choosing an appropriate measure for the space-time stochasticity, but there is no principle for which kind of measure should be chosen. One may do this to obtain a nonlocal quantum field theory as constructed in [56]. There is however some doubt about the new concept of causality that is introduced into these theories [57], and as the propagators change in this theory, but not the vertices as they do in noncommutative quantum field theory [44], we shall not dwell more on these nonlocal constructions.

## 2.5 Noncommutative gauge field theories

In order to construct for instance an extension of the standard model of particle physics to the noncommutative setting, we must naturally define gauge field theory within this approach. Due to the noncommutativity of the star-product, it is expectable that we will encounter some problems with the closure condition of the multiplication of group theory when we make the gauge group "local" and introduce the star-product between elements. This is indeed the case, and groups

such as  $SU(N)$ ,  $SO(N)$  and  $Sp(N)$  that close under the usual multiplication, do not allow for this kind of a minimal extension with the insertion of the noncommutative star-product. Of the groups interesting to physicists, the only group that does close under this extension is  $U_\star(N)$ . It is defined as follows:

Take the algebra  $u_\star(N)$  of  $U_\star(N)$ , which is generated by the  $N \times N$  hermitian matrices in which the elements of the matrices are multiplied by the star-product. Denote the generators of the usual  $u(N)$  by  $T_a, a = 1, \dots, N^2 - 1$  and normalize them as  $\text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab}$ . To these matrices we add the element  $T^0 = \frac{1}{\sqrt{2N}} \mathbf{1}_{N \times N}$  so that all hermitian matrices of  $u_\star(N)$  can be covered by the expansion

$$f = \sum_{A=0}^{N^2-1} f^A(x) T^A. \quad (2.34)$$

Then, we may define the Lie-algebra of  $u_\star(N)$  as

$$[f, g]_\star = f \star g - g \star f, \quad f, g \in u_\star(N), \quad (2.35)$$

which closes in the  $u_\star(N)$  algebra. This construction defines the noncommutative minimally extended  $U_\star(N)$  groups and their algebras. One may consider the groups  $SO(N)$ ,  $Sp(N)$ ,  $USp(N)$  and  $O(N)$  in the noncommutative setting [58, 59] and construct algebras for them. However, no extension of the group  $SU(N)$  has been made and one has therefore mostly been interested in the group  $U_\star(N)$  in the context of noncommutative gauge field theory.

This construction implies for gauge field theory that the gauge transformation for a  $U_\star(n)$  gauge field  $A_\mu = A_\mu^a T_a$  is

$$A_\mu \rightarrow U_\star(n) \star A_\mu \star U_\star^{-1}(n) - i U_\star(n) \star \partial_\mu U_\star^{-1}(n), \quad (2.36)$$

with  $U_\star(n) = e_\star^{i\lambda^a(x)T_a} = 1_{n \times n} + i\lambda^a(x)T_a - \frac{1}{2!}(\lambda^a(x)T_a)^2 + \dots$ ,  $U_\star^{-1}(n)$  its inverse,  $\lambda^a(x)$  a vector function and  $T_a$  the generator of the group  $U_\star(n)$ . From this we obtain the field-strength as

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu - i[A_\mu, A_\nu]. \quad (2.37)$$

The field-strength transforms gauge covariantly and the action

$$S = \int d^4x \text{Tr}(F^{\mu\nu} \star F_{\mu\nu}), \quad (2.38)$$

where the trace is taken over the generators of the group, is gauge invariant.

It should be mentioned that this is the minimal extension of group theory to the noncommutative context and that due to the discovered twisted Poincaré symmetry [25], one might think that this new symmetry could be extended to some form of a "twisted" symmetry principle within noncommutative gauge field theory. This was first attempted in [60] and seemed to result in that one could use any representations for the fields in the gauge algebra, over-riding the no-go theorem [61]<sup>9</sup>. Nevertheless, the proof in [60] assumed that once the fields transform in a given representation of the gauge algebra, their derivatives of any order also transform according to the representations of the gauge algebra, which is certainly not the case [62]. Another attempt at twisting the gauge algebra was made in [63] using covariant derivatives instead of usual ones, but unfortunately it leads to a non associative star-product and therefore this road has been abandoned for now.

One further complication in noncommutative gauge field theories arises when one wishes to construct gauge invariant observables into the theory. Since the field-strength is gauge covariant and other combinations of the gauge potential also are, in their most symmetric state, gauge covariant, the question of how to construct gauge invariant observables into noncommutative gauge field theories appears non-trivial. This problem is overcome in [64] where one notes that a noncommutative Wilson line of momentum  $p_\mu$  is gauge invariant if its length satisfies  $l_\mu = \theta_{\mu\nu} p_\nu$ . These operators can then be generalized [65] to yield gauge invariant operators that reduce in the commutative limit to their corresponding commutative gauge invariant operators. The Wilson lines and this construction of gauge invariant observables will become more relevant to us in section 4.4.

### 2.5.1 Charge quantization

A very interesting phenomenon within noncommutative gauge field theory encountered in [66], is that of charge quantization in noncommutative quantum electrodynamics. We can exemplify this by looking at the minimal noncommutative extension of quantum electrodynamics. In this model, with the gauge group  $U_\star(1)$ ,

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<sup>9</sup>This theorem is discussed more in the following subsection.

we must include a star-product in the covariant derivative between the gauge and matter fields

$$D_\mu \psi = \partial_\mu \psi - iQ_\psi A_\mu \star \psi. \quad (2.39)$$

Here,  $Q_\psi$  represents the charge of the matter field. This covariant derivative is covariant in the fundamental, anti-fundamental and adjoint representations of  $U_\star(1)$  if the charges are  $+1, -1$  and  $0$  respectively<sup>10</sup>. However, if we want to couple the gauge bosons to other matter fields with other charges than  $+1, -1$  or  $0$ , e.g. to quarks, the combination

$$D_\mu \psi^{(n)} = \partial_\mu \psi^{(n)} - inA_\mu \star \psi^{(n)}, \quad (2.40)$$

where  $n$  is a multiple of an integer denoting a charge different from  $+1, -1$  or  $0$ , fails to transform covariantly [66]. This is evident as the noncommutative gauge transformation for  $A_\mu$  dictates in which representation  $\psi^{(n)}$  in (2.40) can be and since we have used up the three possible representations of the matter fields for the charges  $+1, -1$  and  $0$ , none remains for the charges  $n = +1, -1, 0$ . This is referred to as the problem of charge quantization within noncommutative quantum electrodynamics and it places a serious restriction on the particle content of a possible noncommutative standard model of particle physics.

This problem becomes even more complicated [61] when one considers direct products of groups. To see this, we may consider the group  $G = G_1 \times G_2$ , defined as

$$g = g_1 \times g_2 \quad g \in G, g_i \in G_i \quad (2.41)$$

$$g' = g'_1 \times g'_2 \quad g' \in G, g'_i \in G_i. \quad (2.42)$$

In this case

$$g \cdot g' = (g_1 \times g_2) \cdot (g'_1 \times g'_2) = (g_1 \cdot g'_1) \times (g_2 \cdot g'_2), \quad (2.43)$$

where  $\cdot$  is the group multiplication, we can only consider  $G_1 = U_\star(m)$  and  $G_2 = U_\star(n)$  when the matter field is in the fundamental representation of one group e.g.  $U_\star(m)$  and in the antifundamental of the other group,  $U_\star(n)$  in our case. This follows because other choices of representations for the matter fields charged under

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<sup>10</sup>These charges are chosen for compatibility with the commutative limit of this model which should be usual quantum electrodynamics for electrons, positrons and photons.

the two groups would again lead to a non gauge-covariant expression. That is, a matter field can at most be charged under two group factors in a noncommutative gauge field theory. This poses a serious obstacle for the construction of a noncommutative standard model in the approach to noncommutativity taken in this work. However, the result in [67] where all presently known particles are included in a noncommutative standard model of the form  $U_*(3) \times U_*(2) \times U_*(1)$  gives hope to that this may be accomplished, even though the Higgs mechanism used there is not unitary [68] and although corrected in [69] the model suffers from that the reduced trace- $U(1)$  factors that remain after the symmetry reduction imply that they cannot be interpreted as the photon.

In the case of Matrix models the situation is better. The trace- $U(1)$  factors may be absorbed into a dynamical Poisson structure which can be interpreted as an effective metric and consequently, become part of a theory of emergent gravity [71]. These, in turn, can explain the IR behavior of the noncommutative quantum field theories as theories containing gravitons (not photons) that define a nontrivial background for the fields [72]. This explains why the  $U(1)$  sector cannot be disentangled from the other  $SU(N)$ <sup>11</sup> fields. One may even introduce spontaneous symmetry breaking into these theories [73]. They are however not yet viable candidates for an extension of the standard model as there, amongst other things, is at the moment no known mechanism to construct them so that they contain chirality.

## 2.5.2 Seiberg-Witten-map

Although the charge quantization problem [66] and the no-go theorem for noncommutative groups [61] restrict the particle content of a possible noncommutative standard model very strongly, these restrictions have been claimed to be less severe [74] if one uses the Seiberg-Witten (SW)-map [15]. Therefore we shall in the following first present the SW-map in its original form [15] and then briefly review another way of obtaining it [70]. This other method enables one to use fields charged under any Lie-algebra in the noncommutative case instead of only fields charged under the  $U_*(N)$  groups.

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<sup>11</sup>One may indeed speak of  $SU(N)$  valued fields in the Matrix model formulation of noncommutative field theory.

The SW-map is a consequence of open string theory in a background Neveu-Schwartz field. It maps noncommutative gauge fields to commutative gauge fields. More precisely, if we consider open strings ending on  $D$  branes and quantize them in a strong background  $B$ -field, which can be considered a low-energy limit of open string theory, we end up with a noncommutative field theory. This field theory can be expressed with either commutative gauge fields or noncommutative gauge fields depending on the regularization used. Since the theory in this low energy limit should not depend on the regularization used, one is lead to believe that there exists a map between the commutative and noncommutative gauge fields. The SW-map. This requirement can be written

$$A(A) + \delta_{\widehat{\lambda}} A(A) = A(A + \delta_{\lambda} A), \quad (2.44)$$

where  $A$  is the noncommutative gauge field,  $A$  is the commutative gauge field,  $\delta_{\widehat{\lambda}}$  is a variation w.r.t. the noncommutative gauge parameter  $\lambda$  and  $\delta_{\lambda}$  is a variation w.r.t. to the commutative gauge parameter  $\lambda$ . These variations are given by

$$\delta_{\lambda} A_i = \partial_i \lambda + i[\lambda, A_i] \quad (2.45)$$

$$\delta_{\widehat{\lambda}} A_i = \partial_i \lambda + i[\lambda, A_i]_{\star}, \quad (2.46)$$

where the Moyal bracket (2.10) has been used in the expression (2.46). As we have noted previously, one may only obtain spatial noncommutativity from string theory. Therefore the use of the latin indices in equations (2.45) and (2.46) and in the remainder of this section.

To fulfill the condition (2.44) one requires that if the commutative fields  $A$  and  $A'$  are gauge equivalent by a commutative gauge transformation  $e^{i\lambda}$ , then the noncommutative fields  $A$  and  $A'$  are gauge equivalent by a noncommutative gauge transformation  $e^{i\widehat{\lambda}}$ , where  $\lambda$  depends on both  $A$  and  $\lambda$  as a consequence of the SW-map. A perturbative solution to first order in the noncommutativity parameter  $\theta^{ij}$  for the SW-map can then be given [15] as

$$A'_i(A) = -\frac{1}{4}\theta^{kl} A_k, \partial_l A_i + F_{li} + \mathcal{O}(\theta^2) \quad (2.47)$$

$$\lambda'(\lambda, A) = \frac{1}{4}\theta^{kl} \partial_k \lambda, A_l + \mathcal{O}(\theta^2). \quad (2.48)$$

This is the SW-map as presented in [15]. However, as such there is no remedy for the charge quantization problem and one may not consider noncommutative extensions

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of groups such as  $SU(N)$  yet. This is claimed to change [74] if one generalizes the SW-map to hold for general star-products of the Kontsevich type [19] for different Poisson structures, one for the commutative theory and one for the noncommutative theory. These Poisson structures are related and their relation is given precisely by the SW-map. In that case, one may consider the enveloping algebra of the Lie-algebra of the noncommutative group in question instead of its Lie-algebra and this way generalize the noncommutative groups to also involve noncommutative  $SU(N)$  groups [70]. However, the solution to the charge quantization problem presented in [74] where one associates one different noncommutative gauge field with every different charge of the matter fields, and then maps them with the SW-map to their corresponding commutative fields, does not work [75]. Therefore, even by use of the SW-map, the charge quantization problem stands strong, although the SW-map does seem to enable one to consider noncommutative  $SU(N)$  groups.

# Chapter 3

## The Aharonov-Bohm Effect

The Aharonov-Bohm (AB) effect discovered in [76] and independently rediscovered in [77] is an effect which displays the importance of quantum mechanics as a new theory compared to classical mechanics and Maxwellian electrodynamics in a very transparent way. It can also be considered a link to gauge theory and its relation to holonomies of loops or Wilson loops, as we shall see shortly. These are very important in noncommutative gauge field theories as here, the gauge covariant Wilson loops are related to the gauge invariant observables of the theory and Wilson lines (see [78], chapter 4.2 for a review of noncommutative Wilson lines). The AB-effect is also very interesting in relation to the magnetic monopoles that we shall explore in chapter 4, as it is mathematically very closely related as we shall soon see and at the same time experimentally verified [79], whereas no magnetic monopoles have ever been observed [80].

### 3.1 Wilson loops and the AB-effect

The AB-effect is best explained by taking a close look at the experiment in figure 3.1. In this experiment an electron beam coming in from the left is divided into two and then passed through a double-slit as in the figure and the emerging beams form an interference pattern on the screen. The most crucial part here is that there is a magnetic field going through the solenoid, but confined within it. This means that

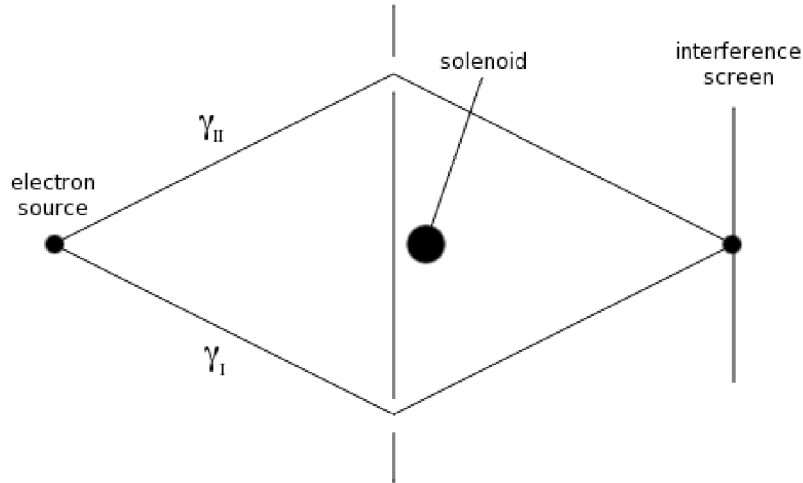


Figure 3.1: The experimental setup for a double-slit Aharonov-Bohm experiment. The electrons take paths  $\gamma_I$  and  $\gamma_{II}$  and an interference pattern is produced when they are reunited at the screen due to the different magnetic fluxes they experience when they pass above and below the solenoid.

the two electron beams  $\gamma_I$  and  $\gamma_{II}$  never experience the magnetic field where they travel. Classically, this would imply that no effect of the solenoid on the beam can be observed as no Lorentz force acts on the beam. However, quantum mechanically there is still something that can happen.

The Hamiltonian for this problem is given by

$$H = -\frac{1}{2m}(\partial_\mu - ieA_\mu)^2 + V(r), \quad (3.1)$$

where  $m$  is the electron mass,  $e$  the electron charge and  $V(r)$  represents the effect of the experimental apparatus. The Schrödinger equation for this Hamiltonian can be solved in the form

$$\psi_i^A(r) = \exp\left(ie \int_{\gamma_i} \vec{A}(r') \cdot dr'\right) \psi_i(r), \quad i = I, II, \quad (3.2)$$

where  $\psi_I(r)$  and  $\psi_{II}(r)$  are the solutions to the Schrödinger equation in the case  $\vec{A}(r) = 0$  along paths  $\gamma_I$  or  $\gamma_{II}$  that begin at the point  $P$  where the electron beam splits into two. If we then consider a superposition of the two wavefunctions along different paths  $\psi_I^A + \psi_{II}^A$ , such that they satisfy  $\psi_I^A(P) = \psi_{II}^A(P)$ , one can write the amplitude at a point  $Q$  on the screen as

$$\begin{aligned} \psi_I(Q) + \psi_{II}(Q) &= \exp\left(ie \int_{\gamma_I} \vec{A}(r') \cdot dr'\right) \psi_I(Q) + \exp\left(ie \int_{\gamma_{II}} \vec{A}(r') \cdot dr'\right) \psi_{II}(Q) \\ &= \exp\left(ie \int_{\gamma_{II}} \vec{A}(r') \cdot dr'\right) \exp\left(ie \int_{\gamma} \vec{A}(r') \cdot dr'\right) \psi_I(Q) + \psi_{II}(Q) , \end{aligned} \quad (3.3)$$

where  $\gamma$  is the loop  $\gamma_I - \gamma_{II}$ . It is clear that although the magnetic field is zero in the region where the electrons travel, the gauge potential need not be and via Stokes theorem

$$\int_{\gamma} \vec{A}(r') \cdot dr' = \int_S (\nabla \times \vec{A}) \cdot dS = \int_S \vec{B} \cdot dS = \Phi, \quad (3.4)$$

where  $S$  is a surface bounded by  $\gamma$ , the magnetic flux is nonzero. Therefore we see clearly that the interference patterns should be the same for two values of the magnetic fluxes  $\Phi_a$  and  $\Phi_b$  whenever

$$e(\Phi_a - \Phi_b) = 2\pi n, \quad n \in \mathbb{Z}. \quad (3.5)$$

Additionally we note that the AB-effect is directly related to the holonomy of a closed loop  $\exp\left(ie \int_{\gamma} \vec{A}(r') \cdot dr'\right)$  of the  $U(1)$  gauge connection. This is a phenomenon that generalizes nicely to the noncommutative AB-effect [I].

The Dirac potential for a magnetic monopole is very closely related to the AB-effect. We can see this if we take the Maxwell equations for a magnetic monopole

$$\nabla \times \vec{B} = 0 \quad (3.6)$$

$$\nabla \cdot \vec{B} = 4g\pi\delta^3(r), \quad (3.7)$$

where  $\vec{B} = \nabla \times \vec{A}$  and  $g$  is the magnetic charge, and note that they may be solved by the potential in spherical coordinates

$$A_x = -g \frac{1 + \cos \theta}{r \sin \theta} \sin \phi, \quad A_y = g \frac{1 + \cos \theta}{r \sin \theta} \cos \phi, \quad A_0 = A_z = 0. \quad (3.8)$$

This potential is singular along the line  $\theta = 0$ , that is along the positive  $z$ -axis. Due to the definition  $\vec{B} = \nabla \times \vec{A}$ ,  $\vec{A}$  cannot be determined everywhere in space because this definition clashes with the equation (3.7), but the potential (3.8) does the job, because the singular line  $\theta = 0$ , also called the Dirac string can be moved by a gauge transformation and is therefore not observable. The potential (3.8) results after careful regularization [81] in the magnetic field

$$\vec{B} = \frac{g\hat{r}}{r^2} + 4\pi g\delta(x)\delta(y)\theta(z)\hat{z}, \quad (3.9)$$

where  $\hat{r}$  is a unit radius vector,  $\hat{z}$  is a unit vector along the  $z$ -axis and  $\theta(z)$  is the Heaviside step-function.

What is interesting to us in this chapter is that since the Dirac string in (3.9) is unobservable because it can be moved by a gauge transformation, we require that it cannot be observed in an AB-experiment around a monopole instead of a solenoid. In this case the wave-function of (3.2) must be single-valued, which happens if the condition

$$\exp\left(ie \int_{\gamma'} \vec{A}(r') \cdot dr'\right) = 1, \quad (3.10)$$

or equivalently

$$2\pi n = e \int_{\gamma'} \vec{A}(r') \cdot dr' = \int_{dS} \vec{B} \cdot dS = 4\pi eg, \quad n \in \mathbb{Z}, \quad (3.11)$$

where the magnetic field (3.9) has been integrated over an infinitesimally small area  $dS$  which the Dirac string passes through, holds. Thus we have arrived at the Dirac Quantization Condition (DQC) [82]

$$\frac{2ge}{\hbar c} = n, \quad n \in \mathbb{Z}, \quad (3.12)$$

where we have restored the units. It can be seen by this derivation that the DQC is closely related to the AB-effect.

One could hope to pursue this analogy exploring magnetic monopoles in the non-commutative case, but it turns out to be less fruitful due to the technical difficulty of the calculation and the definition of a noncommutative source for the noncommutative version of equation (3.7). There is also no guarantee that the complete noncommutative Dirac potential displays a similar Dirac string. Moreover, once one

has a good noncommutative magnetic monopole source, it is more desirable to use the approach [84] to find the DQC because it directly shows that the Dirac string is unphysical. However, it is interesting to note that one can define a noncommutative gauge covariant Aharonov-Bohm phase factor [I], but the DQC for magnetic monopoles becomes a lot more difficult to obtain in the noncommutative case [83].

### 3.2 The noncommutative AB-effect

In the following we shall generalize the AB-effect to the noncommutative case following [I]. The noncommutative AB-effect has been calculated using a path integral approach in [85] and using a shift of the coordinates in [86], often referred to as the Bopp shift. These approaches are however troublesome as they do not preserve the gauge covariance of the AB-phase-factor and this ultimately leads to that the interference pattern calculated from them is not a gauge invariant observable.

The solution to this problem resides in how we treat the noncommutative Hamiltonian

$$H(x) = \frac{1}{2m} \left( P_i + \frac{e}{c} A_i(x) \right)_\star^2, \quad (3.13)$$

where  $P_i = -i \partial_i$ . This Hamiltonian is Weyl ordered, and therefore must be treated with the midpoint prescription in the path integral approach. However, the midpoint prescription fails to remain gauge covariant for the Hamiltonian (3.13) and hence the problem with the AB-effect calculated in [85, 86]. This should be taken as a reminder of how careful one has to be with path integrals in the noncommutative case. The remedy for this situation can be found by solving the noncommutative Schrödinger equation

$$i \frac{\partial}{\partial t} \Psi(x, t) = \frac{1}{2} \left( P_i + \frac{e}{c} A_i(x) \right)_\star^2 \Psi(x, t), \quad (3.14)$$

where the lower index  $x$  of the star indicates on which points the star-product acts. The solution of (3.14) is then given by

$$\Psi(x, x_0, t) = P \exp_{\star_{x_0}} \left[ -i \int_0^1 ds \frac{d\xi_i}{ds} A_i(x_0 + \xi(s)) \right] \star_{x_0} \psi(x, x_0, t). \quad (3.15)$$

The symbol P stands for path-ordering and the parameter  $0 \leq s \leq 1$  parametrizes the path  $C$  so that it has the endpoints  $x_0 + \xi(0) = x_0$  and  $x_0 + \xi(1) = x_0 + l = x$ .

$\psi(x, x_0, t)$  is the solution of the free Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi(x, x_0, t) = i\frac{\partial\psi(x, x_0, t)}{\partial t}. \quad (3.16)$$

The points  $x$  and  $x_0$  appearing here signify the locations of the screen and the source of the electrons for the AB-experiment, respectively. The appearance of the path ordered exponential is not surprising, since it is simply the noncommutative Wilson line [65] and therefore the AB-effect is given by the holonomy of the  $U_\star(1)$ -group (or Wilson loop) in this case. This is the analogy to the commutative case as discussed in the previous section and also to the result of [84], where the non-Abelian phase-factor is given, although in the commutative context.

If we write the solution (3.15) in the form

$$\Psi(x, x_0, t) = \mathcal{U}(x, x_0, C) \star_{x_0} \psi(x, x_0, t), \quad (3.17)$$

the path-ordered exponential is given by

$$\begin{aligned} \mathcal{U}(x, x_0, C) &\equiv \text{P exp}_{\star_{x_0}} \left[ -i \int_0^1 ds \frac{d\xi_i}{ds} A_i(x_0 + \xi(s)) \right] \\ &= 1 + \sum_{n=1}^{\infty} (-i)^n \int_0^1 ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n \frac{d\xi_{i_1}(s_1)}{ds_1} \cdots \frac{d\xi_{i_n}(s_n)}{ds_n} \\ &\quad \times A_{i_1}(x_0 + \xi(s_1)) \star_{x_0} \cdots \star_{x_0} A_{i_n}(x_0 + \xi(s_n)). \end{aligned} \quad (3.18)$$

This shows the difficulty with evaluating the AB-effect in the noncommutative case. We may calculate the Wilson loop in the case of an infinitesimal loop, but when the loop has a finite size, the result depends on the loop chosen. This would be true mathematically speaking also in the commutative case, but here one can relate the line integral around the loop to the magnetic flux which one knows to be confined within the solenoid and hence the form of the loop does not matter. This is due to the gauge invariance of the loop in the commutative case, but in the noncommutative case the Wilson loop is gauge covariant and the result therefore depends on the loop chosen. This means that in practice, we must calculate equation (3.18) explicitly for some finite size loop to evaluate the noncommutative AB-effect. Although the noncommutative interference pattern is gauge invariant and does not depend on the form of the loop, we cannot get to it without first calculating the noncommutative Wilson loop.

One might think that it would be possible to evaluate the AB-effect using an infinitesimal Wilson loop. In the commutative case this does work, because the magnetic field is gauge invariant and we therefore know that it is confined within the solenoid. However, in the noncommutative case we do not speak of the noncommutative magnetic field, but of the noncommutative potential. Due to that the noncommutative Wilson loop is gauge covariant, it is not clear that there is some gauge invariant piece of the noncommutative potential that remains completely confined within the solenoid. Therefore we should calculate the noncommutative AB-effect using a finite size Wilson loop which surrounds both the field within the solenoid and a piece outside of it.

In the case of a noncommutative magnetic monopole that has a Dirac string, we may still calculate the Wilson loop for an AB-experiment using an infinitesimal loop to find the DQC, because the Dirac string is infinitesimal. However, one should first show that the noncommutative Maxwell equations with an appropriate noncommutative source for the magnetic monopole, give a potential that has a Dirac string. In this case it seems easier to follow the approach taken in [III].

## Chapter 4

# Magnetic Monopoles

For a theorist, the magnetic monopole needs no separate introduction (see [87] for good reviews.). It is the one particle that fits so nicely into so many different theories, without ever being observed. Be it as it may with the existence of magnetic monopoles, it is a theoretical formulation that becomes very interesting in the context of a noncommutative space-time. Due to that the Dirac monopole is point-like, one cannot help but wonder whether this object can be placed into a noncommutative space-time. Therefore, the study of magnetic monopoles tells us something about the incorporation of point-like particles in noncommutative space-time. Furthermore, since the UV is mixed with the IR in noncommutative field theory [31], this theory has new IR singularities compared to the commutative version of field theory. Since the magnetic monopole is described by the delta function (a physical singularity), one may indeed think that the study of monopoles in a noncommutative space-time is an interesting one.

Having accepted that the magnetic monopole is an interesting object of study in the noncommutative case, one may wonder why one should start by looking at the Dirac monopoles, when there are monopole formulations much more suitable for quantum field theory such as, the 't Hooft-Polyakov monopole [88] or the monopoles in the BPS (Bogomol'nyi-Prasad-Sommerfield)-limit [89]. Although these objects are interesting in their own right, it is imperative to start the study of magnetic monopoles in noncommutative space-time with the Dirac monopole as it directly tells us what we may call a noncommutative magnetic charge. The existence of the

Dirac monopole is the foundation for calling the 't Hooft-Polyakov soliton or the BPS-solitons with magnetic charge, *magnetic* monopoles, as their gauge potential reduces to that of the Dirac monopole in the asymptotic far, i.e.  $r \rightarrow \infty$ . If the asymptotic limit is something else than the Dirac monopole potential, they are but solitons, without the interpretation of being particles with a noncommutative magnetic charge.

There have been many works devoted to the study of BPS-monopoles in noncommutative space-time and [90–93], gives only a partial list. Some of these works have taken the perturbative approach [90, 91] while some are nonperturbative [92, 93]. Some of them have even claimed to give rise to the DQC [91] given in (3.12). However, all these works share the same assumption: The noncommutative topological charge is the same as the noncommutative magnetic charge. As an example we may take the work [93], where a noncommutative  $U_\star(1)$  soliton is constructed. The soliton of [93] which reduces to the  $U(1)$ -monopole in the commutative limit has a vanishing charge, but this is its topological charge. The question is, does the topological charge represent the magnetic charge in noncommutative space-time? In a commutative space-time this is the case as the non-Abelian potential reduces to the Abelian potential of the Dirac monopole in the asymptotic far. However, in a noncommutative space-time of the type (2.1), the noncommutativity of the space-time does not go away in some physical limit<sup>1</sup>. That is why we start from the minimal noncommutative extension of Maxwell's equations and the obvious minimal noncommutative generalization of magnetic charge, and see whether we can define a DQC respective magnetic charge in the noncommutative context [III]. One could then in principle check whether the noncommutative 't Hooft Polyakov/BPS potentials give the noncommutative magnetic potential of the noncommutative Maxwell's equations in the asymptotic far. Nevertheless, to get this far, one must first solve the noncommutative Maxwell's equations for a noncommutative point-like particle.

Having further accepted that we should solve the noncommutative Maxwell's equations for a point-like particle with noncommutative magnetic charge, we face

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<sup>1</sup>It does go away in the  $\theta^{\mu\nu} \rightarrow 0$ -limit, but the limit  $\theta^{\mu\nu} \rightarrow 0$  is only a mathematical operation and does not represent any physical limit, unless the theory has some extra structure, such as e.g. a spontaneous symmetry breaking of the noncommutative space-time into the commutative space-time.

another problem. Which method of solution should we go for: a perturbative or a nonperturbative solution? Although one can justly argue that a soliton solution of the noncommutative Maxwell's equations should be given in a compact, nonperturbative way, one resorts to perturbation in both [II] and [III]. This follows because the DQC, which we shall be considering as the most important consequence of the theoretical construction of magnetic monopoles, is due to the delta function appearing in the commutative case. As a consequence, it is very hard to imagine a source that is a point-like particle in the noncommutative space-time, but which does either not reduce to the delta-function in the limit  $\theta^{\mu\nu} \rightarrow 0$  but does produce the DQC or then it reduces to the delta function in the commutative limit, but in a nonperturbative way. If we ignore the perturbative approach, the latter of the two cases mentioned seems to be more reasonable, but due to symmetry requirements of the point-particle source, it too, is very difficult to handle. This will be discussed more thoroughly in section 4.4.

In this chapter we shall begin by explaining the importance of [II] for the introduction of a magnetic source into the noncommutative Maxwell's equations. We shall then show that the source discovered in [II] may be used in the first order of the perturbation in  $\theta^{\mu\nu}$  of the Maxwell's equations that ultimately leads to that the DQC holds in the noncommutative setting, perturbatively to first order in the parameter  $\theta^{\mu\nu}$  [III]. We finally discuss the difficulties with the same analysis carried out in second order where the DQC does no longer hold [83].

## 4.1 A source for the magnetic monopole

To discuss magnetic monopoles in the noncommutative setting, it is important to generalize the source of the magnetically charged particle appearing in the Maxwell equations. In the commutative case, it is but a Dirac delta-function, but due to the requirement of gauge covariance of the noncommutative Maxwell's equations (see section 4.4), the noncommutative source must also transform gauge covariantly. Consequently, the noncommutative source cannot remain only a delta function, since it does not transform under noncommutative gauge transformations and it thereby must be different from the commutative source. We shall in the following

give a brief overview of a method [94] to find the DQC, which indicates what the magnetic source looks like if the DQC is preserved.

In [94] it is realized that one may obtain the DQC by requiring the associativity of the gauge invariant translation operators of a certain algebra. This is an example of how three-cocycles, that appear when a group of transformations is non-associative, are related to physics. The algebra considered in [94] is given by

$$[x_i, x_j] = 0, \quad [x_i, \pi_j] = i\hbar\delta_{ij}, \quad (4.1)$$

$$[\pi_i, \pi_j] = i\hbar\frac{e}{c}\epsilon_{ijk}B_k(\vec{x}). \quad (4.2)$$

These are the quantum mechanical brackets for an electrically charged particle with the charge  $e$  moving in a magnetic field  $B_k(\vec{x})$  and  $\pi_i = -i\hbar\partial_i - \frac{e}{c}A_i(\vec{x})$ , where  $A_i(\vec{x})$  is the gauge potential. Together with the Hamiltonian

$$H = \frac{\vec{\pi}^2}{2m}, \quad \vec{\pi} = m\vec{\dot{x}}, \quad (4.3)$$

the brackets (4.2) yield

$$\dot{\vec{x}} = \frac{i}{\hbar}[H, \vec{x}] = \frac{\vec{\pi}}{m}, \quad (4.4)$$

$$\dot{\vec{\pi}} = \frac{i}{\hbar}[H, \vec{\pi}] = \frac{e}{2mc}[\vec{\pi} \times \vec{B} - \vec{B} \times \vec{\pi}]. \quad (4.5)$$

Up to this point nothing has been said about the form of the magnetic field, but once we consider the Jacobi identity

$$\frac{1}{2}\epsilon_{ijk}[[\pi_i, \pi_j], \pi_k] = \frac{e\hbar^2}{c} \cdot \vec{B}, \quad (4.6)$$

we note that the magnetic field has to be source-free. If this is not the case, there will be a loss of the associativity of the translation generators  $T(\vec{a}) \equiv \exp\left(-\frac{i}{\hbar}\vec{a} \cdot \vec{\pi}\right)$  given by

$$\left(T(\vec{a}_1)T(\vec{a}_2)\right)T(\vec{a}_3) = \exp\left(-\frac{ie}{\hbar c}\omega(\vec{x}; \vec{a}_1, \vec{a}_2, \vec{a}_3)\right)T(\vec{a}_1)\left(T(\vec{a}_2)T(\vec{a}_3)\right). \quad (4.7)$$

Here,  $\vec{a}_i$  are constant vectors and the non-trivial phase-factor is the magnetic flux coming out of the tetrahedron formed by the  $\vec{a}_i$  (see figure 4.1<sup>2</sup>). Since the flux

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<sup>2</sup>The figure is taken from [95].

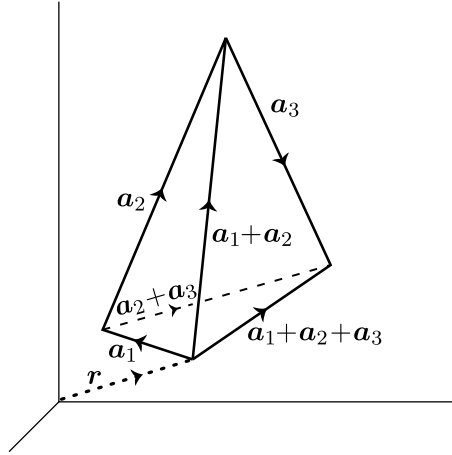


Figure 4.1: The tetrahedron formed by the vectors  $\vec{a}_1$ ,  $\vec{a}_2$  and  $\vec{a}_3$ .

through the tetrahedron in (4.7) is given by

$$\omega(\vec{x}; \vec{a}_1, \vec{a}_2, \vec{a}_3) = \int_{\partial\Delta} \vec{B} \cdot d\vec{S} = \int_{\Delta} \nabla \cdot \vec{B} dV, \quad (4.8)$$

where  $\partial$  signifies the boundary of the tetrahedron and  $\int_{\Delta}$  its volume, the phase-factor is given by

$$\exp\left(-\frac{ie}{\hbar c} \omega(\vec{x}; \vec{a}_1, \vec{a}_2, \vec{a}_3)\right) = \exp\left(-\frac{ie}{\hbar c} \int_{\Delta} \nabla \cdot \vec{B} dV\right). \quad (4.9)$$

This phase-factor must be equal to unity for the condition of associativity of the translation generators (4.7) to hold. For a magnetic monopole of charge  $g$  the requirement of associativity then becomes

$$1 = \exp\left(-\frac{ie}{\hbar c} \int_{\Delta} \nabla \cdot \vec{B} dV\right) = \exp\left(-\frac{4\pi ieg}{\hbar c}\right) \quad (4.10)$$

$$2eg = \hbar cn, \quad n \in \mathbb{Z} \quad (4.11)$$

where the equation  $\nabla \cdot \vec{B} = 4\pi g \delta^3(\vec{r})$ , has been used. Hence, the requirement of associativity of the translation generators, is equivalent to the requirement of the DQC holding for an electron moving in the field of a magnetic monopole. Additionally, we can read of the source from the right hand side of (4.6). We shall use this observation in the construction of the noncommutative Maxwell's equations in section 4.4.

## 4.2 The DQC from a noncommutative quantum mechanical model

The previous review of the method of [94] suggests via (4.6) that one can obtain the noncommutative magnetic source, modulo some constant factor, using a noncommutative quantum mechanical algebra. This has been attempted in [II], where a specific quantum mechanical model is used to derive the DQC. The model used [96] is a model in two dimensions with Lagrangian given in [97], but in [II] it is straight-forwardly generalized to three dimensions with a Lagrangian given by

$$L = \left( P_i + \frac{e}{c} A_i \right) \dot{X}_i - \frac{1}{2} \epsilon_{ijk} P_i \dot{P}_j \theta_k - \frac{1}{2m} \vec{P}^2 + e A_0, \quad (4.12)$$

where  $P_i$  is the momentum,  $\theta_k$  - the noncommutativity parameter, of dimension  $(\text{length})^2/\text{action}$ , and  $A_i, A_0$  - the magnetic and electric potential, respectively. The two dimensional Lagrangian that the Lagrangian (4.12) is an extension of, is the Lagrangian of a free particle in an electromagnetic field, supplemented by a term  $k\epsilon_{ij}\dot{x}_i\dot{x}_j$  that produces the noncommutativity of the model. This Lagrangian can be rewritten by the introduction of new momenta corresponding to each order of derivatives such that to  $\dot{x}$  there is a corresponding momentum and to  $\ddot{x}$  there is a different corresponding momentum. These momenta can then be introduced into the Lagrangian as Lagrange multipliers and after a choice of new variables one arrives at the model [96]. Hence, the unfamiliar form of the Lagrangian (4.12).

The Dirac brackets [98] can be calculated for the model (4.12) and they are

$$\begin{aligned} X_i, X_j &= \frac{\epsilon_{ijk}\theta_k}{1 - \frac{e}{c}\vec{\theta} \cdot \vec{B}}, \\ X_i, P_j &= \frac{\delta_{ij} - \frac{e}{c}B_i\theta_j}{1 - \frac{e}{c}\vec{\theta} \cdot \vec{B}}, \\ P_i, P_j &= \frac{\epsilon_{ijk}\frac{e}{c}B_k}{1 - \frac{e}{c}\vec{\theta} \cdot \vec{B}}. \end{aligned} \quad (4.13)$$

For the quantization of a noncommutativity of this type, one can note that this algebra is of the Moyal type with a constant  $\theta_{ij} = \epsilon_{ijk}\theta_k$  only if we expand it perturbatively to first order in  $\theta^{ij}$ . One might think that this Dirac bracket algebra (4.13) when expanded to second order, could by a coordinate transformation become

of the Moyal type. However, the algebra obtained in this way is very difficult to quantize<sup>3</sup>, as we cannot find appropriate representations for the operators  $X_i$  and  $P_i$  in the quantized brackets. This issue will be discussed some more at the end of this section.

In first order one can however treat the problem perturbatively and we expand the algebra (4.13) to first order in  $\theta_k$  and change each Dirac Bracket to a quantum bracket and promote  $X_i$  and  $P_i$  to operators to obtain:

$$\begin{aligned} [\hat{X}_i, \hat{X}_j] &= i\hbar\epsilon_{ijk}\theta_k + \mathcal{O}(\theta^2), \\ [\hat{X}_i, \hat{P}_j] &= i\hbar\left[\delta_{ij} - \frac{e}{c}B_i(\hat{X})\theta_j + \frac{e}{c}\delta_{ij}\vec{\theta} \cdot \vec{B}(\hat{X})\right] + \mathcal{O}(\theta^2), \\ [\hat{P}_i, \hat{P}_j] &= i\hbar\frac{e}{c}\epsilon_{ijk}B_k(\hat{X})\left[1 + \frac{e}{c}\vec{\theta} \cdot \vec{B}(\hat{X})\right] + \mathcal{O}(\theta^2). \end{aligned} \quad (4.14)$$

We can then choose new definitions for our operators and we choose to set

$$x_i = \hat{X}_i + \frac{1}{2}\epsilon_{ijk}\hat{P}_j\theta_k, \quad (4.15)$$

$$B_i(\hat{X}) = B_i(\vec{x}) - \frac{1}{2}\epsilon_{njk}\theta_k\hat{P}_j\partial_n B_i(\vec{x}) + \mathcal{O}(\theta^2), \quad (4.16)$$

$$p_j = \hat{P}_j - \frac{1}{2}\frac{e}{c}\left(\hat{P}_j(\vec{B} \cdot \vec{\theta}) - \vec{P} \cdot \vec{B}\theta_j\right). \quad (4.17)$$

Here it is important to notice the order in which the equations are given. We first choose the form of the  $x_i$  in equation (4.15) which in turn gives us (4.16) and then one can calculate an intermediate algebra, and finally define the  $p_j$  in (4.17). It is not possible to change this order of doing things and it is important to remember once we look at the problems related to the second order calculation (see section 4.2.1). The quantum algebra now reads

$$\begin{aligned} [x_i, x_j] &= 0 + \mathcal{O}(\theta^2), \\ [x_i, p_j] &= i\hbar\delta_{ij} + \mathcal{O}(\theta^2), \\ [p_i, p_j] &= i\hbar\frac{e}{c}\epsilon_{ijk}B_k - \frac{e}{2c}i\hbar\left(p_{[j}\partial_{i]}(\vec{B} \cdot \vec{\theta}) + p_{[i}\theta_{j]} \cdot \mathbf{B} + \mathbf{p} \cdot \theta_{[i}\partial_{j]}\mathbf{B}\right) \\ &\quad + p_{[j}[p_{i}], \mathbf{B}] \cdot \vec{\theta} + \vec{p} \cdot [\vec{B}, p_{[i}\theta_{j]}] + \mathcal{O}(\theta^2), \end{aligned} \quad (4.18)$$

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<sup>3</sup>In principle one might, if we are dealing with a genuine Poisson structure, quantize the brackets by the Kontsevich method [19], but then we have no explicit representations of the operators and these are what we are looking for in order to complete the construction as sketched in section 4.1.

with lower index brackets indicating anti-symmetrization w.r.t. to the indices. A proper quantization of the algebra (4.18) requires us to specify the representations of the operators appearing in the algebra. This is the part of this method where we run into trouble when we go to second order in  $\theta_k$ , but in first order of  $\theta_k$  these representations are simply given by

$$p_i = -i\hbar\partial_i - \frac{e}{c}A_i(\vec{x}) + T_i(\theta, \vec{x}) + \mathcal{O}(\theta^2), \quad (4.19)$$

where  $x_i$  is the usual commutative coordinate,  $T_i(\theta, \vec{x}) = -\frac{1}{2}\frac{e}{c}\theta_i \cdot \vec{B} + G_i$  and  $\partial_j G_i = \frac{1}{2\hbar}\left(\frac{e}{c}\right)^2 A_j\theta_i \cdot \vec{B}$ . Thus we have the quantized algebra given in the  $x$ -representation as

$$\begin{aligned} [x_i, x_j] &= 0 + \mathcal{O}(\theta^2), \\ [x_i, p_j] &= i\hbar\delta_{ij} + \mathcal{O}(\theta^2), \\ [p_i, p_j] &= i\hbar\frac{e}{c}\epsilon_{ijk}B_k + \frac{e}{2c} (i\hbar\partial_{[i} + \frac{e}{c}A_{[i}\theta_{j]}) \cdot \vec{B} + \mathcal{O}(\theta^2). \end{aligned} \quad (4.20)$$

From (4.20) we then obtain the Jacobi identity

$$\frac{1}{2}\epsilon_{ijk}[[p_i, p_j], p_k] = -\hbar^2\frac{e}{c} \cdot \vec{B} + \frac{i\hbar}{2}\left(\frac{e}{c}\right)^2 \epsilon_{ijk}\partial_k(A_i\theta_j \cdot \vec{B}), + \mathcal{O}(\theta^2). \quad (4.21)$$

which directly shows that the DQC can be satisfied to first order in  $\theta_k$  in this noncommutative quantum mechanical model because the second term on the right hand side does not contribute to the associativity condition (4.10) due to the integral. However, what is more important is that we may read off the noncommutative source from (4.21) in first order of the perturbation. It is, perhaps modulo some constant in front of the first order correction, given by

$$\rho_{NC} = 4\pi g \left( \delta^3(\vec{r}) - \frac{e}{2c\hbar}\epsilon_{ijk}\partial_k(A_i\theta_j\delta^3(\vec{r})) \right) + \mathcal{O}(\theta^2). \quad (4.22)$$

As can be easily checked, this source transforms gauge covariantly to first order in  $\theta_k$  under a noncommutative gauge transformation. This symmetry requirement on the source will be essential in section 4.4.

### 4.2.1 Problems in the second order of $\theta_k$

Having quantized the model (4.12) to first order in **[II]**, one would indeed expect that a generalization to second order is not all that difficult. However, it becomes very tedious, as we shall illustrate in the following.

It might seem that one should start with the simplest approach, that is expand (4.13) to second order and quantize it and then try to find the appropriate representations for the operators. However, as this does not work, we choose to exemplify all the trouble in a more general approach, in which the Lagrangian (4.12) is modified to become

$$L = (P_i + \frac{e}{c}A_i + K_i(X, P))\dot{X}_i - \frac{1}{2}\epsilon_{ijk}P_i\dot{P}_j\theta_k + E_i(X, P)\dot{P}_i - \frac{1}{2m}\vec{P}^2 + eA_0, \quad (4.23)$$

Here, we have inserted two new functions  $K_i(X, P)$  and  $E_i(X, P)$  that we hope to be able to choose in such a way that we can quantize the algebra to second order in  $\theta_k$ . Using the method of [99], the Dirac brackets are easily found to be

$$\begin{aligned} X_i, X_j &= -\epsilon_{ijk}\left(\theta_k\left(1 - \frac{e}{c}B \cdot \theta\right) + \epsilon_{kqm}\partial_q^P E_m(X, P)\right) + \mathcal{O}(\theta^3), \\ X_i, P_j &= \delta_{ij}\left(1 - \frac{e}{c}B \cdot \theta\left(1 - \frac{e}{c}B \cdot \theta\right) + \frac{e}{c}\epsilon_{pkl}B_p\partial_k^P E_l(X, P)\right) \\ &\quad + \frac{e}{c}B_i\theta_j\left(1 - \frac{e}{c}B \cdot \theta\right) - \partial_i^P K_j + \partial_j^X E_i(X, P) - \frac{e}{c}B_i\epsilon_{jkl}\partial_k^P E_l(X, P) + \mathcal{O}(\theta^3), \\ P_i, P_j &= \epsilon_{ijk}\frac{e}{c}\partial_k^P K_l(X, P)B_l + \epsilon_{klm}\partial_l^X K_m(X, P) - \frac{e}{c}B_q\partial_q^X E_k(X, P) \\ &\quad + \frac{e}{c}B_k\left(1 - \partial_l^P K_l(X, P) + \partial_l^X E_l(X, P) - \frac{e}{c}B \cdot \theta\left(1 - \frac{e}{c}B \cdot \theta\right)\right) \\ &\quad + \frac{e}{c}\epsilon_{plm}B_p\partial_l^P E_m(X, P) + \mathcal{O}(\theta^3). \end{aligned} \quad (4.24)$$

Here  $\partial_q^P$  is a differentiation w.r.t. to the variable  $P$  and  $\partial_j^X$  is a differentiation w.r.t. the variable  $X$ .

Next we quantize the algebra, promote  $X$  and  $P$  to operators  $\hat{X}$  and  $\hat{P}$  and take every commutator times  $i\hbar$ . Then we can use the known first order definitions of the operators from the equations (4.15), (4.16) and (4.17) and write

$$x_i = \hat{X}_i - \frac{1}{2}\epsilon_{ijk}\hat{P}_j\theta_k + Q_i(\hat{X}, \hat{P}) + \mathcal{O}(\theta^3), \quad (4.25)$$

$$B_i(\hat{X}) = B_i(x) + \frac{1}{2}\epsilon_{klm}\theta_m\hat{P}_l\partial_k B_i(x) + A_i(\hat{X}, \hat{P}) + \mathcal{O}(\theta^3), \quad (4.26)$$

$$p_i = \hat{P}_i + \frac{e}{2c}\left(\hat{P}_i(B \cdot \theta) - \hat{P} \cdot B\theta_i\right) + C_i(\hat{X}, \hat{P}) + \mathcal{O}(\theta^3), \quad (4.27)$$

where  $Q_i(\hat{X}, \hat{P})$ ,  $A_i(\hat{X}, \hat{P})$  and  $C_i(\hat{X}, \hat{P})$  are new unspecified functions that are all of second order in  $\theta_k$ . The above  $x$ , we want to commute now to second order in

$\theta_k$ . This will set a requirement on the form of the  $Q_i(\hat{X}, \hat{P})$  and the  $E_i(\hat{X}, \hat{P})$ . It is

$$i\hbar\epsilon_{ijk}\left(\frac{e}{4c}\theta_k B \cdot \theta - \epsilon_{kqm}\partial_q^P E_m(\hat{X}, \hat{P})\right) + [\hat{X}_i, Q_j(\hat{X}, \hat{P})] - [\hat{X}_j, Q_i(\hat{X}, \hat{P})] = 0. \quad (4.28)$$

If this requirement is satisfied, then the following is the algebra for  $x$  and  $\hat{P}$

$$[x_i, x_j] = 0 + \mathcal{O}(\theta^3), \quad (4.29)$$

$$\begin{aligned} [x_i, \hat{P}_j] &= i\hbar \delta_{ij} \left(1 - \frac{e}{2c}(1 - \frac{e}{c}B \cdot \theta) + \frac{e}{c}\epsilon_{pkl}B_p\partial_k^P E_l(\hat{X}, \hat{P}) - \frac{e}{4c}\epsilon_{klm}\theta_m\hat{P}_l\partial_k(B \cdot \theta)\right) \\ &\quad + \frac{e}{2c}B_i\theta_j\left(1 - \frac{e}{c}B \cdot \theta\right) - \frac{e}{c}B_i\epsilon_{jkl}\partial_k^P E_l(\hat{X}, \hat{P}) + \frac{e}{4c}\epsilon_{klm}\theta_m\hat{P}_l\partial_k B_i\theta_j \\ &\quad - \partial_i^P K_j(\hat{X}, \hat{P}) + \partial_j E_i(\hat{X}, \hat{P}) + [Q_i(\hat{X}, \hat{P}), \hat{P}_j] + \mathcal{O}(\theta^3), \end{aligned} \quad (4.30)$$

$$[\hat{P}_i, \hat{P}_j] = i\hbar\frac{e}{c}\epsilon_{ijk}B_k\left(1 - \frac{e}{c}B \cdot \theta\right) + i\hbar\frac{e}{2c}\epsilon_{ijk}\epsilon_{qlm}\theta_m\hat{P}_l\partial_q B_k + \mathcal{O}(\theta^2), \quad (4.31)$$

where differentiation w.r.t to the variable  $x$  is now represented by  $\partial_j$ . Observe that the last bracket (4.31) is only given to order  $\mathcal{O}(\theta)$ . The reason is that for the time being, we want to find an algebra for which  $[x_i, x_j] = 0 + \mathcal{O}(\theta^3)$  and  $[x_i, p_j] = i\hbar\delta_{ij} + \mathcal{O}(\theta^3)$  holds. The form of  $[\hat{P}_i, \hat{P}_j]$  is only important to the order given, for this task. It is also good to remember throughout this calculation that the  $P$ -differentiation appearing in these brackets does not necessarily have anything to do with the representations of the operators, they are usual classical derivatives that appear due to the Dirac brackets (4.24) of the Lagrangian (4.23). The hope is to be able to choose them in such a fashion that one can quantize the algebra and find its representations. Then, once the functions  $E_i(X, P)$  and  $K_i(X, P)$  have been chosen in the classical sense, one has to consider the operator ordering of them when they become operators. However, if we cannot choose them even classically, the algebra we are looking for cannot be constructed by way of the Dirac brackets. That is why, although it may look a little strange, the derivatives of the functions  $E_i(\hat{X}, \hat{P})$  and  $K_i(\hat{X}, \hat{P})$  appear also in the brackets.

Next, we observe that the  $(x, \hat{P})$ -algebra is the same to zeroth order in  $\theta_k$  as the  $(\hat{X}, \hat{P})$ -algebra, therefore we may (perturbatively) replace the  $\hat{X}$ :s by the  $x$ :s in the functions  $Q_i(\hat{X}, \hat{P})$ ,  $E_i(\hat{X}, \hat{P})$  and  $K_i(\hat{X}, \hat{P})$ , because these terms are already of second order in  $\theta_k$ . Then we use equation (4.27) to calculate a condition for the

commutator of  $x$  and  $p$ , to become  $[x_i, p_j] = i\hbar\delta_{ij} + \mathcal{O}(\theta^3)$ . It is given by

$$\begin{aligned} i\hbar \delta_{ij} \left( \frac{e}{c} \epsilon_{pkl} B_p \partial_k^P E_l(x, \hat{P}) - \frac{e}{4c} \epsilon_{klm} \theta_m \hat{P}_l \partial_k (B \cdot \theta) \right) - \frac{e}{c} B_i \epsilon_{jkl} \partial_k^P E_l(x, \hat{P}) \\ + \frac{e}{4c} \epsilon_{klm} \theta_m \hat{P}_l \partial_k B_i \theta_j - \partial_i^P K_j(x, \hat{P}) + \partial_j E_i(x, \hat{P}) + [Q_i(x, \hat{P}), \hat{P}_j] \\ + [x_i, S_j(x, \hat{P})] + \mathcal{O}(\theta^3) = 0, \end{aligned} \quad (4.32)$$

where we have used

$$p_i = \hat{P}_i + \frac{e}{2c} \left( \hat{P}_i (B \cdot \theta) - \hat{P} \cdot B \theta_i \right) \left( 1 - \frac{e}{c} B \cdot \theta \right) + S_i(x, \hat{P}) + \mathcal{O}(\theta^3), \quad (4.33)$$

and the function  $C_i(\hat{X}, \hat{P})$  appearing in (4.27) is chosen as  $C_i(\hat{X}, \hat{P}) = -\frac{e}{2c} \left( \hat{P}_i (B \cdot \theta) - \hat{P} \cdot B \theta_i \right) \frac{e}{c} B \cdot \theta + S_i(\hat{X}, \hat{P})$  where  $S_i(\hat{X}, \hat{P})$  is a new unspecified function in second order of  $\theta_k$  and then finally given in the variables  $x$  and  $\hat{P}$  and put into equation (4.33). The problem now consists of solving two equations (4.28) and (4.32). This problem can be written in a simpler form, although one that is not as general as the two equations (4.28) and (4.32)<sup>4</sup>, if one notes that equation (4.28) can be written as

$$\frac{e}{4c} \theta_p B \cdot \theta - \frac{i}{\hbar} \epsilon_{klp} [x_k, Q_l(x, \hat{P})] = \epsilon_{pkl} \partial_k^P E_l(x, \hat{P}). \quad (4.34)$$

This can be substituted into eqn (4.32) to get

$$\begin{aligned} i\hbar \delta_{ij} \left( \frac{e}{c} B_p \left( \frac{e}{4c} \theta_p B \cdot \theta - \frac{i}{\hbar} \epsilon_{klp} [x_k, Q_l(x, \hat{P})] \right) - \frac{e}{4c} \epsilon_{klm} \theta_m \hat{P}_l \partial_k B \cdot \theta \right) - \frac{e}{c} B_i \left( \frac{e}{4c} \theta_j B \cdot \theta \right. \\ \left. - \frac{i}{\hbar} \epsilon_{klj} [x_k, Q_l(x, \hat{P})] \right) + \frac{e}{4c} \epsilon_{klm} \theta_m \hat{P}_l \partial_k B_i \theta_j - \partial_i^P K_j(x, \hat{P}) + \partial_j E_i(x, \hat{P}) \\ + [Q_i(x, \hat{P}), \hat{P}_j] + [x_i, S_j(x, \hat{P})] = 0. \end{aligned} \quad (4.35)$$

This complicated looking equation can be simplified a bit by setting

$$S_j(x, \hat{P}) = -\frac{e^2}{4c^2} \hat{P}_j B \cdot \theta - \hat{P} \cdot B \theta_j B \cdot \theta + S'_j(x, \hat{P}), \quad (4.36)$$

where another unknown function  $S'_j(x, \hat{P})$  has been introduced, and by noting that since  $p_i$  is the same as  $\hat{P}_i$  to zeroth order in  $\theta$ , the representation for the derivative terms is the same for both operators to this order and  $i\hbar\partial_i B_j$  can be identified with

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<sup>4</sup>Equation (4.28) is more general than the condition (4.34), but condition (4.34) is sufficient for equation (4.28) to be satisfied.

$[B_j, \hat{P}_j]$  in terms that are second order in  $\theta_k$ . This gives us the final form of equation (4.35) as

$$\begin{aligned} & \frac{e}{4c} \epsilon_{klm} \theta_m \hat{P}_l \theta_j [B_i, \hat{P}_k] - \delta_{ij} [B \cdot \theta, \hat{P}_k] - i\hbar \partial_i^P K_j(x, \hat{P}) + i\hbar \partial_j E_i(x, \hat{P}) \\ & + [Q_i(x, \hat{P}), \hat{P}_j] + [x_i, S'_j(x, \hat{P})] + \frac{e}{c} \delta_{ij} \epsilon_{pkl} B_p [x_k, Q_l(x, \hat{P})] - \frac{e}{c} B_i \epsilon_{jkl} [x_k, Q_l(x, \hat{P})] = 0. \end{aligned} \quad (4.37)$$

The main problem of this equation resides in the two first terms. If they can be removed without generating other terms containing two or more  $\hat{P}$ :s then the equation can most probably be solved. Unfortunately there are far too many unknown functions in this equation, that one might get a good view of how the equation might be solved, if it can be solved. Furthermore, even if it could not be solved, one could still think that the simultaneous equations (4.28) and (4.32) could have a solution, and that would have to be checked. Therefore this approach seems to become too difficult in the second order of  $\theta_k$  and it is consequently abandoned.

It should be emphasized that there is certainly no final proof that this method cannot work in the second order of  $\theta_k$ , but (4.37) does illustrate how difficult it becomes to construct the algebra in second order of  $\theta_k$ . In section 4.4, we shall see more explicitly what kind of trouble the perturbative expansion to second order of  $\theta_k$  does introduce.

### 4.3 Magnetic monopoles and the Wu & Yang method

The monopole equations due to Dirac [82]:

$$\nabla \cdot \vec{B} = 4\pi g \delta^3(r), \quad (4.38)$$

$$\nabla \times \vec{B} = 0, \quad (4.39)$$

where  $g$  is the magnetic charge, have the peculiar property that the potential  $\vec{A}$  of the magnetic field  $\vec{B} = \nabla \times \vec{A}$  cannot be defined to be non-singular everywhere in the space  $\mathbb{R}^3 - 0$  due to the identity  $\nabla \cdot (\nabla \times \vec{A}) = 0$ . In Dirac's original work this problem is circumvented by showing that one can define a potential that is non-singular everywhere in space except along a semi-infinte line (called a string)

stretching out from the monopole into infinity. This string may be rotated by gauge transformations and therefore cannot be observable. However, the gauge transformations required for the job are singular and one may therefore still think that there could be something rotten with the Dirac monopoles. This situation is changed for the better in [84]. If one abandons the need to solve the equations (4.38) and (4.39) by one gauge potential in the whole of space, the situation improves considerably.

The key insight in [84] is that the Abelian field-strength  $F^{\mu\nu}$  of electromagnetism underdescribes electromagnetic phenomena, the phase  $\frac{e}{\hbar c} \int A_\mu dx^\mu$  overdescribes them, but the phase-factor

$$\exp\left(\frac{ie}{\hbar c} \int A_\mu dx^\mu\right), \quad (4.40)$$

is just enough. This can be seen by the following argument: Consider the Aharonov-Bohm experiment. In a region where  $F^{\mu\nu} = 0$  the electrons are affected by the gauge potential of  $F^{\mu\nu}$  although  $F^{\mu\nu}$  itself is zero. Therefore  $F^{\mu\nu}$  underdescribes electromagnetism. On the other hand the phase of the Aharonov-Bohm experiment  $\frac{e}{\hbar c} \int A_\mu dx^\mu$  overdescribes it, because different phases, may still have the same interference fringes and therefore describe the same physical situation. However, the phase-factor (4.40) does not share these problems.

If one comes to accept that the phase-factor is the best description of electromagnetism, we must take special care when we study gauge potentials that are singular, such as the magnetic monopole potential of Dirac. The phase-factor (4.40) is not well-defined if its path of integration goes through a singularity. A simple resolution of this problem is found by defining the potential in two regions of space. In these regions the potentials are singularity-free, their curls give the magnetic field and they are gauge transformable to each other in the region where they overlap. The regions can be selected in spherical coordinates as

$$\begin{aligned} R^N : 0 \leq \theta < \pi/2 + \delta, \quad r > 0, \quad 0 \leq \phi < 2\pi, \quad t \quad ] - \quad , \quad [ \\ R^S : \pi/2 - \delta < \theta \leq \pi, \quad r > 0, \quad 0 \leq \phi < 2\pi, \quad t \quad ] - \quad , \quad [ \end{aligned} \quad (4.41)$$

where  $\delta > 0$ . The monopole potentials on  $R^N$  and  $R^S$  can then be given by

$$\begin{aligned} A_t^N = A_r^N = A_\theta^N = 0, \quad A_\phi^N &= \frac{g}{r \sin \theta} (1 - \cos \theta), \\ A_t^S = A_r^S = A_\theta^S = 0, \quad A_\phi^S &= -\frac{g}{r \sin \theta} (1 + \cos \theta). \end{aligned} \quad (4.42)$$

In this case we can cover the space  $\mathbb{R}^3 - 0$  as is shown and exaggerated in figure 4.2. One can then take the region of overlap of the potentials  $A^N$  and  $A^S$  and shrink

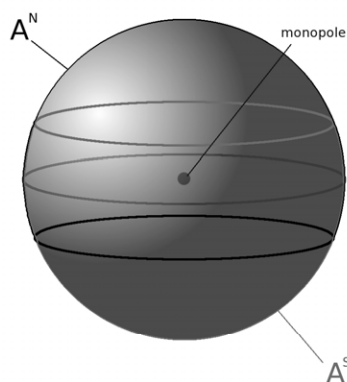


Figure 4.2: The potentials  $A^N$  and  $A^S$  that cover the space  $\mathbb{R}^3 - 0$ .

it until it becomes infinitesimal, i.e.  $\delta \rightarrow 0$ . In this case the gauge transformation connecting the two potentials (4.42) is given by

$$\exp\left(\frac{2ige}{\hbar c}\phi\right), \quad (4.43)$$

where  $\phi$  is the azimuthal angle. If this gauge transformation is to be single-valued we must have

$$\frac{2ge}{\hbar c} = N, \quad N \in \mathbb{Z}, \quad (4.44)$$

which is the DQC.

This formulation is in fact equivalent to taking the  $U(1)$  fibre-bundle and covering the base space  $\mathbb{R}^3 - 0$  with it. That is, when we remove one point from the

manifold  $\mathbb{R}^3$  and consider the Abelian gauge group  $U(1)$  on it [100]<sup>5</sup>, it has a nontrivial topological charge, the first Chern number or winding number, which is directly related to the magnetic charge. Due to this, one can construct models within field theory, such as the 't Hooft-Polyakov or BPS monopoles, that have a topological charge that is isomorphic to the topological charge of the Dirac monopole and say that these non-Abelian solitons are *magnetic* monopoles. The requirement of isomorphism of the topological charges is equivalent to requiring that the non-Abelian potential of the 't-Hooft Polyakov or BPS-like solitons reduce to the  $U(1)$  potential in the asymptotic far. However, for this isomorphism to work in the noncommutative case, we should have to show that noncommutative  $U_\star(1)$  on  $\mathbb{R}^3 - 0$  displays a DQC-like quantization that is equivalent to its topological charge. This is the topic of [III] and the next section.

## 4.4 Noncommutative Maxwell equations with a monopole

To discuss the magnetic monopole issue in the noncommutative context we start by introducing the analogs of the monopole equations (4.38) and (4.39) into noncommutative space-time. Since Maxwell's equations can be generalized into noncommutative space-time from their commutative covariant form [III], we have the noncommutative Maxwell's equations given by

$$\epsilon^{\mu\nu\gamma\delta} D_\nu^\star \mathcal{F}_{\gamma\delta} = 0 \tag{4.45}$$

$$D_\mu^\star \mathcal{F}^{\mu\nu} = J^\nu, \tag{4.46}$$

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<sup>5</sup>This work contains all the necessary material to establish the equivalence of the Dirac monopole with the homotopy classes of the  $U(1)$  fibre bundle and hence the relation of the topological and magnetic charge. It is interesting to note that the work appeared during the same time Dirac published his paper on monopoles [82], but went unnoticed by physicists for some 40 years. This is most likely due to that the topological language and notions of mathematicians were not familiar to physicists at that time.

where  $\mathcal{F}_{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\gamma\delta}F_{\gamma\delta}$  is the dual field strength tensor. Here, the NC  $U_\star(1)$  field strength tensor  $F_{\mu\nu}$  and the covariant derivative  $D_\nu^\star$  are given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu, A_\nu]_\star \quad (4.47)$$

$$D_\nu^\star = \partial_\nu - ie[A_\nu, \cdot]_\star. \quad (4.48)$$

From the above expressions, the monopole equations follow almost trivially.

The Maxwell equations (4.45) and (4.46) have some new interesting properties worth mentioning. To begin with, the equation (4.46) must be covariantly conserved. That is, if we act upon it from the left with the covariant derivative  $D_\nu^\star$ , we have the condition  $D_\nu^\star \star J^\nu = 0$ . Fortunately for us, this does not produce any extra consistency condition on the magnetic monopoles because the monopoles are static and all electric fields are turned off, i.e.  $A_0 = 0$  and all time derivatives vanish. In this case the covariant divergence of the current  $J^\nu$  vanishes trivially.

Another interesting question is, how do we define the noncommutative magnetic or electric fields? In the commutative case these are given by gauge invariant combinations of the potentials. In the noncommutative case, this becomes tricky because at first glance the field strength tensor transforms gauge covariantly

$$F^{\mu\nu} \rightarrow U(x) \star F^{\mu\nu} \star U^{-1}(x). \quad (4.49)$$

We could make it gauge invariant by an integration over space-time, but then we would end up with a number and that is not what we want. Therefore one must define the noncommutative electric and magnetic fields in a form which is very different from their commutative definitions. One possibility inspired by [101] can be given by

$$G^{\mu\nu} = \int d^4k e^{-ikx} \int d^4x F^{\mu\nu} \star W(x, C) \star e^{ikx} \quad (4.50)$$

where  $W(x, C)$  is the noncommutative  $U_\star(1)$  Wilson line:

$$W(x, C) = P_\star \exp \left( ig \int_0^1 d\sigma \frac{d\zeta^\mu}{d\sigma} A_\mu(x + \zeta(\sigma)) \right), \quad (4.51)$$

and where  $C$  is the curve which is parameterized by  $\zeta^\mu(\sigma)$  with  $0 \leq \sigma \leq 1$ ,  $\zeta(0) = 0$ ,  $\zeta(1) = l$  and satisfies the condition  $l^\nu = k_\mu \theta^{\mu\nu}$ , where  $l$  is the length of the curve.

$P_\star$  denotes path ordering with respect to the star product given by

$$W(x, C) = \prod_{n=0}^{\infty} (ig)^n \int_{\sigma_1}^1 d\sigma_1 \int_{\sigma_1}^1 d\sigma_2 \dots \int_{\sigma_{n-1}}^1 d\sigma_n \zeta'_{\mu_1}(\sigma_1) \dots \zeta'_{\mu_n}(\sigma_n) A_{\mu_1}(x + \zeta(\sigma_1)) \star \dots \star A_{\mu_n}(x + \zeta(\sigma_n)). \quad (4.52)$$

That is, the noncommutative magnetic field is now given by  $\epsilon_{ijk} G^{jk}$  and the noncommutative electric field by  $G^{i0}$ . These definitions are not unique and depend obviously on the shape of the curve  $C$  and on the point of attachment of the field strength  $F^{\mu\nu}$  to the Wilson line. One choice might be to take the shape of the curve to be that of a straight line because then the point of attachment of  $F^{\mu\nu}$  to the line does not matter. Fortunately for us once again, we do not need to consider the definition of the magnetic field when we discuss the noncommutative magnetic monopoles, as these are solely dependent on the gauge potential.

As an aside we should mention that the noncommutative Maxwell's equations (4.45) and (4.46) can be interpreted in a very fascinating way if we use the Seiberg-Witten map [102]. In this case the usual Maxwell equations

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0, \quad (4.53)$$

$$\nabla \cdot \vec{B} = 0, \quad (4.54)$$

remain as a consequence of the definition of the  $U(1)$  field strength  $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$  that one maps the noncommutative theory into with aid of the Seiberg-Witten map. However, the remaining corrections to the action resulting from the Seiberg-Witten map, can be collected into the equations

$$\nabla \cdot \vec{D} = \rho_f, \quad (4.55)$$

$$\nabla \times \vec{H} = J_f + \frac{\partial \vec{D}}{\partial t}, \quad (4.56)$$

where  $\rho_f$  is the free charge and  $J_f$  is the free current and  $\vec{H}$  is the magnetic field and  $\vec{D}$  is the displacement field. Constitutive relations for these follow directly from the Seiberg-Witten map and hence we have the relations (to first order in  $\theta$ ):

$$\vec{D} = (1 - \vec{\theta} \cdot \vec{B}) \vec{E} + (\vec{\theta} \cdot \vec{E}) \vec{B} + (\vec{E} \cdot \vec{B}) \vec{\theta} + \mathcal{O}(\theta^2) \quad (4.57)$$

$$\vec{H} = (1 - \vec{\theta} \cdot \vec{B}) \vec{B} + \frac{1}{2} (\vec{E}^2 - \vec{B}^2) \vec{\theta} - (\vec{\theta} \cdot \vec{E}) \vec{E} + \mathcal{O}(\theta^2) \quad (4.58)$$

In this case we may interpret the noncommutativity of space-time as an electromagnetic medium through which e.g. light propagates, by comparison to the usual Maxwell equations in vacuum. In [102] this aspect is studied to conclude, as in [46], that the propagation speed of light in a noncommutative space-time is polarization dependent. The interpretation of the noncommutativity of space-time as an electromagnetic medium through which everything propagates used in equations (4.55) and (4.56) in the case of no Seiberg-Witten map, is not possible due to that the noncommutative  $U_\star(1)$  field strength is not mappable to the commutative  $U(1)$  field strength and consequently the equations (4.54) and (4.53) also receive noncommutative corrections.

#### 4.4.1 The solution in the first order of $\theta_k$

Now that we have introduced the noncommutative Maxwell equations in (4.45) and (4.46), it is time to solve them for a noncommutative magnetic monopole. Since our method of solution is perturbative, it is good to note that the definition for the total noncommutative magnetic charge of a particle in the noncommutative context becomes

$$g_{NC} = \int J^0 d^3x. \quad (4.59)$$

If this series begins with the commutative source term  $J^0 = 4\pi g \delta^3(r) + \mathcal{O}(\theta)$ , the commutative magnetic charge  $g$  behaves like a coupling constant that coincides with the noncommutative magnetic charge in the  $\theta \rightarrow 0$  limit.

Because we will be focusing our attention on point-like particles, we must next generalize the source term of the magnetic charge. In this case one might simply think that the source  $4\pi g \delta^3(r)$  would do. However, the commutative source does not transform gauge covariantly but the left hand side of (4.46) does. Therefore we should find a source that transforms this way. If we keep ourselves to first order in  $\theta$ , a gauge covariant point particle source can be found from [II], where it is given by (4.22). Another source that also satisfies the requirement of gauge covariance to first order is given by

$$\rho'_{NC} = 4\pi g \left( \delta^3(\vec{r}) - \frac{e}{2c\hbar} \epsilon_{ijk} (A_i \theta_j \partial_k \delta^3(\vec{r})) \right) + \mathcal{O}(\theta^2). \quad (4.60)$$

Although the final effect of this source shall be to only make the potential  $A_i$  more singular at the origin  $r = 0$ , it is important to find as such because it makes the equation (4.46) consistent with the noncommutative gauge symmetry.

Finally, we modify the Wu-Yang requirements (see section 4.3) for the potentials into the form:

1. The potentials are gauge transformable to each other in the overlap region of the potentials. For the non-Abelian group  $U_*(1)$  this means that we require

$$A_\mu^{N/S}(x) \quad U(x) \star A_\mu^{N/S}(x) \star U^{-1}(x) - iU(x) \star \partial_\mu U^{-1}(x) = A_\mu^{S/N}(x). \quad (4.61)$$

2. Both potentials satisfy Maxwell's equations with an appropriate source for the magnetic charge.
3. The potentials remain singularity-free in their respective regions of validity. That is, Maxwell's equations are solved in such a way that noncommutativity does not produce new singularities into the potentials.

With this in hand, we may then begin to solve the noncommutative Maxwell equations for a noncommutative magnetic monopole in the Wu-Yang method using the zeroth order (commutative case) potentials in the form

$$\begin{aligned} A_1^{N_0} &= \frac{-y(r-z)}{(x^2+y^2)r}, & A_2^{N_0} &= \frac{x(r-z)}{(x^2+y^2)r}, & A_1^{S_0} &= \frac{y(r+z)}{(x^2+y^2)r}, & A_2^{S_0} &= \frac{-x(r+z)}{(x^2+y^2)r}, \\ A_3^{N_0} &= A_3^{S_0} = A_0^{N_0} = A_0^{S_0} = 0, \end{aligned} \quad (4.62)$$

where  $r = \sqrt{x^2 + y^2 + z^2}$ . After opening up the equations (4.45) and (4.46) perturbatively using the notation  $A_i = A_i^0 + A_i^1 + A_i^2 + \dots$ , where the upper index denotes the order of  $\theta$ , and combining them using the identity  $\vec{\partial} \times (\vec{\partial} \times \vec{B}) + (\vec{\partial} \cdot \vec{B}) \vec{\partial}$  we get Laplace equations for  $B_i^1 = \epsilon_{ijk} \partial_j A_k^1$  in the overlapping region. With the choice  $\theta = \theta^{12}$ , they can be solved by

$$A_1^{N_1} - A_1^{S_1} = \frac{2\theta yz(2r^2 - z^2)}{(r^2 - z^2)^2 r^3}, \quad (4.63)$$

$$A_2^{N_1} - A_2^{S_1} = -\frac{2\theta xz(2r^2 - z^2)}{(r^2 - z^2)^2 r^3}, \quad (4.64)$$

$$A_3^{N_1} - A_3^{S_1} = 0. \quad (4.65)$$

On the other hand, if we calculate the gauge transformation in first order for the gauge potential  $A_i^1$  in the overlapping region, it is

$$A_i^{N_1}(x) - A_i^{S_1}(x) = \theta (\partial_1 \lambda \partial_2 A_i^{S_0}(x) - \partial_2 \lambda \partial_1 A_i^{S_0}(x)) + \frac{\theta}{2} (\partial_1 \lambda \partial_2 \partial_i \lambda - \partial_2 \lambda \partial_1 \partial_i \lambda). \quad (4.66)$$

Inserting the potentials of Wu and Yang (4.62) and  $\lambda = \lambda_0 + \mathcal{O}(\theta^2) = \frac{2ge}{\hbar c} \phi + \mathcal{O}(\theta^2)$ , where  $\phi = \arctan(\frac{y}{x})$ , we recover exactly the equations (4.63), (4.64) and (4.65). That means that requirements 1 and 2 of this construction are satisfied. We may do the same analysis with the choice  $\theta = \theta^{13}$  but we still get a similar agreement [III]. Therefore we must now only prove that we can find a potential that is a solution of the Laplace equations. It can be found [III] and it is given by

$$A_1^{N_1} = \theta \left( \frac{-2x \arctan(\frac{x}{y})}{(r^2 - z^2)^2} + \frac{y}{4} \frac{7}{r^4} - \frac{2}{(r^2 - z^2)r^2} + \frac{4z(2r^2 - z^2)}{(r^2 - z^2)^2 r^3} \right), \quad (4.67)$$

$$A_2^{N_1} = -\theta \left( \frac{2y \arctan(\frac{x}{y})}{(r^2 - z^2)^2} + \frac{x}{4} \frac{7}{r^4} - \frac{2}{(r^2 - z^2)r^2} + \frac{4z(2r^2 - z^2)}{(r^2 - z^2)^2 r^3} \right), \quad (4.68)$$

$$A_3^{N_1} = 0. \quad (4.69)$$

The solution in the southern hemisphere can be found by use of the expressions (4.63), (4.64) and (4.65). Therefore we conclude that the requirements 1,2 and 3 are fulfilled to first order in  $\theta$  and that the DQC receives no corrections in this order [III].

We should point out that the SW-map does not produce the solution (4.67), (4.68) and (4.69) and it is not possible to modify the source in such a way that reconciles the solution the SW-map gives with the DQC to first order [83]. The source of this discrepancy is not well understood, but it may be due to the nontrivial topology of the problem at hand.

#### 4.4.2 The solution in the second order of $\theta_k$

In first order there is no difference in the DQC compared to the commutative case. However, when we expand our approach [III] to second order [83], things become interesting.

To be able to solve Maxwell's equations in second order of the perturbation we must generalize the point-like particle source found in [II] to the second order of

the perturbation [83]. This can be done and again there are two possibilities for us to consider

$$\begin{aligned} \rho = & \rho^0 + \rho^1 + \rho^2 + \mathcal{O}(\theta^3) = \delta^3(r) - \theta^{kl} \partial_k (A_l^0 \delta^3(r)) \\ & - \theta^{ij} A_j^1 \partial_i \delta^3(r) + \theta^{ij} \theta^{kl} \left[ A_j^0 \partial_i (\partial_k A_l^0 \delta^3(r)) + \frac{1}{2} A_i^0 A_k^0 \partial_j \partial_l \delta^3(r) \right] + \mathcal{O}(\theta^3), \end{aligned} \quad (4.70)$$

and

$$\rho = \delta^3(r) - \theta^{kl} A_l^0 \partial_k \delta^3(r) - \theta^{ij} A_j^1 \partial_i \delta^3(r) + \frac{1}{2} \theta^{ij} \theta^{kl} A_i^0 A_k^0 \partial_j \partial_l \delta^3(r) + \mathcal{O}(\theta^3), \quad (4.71)$$

are both gauge covariant to the second order of the perturbation as can be verified by calculating the expression

$$U(x) \star \rho \star U^{-1}(x), \quad (4.72)$$

and comparing it to the expression one obtains by performing the gauge transformations

$$A_i^0(x) \quad A_i^0(x) + \partial_i \lambda, \quad (4.73)$$

$$A_i^1(x) \quad A_i^1(x) + \theta^{kl} \partial_k \lambda \partial_l A_i^0(x) + \frac{\theta^{kl}}{2} \partial_k \lambda \partial_l \partial_i \lambda, \quad (4.74)$$

$$\begin{aligned} A_i^2(x) \quad & A_i^2(x) + \theta^{kl} \partial_k \lambda \partial_l A_i^1 - \frac{1}{2} \theta^{kl} \theta^{pq} \left( \partial_k A_i^0 \partial_p \lambda \partial_q \partial_l \lambda - \right. \\ & \left. \partial_k \partial_p A_i^0 \partial_q \lambda \partial_l \lambda + \frac{1}{3} (\partial_k \partial_p \lambda \partial_l \lambda \partial_q \partial_i \lambda - \partial_k \lambda \partial_p \lambda \partial_l \partial_q \partial_i \lambda) \right), \end{aligned} \quad (4.75)$$

of the sources (4.70) or (4.71). Since the sources in first order are unique up to a constant factor, it is interesting to note that the constant factor is fixed by the second order expression for the source and it does not anymore appear in the expressions (4.70) or (4.71). This suggests that there should be a closed form gauge covariant expression that is responsible for these terms. It has however not yet been found.

Again, the effect of the sources will only be to make the origin  $r = 0$  more singular. Therefore it does not contribute to the potentials that are already singular at the origin in the zeroth order and we end up, by the same method used in the

first order, with the differential equations in second order in the overlap:

$${}^2(B^{N_2} - B^{S_2})_1 = \frac{4\theta^2 xz}{(x^2 + y^2)^3 r^{10}} - 375(x^2 + y^2)^3 + 131z^2(x^2 + y^2)^2 \quad (4.76)$$

$$-2z^4(x^2 + y^2) - 4z^6 - \partial_1 \rho^{N_2} + \partial_1 \rho^{S_2},$$

$${}^2(B^{N_2} - B^{S_2})_2 = \frac{4\theta^2 yz}{(x^2 + y^2)^3 r^{10}} - 375(x^2 + y^2)^3 + 131z^2(x^2 + y^2)^2 \quad (4.77)$$

$$-2z^4(x^2 + y^2) - 4z^6 - \partial_2 \rho^{N_2} + \partial_2 \rho^{S_2},$$

$${}^2(B^{N_2} - B^{S_2})_3 = \frac{4\theta^2}{(x^2 + y^2)^4 r^{10}} \left( 120(x^2 + y^2)^5 - 900(x^2 + y^2)^4 z^2 \right. \quad (4.78)$$

$$\left. -1285(x^2 + y^2)^3 z^4 - 1289(x^2 + y^2)^2 z^6 - 652(x^2 + y^2) z^8 \right.$$

$$\left. -132z^{10} - \partial_3 \rho^{N_2} + \partial_3 \rho^{S_2} \right).$$

These should be compared with the equations for the overlapping region coming from the gauge transformations in second order (4.75). They are

$${}^2(B^{N_2} - B^{S_2})_1^{GT} = \frac{4\theta^2 xz}{(x^2 + y^2)^3 r^{10}} \left( -321(x^2 + y^2)^3 + 205(x^2 + y^2)^2 z^2 \right. \quad (4.79)$$

$$\left. + 26(x^2 + y^2) z^4 + 4z^6 \right),$$

$${}^2(B^{N_2} - B^{S_2})_2^{GT} = \frac{4\theta^2 yz}{(x^2 + y^2)^3 r^{10}} \left( -321(x^2 + y^2)^3 + 205(x^2 + y^2)^2 z^2 \right. \quad (4.80)$$

$$\left. + 26(x^2 + y^2) z^4 + 4z^6 \right),$$

$${}^2(B^{N_2} - B^{S_2})_3^{GT} = \frac{4\theta^2}{(x^2 + y^2)^4 r^{10}} \left( 144(x^2 + y^2)^5 - 564(x^2 + y^2)^4 z^2 \right. \quad (4.81)$$

$$\left. - 455(x^2 + y^2)^3 z^4 - 403(x^2 + y^2)^2 z^6 - 188(x^2 + y^2) z^8 - 36z^{10} \right),$$

where  $B_i^2 = \epsilon_{ijk} \partial_j A_k^2$ . In order for the DQC to be fulfilled, the equations (4.76)-(4.78) and (4.79)-(4.81) need to be satisfied simultaneously. We may simplify this system of equations by subtracting (4.76) from (4.79), (4.77) from (4.80) and (4.78)

from (4.81). The resulting system of equations is given by

$$-\partial_x(\rho^{N_2} - \rho^{S_2}) = \frac{8\theta^2 xz}{(x^2 + y^2)^3 r^8} \left( 27(x^2 + y^2)^2 + 10(x^2 + y^2)z^2 + 4z^4 \right), \quad (4.82)$$

$$-\partial_y(\rho^{N_2} - \rho^{S_2}) = \frac{8\theta^2 yz}{(x^2 + y^2)^3 r^8} \left( 27(x^2 + y^2)^2 + 10(x^2 + y^2)z^2 + 4z^4 \right), \quad (4.83)$$

$$-\partial_z(\rho^{N_2} - \rho^{S_2}) = \frac{2\theta^2}{(x^2 + y^2)^3 r^8} \left( 48(x^2 + y^2)^4 + 624(x^2 + y^2)^3 z^2 + 1036(x^2 + y^2)^2 z^4 + 736(x^2 + y^2)z^6 + 192z^8 \right). \quad (4.84)$$

We can then differentiate equation (4.82) with respect to  $y$  and equation (4.83) with respect to  $x$  and subtract the two:

$$0 = (\partial_x \partial_y - \partial_y \partial_x)(\rho^{N_2} - \rho^{S_2}) = 0, \quad (4.85)$$

where the 0 on the left hand side is due to the partial derivatives commuting<sup>6</sup> and the 0 on the right hand side is due to a calculation of the expression  $(\partial_x \partial_y - \partial_y \partial_x)(\rho^{N_2} - \rho^{S_2})$  by using equations (4.82) and (4.83). We get the following two additional equations in a similar manner:

$$0 = (\partial_x \partial_z - \partial_z \partial_x)(\rho^{N_2} - \rho^{S_2}) = \frac{24\theta^2 x}{(x^2 + y^2)^5 r^8} \left( 41(x^2 + y^2)^4 + 426(x^2 + y^2)^3 z^2 + 704(x^2 + y^2)^2 z^4 + 496(x^2 + y^2)z^6 + 128z^8 \right), \quad (4.86)$$

$$0 = (\partial_y \partial_z - \partial_z \partial_y)(\rho^{N_2} - \rho^{S_2}) = \frac{24\theta^2 y}{(x^2 + y^2)^5 r^8} \left( 41(x^2 + y^2)^4 + 426(x^2 + y^2)^3 z^2 + 704(x^2 + y^2)^2 z^4 + 496(x^2 + y^2)z^6 + 128z^8 \right). \quad (4.87)$$

These equations will only be satisfied when  $(x = y = 0)$  and thus the DQC does not hold, i.e. we cannot gauge transform ourselves from the south to the north or

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<sup>6</sup>We can require that the partial derivatives commute, due to that the right hand sides of equations (4.82), (4.83) and (4.84) are continuous functions in their region of validity. That is, because we want to solve the aforementioned equations by finding an appropriate source in the second order of  $\theta_k$  and the right hand sides of the equations are continuous, this can only be accomplished by a continuous function. Hence, we require  $\rho^{N_2} - \rho^{S_2}$  to be continuous and its mixed partial derivatives commute.

vice versa by requiring the single-valuedness of the gauge transformation and retain the DQC. It also means that this type of a perturbative solution to the noncommutative Maxwell's equations with the gauge parameter unchanged is impossible because the whole procedure of gauge transformation between the hemi-spheres becomes perturbatively impossible. Therefore, the first order solution of the noncommutative Maxwell's equations that we have presented here is valid, because we can gauge transform ourselves from one hemi-sphere to the other in this order. However, if we would find the explicit second order solution from the equations (4.76), (4.77) and (4.78), it would not be a proper solution of the noncommutative Maxwell's equations since one could not take oneself from one hemi-sphere to the other by a gauge transformation. If one did want to find a solution to the noncommutative monopole equations in second order, one could use the Dirac potential containing the Dirac string and solve the corresponding perturbative noncommutative Maxwell's equations starting from it. Alternatively, one could choose to change the gauge parameter in second order, but this would not necessarily give enough freedom to solve the resulting equations.

This result also gives an indication of why the algebra of **[II]** could not be constructed in second order of  $\theta_k$ . It may well be that for the DQC to hold in noncommutative space-time, the translation generators in the algebra of **[II]** will no longer remain associative. This would make it impossible to construct an equivalent of the algebra in [94] in a noncommutative space-time and explain the difficulties encountered in section 4.2.1.

It is interesting to speculate over why there arises a clear difference of result between the first order and the second order expansions. One possible explanation has to do with that the gauge group elements receive perturbative corrections first in the second order of the expansion. I.e.

$$e_{\star}^{i\lambda} = e^{i\lambda} + \frac{\theta^{ij}\theta^{kl}}{8} e^{i\lambda} \partial_j \partial_l \lambda \left( \frac{1}{2} \partial_i \partial_k \lambda + \frac{i}{3} \partial_i \lambda \partial_k \lambda \right) + \mathcal{O}(\theta^3), \quad (4.88)$$

but this issue remains to be better understood. Another possibility is given by the Witten effect [103]. The Witten effect, which also holds in a curved space-time [104], is a result that states that the DQC does not hold in a commutative theory that breaks  $CP$ -invariance. In the noncommutative case, noncommutative quantum

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electrodynamics is known to break  $CP$ -invariance [105], and thus this could be the source of the breakdown of the DQC. However, in the commutative case one adds the  $CP$ -breaking term to the quantum electrodynamics Lagrangian by hand and in noncommutative quantum electrodynamics it is broken by the theory without any additional modification. Therefore, these effects may be unrelated, but it is an interesting possibility that also remains to be better understood. One may also note that the breaking of the DQC cannot only be related to the broken rotational invariance of the noncommutative model, as the rotational invariance is broken already in the first order of the perturbation, but the DQC survives in this order.

Another remark that is good to make is related to the following question: What is the meaning of the perturbative source we have constructed? It certainly satisfies all the necessary symmetry requirements and it behaves the same as a point-like source. It removes one point from the manifold. In the case of a von Neumann-like pointless geometry it may indeed be possible to remove only one point from the manifold and consider the resulting nontrivial topology. One way to see this is to perform the construction of the Moyal star-product in section 2.1 for this nontrivial topology. As long as the Fourier transforms can be taken, which they can for a manifold with one point removed, the star-product is well-defined. This conclusion is also supported by string theory where the constant  $B$  field projects the endpoints of the open strings onto the D-brane as point particles. In other words the noncommutative field theory resulting from open string theory is a noncommutative field theory of point particles. However, the question of the existence of point particles in a true noncommutative geometry in the sense of Connes such as the fuzzy sphere is more complicated. Some partial light on this problem could be shed by the finding of a nonperturbative source for the monopole. However, to find such a source that transforms gauge covariantly is very tricky. As an example we may consider the potential source

$$\rho_{NC}(r) = \frac{1}{(4\pi\theta)^3} \exp\left(\frac{-r^2}{4\theta}\right). \quad (4.89)$$

This source does produce a delta function in the  $\theta \rightarrow 0$  limit and it is not perturbatively expandable. However, it contains no gauge fields and does therefore not transform under gauge transformations and must be abandoned as a candidate. Therefore, if we are to require that we obtain the delta function in the commutative

limit, it seems we should have to mix together the potential and some coordinates in a gauge covariant way. This is a very delicate problem that deserves further investigation, but that we shall leave for the future.

It should be noted that we may of course consider some arbitrary gauge covariant source that does not reduce to the delta function in the commutative limit, but then there must be some very good reason to consider it as a noncommutative particle source. There is at present no principle for this and this should as a consequence be considered as a very speculative road to a possible noncommutative quantization condition.

# Chapter 5

## Conclusions

In this thesis the Aharonov-Bohm effect and the Dirac monopole has been considered in noncommutative space-time. While the Aharonov-Bohm effect can be incorporated into noncommutative quantum mechanics in a satisfactory gauge invariant way, the DQC does not hold in the perturbative expansion. One may argue that the DQC should be investigated for a truly nonperturbative source and therefore it should not yet be concluded that noncommutative space-time is devoid of the DQC. A nonperturbative investigation of noncommutative magnetic monopoles in the noncommutative Maxwell equations is the next step of this analysis. It is currently a work in progress.

This work also shows the importance of the noncommutative Wilson lines that enter the description of the Aharonov-Bohm effect and that can be used in the definition of the noncommutative magnetic or electric fields. These are especially linked to the gauge invariant observables of these theories and a better mathematical understanding of them would be imperative.

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