REGGE CALCULUS WITH TORSION

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1. Introduction to Regge Calculus

Regge Calculus was introduced by T. Regge [1] almost 30 years ago for Riemannian manifolds without torsion. Regge defined the concept of curvature and metric for a simplicial manifold, giving thus up the differentiable structure, and gave us a "discrete" version of Einstein's theory of gravity. This is done for several reasons: for a compact manifold the triangulation is finite, so one has to deal with only a finite number of simplices; the discreteness makes numerical calculations possible, this is helpful for example in strong fields; and finally, Regge calculus is viewed as a possible road to quantum gravity [2].

In two dimensions a differentiable surface is approximated by triangles, whose interior is assumed to be flat. The metric information of the manifold is encoded in the edgelengths of the simplices. For a n-simplex the number of edges is n(n+1)/2, which matches exactly the number of independent components of the Riemannian metric tensor $g_{\mu\nu}$, so that this information is exactly equivalent to specifying the metric. The curvature is measured by carrying vectors around closed loops, which encircle a hinge, which is a (n-2)-simplex. The angle θ by which the parallel transported vector has been rotated is called the deficit angle associated with that hinge and is a measure of the scalar curvature $R:=R_{\mu\nu}g^{\mu\nu}$.

Regge gave a discrete version of the Einstein-Hilbert Lagrangian

$$L_{E\text{-}H}=\int R\sqrt{\det(-g)}\,d^4x \quad \text{as} \quad L_R=\sum_{\text{all hinges } h}\theta_h(l_\mu)V_h(l_\mu)\,.$$

The variation of L_R with respect to the edgelengths l_μ gives the discrete Einstein-Regge equations, one equation for each edge. It is interesting to note that in the variation we have $\delta\theta_h(l_\mu)=0$. This is completely equivalent to the situation in the continuum where the variation δR also does not contribute to the field equations.

Regge's formalism is a so-called second order formalism, because he varies only the metric. A first order formalism, where one varies the connection and the metric (or tetrad) independently, would be closer to gauge theoretic formulations and also would allow the introduction of torsion. This puts us in the need of talking about the idea of a connection in a simplicial manifold.

2. Simplicial Connections

On a pseudo-Riemannian manifold the Levi-Civita connection is the unique connection which is metric and torsion free, and as such is defined by the metric alone through Christoffel's formula. Similarly there is a natural parallel transport defined on a Riemannian simplicial manifold. Because the parallel transport inside a simplex is trivial, one has to define the transport only for crossing the interface σ_{12}, a (n-1)-simplex between two neighboring simplices σ_1 and σ_2. For a vector parallel to σ_{12} the transport is the obvious one: two vectors $v(\sigma_1) \in T(\sigma_1)$ and $v(\sigma_2) \in T(\sigma_2)$ in adjacent simplices are called *natural parallel* if they can be obtained by parallel transport from the same vector in $T(\sigma_{12})$. Vectors orthogonal to σ_{12} remain orthogonal and preserve their orientation and length with respect to the different metrics in σ_1 and σ_2. This completes the operational definition of the Levi-Civita parallel transport. That this transport is torsion free can be seen by considering an elementary torsion parallelogram. Since every Riemannian simplicial manifold can be embedded in a higher dimensional Euclidean space [3] we call this connection *umklapp-connection* for one can think that at σ_{12} the tangent space $T(\sigma_1)$ of σ_1 gets tilted onto $T(\sigma_2)$. The umklapp-connection can be defined more abstractly as a linear isometric orientation preserving map between $T(\sigma_1)$ and $T(\sigma_2)$ which respects natural parallelism.

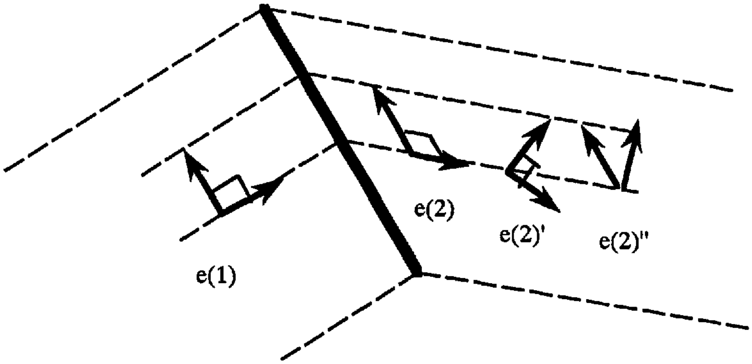

Figure 1: a frame e(1) is transported from σ_1 to σ_2 by an umklapp-connection (e(2)), a metric connection with torsion (e(2)'), and a symmetric, nonmetric connection (e(2)'').

Analogous to the continuum case we generalize this concept and call a linear map between $T(\sigma_1)$ and $T(\sigma_2)$ which is isometric and orientation preserving a *metric simplicial connection*. Because the natural parallelism is not necessarily respected in this case a frame e(1) transported from σ_1 to σ_2 can get rotated by a rotation K when it crosses σ_{12}. The rotation K corresponds to the contortion one-form w_k that appears in the decomposition of a general metric connection one-form w_m into Levi-Civita part w_{LC} and w_k:

$$w_m = w_{LC} + w_k.$$

In the most general case a *simplicial connection* is a linear non-singular orientation preserving map from $T(\sigma_1)$ to $T(\sigma_2)$.

An arbitrary simplicial connection is called *symmetric* if it respects natural parallelism. This completes the definition of simplicial analogues of important differential connections.

3. Torsion

As we noted before torsion can be described via the contortion matrix K which is naturally defined on the (n-1)-dimensional faces between two adjacent simplices. This kind of torsion has been termed interface torsion in [4] and discarded because it would not contribute to L_R. This is not generally true. In the continuum we have the relation

$$R = R_{LC} + Dw_k - w_k \wedge w_k$$

and something similar holds on he simplicial level. If the contortion obeys a cocycle condition, it will define a one-form w_k on the whole n-simplex [5] and contribute with a quadratic term in w_k to the Regge action, similar to Drummond's term arising from "body torsion", which in our opinion is just a special case of our notion of torsion. If one builds these torsion degrees of freedom into the Regge action one should obtain the discrete Einstein-Cartan equations upon variation. The exact formalism will be worked out in a subsequent publication.

4.Dual Lattice

It turns out to be advantageous to formulate the theory on the dual lattice instead of the simplicial lattice itself [6]. The dual to a simplicial lattice is in general not made out of simplices, but consists of polyhedral cells. These cells are called Voronoi cells [7]. The dual to a vertex P is the set of all points that are closer to P than to any other vertex. In this way a n-simplex is dual to a 0-polyhedron (a vertex), and in general a n-k-simplex is dual to a k-polyhedron. Because the connection and torsion are defined on (n-1)-simplices, they correspond to links L(ij) in the dual lattice. The curvature was defined by going around a hinge, which is a 2-polyhedron, a so-called plaquette, in the dual lattice, whose boundary is made up by the connection. This makes Regge calculus almost look like lattice gauge theory, where the gauge fields are defined on links, and the field strength is a plaquette variable, made up by the product of the link variables around a plaquette.

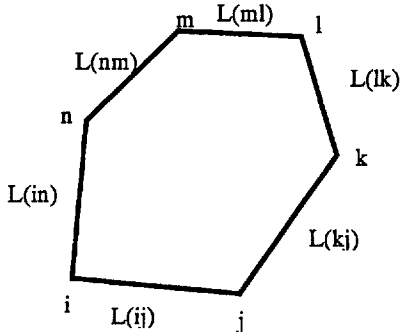

Figure 2: Plaquette in the dual lattice

5. Further Remarks

What remains to be done? The formalism of Regge calculus should be closer to differential geometry. In particular one should be able to derive a simplicial analogue of Cartan's structure equations. To achieve this goal we want to use the natural isomorphism of the de'Rham cohomology onto the simplicial cohomology. In de'Rham cohomology we have the set of closed p-forms $Z^p{}_{DR}:=\{w_p|dw_p=0\}$ and the set of exact p-forms $B^p{}_{DR}:=\{w_p|w_p=d\alpha_{p-1}\}$. The de'Rham p-th cohomology groups are then defined by $H^p{}_{DR}:=Z^p{}_{DR}/B^p{}_{DR}$. In simplicial homology we have the p-cycles $Z_p:=\{\alpha_p|\partial\alpha_p=0\}$ and the p-boundaries $B_p:=\{\partial\alpha_p=0\}$ that define the p-th homology group $H_p:=Z^p/B_p$. De'Rham's theorem makes them naturally isomorphic.

Differentiable manifold	Simplicial manifold
p-forms	p-cochains
exterior derivative d	coboundary operator ∂^*
metric $g_{\mu\nu}$	edge lengths l_μ
Grassmann product $\wedge$	Cup product $\cup$

The motivation is to make use of the simplicial topology as much as possible already for the formulation of the theory in order to concentrate on the algebraic quantities. So we try to use as little as possible of differential geometry, but as much as possible of algebraic topology.

The last remark concerns the idea of a space time point. We saw that the connection links tetrads defined on adjacent simplices. On the dual lattice the connection sits on links and the tetrads are defined on points. This suggests to take the idea of a simplicial manifold very seriously and to use the whole simplex as defining a space time point. Because the tetrad is defined on it it possesses by definition local Lorentz invariance. Because a space time point is then an extended object it can possess internal symmetries which can act as a source of gauge fields. If Regge simplices are physical then they can define a fundamental volume and therefore length, which is presumable of the order of the Planck length.

All presented material will be published in an expanded and more detailed version elsewhere.

Acknowledgements

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