May 12, 1969

CORRECTION TO SLAC-TN-69-8 ("THE HEAD-TAIL EFFECT")

A careless error in the Note "The Head-Tail Effect: An Instability Mechanism in Storage Rings"* has been called to my attention by Dr. Philip Morton.

On page 18 the statement that the first term of Eq. (43) integrates to zero is, evidently, true only for modes other than mode "0". The results which follow--including parts of the discussion--are, therefore, completely valid only for $\mu > 0$. For mode "0" alone there is a real part of the frequency shift; and from Eq. (43) onward the material of the Note needs correction.

Please substitute the following material for pages 17 and onward of your copy of TN-69-8.

MATTHEW SANDS

P. S. -

We are also sending herewith a copy of a second Note on the same subject: SLAC TN-69-10, "Head-Tail Effect: From a Resistive-Wall Wake".

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* SLAC Technical Note: SLAC-TN-69-8, March 28, 1969.

SLAC-TN-69-8
Matthew Sands
March 28, 1969

THE HEAD-TAIL EFFECT:
AN INSTABILITY MECHANISM IN STORAGE RINGS

Abstract:

An analysis is made of an instability mechanism which depends on a short-range interaction among the particles in an electron storage ring and on the interplay of betatron and synchrotron oscillations.

Contents:

I. INTRODUCTION

The lateral instabilities observed at rather low currents in the electron storage rings, ACO and Adone, are not explained by the usual analysis of the resistive-wall effect.[1] There is, rather, growing evidence that the observed instabilities may be caused by a short-range interaction among the particles of a bunch which, in combination with the interplay of betatron and synchrotron oscillations, can lead to unstable motions. One model of such an interaction--called the "head-tail effect"--is analyzed in this note.

First, we recall the standard resistive-wall mechanism, which, for a single circulating bunch, operates in the following way. As a circulating bunch travels along the vacuum chamber it leaves behind electromagnetic fields--called the "wake"--some of whose components are proportional to the lateral coordinate of the center-of-charge of the bunch. As the bunch passes each azimuthal position, it goes through the wakes left behind on previous turns and receives a lateral kick which is proportional to its own lateral displacement on the previous passages. The phase of betatron oscillations is displaced from one revolution to the next, so the effect of the wake is to drive the coherent oscillation of a bunch with a force proportional to the amplitude of displacement but _displaced in phase_. Depending on the sign of the phase displacement the driving term may be stabilizing--leading to increased damping of the coherent oscillation--or may be unstabilizing--leading to an antidamped coherent oscillation of the bunch. The instabilities observed at ACO and Adone _do not display_ this expected dependence on the betatron phase shift per revolution, so another mechanism must be at play.

Consider now the following possible mechanism of a head-tail effect. The particles at the "head" of a bunch deposit fields along the orbit; particles of the bunch which arrive later--those in the "tail"--

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1 A discussion of the resistive-wall effect for bunched beams, and references to earlier work, are found in the paper of E. D. Courant and A. M. Sessler, Rev. Sci. Inst. 37, 1579 (1966).

experience lateral forces from these fields. We postulate that the fields under consideration decay rapidly with time so that although they provide an interaction between the particles of a bunch they have a negligible effect on subsequent passages of the bunch (or of other bunches in the beam). We may call these fields a "fast wake."

If we were to think of a bunch as a single "particle" (with all particles oscillating together), we would not expect the fast wake effects to produce an instability. Wake fields proportional to the displacement of the bunch would merely change the effective restoring force on the bunch and produce a small change in the betatron frequency. Fast wake fields proportional to the lateral velocity of bunch could, in principle, produce an instability; but one can show that for a non-pathological vacuum chamber (one without extra reactive elements) such fields will generally produce a damping effect.

We shall show in this note that fast wake fields together with the internal circulation of the particles of a bunch can cause certain "internal coherent modes" within the bunch to become unstable. In the next Section we give a primitive calculation which illustrates the basic phenomenon and demonstrates the dependence of the effect on the parameters of the guide field. In the subsequent Sections we give a more general analysis of the effect and show a complete solution for one special case.

The material presented here was stimulated by--and draws in part--from the work of C. Pellegrini of Frascati (private communication), who has obtained all of the significant results presented here. The details of the analysis, however, differ somewhat from his. I am grateful to Pellegrini for many informative discussions. A group at CERN--H. Hereward, P. Morton, and K. Schindl--has also been considering the head-tail effect in proton synchrotrons.

II. A SIMPLE MODEL

Consider a highly simplified model of a bunch in which there are just two particles whose synchrotron oscillations have the same amplitude but opposite phases. One-half of the time particle "1" is at the "head" of the bunch and particle "2" is at the "tail"; the other half of the time particle "2" is at the "head." The particles exchange places every one-half of a synchrotron oscillation. We may describe the synchrotron oscillations in terms of the time-displacement τ (defined as the time-of-arrival of a particle at a reference point on the orbit minus the synchronous arrival time) and the energy deviation ΔE. These two coordinates both oscillate at the synchrotron frequency* ω_s, but with a relative phase of $\pi/2$. We show in Fig. 1 the phase and energy coordinates for the particles "1" and "2" at four successive quarter periods of the synchrotron oscillations. The amplitudes τ_{max} and ΔE_{max} are related by

$$\omega_s \tau_{max} = \alpha \frac{\Delta E_{max}}{E} \tag{1}$$

where α is the momentum compaction and E is the mean energy.

Now let's consider the betatron oscillations of the two particles in, say, the radial coordinate x. (The same consideration would apply to vertical oscillations.) The frequency of the betatron oscillations will, in general, depend on the energy deviation of the particle (because, in part, of the different focussing strength on the displaced equilibrium orbit). If ω is the frequency on the nominal orbit and $\Delta\omega$ the shift for an energy-displaced orbit we can write**

$$\frac{\Delta\omega}{\omega} = \xi \frac{\Delta E}{E} \tag{2}$$

where ξ is a measure of the "chromaticity" of the guide field.

Now suppose we start our two particles oscillating in x with the same betatron amplitude and phase at an instant when they both have zero time displacement as in (a) of Fig. 1. Since they have slightly

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* By "frequency" we mean always the "angular frequency."

** This is correct only if ξ is $\gg 1/\alpha$; see Section II.

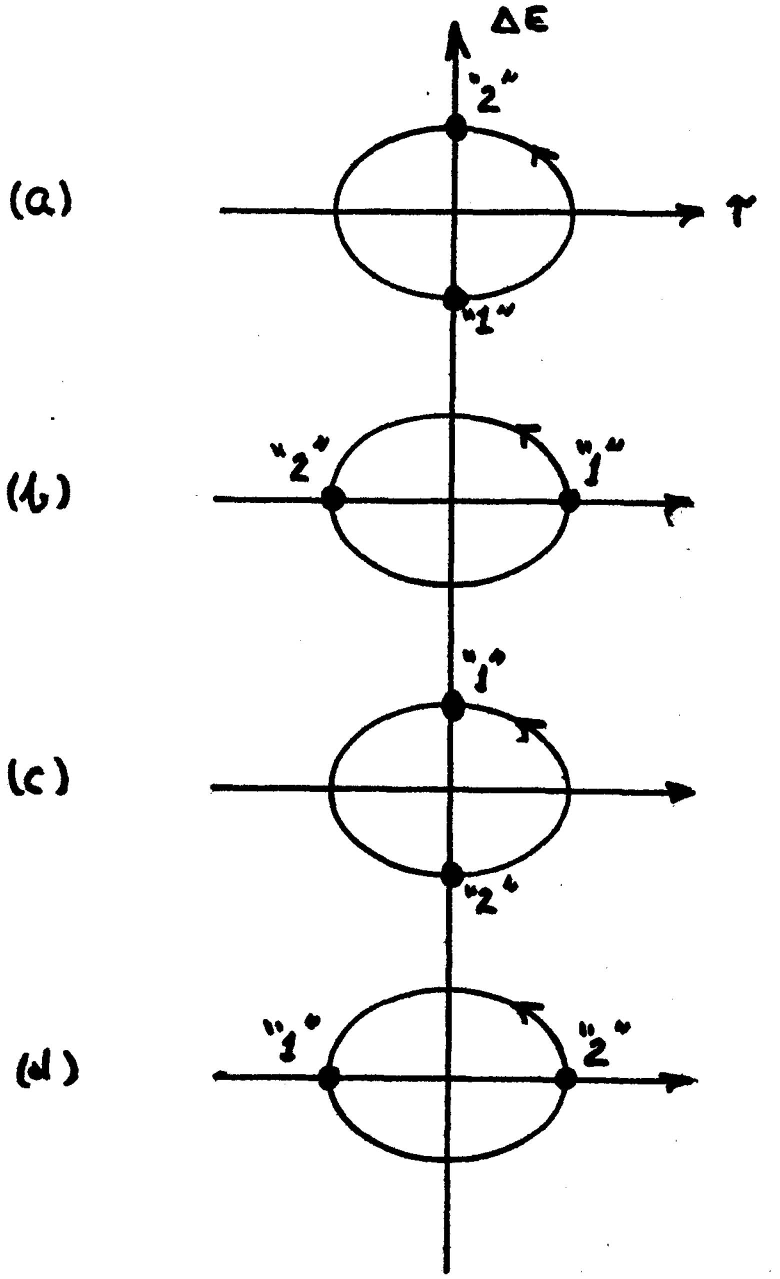

Fig. 1. The synchrotron oscillations of the two particles.

different betatron frequencies--because their energies are different--the phases of their betatron oscillations will drift apart reaching a maximum difference when they arrive at the situation represented by (b) in Fig. 1. The difference between the two betatron phases accumulated between (a) and (b), which we call $\Delta\phi$, is

$$\Delta\phi = 2 \int \Delta\omega dt = 2 \xi \frac{\omega}{E} \int \Delta E dt .$$

The integral of ΔE over a quarter synchrotron period is $\Delta E_{max}/\omega_s$, so we have

$$\Delta\phi = \xi \frac{\Delta E_{max}}{E} \frac{\omega}{\omega_s} . \tag{3}$$

Or, using Eq. (2),

$$\Delta\phi = \frac{\xi}{\alpha} \omega\tau_{max} . \tag{4}$$

When the two particles arrive at the positions shown in (c) of Fig. 1 the phase difference will be again zero, and the process will begin again. Thus the trailing particle always has the same phase displacement with respect to the leading particle. It is this oscillating phase displacement (of the betatron oscillations) which can generate an instability from the fast wake forces.

Suppose that the leading particle leaves behind a fast wake which produces a force on the trailing particle proportional to the displacement of the leading particle. When a particle which is oscillating with the amplitude a and frequency ω is driven by a synchronous oscillating force with the amplitude F and the phase lag $\Delta\phi$, the amplitude a increases at the rate

$$\frac{da}{dt} = \frac{\omega F}{2K} \Delta\phi \tag{5}$$

where K is the restoring force of the oscillator. If we now consider what happens on the average over a synchrotron oscillation, we see that particle "1" will drive particle "2" for one-half the time and particle "2" will drive particle "1" the other half of the time. So on the long-time average the oscillation energy of each particle will increase at half the rate of Eq. (5) (in which we may take $\Delta\phi$ as approximately equal to the peak value calculated earlier). Since we have started both particles with the same amplitude a we can write $F = Sa$, and Eq. (5) becomes:

$$\frac{da}{dt} \approx \frac{\omega S a}{4K} \Delta\phi \ . \tag{6}$$

The amplitudes will vary with time as $e^{\beta t}$, with

$$\beta \equiv \frac{1}{a}\frac{da}{dt} \approx \frac{\omega S}{4K} \Delta\phi \ . \tag{7}$$

Or, taking $\Delta\phi$ from (4), the exponential coefficient is

$$\beta \approx \frac{1}{4}\frac{\xi}{\alpha}\frac{S}{K}\omega^2 \tau_{max} \ . \tag{8}$$

We should make an additional observation. If we had chosen our initial starting conditions for the two particles with opposite betatron displacements, the only effect would be to change the sign of our expression for β.

The most general form of the betatron oscillations for our two particles can be described in terms of two modes--one "symmetric" and the other "asymmetric"--which have opposite signs for β. Depending on the algebraic signs of the two coefficients ξ and S, the first mode will be stable and the other will be unstable, or vice versa.

The detailed calculations of the later Sections yield a similar result for a more general case--see Eq. (43). In Section V we give a detailed discussion of the significance of these results. Here we make only the following qualitative observation. If the interaction between the two particles were symmetric, we would estimate for a rigid bunch model that there would be a (real) frequency shift of the oscillations in the amount

$$\frac{\delta\omega}{\omega} \approx \frac{S}{K} . \tag{9}$$

We see that the effect of the head-tail effect is to convert some of this real frequency shift into a growth constant. We can, indeed, write--using (8) and (9)--

$$\beta = \frac{1}{4} \frac{\xi}{\alpha} \omega\tau \ \delta\omega . \tag{10}$$

The "conversion" from $\delta\omega$ to β is just proportional to the machine parameter ξ/α and to $\omega\tau$, which is the length of the bunch in units of the reduced betatron wavelength.

III. THE GENERAL EQUATIONS

We begin by considering the combined betatron and synchrotron motion of an unperturbed single particle in a storage ring. We consider only betatron motion in the axial direction, which we call x . The treatment is nearly identical for horizontal oscillations, and we assume a negligible coupling between the two. In general, the revolution time T and the betatron number ν of the oscillations in x depend on the instantaneous energy E . For small deviations ΔE from the synchronous energy E_o we may keep only the linear part of the dependence and write

$$T = T_o \left(1 + \alpha \frac{\Delta E}{E_o}\right) \tag{11}$$

and

$$\nu = \nu_o \left(1 + \xi \frac{\Delta E}{E_o}\right) . \tag{12}$$

(The number α is the usual "momentum compaction", and we may call ξ

the "chromaticity"; both are properties of the magnetic guide field.)

We approximate the betatron motion by a harmonic oscillator whose frequency ω is given by $2\pi\nu/T$. Keeping only first order terms in $\Delta E/E_o$, (11) and (12) give

$$\omega = \omega_o \left\{ 1 + (\xi - \alpha) \frac{\Delta E}{E_o} \right\} . \tag{13}$$

(Note that we are using ω_o for the synchronous <u>betatron</u> frequency and <u>not</u> for the rotation frequency.)

Typically, ξ may be of the order of 1 or less; α is approximately $1/\nu_o{}^2$--which is much less than 1 for a strong focussing machine--and $\Delta E/E_o$ is typically 10^{-2} or 10^{-3}; so our linearization is quite appropriate.

We chose to describe the synchrotron oscillations of a particle in terms of its time-of-arrival τ at some azimuth, measured with respect to the time-of-arrival of a synchronous particle of the same bunch.*
We assume a linear restoring force for the synchrotron oscillations so that τ oscillates harmonically at the synchrotron frequency ω_s. Synchrotron action associates with this time-oscillation a variation ΔE of the particle energy. From Eq. (11) τ and ΔE are related by

$$\frac{\Delta E}{E} = -\frac{\dot{\tau}}{\alpha} \quad ; \quad \dot{\tau} \equiv \frac{d\tau}{dt} \tag{14}$$

To make later equations less cumbersome, we define

$$\eta \equiv (1 - \frac{\xi}{\alpha}) , \tag{15}$$

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* That is, the particle arrives the time τ <u>in advance</u> of a synchronous Particle. In terms of the usual phase variable ψ, $\tau = (T_o/2\pi h)\psi$, where h is the harmonic number of the accelerating voltage.

so that the varying betatron frequency of Eq. (13) becomes

$$\omega = \omega_o \; (1 + \eta\dot{\tau}) \; . \tag{16}$$

A word about orders of magnitude is now in order. We shall assume for the machines of interest (and characteristic of typical practical machines) that η is of the order of 1 or less, that the synchrotron frequency ω_s, is much less than the betatron frequency ω_o, and that the typical magnitude of τ is less than $1/\omega_o$ (the bunch length is less than the betatron wavelength). Then, $\eta\dot{\tau} \simeq \eta\omega_s\tau << 1$; and $\ddot{\tau} \simeq \omega_s\dot{\tau} << \omega_o\dot{\tau}$. We are, therefore, justified in consistently ignoring terms in $\ddot{\tau}/\omega_o$ and $\dot{\tau}^2$ in comparison with the dominant small terms in $\dot{\tau}$.

We can, then, say that in the absence of perturbing forces on the particles Eq. (16) implies that the betatron motion is described by the differential equation

$$\ddot{x} + \omega_o^{\,2}(1 + \eta\dot{\tau})^2 x = 0 \; , \tag{17}$$

which, in turn, is satisfied in our approximation by

$$x = Ze^{i\omega_o(t \,+\, \eta\tau)} \tag{18}$$

(real part understood); where Z is an arbitrary complex constant and the explicitly shown phase is just the time integral of the ω of Eq. (16)--with the initial phase absorbed into Z . (More accurately, x would have also a small periodic modulation of amplitude at the synchrotron frequency, but since this modulation is small we may ignore it.) We should, however, for consistency rewrite Eq. (17) as

$$\ddot{x} + \omega_o^{\,2}(1 + 2\eta\dot{\tau})x = 0 \tag{19}$$

Now consider the interaction between two particles which circulate in the same bunch. Imagine that as a leading particle k travels along it continuously* leaves behind a wake field which is traversed by a trailing particle ℓ. We take that the force $F_{\ell k}(t)$ on particle ℓ at the time t is proportional to the betatron displacement of particle k at the earlier time $t' = t - \tau_k + \tau_\ell$, modified by some "wake function" $\rho(\tau_k - \tau_\ell)$, which describes the transient decay of the wake. Note that we are assuming that the force on particle ℓ is independent of its lateral position. This is generally justified for wakes which are mediated by the walls of the vacuum chamber. A more subtle complication arises if one wishes to consider the head-tail effect for radial betatron oscillations. Strictly speaking, the wake effect must then depend on the sum of the radial betatron and synchrotron displacements. But the only effect of the extra term is to induce a very small forced betatron oscillation modulated at the synchrotron frequence. Neglecting this effect, the rest of the analysis is equally applicable to vertical and radial instabilities.

In any case we shall take that the wake force on particle ℓ due to particle k is given by

$$F_{k\ell}(t) = x_k(t - \tau_k + \tau_\ell)\rho(\tau_k - \tau_\ell) \ . \tag{20}$$

For the wake function ρ, we ask only that it go to zero at sufficiently large arguments so that there are no forces on the particles of the following bunch.

We are now ready to consider the effects of a whole bunch. We may let k be a running index so that the total force on particle ℓ becomes the sum of the force in (20) for all of the particles of the bunch. We use this total force as a driving term in Eq. (19), and get

$$\ddot{x}_\ell + \omega_o{}^2(1 + 2\eta\dot{\tau}_\ell)x_\ell = \sum_k x_k(t - \tau_k + \tau_\ell)\rho(\tau_k - \tau_\ell) \ . \tag{21}$$

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* We could, alternatively, consider an impulsive wake--the leading particle excites fields in localized elements of the chamber which give impulsive forces to the trailing particle; however, we show in the Appendix that the significant effects are identical, so we treat in detail only the case of a continuous wake.

(In keeping with our other approximations, we are neglecting a small modulation of the "effective mass" of the betatron oscillation which should multiply the force term.)

We wish to consider only the situation in which the total driving force is small in comparison with the restoring force, so that the solutions of Eq. (21) will differ only slightly from the free solution of Eq. (18). It is, then, convenient to write

$$x_\ell(t) = Z_\ell(t)e^{i\omega_o(t + \eta\tau_\ell)} , \qquad (22)$$

where now $Z_\ell(t)$ is a slowly varying function of t--including both the amplitude and phase perturbation on the betatron motion. Substituting this form into Eq. (21), and dropping terms of second order smallness (namely in $\ddot{Z}$, $\dot{Z}\rho$, $\dot{Z}\dot{\tau}$, $\dot{\tau}^2$, and $\ddot{\tau}$), we find that the amplitude $Z_\ell(t)$ satisfies

$$2i\omega_o\dot{Z}_\ell(t) = \sum_k Z_k(t)e^{-i\omega_o(1 - \eta)(\tau_k - \tau_\ell)}\rho(\tau_k - \tau_\ell) , \qquad (23)$$

where all τ's are to be evaluated at t . Eq. (23) is the basic equation of the head-tail effect--and it is the imaginary part of the exponential on the right which can give rise to unstable solutions.

Before considering the solution for a particular case, we can make some further simplifications. Let's write Eq. (23) as

$$i\dot{Z}_\ell(t) = \sum_k W_{\ell k}(t)Z_k(t) \qquad (24)$$

with

$$W_{\ell k}(t) = \frac{1}{2\omega_o} e^{-i\omega_o(1 - \eta)(\tau_k - \tau_\ell)}\rho(\tau_k - \tau_\ell) \qquad (25)$$

We also write explicitly that

$$\tau_\ell = A_\ell \cos(\omega_s t + \phi_\ell) ,$$
$$\tau_k = A_k \cos(\omega_s t + \phi_k) . \tag{26}$$

Since we are generally interested in small perturbations which give only small changes in Z and $\dot{Z}$ in one synchrotron period, we can for most purposes replace $W_{k\ell}(t)$ by its average value over a synchrotron period. We define

$$\overline{W}_{\ell k} = \frac{1}{T_s} \int_{T_s} W_{\ell k}(t) dt . \tag{27}$$

(The integral is to be taken over a time interval $T_s = 2\pi/\omega_s$.) The matrix $\overline{W}_{\ell k}$ is then independent of time and depends only on the two amplitudes A_k and A_ℓ and the relative phase $(\phi_k - \phi_\ell)$;

$$\overline{W}_{\ell k} = \overline{W}(A_\ell, A_k, \phi_k - \phi_\ell) .$$

We can then write Eq. (24) as

$$i\dot{Z}_\ell(t) = \sum_k \overline{W}_{\ell k} Z_k(t) , \tag{28}$$

where it is understood that we have now suppressed a very small modulation of Z at the synchrotron frequency.

Given the form of $\rho(\tau)$ and the synchrotron coordinates of all particles, the matrix $\overline{W}_{\ell k}$ is determined, and the set of differential equations (24) can in principle be solved. We shall consider in the next Section a solution for a particular $\rho(t)$ and for a special distribution of particles in synchrotron phase space.

IV. AN ILLUSTRATIVE EXAMPLE

The head-tail effect can produce an instability only if $\rho(\tau)$ is an asymmetric function. In the following, we will take a particularly simple model for $\rho(\tau)$, namely

$$\rho(\tau) = \left\{ \begin{matrix} 0; & \tau < 0 \\ S; & \tau > 0 \end{matrix} \right\} \quad . \tag{29}$$

We are imagining a wake which is zero in front of, and constant behind, the particle.* (We must also assume that it eventually drops to zero after all the particles of a bunch have passed.)

We shall also make an approximation for $W_{\ell k}$ which is quite appropriate for electron storage rings for which $\omega_o \tau \ll 1$. The exponent in $W_{\ell k}$ is then small, so Eq. (25) takes on the simpler form (we have also replaced η by its equivalent from Eq. (15))

$$W_{\ell k}(t) = \frac{1}{2\omega_o} \left\{ 1 - i\omega_o \frac{\xi}{\alpha}(\tau_k - \tau_\ell) \right\} \rho(\tau_k - \tau_\ell) \ . \tag{30}$$

We are now ready to evaluate $\overline{W}_{\ell k}$. The phase difference $(\tau_k - \tau_\ell)$ can be written--see (26)--as

$$\tau_k - \tau_\ell = R_{\ell k} \cos(\omega_s t + \phi_{\ell k}) \ , \tag{31}$$

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* We have also considered a wake which corresponds to the resistive-wall effect; the results are qualitatively similar. See SLAC-TN-69-10.

with

$$R_{\ell k}^2 = A_k^2 + A_\ell^2 - 2A_k A_\ell \cos(\phi_k - \phi_\ell) \ . \tag{32}$$

Since we are going to integrate over a full synchrotron period, we do not need to evaluate $\phi_{\ell k}$. The behavior of $(\tau_k - \tau_\ell)$ and of $\rho(\tau_k - \tau_\ell)$ are shown in Fig. 2. Since $(\tau_k - \tau_\ell)$ is a sinusoidal function, the mean value of ρ is just $S/2$; and the mean value of $(\tau_k - \tau_\ell)\cdot\rho$ is $SR_{\ell k}/\pi$. We get

$$\overline{W}_{\ell k} = \frac{S}{2\omega_o} \left(\frac{1}{2} - i \frac{\xi}{\pi\alpha} \omega_o R_{\ell k} \right) \tag{33}$$

Now we suppose that we are given the distribution of particles in synchrotron phase space. Let $n(A,\phi)AdAd\phi$ be the number of particles in an element of area at A and ϕ . The sum in Eq. (28) becomes an integral, and we have

$$i\dot{Z}(A,\phi,t) = \int \overline{W}(A,A',\phi-\phi')Z(A',\phi',t)n(A',\phi')A'dA'd\phi' \tag{34}$$

(where A,ϕ identify the particle ℓ and A',ϕ' identify the particle k). This equation would be the starting point for a complete treatment of a real bunch. We will, however, not consider further the general case, but rather show a solution of (34) for an artificial model of a bunch.

We consider now a model of a bunch in which there are N particles all with the same synchrotron amplitude A , but distributed uniformly in the variable ϕ of the synchrotron oscillations. Equation (34) then simplifies to

$$i\dot{Z}(\phi,t) = \frac{N}{2\pi} \int_{-\pi}^{+\pi} \overline{W}(\phi-\phi')Z(\phi',t)d\phi' \tag{35}$$

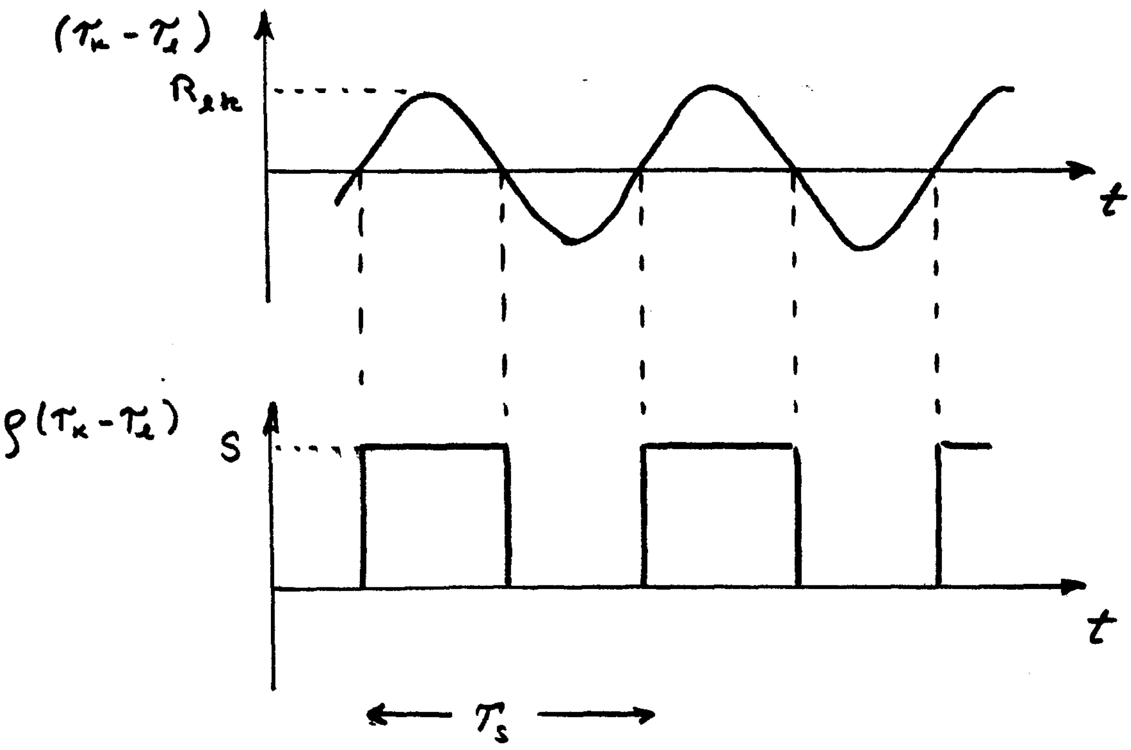

Fig. 2. The functions $(\tau_k - \tau_\ell)$ and $\rho(\tau_k - \tau_\ell)$.

This equation is readily solved by expanding Z in a Fourier series in ϕ. Let

$$Z(\phi,t) = \sum_{n=0}^{\infty} a_n(t)e^{in\phi} ,$$

with $n = 0,1,2,3,\ldots$. Substitution in (35) gives

$$i \sum_n \dot{a}_n(t)e^{in\phi} = \frac{N}{2\pi} \int_{-\pi}^{+\pi} \overline{W}(\phi-\phi') \sum_m a_m(t)e^{im\phi'}d\phi' \tag{36}$$

The right-hand side can be written as

$$\frac{N}{2\pi} \sum e^{im\phi}a_m(t)\int \overline{W}(\phi-\phi')e^{im(\phi'-\phi)}d\phi' \tag{37}$$

The integral is a function only of $(\phi'-\phi)$, so the integral is just a number. We define

$$\Delta\omega_m = - \frac{N}{2\pi} \int_{-\pi}^{+\pi} \overline{W}(-\psi)e^{im\psi}d\psi , \tag{38}$$

and Eq. (36) becomes

$$i \sum_n \dot{a}_n(t)e^{in\phi} = - \sum_m e^{im\phi}\Delta\omega_m a_m(t) . \tag{39}$$

Now multiply this equation by $e^{-i\mu\phi}$ (with $\mu = 0,1,2,\ldots$) and integrate over all ϕ. In each sum only the μ term remains, and we have

$$i\dot{a}_\mu(t) = - \Delta\omega_\mu a_\mu \tag{40}$$

The solutions are, evidently,

$$a_\mu(t) = a_\mu(0)e^{i\Delta\omega_\mu t}$$

We have found that for the special distribution under consideration there are normal modes (mode number μ) in which the amplitude function depends on μ and t according to

$$Z_\mu(\phi,t) = a_\mu(0)e^{i(\Delta\omega_\mu t+\mu\phi)} \quad . \tag{41}$$

We must now evaluate $\Delta\omega_\mu$ from the integral of Eq. (38). For our assumed distribution Eq. (32) gives

$$R_{\ell k} = \sqrt{2A} \; \{ 1 - \cos(\phi_k - \phi_\ell) \}^{1/2}$$

which we can write more conveniently as

$$R_{\ell k} = 2A\left|\sin\left(\frac{\phi_k - \phi_\ell}{2}\right)\right| . \tag{42}$$

Then, using (33) with (38) we get

$$\Delta\omega_\mu = -\frac{NS}{4\pi\omega_o}\int_{-\pi}^{+\pi}\left\{\frac{1}{2} - i\,\frac{2\xi\omega_o A}{\pi\alpha}\left|\sin\frac{\psi}{2}\right|\right\} e^{i\mu\psi}d\psi \tag{43}$$

The first term in the brackets integrates to π for mode "0" , and to zero for all other modes. For the second term, only the even part of the exponential will contribute; we must evaluate

$$\int_{-\pi}^{+\pi} \left| \sin \frac{\psi}{2} \right| \cos \mu\psi d\psi = 2 \int_{0}^{\pi} \sin \frac{\psi}{2} \cos \mu\psi d\mu \quad ,$$

which is just

$$- \frac{4}{4\mu^2 - 1} \quad .$$

We get, finally, that*

$$\Delta\omega_o = - \frac{NS}{4\omega_o} + \frac{2i}{\pi^2} \frac{NS\xi A}{\alpha}$$

$$\Delta\omega_\mu = - \frac{2i}{\pi^2} \frac{NS\xi A}{\alpha(4\mu^2 - 1)} \quad ; \quad \mu > 0 \qquad (44)$$

For our particular model $\Delta\omega_\mu$ is purely imaginary for $\mu > 0$. (Other forms of ρ would, in general, give a complex $\Delta\omega_\mu$ for all modes.) Suppose we write

$$\Delta\omega_\mu = \alpha_\mu - i\beta_\mu$$

where α_μ and β_μ are real numbers. Then α_μ is the real frequency change, and β_μ is the exponential growth constant of each mode.

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* A closely similar result was first obtained by C. Pellegrini of Frascati from a somewhat different analysis.

We have

$$\alpha_o = -\frac{NS}{4\omega_o} \quad ;$$

$$\alpha_\mu = 0 \quad , \quad \mu > 0 \quad ; \tag{45}$$

$$\beta_\mu = \frac{2}{\pi^2}\,\frac{NS\xi A}{\alpha(4\mu^2 - 1)} \quad .$$

V. DISCUSSION

We have solved the head-tail effect for a special distribution of synchrotron oscillations in a bunch--namely, one in which all particles have the same amplitude A of synchrotron oscillations but are uniformly distributed in the phase ϕ of these oscillations. For such a distribution we have found that the betatron oscillations within the bunch can be described as a sum of discrete normal modes. In each mode--mode number μ--all of the particles have the same (real) amplitude $|a_\mu|$ of betatron oscillation, but have (time average) relative phase shifts of the betatron oscillation equal to $\mu\phi$. The amplitude of each mode varies with time as

$$a_\mu(0)e^{i\Delta\omega_\mu t} \quad ,$$

where $\Delta\omega_\mu$ is given by Eq. (44). We recall that the symbols used there are

- N ; number of electrons in a bunch.
- S ; wake coupling strength. (It is the wake force divided by γm_o the effective mass of the oscillator.)
- ω_o; the unperturbed betatron (angular) frequency.
- A ; amplitude (in time units) of the synchrotron phase oscillations.
- α ; momentum compaction.
- ξ ; chromaticity, $(E/\nu)\cdot(d\nu/dE)$.

Our solution has the following properties. If $\xi S/\alpha$ is positive, mode "0" will be stable, and all other modes may be unstable. If $\xi S/\alpha$ is negative, the opposite is true. Since the chromaticity ξ is proportional to the effective sextupole terms in the guide field, a change in the sign of ξ can be made by a sextupole correction, and the relative stability of the modes can be reversed. Or, indeed, ξ can be made zero so that all modes should be stable.

Our analysis would predict that all modes with $\beta_\mu > 0$ would be unstable and would grow exponentially with a growth constant β_μ --which is for all modes proportional to N , the number of particles. But we have, of course, neglected the effects of Landau damping due to the spread of unperturbed betatron frequencies in a bunch. We have not carried out a detailed analysis, but we may make the following qualitative observations. The "unstable" modes will be stabilized by the Landau damping until the magnitude of the frequency shift $|\Delta\omega_\mu|$ is roughly comparable with the frequency spread of betatron oscillations due to octupole nonlinearities in the guide field. For a given nonlinearity, an unstable motion will appear at a particular value of $\Delta\omega_\mu$; and each mode will, therefore, be stable until the number of particles N reaches the level which produces this $\Delta\omega_\mu$. Since the coefficients in $\Delta\omega_\mu$ decrease with increasing μ , we would expect the instability to appear first in either mode "0" or mode "1" depending on the algebraic signs of the various parameters. In particular, if the wake field follows our model (decays only slowly with time), we would expect all modes with $\mu > 0$ to have a threshold beam current (the current is proportional to N) which varies inversely with the bunch length. For mode "0" the behavior is less clear since it depends on the relative magnitudes of the real and imaginary parts of $\Delta\omega$. If β_0 dominates, the threshold would vary inversely with the bunch length; but if α_0 dominates the threshold would be relatively independent of bunch length.

Some of these results may explain the apparently different behavior observed in the various electron storage rings. The Princeton-Stanford Ring and the Novosibirsk Rings (Vep 1 and Vepp 2) are weak focussing and so have an $\alpha \approx 1$. Also, the Princeton-Stanford Ring was designed to have $\xi = 0$. On the other hand, ACO and Adone have $\alpha \approx 0.1$ and 0.06 , and $\xi \approx 1$. The head-tail effect would explain the observed fact that these latter machines have much lower thresholds for lateral instabilities.

It is also reported that the ACO instability has little visible coherent motion, while at Adone a large unstable coherent motion is observed which can be stabilized by an external feedback system. A large coherent amplitude is expected for the mode $\mu = 0$ but not for other modes. The observed disparity in the behavior of ACO and Adone would be expected if it happened that ξS were positive for the former and negative for the latter.

The bunch length dependence we have found agrees only in part with the observations at Adone; there, the threshold current was proportional to bunch length in some circumstances and independent of bunch length in others. Such a behavior might be explained by changes in the form of the wake function ρ . (For example, a wake function which corresponds to the resistive-wall effect gives a similar result for β_μ , but with A replaced by $\sqrt{A}$.)

If the wake field were independent of the particle energy, the parameter S would (as we have defined it) be inversely proportional to E . If the instability threshold is controlled by Landau damping, which we expect to go as E^2 , the threshold current would, for our model, vary as E^3 or as E^3/A depending on whether α_μ or β_μ dominates. For an ideal ring with constant r-f voltage A should vary as $E^{3/2}$, so the threshold would vary as E^3 or $E^{3/2}$. The lateral instability observed at Adone has an energy dependence close to E^3 , as might be expected for mode "0".

Finally, we have said nothing about the mechanism of the production of fast wakes. It has been suggested that they may be due to discontinuities or other effects from the vacuum chamber. We have, as yet, no theory of such effects which can give a quantitative explanation of the observed thresholds.

APPENDIX

We wish to show that Eq. (23) is also obtained for an impulsive wake--one which is generated in a localized region of the orbit and gives an impulsive force when traversed by a particle.

Write the displacement of the particle ℓ as

$$x_\ell = Re\ Z_\ell e^{i\omega_o(t + \eta\tau_\ell)}$$

where Z_ℓ is a (complex) constant during any one revolution but which changes discontinuously by the amount δZ_ℓ when the particle goes through a localized field. Say that the impulsive force of such a field produces a change in the slope of x_ℓ at the time t_i in the amount $\delta\dot{x}_i$. Then

$$\delta Z_\ell(t_i) = \frac{i}{\omega}\,\delta\dot{x}_i e^{-i\omega_o(t_i + \eta\tau_\ell)} \quad , \tag{A-1}$$

where τ_ℓ is to be evaluated at t_i , and $\omega \equiv \omega_o + \eta\dot{\tau}_\ell$, which can be approximated by ω_o .

Now let $\delta\dot{x}_i$ be proportional to the displacement x of the particle k on its i-th traversal of the same spot and to a wake factor $\sigma(\tau_k - \tau_\ell)$. We write

$$\begin{aligned} \delta\dot{x}_i &= Re\ Z_k e^{i\omega_o(t_i' + \eta\tau_k)}\sigma(\tau_k - \tau_\ell) \\ &= \frac{1}{2}\left\{ Z_k e^{i\omega_o(t_i' + \eta\tau_k)} + Z_k^* e^{-i\omega_o(t_i' + \eta\tau_k)} \right\} \sigma(\tau_k - \tau_\ell) \end{aligned} \tag{A-2}$$

where t_i' is the time of passage of the particle k , and τ_k is to be evaluated at t_i .

Substituting in (A-1),

$$\delta x_i = -\frac{i}{2\omega}\left\{ Z_k e^{-i\omega_o \left[t_i - t_i' + \eta(\tau_\ell - \tau_k)\right]} \right.$$

$$\left. + Z_k^* e^{-i\omega_o \left[t_i + t_i' + \eta(\tau_\ell + \tau_k)\right.} \left.\left. \sigma(\tau_k - \tau_\ell)\right]\right\}$$

Now $(t_i - t_i')$ is just equal to $-(\tau_\ell - \tau_k)$, so that the first term in the curly brackets of (A-3) is just

$$Z_k e^{-i\omega_o\, t + (1 - \eta)\,(\tau_k - \tau_\ell)} \quad . \tag{A-4}$$

Next, we take into account that we are only interested in the <u>average</u> of δZ_ℓ over many revolutions. Since the τ's vary only slowly, and since $\omega_o(t_{i+1} - t_i)$ is not generally close to an integral multiple of 2π , there is from t_i to t_{i+1} a large, non-resonant phase advance of the second term in (A-3); this term will then average to zero.

We can write that $\dot{Z}_\ell = \delta \overline{Z}_i / T$, where T is the mean revolution time. We get

$$\dot{Z}_\ell = -\frac{i}{2\omega_o T} Z_k e^{-i\omega_o\left[t + (1 - \eta)\,(\tau_k - \tau_\ell)\right.} \left. \sigma(\tau_k - \tau_\ell)\right],$$

which is the same as Eq. (23) if we make the identification

$$\rho(\tau_k - \tau_\ell) = \frac{\sigma(\tau_k - \tau_\ell)}{T} \quad .$$