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Simple Derivation of Distortion Functions

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I. INTRODUCTION

The distortion functions introduced by Collins[1] are important functions for a lattice. Just as the betatron functions β_x and β_y that specify the elliptic evelopes of betatron oscillations in both transverse planes, the distortion functions specify the deviations from these ellipses. Knowing the distortion functions, the nonlinear smears as well as betatron tunes dependence on amplitudes can be computed. Inversely, if the distortion functions can be controlled in the design of the lattice or placement of higher multipoles, the nonlinearity of the lattice can be very much reduced.

Take the contribution of the sextupoles as an example, each pair of distortion functions (B_α, A_α) can be computed using the following three criteria. (1) In the region between two sextupoles, (B_α, A_α) rotates like a vector by the angle α. (2) On passing a 'thin' sextupole, B is continuous while A jumps by an amount m_α which may be $s/4$ or $\bar{s}/4$. (3) (B_α, A_α) have to close after one revolution of the ring. In the above, the sextupole strengths s and $\bar{s}$ are defined as

$$s = \left(\frac{\beta_x^3}{\beta_0}\right)^{1/2} S \qquad \bar{s} = \left(\frac{\beta_x \beta_y^2}{\beta_0}\right)^{1/2} S \ , \tag{1.1}$$

where

$$S = \lim_{\ell \to 0} \left[\frac{B_y'' \ell}{2(B\rho)}\right] \ , \tag{1.2}$$

with $B_y'' x$ the local gradient of sextupole field, ℓ its length, and $(B\rho)$ the magnetic rigidity of the particle. The angle α stands for either, ψ_x, $3\psi_x$, $2\psi_y + \psi_x$, or $2\psi_y - \psi_x$. However, this derivation includes an *a priori* assumption of the closure (or periodicity) for the distortion functions.

An alternative derivation starts from the Hamiltonian. We solve for the amplitudes of betatron oscillations in both transverse planes, pick out the functions that are periodic and amplitude independent and called them the distortion functions. This approach was performed in Refs. 2, 3, 4, and 5. There, we expanded the Hamiltonian into harmonics and then resummed the harmonics at the end. The expansion into harmonics is only necessary if we wish to identify the harmful harmonics, which are usually only a few, and avoid them by suitable arrangement of the higher multipoles. If we are just interested in the derivation of the betatron amplitudes in terms of the distortion functions, the expansion and resummation are in fact unnecessary. The derivation will become simpler and more elegant. Such approach has already been used in the analytic computation of the horizontal smear.[6]

II. DERVATION

We demonstrate here the derivation including only normal sextupoles as nonlinear elements. The Hamiltonian is

$$H_1 = \frac{1}{2}[P_x^2 + K_x X^2] + \frac{1}{2}[P_y^2 + K_y Y^2] + \frac{B_y''}{6(B\rho)}(X^3 - 3XY^2) \; , \tag{2.1}$$

where P_x and P_y are the canonical momenta conjugate to the horizontal and vertical displacements X and Y, $K_x(s)$ and $K_y(s)$ are proportional to the restoring forces due to the ring's curvature and the field gradients of the quadrupoles. In the Floquet space, this Hamiltonian becomes

$$\begin{aligned} H_2 = \frac{R}{2\beta_x}\left(\beta_0 p_x^2 + \frac{x^2}{\beta_0}\right) + \frac{R}{2\beta_y}\left(\beta_0 p_y^2 + \frac{y^2}{\beta_0}\right) & \\ + \frac{RB_y''}{6(B\rho)}\left[\left(\frac{\beta_x}{\beta_0}\right)^{3/2} x^3 - 3\left(\frac{\beta_x\beta_y^2}{\beta_0^3}\right)^{1/2} xy^2\right] \; , & \end{aligned} \tag{2.2}$$

where R is the average radius of the ring, θ defined as path length along designed orbit divided by R has been chosen as the independent variable, and β_0 is some arbitrary reference value for the betatron functions.

This Hamiltonian is now solved exactly to zero order in sextupole strength by canonical transformation to the action-angle variables I_x, a_x and I_y, a_y. The generating function

$$G_2(a_x, p_x, a_y, p_y; \theta) = \sum_{u=x,y} \frac{1}{2}\beta_0 p_u^2 \cot[Q_u(\theta) + a_u] \tag{2.3}$$

is used to obtain the transformation ($u = x$ or y)

$$\left\{ \begin{array}{rcl} u & = & (2I_u\beta_0)^{1/2}\cos[Q_u(\theta) + a_u] \; , \\ \beta_0 p_u & = & -(2I_u\beta_0)^{1/2}\sin[Q_u(\theta) + a_u] \; , \end{array} \right. \tag{2.4}$$

where $Q_u(\theta) = \psi_u(\theta) - \nu_u\theta$ and $\beta_0 p_u = du/d\psi_u$. After the transformation, the new Hamiltonian becomes

$$H_3 = \nu_x I_x + \nu_y I_y + \Delta H_3 \; , \tag{2.5}$$

where ν_x and ν_y are the unperturbed horizontal and vertical betatron tunes and the sextupole contribution is included in

$$\Delta H_3 = (2I_x\beta_x)^{3/2}\frac{RB''_y}{24(B\rho)}[\cos 3(Q_x + a_x) + 3\cos(Q_x + a_x)]$$

$$-(2I_x\beta_x)^{1/2}(2I_y\beta_y)\frac{RB''_y}{8(B\rho)}[2\cos(Q_x + a_x)$$

$$+\cos(2Q_y + Q_x + 2a_y + a_x) + \cos(2Q_y - Q_x + 2a_y - a_x)] \quad (2.6)$$

The equations of motion for δI_x and δI_y, the deviations due to sextupoles, are:

$$\frac{d\delta I_x}{d\theta} = -\frac{\partial \Delta H}{\partial a_x} = (2I_x\beta_x)^{3/2}\frac{RB''_y}{8(B\rho)}[\sin 3(Q_x + a_x) + \sin(Q_x - a_x)]$$

$$-(2I_x\beta)^{1/2}(2I_y\beta_y)\frac{RB''_y}{8(B\rho)}[2\sin(Q_x + a_x)$$

$$+\sin(2Q_y + Q_x + 2a_y + a_x) - \sin(2Q_y - Q_x + 2a_y - a_x)] \quad (2.7)$$

$$\frac{d\delta I_y}{d\theta} = -\frac{\partial \Delta H}{\partial a_y} = -(2I_x\beta_x)^{1/2}(2I_y\beta_y)\frac{RB''_y}{4(B\rho)}\times$$

$$\times\left[\sin(2Q_y + Q_x + 2a_y + a_x) + \sin(2Q_y - Q_x + 2a_y - a_x)\right] \quad (2.8)$$

We need to solve Eqs. (2.7) and (2.8) up to first order in sextupole strength. For this, we only need the zero-order dependences of a_x and a_y on θ, which are simply

$$\begin{cases} a_x = \nu_x\theta + \varphi_x \ , \\ a_y = \nu_y\theta + \varphi_y \ , \end{cases} \quad (2.9)$$

where φ_x and φ_y are initial betatron phases. Now, Eqs. (2.7) and (2.8) can be integrated easily. For simplicity, let us consider only the first term in the squared brackets of Eq. (2.7). Noting that Q_x is periodic in θ, the integration gives

$$\delta I_x(\theta) = -\int_\theta^{\theta+2\pi} d\theta'(2I_x\beta_x)^{3/2}\frac{RB''_y}{8(B\rho)}\frac{\cos(Q_x + a_x - \pi\nu_x)}{2\sin\pi\nu_x} \ . \quad (2.10)$$

Since the sextupoles are assumed to be 'thin', in terms of the sextupole strengths defined in Eq. (1.1), we can rewrite Eq. (2.10) as

$$\delta\mathcal{A}_x(\theta) = -\mathcal{A}_x^2\sum_k \frac{s_k}{4}\frac{\cos(\psi'_{xk} + \varphi_x - \pi\nu_x)}{2\sin\pi\nu_x} \ , \quad (2.11)$$

where the summation over k is over each sextupole around the ring located at the 'modified' phase advance ψ'_{xk} which is related to the usual periodic Floquet phase advance ψ_{xk} by

$$\psi'_{xk} = \begin{cases} \psi_{xk} & \text{if } \theta_k \geq \theta \\ \psi_{xk} + 2\pi & \text{if } \theta_k < \theta \ , \end{cases} \tag{2.12}$$

and

$$\mathcal{A}_x = \sqrt{2I_x\beta_0} \ , \qquad \mathcal{A}_y = \sqrt{2I_y\beta_0} \tag{2.13}$$

are the betatron amplitudes in the horizontal and vertical plane. The *instantaneous* horizontal betatron phase at position θ is

$$\phi_x(\theta) = \psi_x(\theta) + \varphi_x \ . \tag{2.14}$$

With this the disortion of the amplitude becomes

$$\delta\mathcal{A}_x(\theta) = \mathcal{A}_x^2[A_1 \sin\phi_x - B_1 \cos\phi_x] \ , \tag{2.15}$$

with

$$B_1(\theta) = \sum_k \frac{s_k}{4} \frac{\cos[\psi'_{xk} - \psi_x(\theta) - \pi\nu_x]}{2\sin\pi\nu_x} \ , \tag{2.16}$$

$$A_1(\theta) = \sum_k \frac{s_k}{4} \frac{\sin[\psi'_{xk} - \psi_x(\theta) - \pi\nu_x]}{2\sin\pi\nu_x} \ , \tag{2.17}$$

Here, $B_1(\theta)$ and $A_1(\theta)$ are periodic in θ and are amplitude independent. They are just one set of distortion functions. It is easy to verify that all the three criteria listed in Sec. 1 are satisfied. In the same way, all the other four sets of distortion functions can be derived easily. For example, the distortion functions corresponding to the sum and difference resonances are

$$B_\pm(\theta) = \sum_k \frac{\bar{s}_k}{4} \frac{\cos[\psi'_{\pm k} - \psi_\pm(\theta) - \pi\nu_\pm]}{2\sin\pi\nu_\pm} \ , \tag{2.18}$$

$$A_\pm(\theta) = \sum_k \frac{\bar{s}_k}{4} \frac{\sin[\psi'_{\pm k} - \psi_\pm(\theta) - \pi\nu_\pm]}{2\sin\pi\nu_\pm} \ , \tag{2.19}$$

where $\nu_\pm = 2\nu_y \pm \nu_x$ and $\psi'_\pm = 2\psi'_y \pm \psi'_x$, with ψ'_y defined similar to Eq. (2.12). If we wish, we could remove the *prime* on ψ and write the argument of the cosine of Eq. (2.18) or the sine of Eq. (2.19) and rewrite it instead as

$$2|\psi_{yk} - \psi_y(\theta)| \pm |\psi_{xk} - \psi_x(\theta)| - \pi\nu_\pm \ . \tag{2.20}$$

The way that it was written in Refs. 2, 3, and 4 is incorrect.

REFERENCES

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6. L. Merminga and K. Y. Ng, Fermilab Report FN-505 (1988) or SSC Report SSC-N-594 (1988). Note that φ in Eq. (4.13) of this reference is the initial phase, whereas in Eq. (4.16) and afterwards, it denotes the *instantaneous* betatron phase.