Cosmological Perturbations in Inflation

Jian-Hua He[1,*] and Bin Wang[2]

[1] *INAF-Observatorio Astronomico, di -Brera, Via Emilio -Bianchi, 46, I-23807, Merate (LC), Italy*

[2] *Department of Physics and Astronomy, Shanghai Jiao Tong University, Shanghai 200240, China*

In this review paper, we go over the single-field theory of inflation. We focus on the cosmological perturbation theory and discuss the measurable predictions of inflationary models. We also discuss implications of the latest cosmological observations on cosmic inflation.

PACS numbers: 98.80.-k,04.50.Kd

1. Introduction

Cosmological inflation nowadays is considered as one of the standard paradigms that the Universe has experienced at the early time of its expansion history. The inflationary theories were developed in a series of papers by [1–7]. The theory solves several major problems that plague the Standard Big-Bang Theory. Inflation explains the origin of the large-scale structure of the cosmos. Inflation also explains the horizon and flatness problem as to why the temperatures and curvatures of different regions are so nearly equal today (see Refs. [8–17] for details).

Despite the underlining particle physics mechanism responsible for inflation is not known, people usually think that the inflation is driven by a field called the inflaton. The inflaton field dominates the Universe during inflation. The inflaton field is not perfectly homogenous but has small quantum fluctuations. These small quantum fluctuations are the seeds for the growth of structure we observe in the universe today. On the large scales, the inflaton field can still be seen as homogenous and responsible for the rapid expansion of the Universe during the inflation. The inflaton field makes a number of predictions that have been partially confirmed by observations.

Most strikingly, BICEP2 collaboration [18] announced a landmark discovery of the B-mode in the cosmic microwave background (CMB) polarization spectra at a significance of $> 5\sigma$ [18]. These findings provide the first detection of gravitational waves ever since the birth of general relativity in 1915. The finding of B-mode also provides a clear experimental evidence for the theory of inflation because the B-mode is mainly produced by the primordial gravitational waves which is further generated by the quantum fluctuation of gravitational field during inflation at the early time of the Universe. If the existence of such B-mode is further confirmed, it will be a great evidence for gravitational waves and its quantization. The findings will have profound impact on both modern cosmology and fundamental physics.

Inspired by these exciting findings, the aim of this paper is to provide a brief pedagogical review on the single-field theory of inflation. We will particularly focus on the cosmological perturbation theory and address the measurable predictions made by inflationary models. We will then focus on a specific model and compare it with BICEP2 and *Planck* [19] observations.

This review paper is originated as follows: In Sec. 2, we introduce the background expansion of Universe during the inflation. In Sec. 3, we introduce cosmological perturbation theory. In Sec. 4, we present the cosmological perturbation theory in Fourier space. In Sec. 5, we introduce the slow-roll limit. In Sec. 6, we present the primordial gravitational waves and its quantization. In Sec. 7, we discuss the cosmological constraints on inflationary models. In Sec. 8, we summarize and conclude this paper.

*Email address: jianhua.he@brera.inaf.it

2. The Background Expansion

In this section, we will go over the background expansion of the inflaton field and then turn to the cosmological perturbations.

The action of the inflaton field is given by

$$S = \int d^4x\sqrt{-g}\mathcal{L}_\phi = -\int d^4x\sqrt{-g}\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi)\right], \quad (1)$$

where g is the determinant of the metric $g_{\mu\nu}$ and the Lagrangian density of the field ϕ is given by

$$\mathcal{L}_\phi = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\,. \quad (2)$$

From the Lagrangian density $\mathcal{L}_\phi$, we can obtain the stress energy-momentum tensor as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{(\delta\sqrt{-g}\mathcal{L}_\phi)}{\delta g^{\mu\nu}} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)\right). \quad (3)$$

We consider the spatially flat Friedmann-Lemaitre-Robertson-Walker(FLRW) metric

$$ds^2 = -dt^2 + a(t)^2 d\vec{x}^2 = -a(\eta)^2 d\tau^2 + a(\eta)^2 d\vec{x}^2\,, \quad (4)$$

where t is cosmic time and η is conformal time. The background energy density ρ_ϕ and the pressure P_ϕ for the scalar field ϕ in conformal time η are given by

$$\rho_\phi = \frac{1}{2}\frac{(\phi')^2}{a^2} + V(\phi)\,,$$
$$P_\phi = \frac{1}{2}\frac{(\phi')^2}{a^2} - V(\phi)\,,$$
$$\rho_\phi + p_\phi = \frac{(\phi')^2}{a^2}\,, \quad (5)$$

where prime denotes the time derivative with respect to the conformal time η.

Using Eq. (5), the Freedamann equations read

$$\mathcal{H}^2 = \frac{\kappa^2}{3}\left(\frac{1}{2}(\phi')^2 + a^2V\right), \quad (6)$$
$$\mathcal{H}^2 - 2\frac{a''}{a} = \kappa^2\left(\frac{1}{2}(\phi')^2 - a^2V\right). \quad (7)$$

Combing Eq. (6) and Eq. (7) to eliminate the potential V, we obtain

$$\mathcal{H}^2 - \mathcal{H}' = \frac{\kappa^2}{2}(\phi')^2\,, \quad (8)$$

where

$$\mathcal{H}' = \frac{a''}{a} - \mathcal{H}^2\,. \quad (9)$$

Taking the derivative of Eq. (8) and then using Eq. (8) again to eliminate κ^2, we obtain an useful expression

$$\frac{a'''}{a} = \left(2\frac{a''}{a} - 4\mathcal{H}^2\right)\frac{\phi''}{\phi'} - 4\mathcal{H}^3 + 5\mathcal{H}\frac{a''}{a}\,, \quad (10)$$

which will be used in the next section.

The Klein-Gordon (KG) equation in the cosmological background reads

$$\partial^\mu\partial_\mu\phi = \frac{\partial V}{\partial\phi}\,. \quad (11)$$

Noting that

$$\partial^\mu\partial_\mu\phi = \frac{1}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right), \quad (12)$$

and inserting Eq. (4), we obtain

$$\phi'' + 2\mathcal{H}\phi' + a^2\frac{dV}{d\phi} = 0\,. \quad (13)$$

Next we define two parameters

$$\epsilon \equiv \frac{3}{2}(1+w_\phi) = \frac{\frac{3}{2}\frac{(\phi')^2}{a^2}}{\frac{1}{2}\frac{(\phi')^2}{a^2} + V}\,, \quad (14)$$
$$\delta \equiv \frac{\phi''}{\mathcal{H}\phi'} - 1\,. \quad (15)$$

We can present Eq. (6) and Eq. (7) in terms of ϵ and δ

$$\mathcal{H}^2 = 4\pi G\frac{(\phi')^2}{\epsilon}\,, \quad (16)$$
$$\mathcal{H}' = \mathcal{H}^2(1-\epsilon)\,. \quad (17)$$

The above equations are exact. Similarly, the Klein-Gordon equation Eq. (13) can be presented as

$$\phi'(3+\delta)\mathcal{H} = -a^2\frac{\partial V}{\partial\phi}\,. \quad (18)$$

It can also be shown that

$$\epsilon' = 2\epsilon(\delta+\epsilon)\mathcal{H}\,. \quad (19)$$

3. The Cosmological Perturbation

As pointed out at the beginning of this paper, the inflaton field ϕ also generates small quantum fluctuations $\delta\phi$. These small quantum fluctuations $\delta\phi$, in turn, will have impact on the spacetime through the coupled Einstein equations of gravity. For a full treatment, we need to consider the perturbation of the Einstein equations as well as the perturbation of the scalar field equations for the inflaton field.

We begin by introducing the cosmological perturbation theory (See Ref. [20] for reviews). The total metric of the spacetime can be divided into the background unperturbed metric and a small perturbation

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu}\,. \tag{20}$$

The perturbation $\delta g_{\mu\nu}$ can be presented as

$$\begin{aligned}&\delta g_{\mu\nu} dx^\mu dx^\nu \\ =\;& a^2[-2Ad\tau^2 - 2B_i d\tau dx^i + 2H_{ij}dx^i dx^j]\,,\end{aligned} \tag{21}$$

where the vector field B_i can be decomposed into two parts

$$B_i = \nabla_i B + B_i^*, \quad \nabla^i B_i^* = 0 \quad . \tag{22}$$

B is a scalar field and it is the spin-0 component of the vector field B_i. B_i^* is a vector field. It is the spin-1 component of the vector field B_i. Similarly, the spatial symmetric tensor field H_{ij} can be decomposed into

$$H_{ij} = H_L\delta_{ij} + D_{ij}H_T + \frac{1}{2}(\nabla^j H_i^V + \nabla^i H_j^V) + H_{ij}^T\,, \tag{23}$$

where

$$\nabla^i H_i^V = 0 \quad, D_{ij} = (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2). \tag{24}$$

and H_{ij}^T is a traceless ($H_i^{Ti} = 0$), divergence free ($\nabla^i H_{ij}^T = 0$), symmetric ($H_{ij}^T = H_{ji}^T$) tensor field. H_L and $2H_T$ are spin-0 components, H_i^V is spin-1 component and H_{ij}^T is spin-2 component.

The scalar,vector and tensor perturbations refer to the components in the metric perturbation with spin-0, spin-1, spin-2 respectively. It can be shown that the scalar,vector and tensor perturbations evolve independently during the expansion of the Universe.

$$\begin{aligned}\delta g_{\mu\nu}^{(S)} dx^\mu dx^\nu &= a^2[-2Ad\tau^2 - 2\partial_i B d\tau dx^i \\ &\quad + 2H_L\delta_{ij}dx^i dx^j + 2D_{ij}H_T dx^i dx^j]\,. \\ \delta g_{\mu\nu}^{(V)} dx^\mu dx^\nu &= a^2[-B_i^* d\tau dx^i \\ &\quad + \frac{1}{2}(\nabla^j H_i^V + \nabla^i H_j^V)dx^i dx^j]\,. \\ \delta g_{\mu\nu}^{(T)} dx^\mu dx^\nu &= a^2 H_{ij}^T dx^i dx^j\,.\end{aligned} \tag{25}$$

In this section, we will focus on the scalar perturbation. The vector perturbation will decay and eventually die out. We will therefore not consider the vector perturbation in this review paper. The tensor perturbation will be discussed in the next few sections.

3.1 *Perturbed Energy-Momentum Tensor*

The perturbed energy-momentum tensor is given by

$$\begin{aligned}\delta T^{\mu\nu} &= (p_\phi + \rho_\phi)\delta U^\mu U^\nu + (p_\phi + \rho_\phi)U^\mu \delta U^\nu \\ &\quad + (\delta p_\phi + \delta\rho_\phi)U^\mu U^\nu + \delta p_\phi g^{\mu\nu} + p_\phi \delta g^{\mu\nu}\,.\end{aligned} \tag{26}$$

The perturbed conservation equations are given by

$$\begin{aligned}\delta\nabla_\mu T^{\mu\nu} &= \partial_\mu \delta T^{\mu\nu} + \delta\Gamma^\mu_{\mu\tau}T^{\tau\nu} + \Gamma^\mu_{\mu\tau}\delta T^{\tau\nu} \\ &\quad + \delta\Gamma^\nu_{\mu\tau}T^{\mu\tau} + \Gamma^\nu_{\mu\tau}\delta T^{\mu\tau} \\ &= \delta Q^\nu\,.\end{aligned} \tag{27}$$

Inserting Eq. (25) into the above two equations and noting that $\delta U^0 = -\frac{A}{a}$, $\delta U^i = \frac{\partial^i v}{a}$, we obtain,

$$\begin{aligned}\delta T^{00} &= \frac{1}{a^2}(\delta\rho_\phi - 2A\rho_\phi)\,, \\ \delta T^{i0} &= \frac{1}{a^2}[(p_\phi + \rho_\phi)\partial^i v_\phi - p_\phi \partial^i B]\,, \\ \delta T^{ij} &= \frac{1}{a^2}[\delta p_\phi \delta^{ij} - p_\phi(2H_L\delta^{ij} + 2D^{ij}H_T)]\,, \\ \delta T^{0i} &= \delta T^{i0}\,,\end{aligned} \tag{28}$$

where

$$\theta_\phi = \nabla^2 v_\phi \,.$$

v_ϕ is the scalar potential of three velocity. Eq. (28) can be equivalently presented as

$$\begin{aligned}
\delta T^0{}_0 &= -\delta\rho_\phi \,,\\
\delta T^0{}_i &= (p_\phi+\rho_\phi)(\partial_i v_\phi - \partial_i B)\,,\\
\delta T^i{}_0 &= -(p_\phi+\rho_\phi)\partial^i v \,,\\
\delta T^i{}_j &= \delta p_\phi \delta^i{}_j + \Pi^i_{\phi j} \,. && (29)
\end{aligned}$$

Further noting that

$$\begin{aligned}
\delta\rho_\phi &= \frac{1}{a^2}\left(\delta\phi'\phi' - A\phi'\right) + \delta\phi V(\phi)\,,\\
\delta p_\phi &= \frac{1}{a^2}\left(\delta\phi'\phi' - A\phi'\right) - \delta\phi V(\phi)\,,\\
v_\phi &= -\frac{\delta\phi}{\phi'} + B\,, && (30)
\end{aligned}$$

the perturbed energy-momentum tensor for the scalar field can be presented as

$$\delta T^0{}_0 = -\left[\frac{1}{a^2}\left(\delta\phi'\phi' - A\phi'\right) + \delta\phi V(\phi)\right], \qquad (31)$$

$$\begin{aligned}
\delta T^0{}_i &= -\frac{\phi'}{a^2}\partial_i\delta\phi\,,\\
\delta T^i{}_0 &= -\frac{(\phi')^2}{a^2}\left(\partial^i B - \frac{1}{\phi'}\partial^i\delta\phi\right)\,. && (32)
\end{aligned}$$

The above results are consistent with the perturbations obtained from the energy-momentum tensor of the scalar field directly.

$$\begin{aligned}
&\delta T_{\mu\nu}\\
&= \partial_\mu\delta\phi\partial_\nu\phi + \partial_\mu\phi\partial_\nu\delta\phi\\
&\quad -\delta g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)\right)\\
&\quad -g_{\mu\nu}\left(\frac{1}{2}\delta g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + g^{\alpha\beta}\partial_\alpha\delta\phi\partial_\beta\phi + \frac{\partial V}{\partial\phi}\phi\right)\,. \qquad (33)
\end{aligned}$$

3.2 *Perturbed Einstein's Equations*

We calculate the perturbed Ricci tensor $\delta R_{\mu\nu}$ as follows

$$\begin{aligned}
\delta R_{\mu\nu} &= \delta\Gamma^\alpha_{\mu\nu,\alpha} - \delta\Gamma^\alpha_{\nu\alpha,\mu} + \delta\Gamma^\alpha_{\sigma\alpha}\Gamma^\sigma_{\mu\nu} + \Gamma^\alpha_{\sigma\alpha}\delta\Gamma^\sigma_{\mu\nu}\\
&\quad -\delta\Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\mu\alpha} - \Gamma^\alpha_{\sigma\nu}\delta\Gamma^\sigma_{\mu\alpha}\,,
\end{aligned}$$

where the perturbed affine connections $\delta\Gamma^\alpha_{\mu\nu}$ can be computed from the perturbation of the metric $\delta g_{\mu\nu}$

$$\delta\Gamma^\alpha_{\mu\nu} = \delta g^{\alpha\tau}g_{\tau\sigma}\Gamma^\sigma_{\mu\nu} + \frac{1}{2}g^{\alpha\tau}(\delta g_{\tau\mu,\nu} + \delta g_{\nu\tau,\mu} - \delta g_{\mu\nu,\tau})\,.$$

$\delta g^{\tau\nu}$ is further defined by

$$\delta g^{\tau\nu} = -g^{\tau\mu}g^{\nu\sigma}\delta g_{\sigma\mu}\,.$$

The perturbed Einstein equations in components read

$$\begin{aligned}
-4\pi Ga^2\delta\rho_\phi &= \nabla^2 H_L + 3\mathcal{H}\left(\mathcal{H}A - H_L'\right)\\
&\quad -\mathcal{H}\nabla^2 B - \frac{1}{2}\partial_k\partial^i D_i{}^k H_T, && (34)
\end{aligned}$$

$$\begin{aligned}
&-4\pi Ga^2(\rho_\phi+p_\phi)\theta_\phi\\
&= \mathcal{H}\nabla^2 A - \nabla^2 H_L' - 2\mathcal{H}^2\nabla^2 B\\
&\quad +\frac{a''}{a}\nabla^2 B + \frac{1}{2}\partial_k\partial^i D_i{}^k H_T', \qquad (35)
\end{aligned}$$

$$\begin{aligned}
&4\pi Ga^2(\rho_\phi+p_\phi)(\theta_\phi - \nabla^2 B)\\
&= -\mathcal{H}\nabla^2 A + \nabla^2 H_L' - \frac{1}{2}\partial_k\partial^i D_i{}^k H_T', \qquad (36)
\end{aligned}$$

$$\begin{aligned}
8\pi Ga^2\delta p_\phi &= 2\mathcal{H}A' + 4\frac{a''}{a}A - 2\mathcal{H}^2 A + \frac{2}{3}\nabla^2 A\\
&\quad +\frac{2}{3}\nabla^2 H_L - 4\mathcal{H}H_L' - 2H_L''\\
&\quad -\frac{4}{3}\mathcal{H}\nabla^2 B - \frac{2}{3}\nabla^2 B' - \frac{1}{3}\partial_k\partial^i D_i{}^k H_T, && (37)
\end{aligned}$$

$$\begin{aligned}
8\pi Ga^2\Pi^i_{\phi j} &= -\partial^i\partial_j A - \partial^i\partial_j H_L + 2\frac{1}{2}D^i{}_j H_T''\\
&\quad +2\mathcal{H}D^i{}_j H_T' + \frac{1}{3}\nabla^2 D^i{}_j H_T\\
&\quad +2\mathcal{H}\partial^i\partial_j B + \partial^i\partial_j B'. && (38)
\end{aligned}$$

3.3 *Perturbed Klein-Gordon Equation*

Noting that

$$\begin{aligned}\delta\partial^{\mu}\partial_{\mu}\phi &= \delta\left(\frac{1}{\sqrt{-g}}\right)\partial_{\nu}(\sqrt{-g}g^{\mu\nu}\partial_{\nu}\phi) \\ &+\frac{1}{\sqrt{-g}}\partial_{\nu}\left[(\delta\sqrt{-g})g^{\mu\nu}\partial_{\nu}\phi\right] \\ &+\frac{1}{\sqrt{-g}}\partial_{\nu}(\sqrt{-g}\delta g^{\mu\nu}\partial_{\nu}\phi) \\ &+\frac{1}{\sqrt{-g}}\partial_{\nu}(\sqrt{-g}g^{\mu\nu}\partial_{\nu}\delta\phi) \\ &= \frac{\partial^2 V}{\partial^2\phi}\delta\phi\,, \end{aligned} \tag{39}$$

where

$$\begin{aligned}\delta\frac{1}{\sqrt{-g}} &= \frac{\delta\sqrt{-g}}{g}\,, \\ \delta\sqrt{-g} &= -\frac{\delta g}{2\sqrt{-g}}\,, \\ \delta g &= gg^{\mu\nu}\delta g_{\mu\nu}\,. \end{aligned} \tag{40}$$

The perturbed Klein-Gordon equation reads

$$\begin{aligned}&-\delta\phi^{''}-2\mathcal{H}\delta\phi'+\nabla^2\delta\phi+2A\phi^{''}+A'\phi' \\ &+4A\mathcal{H}\phi'-3\phi' H_L'-\nabla^2 B\phi' \\ =\ & a^2\frac{\partial^2 V}{\partial\phi^2}\delta\phi\,. \end{aligned} \tag{41}$$

3.4 *Gauge Issue*

3.4.1 *Infinitesimal Gauge Transformation*

The equations in General relativity should be generally covariant. The perturbation equation should preserve its form under a general coordinate transformation. In order to see this point, we consider an infinitesimal coordinate transformation

$$\begin{aligned}\tilde{x}^{\mu} &= x^{\mu}+\delta x^{\mu}\,, \\ \delta x^0 &= \xi^0(x^{\mu})\,, \\ \delta x^i &= \partial^i\beta(x^{\mu})+v_*^i(x^{\mu})(\partial_i v_*^i=0)\,. \end{aligned} \tag{42}$$

For the scalar perturbation, the Lie derivative is defined as

$$\begin{aligned}&\mathcal{L}_{\delta x}g_{\mu\nu} \\ =\ & g_{\mu\sigma}\nabla_{\nu}\delta x^{\sigma}+g_{\sigma\nu}\nabla_{\mu}\delta x^{\sigma} \\ =\ & \delta x^{\sigma}\partial_{\sigma}g_{\mu\nu}+g_{\mu\sigma}\partial_{\nu}\delta x^{\sigma}+g_{\sigma\nu}\partial_{\mu}\delta x^{\sigma} \end{aligned} \tag{43}$$

Inserting the background metric $g_{\mu\nu}$ and Eq. (42), we obtain

$$\begin{aligned}&\mathcal{L}_{\delta x}g_{\mu\nu}dx^{\mu}dx^{\nu} \\ =\ & a^2\{-2(\frac{a'}{a}\xi^0+\xi^{0'})d\tau^2+2(\partial_i\beta'-\partial_i\xi^0)d\tau dx^i \\ &+2\frac{a'}{a}\xi^0\delta_{ij}dx^i dx^j+(\partial_i\partial_j\beta+\partial_j\partial_i\beta)dx^i dx^j\}\,. \end{aligned} \tag{44}$$

Recall that

$$\begin{aligned}&\delta g_{\mu\nu}dx^{\mu}dx^{\nu} \\ =\ & a^2\left[-2Ad\tau^2-2\partial_i Bd\tau dx^i+2H_L\delta_{ij}dx^i dx^j\right. \\ &\left.+2D_{ij}H_T dx^i dx^j\right]\,, \end{aligned} \tag{45}$$

the infinitesimal gauge transformation for the perturbed quantities can be presented as

$$\delta\tilde{g_{\mu\nu}}=\delta g_{\mu\nu}-\mathcal{L}_{\delta x}g_{\mu\nu}\,. \tag{46}$$

In components, we obtain

$$\begin{aligned}d\tau^2 &\longrightarrow A-\xi^{0'}-\frac{a'}{a}\xi^0\,, \\ d\tau dx^i &\longrightarrow -B+\xi^0-\beta'\,, \\ dx^i dx^j &\longrightarrow 2(H_L-\frac{a'}{a}\xi^0-\frac{1}{3}\nabla^2\beta)\delta_{ij} \\ &\qquad +2(\partial_i\partial_j-\frac{1}{3}\delta_{ij}\nabla^2)(H_T-\beta)\,. \end{aligned}$$

Thus, the perturbed quantities behave as

$$\begin{aligned}\tilde{A} &= A-\xi^{0'}-\frac{a'}{a}\xi^0\,, \\ \tilde{B} &= B-\xi^0+\beta'\,, \\ \tilde{H_L} &= H_L-\frac{1}{3}\nabla^2\beta-\frac{a'}{a}\xi^0\,, \\ \tilde{H_T} &= H_T-\beta\,, \\ \tilde{v} &= v+\beta'\,, \end{aligned} \tag{47}$$

where the tildes denote the quantities in the transformed coordinates. Inserting Eq.(47) back into Eqs.(65, 35, 67, 37, 38), and further noting that

$$\begin{aligned}\mathcal{H}^2 &= \frac{8\pi G\rho_{\phi}a^2}{3}\,, \\ \mathcal{H}' &= -\frac{4\pi G}{3}(\rho_{\phi}+3p_{\phi})a^2\,, \end{aligned} \tag{48}$$

we can show that the transformed perturbed equations have the same forms as the untransformed ones. The perturbed Einstein equations are therefore covariant. Similarly, we can show that Eq. (41) is covariant as well.

3.4.2 *Fixing Gauge*

The density contrast δ in the relativistic perturbation theory depends on the choice of gauge. Even for an ideal homogenous background field, it could still be viewed as a perturbed field in certain coordinates. The background field is homogenous only in the sense that the spatial distribution of the density field at a given time t is homogenous. Actually, it is not homogenous in the temporal distribution if we investigate the density field at a given spatial point along the time line. It is possible that if we take a wrapped coordinate, which could mix the homogenous spatial slices at different times in the original coordinate, in the new coordinate, the density field looks like a perturbed one. The density contrast δ as well as other perturbed quantities therefore only have their physical meanings in specified coordinates. However, if the gauge transformation is completely fixed, different perturbations in different gauges will have one-to-one map to each other and they are therefore equivalent.

Fixing gauge means that we constrain the general infinitesimal gauge transformation in certain ways that the coordinate transformation is unambiguous. To completely fix the gauge, we need to specify both the temporal and spatial infinitesimal transformations. For scalar perturbation, that means we need to specify ξ^0 and β. For instance, the condition $H_L = H_T = 0$ imply that

$$\xi^0 = -(\tilde{H}_L - \frac{1}{3}\nabla^2 \tilde{H}_T)\mathcal{H}^{-1}\,, \qquad (49)$$

$$\beta = -\tilde{H}_T\,. \qquad (50)$$

The infinitesimal coordinate transformation is completely fixed by the gauge condition $H_L = H_T = 0$.

4. Fourier Space and Covariant Perturbation Equations

4.1 *Covariant Perturbation Equations*

It is more convenient to study the perturbation equations in the Fourier space. In the Fourier space, the perturbed metric can be decomposed by the scalar harmonics

$$\begin{aligned} &\delta g_{\mu\nu}dx^\mu dx^\nu \\ =\ & a^2\Big\{-2AdY^{(s)}d\tau^2 - 2BY_i^{(s)}d\tau dx^i \\ &+2H_L Y^{(s)}\delta_{ij}dx^i dx^j + 2H_T Y_{ij}^{(s)}dx^i dx^j\Big\}\,, \end{aligned} \qquad (51)$$

where the scalar harmonics are defined as

$$\begin{aligned} &(\triangle + k^2)Y^{(s)} = 0\,, \\ &Y_j^{(s)} \equiv -\frac{1}{k}\partial_j Y^{(s)}\,, \\ &Y_{ij}^{(s)} \equiv \frac{1}{k^2}\partial_i\partial_j Y^{(s)} + \frac{1}{3}\delta_{ij}Y^{(s)}\,. \end{aligned} \qquad (52)$$

The Lie derivative reads

$$\begin{aligned} \mathcal{L}_{\delta x}g_{\mu\nu} =\ & a^2\{-2(\frac{a'}{a}\xi^0 + \xi^{0\prime})Y^{(s)}d\tau^2 \\ &+2(\beta' + k\xi^0)Y_i^{(s)}d\tau dx^i \\ &+2\frac{a'}{a}\xi^0 Y^{(s)}\delta_{ij}dx^i dx^j \\ &+\beta(Y_{i,j}^{(s)} + Y_{j,i}^{(s)})dx^i dx^j\}\,. \end{aligned} \qquad (53)$$

From the infinitesimal gauge transformation

$$\delta\tilde{g_{\mu\nu}} = \delta g_{\mu\nu} - \mathcal{L}_{\delta x}g_{\mu\nu}\,, \qquad (54)$$

we obtain

$$\begin{aligned} d\tau^2 &\longrightarrow (A - \xi^{0\prime} - \frac{a'}{a}\xi^0)Y^{(s)}\,, \\ d\tau dx^i &\longrightarrow (-B - k\xi^0 - \beta')Y_i^{(s)}\,, \\ dx^i dx^j &\longrightarrow 2(H_L - \frac{a'}{a}\xi^0 - \frac{1}{3}k\beta)\delta_{ij}Y^{(s)} \\ &\qquad +2(H_T + k\beta)Y_{ij}^{(s)}\,, \end{aligned}$$

where

$$\begin{aligned} &2H_T Y_{ij}^{(s)} - 2\beta Y_{i,j}^{(s)} + \frac{2}{3}k\beta\delta_{ij}Y^{(s)} \\ =\ & 2H_T Y_{ij}^{(s)} - 2\beta k(-Y_{ij}^{(s)} + \frac{1}{3}\delta_{ij}Y^{(s)}) \\ &+\frac{2}{3}k\beta\delta_{ij}Y^{(s)} \\ =\ & (2H_T + 2k\beta)Y_{ij}^{(s)}\,. \end{aligned}$$

Thus the perturbed quantities in the Fourier space, under the infinitesimal gauge transformation, behavior as

$$
\begin{aligned}
\tilde{A}Y^{(s)} &= (A - \xi^{0\prime} - \frac{a'}{a}\xi^0)Y^{(s)}\,,\\
\tilde{B}Y_i^{(s)} &= (B + k\xi^0 + \beta')Y_i^{(s)}\,,\\
\tilde{H_L}Y^{(s)} &= (H_L - \frac{1}{3}k\beta - \frac{a'}{a}\xi^0)Y^{(s)}\,,\\
\tilde{H_T}Y_{ij}^{(s)} &= (H_T + k\beta)Y_{ij}^{(s)}\,,\\
\tilde{v}Y^{(s)} &= (v + \beta')Y^{(s)}\,,
\end{aligned}
\tag{55}
$$

where we have used

$$
\partial^i\partial^j D_{ij}H_T = \frac{2}{3}[\nabla^2]^2 H_T \longrightarrow \frac{2}{3}k^2 H_T Y^{(s)}\,. \tag{56}
$$

The perturbed Einstein equations in the Fourier space read

$$
\begin{aligned}
-4\pi G a^2\delta\rho_\phi &= -k^2 H_L + 3\mathcal{H}\left(\mathcal{H}A - H_L'\right) - \mathcal{H}kB - \frac{1}{3}k^2 H_T\,,\\
-4\pi G a^2(\rho_\phi + p_\phi)kv_\phi &= -k^2\mathcal{H}A + k^2 H_L' - 2\mathcal{H}^2 kB + \frac{a''}{a}kB + \frac{1}{3}k^2 H_T'\,,\\
4\pi G a^2(\rho_\phi + p_\phi)\frac{v_\phi - B}{k} &= \mathcal{H}A - H_L' - \frac{1}{3}H_T'\,,\\
8\pi G a^2\delta p_\phi &= 2\mathcal{H}A' + 4\frac{a''}{a}A - 2\mathcal{H}^2 A - \frac{2}{3}k^2 A - \frac{2}{3}k^2 H_L - 4\mathcal{H}H_L' - 2H_L'' - \frac{4}{3}\mathcal{H}kB - \frac{2}{3}kB' - \frac{2}{9}k^2 H_T\,,\\
8\pi G a^2 p_\phi\Pi_\phi &= -k^2 A - k^2 H_L + H_T'' + 2\mathcal{H}H_T' - \frac{1}{3}k^2 H_T - 2\mathcal{H}kB - kB'\,.
\end{aligned}
$$

We can show that the perturbed Einstein equations are covariant under the infinitesimal gauge transformation.

The perturbed KG equation reads

$$
\begin{aligned}
&\delta\phi'' + 2\mathcal{H}\delta\phi' + (k^2 + a^2\frac{\partial^2 V}{\partial\phi})\delta\phi\\
&+(kB + 3H_L' - A')\phi' + 2a^2 A\frac{\partial V}{\partial\phi}\\
&= 0\,.
\end{aligned}
\tag{57}
$$

The perturbation for the density field and pressure field read

$$
\delta\rho_\phi = \frac{1}{a^2}\left(\delta\phi'\phi' - A\phi'\right) + \delta\phi V(\phi)\,, \tag{58}
$$

$$
\delta p_\phi = \frac{1}{a^2}\left(\delta\phi'\phi' - A\phi'\right) - \delta\phi V(\phi)\,, \tag{59}
$$

$$
v_\phi = \frac{k\delta\phi}{\phi'} + B\,. \tag{60}
$$

Inserting the above equations into

$$
\begin{aligned}
\delta T^0{}_0 &= -\delta\rho_\phi\,,\\
\delta T^0{}_i &= (p_\phi + \rho_\phi)(v_\phi - B)Y_i\,,\\
\delta T^i{}_0 &= -(p_\phi + \rho_\phi)v_\phi Y^i\,,
\end{aligned}
\tag{61}
$$

we obtain

$$
\delta T^0{}_0 = -\left[\frac{1}{a^2}\left(\delta\phi'\phi' - A\phi'\right) + \delta\phi V(\phi)\right]Y\,, \tag{62}
$$

$$
\begin{aligned}
\delta T^0{}_i &= \frac{k\phi'\delta\phi}{a^2}Y_i\,,\\
\delta T^i{}_0 &= -\left[\frac{B(\phi')^2}{a^2} + \frac{k\phi'\delta\phi}{a^2}\right]Y^i\,.
\end{aligned}
\tag{63}
$$

4.2 $H_L = H_T = 0$ *Gauge*

We work in a particular gauge $H_L = H_T = 0$, which could significantly simplify the perturbation equations. As discussed in the previous section, the gauge condition $H_L = H_T = 0$ completely fix the infinitesimal coordinate transformations. The perturbed KG equation in $H_L = H_T = 0$ gauge reduces to

$$\begin{aligned} &\delta\phi'' + 2\mathcal{H}\delta\phi' + (k^2 + a^2\frac{\partial^2 V}{\partial\phi})\delta\phi \\ &+(kB - A')\phi' + 2a^2 A\frac{\partial V}{\partial\phi} \\ =&\ 0\,. \end{aligned} \tag{64}$$

The perturbed Einstein's equations reduce to

$$\begin{aligned} &\frac{\kappa^2}{2}(\phi')^2 A - \frac{\kappa^2}{2}\delta\phi'\phi' - \frac{\kappa^2}{2}\frac{dV}{d\phi}\delta\phi a^2 \\ =&\ -kB\mathcal{H} + 3A\mathcal{H}^2\,, \end{aligned} \tag{65}$$

$$\begin{aligned} &\frac{\kappa^2}{2}(\phi')^2 B + \frac{\kappa^2}{2}k\phi'\delta\phi \\ =&\ -B\frac{a''}{a} + k\mathcal{H}A + 2B\mathcal{H}^2\,, \end{aligned} \tag{66}$$

$$\frac{\kappa^2}{2}\phi'\delta\phi = A\mathcal{H}\,, \tag{67}$$

$$\begin{aligned} &-\kappa^2(\phi')^2 A + \kappa^2\delta\phi'\phi' - \kappa^2\frac{dV}{d\phi}\delta\phi a^2 \\ =&\ 4\frac{a''}{a}A - \frac{4}{3}k\mathcal{H}B - \frac{2}{3}kB' - 2A\mathcal{H}^2 \\ &+2\mathcal{H}A' - \frac{2}{3}Ak^2\,. \end{aligned} \tag{68}$$

Using Eq. (65) and Eq. (67) to eliminate the metric perturbation A and B in Eq. (64), we arrive at a second order differential equation with respect to $\delta\phi$.

$$\delta\phi'' + 2\mathcal{H}\delta\phi' + [k^2 + f(\eta)]\delta\phi = 0\,. \tag{69}$$

The coefficient of $\delta\phi$ in Eq. (69) only consists of the background quantities

$$\begin{aligned} f(\eta) =&\ -\frac{\kappa^2}{2}\frac{\phi''\phi'}{\mathcal{H}} + \kappa^2(\phi')^2 + a^2\frac{d^2V}{d\phi^2} \\ &+a^2\frac{3\kappa^2}{2}\frac{\phi'}{\mathcal{H}}\frac{dV}{d\phi} + \frac{\kappa^2}{2}\frac{(\phi')^2}{\mathcal{H}^2}\frac{a''}{a} \\ &-\frac{\kappa^4}{4}\frac{(\phi')^4}{\mathcal{H}^2}\,. \end{aligned} \tag{70}$$

Using Eq. (10), Eq. (16) and Eq. (17), $f(\eta)$ can be presented in a compact form

$$f(\eta) = -\mathcal{H}^2\left[3(\epsilon+\delta) + (\delta+2\epsilon)(\delta+\epsilon)\right] - \mathcal{H}\delta'\,. \tag{71}$$

Eq. (69) with $f(\eta)$ given by Eq. (71) is an exact equation. Next we use a new quantity $u = a\delta\phi$ to get rid off the background expansion. We obtain [21]

$$u'' + [k^2 + g(\eta)]u = 0\,. \tag{72}$$

where $g(\eta)$ is explicitly given by

$$\begin{aligned} g(\eta) =&\ f(\eta) + \mathcal{H}^2(\epsilon - 2) \\ =&\ -\mathcal{H}^2\left[2 + 2\epsilon + 3\delta + (\delta+2\epsilon)(\delta+\epsilon)\right] - \mathcal{H}\delta' \\ =&\ -\frac{z''}{z}\,, \end{aligned} \tag{73}$$

where z is defined as

$$z = \frac{a\phi'}{\mathcal{H}}\,. \tag{74}$$

Given the background, Eq. (72) can be solved to yield the evolution of the scalar field fluctuation. The exact numerical treatment for Eq. (72) can be found in Refs. [22–25]. However, in this review paper, we will solve Eq. (72) based on slow-roll approximations.

Recall that the curvature perturbation ζ relates to the perturbation $\delta\phi$ in $H_L = H_T = 0$ gauge by

$$\zeta = -\mathcal{H}\frac{\delta\phi}{\phi'}\,. \tag{75}$$

The curvature ζ power spectrum is defined by the supper-horizon limit

$$\Delta^2_\zeta(k) \equiv \frac{k^3}{2\pi^2}\lim_{k\eta\to 0}|\zeta|^2 = \frac{k^3}{2\pi^2}\lim_{k\eta\to 0}\left|\frac{u}{z}\right|^2\,. \tag{76}$$

5. Slow-Roll Limit

During inflation, the scalar field is slowly rolling down its potential. The kinetic term ϕ' is very small. This period is called the *slow-roll*. In slow-roll limit, $\epsilon \ll 1$, $\delta \ll 1$, and $\delta' \ll \frac{a'}{a}$, the parameters ϵ and δ can be approximated as

$$\epsilon \approx \frac{1}{16\pi G}\left(\frac{\frac{\partial V}{\partial\phi}}{V}\right)^2 = \frac{M_{\rm pl}^2}{2}\left(\frac{\frac{\partial V}{\partial\phi}}{V}\right)^2, \tag{77}$$

$$\delta \approx \epsilon - \frac{1}{8\pi G}\frac{\frac{\partial^2 V}{\partial\phi^2}}{V} = \epsilon - \frac{M_{\rm pl}^2\frac{\partial^2 V}{\partial\phi^2}}{V}\,, \tag{78}$$

$$\eta_V \equiv \epsilon - \delta \approx M_{\rm pl}^2\frac{\frac{\partial^2 V}{\partial\phi^2}}{V}\,, \tag{79}$$

where $M_{\rm pl}^{-2} = 8\pi G$ is the reduced Planck mass. Eq. (72) can be solved analytically in slow-roll limit

$$u'' + \left[k^2 - 2\left(\frac{a'}{a}\right)^2\right] u = 0\,. \quad (80)$$

During the de Sitter expansion, the scale factor grows exponentially as $a \sim e^{Ht}$ and the conformal factor reads

$$\tilde{\eta} = \eta - \eta_{\rm end} = \int_{a_{\rm end}}^{a} \frac{da}{Ha^2} \approx -\frac{1}{aH} = -\frac{1}{\mathcal{H}}\,. \quad (81)$$

Eq. (80) reduces to

$$u'' + \left[k^2 - \frac{2}{\tilde{\eta}^2}\right] u = 0\,. \quad (82)$$

The exact solution to this equation reads

$$u = A(k \pm \frac{i}{\tilde{\eta}})e^{\mp ik\tilde{\eta}}\,. \quad (83)$$

Recall that the inflation theory is essentially quantum theory rather than classical theory, we need to quantize the above field. The parameter A can be determined by quantum fluctuations of free field. In order to do this, we consider the sub-horizon approximation $k\tilde{\eta} \gg 1$, Eq. (80) and Eq. (83) reduce to

$$\begin{aligned} u'' \;+\; k^2 u &= 0\,, && (84)\\ u \;&=\; Ake^{\mp ik\tilde{\eta}}\,. && (85) \end{aligned}$$

The field goes back to the standard case in Minkowski's space. We follow the standard procedure of quantizing a scalar field. The field u in Eq. (84) can be quantized as

$$\hat{u} = v(k,\eta)\hat{a} + v^*(k,\eta)\hat{a}^\dagger\,, \quad (86)$$

where

$$v(k,\eta) = \frac{e^{-ik\tilde{\eta}}}{\sqrt{2k}}\,. \quad (87)$$

$\hat{a}$ and $\hat{a}^\dagger$ are annihilation and creation operators which satisfy $[\hat{a}, \hat{a}^\dagger] = 1$ and $\hat{a}|0>= 0$. It can be shown that

$$\begin{aligned} &< \hat{u}^\dagger(\vec{k},\eta)\hat{u}(\vec{k}',\eta) > \\ =\;& |v(k,\eta)|^2(2\pi)^3\delta^3(\vec{k}-\vec{k}')\\ \equiv\;& (2\pi)^3\frac{2\pi^2}{k^3}P_u(k,\eta)\delta^3(\vec{k}-\vec{k}')\,, \quad (88) \end{aligned}$$

where $P_u(k,\eta) = \frac{k^3}{2\pi^2}|v(k,\eta)|^2$ is the power spectrum. The normalized wave function for Eq. (84) is

$$u(k,\tilde{\eta}) = \frac{1}{\sqrt{2k}}e^{-ik\tilde{\eta}}\,. \quad (89)$$

Comparing the above equation with Eq. (85), the normalization A is given by

$$A = \frac{1}{\sqrt{2k^3}}\,. \quad (90)$$

Next we discuss the super-horizon approximation $|k\tilde{\eta}| \ll 1$. The expression Eq. (83) can be written as

$$u = \pm\frac{i}{\tilde{\eta}}A = \pm iHaA\,, \quad (91)$$

Recall that $u = a\delta\phi$

$$\delta\phi = \pm iHA\,. \quad (92)$$

Inserting the above equation into Eq. (75), we obtain

$$\zeta = \mp iHA\mathcal{H}\frac{1}{\phi'}\,. \quad (93)$$

From Eq. (16), we obtain

$$\frac{\mathcal{H}^2}{(\phi')^2} = \frac{4\pi G}{\epsilon}. \quad (94)$$

Recall that Eq. (76), the curvature ζ power spectrum finally is given by

$$\begin{aligned} \Delta_\zeta^2(k) \;&=\; \frac{k^3}{2\pi^2}\lim_{k\eta\to 0}|\zeta|^2\\ &=\; \frac{2k^3GH^2A^2}{\pi\epsilon}\\ &=\; \frac{G}{\pi}\frac{H^2}{\epsilon}\\ &\approx\; \frac{V}{24\pi^2M_{\rm pl}^4\epsilon}\,. \quad (95) \end{aligned}$$

The last equality comes from the slow-roll approximation $3H^2 \approx V/M_{\rm pl}^2$.

We define the scalar spectral index n_s as

$$n_s - 1 \equiv \frac{d\ln\Delta_\zeta^2}{d\ln k} = 2\frac{d\ln H}{d\ln k} - \frac{d\ln\epsilon}{d\ln k}\,. \quad (96)$$

There are two logarithmic derivative terms. We evaluate these two terms at the horizon crossing

$k = aH$. From Eq. (17), using $a^2\dot{H} = \mathcal{H}' - \mathcal{H}^2$, we obtain

$$\frac{dH}{d\eta} = -aH^2\epsilon\,, \tag{97}$$

where the dot denotes the derivative with respect to the proper cosmic time dt. The logarithmic derivative of the Hubble rate at horizon crossing can be calculated by

$$\frac{d\ln H}{d\ln k}|_{aH=k} = \frac{k}{H}\frac{dH}{d\eta} \times \frac{d\eta}{dk}|_{aH=k}\,. \tag{98}$$

From Eq. (81), we obtain $d\eta = d\tilde{\eta} = -d(aH)^{-1} = d(aH)/(aH)^2$ and

$$\frac{d\eta}{dk}|_{aH=k} = \frac{1}{k^2}\,. \tag{99}$$

Therefore, it follows that

$$\frac{d\ln H}{d\ln k}|_{aH=k} = -\epsilon\,. \tag{100}$$

The second term in Eq. (96) is given by

$$\frac{d\ln \epsilon}{d\ln k}|_{aH=k} = \frac{k}{\epsilon}\frac{d\epsilon}{d\eta} \times \frac{d\eta}{dk}|_{aH=k} = 2(\delta+\epsilon)\,. \tag{101}$$

In the above expression, we have used Eq. (19). The relationship between n_s and the slow-roll parameters follow immediately

$$n_s = 1 - 4\epsilon - 2\delta = 1 + 2\eta_V - 6\epsilon\,. \tag{102}$$

The above expression holds at $aH = k$ under the slow-roll approximation.

Next we calculate the logarithmic derivative of n_s. From Eq. (18), in the slow-roll limit, we have $\phi' \approx -\frac{a^2V_\phi}{3\mathcal{H}}$, $3\mathcal{H}^2 \approx a^2V/M_{\rm pl}^2$. The logarithmic derivative of ϕ can be evaluated by

$$\begin{aligned}\frac{d\phi}{d\ln k}|_{aH=k} &= \frac{d\phi}{d\tilde{\eta}}\frac{d\tilde{\eta}}{d\ln k}|_{aH=k} \\ &\approx -\frac{a^2V_\phi}{3\mathcal{H}} \times \frac{1}{\mathcal{H}}|_{aH=k} = -M_{\rm pl}^2\frac{V_\phi}{V}|_{aH=k}\,.\end{aligned} \tag{103}$$

Using the above equation, we can find that

$$\begin{aligned}&\frac{d\eta_V}{d\ln k}|_{aH=k} \\ \approx\ & -M_{\rm pl}^4\frac{V_{\phi\phi\phi}V_\phi}{V^2}|_{aH=k} \\ & +M_{\rm pl}^2\left(\frac{V_{\phi\phi}}{V}\right)M_{\rm pl}^2\left(\frac{V_\phi}{V}\right)^2|_{aH=k} \\ =\ & 2\epsilon\eta_V - \xi^2\,,\end{aligned} \tag{104}$$

where

$$\xi^2 \equiv M_{\rm pl}^4\frac{V_{\phi\phi\phi}V_\phi}{V^2}\,. \tag{105}$$

Taking the derivative of Eq. (102) and combining Eq. (103) and Eq. (104), we obtain

$$\frac{dn_s}{d\ln k} = 16\epsilon\eta_V - 2\xi^2 - 24\epsilon^2\,. \tag{106}$$

6. Primordial Gravitational Waves

In this section we introduce the generation of the quantum fluctuations in the gravitational fields. The perturbed metric of the tenser perturbation can be presented as

$$\delta g_{\mu\nu}^{(T)}dx^\mu dx^\nu = a^2H_{ij}^Tdx^idx^j\,, \tag{107}$$

where H_{ij}^T is a traceless($H_i^{Ti} = 0$), divergence free ($\nabla^iH_{ij}^T = 0$), symmetric ($H_{ij}^T = H_{ji}^T$) tensor field. H_{ij}^T has only 2 physical degrees of freedom, which can be decomposed as

$$H_{ij}^T = H_+^Te_{ij}^+ + H_\times^Te_{ij}^\times\,, \tag{108}$$

where H_+^T is diagonal component and $H_\times^T$ is the off diagonal component. e_{ij}^+ and $e_{ij}^\times$ are the basis of different modes for polarizations with the normalization that

$$e_{ij}^+e^{ij,+} = e_{ij}^\times e^{ij,\times} = 2\,. \tag{109}$$

H_+^T and $H_\times^T$ satisfy the same perturbed Einstein equation

$$H_T'' + 2\mathcal{H}H_T' + k^2H_T = 0\,. \tag{110}$$

We denote H_+^T and $H_\times^T$ by H_T hereafter. The above equation is a wave equation, and the corresponding solutions are called gravity waves. Eq. (110) can be easily quantized. Denoting

$$h_T = \frac{aH_T}{\sqrt{16\pi G}}\,, \tag{111}$$

Eq. (110) can be rewritten as

$$h_T'' + (k^2 - \frac{a''}{a})h_T = 0\,. \tag{112}$$

Noting that

$$\frac{a''}{a} \approx -\frac{1}{a}\frac{d}{d\eta}\left(\frac{a}{\eta}\right)\,, \tag{113}$$

we obtain

$$h_T'' + (k^2 - \frac{2}{\eta^2})h_T = 0\,, \tag{114}$$

which is similar to Eq. (82). We quantize the field h_T as

$$\hat{h}_T(k,\eta) = v_T(k,\eta)\hat{a} + v_T^*(k,\eta)\hat{a}^\dagger\,, \tag{115}$$

where $v_T(k,\eta)$ satisfies

$$v_T'' + (k^2 - \frac{2}{\eta^2})v_T = 0\,. \tag{116}$$

From Eq. (83) the solution to the above equation reads

$$v_T = \frac{1}{\sqrt{2k^3}}(k \pm \frac{i}{\tilde{\eta}})e^{\mp ik\tilde{\eta}}\,. \tag{117}$$

The power spectrum of the gravitational wave is given by

$$\begin{aligned}\Delta_H^2 &\equiv \frac{k^3}{2\pi^2}\sum_\lambda \lim_{k\eta\to 0}|H_T|^2 \\ &= 4\times\frac{k^3}{2\pi^2}\frac{16\pi G}{a^2}\lim_{k\eta\to 0}|v_T|^2 \\ &= 4\times\frac{k^3}{2\pi^2}\frac{16\pi G}{a^2}\frac{1}{2k^3\tilde{\eta}^2} \\ &= 4\times\frac{k^3}{2\pi^2}\frac{16\pi G}{a^2}\frac{a^2H^2}{2k^3} \\ &= 4\times\frac{4GH^2}{\pi}\,,\end{aligned} \tag{118}$$

where λ denotes the polarization states. Note that there is a factor 4 in the second equality by definition [27]. The tensor spectral index n_T is defined by

$$\frac{d\ln\Delta_H^2}{d\ln k} \equiv n_T = 2\frac{d\ln H}{d\ln k} = -2\epsilon\,, \tag{119}$$

where we have used Eq. (100). The tensor index n_T is proportional to ϵ. Combing Eq. (95) and Eq. (118), we obtain

$$\frac{\Delta_H^2}{\Delta_\zeta^2} = 16\epsilon\,. \tag{120}$$

From the above equation, we can find a consistant relation between tensor-to-scalar ratio and tensor tilt [12]

$$r \equiv \frac{\Delta_H^2}{\Delta_\zeta^2} = -8n_T\,, \tag{121}$$

where r is the tensor-to-scalar ratio.

7. Cosmological Constrains on Slow-Roll Inflationary Models

In this section, we compare the predictions of single-field inflationary models to observations. From Eqs. (77, 78, 79, 105), the inflationary model, under the slow-roll approximation, can predict the exact values of measurable quantities such as n_s, $\frac{dn_s}{d\ln k}$ and r as long as the potential $V(\phi)$ is specified. Combining Eqs. (77, 78, 79, 121) and Eq. (102), we obtain [26]

$$r = \frac{8}{3}(1-n_s) + \frac{16}{3}\eta_V = \frac{8}{3}(1-n_s) + \frac{16}{3}\frac{M_{\rm pl}^2V_{\phi\phi}}{V}\,. \tag{122}$$

This equation indicates that it is the *curvature* of the potential that divides models on the $n_s - r$ plane. In this review paper, we focus on a specific inflationary model, namely, the model with a power law potential

$$V(\phi) = \lambda M_{\rm pl}^4\left(\frac{\phi}{M_{\rm pl}}\right)^p\,. \tag{123}$$

The motivation behind this is twofold. First, this is one of the simplest classes of inflationary model that includes the simplest chaotic models. Second, the model can predict a larger tensor-to-scalar ratio r, which is in favor of the latest BICEP2 [18] results.

Noting that $\frac{V_\phi}{V} = \frac{p}{\phi}$ and $\frac{V_{\phi\phi}}{V} = \frac{p(p-1)}{\phi}$, the slow-roll parameters in Eqs. (77, 79, 105) can be written as

$$\eta_V \approx \frac{M_{\rm pl}^2p(p-1)}{\phi^2}\,, \tag{124}$$

$$\epsilon \approx \frac{M_{\rm pl}^2p^2}{2\phi^2}\,, \tag{125}$$

$$\xi^2 \approx \frac{M_{\rm pl}^2p^2(p-1)(p-2)}{\phi^4}\,. \tag{126}$$

The number of e-folds before the end of inflation, N_*, at which the pivot scale k_* exits from the Hubble radius is

$$N_* \approx \frac{1}{M_{\rm pl}^2}\int_{\phi_e}^{\phi_*}\frac{V}{V_\phi}d\phi\,, \tag{127}$$

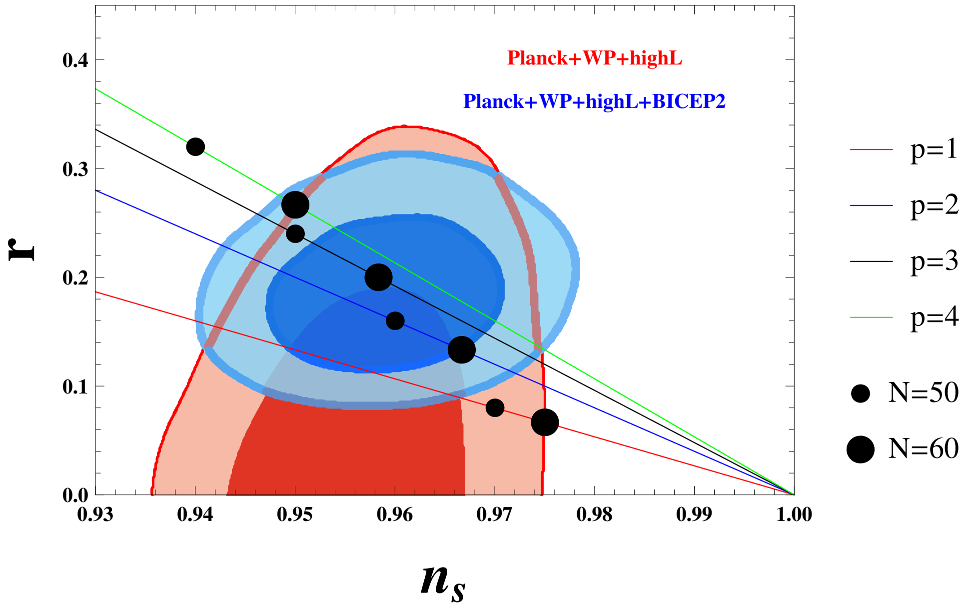

Fig. 1. Marginalized joint 68% and 95% CL regions for n_s and r from *Planck* and BICEP2 in combination with other data sets compared to the theoretical predictions of the inflationary model. The plot is from Ref. [28].

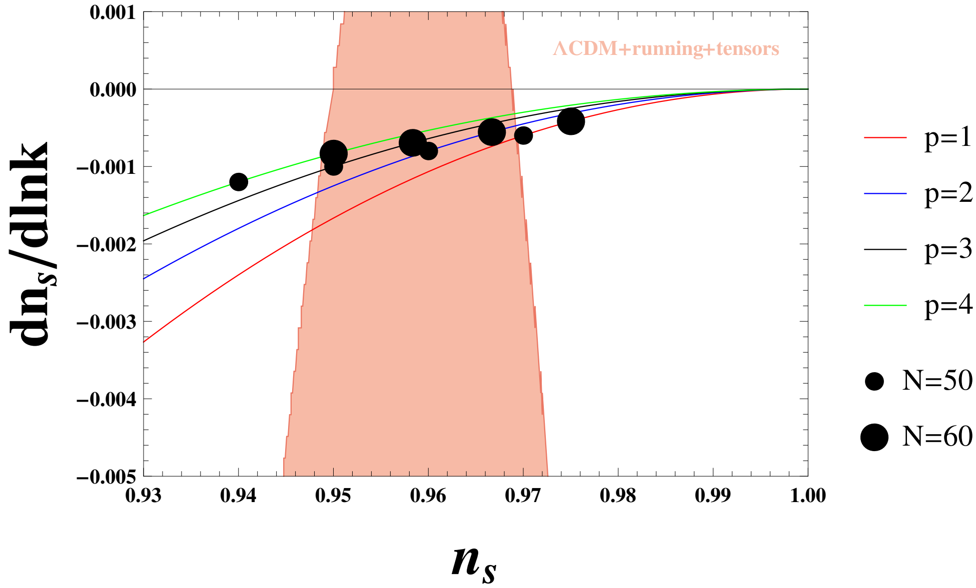

Fig. 2. Marginalized joint 68% and 95% CL regions for n_s and $\frac{dn_s}{d\ln k}$ from *Planck* and BICEP2 in combination with other data sets compared to the theoretical predictions of the inflationary model. The plot is from Ref. [28].

where subscript e denotes the end of inflation in which the slow-roll approximation no longer holds $\epsilon \approx 1$. From Eq. (125), $\epsilon \approx 1$ gives

$$\phi_e^2 \approx \frac{p^2 M_\mathrm{pl}^2}{2} \,. \tag{128}$$

Inserting the above expression back to Eq. (127), we could find the relation between N_* and ϕ_*

$$\phi_*^2 \approx 2pM_{\rm pl}^2 N_* + \phi_e^2 = pM_{\rm pl}^2(4N_* + p)/2\,. \quad (129)$$

Combing Eqs. (102, 106) and Eq. (121), n_s, $\frac{dn_s}{d\ln k}$, and r can be simply presented in terms of N_* and p

$$n_s \approx 1 - p(p+2)\frac{M_{\rm pl}^2}{\phi_*^2} = 1 - \frac{2p+4}{4N_*+p}\,, \quad (130)$$

$$\frac{dn_s}{d\ln k} \approx -M_{\rm pl}^4\frac{2p^2(p+2)}{\phi_*^4} = -\frac{8(p+2)}{(4N_*+p)^2}\,, \quad (131)$$

$$r \approx 8p^2\frac{M_{\rm pl}^2}{\phi_*^2} = \frac{16p}{4N_*+p}\,. \quad (132)$$

From the above set of equations, we can further find that

$$\frac{dn_s}{d\ln k} \approx -\frac{2}{p+2}(1-n_s)^2\,, \quad (133)$$

$$r \approx 8(1-n_s)\frac{p}{p+2}\,. \quad (134)$$

The inflationary model with the potential Eq. (123) predicts a negative running for the scalar spectral index $\frac{dn_s}{d\ln k} < 0$ as well as a linear relationship between r and n_s on the $r - n_s$ plane. We assume that the number of e-folds N_* to the end of inflation lies in the interval $50 < N_* < 60$. The prediction of such model on n_s, $\frac{dn_s}{d\ln k}$ and r will be line segments on both $n_s - r$ and $n_s - \frac{dn_s}{d\ln k}$ planes, which can be compered to observations directly.

For cosmological observations, we discuss the implications of the *Planck* [19, 29] and BICEP2 [18] data for cosmic inflation. *Planck* [19, 29] data has placed very tight constraints on $n_s = 0.957 \pm 0.015$,$dn_s/d\ln k = -0.022^{+0.020}_{-0.021}$ and $r_{0.002} < 0.263$ at the 95% confidence level, where $r_{0.002}$ is evaluated at the pivot scale of $k_* = 0.002\text{Mpc}^{-1}$. *Planck* favors a negative running for the scalar spectral index $dn_s/d\ln k < -0.002$ at a 95% confidence level. The BICEP2 data places a constraint on $r = 0.20^{+0.07}_{-0.05}$ for the ΛCDM model. Fig. 1 shows the two dimensional joint likelihood for n_s and $r_{0.002}$ and the theoretical predictions of the inflationary model with $p = 1, 2, 3, 4$, respectively. The theoretical predictions are indicated by the line segments between the smaller solid points and larger solid points. On $n_s - r$ plane, only the models with $2 < p < 3$ are within the 1σ range of BICEP2 constraints. On $n_s - dn_s/d\ln k$ plane, as shown in Fig. 2 the models with $2 < p < 3$ over predicts the value of the running index $dn_s/d\ln k$. $dn_s/d\ln k$ predicted by the inflationary models Eq. (123) only marginally satisfies the constraints from *Planck* data at the 95% confidence level as reported in Ref. [28]. Similar results can also be found in the natural inflation model with potential $V(\phi) = \Lambda^4[1 + \cos(\phi/f)]$ [30] where f is a scale that determines the slope of the potential. $dn_s/d\ln k$ predicted by this model only marginally satisfies the constraints from *Planck* [28].

8. Conclusions

In this paper, we have reviewed the cosmological perturbation theory for inflation. We have discussed the theoretic predictions of inflationary models under the slow-roll approximation on the measurable quantities n_s, $dn_s/d\ln k$ and r. We have also discussed the constraints from *Planck* and BICEP2 data on cosmic inflationary models. We focus on the power law model that can generate a larger value of tensor-to-scalar ratio r. Although on $n_s - r$ plane, the model is consistent with the observations from BICEP2 data, the prediction of $dn_s/d\ln k$ only marginally satisfies the constraints by *Planck* data at the 95% confidence level on $n_s - dn_s/d\ln k$ plane. The predictions from inflationary model under the slow-roll approximation is not fully consistent with the latest constraints from the joint analysis of *Planck* and BICEP2 data. However, it is still too early to make any conclusions as to whether the inflationary model is consistent with observations or not. A global fitting with the initial power spectrum calculated by integrating Eq. (72) numerically (e.g. [25], [29]) is called for to fully explore the validity of inflationary models against the current observations.

References

[1] R. Brout, F. Englert, E. Gunzig, Annals Phys., **115**, 78 (1978).

[2] A. A. Starobinsky, Phys. Lett. **B91**, 99 (1980).

[3] D. Kazanas, Astrophys. J., **241**, L59 (1980).

[4] K. Sato, MNRAS, **195**, 467 (1981).

[5] A. H. Guth, Phys. Rev. **D23**, 347 (1981).

[6] A. D. Linde, Phys. Lett. **B108**, 389 (1982).

[7] A. D. Linde, Phys. Lett. **B129**, 177 (1983).

[8] Edward W. Kolb, hep-ph/9910311v2.

[9] V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, Phys. Rept., **215**, 203 (1992).

[10] D. H. Lyth and A. Riotto, Phys. Rept., **314**, 1 (1999).

[11] A. R. Liddle, D. H. Lyth, Phys. Rept., **231**, 1 (1993).

[12] A. R. Liddle, D. H. Lyth, Cosmological inflation and large-scale structure (Cambridge University Press).

[13] A. D. Linde, Particle Physics and Inflationary Cosmology, Harwood Academic, Switzerland (1990).

[14] A. Ritto, hep-ph/0210162.

[15] W. Hu, astro-ph/0402060.

[16] Y. Wang, arXiv:1303.1523.

[17] D. Langlois, arXiv:1001.5259.

[18] P. Ade *et al.*, BICEP2 I: Detection of B-Mode Polarization at Degree Angular Scales, Phys. Rev. Lett., **112**, 241101 (2014), arXiv:1403.3985.

[19] P. Ade *et al.*, Planck 2013 results. XVI. Cosmological parameters, arXiv:1303.5076.

[20] H. Kodama, M. Sasaki, Prog. Theor. Phys. Suppl., **78**, 1 (1984).

[21] V. F. Mukhanov, JETP Lett., **41**, 493 (1985).

[22] J. Adams, B. Cresswell, R. Easther, Phys. Rev. **D64**, 123514 (2001).

[23] H. V. Peiris, E. Komatsu, L. Verde, *et al.*, ApJS, **148**, 213 (2003).

[24] M. J. Mortonson, C. Dvorkin, H. V. Peiris, W. Hu, Phys. Rev. **D79**, 103519 (2009).

[25] J. Lesgourgues, W. Valkenburg, Phys. Rev. **D75**, 123519, (2007).

[26] A. D. Linde, Phys. Lett. **B129**, 177 (1983).

[27] E. Komatsu, *et al.*, Astrophys. J. Suppl., **192**, 18 (2011).

[28] Q. Gao, Y. Gong, Phys. Lett., **B734**, 41 (2014).

[29] P. Ade *et al.*, Planck 2013 results. XXII. Constraints on inflation, arXiv:1303.5082.

[30] K. Freese, J. A. Frieman, and A. V. Olinto, Phys. Rev. Lett., **65**, 3233 (1990).