Non-relativistic strings and membranes

Niet-relativistische snaren en membranen
(met een samenvatting in het Nederlands)

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Abstract

The main focus of this thesis is the derivation of non-relativistic particle, string and membrane actions and equations of motion. In particular, the theories we consider are based on (generalizations of) the Galilean algebra and Newton-Cartan gravity. Our starting point will be computing the beta functions of a non-relativistic string theory with Torsional Newton Cartan symmetries in the target space. In analogy with usual relativistic string theory, the equations obtained by setting these beta functions to zero are then interpreted as the target space equations of motion for (Type I) Torsional Newton Cartan gravity. Subsequently, we derive a target space action for this theory, as well as for other non-Riemannian theories that are closely related to it: Carrollian and Stringy Newton Cartan gravity. These actions correspond to different non-Riemannian limits of the bosonic sector of the usual ten-dimensional supergravity actions. Finally, we study a non-relativistic limit of M-Theory, whose low energy limit gives a theory that we dub Membrane Newton Cartan gravity, which should be thought of as the non-relativistic limit of the bosonic sector of eleven-dimensional supergravity. Two conceptually different dimensional reductions can then be performed on MNC gravity: one of them turns out to be precisely the same SNC gravity mentioned above, while the other one is a novel type of non-relativistic theory associated to $D2$ branes.

Contents

Publications

My first publication concerned the computation of the beta functions for a string theory describing a non-relativistic (Newton-Cartan) target space:

- A.D. Gallegos, U. Gürsoy and N. Zinnato
  Torsional Newton Cartan gravity from non-relativistic strings,
  JHEP **09** (2020) 172, [1906.01607].

This approach allowed us to obtain the equations of motion for the bosonic sector of non-relativistic supergravity in ten dimensions. As a natural next step, we were interested in finding the actions corresponding to these equations of motion. To this end, we embedded various non-Riemannian geometries into the framework of Double Field Theory:

- A.D. Gallegos, U. Gürsoy, S. Verma and N. Zinnato
  Non-Riemannian gravity actions from double field theory,
  JHEP **06** (2021) 173, [2012.07765].

Once we obtained the actions for (the bosonic sector) of non-relativistic SUGRA, we asked the question whether we could do something similar for eleven-dimensional supergravity/M-theory. This led to the third and final paper that will be discussed in this thesis, where we found the action and equations of motion of non-relativistic supergravity in eleven dimensions, as well as a novel type of non-relativistic ten-dimensional supergravity:

- C. D. A. Blair, A.D. Gallegos and N. Zinnato
  A non-relativistic limit of M-theory and 11-dimensional membrane Newton-Cartan geometry,
  JHEP **10** (2021) 015, [2104.07579].

For the past four years I have been closely collaborating with Domingo Gallegos, who is (at the time of writing) a PhD candidate at Utrecht University.

While both of us made essential contributions to these papers, Domingo has not included these publications in his thesis, so that there is no overlap between my thesis and his.

My original reason for studying non-relativistic physics was very simple: I was hoping to gain some insights into the workings of relativistic quantum gravity. My interest in quantum gravity led me to briefly work on a different area of high energy physics as well, that of holographic quark-gluon plasma:

- U. Gürsoy, M. Järvinen, G. Policastro and N. Zinnato
  Analytic long-lived modes in charged critical plasma,
  preprint: [2112.04296].

This paper will not be discussed in this thesis.

Chapter 1

Introduction

It is a very well known fact that the universe we live in is relativistic. Nonetheless, there are many settings in nature that are well described by non-relativistic physics, ranging from condensed matter systems to gravitational systems in the post-Newtonian approximation, to non-relativistic holography. Besides practical applications, what makes non-relativistic physics worth studying is the hope that it will give us some insights about how our universe works at a fundamental level. One of the biggest challenges in modern physics is how to go beyond Quantum Field Theory and General Relativity to obtain a theory of *relativistic* quantum gravity. This has famously proven extremely difficult. A different approach, however, would be to take a step back and ask a simpler question: can we at least build a consistent theory of *non-relativistic* quantum gravity? This is still not an easy question to answer, but hopefully this thesis will help lay the foundation for finding such an answer. This builds on a recent revival of interest in non-relativistic versions of string theory, see e.g. [1–16].

A great deal has been learned about string theory from the exploration of special limits of the theory. There are many examples. In the $\alpha' \to 0$ limit, string theory predicts Einstein gravity, extended to supergravity in ten dimensions, via the one-loop beta functionals of the worldsheet [17]. When compactified on a circle of radius R, T-duality relates the $R \to 0$ limit of one string theory to the $R \to \infty$ limit of another. The strong coupling limit of the type IIA theory leads to the eleven-dimensional description in terms of M-theory, from the perspective of which we can view all the different dual versions of ten-dimensional string theories again as different limits [18, 19]. A different limit of M-theory is its low energy effective theory:

eleven-dimensional supergravity [20]. Another interesting class of limits are those which decouple degrees of freedom, and which may again lead to new geometric perspectives or to different dual descriptions (the most famous example being the original derivation of the AdS/CFT correspondence [21]).

The mathematics of non-relativistic physics is based on the original work by Cartan [22, 23], who introduced what today is known as the *Newton-Cartan* (NC) geometry. This has been later generalized to a (Type I) Torsional Newton-Cartan (TNC) geometry, which was found to be the boundary geometry in Lifshitz holography [2, 3, 24]. Finally, a new kind of torsional Newton-Cartan geometry was recently built, dubbed Type II Torsional Newton-Cartan, which was obtained by considering a careful expansion of the metric field for large speed of light. While this theory has many interesting properties, in that it is possible to obtain all the equations of motion from an action principle[1] and that it passes the three classical tests of General Relativity (see [25–27] for detailed discussions of Type II TNC), we will only discuss Type I TNC in this thesis. The reason for this is that Type I TNC is much easier to embed in string theory, as we will later see, and in this thesis we will focus precisely on 'critical' limits of string theory and M-theory, in which the ten- or eleven-dimensional geometry becomes non-relativistic [28–31]. However, we will briefly discuss the main mathematical differences between Type I and Type II TNC in the next section.

The first step in understanding non-relativistic physics is of course studying how particles propagate in such a spacetime. We will therefore give a brief introduction to non-relativistic particles in this chapter. Many of the properties and relations studied here will also be relevant when we will move our attention to non-relativistic strings and membranes.

1.1 Non-relativistic particles

Probably the most intuitive way of constructing a theory of non-relativistic particles is to simply expand the metric in a large c limit, with c being the

[1]We will see that this is not possible for Type I TNC and related theories.

speed of light. For a Minkowski spacetime in four dimensions this yields

$$\eta_{\mu\nu} = \begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \Leftrightarrow \qquad E_\mu^a = c\tau_\mu \delta_0^a + e_\mu^{A'} \delta_{A'}^a \,, \tag{1.1}$$

where E_μ^a denotes the vielbein associated with the relativistic metric $\eta_{\mu\nu}$, i.e. $E_\mu^a E_{\nu a} = \eta_{\mu\nu}$. Note that this implies

$$\tau_\mu \tau_\nu = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \qquad e_\mu^{A'} e_{\nu A'} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{1.2}$$

The index a runs over the four flat directions, while the index A' runs over the three transverse/space-like flat directions. Similarly, we can introduce the inverse vielbeins,

$$E_a^\mu = -\tfrac{1}{c} \upsilon^\mu \delta_a^0 + e_{A'}^\mu \delta_a^{A'} \,, \tag{1.3}$$

from which we can find the following relations by requiring $E_a^\mu E_\mu^b = \delta_a^b$:

$$\tau_\mu e_{A'}^\mu = \upsilon^\mu e_\mu^{A'} = 0 \,, \qquad \tau_\mu \upsilon^\mu = -1 \,, \qquad h^{\mu\rho} h_{\rho\nu} = \delta_\nu^\mu + \upsilon^\mu \tau_\nu \,, \tag{1.4}$$

where we defined $h^{\mu\nu} \equiv e_{A'}^\mu e^{\nu A'}$ and an analogous formula holds for $h_{\mu\nu}$. We will refer to these kinds of relations as *Newton-Cartan structure*. This expansion can be easily generalized to more complicated curved backgrounds with arbitrary metric $g_{\mu\nu}$. In general, the resulting non-relativistic theory will have a non-vanishing intrinsic torsion, in which case it is given the name of Torsional Newton-Cartan (TNC) geometry.

Let's analyze what we have obtained by performing this expansion. We have essentially doubled the number of fields we originally had, since we started with a single metric $g_{\mu\nu}$ and we ended up with a *metric complex* $(\tau_\mu, h_{\mu\nu})$. Note that the number of degrees of freedom is unchanged because of the conditions (1.4). There is no concept of 'metric' anymore, since the new objects we built are degenerate matrices. The spacetime directions are now split in *longitudinal* (τ_μ) and *transverse* $(h_{\mu\nu})$ components.

The same expansion can also be performed on the Poincaré algebra directly. First recall the commutation relations

$$\begin{aligned} [P_a, P_b] &= 0\,, \\ [J_{bc}, P_a] &= -2\eta_{a[b}P_{c]}\,, \\ [J_{ab}, J_{cd}] &= 4\eta_{[a[c}J_{d]b]}\,, \end{aligned} \tag{1.5}$$

where P_a is the generator of translations and J_{ab} is the generator of spacetime rotations. Next we introduce a $U(1)$ extension of this algebra, whose generator, which we denote by M, commutes with all other generators. Finally we perform a contraction of the resulting algebra [32]:

$$P_0 \to Mc^2 + H\,, \qquad P_i \to cP_{A'}\,, \qquad J_{A'0} \to c\,G_i\,, \qquad c \to \infty\,. \tag{1.6}$$

The contraction of P_0 is inspired by the non-relativistic approximation of the energy of a relativistic particle:

$$P_0 = \sqrt{c^2 P_i P^i + M^2 c^4} \approx Mc^2 + \frac{P_i P^i}{2M}\,. \tag{1.7}$$

Substituting in (1.5) we find the *Bargmann algebra*, whose non-zero commutation relations are

$$\begin{aligned} [J_{A'B'}, J_{C'D'}] &= 4\delta_{[A'[C'}J_{D']B']}\,, \\ [J_{A'B'}, G_{C'}] &= -2\delta_{C'[A'}G_{B']}\,, \\ [G_{A'}, P_{B'}] &= -\delta_{A'B'}M\,, \\ [J_{A'B'}, G_{C'}] &= -2\delta_{C'[A'}P_{B']}\,, \\ [G_{A'}, H] &= -P_{A'}\,. \end{aligned} \tag{1.8}$$

This is a centrally extended Galilean algebra, with the central generator given by M. We will often refer to the gauge field associated to the generator M as m_μ. Note that in the simple example (1.1)-(1.4) there was no m_μ, but this gauge field will always appear in more general settings.

For an arbitrary curved spacetime, we can think of generalizing the expansion of the metric as follows:

$$\begin{aligned} g_{\mu\nu} &= -c^2\tau_\mu\tau_\nu + \bar{h}_{\mu\nu} + c^{-2}\bar{\Phi}_{\mu\nu} + \dots\,, \\ g^{\mu\nu} &= h^{\mu\nu} - c^{-2}\left(\hat{\upsilon}^\mu\hat{\upsilon}^\nu + h^{\mu\rho}h^{\nu\sigma}\bar{\Phi}_{\rho\sigma}\right) + c^{-4}\left(2\hat{\upsilon}^\mu\hat{\upsilon}^\nu\Phi + Y^{\mu\nu}\right) + \dots\,, \end{aligned} \tag{1.9}$$

with

$$\begin{aligned} \bar{h}_{\mu\nu} &\equiv h_{\mu\nu} - 2\tau_{(\mu}m_{\nu)}\,, \\ \hat{\upsilon}^{\mu} &\equiv \upsilon^{\mu} - h^{\mu\rho}m_{\rho}\,, \\ \Phi &\equiv -\upsilon^{\mu}m_{\mu} + \frac{1}{2}h^{\mu\nu}m_{\mu}m_{\nu}\,, \end{aligned} \tag{1.10}$$

which are nothing but combinations of the fields introduced previously, with the addition of m_μ. The full expansion (1.9) leads to what is known as a *Type II Torsional Newton-Cartan* geometry. The corresponding algebra is an *expansion* of the Poincaré algebra, rather than a contraction as the one we saw earlier in (1.8). In this thesis we will consider the theory obtained by the truncation of this expansion at a given power of c, in particular the case where $\bar{\Phi}_{\mu\nu} = Y_{\mu\nu} = 0$, which leads to a *Type I Torsional Newton-Cartan* geometry (we will often refer to this simply as 'TNC')[2]:

$$\begin{aligned} g_{\mu\nu} &= -c^2\tau_\mu\tau_\nu + \bar{h}_{\mu\nu}\,, \\ g^{\mu\nu} &= h^{\mu\nu} - c^{-2}\hat{\upsilon}^{\mu}\hat{\upsilon}^{\nu}\,. \end{aligned} \tag{1.11}$$

One can check that $g^{\mu\rho}g_{\rho\nu} = \delta^\mu_\nu$ (up to terms going to zero when $c \to 0$) if we impose the following constraints on the TNC fields:

$$\tau_\mu h^{\mu\nu} = 0\,, \qquad \tau_\mu\hat{\upsilon}^{\mu} = -1\,, \qquad h^{\mu\rho}\bar{h}_{\rho\nu} = \delta^\mu_\nu + \hat{\upsilon}^{\mu}\tau_\nu\,. \tag{1.12}$$

Note that these are equivalent to (1.4), except they are rewritten in terms of boost invariant quantities thanks to the presence of m_μ. Boost transformations correspond to the generator G_i in (1.8) and they are explicitly:

$$\delta_\lambda\tau_\mu = 0\,, \qquad \delta_\lambda e^{A'}_\mu = \lambda^{A'}\tau_\mu\,, \qquad \delta_\lambda m_\mu = \lambda_{A'}e^{A'}_\mu\,, \tag{1.13}$$

where $\lambda^{A'}$ is the boost parameter.

It turns out that there is a different way of deriving a theory involving precisely the same relations as (1.12) and thus the same TNC algebra. We start by considering the following metric:

$$ds^2 = 2\tau_\mu dx^\mu du + \bar{h}_{\mu\nu}dx^\mu dx^\nu\,, \tag{1.14}$$

[2]To be precise, this is only possible when we have vanishing torsion, i.e. Type II TNC can reduce to Type I TNC only when time is absolute.

and we assume that there is a null isometry along the u direction, i.e. ∂_u is a Killing vector (note that $g_{uu} = 0$). Upon reduction along this null direction, we are left with the two matrices

$$g_{\mu\nu} = \bar{h}_{\mu\nu}\,, \qquad\qquad g^{\mu\nu} = h^{\mu\nu}\,, \tag{1.15}$$

as well as the two vectors

$$g_{\mu u} = \tau_\mu\,, \qquad\qquad g^{\mu u} = -\hat{v}^\mu\,, \tag{1.16}$$

and the scalar

$$g^{uu} = 2\Phi\,. \tag{1.17}$$

These are precisely the same fields that appear in (1.12)! This way of obtaining a TNC geometry from a null reduction of General Relativity will turn out to be extremely useful in Chapters 2 and 3 when we embed TNC in string theory.

Up to this point we have essentially only expanded a metric, which could be considered at most a fun exercise, so it is natural to ask the question whether this type of expansion actually leads to an interesting physical theory, i.e. can we obtain an action for this theory? As can be inferred by the fact that this thesis exists, the answer to this question is yes. Perhaps the easiest case to consider is obtained by starting with the usual action for a relativistic massless point-particle and substituting the ansatz (1.14) (see e.g. [33]):

$$S_{rel} \propto \int \frac{\dot{x}^M \dot{x}^N g_{MN}}{e} ds \qquad \Rightarrow \qquad S_{nr} \propto \int \frac{\dot{x}^\mu \dot{x}^\nu}{\tau_\rho \dot{x}^\rho} \left(h_{\mu\nu} - 2m_\mu \tau_\nu\right) ds\,, \tag{1.18}$$

where g_{MN} is the metric of the geometry (1.14), e is the determinant of the vielbein and a dot denotes differentiation with respect to the parameter s. We also remark again that we introduced the $U(1)$ central extension of the Galilei algebra, whose generator is denoted by m_μ. This action now describes a *massive non-relativistic* particle. A similar expansion can be performed on the Einstein-Hilbert action, which would yield a non-relativistic version of General Relativity; or on the equations of motion obtained from the Einstein-Hilbert action, which would then produce the equations of motion for a particle propagating in a non-relativistic curved background. In fact, this will largely be the focus of this thesis, that is we will employ non-relativistic parametrizations to find novel types of non-relativistic actions and equations of motion.

1.2 Outline of the thesis

In the previous section we briefly discussed some basic properties of non-relativistic particles. This means we divided the spacetime in one longitudinal component and the remaining transverse components. The analysis can be generalized to multiple longitudinal directions, in which case we would have non-relativistic strings, non-relativistic membranes, etc.

In this thesis we will only consider cases where the longitudinal space is one-, two- or three-dimensional and we will often assume that the full spacetime is either ten- or eleven-dimensional (for strings and membranes, respectively). For example, the target space geometry that appears in the two-dimensional case extends the generally covariant but non-relativistic NC geometry to what is usually called a 'stringy Newton-Cartan' (SNC) geometry. The full ten-dimensional Lorentz symmetry is absent, and there is instead a split into two longitudinal directions (including time) and eight purely spatial transverse directions which transform into each other only under Galilean boosts. Correspondingly, one can describe the target space geometry in terms of a pair of mutually orthogonal vielbeins, ${\tau_\mu}^A$, ${e^\mu}_{A'}$, such that ${\tau_\mu}^{A'}{e^\mu}_{A'} = 0$, where $A = 0, 1$ indexes the longitudinal tangent space directions and $A' = 2, \ldots$ indexes the transverse tangent space directions.

In Chapter 2 we will study the Torsional Newton Cartan parametrization of string theory, with the goal of computing the beta functions of the theory, which we will then set to zero to obtain the target-space equations of motion. This computation closely follows that of relativistic bosonic string theory, although with some extra technical difficulties. In particular, we can only perform this analysis when a certain constraint on torsion is imposed, so the resulting equations will be a subset of the most general TNC equations of motion.

Luckily, as is usually the case in physics, there are different routes leading to the same goal. In Chapter 3 we will use a different framework (Double Field Theory) to obtain the actions and equations of motion of three different, albeit related, non-relativistic and ultra-relativistic theories of propagating strings/particles: the theories we study exhibit Torsional Newton Cartan, Carrollian and stringy Newton Cartan symmetries. More concretely, we will be able to find the most general TNC equations of motion that we could not find in the previous chapter by only computing the worldsheet beta functions.

In Chapter 4 we will take this discussion one step further and apply the non-relativistic expansion to membranes, from which we will find the non-relativistic version of eleven-dimensional supergravity (or rather, its bosonic sector). Once we arrive at the eleven-dimensional action, we can dimensionally reduce it along a longitudinal direction to once again obtain the theories studied in the previous chapters, or we can reduce it along a transverse direction to obtain a novel type of ten-dimensional non-relativistic theory, related to a $D2$ brane rather then to the fundamental string.

Finally, a word of caution about notations. In this thesis we will deal with several different types of indices. For example, in Chapter 2 we will deal with curved/flat worldsheet indices and curved/flat spacetime indices that are further split in longitudinal/transverse components. In the following chapters we will have to introduce even more kinds of indices, so it is almost impossible to use a uniform notation for all chapters. As a rule of thumb, greek indices will refer to curved directions, while latin indices will refer to flat directions. We will give a summary of the notations used at the beginning of each chapter.

Chapter 2

Non-relativistic strings: the worldsheet approach

In this chapter we ask the question whether a TNC geometry can be UV completed in a consistent theory of quantum gravity and take a few first steps in answering this question in the context of bosonic string theory. One of the triumphs of the ordinary (relativistic) string theory has been the derivation of Einstein's equations in the weak gravity limit by demanding Weyl invariance of the worldsheet sigma model [34]. In our case of string propagating on a manifold with local Galilean invariance, we similarly expect that the demand of quantum Weyl invariance on the worldsheet yields Newton's law/Poisson's equation. This is what we mean precisely by the consistency of the TNC geometry with quantum gravity.

Various proposals to realize the Galilean symmetries in string theory exist in the literature. The Newton-Cartan geometry has only recently been embedded in string theory at the classical level, that is at the tree level of the worldsheet non-linear sigma model [6,8,35]. Non-relativistic string theory on a TNC background with $R \times S^2$ topology has been studied in [36–39]. As mentioned in the introduction, a parallel and separate line of work [1,7,40,41] which started by the original paper of Gomis and Ooguri [28] realized the Galilean symmetry in the context of closed string theory in a particular contraction limit and continued by [9, 13], that ask the same question we ask here but in the context of the Gomis-Ooguri theory.

We will follow the route taken by the papers [6, 8] where a Polyakov type action for a string propagating in the TNC geometry was constructed. Taking this Polyakov action as our starting point, we extend it to include bosonic target space matter[1], i.e. the Kalb-Ramond field $\bar{B}_{\mu\nu}$ and dilaton ϕ, as well as an extra Kalb-Ramond one-form $\aleph_\mu$, and we determine both the target space and worldsheet symmetries of this action at the classical level. We then go beyond the tree level and construct the worldsheet perturbation theory in powers of the string length l_s, assuming that the target TNC space is weakly curved. We then obtain the target space equations of motion from quantum Weyl invariance of the non-linear sigma model proposed in [8] and its generalizations including the Kalb-Ramond and the dilaton fields.

This chapter is organized as follows. We begin, in section 2.1, by reviewing the Polyakov-type action we used for the closed bosonic string moving in a TNC background and then generalize it to include the Neveu-Schwarz background matter, i.e. the dilaton and the Kalb-Ramond fields. We then discuss how the target space and worldsheet symmetries are realized at the classical level. Section 2.2 constitutes the core of the chapter, where we introduce the covariant background field expansion. This expansion coincides with the derivative expansion in the target space. We truncate this series at the second order both in the target-space derivatives and in the quantum fluctuations. Using this quantum effective action at the quadratic level, we then compute the one loop contribution to the Weyl anomaly and obtain the equations of motion for the TNC geometry arising from the vanishing of the beta functions.

We will use greek letters coming from the second half of the alphabet, $\{\lambda, \mu, \nu, \dots\}$, to denote target-space dimensions, while the first letters of the alphabet, $\{\alpha, \beta, \dots\}$, will be used to denote their pull-backs onto the worldsheet. Primed capital letters, $\{A', B', \dots\}$, will refer to spatial flat directions in target space. Finally, we use $\{a, b, \dots\}$ to describe flat worldsheet directions.

2.1 The Type I TNC string action and its symmetries

2.1.1 The Polyakov action without matter

The geometric data of the TNC geometry in the absence of matter fields is encoded in a pair of vielbeins $(\tau_\mu, e_\mu^{A'})$ and a $U(1)$ connection m_μ collectively

[1]See also [27] for a discussion on the coupling of matter to non-relativistic gravity.

referred as the TNC metric complex. The vielbeins $e_\mu^{A'}$ define a degenerate spatial metric through $h_{\mu\nu} = e_\mu^{A'} e_\nu^{B'} \delta_{A'B'}$ and it is possible to use the inverse of the square matrix $(\tau_\mu, e_\nu^{A'})$, denoted as $(-\upsilon^\mu, e^\nu_{A'})$ with $\upsilon^\mu \tau_\mu = -1$ and $\tau_\mu e^\mu_{A'} = 0$, to define an independent spatial inverse metric $h^{\mu\nu} = e^\mu_{A'} e^\nu_{B'} \delta^{A'B'}$. These spatial metrics together with the temporal coframes, τ_μ and υ^μ, are subject to the completeness relation

$$\delta^\mu_\nu = -\upsilon^\mu \tau_\nu + h^{\mu\rho} h_{\rho\nu}. \tag{2.1}$$

Quite conveniently, the TNC geometry with this geometric data can be derived from a higher dimensional relativistic spacetime with an isometry in the extra null direction—which we will denote as the u-direction—via the procedure of null reduction [42]. In particular we consider the TNC manifold to be $d+1$-dimensional and the relativistic one will be $d+2$-dimensional. The metric of such relativistic spacetimes can always be written as

$$\bar{g}_{\hat{\mu}\hat{\nu}} dx^{\hat{\mu}} dx^{\hat{\mu}} = 2\tau\,(du - m) + h_{\mu\nu} dx^\mu dx^\nu, \tag{2.2}$$

with ∂_u the corresponding null Killing vector. We label indices of the $d+2$ dimensional space as $\hat{\mu} = \{u, \mu\}$. We also define $\tau = \tau_\mu dx^\mu$, $m = m_\mu dx^\mu$ with x^μ the coordinates of the $(d+1)$-TNC manifold. It is now possible to derive the worldsheet action for a string moving in the TNC geometry [6,8] starting from the ordinary Polyakov action in the relativistic target space (2.2):

$$\mathcal{L} = -\frac{\sqrt{-\gamma}}{4\pi l_s^2} \gamma^{\alpha\beta} \left(h_{\alpha\beta} - \tau_\alpha m_\beta - m_\alpha \tau_\beta\right) - \frac{\sqrt{-\gamma}}{2\pi l_s^2} \gamma^{\alpha\beta} \tau_\alpha \partial_\beta X^u, \tag{2.3}$$

where γ is the determinant of the worldsheet metric $\gamma_{\alpha\beta}$, and where $h_{\alpha\beta} = h_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu$ and $\tau_\alpha = \tau_\mu \partial_\alpha X^\nu$ are the pullbacks of $h_{\mu\nu}$ and τ_μ respectively.

We consider a closed string without winding, i.e. $X^\mu(\sigma^0, \sigma^1 + 2\pi) = X^\mu(\sigma^0, \sigma^1)$, and with non zero momentum P along X^u

$$P = \int_0^{2\pi} d\sigma^1 P_u^0\,, \tag{2.4}$$

with momentum current

$$P_u^\alpha = \frac{\partial \mathcal{L}}{\partial \partial_\alpha X^u} = -\frac{\sqrt{-\gamma}\gamma^{\alpha\beta}\tau_\beta}{2\pi l_s^2}\,. \tag{2.5}$$

Following [8], it is possible to rewrite (2.3) in a dual formulation where the conservation of the momentum current (2.5) is implemented off-shell through the classically equivalent Lagrangian

$$\mathcal{L} = -\frac{\sqrt{-\gamma}\gamma^{\alpha\beta}\bar{h}_{\alpha\beta}}{4\pi l_s^2} - \frac{1}{2\pi l_s^2}\left(\sqrt{-\gamma}\gamma^{\alpha\beta}\tau_\beta - \epsilon^{\alpha\beta}\partial_\alpha\eta\right)A_\beta\,, \tag{2.6}$$

where A_α is a Lagrange multiplier that enforces conservation of $P^\alpha_u = \frac{\epsilon^{\alpha\beta}\partial_\beta\eta}{2\pi l_s^2}$ off-shell and we defined the combination

$$\bar{h}_{\alpha\beta} \equiv h_{\alpha\beta} - \tau_\alpha m_\beta - m_\alpha\tau_\beta\,. \tag{2.7}$$

The significance of this combination will become clear when we discuss the symmetries of the theory below.

This procedure introduces a novel degree of freedom, a scalar field η on the world sheet. To see that (2.6) and (2.3) are equivalent, one uses the equation of motion for η which gives $A_\alpha = \partial_\alpha\chi$ for some world sheet scalar χ and identifies the latter with the u-direction $\chi = X^u$ recovering the original Lagrangian (2.3). Following [8] we introduce the worldsheet zweibein e^a_α and its inverse $e^\alpha_a = \epsilon^{\alpha\beta}e^b_\beta\epsilon_{ba}$, satisfying $e^a_\alpha e^b_\beta\eta_{ab} = \gamma_{\alpha\beta}$ and $e^\alpha_a e^\beta_b\eta_{ab} = \gamma^{\alpha\beta}$, to rewrite the constraints as

$$\begin{aligned}\epsilon^{\alpha\beta}\left(e^0_\alpha + e^1_\alpha\right)(\tau_\beta + \partial_\beta\eta) = 0\,,\\ \epsilon^{\alpha\beta}\left(e^0_\alpha - e^1_\alpha\right)(\tau_\beta - \partial_\beta\eta) = 0\,.\end{aligned} \tag{2.8}$$

A final field redefinition,

$$A_\alpha = m_\alpha + \frac{1}{2}(\lambda_+ - \lambda_-)\,e^0_\alpha + \frac{1}{2}(\lambda_+ + \lambda_-)\,e^1_\alpha\,, \tag{2.9}$$

yields the Lagrangian

$$\mathcal{L} = \frac{-1}{4\pi l_s^2}\left[2\epsilon^{\alpha\beta}m_\alpha\partial_\beta\eta + e\eta^{ab}e^\alpha_a e^\beta_b h_{\alpha\beta} - \lambda_+ e^\beta_-(\partial_\beta\eta + \tau_\beta) - \lambda_-(\partial_\beta\eta - \tau_\beta)\right], \tag{2.10}$$

where $e^\alpha_\pm = e^\alpha_0 \pm e^\alpha_1$. This is the Polyakov-type Lagrangian for a string moving in a TNC geometry proposed in [8]. We further use the constraints to rewrite (2.10) in a more convenient way for quantization[2]

$$\mathcal{L} = \frac{e}{4\pi l_s^2}\left[e^\alpha_+ e^\beta_-\bar{h}_{\alpha\beta} + \lambda_+ e^\beta_-(\partial_\beta\eta + \tau_\beta) + \lambda_- e^\beta_+(\partial_\beta\eta - \tau_\beta)\right], \tag{2.11}$$

[2]One should think of implementing these constraints inside the Polyakov path integral to ensure equivalence of the quantum path integrals based on the Lagrangians (2.10) and (2.11).

We will examine the quantum path integral defined by this Lagrangian in the rest of the chapter, but we will first extend it to include Neveu-Schwarz matter, i.e. the Kalb-Ramond field and dilaton and then discuss the symmetries of this generalized action both on the worldsheet and in the target space.

2.1.2 The Polyakov action with matter

It is straightforward to generalize the action (2.11) to include standard Neveu-Schwarz matter, i.e. a Kalb-Ramond field $\mathcal{B}_{\hat{\mu}\hat{\nu}}$ and a dilaton ϕ. Let us first consider the B-field. Once again, to derive the corresponding Lagrangian we can start from its null uplifted version. We then obtain the following action by rearranging the terms that follow from the null reduction of the relativistic d+2-dimensional bosonic Polyakov action with the presence of the B-field:

$$\mathcal{L} = -\frac{1}{4\pi l_s^2}\left(\sqrt{-\gamma}\gamma^{\alpha\beta}\bar{h}_{\alpha\beta} + \epsilon^{\alpha\beta}\bar{B}_{\alpha\beta}\right) - \frac{1}{4\pi l_s^2}\left(\sqrt{-\gamma}\gamma^{\alpha\beta}\tau_\alpha - \epsilon^{\alpha\beta}\aleph_\alpha\right)\partial_\beta X^u\,, \tag{2.12}$$

where we defined

$$\aleph_\alpha \equiv \mathcal{B}_{u\alpha} = -\mathcal{B}_{\alpha u}\,, \tag{2.13}$$

$$\bar{B}_{\alpha\beta} \equiv \mathcal{B}_{\alpha\beta}\,. \tag{2.14}$$

Following the same procedure as in [8] described in section 2.1 we compute the momentum along X^u

$$P_u^\alpha = -\frac{1}{2\pi l_s^2}\left(\sqrt{-\gamma}\gamma^{\beta\alpha}\tau_\beta - \epsilon^{\beta\alpha}\aleph_\beta\right) \tag{2.15}$$

and implement its conservation off-shell via

$$\begin{aligned}\mathcal{L} = -\frac{1}{4\pi l_s^2}&\left(\sqrt{-\gamma}\gamma^{\alpha\beta}\bar{h}_{\alpha\beta} + \epsilon^{\alpha\beta}\bar{B}_{\alpha\beta}\right)\\ &- \frac{1}{2\pi l_s^2}\left(\sqrt{-\gamma}\gamma^{\alpha\beta}\tau_\alpha - \epsilon^{\alpha\beta}\aleph_\alpha - \epsilon^{\alpha\beta}\partial_\alpha\eta\right)A_\beta.\end{aligned} \tag{2.16}$$

Making, once again, the field redefinition

$$A_\alpha = m_\alpha + \frac{1}{2}\left(\lambda_+ - \lambda_-\right)e_\alpha^0 + \frac{1}{2}\left(\lambda_+ + \lambda_-\right)e_\alpha^1\,, \tag{2.17}$$

integration over the worldsheet fields $\lambda_\pm$ now imposes the constraints

$$\begin{aligned}\epsilon^{\alpha\beta}\left(e^0_\alpha + e^1_\alpha\right)\left(\tau_\beta + \aleph_\beta + \partial_\beta \eta\right) = 0\,,\\ \epsilon^{\alpha\beta}\left(e^0_\alpha - e^1_\alpha\right)\left(\tau_\beta - \aleph_\beta - \partial_\beta \eta\right) = 0\,.\end{aligned} \tag{2.18}$$

We can cast (2.16) in Polyakov form:

$$\begin{aligned}\mathcal{L} = \frac{1}{4\pi l_s^2} e \Big[e^\alpha_+ e^\beta_- \left(\bar{h}_{\alpha\beta} + \bar{B}_{\alpha\beta}\right) + \lambda_+ e^\beta_- \left(\partial_\beta \eta + \aleph_\beta + \tau_\beta\right) \\ + \lambda_- e^\beta_+ \left(\partial_\beta \eta + \aleph_\beta - \tau_\beta\right)\Big]\,,\end{aligned} \tag{2.19}$$

where just as in (2.11) the constraints (2.18) have been used. Lagrangian (2.19) is still invariant under (2.35) and the contribution of the B-field to the anomaly can in principle be computed in a similar manner as performed for (2.11).

When the worldsheet is non-flat, in addition to the B-field, it is also possible to include a dilaton contribution of the form

$$\mathcal{L}_\phi = \frac{1}{16\pi}\sqrt{-\gamma}\mathcal{R}\phi\,, \tag{2.20}$$

where $\mathcal{R}$ is the worldsheet Ricci scalar. The Polyakov path integral then involves a sum over worldsheet topologies that is organized in powers of $\exp(\phi)$ as usual.

2.1.3 Symmetries of the Polyakov action

We will now discuss both the target space and worldsheet symmetries of the worldsheet action (2.19) and (2.20).

Spacetime symmetries

The fields in the TNC metric complex, without matter, transform under diffeomorphisms ξ, local Galilean boosts $\lambda^{A'}$, local rotations $\lambda^{A'B'}$ and local $U(1)$ gauge transformation σ, with the Lagrangian (2.6) being invariant under these transformations [8]. These transformations are easily generalized in the presence of matter. All in all, the *spacetime* transformations of the objects that enter the calculations read

$$\begin{aligned}
\delta\tau_\mu &= \mathcal{L}_\xi \tau_\mu\,,\\
\delta e_\mu^{A'} &= \mathcal{L}_\xi e_\mu^{A'} + \lambda^{A'}\tau_\mu + \lambda^{A'}_{B'} e_\mu^{B'}\,,\\
\delta\upsilon^\mu &= \mathcal{L}_\xi \upsilon^\mu + \lambda^{A'} e^\mu_{A'}\,,\\
\delta e^\mu_{A'} &= \mathcal{L}_\xi e^\mu_{A'} + \lambda_{A'}{}^{B'} e^\mu_{B'}\,,\\
\delta m_\mu &= \mathcal{L}_\xi m_\mu + \lambda_{A'} e_\mu^{A'} + \partial_\mu\sigma\,,\\
\delta\bar{B}_{\mu\nu} &= \mathcal{L}_\xi \bar{B}_{\mu\nu} + 2\aleph_{[\mu}\partial_{\nu]}\sigma\,,\\
\delta\aleph_\mu &= \mathcal{L}_\xi \aleph_\mu\,,\\
\delta\phi &= \mathcal{L}_\xi \phi\,,\\
\delta X^\mu &= \mathcal{L}_\xi X^\mu\,.
\end{aligned} \tag{2.21}$$

In particular, the combination $\bar{h}_{\mu\nu}$ defined in (2.7) and (2.14) is invariant under local Galilean boosts and transforms under local $U(1)$ mass transformations as

$$\delta_\sigma \bar{h}_{\mu\nu} = -2\tau_{(\mu}\partial_{\mu)}\sigma\,. \tag{2.22}$$

Now, it is straightforward to check that the actions based on (2.19) and (2.20) are invariant under diffeomorphisms, local Gallilean boosts, local rotations and local $U(1)_m$ transformations. When starting with the explicitly Galilean boost invariant form (2.19) it is crucial to use the constraints (2.18) to show invariance under local $U(1)$ mass transformations. However, the classical equations of motion will not be invariant under this $U(1)$ symmetry, see (A.20). To fix this we will ask that the Lagrange multipliers transform under the symmetry as

$$\delta\lambda_+ = -e_+^\alpha\partial_\alpha\sigma\,, \qquad\qquad \delta\lambda_- = e_-^\alpha\partial_\alpha\sigma. \tag{2.23}$$

Taking this into account, both the action and the equations of motion can be shown to be $U(1)$ mass invariant off-shell. In what follows, in addition to $\bar{h}_{\mu\nu}$ and $\bar{B}_{\mu\nu}$ defined in (2.7) and (2.14), it will prove useful to introduce the following combinations

$$\hat{\upsilon}^\mu \equiv \upsilon^\mu - h^{\mu\rho}m_\rho\,, \tag{2.24}$$

$$\Phi \equiv -\upsilon^\rho m_\rho + \frac{1}{2}h^{\rho\sigma}m_\rho m_\sigma\,, \tag{2.25}$$

which are invariant under local Galilean boost and rotations as one can easily check using (2.21). They do transform under local $U(1)$ mass transformations though:

$$\delta_\sigma \hat{\upsilon}^\mu = -h^{\mu\nu}\partial_\mu\sigma\,, \tag{2.26}$$

$$\delta_\sigma \Phi = -\hat{\upsilon}^\nu\partial_\nu\sigma\,. \tag{2.27}$$

Even though they do not appear in the action at the classical level, we have introduced $\hat{\upsilon}^\mu$ as the local Galilean boost and rotations invariant version of υ^μ (the inverse of τ_μ), and the target space scalar Φ which will play the role of the Newton's gravitational potential below. They will become important when we discuss quantum corrections in the theory. We remind the reader that $\hat{\upsilon}^\mu$, τ_μ, $\bar{h}_{\mu\nu}$ and $h^{\mu\nu}$ are subject to the completeness relation $\delta^\mu_\nu = -\hat{\upsilon}^\mu\tau_\nu + h^{\mu\rho}\bar{h}_{\rho\nu}$. Finally, we note that because of the non-trivial $U(1)$ mass transformation of $\bar{B}_{\mu\nu}$ in (2.21), i.e. $\delta_\sigma\bar{B} = \aleph\wedge d\sigma$, the field strength, $H = d\bar{B}$ will transform under mass $U(1)$ as

$$\delta_\sigma H_{\mu\nu\rho} = b_{\mu\nu}\partial_\rho\sigma + b_{\nu\rho}\partial_\mu\sigma + b_{\rho\mu}\partial_\nu\sigma\,, \tag{2.28}$$

with

$$b_{\mu\nu} \equiv \partial_\mu\aleph_\nu - \partial_\nu\aleph_\mu \tag{2.29}$$

being the field strength of $\aleph$. Notice in particular that setting $H_{\mu\nu\rho} = 0$ would not be a mass $U(1)$ invariant condition unless $b_{\mu\nu} = 0$.

$U(1)_B$ one-form symmetry

In the presence of the Kalb-Ramond field there is also a $U(1)$ one-form symmetry. It is well-known that the transformation

$$\delta_\Lambda \mathcal{B}_{\hat{\mu}\hat{\nu}} = \partial_{\hat{\mu}}\Lambda_{\hat{\nu}} - \partial_{\hat{\nu}}\Lambda_{\hat{\mu}}, \tag{2.30}$$

where $\partial_{\hat{\mu}}$ is the partial derivative in the target space, is a symmetry of the $d+2$-dimensional worldsheet action with relativistic target space.

After null reduction, the resulting TNC geometry with Kalb-Ramond matter has a $U(1)$ one-form symmetry:

$$\delta_\Lambda \bar{B}_{\mu\nu} = \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu, \tag{2.31}$$

$$\delta_\Lambda \aleph_\mu = \partial_\mu\Lambda_u\,. \tag{2.32}$$

We see that in the TNC geometry $\aleph$ acquires a new local $U(1)$ symmetry, whereas B transforms under a local one-form symmetry. It is now straightforward to check that the action (2.19) is invariant under (2.31) upon use of the constraint equations (2.18). Invariance of (2.19) under (2.32) however requires a non-trivial transformation of the worldsheet field η:

$$\delta_\Lambda \eta = -\Lambda_u \,, \tag{2.33}$$

which is a trivial shift in the quantum path integral where η is path integrated. Therefore, we conclude that the action, at least at the tree-level, is invariant under both the local one-form symmetry Λ_μ and the new local $U(1)$ symmetry Λ_u. The fact that η is charged under the $U(1)$ that comes from the B-field, i.e. eq. (2.33), is expected as one can think of η as the direction dual to u [8]. In this sense the gauge fields m and $\aleph$ can be considered as dual to each other.

Local worldsheet symmetries

The actions (2.11) and (2.19) are clearly invariant under the worldsheet diffeomorphisms. These symmetries allow us to cast the worldsheet metric in a diagonal form $\gamma^{ab} = e^{-2\rho}\eta^{ab}$ where the conformal factor ρ determines the Ricci curvature of the worldsheet $\mathcal{R}$ (locally) as

$$\sqrt{-\gamma}\mathcal{R} = -2\partial^2\rho \,. \tag{2.34}$$

We will refer to this choice of gauge as the conformal gauge. The reparametrization gauge-fixed Polyakov Lagrangians (2.11) and (2.19) further exhibit a residual Lorentz/Weyl gauge invariance of the form (as can be checked straightforwardly)

$$e^\alpha_\pm \to f_\pm e^a_\pm \,, \qquad \lambda_\pm \to f_\pm \lambda_\pm \,, \tag{2.35}$$

for any worldsheet function $f_\pm$. For $f_+ = f_-$ this is a local Weyl transformation and for $f_+ = -f_-$ it constitutes a local Lorentz transformation. Once we have used diffeomorphism invariance to go to conformal gauge it is possible to use local Weyl invariance to fix the mode ρ and completely fix the worldsheet metric $\gamma^{\alpha\beta}$.

The main purpose of this chapter is to discuss the fate of these residual gauge invariances at the quantum level. Here it suffices to note that, in the case where matter is absent, the condition for invariance of the Polyakov action $S(e,\lambda,X)$ under the gauge transformations (2.35) at the classical level takes the form

$$\frac{\delta S}{\delta f_\pm} = e_c^\gamma \tau_\gamma^c + C^+\lambda_+ + C^-\lambda_- = 0\,, \tag{2.36}$$

where the energy momentum one form[3] τ_γ^c and constraint functions $C^\pm$ are defined as

$$\begin{aligned}\tau_\gamma^c &\equiv -\frac{2\pi l_s^2}{e}\frac{\delta S}{\delta e_c^\gamma} \\ &= \frac{2\pi l_s^2 \mathcal{L}}{e} e_\gamma^c + \frac{1}{2}\Big[2e_b^\beta \eta^{cb}\bar{h}_{\gamma\beta} - \lambda_+\left(\delta_0^c - \delta_1^c\right)\left(\partial_\gamma\eta + \tau_\gamma\right) \\ &\qquad\qquad - \lambda_-\left(\delta_0^c + \delta_1^c\right)\left(\partial_\gamma\eta - \tau_\gamma\right)\Big]\,,\end{aligned} \tag{2.37}$$

$$C^\pm \equiv -\frac{2\pi l_s^2}{e}\frac{\delta S}{\delta\lambda_\pm} = -\frac{1}{2}e_\mp^\beta\left(\partial_\beta\eta \pm \tau_\beta\right). \tag{2.38}$$

Equation (2.36) is nothing but a constrained traceless condition for the energy momentum tensor, and from (2.37) and (2.38) it is clear that this conditions holds for the Polyakov action (2.11). The rest of our work will concern the computation of (2.36) at the quantum level, in particular, at the one-loop level in the perturbative expansion in l_s^2.

2.2 Quantum weyl invariance of the TNC string

2.2.1 Background field quantization

The quantum partition function that follows from the action (2.11) is defined by the Polyakov path integral[4]. As for the bosonic strings [34], it will be very helpful to introduce the background field formalism to organize the perturbative l_s^2 expansion to study the quantum properties of the worldsheet sigma model. To this end, we expand the worldsheet fields $\{X^\mu, \lambda_\pm, \eta\}$ around a

[3] Even though it is possible to define an energy momentum tensor from τ_γ^c via $T_{\alpha\beta} = \eta_{cd} e_\alpha^d \tau_\beta^c$, it feels more natural to define the traceless condition in terms of the energy momentum one form.

[4] It is crucial to include the contribution from the Faddeev-Popov ghosts that come from the gauge fixing but we will not explicitly show them here. The gauge fixing procedure is discussed in [43].

classical configuration $\Psi_0 \equiv \{X_0^\mu, \lambda_\pm^0, \eta_0\}$ as

$$\begin{aligned} X^\mu &= X_0^\mu + l_s \bar{Y}^\mu \,, \\ \lambda_\pm &= \lambda_\pm^0 + l_s \bar{\Lambda}_\pm \,, \\ \eta &= \eta_0 + l_s \bar{H} \,, \end{aligned} \tag{2.39}$$

where $\Psi \equiv \{\bar{Y}^\mu, \bar{\Lambda}_\pm, \bar{H}\}$ below will collectively denote the quantum fields. Using this expansion, the one loop effective effective action $\Gamma[\Psi_0]$ for the background fields can be expressed [44] as a path integral over the quantum fields as

$$e^{i\bar{\Gamma}[\Psi_0](0)} = \int D\Psi \; e^{i\bar{S}[\Psi_0,\Psi](0)} \,. \tag{2.40}$$

where $\bar{S}[\Psi_0, \Psi](0)$ is the $\mathcal{O}\left(l_\sigma^0\right)$ term that arises from substituting (2.39) in (2.11). In (2.40) the zweibeins are completely fixed by the Faddeev-Popov procedure, see [43], using the reparametrization invariance and Weyl symmetry. This in particular fixes the function ρ. If the symmetry (2.35) is to be consistent at the one loop level then any change of ρ should leave the effective action invariant, this means that the Weyl invariance condition (2.36) at one loop level becomes[5]

$$\delta_\psi \bar{\Gamma}\left[\Psi_0\right](0) = 0 \,, \qquad \delta_\psi \rho = \psi \,. \tag{2.41}$$

The goal of this section is to express $\bar{S}[\Psi_0, \Psi](0)$ as an action over TNC covariant fields, for this we first note that $\bar{Y}^\mu$ does not transform as a vector under general coordinate transformations. To get covariant expressions we first need to rewrite $\bar{Y}^\mu$ covariantly. This is achieved [44] by considering a geodesic connecting X_0^μ and $X_0^\mu + \bar{Y}^\mu$ to rewrite $\bar{Y}^\mu$ as

$$\bar{Y}^\mu = Y^\mu - \frac{l_s}{2}\left(\Gamma^\mu_{\rho\sigma} + G^\mu_{\rho\sigma}\right)_0 Y^\rho Y^\sigma + \mathcal{O}\left(l_s^2\right) \,, \tag{2.42}$$

where Y^μ is the tangent vector along the geodesic, $()_0$ indicates the corresponding expression is evaluated at X_0, $\Gamma^\mu_{\rho\sigma}$ is the TNC connection characterising the non-covariant part of $\bar{Y}^\mu$, and $G^\mu_{\rho\sigma}$ is a tensor symmetric in its

[5]We are assuming that a path integral measure invariant under the target spacetime symmetries exists.

lower indices and the solution to the tensor equation[6]

$$\tau_{(\rho}G^{\lambda}_{\mu\nu)} = \tau_\sigma G^{\sigma}_{(\mu\nu}\delta^{\lambda}_{\rho)} - \frac{1}{2}\bar{h}_{(\mu\nu}F_{\rho)\sigma}h^{\sigma\lambda}\,, \tag{2.43}$$

with

$$F \equiv d\tau\,, \tag{2.44}$$

characterising the spacetime torsion. The derivation of (2.42) and (2.43) from the geodesic equation of a particle evolving in a TNC background is shown in appendix A.1. We reproduce below the connection for a generic TNC geometry [45, 46]

$$\Gamma^{\mu}_{\rho\sigma} \equiv -\hat{v}^\mu\partial_\rho\tau_\sigma + \frac{1}{2}h^{\mu\lambda}\left(\partial_\rho\bar{h}_{\sigma\lambda} + \partial_\sigma\bar{h}_{\rho\lambda} - \partial_\lambda\bar{h}_{\rho\sigma}\right)\,. \tag{2.45}$$

It is compatible with the metrics τ_μ and $h^{\mu\nu}$ and exhibits a torsion component $T^{\mu}{}_{\rho\sigma} \equiv 2\Gamma^{\mu}_{[\rho\sigma]} = -2\hat{v}^\mu\partial_{[\rho}\tau_{\sigma]} = -\hat{v}^\mu F_{\rho\sigma}$. While it is of course possible to proceed in the computation by using the connection $\Gamma^{\lambda}_{\mu\nu}$, the solution to the geodesic equation (2.42) suggests that a more natural connection to consider will be the one given by

$$\mathring{\Gamma}^{\lambda}_{\mu\nu} \equiv \Gamma^{\lambda}_{\mu\nu} + \frac{1}{2}\hat{v}^\lambda F_{\mu\nu} + G^{\lambda}_{\mu\nu}. \tag{2.46}$$

This new connection is symmetric and $U(1)$ mass invariant. Although it is not compatible with τ_μ and $h^{\mu\nu}$, the action of the new covariant derivative on these two tensors is quite simple:

$$\mathring{D}_\mu\tau_\nu = \frac{1}{2}F_{\mu\nu}, \qquad \mathring{D}_\rho h^{\mu\nu} = a_\lambda h^{\lambda(\mu}\delta^{\nu)}_{\rho}. \tag{2.47}$$

Where $\mathring{D}$ denotes a covariant derivative with respect to the symmetric $U(1)$ mass invariant connection $\mathring{\Gamma}$, the symbol D will be reserved for the covariant derivative with respect to the standard TNC connection Γ.

From (2.39) and (2.42) it follows that

$$\begin{aligned}\partial_\alpha X^\mu = {}& \partial_\alpha X^\mu_0 + l_s\mathring{\nabla}_\alpha Y^\mu - l_s\left(\mathring{\Gamma}^{\mu}_{\lambda\sigma}\right)_0 Y^\sigma\partial_\alpha X^\lambda_0 - l_s^2\left(\mathring{\Gamma}^{\mu}_{\rho\sigma}\right)_0\mathring{\nabla}_\alpha Y^\rho Y^\sigma \\ & - \frac{l_\sigma^2}{2}\left[\partial_\lambda\mathring{\Gamma}^{\mu}_{\rho\sigma} - 2\mathring{\Gamma}^{\mu}_{\nu\sigma}\mathring{\Gamma}^{\nu}_{\lambda\rho}\right]_0 Y^\rho Y^\sigma\partial_\alpha X^\lambda_0 + \mathcal{O}\left(l_s^3\right)\,,\end{aligned} \tag{2.48}$$

[6] A solution to (2.43) exists as long as the torsion is taken to be twistless, namely as long as $F_{\mu\nu}h^{\mu\rho}h^{\nu\sigma} = 0$.

where $\mathring{\nabla}_\alpha \equiv \partial_\alpha X_0^\nu \mathring{D}_\nu Y^\mu = \partial_\alpha Y^\mu + \left(\mathring{\Gamma}^\mu_{\rho\sigma}\right)_0 \partial_\alpha X_0^\rho Y^\sigma$ is the pullback of the TNC spacetime covariant derivative $\mathring{D}_\nu$ onto the worldsheet. To compute $\bar{S}[\Psi_0, \Psi](0)$ we will also need the quantum expansion of the non-linear couplings $\bar{h}_{\mu\nu}(X), \bar{B}_{\mu\nu}(X), \aleph_\mu(X)$ and $\tau_\mu(X)$. This can be achieved by noting that any vector $V_\mu(X)$ and tensor $W_{\mu\nu}(X)$ can be expanded as

$$\begin{aligned} W_{\mu\nu} &= \left(W_{\mu\nu}\right)_0 + \left(\partial_\rho W_{\mu\nu}\right)_0 l_s Y^\rho + \frac{1}{2}\left(\partial_\rho\partial_\sigma W_{\mu\nu} - \mathring{\Gamma}^\lambda_{\rho\sigma}\partial_\lambda W_{\mu\nu}\right)_0 l_s^2 Y^\rho Y^\sigma \,, \\ V_\mu &= \left(V_\mu\right)_0 + \left(\partial_\rho V_\mu\right)_0 l_s Y^\rho + \frac{1}{2}\left(\partial_\rho\partial_\sigma V_\mu - \mathring{\Gamma}^\lambda_{\rho\sigma}\partial_\lambda V_\mu\right)_0 l_s^2 Y^\rho Y^\sigma \,, \end{aligned} \tag{2.49}$$

where we have made use of (2.42). It is also straightforward to show that the pullback of any vector $V_\mu(X)$ and tensor $W_{\mu\nu}(X)$ can be written in the TNC covariant form

$$\begin{aligned} \frac{W_{\alpha\beta}}{l_s^2} &= W_{\mu\nu}\mathring{\nabla}_\alpha Y^\mu \mathring{\nabla}_\beta Y^\nu + \mathring{D}_\sigma W_{\mu\nu}\mathring{\nabla}_\alpha Y^\mu Y^\sigma \partial_\beta X_0^\nu + \mathring{D}_\sigma W_{\mu\nu}\mathring{\nabla}_\beta Y^\nu Y^\sigma \partial_\alpha X_0^\mu \\ &\quad + \frac{1}{2}\left(\mathring{D}_\rho \mathring{D}_\sigma W_{\mu\nu} + \mathring{R}^\lambda_{\sigma\rho\mu} W_{\lambda\nu} + \mathring{R}^\lambda_{\sigma\rho\nu} W_{\mu\lambda}\right) Y^\rho Y^\sigma \partial_\alpha X_0^\mu \partial_\beta X_0^\nu + \mathcal{O}\left(l_s\right) \,, \end{aligned} \tag{2.50}$$

$$\begin{aligned} \frac{V_\alpha}{l_s^2} &= \frac{V_\mu \mathring{\nabla}_\alpha Y^\mu + \mathring{D}_\rho V_\mu Y^\rho \partial_\alpha X_0^\mu}{l_s} \\ &\quad + \left[\mathring{D}_\mu V_\nu Y^\mu \mathring{\nabla}_\alpha Y^\nu + \frac{1}{2}\left(\mathring{D}_\rho \mathring{D}_\sigma V_\mu + \mathring{R}^\lambda_{\sigma\rho\mu} V_\lambda\right) Y^\rho Y^\sigma \partial_\alpha X_0^\mu\right] + \mathcal{O}\left(l_s\right) \,, \end{aligned} \tag{2.51}$$

where $\mathring{R}^\lambda_{\sigma\rho\mu} \equiv \partial_\rho \mathring{\Gamma}^\lambda_{\mu\sigma} - \partial_\mu \mathring{\Gamma}^\lambda_{\rho\sigma} + \mathring{\Gamma}^\lambda_{\rho\kappa}\mathring{\Gamma}^\kappa_{\mu\sigma} - \mathring{\Gamma}^\lambda_{\mu\kappa}\mathring{\Gamma}^\kappa_{\rho\sigma}$ is the Riemann tensor defined in the usual way from the connection (2.46) and where to avoid cluttering we have dropped the zero index on the background tensor fields. Making use of (2.50) and (2.51) we can rewrite the Polyakov action (2.11) in the TNC covariant way, see appendix A.3 for its derivation,

$$
\begin{aligned}
\bar{S}_0 = -\int \frac{d^2\sigma e}{4\pi} \Big[& \bar{h}_{\mu\nu} \mathring{\nabla}_\alpha Y^\mu \nabla^\alpha Y^\nu - \bar{\Lambda}_+ e_-^\beta \left(\mathring{\nabla}_\beta \hat{H} + \mathring{\nabla}_\beta \left(\tau_\mu Y^\mu\right)\right) \\
& - \bar{\Lambda}_- e_+^\beta \left(\mathring{\nabla}_\beta \hat{H} - \mathring{\nabla}_\beta \left(\tau_\mu Y^\mu\right)\right) \Big] \\
- \int \frac{d^2\sigma e}{4\pi} \Big[& \bar{\Lambda}_+ Y^\rho \left(F_{\mu\rho} + b_{\mu\rho}\right) e_-^\beta \partial_\beta X_0^\mu - \bar{\Lambda}_- Y^\rho \left(F_{\mu\rho} - b_{\mu\rho}\right) e_+^\beta \partial_\beta X_0^\mu \Big] \\
- \int \frac{d^2\sigma e}{4\pi} \Big[& \left(\gamma^{\alpha\beta} A_{\sigma\mu\nu} + \epsilon^{\alpha\beta} \bar{A}_{\sigma\mu\nu}\right) Y^\sigma \mathring{\nabla}_\alpha Y^\mu \partial_\beta X_0^\nu \\
& + \frac{1}{2} \left(\Delta\lambda^\beta F_{\mu\nu} - \Sigma\lambda^\beta b_{\mu\nu}\right) Y^\mu \mathring{\nabla}_\alpha Y^\nu \Big] \\
- \int \frac{d^2\sigma e}{4\pi} \Big[& \left(\gamma^{\alpha\beta} C_{\rho\sigma\mu\nu} + \epsilon^{\alpha\beta} \bar{C}_{\rho\sigma\mu\nu}\right) Y^\rho Y^\sigma \partial_\alpha X_0^\mu \partial_\beta X_0^\nu \\
& + \left(\Delta\lambda^\alpha B_{\rho\sigma\mu} + \Sigma\lambda^\alpha \bar{B}_{\rho\sigma\mu}\right) Y^\rho Y^\sigma \partial_\alpha X_0^\mu \Big] \,,
\end{aligned}
\tag{2.52}
$$

where $\hat{H} = \bar{H} + \aleph_\mu Y^\mu$, $H = d\bar{B}, F = d\tau$, $b = d\aleph$, $\Delta\lambda^\beta \equiv \lambda_-^0 e_+^\beta - \lambda_+^0 e_-^\beta$, $\Sigma\lambda^\alpha \equiv \lambda_-^0 e_+^\beta + \lambda_+^0 e_-^\beta$ and where the coefficients $\{A, \bar{A}, B, \bar{B}, C, \bar{C}\}$ are given by

$$
\begin{aligned}
A_{\sigma\mu\nu} &= 2\mathring{D}_\sigma \bar{h}_{\mu\nu} \,, \\
\bar{A}_{\sigma\mu\nu} &= H_{\sigma\mu\nu} \,, \\
C_{\rho\sigma\mu\nu} &= \frac{1}{2}\mathring{D}_\rho \mathring{D}_\sigma \bar{h}_{\mu\nu} + \mathring{R}^\lambda{}_{(\rho\sigma)(\mu}\bar{h}_{\nu)\kappa} \,, \\
\bar{C}_{\rho\sigma\mu\nu} &= \frac{1}{2}\mathring{D}_\rho H_{\sigma\mu\nu} \,, \\
B_{\rho\sigma\mu} &= \frac{1}{2}\mathring{D}_\rho F_{\sigma\mu} \,, \\
\bar{B}_{\rho\sigma\mu} &= -\frac{1}{2}\mathring{D}_\rho b_{\sigma\mu} \,.
\end{aligned}
\tag{2.53}
$$

We note that (2.52) is manifestly invariant under the $U(1)_{\mathcal{B}}$ zero- and one-form transformations as it is written exclusively in terms of $b_{\mu\nu}$ and $H_{\mu\nu\rho}$ instead of $\aleph_\mu$ and $\bar{B}_{\mu\nu}$ (the coefficient $\bar{B}_{\mu\nu\rho}$ should not be confused with the Kalb-Ramond field $\bar{B}_{\mu\nu}$). Ideally one would like to do the same for the

$U(1)$ mass symmetry, i.e. express the action in terms of the field strength of m_μ, however this would give us an action which is manifestly $U(1)$ mass invariant, but not manifestly Galileian invariant. In other words, while the action is both $U(1)_m$ and Galilean invariant, we can only choose one of these symmetries to be *manifest*. Although the action will be kept in its explicit Galilean invariant form, as written in (2.52), it can be shown that it is still invariant off-shell under the $U(1)$ mass symmetry after making use of the classical equations of motion (A.20) for the background fields, as well as the transformation rules for the quantum Lagrange multipliers, derived from (2.23):

$$\delta\Lambda_\pm = \mp e^\alpha_\pm \left(\mathring{D}_\rho \sigma \mathring{\nabla}_\alpha Y^\rho + \mathring{D}_\rho \mathring{D}_\sigma \sigma Y^\rho \partial_\alpha X^\sigma_0\right) . \tag{2.54}$$

The preservation of this symmetry at the quantum level is then expected to be non-trivial.

2.2.2 Weyl invariance at one loop

From (2.52) we observe that $\Gamma[\Psi_0](0)$ is a free theory with a background dependent normalization for the kinetic and mass terms. Nevertheless, since we are looking at contributions up to $\mathcal{O}\left(D^2\right)$ in target spacetime derivatives we can treat (2.40) perturbatively as long as we can renormalize the $\mathcal{O}\left(D^0\right)$ terms appropriately. One can move these background dependent norms to terms higher order in spacetime derivatives through the following coordinate transformation

$$\begin{aligned} Y^\mu &= -\hat{\upsilon}^\mu \left(\tau_\sigma Y^\sigma\right) + e^\mu_{A'}\left(\delta^{A'B'} e^\rho_{B'} \bar{h}_{\rho\sigma} Y^\sigma\right) \equiv -\hat{\upsilon}^\mu \frac{Y^0}{\sqrt{2\Phi}} + e^\mu_{A'} Y^{A'} \equiv e^\mu_I Y^I , \\ \hat{H} &= \frac{H}{\sqrt{2\Phi}}, \qquad \bar{\Lambda}_\pm = \sqrt{2\Phi}\Lambda_\pm , \end{aligned} \tag{2.55}$$

with Φ defined in (2.25), $Y^I = \{Y^0, Y^{A'}\}$ and the normalizations are judiciously chosen such that the normalization of the first term in (2.52) becomes canonical, i.e. it yields the first two terms in the zeroth order action below. To see this, one needs to use the identity $\bar{h}_{\mu\nu}\hat{\upsilon}^\nu = 2\Phi\tau_\mu$ and in particular we can identify the spacetime inverse vielbeins $e^\mu_I = \{\frac{-\hat{\upsilon}^\mu}{\sqrt{2\Phi}}, e^\mu_{A'}\}$ satisfying $\bar{h}_{\mu\nu} e^\mu_I e^\nu_J = \eta_{IJ}$. The effective action S_0 is now expressed in terms of flat indices and can be expanded as

$$S_0 = S_0^{[0]} + S_0^{[1]} + S_0^{[2]}\,, \tag{2.56}$$

with $S_0^{[a]}$ denoting the $\mathcal{O}\left(D^a\right)$ in target spacetime derivatives. In particular the $\mathcal{O}\left(D^0\right)$ action is given by the free action with constraints

$$\begin{aligned} S_0^{[0]} = -\int \frac{d^2\sigma e}{4\pi}\Big[&\eta_{IJ}\partial_\alpha Y^I \partial^\alpha Y^J - \Lambda_+ e_-^\beta\left(\partial_\beta H + \partial_\beta Y^0\right) \\ &-\Lambda_- e_+^\beta\left(\partial_\beta H - \partial_\beta Y^0\right)\Big]\,. \end{aligned} \tag{2.57}$$

Assuming a diffeomorphism invariant measure, the path integration over the fields $\{Y^\mu, \bar{\Lambda}, \bar{H}\}$ can be changed to an integration over $\{Y^0, Y^{A'}, \Lambda, H\}$. After this change of coordinates, the following propagators for $S_0^{[0]}$ can be constructed

$$\begin{aligned} \langle Y^I(\sigma) Y^J(\sigma')\rangle_0 &= \Delta_2\left(\eta^{IJ} + \delta_0^I \delta_0^{B'}\right)\ln\left(|\Delta\sigma|^2\right)\,, \\ \langle Y^I(\sigma)\Lambda_\pm(\sigma')\rangle_0 &= \delta_0^I \frac{\mp 2\Delta_2}{(\sigma-\sigma')_\pm}\,, \\ \langle H(\sigma)\Lambda_\pm(\sigma')\rangle_0 &= \frac{-2\Delta_2}{(\sigma-\sigma')_\pm}\,, \\ \langle \Lambda_\pm(\sigma)\Lambda_\pm(\sigma')\rangle_0 &= \frac{4\Delta_2}{(\sigma-\sigma')_\pm}\,, \\ \langle \Lambda_+(\sigma)\Lambda_-(\sigma')\rangle_0 &= -4\pi\Delta_2\delta(\sigma-\sigma')\,, \end{aligned} \tag{2.58}$$

where $\langle\rangle_0$ denotes the correlation function computed with respect to the action $S_0^{[0]}$ and where Δ_2 is an unimportant overall factor. At first and second order in covariant derivatives we can perform the further decomposition

$$\begin{aligned} S_0^{[1]} &= \mathcal{S}_1 + \widetilde{\mathcal{S}}_1\,, \\ S_0^{[2]} &= \mathcal{S}_2 + \widetilde{\mathcal{S}}_2\,, \end{aligned} \tag{2.59}$$

where we make a distinction between the contributions coming directly from coefficients $\{A, \bar{A}, C, \bar{C}, B, \bar{B}\}$ and the contributions coming from the non-compatibility of the vielbeins $\{\frac{-\hat{v}^\mu}{\sqrt{2\Phi}}, e^\mu_{A'}\}$ by considering the former in $\mathcal{S}$ and the latter in $\widetilde{\mathcal{S}}$. In detail we find for the $\mathcal{S}$ components

$$\mathcal{S}_1 = -\int \frac{d^2\sigma e}{4\pi}\left[\bar{\Lambda}_+ Y^I e_I^\rho \left(F_{\mu\rho}+b_{\mu\rho}\right) e_-^\beta \partial_\beta X_0^\mu - \bar{\Lambda}_- Y^I e_I^\rho \left(F_{\mu\rho}-b_{\mu\rho}\right) e_+^\beta \partial_\beta X_0^\mu\right] \tag{2.60}$$
$$-\int \frac{d^2\sigma e}{4\pi}\left[\left(\gamma^{\alpha\beta}A_{\sigma\mu\nu}+\epsilon^{\alpha\beta}\bar{A}_{\sigma\mu\nu}\right) e_I^\sigma e_J^\mu Y^I \partial_\alpha Y^J \partial_\beta X_0^\nu\right]$$
$$-\int \frac{d^2\sigma e}{4\pi}\left[\frac{1}{2}\left(\Delta\lambda^\beta F_{\mu\nu} - \Sigma\lambda^\beta b_{\mu\nu}\right) e_I^\mu e_J^\nu Y^I \partial_\alpha Y^J\right],$$

$$\mathcal{S}_2 = -\int \frac{d^2\sigma e}{4\pi}\left[\left(\gamma^{\alpha\beta}C_{\rho\sigma\mu\nu}+\epsilon^{\alpha\beta}\bar{C}_{\rho\sigma\mu\nu}\right) e_I^\rho e_J^\sigma Y^I Y^J \partial_\alpha X_0^\mu \partial_\beta X_0^\nu\right] \tag{2.61}$$
$$-\int \frac{d^2\sigma e}{4\pi}\left[\left(\Delta\lambda^\alpha B_{\rho\sigma\mu}+\Sigma\lambda^\alpha \bar{B}_{\rho\sigma\mu}\right) e_I^\rho e_J^\sigma Y^I Y^J \partial_\alpha X_0^\mu\right],$$

and for the $\widetilde{\mathcal{S}}$ components

$$\widetilde{\mathcal{S}}_1 = -\int \frac{d^2\sigma e}{4\pi}\left[\frac{\Lambda_+ e_-^\beta H_+ + \Lambda_- e_+^\beta H_-}{2}\partial_\alpha X_0^\mu \mathring{D}_\mu \ln\Phi \right.$$
$$\left. + \left(2\bar{h}_{\rho\sigma} e_I^\rho \mathring{D}_\mu e_J^\sigma\right) Y^I \partial_\alpha Y^J \partial^\alpha X_0^\mu\right], \tag{2.62}$$

$$\widetilde{\mathcal{S}}_2 = -\int \frac{d^2\sigma e}{4\pi}\left[\left(\bar{h}_{\rho\sigma}\mathring{D}_\mu e_I^\rho \mathring{D} e_J^\sigma\right)\partial_\alpha X_0^\mu \partial^\alpha X_0^\nu\right]$$
$$-\int \frac{d^2\sigma e}{4\pi}\left[\left(A_{\sigma\rho\nu}\gamma^{\alpha\beta}+\bar{A}_{\sigma\rho\nu}\epsilon^{\alpha\beta}\right) e_I^\sigma \mathring{D}_\mu e_J^\rho Y^I Y^J \partial_\alpha X_0^\mu \partial_\beta X_0^\nu\right] \tag{2.63}$$
$$-\int \frac{d^2\sigma e}{4\pi}\left[\left(\frac{\Delta\lambda^\alpha F_{\sigma\rho}}{2} - \frac{\Sigma\lambda^\alpha b_{sr}}{2}\right) e_I^\sigma \mathring{D}_\mu e_J^\rho Y^I Y^J \partial_\alpha X_0^\mu\right],$$

where we have explicitly broken the manifest covariance of the theory by using $\mathring{\nabla}_\alpha Y^I = \partial_\alpha Y^I + \omega^I{}_{J\alpha} Y^J$ with $\omega^{IJ}{}_\alpha$ the spin connection[7], and where the covariant derivative $\mathring{D}_\mu e_I^\lambda$ is taken only with respect to the curved spacetime indices. The effective action (2.40) can be now be treated perturbatively and its corresponding Weyl variation, (2.41), can be computed as

$$\delta_\psi \bar{\Gamma}[\Psi_0](0) = \delta_\psi \left\langle S^{[1]} + S^{[2]}\right\rangle_0 + \frac{i}{2}\delta_\psi \left\langle S^{[1]} S^{[1]}\right\rangle_0$$
$$- i\delta_\psi \left\langle S^{[1]}\right\rangle_0 \left\langle S^{[1]}\right\rangle_0 - i\delta_\psi \log(Z_0 Z_{FP}) + \mathcal{O}\left(D^3\right), \tag{2.64}$$

[7]The spin connection is not gauge invariant and consequently it will not contribute to the beta functions.

where Z_{FP} is the partition function for the Fadeev-Popov ghosts arising from the gauge fixing procedure, see [43], and where Z_0 denotes the partition function with respect to the action $S_0^{[0]}$. By dimensional considerations we expect $\delta_\psi \log(Z_0 Z_{FP}) = c_T \mathcal{R}$ with c_T a proportionality constant[8]. The coefficient c_T is independent of the background fields and depends only on the dimensionality of the TNC spacetime. Therefore, as in the case of the ordinary string, the requirement $c_T = 0$ fixes the dimensionality of the background geometry. This is the requirement of invariance under conformal reparametrizations, hence the quantum consistency of the theory in the absence of extra dynamical fields. The requirement $c_T = 0$ fixes the critical dimension of the $d+1$ dimensional TNC geometry to be [43]

$$d_c + 1 = 25 \,. \tag{2.65}$$

This result is somewhat expected, as quantum consistency of the ordinary bosonic string sets $d+2 = 26$ and we obtain the TNC geometry by reduction of this 26 dimensional background on a null direction. Nevertheless, it is still a non-trivial result, as we cannot find a simple argument as to why quantization and null reduction should commute. Taking the dimension to be critical, we expect the right hand side of (2.64) to take the form

$$\begin{aligned}\delta_\psi \bar{\Gamma}[\Psi_0](0) = -\int d^2\sigma \frac{\psi}{4\pi} \Big[&\beta_{\rho\sigma}\eta^{\alpha\beta}\partial_\alpha X^\rho \partial_\beta X^\sigma + \bar{\beta}_{\rho\sigma}\epsilon^{\alpha\beta}\partial_\alpha X^\rho \partial_\beta X^\sigma \\ &+\beta_\mu \Delta\lambda^\beta \partial_\beta X_0^\mu + \bar{\beta}_\mu \Sigma\lambda^\beta \partial_\beta X_0^\mu + \beta\lambda_+^0\lambda_-^0\Big] \,,\end{aligned} \tag{2.66}$$

where $\{\beta, \beta_{\rho\sigma}, \bar{\beta}_{\rho\sigma}, \beta_\mu, \bar{\beta}_m\}$ will correspond to the beta functions. We will exemplify the computation of the beta functions by taking the background solution to be $\partial_\alpha X_0^\mu = 0$ so that we can easily compute the scalar beta function β. Under this assumption and making use of (2.64), (2.57), and (2.59) we find

$$\begin{aligned}&-\delta_\psi \bar{\Gamma}\left[\Psi_0\right](0) = \delta_\psi \int \frac{d^2\sigma e}{4\pi}\left[\frac{1}{2}\left(\Delta\lambda^\beta F_{\mu\nu} - \Sigma\lambda^\beta b_{\mu\nu}\right)\right]\Delta^{\mu\nu} \\ &+\delta_\psi \int \frac{d^2\sigma d^2\sigma' i e^2}{64\pi^2}\left[\left(F_{\rho\sigma}F_{\lambda\kappa} - b_{\rho\sigma}b_{\lambda\kappa}\right)\gamma^{\alpha\beta} + F_{\rho\sigma}b_{\lambda\kappa}\epsilon^{\alpha\beta}\right]\lambda_+\lambda_-\gamma^{\alpha\beta}\Delta_{\alpha\beta}^{\rho\sigma\lambda\kappa} \,,\end{aligned} \tag{2.67}$$

[8] At one loop level this is the only contribution to the anomaly proportional to $\mathcal{R}$.

where for simplicity we have defined

$$\begin{aligned} \Delta^{\mu\nu}_{\alpha}(\sigma) &\equiv e^{\mu}_{I} e^{\nu}_{J} \langle Y^I(\sigma) \partial_\alpha Y^J(\sigma) \rangle_0 \,, \\ \Delta^{\rho\sigma\lambda\kappa}_{\alpha\beta}(\sigma, \sigma') &\equiv e^{\rho}_{I} e^{\sigma}_{J} e^{\lambda}_{K} e^{\kappa}_{L} \langle Y^I(\sigma) \partial_\alpha Y^J(\sigma) Y^K(\sigma') \partial_\beta Y^L(\sigma') \rangle_0 \,. \end{aligned} \tag{2.68}$$

The propagators in (2.68) can be computed by making use of the zeroth order action (2.57), and in particular the following identities follow from it

$$\begin{aligned} \delta_\psi \int d^2\sigma J^{\alpha}_{\rho\sigma} \Delta^{\rho\sigma}_{\alpha}(\sigma) &= -\frac{1}{2} \int d^2\sigma \, e \, \psi \, \partial_\alpha \left(h^{\rho\sigma} J^{\alpha}_{\rho\sigma} \right) \,, \\ \delta_\psi \int d^2\sigma d^2\sigma' J^{\alpha\beta}_{\rho\sigma\lambda\kappa} \Delta^{\rho\sigma\lambda\kappa}_{\alpha\beta}(\sigma, \sigma') &= (-2\pi i) \int d^2\sigma \, e \, \psi \, J^{\alpha\beta}_{\rho\sigma\lambda\kappa} h^{\rho\lambda} h^{\sigma\kappa} \gamma_{\alpha\beta} \,, \end{aligned} \tag{2.69}$$

where $J^{\alpha}_{\rho\sigma}$, $J^{\alpha\beta}_{\rho\sigma\lambda\kappa}$ are arbitrary tensors. By using (2.69) and (2.67) we finally find β to be

$$\beta = \frac{1}{4} \left(b_{\rho\sigma} b_{\lambda\kappa} - F_{\rho\sigma} F_{\lambda\kappa} \right) h^{\rho\lambda} h^{\sigma\kappa} \,. \tag{2.70}$$

Our analysis depends on the existence of a solution $G^{\lambda}_{\rho\sigma}$ to the geodesic equation (2.43). It is easy to show that such solution exists as long as the torsion is twistless, i.e. it satisfies the constraint

$$F_{\rho\sigma} h^{\rho\lambda} h^{\sigma\kappa} = 0 \,, \tag{2.71}$$

with corresponding solution to the geodesic equation given by

$$G^{\lambda}_{\mu\nu} = \frac{1}{2} \bar{h}_{\mu\nu} F_{\rho\sigma} \hat{\upsilon}^{\rho} h^{\sigma\lambda} \,. \tag{2.72}$$

For the rest of this chapter we will use (2.72) and assume (2.71) holds for the TNC background. This requirement together with the Weyl invariance condition $\beta = 0$ implies that, just as F, the field strength b is forced to be twistless. This condition can be made explicit by expressing F and b in terms of the decomposition [47]

$$\begin{aligned} F_{\rho\sigma} &\equiv a_\rho \tau_\sigma - \tau_\rho a_\sigma \,, \\ b_{\rho\sigma} &\equiv \mathfrak{e}_\rho \tau_\sigma - \tau_\rho \mathfrak{e}_\sigma \,, \end{aligned} \tag{2.73}$$

with $a_\rho = \hat{\upsilon}^\lambda F_{\lambda\rho}$ the acceleration and $\mathfrak{e}_\rho = \hat{\upsilon}^\lambda b_{\lambda\rho}$ an electric-type field, and where both vectors satisfy $a_\lambda \hat{\upsilon}^\lambda = \mathfrak{e}_\lambda \hat{\upsilon}^\lambda = 0$. For simplicity, from now on we will assume (2.73) holds for the computation of the remaining beta functions. It is important to note that we should think of (2.71) not as an equation of motion arising from Weyl invariance but rather as a constraint to ensure both general covariance and $U(1)$ mass invariance at the quantum level.

Taking $\partial_\alpha X_0^\mu$ satisfying (A.20) and following a similar procedure as the one just outlined for the computation β, the remaining beta functions are found to be

$$\beta_\mu = \left[\frac{1}{2}\mathring{D}\cdot a + \left(\frac{d_c}{4}+\frac{1}{2}\right)a^2 - \mathfrak{e}^2 - a\cdot\mathring{D}\phi\right]\tau_\mu\,, \tag{2.74}$$

$$\bar{\beta}_\mu = -\left[\frac{1}{2}\mathring{D}\cdot\mathfrak{e} + \frac{d_c}{4}a\cdot\mathfrak{e} - \mathfrak{e}\cdot\mathring{D}\phi\right]\tau_\mu\,, \tag{2.75}$$

$$\begin{aligned}\beta_{\mu\nu} = {} & -\mathring{R}_{\mu\nu} + \frac{1}{4}H_{\rho\sigma(\mu}H_{\nu)\lambda\kappa}h^{\rho\lambda}h^{\sigma\kappa} - 2\mathring{D}_{(\mu}\mathring{D}_{\nu)}\phi - \mathfrak{e}_\rho h^{\rho\sigma}\left(\Delta_T\right)_{(\mu}^{\lambda}H_{\nu)\lambda\sigma} \\ & + \frac{\mathfrak{e}^2\left(2\Phi\tau_\mu\tau_\nu + \bar{h}_{\mu\nu}\right) - \mathfrak{e}_\mu\mathfrak{e}_\nu}{2} - \beta_\lambda\hat{\upsilon}^\lambda\bar{h}_{\mu\nu}\,,\end{aligned} \tag{2.76}$$

$$\begin{aligned}\bar{\beta}_{\mu\nu} = {} & \frac{1}{2}h^{\rho\sigma}\mathring{D}_\rho H_{\sigma\mu\nu} + \frac{d_c}{4}a_\rho h^{\rho\sigma}H_{\sigma\mu\nu} - \left(\Delta_S\right)_{[\mu}^{\lambda}\mathring{D}_{\nu]}\mathfrak{e}_\lambda + \left(\Delta_T\right)_{[\mu}^{\rho}\mathring{D}_{\nu]}\mathfrak{e}_\rho + a_{[\mu}\mathfrak{e}_{\nu]} \\ & + \frac{\mathring{D}_\lambda\upsilon^\lambda}{2}b_{\mu\nu} - \left(\hat{\upsilon}^\rho b_{\mu\nu} + h^{\rho\lambda}H_{\lambda\mu\nu}\right)\mathring{D}_\rho\phi\,,\end{aligned} \tag{2.77}$$

with $a^2 = a_\rho a_\sigma h^{\rho\sigma}$, $\mathfrak{e}^2 = \mathfrak{e}_\rho\mathfrak{e}_\sigma h^{\rho\sigma}$, $\mathring{R}_{\mu\nu}$ the Ricci tensor, $\cdot$ denoting an 'inner product' with respect to $h^{\rho\sigma}$, and where the time projector $\left(\Delta_T\right)_\mu^\lambda$ and the space projector $\left(\Delta_S\right)_\mu^\lambda$ are defined as

$$\left(\Delta_T\right)_\mu^\lambda = -\hat{\upsilon}^\lambda\tau_\mu\,, \qquad \left(\Delta_S\right)_\mu^\lambda = h^{\lambda\rho}\bar{h}_{\rho\mu}\,, \tag{2.78}$$

satisfying the projector identities

$$\begin{aligned}\left(\Delta_T\right)_\mu^\lambda + \left(\Delta_S\right)_\mu^\lambda &= \delta_\mu^\lambda\,, \\ \left(\Delta_{T/S}\right)_\mu^\lambda\left(\Delta_{T/S}\right)_\kappa^\mu &= \left(\Delta_{T/S}\,,\right)_\kappa^\lambda \\ \left(\Delta_T\right)_\mu^\lambda\left(\Delta_S\right)_\kappa^\mu &= 0\,.\end{aligned} \tag{2.79}$$

The details of the derivation of (2.74)-(2.77) can be found in appendix A.4. The Weyl invariance of the theory at one loop will follow from the vanishing of the beta functions. These constraints will be interpreted as the gravitational equations of motion for the TNC background, such equations are discussed in the following section. Before finalizing this section we comment on the $U(1)$ mass covariance of the beta functions (2.74)-(2.77) by noting that

$$\begin{aligned} \delta_\sigma \beta &= 0\,, \\ \delta_\sigma \beta_\mu &= 0\,, \\ \delta_\sigma \bar{\beta}_\mu &= 0\,, \\ \delta_\sigma \beta_{\mu\nu} &= 2\left(\beta_\lambda \hat{v}^\lambda\right) \tau_{(\mu} \mathring{D}_{\nu)} \sigma\,, \\ \delta_\sigma \bar{\beta}_{\mu\nu} &= 2\left(\bar{\beta}_\lambda \hat{v}^\lambda\right) \tau_{[\mu} \mathring{D}_{\nu]} \sigma\,, \end{aligned} \tag{2.80}$$

where the transformation rules (2.22), (2.26), (2.27), and (2.28) have been used. From (2.80) we note that the vanishing of the beta functions is a $U(1)_m$ invariant condition.

2.2.3 TNC equations of motion

The gravitational equations for the TNC background will arise from the condition (2.71), and by setting (2.70),(2.74)-(2.77) to zero. The resulting equations can be categorized into two twistless constraints:

$$F_{\rho\sigma} = a_\rho \tau_\sigma - \tau_\rho a_\sigma\,, \tag{2.81}$$

$$b_{\rho\sigma} = \mathfrak{e}_\rho \tau_\sigma - \tau_\rho \mathfrak{e}_\sigma\,, \tag{2.82}$$

two scalar equations:

$$D\cdot a + a^2 = 2\mathfrak{e}^2 + 2\left(a\cdot D\phi\right)\,, \tag{2.83}$$

$$D\cdot \mathfrak{e} = 2\left(\mathfrak{e}\cdot D\phi\right)\,, \tag{2.84}$$

and two tensor equations:

$$\begin{aligned} R_{(\mu\nu)} - \frac{1}{4} H_{\rho\sigma(\mu} H_{\nu)\lambda\kappa} h^{\rho\lambda} h^{\sigma\kappa} + 2D_{(\mu} D_{\nu)}\phi &= \left(\Delta_S\right)^\lambda_{(\mu} D_{\nu)} a_\lambda + \frac{a_\mu a_\nu}{2} - a^2\, \Phi \tau_\mu \tau_\nu \\ &+ \frac{\mathfrak{e}^2\left(2\Phi\tau_\mu\tau_\nu - \bar{h}_{\mu\nu}\right) - \mathfrak{e}_\mu \mathfrak{e}_\nu}{2} - \mathfrak{e}_\rho h^{\rho\sigma} \left(\Delta_T\right)^\lambda_{(\mu} H_{\nu)\lambda\sigma}, \end{aligned} \tag{2.85}$$

$$\frac{1}{2}h^{\rho\sigma}D_\rho H_{\sigma\mu\nu} - h^{\rho\sigma}H_{\sigma\mu\nu}D_\rho\phi = (\Delta_S)^{\lambda}_{[\mu}\, D_{\nu]}\mathfrak{e}_\lambda - (\Delta_T)^{\rho}_{[\mu}\, D_{\nu]}\mathfrak{e}_\rho - a_{[\mu}\mathfrak{e}_{\nu]} \\ - \frac{1}{2}a_\rho h^{\rho\sigma}H_{\sigma\mu\nu} + \left(\hat{\upsilon}^\lambda D_\lambda\phi - \frac{D_\lambda \upsilon^\lambda}{2}\right) b_{\mu\nu}\,. \tag{2.86}$$

In (2.81)-(2.86) we have used the original TNC connection (2.45) and used D to denote its corresponding covariant derivative. The Ricci tensor associated to the standard TNC connection can be read off from the following relation

$$\mathring{R}_{\mu\nu} - R_{\mu\nu} = -\frac{1}{2}a_\mu a_\nu - D_\nu a_\mu + \frac{1}{2}\left(D\cdot a + a^2\right)\bar{h}_{\mu\nu} + (\Delta_T)^{\rho}_{[\mu}\, D_\rho a_{\nu]} \\ + (\Delta_T)^{\rho}_{(\mu}\, D_{\nu)}a_\rho - \frac{1}{2}D_\rho\hat{\upsilon}^\rho F_{\mu\nu} + a^2\Phi\tau_\mu\tau_\nu\,. \tag{2.87}$$

Notice that it is not symmetric in the presence of torsion and, as discussed earlier, the TNC connection is not $U(1)$ mass invariant unless $\Gamma^{\rho}_{[\mu\nu]} = 0$. Consequently the $U(1)$ mass invariance of equations (2.81)-(2.86) is harder to verify in this form, however from (2.80) we know they are indeed invariant. We can also note that unlike the expressions (2.74)-(2.77), where the $U(1)$ mass invariant connection has been used, equations (2.81)-(2.86) have no explicit dependence on the critical dimension d_c. At this point it is convenient to introduce the extrinsic curvature tensor $\mathcal{K}_{\mu\nu}$ as [48]

$$\mathcal{K}_{\mu\nu} \equiv -\frac{1}{2}\mathcal{L}_{\hat{\upsilon}}\bar{h}_{\mu\nu} = -\frac{1}{2}\left[\hat{\upsilon}^\lambda D_\lambda\bar{h}_{\mu\nu} + \bar{h}_{\mu\lambda}D_\nu\hat{\upsilon}^\lambda + \bar{h}_{\nu\lambda}D_\mu\hat{\upsilon}^\lambda - 4\Phi a_{(\mu}\tau_{\nu)}\right]\,, \tag{2.88}$$

and use the TNC identity

$$D_\mu\bar{h}_{\rho\sigma} = 2\tau_{(\rho}\bar{h}_{\sigma)\lambda}D_\mu\hat{\upsilon}^\lambda - 2\tau_\rho\tau_\sigma D_\mu\Phi\,, \tag{2.89}$$

to derive the following contractions of the extrinsic curvature

$$h^{\rho\sigma}\mathcal{K}_{\rho\sigma} = -D_\lambda\hat{\upsilon}^\lambda\,, \\ \mathcal{K}_{\rho\sigma}\mathcal{K}_{\lambda\kappa}h^{\rho\lambda}h^{\sigma\kappa} = D_\mu\hat{\upsilon}^\nu D_\nu\hat{\upsilon}^\mu\,. \tag{2.90}$$

We can then see that $\mathcal{K}_{\rho\sigma}h^{\rho\sigma}$ shows up in the antisymmetric beta function (2.86). To further show the role of $\mathcal{K}_{\mu\nu}$ in equations (2.81)-(2.86) it is instructive to look at the time-time projection of equation (2.85) to write down

Newton's law in a general TNC spacetime. For this it will be necessary to use the $\hat{\upsilon}^\mu \hat{\upsilon}^\nu$ projection of the TNC identity (A.40)

$$\hat{\upsilon}^\nu D_\nu \hat{\upsilon}^\lambda = h^{\lambda\sigma} \left(D_\sigma \Phi + 2 a_\sigma \Phi \right) , \tag{2.91}$$

the scalar equation (2.83), and the extrinsic curvature contractions (2.90) to find that Newton's law takes the form

$$D^2 \Phi + 3 \left(a \cdot D\Phi \right) + m_\Phi^2 \, \Phi = \rho_\kappa + \rho_{\mathrm{m}} , \tag{2.92}$$

with $D^2 \equiv h^{\rho\sigma} D_\rho D_\sigma$, and where the Newton's potential mass m_Φ^2, matter density ρ_{m}, and curvature density $\rho_{\mathcal{K}}$ are defined as

$$m_\Phi^2 \equiv a^2 + 2 \, \mathfrak{e}^2 + 4 \, a \cdot D\phi , \tag{2.93}$$

$$\rho_{\mathcal{K}} \equiv \mathcal{K}_{\rho\sigma} \mathcal{K}_{\lambda\kappa} h^{\rho\lambda} h^{\sigma\kappa} - \hat{\upsilon}^\nu D_\nu \left(\mathcal{K}_{\rho\sigma} h^{\rho\sigma} \right) , \tag{2.94}$$

$$\rho_{\mathrm{m}} \equiv \frac{1}{4} \hat{\upsilon}^\mu \hat{\upsilon}^\nu h^{\rho\lambda} h^{\sigma\kappa} H_{\rho\sigma\mu} H_{\lambda\kappa\nu} - 2 \hat{\upsilon}^\mu \hat{\upsilon}^\nu D_\mu D_\nu \phi . \tag{2.95}$$

From (2.92) we can observe that the extrinsic curvature enters Newton's law in the form of a matter density distribution. In contrast, we can note that the presence of torsion modifies considerably the classical gravitational equation of motion by adding both a mass term[9] via its coupling with matter through (2.83), and an advection term via the coupling $a \cdot D\Phi$. Equation (2.92) is nothing but the temporal trace of the Ricci tensor, however it is also instructive to compute its spatial trace $\mathcal{R}_S \equiv R_{\mu\nu} h^{\mu\nu}$ to find that

$$\mathcal{R}_S = \frac{1}{4} H_S^2 - 2 D^2 \phi + \frac{m_\Phi^2}{2} - \frac{(d_c - 1) \, \mathfrak{e}^2}{2} - a^2 , \tag{2.96}$$

with $H_S^2 \equiv H_{\rho\sigma\mu} H_{\lambda\kappa\nu} h^{\rho\lambda} h^{\sigma\kappa} h^{\mu\nu}$. In addition, the electric Maxwell equation (2.84) reduces to Gauss' law only for a vanishing dilaton while the two-form Maxwell equation (2.86) is not only sourced by ϕ and $\mathfrak{e}$ but also by torsion via the coupling $a_\rho h^{\rho\sigma} H_{\sigma\mu\nu}$.

We should also mention a few properties of torsion and what role it plays in the equations of motion. First of all we recall that the conditions $\Gamma^\lambda_{[\mu\nu]} = 0$ and $a_\mu = 0$ are completely equivalent as long as torsion is forced to be twistless. In the torsionless case (i.e. when the acceleration vanishes) we

[9]From (2.83) we note that whenever torsion vanishes the electric field $\mathfrak{e}$ also vanishes and cosequently $m_\Phi^2 = 0$.

notice that the electric field $\mathfrak{e}$ is also forced to vanish. On the other hand, a non vanishing electric field forces torsion and the Kalb-Ramond field strength to be non-zero. The first property can be read off explicitly from (2.83), while the second one is a consequence of the $U(1)$ mass transformation (2.28). Hence in the absence of torsion the mass and advection terms in Newton's law vanish, yielding the more familiar Poisson equation

$$D^2\Phi = \rho_\kappa + \rho_{\rm m}\,. \tag{2.97}$$

Lastly, we notice that for vanishing torsion the TNC equations of motion assume the same form as the usual equations derived from relativistic string theory.

Chapter 3

Non-relativistic strings: the Double Field Theory approach

In this chapter we will once again derive the target-space equations of motion for some selected non-relativistic theories. The framework we will use is Double Field Theory and will allow us to find the target space equations of motion without the need to impose constraints on the geomerty (e.g. the torsionless constraints, as in the previous chapter). Another important difference with the approach of the previous chapter is that this framework will allow us to derive target-space actions for the theories we study (which should be regarded as the bosonic sector of non-relativistic supergravity actions). The geometries we consider are Type I Torsional Newton Cartan, stringy Newton Cartan and Carrollian. While the latter is technically an *ultra-relativistic* geometry, rater than a non-relativistic one, it will be interesting to study it nonetheless because of its relation to Type I TNC.

Constructing actions for non-relativistic gravitational theories based on local Galilean and Carrollian symmetries has proven to be difficult. It was not until recently that successful attempts have been made [27, 40, 48] for certain classes and limits of non-relativistic geometries. Importance of understanding dynamics of non-Riemannian gravity is underpinned by the far-reaching possibilities this entails: as mentioned in the introduction, theories based on Galilean symmetry play a role on truncations of string theory [8, 11, 28], post-Newtonian physics [47, 49–51], and provide a natural setting to study response in condensed matter systems with Galilean symmetry [52–55]. Carrollian symmetry, on the other hand, is relevant to description of excitations in the near horizon geometry of black holes [56–58], and is instrumental in flat space holography [59, 60].

In the particular cases of the Bargmann and Carroll symmetries, it has been known that the corresponding algebra of generators of the non-Riemannian space-time symmetries can be constructed by considering a parent relativistic theory with a null isometry [61–64]. In the context of string theory they can be further related by means of T-duality transformations [7,65–67]. When T-duality is applied in the null direction of the parent relativistic theory a mapping between non-Riemannian geometries, in particular Torsional Newton-Cartan (TNC) or Carroll type, can be established [7, 28, 66]. We note that this can be done for algebras obtained via group contractions of the Poincaré group, while algebras obtained from large speed of light expansions, i.e. expansions in $1/c$, will fall out of the scope of this construction [26, 27].

This seems to indicate that we should be able to describe both types of geometries, Riemannian and non-Riemannian, in a T-invariant formulation of gravity. Indeed, such a formulation exists. It is based on doubling the degrees of freedom by treating D space-time coordinates and the corresponding D space-time momenta of the compact directions on equal footing [68–72]. This results in a local $O(D, D)$ invariant theory and the aforementioned T-duality transformation becomes an $O(D, D)$ rotation. This Double Field Theory (DFT) is based on a generalized metric $\mathcal{H}$ and a generalized dilaton d encompassing the degrees of freedom in the NS sector of string theory, namely the matter content of the theory consists of metric, the Kalb-Ramond field and dilaton. This generalized metric $\mathcal{H}$ is required to be an $O(D, D)$ tensor. Parametrization of the generalized metric in terms of the original relativistic content $\{g, B, \phi\}$ is known since the postulation of Double Field Theory as a T-duality invariant generalization of string gravity [70]. Importantly however, the generalized metric is not restricted to this form and in particular admits non-Riemannian parametrizations, where the TNC and Carroll limits appear as particular cases [66]. This means that the gravitional dynamics of such non-Riemannian geometries can be obtained by simply considering the double field equations of motion[1]

$$\mathcal{R}_{MN} = 0\,, \qquad\qquad \mathcal{R} = 0\,, \tag{3.1}$$

with $\mathcal{R} \neq \eta^{MN}\mathcal{R}_{MN}$ being the generalization of the Ricci tensor and Ricci scalar to a local $O(D, D)$ geometry. The tensor and curvature scalar appearing in (3.1) can be written in terms of the generalized metric and dilaton

[1] These equations can also be unified into a single master equation, as shown in [73, 74].

allowing us to fix a non-Riemannian parametrization of $\mathcal{H}$ and obtaining the corresponding gravitational equations of motion.

The goal of this chapter is to derive the actions and corresponding equations of motion of certain type of non-Riemannian geometries by means of their embedding in Double Field Theory. The relation between DFT and non-Riemannian geometries has already been explored in some detail in the literature, see e.g. [75,76]. We will employ the general parametrization for $\mathcal{H}$ given by Park and Morand [66] and derive the corresponding general equations of motion. We will then specify to particular cases of interest: Torsional Newton-Cartan (TNC) theory, Carrollian theory and stringy Newton-Cartan (SNC) theory. Some or part of these equations of motion have been obtained from worldsheet beta-functions of string theory, see previous chapter and [9, 77], or from reductions of ordinary Einstein's equations. However, in all of these approaches some extra geometric constraints arise. DFT formulation is free of these constraints which will allow us to generalize and complete the existing studies.

The structure of this chapter is as follows. In section 3.1 we give a brief introduction to Double Field Theory, presenting the basic tensors which will be used to construct the non-relativistic actions. In sections 3.2, 3.3 and 3.4 we determine, respectively, embeddings of TNC, Carroll and SNC theories in Double Field Theory. Using these embeddings we write down the respective actions and compute the equations of motion. In section 3.5 we compare and discuss the equations of motion found in the current chapter with the ones already found in the literature and in the previous chapter.

Throughout the chapter, greek indices $\{\mu, \nu, \rho, \dots\}$ denote curved spacetime directions of TNC, Carroll and SNC. The first capital letters of the latin alphabet (both primed and unprimed) refer to flat directions, e.g. the TNC inverse transverse metric is given by $h^{\mu\nu} = e^{\mu}_{A'} e^{\nu}_{B'} \delta^{A'B'}$ and the SNC longitudinal vielbein is τ_{μ}^{A}. Capital latin letters from the middle of the alphabet $\{M, N, \dots\}$ are reserved for the DFT directions, e.g. for TNC we will have $M = 0, 1, \dots, 2d+1$, where $d = D-1$ is the dimension of the TNC spacetime. Their lowercase counterparts refer to half of the DFT directions, i.e. $m = 0, 1, \dots, d$ for the TNC case.

3.1 Doubled Gravity

In this section we review the necessary ingredients of double field gravity. We will not be thorough in our discussion and will only discuss the key ingredients. For a broader discussion see for example [66, 68–70, 78, 79]. A Double Field Theory of gravity should be invariant under $O(D, D)$ rotations and under double diffeomorphisms. As mentioned earlier, the basic ingredients are the generalized metric $\mathcal{H}$, generalized dilaton d, and the $O(D, D)$ invariant metric η, given below. Coordinates on double geometry are denoted by X^M and can be decomposed as $X^M = (X^\mu, \bar{X}_\nu)$ with the invariant metric

$$\eta^{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{3.2}$$

Double diffeomorphisms are generated via the generalized Lie derivative $\hat{\mathcal{L}}_\xi$ acting on an arbitrary tensor density with weight ω as [66]

$$\begin{aligned} \mathcal{L}_\xi T_{M_1..M_n} = {} & \xi^N \partial_N T_{M_1...M_n} + \omega \partial_N \xi^N T_{M_1...M_n} \\ & + \sum_{i=1}^{n} \left(\partial_{M_i} \xi_N - \partial_N \xi_{M_i}\right) T_{M_1...M_{i-1}}{}^{N}{}_{M_{i+1}...M_n} \end{aligned} \tag{3.3}$$

with $\xi^M = (\xi^\mu, \tilde{\xi}_\mu)$ a generalized vector. The form (3.3) was originally devised such that it reduces to the one form symmetry for the B-field in addition to the standard Lie derivative $\mathcal{L}_\xi$ on the X^μ coordinates after the so called "section" or strong constraint $\bar{\partial}^\mu = 0$ has been imposed [80]. This condition arises from the covariant constraint $\eta_{MN} \partial^M \partial^N = 0$ introduced to reduce the degrees of freedom to their original value and in fact follows from the requirement that the Lie derivatives form a closed symmetry algebra. Under (3.3) it can be shown that after exponentiation the generalized dilaton, e^{-2d}, will act as a scalar density of unit weight and consequently as the integral measure, while, by construction, $\mathcal{H}$ will be an $O(D, D)$ symmetric tensor satisfying

$$\mathcal{H}_{AC} \eta^{CD} \mathcal{H}_{BD} = \eta_{AB}. \tag{3.4}$$

3.1.1 Generalized dilaton and Metric

When considering a Riemannian manifold the generalized dilaton has a simple expression in terms of the usual dilaton ϕ and ordinary "undoubled" metric $g_{\mu\nu}$ which will be useful for example when considering TNC and Carrollian geometries:

$$e^{-2d} = e^{-2\phi}\sqrt{-\det g_{\mu\nu}} \equiv e. \tag{3.5}$$

The most general form of $\mathcal{H}$ that is symmetric and compatible with (3.4) is classified by two non-negative integers, $(n, \bar{n})$ with $n + \bar{n} < D$, and is of the form [66]

$$\mathcal{H}_{MN} = \tag{3.6}$$

$$\begin{pmatrix} K_{\mu\nu} - \mathcal{B}_{\mu\rho}H^{\rho\sigma}\mathcal{B}_{\sigma\nu} + 2x^i_{(\mu}\mathcal{B}_{\nu)\rho}y^\rho_i - 2\bar{x}^{\bar{\imath}}_{(\mu}\mathcal{B}_{\nu)\rho}\bar{y}^\rho_{\bar{\imath}} & -H^{\nu\rho}\mathcal{B}_{\rho\mu} + y^\nu_i x^i_\mu - \bar{y}^\nu_{\bar{\imath}}\bar{x}^{\bar{\imath}}_\mu \\ -H^{\mu\rho}\mathcal{B}_{\rho\nu} + y^\nu_i x^i_\mu - \bar{y}^\nu_{\bar{\imath}}\bar{x}^{\bar{\imath}}_\mu & H^{\mu\nu} \end{pmatrix} \tag{3.7}$$

with $1 \leq i \leq n$ and $1 \leq \bar{\imath} \leq \bar{n}$. Here, $\mathcal{B}$ is a skew-symmetric matrix that is identified with the Kalb-Ramond field of string theory, while K and H are symmetric matrices[2]. They have $n + \bar{n}$ null eigenvalues each,

$$H^{\mu\nu}x^i_\nu = H^{\mu\nu}\bar{x}^{\bar{\imath}}_\nu = 0, \qquad K_{\mu\nu}y^\nu_j = K_{\mu\nu}\bar{y}^\nu_{\bar{\jmath}} = 0\,, \tag{3.8}$$

The corresponding null eigenvectors of H and K are denoted as x, $\bar{x}$ and y, $\bar{y}$ respectively. They are subject to the following completeness relation

$$H^{\mu\rho}K_{\rho\nu} + y^\mu_i x^i_\nu + \bar{y}^\mu_{\bar{\imath}}\bar{x}^{\bar{\imath}}_\nu = \delta^\mu_\nu\,, \tag{3.9}$$

from which the following identities can be inferred

$$\begin{aligned} y^\mu_i x^j_\mu &= \delta^j_i, \qquad \bar{y}^\mu_{\bar{\imath}}\bar{x}^{\bar{\jmath}}_\mu = \delta^{\bar{\jmath}}_{\bar{\imath}}, \qquad y^\mu_i\bar{x}^{\bar{\jmath}}_\mu = \bar{y}^\mu_{\bar{\imath}}x^j_\mu = 0, \\ H^{\rho\mu}K_{\mu\nu}H^{\nu\sigma} &= H^{\rho\sigma}, \qquad K_{\rho\mu}H^{\mu\nu}K_{\nu\sigma} = K_{\rho\sigma}\,. \end{aligned} \tag{3.10}$$

Once the "section" condition is imposed, the generalized Lie derivative (3.3) reduces to (up to $GL(n) \times GL(\bar{n})$ transformations and Milne shifts)

$$\begin{aligned} \delta x^i_\mu &= \mathcal{L}_\xi x^i_\mu\,, \qquad \delta\bar{x}^{\bar{\imath}}_\mu = \mathcal{L}_\xi\bar{x}^{\bar{\imath}}_\mu\,, \qquad \delta y^\nu_j = \mathcal{L}_\xi y^\nu_j\,, \qquad \delta\bar{y}^\nu_{\bar{\jmath}} = \mathcal{L}_\xi\bar{y}^\nu_{\bar{\jmath}}\,, \\ \delta H^{\mu\nu} &= \mathcal{L}_\xi H^{\mu\nu}\,, \qquad \delta K_{\mu\nu} = \mathcal{L}_\xi K_{\mu\nu}\,, \\ \delta\mathcal{B}_{\mu\nu} &= \mathcal{L}_\xi\mathcal{B}_{\mu\nu} + 2\partial_{[\mu}\xi_{\nu]}\,. \end{aligned} \tag{3.11}$$

[2]These matrices are identified with the target space metric and inverse metric respectively, when the target space is Riemannian.

Note that the trace $\mathcal{H}^M_M = 2(n-\bar{n})$[3] is an $O(D,D)$ invariant scalar and also that the B-field acts as an $O(D,D)$ transformation, i.e. its contribution to the generalized metric can be factorized as follows:

$$\mathcal{H}_{AB} = \begin{pmatrix} 1 & \mathcal{B} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} K & Z \\ Z^T & H \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\mathcal{B} & 1 \end{pmatrix} , \tag{3.12}$$

where we defined

$$Z^\mu{}_\nu \equiv y^\mu_i x^i_\nu - \bar{y}^\mu_{\bar{\imath}} \bar{x}^{\bar{\imath}}_\nu \,. \tag{3.13}$$

The generalized metric (3.6) as well as the relations (3.8)-(3.10) are invariant under the $GL(n) \times GL(\bar{n})$ rotations

$$\left(x^i_\mu, y^\mu_i, \bar{x}^{\bar{\imath}}_\mu, \bar{y}^\nu_{\bar{\imath}}\right) \to \left(x^j_\mu R^i_j, \left(R^{-1}\right)_i{}^j y^\nu_j, \bar{x}^{\bar{\jmath}}_\mu \bar{R}^{\bar{\imath}}_{\bar{\jmath}}, \left(\bar{R}^{-1}\right)_{\bar{\imath}}{}^{\bar{\jmath}} \bar{y}^\nu_{\bar{\jmath}}\right) , \tag{3.14}$$

and under the generalized shift symmetry

$$\begin{aligned}
&(y^\mu_i)' = y^\mu_i + V^\mu_i \,, \\
&(\bar{y}^\mu_{\bar{\imath}})' = \bar{y}^\mu_{\bar{\imath}} + \bar{V}^\mu_{\bar{\imath}} \,, \\
&K'_{\mu\nu} = K_{\mu\nu} - 2x^i_{(\mu}K_{\nu)\rho}V^\rho_i - 2\bar{x}^{\bar{\imath}}_{(\mu}K_{\nu)\rho}\bar{V}^\rho_{\bar{\imath}} + \left(x^i_\mu V_{\rho i} + \bar{x}^{\bar{\imath}}_\mu \bar{V}_{\rho\bar{\imath}}\right)\left(x^i_\nu V^\rho_i + \bar{x}^{\bar{\imath}}_\nu \bar{V}^\rho_{\bar{\imath}}\right) , \\
&\mathcal{B}'_{\mu\nu} = \mathcal{B}_{\mu\nu} - 2x^i_{[\mu}V_{\nu]i} + 2\bar{x}^{\bar{\imath}}_{[\mu}\bar{V}_{\nu]\bar{\imath}} + 2x^i_{[\mu}\bar{x}^{\bar{\jmath}}_{\nu]}\left(y^\rho_i \bar{V}_{\rho\bar{\jmath}} + \bar{y}^\rho_{\bar{\jmath}} V_{\rho i} + V_{\rho i}\bar{V}^\rho_{\bar{\jmath}}\right) ,
\end{aligned} \tag{3.15}$$

with $V_{\mu i}$ and $\bar{V}_{\mu\bar{\imath}}$ being the transformation parameters and we defined $V^\mu_i \equiv H^{\mu\rho}V_{\rho i}$, $\bar{V}^\mu_{\bar{\imath}} \equiv H^{\mu\rho}\bar{V}_{\rho\bar{\imath}}$.

Finally, we note that in this thesis we do not include a cosmological constant term in the DFT action. However this generalization is straightforward as one only needs to add a term proportional to $e^{-2d}\Lambda_{DFT}$ to the action [82]. This term could potentially be important when considering non-relativistic holography.

3.1.2 Connection and Curvature

In analogy with general relativity, we will introduce a connection Γ_{CAB} that will allow us to covariantize derivative interactions. The following unique

[3] In [81] it was shown that, upon BRST quantization, a critical bosonic string theory can only be anomaly-free if the trace of the generalized metric is zero, i.e. we have to impose $n = \bar{n}$ at the quantum level for a critical theory to be consistent.

Christoffel connection is found [79]

$$\Gamma_{RMN} = 2\left(P\partial_C P\bar{P}\right)_{[MN]} + 2\left(\bar{P}_{[M}{}^{S}\bar{P}_{N]}{}^{Q} - P_{[M}{}^{S}P_{N]}{}^{Q}\right)\partial_S P_{QR}$$
$$- 4\left(\frac{1}{\mathrm{tr}\,(P)-1}P_{R[M}P_{N]}{}^{S} + \frac{1}{\mathrm{tr}\,(\bar{P})-1}\bar{P}_{R[M}\bar{P}_{N]}{}^{S}\right)\left(\partial_S d + \left(P\partial^Q P\bar{P}\right)_{[QS]}\right)\,, \tag{3.16}$$

with P_{MN} and $\bar{P}_{MN}$ the projector operators defined as

$$P_{MN} = \frac{1}{2}\left(\eta_{MN} + \mathcal{H}_{MN}\right)\,, \tag{3.17}$$

$$\bar{P}_{MN} = \frac{1}{2}\left(\eta_{MN} - \mathcal{H}_{MN}\right)\,, \tag{3.18}$$

and satisfying the standard properties $P^2 = P, \bar{P}^2 = \bar{P}, P\bar{P} = 0, P + \bar{P} = \eta$, and with the corresponding traces

$$\mathrm{tr}\,(P) = P_M{}^M = D + n - \bar{n}\,, \qquad \mathrm{tr}\,(\bar{P}) = \bar{P}_M{}^M = D - n + \bar{n}\,. \tag{3.19}$$

The connection (3.16) is determined uniquely after imposing compatibility with $\mathcal{H}$, d, and the Lie derivative (3.3) as well as the additional set of projection constraints[4]

$$\mathcal{P}_{MNP}{}^{QRS}\Gamma_{QRS} = 0, \qquad \bar{\mathcal{P}}_{MNP}{}^{QRS}\Gamma_{QRS} = 0\,, \tag{3.20}$$

with

$$\mathcal{P}_{MNP}{}^{QRS} \equiv P_M{}^Q P_{[N}{}^{[R}P_{P]}{}^{S]} + \frac{2}{\mathrm{tr}\,(P)-1}P_{M[N}P_{P]}{}^{[R}P^{S]Q}\,, \tag{3.21}$$

$$\bar{\mathcal{P}}_{MNP}{}^{QRS} \equiv \bar{P}_M{}^Q \bar{P}_{[N}{}^{[R}\bar{P}_{P]}{}^{S]} + \frac{2}{\mathrm{tr}\,(\bar{P})-1}\bar{P}_{M[N}\bar{P}_{P]}{}^{[R}\bar{P}^{S]Q}\,, \tag{3.22}$$

A field strength R_{MNPQ} for the connection Γ can be constructed as usual:

$$R_{PQMN} = \partial_M \Gamma_{NPQ} - \partial_N \Gamma_{MPQ} + \Gamma_{MP}{}^R\Gamma_{NRQ} - \Gamma_{NP}{}^R\Gamma_{MRQ}\,. \tag{3.23}$$

However, (3.23) is not a covariant object, in fact no fully covariant four-index Riemann curvature can be constructed in DFT [78, 80]. Nevertheless we can build the semi-covariant curvature [78–80] $\mathcal{R}_{MNPQ}$

$$\mathcal{R}_{MNPQ} = \frac{1}{2}\left(R_{MNPQ} + R_{PQMN} - \Gamma^R{}_{MN}\Gamma_{RPQ}\right)\,, \tag{3.24}$$

[4] If these projections are not enforced the connection can not be fully determined [80]. However the relevant covariant curvatures constructed from it will be unique.

satisfying symmetry properties

$$\mathcal{R}_{MNPQ} = \mathcal{R}_{PQMN} = \mathcal{R}_{[MN][PQ]}\,, \qquad \mathcal{R}_{M[NPQ]} = 0\,, \tag{3.25}$$

and the semi-covariant transformation rule

$$\begin{aligned} \delta_\xi \mathcal{R}_{MNPQ} &= \hat{\mathcal{L}}_\xi \mathcal{R}_{MNPQ} + 2\nabla_{[M}\left(\left(\mathcal{P}+\bar{\mathcal{P}}\right)_{N][PQ]}{}^{RST}\,\partial_R\partial_S\xi_T\right) \\ &\quad +2\nabla_{[P}\left(\left(\mathcal{P}+\bar{\mathcal{P}}\right)_{Q][MN]}{}^{RST}\,\partial_R\partial_S\xi_T\right)\,. \end{aligned} \tag{3.26}$$

Even though the curvature defined in (3.25) is not covariant we can build the following covariant contraction

$$\mathcal{R} \equiv \left(P^{MP}P^{NQ} - \bar{P}^{MP}\bar{P}^{NQ}\right)\mathcal{R}_{MNPQ}\,, \tag{3.27}$$

which we will call the *doubled Ricci scalar*. It can be shown that it can be written in terms of the double fields $\mathcal{H}$ and d as

$$\begin{aligned} \mathcal{R} = {}& 4\mathcal{H}^{MN}\partial_M\partial_N d - \partial_M\partial_N\mathcal{H}^{MN} - 4\mathcal{H}^{MN}\partial_M d\partial_N d + 4\partial_M\mathcal{H}^{MN}\partial_N d \\ &+ \frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_K\mathcal{H}_{NL}\,. \end{aligned} \tag{3.28}$$

Equipped with (3.27), we can write down the DFT equivalent of the Einstein-Hilbert action

$$S_{DFT} = \int d^D x d^D\tilde{x}\, e^{-2d}\mathcal{R}\,. \tag{3.29}$$

The equations of motion are then found by varying (3.29), and given by[5]

$$\delta S = \int d^D x d^D\tilde{x} e^{-2d}\left[\delta\mathcal{H}^{MN}\mathcal{K}_{MN} - 2\mathcal{R}\delta d\right]\,, \tag{3.30}$$

with $\mathcal{K}_{MN}$ defined as

$$\begin{aligned} \mathcal{K}_{MN} = {}& \frac{1}{8}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \frac{1}{4}\left(\partial_L - 2\left(\partial_L d\right)\right)\left(\mathcal{H}^{LK}\partial_K\mathcal{H}_{MN}\right) + 2\partial_M\partial_N d \\ &- \frac{1}{2}\partial_{(M}\mathcal{H}^{KL}\partial_L\mathcal{H}_{N)K} + \frac{1}{2}\left(\partial_L - 2\left(\partial_L d\right)\right)\left(\mathcal{H}^{KL}\partial_{(M}\mathcal{H}_{N)K} + \mathcal{H}^K_{(M}\partial_K\mathcal{H}^L_{N)}\right)\,. \end{aligned} \tag{3.31}$$

[5]It is important to note that the variation cannot be performed using the parametrization (3.6) for fixed $(n,\bar{n})$. This would miss $n\times\bar{n}$ equations [83].

We need to ensure δH^{MN} satisfies (3.4), this can be achieved as long as we assume the variation takes the form [70]

$$\delta H^{MN} = P_L^M \delta\mathcal{M}^{LK} \bar{P}_K^N + \bar{P}_L^M \delta\mathcal{M}^{LK} P_K^N \,. \tag{3.32}$$

After assuming the form (3.32) we find that the equations of motion associated to the action (3.29) are nothing but[6]

$$\begin{aligned} \mathcal{R}_{MN} &= 0\,, \\ \mathcal{R} &= 0\,, \end{aligned} \tag{3.33}$$

where the double Ricci tensor is now expressed in terms of $\mathcal{K}$:

$$\mathcal{R}_{MN} = \bar{P}_M^L \mathcal{K}_{LK} P_N^K + P_M^L \mathcal{K}_{LK} \bar{P}_N^K. \tag{3.34}$$

Calling $\mathcal{R}_{MN}$ as generalized Ricci tensor is appropriate as it can be related to the semi-covariant curvature via $\mathcal{R}_{MN} = P_M^R \mathcal{R}^Q{}_{RQS} \bar{P}_N^S$. Note however that $\mathcal{R} \neq \eta^{MN}\mathcal{R}_{MN} = 0$.

In the following sections we will consider the action (3.29) and equations of motion (3.1) for some choices of (3.6), in particular TNC, Carroll and SNC parametrizations. This will allow us to write non-Riemannian gravitational equations of motion by means of the parametrization (3.6). Our usage of the relation between DFT and the previously mentioned non-relativistic geometries will not go further than (3.6). A detailed analysis pertaining the symmetries and related aspects of non-Riemann geometry following from (3.6) has been done by Blair et al. in [85].

3.2 Type I TNC geometry

3.2.1 Basics

We already introduced Type I TNC geometries in the previous chapters, however let us briefly recall some important basic properties that will be

[6] The same result could also be obtained by using the following property of the Riemann tensor [84]:

$$\delta\mathcal{R}_{MNPQ} = \nabla_{[M}\delta\Gamma_{N]PQ} + \nabla_{[P}\delta\Gamma_{Q]MN}.$$

useful in this chapter as well. The geometry and symmetries can be obtained from *null* reduction of a relativistic theory described by the line element

$$ds^2 = 2\tau_\mu dx^\mu(du - m_\nu dx^\nu) + h_{\mu\nu}dx^\mu dx^\nu, \tag{3.35}$$

where u is the direction associated to the null isometry, i.e. ∂_u is a null Killing vector. Note that the null reduction of the relativistic connection will not give exactly the connection we use in this text, (3.39), however the difference between the two is simply given by the torsion component.

Instead of the set of variables $\{\tau_\mu, \upsilon^\nu, h^{\mu\nu}, h_{\mu\nu}, m_\mu\}$ it will be more convenient for us to work with the boost-invariant set $\{\tau_\mu, \hat{\upsilon}^\mu, \bar{h}_{\mu\nu}, h^{\mu\nu}, \Phi\}$ where $\{\hat{\upsilon}^\mu, \bar{h}_{\mu\nu}, \Phi\}$ are defined as

$$\begin{aligned} \bar{h}_{\mu\nu} &\equiv h_{\mu\nu} - \tau_\mu m_\nu - \tau_\nu m_\mu, \\ \hat{\upsilon}^\mu &\equiv \upsilon^\mu - h^{\mu\nu} m_\nu\,, \\ \Phi &\equiv -\upsilon^\rho m_\rho + \frac{1}{2} h^{\rho\sigma} m_\rho m_\sigma\,, \end{aligned} \tag{3.36}$$

which are the boost-invariant combinations with the physical interpretation of a spatial metric, an "inverse clock", and Newton gravitational potential (i.e. the scalar which will appear in Poisson's equation). While these variables are explicitly invariant under Galilean boosts, they still transform under the $U(1)$ extension as

$$\delta_\sigma \bar{h}_{\mu\nu} = -2\tau_{(\mu}\partial_{\nu)}\sigma\,, \qquad \delta_\sigma \hat{\upsilon}^\mu = -h^{\mu\nu}\partial_\nu\sigma\,, \qquad \delta_\sigma\Phi = -\hat{\upsilon}^\rho\partial_\rho\sigma\,. \tag{3.37}$$

The Galilean invariant fields are subject to the identities

$$\bar{h}_{\mu\rho}h^{\rho\nu} - \hat{\upsilon}^\nu\tau_\mu = \delta^\nu_\mu\,, \qquad \hat{\upsilon}^\mu\tau_\mu = -1\,, \qquad \hat{\upsilon}^\mu\bar{h}_{\mu\nu} = 2\Phi\tau_\nu\,. \tag{3.38}$$

The connection we will use is[7]

$$\Gamma^\rho_{\mu\nu} = -\hat{\upsilon}^\rho\partial_\mu\tau_\nu + \frac{1}{2}h^{\rho\sigma}\left(\partial_\mu\bar{h}_{\nu\sigma} + \partial_\nu\bar{h}_{\mu\sigma} - \partial_\sigma\bar{h}_{\mu\nu}\right) \tag{3.39}$$

which is manifestly boost invariant, but not manifestly $U(1)$ invariant. Note that this connection is compatible with $h^{\mu\nu}$ and τ_μ. The antisymmetric part of this connection is poportional to the torsion tensor of TNC

$$F_{\mu\nu} \equiv 2\partial_{[\mu}\tau_{\nu]}, \tag{3.40}$$

[7]See also [86, 87] for a classification of TNC connections.

in terms of which we define the acceleration

$$a_\mu \equiv \hat{v}^\rho F_{\rho\mu}. \tag{3.41}$$

In the following section we will reformulate this theory by embedding it in DFT.

3.2.2 Embedding in DFT

By examining (3.35) we can identify the following metric and inverse relativistic metrics:

$$g_{\mu\nu} = \begin{pmatrix} \bar{h}_{\mu\nu} & \tau_\mu \\ \tau_\mu & 0 \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} h^{\mu\nu} & -\hat{v}^\mu \\ -\hat{v}^\mu & 2\Phi \end{pmatrix}, \tag{3.42}$$

which can be embedded in DFT as

$$\mathcal{H}_{MN} = \begin{pmatrix} \bar{h}_{\mu\nu} & \tau_\mu & 0 & 0 \\ \tau_\mu & 0 & 0 & 0 \\ 0 & 0 & h^{\mu\nu} & -\hat{v}^\mu \\ 0 & 0 & -\hat{v}^\mu & 2\Phi \end{pmatrix}. \tag{3.43}$$

We now apply a T-duality transformation to swap the null direction u with the dual null direction $\bar{u}$,

$$\mathcal{T}^M{}_N = \begin{pmatrix} \delta^\mu{}_\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \delta_\mu{}^\nu & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{3.44}$$

which results in the following *TNC generalized metric*:

$$\mathcal{H}_{MN} = \begin{pmatrix} \bar{h}_{\mu\nu} & 0 & 0 & \tau_\mu \\ 0 & 2\Phi & -\hat{v}^\nu & 0 \\ 0 & -\hat{v}^\mu & h^{\mu\nu} & 0 \\ \tau_\nu & 0 & 0 & 0 \end{pmatrix}. \tag{3.45}$$

The generalized metrics (3.43) and (3.45) will produce the same actions and equations of motion, however note that the lower right block in (3.45) is now degenerate, meaning that the latter has a clearer non-relativistic interpretation. As shown in [12], after adding matter to this parametrization we find that the embedding is given by[8]

$$\begin{aligned} K_{mn} &= \begin{pmatrix} h_{\mu\nu} & 0 \\ 0 & 0 \end{pmatrix} , \\ H^{mn} &= \begin{pmatrix} h^{\mu\nu} & h^{\mu\rho}\aleph_\rho \\ h^{\nu\rho}\aleph_\rho & h^{\rho\sigma}\aleph_\rho\aleph_\sigma \end{pmatrix} , \\ \mathcal{B}_{mn} &= \begin{pmatrix} \bar{B}_{\mu\nu} & -m_\mu \\ m_\nu & 0 \end{pmatrix} , \end{aligned} \tag{3.46}$$

and

$$\begin{aligned} x_m &= \frac{1}{\sqrt{2}} \begin{pmatrix} \tau_\mu - \aleph_\mu \\ 1 \end{pmatrix} , & \bar{x}_m &= \frac{1}{\sqrt{2}} \begin{pmatrix} \tau_\mu + \aleph_\mu \\ -1 \end{pmatrix} , \\ y^m &= \frac{1}{\sqrt{2}} \begin{pmatrix} -\upsilon^\mu \\ 1 - \upsilon^\mu \aleph_\mu \end{pmatrix} , & \bar{y}^m &= \frac{1}{\sqrt{2}} \begin{pmatrix} -\upsilon^\mu \\ -1 - \upsilon^\mu \aleph_\mu \end{pmatrix} , \end{aligned} \tag{3.47}$$

where $\aleph_\mu \equiv -B_{\mu u}$ appears from the dimensional reduction of the B-field. We also define

$$b_{\mu\nu} \equiv \partial_{[\mu}\aleph_{\nu]} , \qquad \mathfrak{e}_\mu \equiv \hat{\upsilon}^\rho b_{\rho\mu} . \tag{3.48}$$

The tensors $K_{\mu\nu}$ and $H^{\mu\nu}$ both have $2 = 1 + 1$ null eigenvectors and the trace of the generalized metric is $\mathcal{H}_M^M = 0$, implying that a TNC geometry corresponds to a $(1, 1)$ theory in the DFT framework.

3.2.3 Action and equations of motion

Having obtained the embedding of TNC geometry in Double Field Theory, we can now immediately write down its action using the DFT action in equation (3.29). We first introduce the following notation. Given an arbitrary tensor $A_{\mu\nu\dots}$ we will define for brevity

$$A^{\mu\nu\dots} \equiv A_{\rho\sigma\dots} h^{\mu\rho} h^{\nu\sigma} \dots \tag{3.49}$$

[8] Note that now we are briefly using m, n as indices of the DFT tensors, as a way to avoid introducing new symbols. We have $m = 0, 1, \dots, d+1$ and $\mu = 0, 1, \dots, d$ where d is the dimension of the TNC spacetime and $d+1 = D$ is the dimension of its uplift, which includes the null direction u.

Note that the only TNC fields which naturally have upper indices are $\hat{\upsilon}^\mu$ and $h^{\mu\nu}$, hence any other tensor with upper indices is to be understood as defined via (3.49).

Now, the TNC action is given by[9]

$$S = \int d^dx\, e \left[\mathcal{R} + \frac{1}{2}a^\mu a_\mu + \frac{1}{2}\mathfrak{e}^\mu \mathfrak{e}_\mu - 4a^\mu D_\mu\phi + 4D^\mu\phi D_\mu\phi - \frac{1}{12}H^{\mu\nu\rho}H_{\mu\nu\rho} \right. \\ \left. -\frac{1}{2}\hat{\upsilon}^\rho H_{\rho\mu\nu}b^{\mu\nu} - \frac{1}{2}\left(F^{\mu\nu}F_{\mu\nu} + b^{\mu\nu}b_{\mu\nu}\right)\Phi\right], \tag{3.50}$$

with $H = dB$, $b = d\aleph$ and

$$e \equiv \sqrt{\frac{\det\bar{h}}{2\Phi}}e^{-2\phi}\,. \tag{3.51}$$

We choose the independent fields of our theory to be $\Phi, \hat{\upsilon}^\mu, h^{\mu\nu}, B_{\mu\nu}$ and $\aleph_\mu$, see Appendix B.1 for useful identites.

The variation of the action (3.50) with respect to Φ imposes a generalized twistlessness constraint on torsion:

$$F_{\mu\nu}F^{\mu\nu} = b_{\mu\nu}b^{\mu\nu}. \tag{3.52}$$

The equations for the matter fields are

$$D_\mu D^\mu\phi + a^\mu D_\mu\phi - 2D_\mu\phi D^\mu\phi = \frac{1}{2}\mathfrak{e}^2 - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} - \frac{1}{2}\hat{\upsilon}^\lambda H_{\lambda\mu\nu}b^{\mu\nu} \\ - \frac{1}{2}F_{\mu\nu}F^{\mu\nu}\Phi\,, \tag{3.53}$$

$$D^\rho b_{\rho\mu} - 2D^\rho\phi\, b_{\rho\mu} = \frac{1}{2}F^{\rho\sigma}H_{\rho\sigma\mu}\,, \tag{3.54}$$

$$D^\rho H_{\rho\mu\nu} + a^\rho H_{\rho\mu\nu} - 2D^\rho\phi\, H_{\rho\mu\nu} = 2D_{[\mu}\mathfrak{e}_{\nu]} - 2\hat{\upsilon}^\lambda D_\lambda b_{\mu\nu} + 2a_{[\mu}\mathfrak{e}_{\nu]} + 4b^\rho{}_{[\mu}F_{\nu]\rho}\Phi \\ + \left(2\hat{\upsilon}^\rho D_\rho\phi - D_\rho\hat{\upsilon}^\rho\right)b_{\mu\nu}, \tag{3.55}$$

[9] It is worth noting that this same action can also be obtained as the dimensional reduction of the standard NS-NS sector of supergravity

$$S_{NS-NS} = \int d^Dx\, e^{-2\phi}\sqrt{-\det g}\left(\mathcal{R} - \frac{1}{12}H^2 + 4\left(\partial\phi\right)^2\right),$$

where g is the Riemannian metric appearing in (3.42), $H_{\mu\nu\rho}$ is the field-strength of the B-field and $D = d+1$ with d the dimension of the TNC spacetime. Note however that this null reduction is not fully consistent, as one needs to impose Poisson's equation on-shell, rather than deriving it from the (null-reduced) action.

while the equation for τ_μ is

$$D^\rho F_{\rho\mu} + a^\rho F_{\rho\mu} - 2D^\rho \phi F_{\rho\mu} = \frac{1}{2} b^{\rho\sigma} H_{\rho\sigma\mu} + \mathfrak{e}^\rho b_{\rho\mu} \,. \tag{3.56}$$

The equation obtained from the variation with respect to $h^{\mu\nu}$ requires more care. First of all, notice that we have $\tau_\mu \tau_\nu \delta h^{\mu\nu} = 0$. This means that the most general variation of $h^{\mu\nu}$ is given by

$$\delta h^{\mu\nu} = \left(\Delta_S\right)^\mu_\rho \delta\mathcal{M}^{\rho\nu} + \delta\mathcal{M}^{\mu\rho} \left(\Delta_S\right)^\nu_\rho \,, \tag{3.57}$$

where $\delta\mathcal{M}^{\mu\nu}$ is an arbitrary symmetric tensor and $\left(\Delta_S\right)^\mu_\nu = h^{\mu\rho}\bar{h}_{\rho\nu}$. It also follows that the time-time projection of the equation obtained from this variation will be trivially zero, i.e. we will not be able to obtain Newton's law from this variation. This will also be true for variation of the Carrollian and SNC actions and is, in fact, a general property inherited from Double Field Theory [83]. By imposing an ansatz on the generalized metric and then computing the variation of the resulting action we will end up with $n \times \bar{n}$ equations less than we would initially expect. However, this problem can be easily avoided by taking the variation of the DFT action *first* and imposing the TNC ansatz on the resulting equations of motion (3.1). With this in mind, we find that the variation of (3.50) with respect to $h^{\mu\nu}$ produces the equation

$$\begin{aligned} \mathcal{R}_{(\mu\nu)} + 2D_{(\mu}D_{\nu)}\phi - \frac{1}{4} h^{\rho\sigma} h^{\lambda\kappa} H_{\mu\rho\lambda} H_{\nu\sigma\kappa} &= \frac{a_\mu a_\nu - \mathfrak{e}_\mu \mathfrak{e}_\nu}{2} - \hat{\upsilon}^\rho D_{(\mu} F_{\nu)\rho} \\ &- \left(F_{\mu\rho} F_{\nu\sigma} h^{\rho\sigma} - b_{\mu\rho} b_{\nu\sigma} h^{\rho\sigma}\right) \Phi + \mathfrak{R}_{\rho\sigma} \left(\Delta_T\right)^\rho_\mu \left(\Delta_T\right)^\sigma_\nu \,, \end{aligned} \tag{3.58}$$

where we defined

$$\mathfrak{R}_{\mu\nu} \equiv \mathcal{R}_{\mu\nu} + 2D_\mu D_\nu \phi - \frac{1}{4} h^{\rho\sigma} h^{\lambda\kappa} H_{\mu\rho\lambda} H_{\nu\sigma\kappa} + a^2 \tau_\mu \tau_\nu \Phi - \mathfrak{e}^2 \tau_\mu \tau_\nu \Phi \,. \tag{3.59}$$

Note that, because of the presence of $\mathfrak{R}_{\mu\nu}$ in (3.58), the time-time projection of Einstein's equations is identically zero. As we already explained, Newton's law can be found by imposing the TNC ansatz on the DFT equations (3.1). The resulting equation is

$$D^\mu D_\mu \Phi + 3a^\mu D_\mu \Phi + m_\Phi^2 \Phi - 2F_{\mu\nu} F^{\mu\nu} \Phi^2 = \rho_{\mathcal{K}} + \rho_m \,, \tag{3.60}$$

with

$$\begin{aligned} m_\Phi^2 &= a^2 + \mathfrak{e}^2 + 4a^\mu D_\mu \phi - \hat{\upsilon}^\rho H_{\rho\mu\nu} b^{\mu\nu} \,, \\ \rho_{\mathcal{K}} &= \hat{\upsilon}^\mu D_\mu D_\nu \hat{\upsilon}^\nu + D_\mu \hat{\upsilon}^\nu D_\nu \hat{\upsilon}^\mu = -\hat{\upsilon}^\mu D_\mu \mathcal{K}_\nu^\nu + \mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu} \,, \\ \rho_m &= \frac{1}{4}\hat{\upsilon}^\mu \hat{\upsilon}^\nu H_{\mu\rho\sigma} H_\nu{}^{\rho\sigma} - 2\hat{\upsilon}^\mu \hat{\upsilon}^\nu D_\mu D_\nu \phi \,. \end{aligned} \tag{3.61}$$

Notice that by using the identities

$$\begin{aligned} \hat{\upsilon}^\mu \hat{\upsilon}^\nu \mathcal{R}_{\mu\nu} &= -\hat{\upsilon}^\mu D_\mu D_\nu \hat{\upsilon}^\nu + \hat{\upsilon}^\mu D_\nu D_\mu \hat{\upsilon}^\nu + a^\mu D_\mu \Phi + 2a^2 \Phi \,, \\ D_\mu \hat{\upsilon}^\rho D_\rho \hat{\upsilon}^\nu + \hat{\upsilon}^\rho D_\mu D_\rho \hat{\upsilon}^\nu &= h^{\nu\sigma} \left(D_\mu D_\sigma \Phi + 2D_\mu a_\sigma \, \Phi + 2a_\sigma D_\mu \Phi \right) \,, \end{aligned} \tag{3.62}$$

we can rewrite (3.60) as

$$\hat{\upsilon}^\mu \hat{\upsilon}^\nu \mathcal{R}_{\mu\nu} = \frac{1}{4}\hat{\upsilon}^\mu \hat{\upsilon}^\nu H_{\mu\rho\sigma} H_\nu{}^{\rho\sigma} - 2\hat{\upsilon}^\mu \hat{\upsilon}^\nu D_\mu D_\nu \phi - a^2 \Phi + \mathfrak{e}^2 \Phi \,, \tag{3.63}$$

so that the full Einstein's equations can compactly be written as

$$\begin{aligned} \mathcal{R}_{(\mu\nu)} + 2D_{(\mu} D_{\nu)} \phi - \frac{1}{4} h^{\rho\sigma} h^{\lambda\kappa} H_{\mu\rho\lambda} H_{\nu\sigma\kappa} = \frac{a_\mu a_\nu - \mathfrak{e}_\mu \mathfrak{e}_\nu}{2} - \hat{\upsilon}^\rho D_{(\mu} F_{\nu)\rho} \\ + \hat{\upsilon}^\lambda h^{\rho\sigma} b_{\rho(\mu} H_{\nu)\lambda\sigma} - \left(F_{\mu\rho} F_{\nu\sigma} h^{\rho\sigma} - b_{\mu\rho} b_{\nu\sigma} h^{\rho\sigma} \right) \Phi \,. \end{aligned} \tag{3.64}$$

In summary, the TNC equations of motion are given by

$$F_{\mu\nu} F^{\mu\nu} = b_{\mu\nu} b^{\mu\nu} \,, \tag{3.65}$$

$$D^\rho F_{\rho\mu} + a^\rho F_{\rho\mu} - 2D^\rho \phi F_{\rho\mu} = \frac{1}{2} b^{\rho\sigma} H_{\rho\sigma\mu} + \mathfrak{e}^\rho b_{\rho\mu} \,, \tag{3.66}$$

$$D^\rho b_{\rho\mu} - 2D^\rho \phi \, b_{\rho\mu} = \frac{1}{2} F^{\rho\sigma} H_{\rho\sigma\mu} \,, \tag{3.67}$$

$$\begin{aligned} D_\mu D^\mu \phi + a^\mu D_\mu \phi - 2D_\mu \phi D^\mu \phi &= \frac{1}{2}\mathfrak{e}^2 - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{1}{2}\hat{\upsilon}^\lambda H_{\lambda\mu\nu} b^{\mu\nu} \\ &\quad - \frac{1}{2} F_{\mu\nu} F^{\mu\nu} \Phi \,, \end{aligned} \tag{3.68}$$

$$\begin{aligned} D^\rho H_{\rho\mu\nu} + a^\rho H_{\rho\mu\nu} - 2D^\rho \phi \, H_{\rho\mu\nu} &= 2D_{[\mu} \mathfrak{e}_{\nu]} - 2\hat{\upsilon}^\lambda D_\lambda b_{\mu\nu} + 2a_{[\mu} \mathfrak{e}_{\nu]} + 4b^\rho{}_{[\mu} F_{\nu]\rho} \Phi \\ &\quad + \left(2\hat{\upsilon}^\rho D_\rho \phi - D_\rho \hat{\upsilon}^\rho \right) b_{\mu\nu} , \end{aligned} \tag{3.69}$$

$$\begin{aligned} \mathcal{R}_{(\mu\nu)} + 2D_{(\mu} D_{\nu)} \phi &= \frac{1}{4} H_\mu{}^{\rho\sigma} H_{\nu\rho\sigma} + \frac{a_\mu a_\nu - \mathfrak{e}_\mu \mathfrak{e}_\nu}{2} - \hat{\upsilon}^\rho D_{(\mu} F_{\nu)\rho} \\ &\quad + \hat{\upsilon}^\lambda b^\rho{}_{(\mu} H_{\nu)\lambda\rho} - \left(F_{\mu\rho} F_{\nu\sigma} h^{\rho\sigma} - b_{\mu\rho} b_{\nu\sigma} h^{\rho\sigma} \right) \Phi \,, \end{aligned} \tag{3.70}$$

where we remind the reader of the short-hand notation (3.49), as well as the following definitions

$$\begin{aligned} a_\mu &= \hat{\upsilon}^\rho F_{\rho\mu} = 2\hat{\upsilon}^\rho \partial_{[\rho}\tau_{\mu]}, \\ \mathfrak{e}_\mu &= \hat{\upsilon}^\rho b_{\rho\mu} = 2\hat{\upsilon}^\rho \partial_{[\rho}\aleph_{\mu]}\,, \end{aligned} \tag{3.71}$$

and Φ is Newton's potential, not to be confused with the dilaton ϕ.

Note that these equations are *manifestly* invariant under almost all transformations described by the Bargmann algebra. The only nontrivial transformation corresponds to the $U(1)$ generator m_μ. However, a straightforward (but tedious) computation shows that these equations are indeed invariant under mass $U(1)$, although not manifestly so. We will discuss some properties of these equations and their relation to known results in section 3.5.

3.3 Carrollian geometry

3.3.1 Basics

The Carroll algebra can be obtained by considering a particular contraction ($c \to 0$) of the Poincaré algebra [88–90]. In [91] it was suggested that this algebra could play an important role in flat space holography, hence it would be interesting to study how field theories couple to Carrollian spacetime, see e.g. [92–94]. To this end, one needs a gravitational action coupled to matter, and this is precisely what we will compute below.

We will describe a Carrollian geometry using the same set of symbols we already introduced for the TNC case above. However, these fields will now transform under Carrollian boosts rather than Galilean boosts:

$$\delta_{\mathcal{C}}\tau_\mu = \lambda_\mu\,, \qquad\qquad \delta_{\mathcal{C}}h^{\mu\nu} = 2h^{\rho(\mu}\upsilon^{\nu)}\lambda_\rho\,. \tag{3.72}$$

Moreover, we replace the gauge field m_μ by a contravariant vector M^μ, which transforms as

$$\delta M^\mu = \mathcal{L}_\xi M^\mu + e^\mu_a\lambda^a = \mathcal{L}_\xi M^\mu + h^{\mu\nu}\lambda_\nu\,. \tag{3.73}$$

This allows us to build the following manifestly boost-invariant tensors

$$\begin{aligned} \hat{\tau}_\mu &= \tau_\mu - h_{\mu\nu}M^\nu, \\ \hat{h}^{\mu\nu} &= h^{\mu\nu} - 2M^{(\mu}\upsilon^{\nu)} + 2\Phi\upsilon^\mu\upsilon^\nu \equiv \bar{h}^{\mu\nu} + 2\Phi\upsilon^\mu\upsilon^\nu, \\ \Phi &= -M^\mu\tau_\mu + \frac{1}{2}h_{\mu\nu}M^\mu M^\nu = -M^\mu\hat{\tau}_\mu - \frac{1}{2}h_{\mu\nu}M^\mu M^\nu\,, \end{aligned} \tag{3.74}$$

which we interpret as the boost-invariant clock one-form, inverse spatial metric and Newton's potential respectively. They satisfy the following orthogonality and completeness relations

$$\hat{h}^{\mu\rho}h_{\rho\nu} - \upsilon^{\mu}\hat{\tau}_{\nu} = \delta^{\mu}_{\nu}, \qquad \hat{\tau}_{\mu}\hat{h}^{\mu\rho} = 0 = \upsilon^{\mu}h_{\mu\rho}\,. \tag{3.75}$$

It is possible to embed a Carrollian geometry in a Lorentzian one just as we did for TNC in (3.35):

$$ds^2 = du(2\Phi\, du - 2\hat{\tau}_{\mu}dx^{\mu}) + h_{\mu\nu}dx^{\mu}dx^{\nu}\,. \tag{3.76}$$

A connection compatible with $\hat{\tau}_{\mu}, \upsilon^{\mu}, h_{\mu\nu}$ and $\hat{h}^{\mu\nu}$ can be constructed [89,95]:

$$\widetilde{\Gamma}^{\rho}_{\mu\nu} = -\upsilon^{\rho}\partial_{\mu}\hat{\tau}_{\nu} + \frac{1}{2}\hat{h}^{\mu\lambda}\left(\partial_{\mu}h_{\nu\lambda} + \partial_{\nu}h_{\mu\lambda} - \partial_{\lambda}h_{\mu\nu}\right) - \hat{h}^{\mu\lambda}\mathcal{K}_{\lambda\mu}\hat{\tau}_{\nu}\,, \tag{3.77}$$

where we introduced the extrinsic curvature

$$\mathcal{K}_{\mu\nu} = -\frac{1}{2}\mathcal{L}_{\upsilon}h_{\mu\nu} = -\frac{1}{2}\left(\upsilon^{\rho}\partial_{\rho}h_{\mu\nu} + \left(\partial_{\mu}\upsilon^{\rho}\right)h_{\rho\nu} + \left(\partial_{\nu}\upsilon^{\rho}\right)h_{\mu\rho}\right)\,. \tag{3.78}$$

However, we will find it more convenient to use a slightly different connection,

$$\Gamma^{\rho}_{\mu\nu} \equiv \widetilde{\Gamma}^{\rho}_{\mu\nu} + \hat{h}^{\rho\lambda}\mathcal{K}_{\lambda\mu}\hat{\tau}_{\nu}\,. \tag{3.79}$$

Using $\hat{\tau}_{\mu}\hat{h}^{\mu\nu} = 0$, it is easy to show that this connection is still compatible with $\hat{\tau}_{\mu}$ and $\hat{h}^{\mu\nu}$, but now we have

$$D_{\mu}\upsilon^{\nu} = -\hat{h}^{\nu\lambda}\mathcal{K}_{\lambda\mu}, \qquad D_{\rho}h_{\mu\nu} = -2\mathcal{K}_{\rho(\mu}\hat{\tau}_{\nu)}\,, \tag{3.80}$$

where we used

$$\upsilon^{\mu}\mathcal{K}_{\mu\nu} = \upsilon^{\rho}D_{\rho}\upsilon^{\mu} = 0\,. \tag{3.81}$$

We also define the following tensors in analogy with TNC:

$$F_{\mu\nu} \equiv 2\partial_{[\mu}\hat{\tau}_{\nu]}\,, \qquad a_{\mu} \equiv \upsilon^{\rho}F_{\rho\mu}\,. \tag{3.82}$$

3.3.2 Embedding in DFT

Given the relativistic geometry (3.76), we can construct the generalized metric

$$\mathcal{H}_{MN} = \begin{pmatrix} h_{\mu\nu} & -\hat{\tau}_{\mu} & 0 & 0 \\ -\hat{\tau}_{\mu} & 2\Phi & 0 & 0 \\ 0 & 0 & \bar{h}^{\mu\nu} & \upsilon^{\mu} \\ 0 & 0 & \upsilon^{\mu} & 0 \end{pmatrix}. \tag{3.83}$$

There are clearly some similarities between this metric and the TNC one (3.43). In fact when the scalars $\Phi^{(\mathcal{C})}$ and $\Phi^{(TNC)}$ are both zero (which implies $M^\mu = m_\mu = 0$) it is easy to see that the two generalized metrics become identical up to some obvious identifications. When the two scalars are not zero we still have a relation between the two geometries, but it is a bit more involved. To see this we can start from (3.83) and then apply a T-duality transformation that swaps the Carrollian directions μ with their dual ones $\bar{\mu}$ and arrive at

$$\mathcal{H}_{MN} \to \begin{pmatrix} \bar{h}^{\mu\nu} & 0 & 0 & \upsilon^\mu \\ 0 & 2\Phi & -\hat{\tau}_\nu & 0 \\ 0 & -\hat{\tau}_\mu & h_{\mu\nu} & 0 \\ \upsilon^\nu & 0 & 0 & 0 \end{pmatrix}. \tag{3.84}$$

This is equivalent to (3.45) if we make the identifications

$$\bar{h}^{\mu\nu}_{(\mathcal{C})} \leftrightarrow \bar{h}^{(TNC)}_{\mu\nu}, \qquad \hat{\tau}^{(\mathcal{C})}_\mu \leftrightarrow \hat{\upsilon}^\mu_{(TNC)}, \qquad \Phi^{(\mathcal{C})} \leftrightarrow \Phi^{(TNC)}, \tag{3.85}$$

$$h^{(\mathcal{C})}_{\mu\nu} \leftrightarrow h^{\mu\nu}_{(TNC)}, \qquad \upsilon^\mu_{(\mathcal{C})} \leftrightarrow \tau^{(TNC)}_\mu, \qquad M^\mu_{(\mathcal{C})} \leftrightarrow m^{(TNC)}_\mu. \tag{3.86}$$

These are the same transformations that were proposed and discussed in [89, 95, 96], however when working in the DFT framework it is more clear what the interpretation of this relationship is, i.e. at least part of it is a T-duality following from the fact that this theory is embedded in string theory! More precisely, this relation between TNC and Carrollian generalized metrics involves the exchange of covariant tensors with contravariant ones (and vice versa), including the generalized metric, so that it cannot *only* be a T-duality. In fact, it is not even clear that the transformations in (3.85)-(3.86) are allowed within the framework of DFT, although at the very least they seem to suggest that there may be a deep relationship between TNC and Carrollian geometries. Although it is likely not correct to call this relationship a 'duality', we will do so anyway for brevity.

Note that this 'duality' is mapping two theories that are in principle really different, since one is a non-relativistic theory while the other is an ultra-relativistic theory. It is also interesting to note that when acting with this transformation on the Carrollian side we are effectively generating massive particles, which will correspond to the eigenstates of the $U(1)$ generator on the TNC side.

Moreover we remark that this duality only maps the TNC generalized metric to the Carrollian. To map the full actions to one another we would need to transform the partial derivatives as well, i.e. we would need a transformation of the form

$$(\partial_\mu)^{(Car)} \leftrightarrow (\partial^\mu)^{(TNC)}, \tag{3.87}$$

but it is not clear what a partial derivative with upper index means in a non-relativistic (or ultra-relativistic) theory. It will be interesting to explore this duality in more detail in future works, to understand if the two actions can indeed be mapped to each other.

3.3.3 Action and equations of motion

Once again, given an arbitrary tensor $A_{\mu\nu\dots}$ we define

$$A^{\mu\nu\dots} \equiv A_{\rho\sigma\dots}\hat{h}^{\mu\rho}\hat{h}^{\nu\sigma}\dots \tag{3.88}$$

The only Carroll fields which naturally have upper indices are $\upsilon^\mu, \bar{h}^{\mu\nu}, \hat{h}^{\mu\nu}$ and M^μ, hence any other tensor with upper indices is to be understood as defined via (3.88).

The action for a Carrollian gravitational theory is given by

$$\begin{aligned} S = \int d^dx\, e\, \Bigg[&\mathcal{R} + \frac{1}{2}(a^\mu a_\mu + \mathfrak{e}^\mu \mathfrak{e}_\mu) - 4a^\mu D_\mu\phi + 4D^\mu\phi D_\mu\phi + 4\upsilon^\mu D_\mu\phi\, \upsilon^\nu D_\nu\Phi \\ &+2\Phi\left(\mathcal{K}^{\mu\nu}\mathcal{K}_{\mu\nu} - \mathcal{K}^2 - 4\upsilon^\mu D_\mu\phi\, \upsilon^\nu D_\nu\phi - 4\mathcal{K}\upsilon^\mu D_\mu\phi + \frac{1}{4}\upsilon^\rho\upsilon^\sigma H^{\mu\nu}{}_\rho H_{\mu\nu\sigma}\right) \\ &+2\mathcal{K}\upsilon^\mu D_\mu\Phi - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + \frac{1}{2}b^{\mu\nu}H_{\mu\nu\rho}\upsilon^\rho \Bigg], \end{aligned} \tag{3.89}$$

with $\mathcal{K} \equiv \hat{h}^{\mu\nu}\mathcal{K}_{\mu\nu} = -D_\mu\upsilon^\mu$, $\mathcal{R} \equiv \hat{h}^{\mu\nu}\mathcal{R}_{\mu\nu}$ and the measure is defined as

$$e = e^{-2\phi}\sqrt{\frac{2\Phi}{\det \bar{h}^{\mu\nu}}} = e^{-2\phi}\sqrt{2\Phi \det \bar{h}_{\mu\nu}}\ , \tag{3.90}$$

where $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{\hat{\tau}_\mu\hat{\tau}_\nu}{2\Phi}$. We choose the independent fields of our theory to be $\Phi, \upsilon^\mu, \hat{h}^{\mu\nu}, B_{\mu\nu}$ and $\aleph_\mu$, see Appendix B.2 for useful identites.

As for TNC, this action can be obtained as the dimensional reduction of the standard NS-NS sector of the supergravity action, with (relativistic) metric given by $g_{\mu\nu}$ appearing in (3.76). Furthermore note that when $\Phi^{(Carroll)} = 0$ this action correctly reduces to (3.50) with $\Phi^{(TNC)} = 0$. An important point to note here is that a null reduction should not yield a Carrollian geometry (as that always gives TNC instead[10]), so, although the action (3.89) is invariant under the correct boost transformations (3.72) and (3.73), it is unclear whether it can truly be interpreted as the action of an ultra-relativistic theory. However, in this thesis we are only interested in (3.89) because of its relationship to TNC, so we leave the precise interpretation of this theory as an open question for future work.

The variation of the action with respect to Φ gives

$$\mathcal{R}_{\mu\nu}\upsilon^\mu\upsilon^\nu = \frac{1}{4}\upsilon^\rho\upsilon^\sigma H^{\mu\nu}{}_\rho H_{\mu\nu\sigma} - 2\upsilon^\mu\upsilon^\nu D_\mu D_\nu\phi\,. \tag{3.91}$$

Note that contrary to what happened for TNC, this equation does not impose any constraint on the antisymmetric part of the connection, however it does impose a constraint on intrinsic torsion, which in the Carrollian case is given by the extrinsic curvature [97].

The equations for the matter fields are

$$\begin{aligned} D_\mu D^\mu\phi + a^\mu D_\mu\phi - 2D_\mu\phi D^\mu\phi &= 2\upsilon^\mu\upsilon^\nu\left(\Phi D_\mu D_\nu\phi - 2\Phi D_\mu\phi D_\nu\phi + D_\mu\phi D_\nu\Phi\right) \\ &- 2\mathcal{K}\Phi\upsilon^\mu D_\mu\phi - \frac{1}{12}H^{\mu\nu\rho}H_{\mu\nu\rho} + \frac{1}{2}\upsilon^\rho\upsilon^\sigma\Phi H_\rho{}^{\mu\nu}H_{\sigma\mu\nu} + \frac{\mathfrak{e}^\mu\mathfrak{e}_\mu}{2} + \frac{1}{2}\upsilon^\rho b^{\mu\nu}H_{\rho\mu\nu}\,, \end{aligned} \tag{3.92}$$

$$\begin{aligned} D^\rho b_{\rho\mu} - 2D^\rho\phi\, b_{\rho\mu} &= -2\Phi\upsilon^\rho D_\rho\mathfrak{e}_\mu - 2\Phi\mathfrak{e}^\rho\mathcal{K}_{\rho\mu} + 2\Phi\mathfrak{e}_\mu\mathcal{K} + 4\Phi\upsilon^\rho D_\rho\Phi\,\mathfrak{e}_\mu \\ &+ \frac{1}{2}F^{\rho\sigma}H_{\rho\sigma\mu} + 2D^\rho\Phi\,\upsilon^\sigma H_{\rho\sigma\mu} - 2\Phi\upsilon^\rho a^\sigma H_{\rho\sigma\mu}\,, \end{aligned} \tag{3.93}$$

$$\begin{aligned} D^\rho H_{\rho\mu\nu} + a^\rho H_{\rho\mu\nu} - 2D^\rho\phi\, H_{\rho\mu\nu} &= \upsilon^\rho D_\rho b_{\mu\nu} - 2\upsilon^\rho D_\rho\phi\, b_{\mu\nu} - 2\mathcal{K}_{[\mu}{}^\rho b_{\nu]\rho} - \mathcal{K}b_{\mu\nu} \\ &+ 2\Phi\upsilon^\rho\upsilon^\sigma\left(D_\rho H_{\sigma\mu\nu} - 2D_\rho\phi\, H_{\sigma\mu\nu}\right) + 2\upsilon^\rho\upsilon^\sigma D_\rho\Phi\, H_{\sigma\mu\nu} \\ &+ 4\Phi\upsilon^\sigma\mathcal{K}_{[\mu}{}^\rho H_{\nu]\rho\sigma} - 2\mathcal{K}\Phi\upsilon^\rho H_{\rho\mu\nu}\,. \end{aligned} \tag{3.94}$$

The equation for υ^μ is

$$\begin{aligned} D^\rho F_{\rho\mu} + a^\rho F_{\rho\mu} - 2D^\rho\phi\, F_{\rho\mu} &= -4\Phi\upsilon^\rho D_{(\rho}a_{\mu)} + 2\Phi a_\mu\mathcal{K} + 4\Phi\upsilon^\rho D_\rho\phi\, a_\mu \\ &+ 2\upsilon^\rho D_\rho D_\mu\Phi + 2\mathcal{K}_{\rho\mu}D^\rho\Phi - 2\mathcal{K}D_\mu\Phi - 4\upsilon^\rho D_\rho\phi\, D_\mu\Phi \\ &+ \frac{1}{2}b^{\rho\sigma}H_{\rho\sigma\mu} - \mathfrak{e}^\rho b_{\rho\mu} - 2\Phi\upsilon^\rho\mathfrak{e}^\sigma H_{\rho\sigma\mu} - 2\upsilon^\rho D_\rho\Phi\, a_\mu\,. \end{aligned} \tag{3.95}$$

[10] I would like to thank Jelle Hartong for pointing this out.

Einstein's equations are given by

$$\begin{aligned}\mathcal{R}_{(\mu\nu)} + 2D_{(\mu}D_{\nu)}\phi - \frac{1}{4}H_{\mu}{}^{\rho\sigma}H_{\nu\rho\sigma} = \frac{a_\mu a_\nu - \mathfrak{e}_\mu\mathfrak{e}_\nu}{2} + D_{(\mu}a_{\nu)} - F_{(\mu}{}^{\rho}\mathcal{K}_{\nu)\rho} \\ + 2\upsilon^\rho\mathcal{K}_{\mu\nu}D_\rho\Phi - 2\Phi\mathcal{K}\,\mathcal{K}_{\mu\nu} + 2\Phi\upsilon^\rho D_\rho\mathcal{K}_{\mu\nu} \\ - 4\Phi\upsilon^\rho\mathcal{K}_{\mu\nu}D_\rho\phi - \Phi\upsilon^\lambda\upsilon^\kappa H_{\mu\lambda}{}^{\rho}H_{\nu\kappa\rho} + \upsilon^\rho b_{(\mu}{}^{\sigma}H_{\nu)\rho\sigma}\,.\end{aligned} \tag{3.96}$$

Once again one equation is missing, but it can be found directly from DFT:

$$\begin{aligned}D^\mu D_\mu\Phi - a^\mu D_\mu\Phi - 2D^\mu\phi D_\mu\Phi + (a^2 - \mathfrak{e}^2)\Phi = \frac{1}{4}\left(F^{\mu\nu}F_{\mu\nu} - b^{\mu\nu}b_{\mu\nu}\right) \\ - 2\mathcal{K}\Phi\upsilon^\rho D_\rho\Phi + 2\Phi\upsilon^\rho\upsilon^\sigma\left(D_\rho D_\sigma\Phi - 2D_\rho\phi D_\sigma\Phi\right)\,.\end{aligned} \tag{3.97}$$

It is also possible to rewrite these equations in terms of $\bar{h}^{\mu\nu} = \hat{h}^{\mu\nu} + 2\Phi\upsilon^\mu\upsilon^\nu$, rather than $\hat{h}^{\mu\nu}$. The new connection $\bar{\Gamma}^\rho_{\mu\nu}$ is defined as

$$\Gamma^\rho_{\mu\nu} = \bar{\Gamma}^\rho_{\mu\nu} + 2\Phi\upsilon^\rho\mathcal{K}_{\mu\nu}\,, \tag{3.98}$$

where quantities with a bar on them are understood to be defined with $\bar{h}^{\mu\nu}$ instead of $\hat{h}^{\mu\nu}$, i.e.

$$\bar{\Gamma}^\rho_{\mu\nu} = -\upsilon^\rho\partial_\mu\hat{\tau}_\nu + \frac{1}{2}\bar{h}^{\mu\lambda}\left(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}\right). \tag{3.99}$$

The new Ricci tensor is related to the old one via

$$\mathcal{R}_{\mu\nu} = \bar{\mathcal{R}}_{\mu\nu} - 2\upsilon^\rho\bar{D}_\rho\Phi\,\mathcal{K}_{\mu\nu} + 2\Phi\mathcal{K}\,\mathcal{K}_{\mu\nu} - 2\Phi\upsilon^\rho\bar{D}_\rho\mathcal{K}_{\mu\nu}\,. \tag{3.100}$$

The action and equations will look nicer when using this connection, however $\bar{h}^{\mu\nu}, \hat{h}^{\mu\nu}$ and $\hat{\tau}_\mu$ are not compatible now:

$$\begin{aligned}\bar{D}_\mu\bar{h}^{\rho\sigma} &= 2\upsilon^\rho\upsilon^\sigma\bar{D}_\mu\Phi - 8\mathcal{K}_{\mu\lambda}\bar{h}^{\lambda(\rho}\upsilon^{\sigma)}\Phi\,, \\ \bar{D}_\mu\hat{h}^{\rho\sigma} &= -4\mathcal{K}_{\mu\lambda}\bar{h}^{\lambda(\rho}\upsilon^{\sigma)}\Phi\,, \\ \bar{D}_\mu\hat{\tau}_\nu &= -2\mathcal{K}_{\mu\nu}\Phi\,.\end{aligned} \tag{3.101}$$

The action can then be rewritten as

$$\begin{aligned}S = \int d^dx\, e\,\Big[\bar{\mathcal{R}} + \frac{1}{2}a^\mu a_\mu + \frac{1}{2}\mathfrak{e}^\mu\mathfrak{e}_\mu - 4a^\mu\bar{D}_\mu\phi + 4\bar{D}^\mu\phi\bar{D}_\mu\phi + 4\upsilon^\mu\upsilon^\nu\bar{D}_\mu\phi\,\bar{D}_\nu\Phi \\ -8\Phi\mathcal{K}\upsilon^\rho\bar{D}_\rho\phi - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + \frac{1}{2}b^{\mu\nu}\upsilon^\rho H_{\mu\nu\rho}\Big]\,,\end{aligned} \tag{3.102}$$

where now the tensors with upper indices are defined using $\bar{h}^{\mu\nu}$ and we will always write any expression such that no derivatives act on $\bar{h}^{\mu\nu}$, e.g. $\bar{D}^{\mu}\phi = \bar{h}^{\mu\nu}\bar{D}_{\nu}\phi \neq \bar{D}_{\nu}\left(\bar{h}^{\mu\nu}\phi\right)$. The resulting equations of motion for the geometric fields are

$$\bar{\mathcal{R}}_{\mu\nu}\upsilon^{\mu}\upsilon^{\nu} = \frac{1}{4}\upsilon^{\rho}\upsilon^{\sigma}H^{\mu\nu}{}_{\rho}H_{\mu\nu\sigma} - 2\upsilon^{\mu}\upsilon^{\nu}\bar{D}_{\mu}\bar{D}_{\nu}\phi\,, \tag{3.103}$$

$$\begin{aligned}\bar{D}^{\rho}F_{\rho\mu} + a^{\rho}F_{\rho\mu} - 2\bar{D}^{\rho}\phi\,F_{\rho\mu} = \frac{1}{2}b^{\rho\sigma}H_{\rho\sigma\mu} + \mathfrak{e}^{\rho}b_{\mu\rho} + 2\bar{D}^{\rho}\mathcal{K}_{\rho\mu} - 2\mathcal{K}\bar{D}_{\mu}\Phi \\ + 2\upsilon^{\rho}\bar{D}_{\mu}\bar{D}_{\rho}\Phi - 4\upsilon^{\rho}\bar{D}_{\rho}\phi\,D_{\mu}\Phi\,,\end{aligned} \tag{3.104}$$

$$\begin{aligned}\bar{\mathcal{R}}_{(\mu\nu)} + 2\bar{D}_{(\mu}\bar{D}_{\nu)}\phi - \frac{1}{4}H_{\mu}{}^{\rho\sigma}H_{\nu\rho\sigma} = \frac{a_{\mu}a_{\nu} - \mathfrak{e}_{\mu}\mathfrak{e}_{\nu}}{2} + D_{(\mu}a_{\nu)} - F_{(\mu}{}^{\rho}\mathcal{K}_{\nu)\rho} \\ + \upsilon^{\rho}b_{(\mu}{}^{\sigma}H_{\nu)\rho\sigma} + 4\Phi\upsilon^{\rho}\left(\bar{D}_{\rho}\mathcal{K}_{\mu\nu} - \mathcal{K}\,\mathcal{K}_{\mu\nu} + \frac{\bar{D}_{\rho}\Phi}{\Phi}\mathcal{K}_{\mu\nu}\right)\,.\end{aligned} \tag{3.105}$$

The equations for the matter fields are

$$\begin{aligned}\bar{D}_{\mu}\bar{D}^{\mu}\phi + a^{\mu}\bar{D}_{\mu}\phi - 2\bar{D}_{\mu}\phi\bar{D}^{\mu}\phi = 2\upsilon^{\mu}\upsilon^{\nu}D_{\mu}\Phi D_{\nu}\phi - \frac{1}{2}H^{\rho\mu\nu}H_{\rho\mu\nu} \\ + \frac{1}{2}\mathfrak{e}^{\mu}\mathfrak{e}_{\mu} + \frac{1}{2}\upsilon^{\rho}b^{\mu\nu}H_{\rho\mu\nu}\,,\end{aligned} \tag{3.106}$$

$$\bar{D}^{\rho}b_{\rho\mu} - 2\bar{D}^{\rho}\phi\,b_{\rho\mu} = \frac{1}{2}F^{\rho\sigma}H_{\rho\sigma\mu} + 2\upsilon^{\rho}\bar{D}^{\sigma}\Phi\,H_{\rho\sigma\mu}\,, \tag{3.107}$$

$$\begin{aligned}\bar{D}^{\rho}H_{\rho\mu\nu} + a^{\rho}H_{\rho\mu\nu} - 2\bar{D}^{\rho}\phi\,H_{\rho\mu\nu} = 2b_{[\mu}{}^{\rho}\mathcal{K}_{\nu]\rho} - \mathcal{K}b_{\mu\nu} \\ + \upsilon^{\rho}\left(\bar{D}_{\rho}b_{\mu\nu} - 2\bar{D}_{\rho}\phi\,b_{\mu\nu} + 2\upsilon^{\sigma}\bar{D}_{\sigma}\Phi\,H_{\rho\mu\nu}\right)\,.\end{aligned} \tag{3.108}$$

The ‘missing’ equation is

$$\bar{D}^{\mu}\bar{D}_{\mu}\Phi - a^{\mu}\bar{D}_{\mu}\Phi - 2\bar{D}^{\mu}\phi\,\bar{D}_{\mu}\Phi = \frac{1}{4}\left(F^{\mu\nu}F_{\mu\nu} - b^{\mu\nu}b_{\mu\nu}\right)\,. \tag{3.109}$$

As in the case of TNC, we note that the equations (3.103)-(3.108) are manifestly invariant under transformations corresponding to the Carroll algebra. We also recall the following definitions:

$$\begin{aligned}a_\mu &= \upsilon^\rho F_{\rho\mu} = 2\upsilon^\rho \partial_{[\rho}\hat{\tau}_{\mu]}\,,\\ \mathfrak{e}_\mu &= \upsilon^\rho b_{\rho\mu} = 2\upsilon^\rho \partial_{[\rho}\aleph_{\mu]}\,,\\ \mathcal{K}_{\mu\nu} &= -\frac{1}{2}\mathcal{L}_\upsilon h_{\mu\nu} = -\frac{1}{2}\left(\upsilon^\rho\partial_\rho h_{\mu\nu} + (\partial_\mu \upsilon^\rho)\, h_{\rho\nu} + (\partial_\nu \upsilon^\rho)\, h_{\mu\rho}\right)\,,\end{aligned} \tag{3.110}$$

which have the physical interpretation of the acceleration field, the electric field and the extrinsic curvature of the geometry.

3.4 Stringy Newton-Cartan geometry

3.4.1 Basics

In [28, 29, 98–100] a non-relativistic string theory was formulated, which was then found to correspond to a target space geometry called Stringy Newton-Cartan (SNC) [1, 7, 11, 31, 101, 102]. A D-dimensional SNC spacetime naturally splits into two *longitudinal* directions and $D-2$ transverse directions, that are mapped to each other by means of *string* Galilean boosts. Recent works have delved deeper into the quantum aspects of such non-relativistic string theory, in particular studying the Weyl symmetry and computing the beta functions, that are required to vanish for the theory to be anomaly-free [9, 77].

The basic geometric fields are the longitudinal and transverse vielbeins, τ_μ^A, υ_A^μ, $E_\mu^{A'}$, $E_{A'}^\mu$ where the index A runs over the two longitudinal directions and A' runs over the remaining transverse directions. Using the transverse vielbeins we can build, as usual, two tensors $h^\perp_{\mu\nu}$ and $h^{\mu\nu}$. These fields satisfy the following completeness relations

$$h^{\mu\rho}h^\perp_{\rho\nu} - \upsilon^\mu_A \tau^A_\nu = \delta^\mu_\nu, \qquad \upsilon^\mu_A \tau^B_\mu = -\delta^B_A, \qquad \tau^A_\mu h^{\mu\rho} = \upsilon^\mu_A h^\perp_{\mu\rho} = 0\,, \tag{3.111}$$

and transform under string boosts with parameter $\Sigma_A^{A'}$ as

$$\delta_\Sigma \upsilon^\mu_A = -E^\mu_{A'}\Sigma^{A'}_A, \qquad \delta_\Sigma E^{A'}_\mu = -\tau^A_\mu \Sigma^{A'}_A, \qquad \delta E^\mu_{A'} = \delta_\Sigma \tau^A_\mu = 0\,. \tag{3.112}$$

The metric of the longitudinal space is $\eta_{AB} = \mathrm{diag}\,(-1, 1)$ and our convention for the longitudinal Levi-Civita symbol is $\epsilon_{01} = +1$.

Furthermore, we can introduce a Z_A symmetry in the theory via the gauge field m_μ^A, which transforms under boosts as

$$\delta_\Sigma m_\mu^A = E_\mu^{A'} \Sigma_{A'}^A \,. \tag{3.113}$$

This allows us to build the following manifestly boost invariant quantities:

$$\begin{aligned} \bar{h}_{\mu\nu} &= h^\perp_{\mu\nu} + 2\eta_{AB} m^A_{(\mu}\tau^B_{\nu)} \,, \\ u^\mu_A &= \upsilon^\mu_A + h^{\mu\rho} m_{\rho A} \,, \\ \Phi^{AB} &= 2u^{\rho(A} m_\rho{}^{B)} - h^{\mu\nu} m^A_\mu m^B_\nu = -u^{\mu A} u^{\nu B} \bar{h}_{\mu\nu} \,, \end{aligned} \tag{3.114}$$

which satisfy

$$h^{\mu\rho}\bar{h}_{\rho\nu} - u^\mu_A \tau^A_\nu = \delta^\mu_\nu, \qquad u^\mu_A \tau^B_\mu = -\delta^B_A, \qquad u^\mu_A \bar{h}_{\mu\rho} = \Phi_{AB}\tau^B_\rho \,. \tag{3.115}$$

The Z_A transformations of the fields are

$$\begin{aligned} \delta_Z m^A_\mu &= D_\mu \sigma^A \,, \\ \delta_Z \bar{h}_{\mu\nu} &= 2\eta_{AB} \tau^{AB}_{(\mu} D_{\nu)} \sigma^B \,, \\ \delta_Z u^\mu_A &= h^{\mu\rho} D_\rho \sigma^A \,, \\ \delta_Z \Phi^{AB} &= 2u^{\rho(A} D_\rho \sigma^{B)} \,. \end{aligned} \tag{3.116}$$

Note that at this point this transformation is a symmetry of SNC only if the foliation constraint is imposed. However, one can avoid imposing this constraint by requiring the B-field to transform as well [10], the full Z_A transformations are then given by the upcoming (3.149).

Using these fields it is also possible to construct the following boost-invariant connection:

$$\begin{aligned} \Gamma^\rho_{\mu\nu} &= -u^\rho_A \left(\partial_\mu \tau^A_\nu + \omega_\mu \epsilon^A{}_B \tau^B_\nu\right) + \frac{1}{2} h^{\rho\sigma}\left(\partial_\mu \bar{h}_{\nu\sigma} + \partial_\nu \bar{h}_{\mu\sigma} - \partial_\sigma \bar{h}_{\mu\nu}\right) \\ &= -u^\rho_A \nabla_\mu \tau^A_\nu + \frac{1}{2} h^{\rho\sigma}\left(\partial_\mu \bar{h}_{\nu\sigma} + \partial_\nu \bar{h}_{\mu\sigma} - \partial_\sigma \bar{h}_{\mu\nu}\right) \,, \end{aligned} \tag{3.117}$$

where we introduced the spin connection $\omega_\mu{}^{AB} \equiv \omega_\mu \epsilon^{AB}$, associated with the longitudinal covariant derivative ∇_μ. The connection (3.117) is compatible with $h^{\mu\nu}$ and τ^A_μ and has an antisymmetric component

$$2\Gamma^\rho_{[\mu\nu]} \equiv -u^\rho_A F^A_{\mu\nu}, \tag{3.118}$$

where we defined the 'torsion' tensors as[11]

$$F^A_{\mu\nu} \equiv 2\partial_{[\mu}\tau_{\nu]}{}^A + 2\epsilon^A{}_B\omega_{[\mu}\tau_{\nu]}{}^B = 2\nabla_{[\mu}\tau^A_{\nu]}\,. \tag{3.119}$$

We can also decompose these tensors as

$$F^A_{\mu\nu} = f^A\epsilon_{BC}\tau^B_\mu\tau^C_\nu + 2a_{[\mu}{}^{BA}\tau_{\nu]B} + \widetilde{F}^A_{\mu\nu}\,, \tag{3.120}$$

where we defined the acceleration $a_{\mu AB}$, the temporal part of torsion f^A and the transverse torsion tensor $\widetilde{F}^A_{\mu\nu}$ as

$$u^\rho_A F_{\rho\mu B} \equiv a_{\mu AB}\,, \qquad u^{\mu A}a_\mu{}^{BC} = \epsilon^{AB}f^C\,, \qquad u^{\mu A}\widetilde{F}_{\mu\nu B} = 0\,. \tag{3.121}$$

Finally, we define the extrinsic curvature in the usual way:

$$\mathcal{K}^A_{\mu\nu} = -\frac{1}{2}\mathcal{L}_u\bar{h}_{\mu\nu}\,, \tag{3.122}$$

which satisfies in particular

$$\begin{aligned} \mathcal{K}^A_{\mu\nu}h^{\mu\nu} &= -D_\mu u^{\mu A}, \\ h^{\mu\nu}h^{\rho\sigma}\mathcal{K}^A_{\mu\rho}\mathcal{K}^B_{\nu\sigma} &= D_\mu u^{\nu A}D_\nu u^{\mu B} + \frac{1}{4}h^{\mu\nu}h^{\rho\sigma}F_{\mu\nu C}F_{\rho\sigma D}\Phi^{AC}\Phi^{BD}. \end{aligned} \tag{3.123}$$

We should also mention that our notation for SNC in this chapter is somewhat different from the one usually found in the literature. We have adapted our notation so that it will be easier to match and compare results with TNC and Carroll. In the next chapter we will conform our notation and conventions to the more standard ones.

3.4.2 Embedding in DFT

To find the embedding of SNC in DFT we will use the same approach used for TNC [12], namely we will compare the SNC worldsheet action [9],

$$\begin{aligned} S_{SNC} = &-\frac{1}{2}\int d^2\sigma\left[e\,\partial X^\mu\partial X^\nu\,\bar{h}_{\mu\nu} + \epsilon^{\alpha\beta}\left(\lambda e_\alpha\tau_\mu + \bar{\lambda}\bar{e}_\alpha\bar{\tau}_\mu\right)\partial_\beta X^\mu\right] \\ &-\frac{1}{2}\int d^2\sigma\,\varepsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu B_{\mu\nu}\,, \end{aligned} \tag{3.124}$$

[11] Note that $F^A_{\mu\nu} = 0$ is the so called foliation constraint.

with the worldsheet action written in terms of DFT fields[12],

$$\begin{aligned} S_{DFT} = & -\frac{1}{2}\int d^2\sigma \left[e\,\partial X^\mu \partial X^\nu\, K_{\mu\nu} + 2\epsilon^{\alpha\beta}\left(\beta_{\alpha a}x^a_\mu + \bar\beta_{\alpha\bar a}\bar x^{\bar a}_\mu\right)\partial_\beta X^\mu\right] \\ & -\frac{1}{2}\int d^2\sigma \left[\varepsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \mathcal{B}_{\mu\nu} - 2\gamma^{\alpha\beta}\left(\beta_{\alpha a}x^a_\mu - \bar\beta_{\alpha\bar a}\bar x^{\bar a}_\mu\right)\partial_\beta X^\mu\right]\,. \end{aligned} \tag{3.125}$$

To make the identifications needed to define the SNC embedding we need to rewrite (3.124) in a more suitable way. In particular there are two issues we need to address:

1. We need to have a term proportional to $\gamma^{\alpha\beta}\lambda\partial_\beta X^\mu$ and $\gamma^{\alpha\beta}\bar\lambda\partial_\beta X^\mu$ to be able to properly identity the null eigenvectors of DFT;

2. We need to make sure that $K_{\mu\nu}$ is a degenerate matrix, which is not the case if we naively identify $K_{\mu\nu} = \bar h_{\mu\nu}$.

The first issue is simply solved by the following field redefinition:

$$A_\alpha = \frac{1}{2}(\lambda-\bar\lambda)e^0_\alpha + \frac{1}{2}(\lambda+\bar\lambda)e^1_\alpha = \frac{1}{2}\lambda e_\alpha - \frac{1}{2}\bar\lambda\bar e_\alpha\,. \tag{3.126}$$

In terms of these new Lagrange multipliers, the SNC action becomes

$$\begin{aligned} S_{SNC} = & -\frac{1}{2}\int d^2\sigma \left[e\,\partial X^\mu \partial X^\nu\, \bar h_{\mu\nu} + 2\epsilon^{\alpha\beta}\tau^1_\mu A_\alpha \partial_\beta X^\mu\right] \\ & -\frac{1}{2}\int d^2\sigma \left[\epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu} + 2e\gamma^{\alpha\beta}\tau^0_\mu A_\alpha \partial_\beta X^\mu\right]\,, \end{aligned} \tag{3.127}$$

This form of the action makes it easy for us to identify the null eigenvectors of DFT, however we still have to solve problem (2), i.e. the fact that $K_{\mu\nu} = \bar h_{\mu\nu}$ is not a degenerate matrix. To solve this issue we can make use of the

[12] We define $e_\alpha \equiv e^0_\alpha + e^1_\alpha$, $\bar e_\alpha \equiv e^0_\alpha - e^1_\alpha$, and similarly for τ and $\bar\tau$. Notice that $\gamma^{\alpha\beta} = \eta^{ab}e^\alpha_a e^\beta_b = -e^\alpha_0 e^\beta_0 + e^\alpha_1 e^\beta_1 = -4e^{(\alpha}\bar e^{\beta)}$ where $e^\alpha \equiv \frac{1}{2}(e^\alpha_0 + e^\alpha_1)$, $\bar e^\alpha \equiv \frac{1}{2}(e^\alpha_0 - e^\alpha_1)$, such that $e^\alpha e_\alpha = 1 = \bar e^\alpha \bar e_\alpha$ and $e^\alpha \bar e_\alpha = 0$. Moreover we have $\varepsilon^{\alpha\beta} = e\epsilon^{ab}e^\alpha_a e^\beta_b$ with $\varepsilon^{01} = -\varepsilon_{01} = +1$ and $e^\alpha_a = e^{-1}\varepsilon^{\alpha\beta}e^b_\beta \varepsilon_{ba}$, which implies $e^\alpha = -\frac{1}{2e}\varepsilon^{\alpha\beta}\bar e_\beta$ and $\bar e^\alpha = \frac{1}{2e}\varepsilon^{\alpha\beta}e_\beta$. Note that our convention for the worldsheet Levi-Civita symbol $\varepsilon^{\alpha\beta}$ is the opposite of the one for the longitudinal SNC Levi-Civita symbol ϵ^{AB}.

Stueckelberg symmetry of the SNC action. The action (3.124) is invariant under the following transformations with parameters C^A [10, 11]:

$$\delta\bar{h}_{\mu\nu} = 2C^A_{(\mu}\tau^B_{\nu)}\eta_{AB}\,, \qquad \delta B_{\mu\nu} = -2C^A_{[\mu}\tau^B_{\nu]}\epsilon_{AB}\,, \tag{3.128}$$

$$\lambda' = \lambda + e^{-1}\epsilon^{\alpha\beta}\bar{e}_\alpha\bar{C}_\beta\,, \qquad \bar{\lambda}' = \bar{\lambda} + e^{-1}\epsilon^{\alpha\beta}e_\alpha C_\beta\,, \tag{3.129}$$

where $C_\mu = C^0_\mu + C^1_\mu$ and $\bar{C}_\mu = C^0_\mu - C^1_\mu$. This means we can rewrite the action in terms of $h^\perp_{\mu\nu}$ and $\bar{B}_{\mu\nu}$ by choosing $C^A_\mu = -m^A_\mu$. After redefining the Lagrange multipliers one more time we arrive at the action

$$\begin{aligned} S_{SNC} = &-\frac{1}{2}\int d^2\sigma\left[e\,\partial X^\mu\partial X^\nu\, h^\perp_{\mu\nu} + 2\epsilon^{\alpha\beta}\tau^1_\mu A_\alpha\partial_\beta X^\mu\right] \\ &-\frac{1}{2}\int d^2\sigma\left[\epsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\bar{B}_{\mu\nu} + 2e\gamma^{\alpha\beta}\tau^0_\mu A_\alpha\partial_\beta X^\mu\right], \end{aligned} \tag{3.130}$$

where now all the tensors appearing in (3.130) are manifestly Z_A invariant. This allows us to make the following identifications:

$$K_{\mu\nu} = h^\perp_{\mu\nu}, \qquad \mathcal{B}_{\mu\nu} = \bar{B}_{\mu\nu}\,, \tag{3.131}$$

and

$$\begin{aligned} \left(\beta_\alpha x_\mu + \bar{\beta}_\alpha\bar{x}_\mu\right) &= \left(\tau_\mu + \bar{\tau}_\mu\right)A_\alpha = 2\tau^0_\mu A_\alpha\,, \\ \left(\beta_\alpha x_\mu - \bar{\beta}_\alpha\bar{x}_\mu\right) &= -\left(\tau_\mu - \bar{\tau}_\mu\right)A_\alpha = -2\tau^1_\mu A_\alpha\,. \end{aligned} \tag{3.132}$$

This is solved by (note the mismatch of bars between DFT and SNC fields)

$$\begin{aligned} x_\mu &= \frac{1}{\sqrt{2}}\bar{\tau}_\mu\,, \qquad & \beta_\alpha &= \frac{1}{\sqrt{2}}A_\alpha\,, \\ \bar{x}_\mu &= \frac{1}{\sqrt{2}}\tau_\mu\,, \qquad & \bar{\beta}_\alpha &= \frac{1}{\sqrt{2}}A_\alpha\,. \end{aligned} \tag{3.133}$$

By requiring $x_\mu y^\mu = 1 = \bar{x}_\mu\bar{y}^\mu$, we find

$$y^\mu_a = -\sqrt{2}\,\bar{\upsilon}^\mu, \qquad \bar{y}^\mu_{\bar{a}} = -\sqrt{2}\,\upsilon^\mu\,. \tag{3.134}$$

Moreover we have

$$H^{\mu\nu} = h^{\mu\nu} \tag{3.135}$$

and

$$Z_\mu^\nu = \tau_\mu\upsilon^\nu - \bar{\tau}_\mu\bar{\upsilon}^\nu\,. \tag{3.136}$$

In summary, stringy Newton-Cartan can be embedded in DFT via

$$K_{\mu\nu} = h^{\perp}_{\mu\nu}\,, \qquad H^{\mu\nu} = h^{\mu\nu}\,, \tag{3.137}$$
$$\mathcal{B}_{\mu\nu} = B_{\mu\nu}\,, \qquad Z_{\mu}^{\nu} = -\tau_{\mu}\upsilon^{\nu} + \bar{\tau}_{\mu}\bar{\upsilon}^{\nu}\,, \tag{3.138}$$

and the eigenvectors are

$$\begin{aligned} x_{\mu} &= \frac{1}{\sqrt{2}}\bar{\tau}_{\mu}\,, \qquad & y^{\mu} &= -\sqrt{2}\,\bar{\upsilon}^{\mu}\,, \\ \bar{x}_{\mu} &= \frac{1}{\sqrt{2}}\tau_{\mu}\,, \qquad & \bar{y}^{\mu} &= -\sqrt{2}\,\upsilon^{\mu}\,. \end{aligned} \tag{3.139}$$

An issue with this parametrization is that it does not include the gauge field m^{A}_{μ}. To reinstate it we can once again make a Stueckelberg transformation (or a shift transformation (3.15) from the point of view of DFT):

$$\begin{aligned} (y^{\mu})' &= y^{\mu} + H^{\mu\rho}V_{\rho}\,, \\ (\bar{y}^{\mu})' &= \bar{y}^{\mu} + H^{\mu\rho}\bar{V}_{\rho}\,, \\ (K_{\mu\nu})' &= K_{\mu\nu} - 2x_{(\mu}K_{\nu)\rho}V^{\rho} - 2\bar{x}_{(\mu}K_{\nu)\rho}\bar{V}^{\rho} + \left(x_{\mu}V_{\rho} + \bar{x}_{\mu}\bar{V}_{\rho}\right)\left(x_{\nu}V^{\rho} + \bar{x}_{\nu}\bar{V}^{\rho}\right)\,, \\ (\mathcal{B}_{\mu\nu})' &= \mathcal{B}_{\mu\nu} - 2x_{[\mu}V_{\nu]} + 2\bar{x}_{[\mu}\bar{V}_{\nu]} + 2x_{[\mu}\bar{x}_{\nu]}\left(y^{\rho}\bar{V}_{\rho} + \bar{y}^{\rho}V_{\rho} + V_{\rho}\bar{V}^{\rho}\right)\,, \end{aligned} \tag{3.140}$$

where $V_{\mu}, \bar{V}_{\mu}$ are two arbitrary local parameters and we set $V^{\rho} \equiv H^{\rho\mu}V_{\mu}$ for brevity. The choice $V_{\mu} = \frac{1}{2}m_{\mu}, \bar{V}_{\mu} = \frac{1}{2}\bar{m}_{\mu}$ gives

$$\begin{aligned} (y^{\mu})' &= -\bar{\upsilon}^{\mu} + \frac{1}{2}m^{\mu}\,, \\ (\bar{y}^{\mu})' &= -\upsilon^{\mu} + \frac{1}{2}\bar{m}^{\mu}\,, \\ (K_{\mu\nu})' &= h_{\mu\nu} - \frac{1}{2}\bar{\varphi}\tau_{\mu}\tau_{\nu} - \frac{1}{2}\varphi\bar{\tau}_{\mu}\bar{\tau}_{\nu} - \mathcal{T}\tau_{(\mu}\bar{\tau}_{\nu)}\,, \\ (\mathcal{B}_{\mu\nu})' &= B_{\mu\nu} + \mathcal{T}\tau_{[\mu}\bar{\tau}_{\nu]}\,, \end{aligned} \tag{3.141}$$

with

$$\begin{aligned} \varphi &\equiv 2\bar{u}^{\mu}m_{\mu} + \frac{1}{2}m^{2}\,, \\ \bar{\varphi} &\equiv 2u^{\mu}\bar{m}_{\mu} + \frac{1}{2}\bar{m}^{2}\,, \\ \mathcal{T} &\equiv u^{\mu}m_{\mu} + \bar{u}^{\mu}\bar{m}_{\mu} + \frac{1}{2}m\cdot\bar{m}\,, \end{aligned} \tag{3.142}$$

and

$$u^\mu \equiv \upsilon^\mu - \frac{1}{2} h^{\mu\rho} \bar{m}_\rho \,, \qquad \bar{u}^\mu \equiv \bar{\upsilon}^\mu - \frac{1}{2} h^{\mu\rho} m_\rho \,, \tag{3.143}$$

$$\bar{h}_{\mu\nu} = h^\perp_{\mu\nu} - \tau_{(\mu}\bar{m}_{\nu)} - \bar{\tau}_{(\mu} m_{\nu)} \,, \qquad B_{\mu\nu} = \bar{B}_{\mu\nu} + \tau_{[\mu}\bar{m}_{\nu]} - \bar{\tau}_{[\mu} m_{\nu]} \,. \tag{3.144}$$

Changing back from lightcone coordinates we finally find

$$\begin{aligned} K_{\mu\nu} &= \bar{h}_{\mu\nu} + \Phi_{AB}\tau_\mu^A \tau_\nu^B \,, \\ H^{\mu\nu} &= h^{\mu\nu} \,, \\ Z^\mu_\nu &= -\epsilon_{AB} u^{\mu A} \tau_\nu^B \,, \\ \mathcal{B}_{\mu\nu} &= B_{\mu\nu} - \frac{1}{2}\Phi \epsilon_{AB}\tau_\mu^A \tau_\nu^B \,, \end{aligned} \tag{3.145}$$

where we defined $\Phi \equiv \Phi_A^A$.

3.4.3 Action and equations of motion

Given an arbitrary tensor $A_{\mu\nu\ldots}$ we will define

$$A^{\mu\nu\ldots} \equiv A_{\rho\sigma\ldots} h^{\mu\rho} h^{\nu\sigma} \ldots \tag{3.146}$$

The only SNC fields which naturally have upper *curved* indices are u^μ_A and $h^{\mu\nu}$, hence any other tensor with upper indices is to be understood as defined via (3.146). Similarly, we will use the longitudinal vielbeins τ and u do interchange curved and flat longitudinal indices, i.e. given an arbitrary tensor A we have for example $A^A{}_B = A_\mu{}^\nu u^{\mu A} \tau_{\nu B}$.

Using the parametrization (3.145) we find that the SNC action is given by

$$\begin{aligned} S = \int d^D x \, e \, \Big[&\mathcal{R} - a^{\mu AB}(a_{\mu(AB)} - \frac{1}{2}\eta_{AB} a_\mu) + (a^\mu - 2D^\mu\phi)(a_\mu - 2D_\mu\phi) \\ &- \frac{1}{2} F^{\mu\nu A} F^B_{\mu\nu}(\Phi_{AB} - \frac{1}{2}\eta_{AB}\Phi) + \frac{1}{2}\epsilon_{AB} u^{\rho A} F^{\mu\nu B} H_{\rho\mu\nu} - \frac{1}{12} H^{\rho\mu\nu} H_{\rho\mu\nu} \Big] \,, \end{aligned} \tag{3.147}$$

where we defined $a_\mu \equiv a_{\mu A}{}^A$, $\Phi \equiv \Phi_A^A$ and the invariant measure is given by

$$e \equiv e^{-2\phi} \sqrt{\frac{\det \bar{h}}{\det \Phi}} . \tag{3.148}$$

This action is invariant under the Z_A transformations

$$\delta u^{\mu A} = h^{\mu\nu} D_\nu \sigma^A , \qquad \delta\Phi^{AB} = 2u^{\mu(A} D_\mu \sigma^{B)} \,. \tag{3.149}$$

The equations of motion can be found by varying the action with respect to the independent fields $h^{\mu\nu}$, u^μ_A and Φ_{AB}[13], also see Appendix B.3 for useful identities. Sometimes it will be useful to further decompose the acceleration in its antisymmetric and traceless symmetric components:

$$a_\mu{}^{AB} = \mathcal{S}^{AB}_\mu + \frac{1}{2}\eta^{AB} a_\mu + \epsilon^{AB}\mathcal{A}_\mu \,, \tag{3.150}$$

with $\mathcal{S}_\mu{}^{AB}\eta_{AB} = 0$ and we recall $a_\mu = a_\mu{}^{AB}\eta_{AB}$.

Taking the variation of (3.147) with respect to Φ_{AB} we find the equation[14]

$$F^{\mu\nu A} F_{\mu\nu A} = 0 \,, \tag{3.151}$$

which we will use when writing the remaining equations.

The space projection of the equation of motion for u^μ_A is

$$D^\rho F^\mu{}_{\rho A} + a_{\rho A}{}^B F^{\mu\rho}{}_B - 2F^\mu{}_{\rho A} D^\rho \phi = \frac{1}{2}\epsilon_{AB} F^{\rho\sigma B} H^\mu{}_{\rho\sigma} \,, \tag{3.152}$$

while the time projection is

$$D^\rho \mathcal{S}_\rho{}^{AB} + a^\rho \mathcal{S}_\rho{}^{AB} - 2\mathcal{S}^{\rho AB} D_\rho \phi = -\frac{1}{4} F^{\mu\nu A} F^B_{\mu\nu}\Phi + 2\epsilon^{(A}{}_C \mathcal{F}^{B)C} \,, \tag{3.153}$$

where we defined

$$\mathcal{F}^{BC} \equiv -\mathcal{A}^\rho \mathcal{S}_\rho{}^{BC} + \frac{1}{4} u^{\rho(B} F^{C)}_{\mu\nu} H_\rho{}^{\mu\nu} \,. \tag{3.154}$$

The equation for the B-field is

$$D^\rho H_\rho{}^{\mu\nu} + a^\rho H_\rho{}^{\mu\nu} - 2D^\rho\phi \, H_\rho{}^{\mu\nu} = 2\Omega^{\mu\nu} + \epsilon_{AB}\mathcal{H}^{\mu\nu AB} \,, \tag{3.155}$$

[13] Note that, despite appearances, the action does not depend on the spin connection, i.e. $\delta S/\delta\omega_\mu = 0$.

[14] The action actually only depends on the *trace* of the tensor Φ_{AB}, as can be seen by expanding the Ricci scalar and noticing that the $F^{\mu\nu A} F_{\mu\nu}{}^B \Phi_{AB}$ terms cancel out. This is why the corresponding equation is a singlet and not a tensor.

where we defined

$$\begin{aligned}\mathcal{H}^{\mu\nu AB} \equiv 2u^{\rho A}D_\rho F^{\mu\nu B} - 2u^{\rho A}F^{\mu\nu B}D_\rho\phi - 2D^{[\mu}a^{\nu]AB} - 2\mathcal{S}^{\mu A}{}_C\mathcal{S}^{\nu BC} \\ - \mathcal{K}^A F^{\mu\nu B} + 2F^{[\mu}{}_{\rho C}F^{\nu]\rho A}\Phi^{BC}\,,\end{aligned} \tag{3.156}$$

as well as $\Omega_{\mu\nu} \equiv 2\partial_{[\mu}\omega_{\nu]}$ and $\mathcal{K}^A \equiv h^{\mu\nu}\mathcal{K}^A_{\mu\nu}$.

The dilaton equation is

$$\begin{aligned}D^\mu D_\mu\phi + 2a_\mu D^\mu\phi - 2D^\mu\phi\, D_\mu\phi = \frac{1}{2}D^\mu a_\mu + \frac{1}{2}a_\mu a^\mu \\ + \frac{1}{4}\epsilon_{AB}u^{\rho A}F^{\mu\nu B}H_{\rho\mu\nu} - \frac{1}{12}H^{\rho\mu\nu}H_{\rho\mu\nu}\,.\end{aligned} \tag{3.157}$$

The space-space projection of Einstein's equations is

$$\begin{aligned}\mathcal{R}^{(\mu\nu)} + 2D^{(\mu}D^{\nu)}\phi - \frac{1}{4}H^\mu{}_{\rho\sigma}H^{\nu\rho\sigma} = u^\rho_A D^{(\mu}F_\rho{}^{\nu)A} + \mathcal{S}^{\mu AB}\mathcal{S}^\nu{}_{AB} \\ - \epsilon^{AB}u^\sigma{}_A F^{\rho(\mu}{}_B H^{\nu)}{}_{\rho\sigma} - \frac{1}{2}F^{\mu\rho A}F^\nu{}_{\rho A}\Phi\,.\end{aligned} \tag{3.158}$$

The time-space projection is

$$\begin{aligned}h^{\mu\rho}u^{\sigma A}\mathcal{R}_{(\rho\sigma)} + 2u^{\rho A}D^\mu D_\rho\phi + a^\mu u^{\rho A}D_\rho\phi - \frac{1}{4}H^\mu{}_{\rho\sigma}H_\lambda{}^{\rho\sigma}u^{\lambda A} = \frac{1}{2}u^{\rho A}u^\sigma{}_B D_\rho F_\sigma{}^{\mu B} \\ - u^\rho{}_B D^\mu a_\rho{}^{(AB)} + u^{\rho A}D^\mu a_\rho - \frac{1}{2}u^{\rho B}D_\rho a^\mu_{BA} + \frac{1}{2}u^{\rho A}D_\rho a^\mu \\ - 2a^{\mu[AB]}u^\rho{}_B D_\rho\phi - a^\mu u^{\rho A}D_\rho\phi - \frac{1}{2}a^\mu\mathcal{K}^A + \frac{1}{2}a^{\mu BA}\mathcal{K}_B \\ - F^\mu{}_{\rho C}a^{\rho B(A}\Phi_B{}^{C)} + a^\rho F^\mu{}_{\rho C}\Phi^{AC} + \frac{1}{2}a^\rho F_\rho{}^{\mu A}\Phi + \frac{1}{2}F^{\mu\rho B}D_\rho\Phi^A{}_B \\ - \frac{1}{2}F^{\mu\rho A}D_\rho\Phi + \epsilon^{AB}\mathcal{P}^\mu{}_B\,,\end{aligned} \tag{3.159}$$

where

$$\begin{aligned}\mathcal{P}^{\mu B} = -\frac{1}{2}u^{\sigma B}D^\rho H^\mu{}_{\rho\sigma} + u^{\rho B}H^\mu{}_{\rho\sigma}D^\sigma\phi + a^\rho u^{\lambda B}H^\mu{}_{\rho\lambda} + \frac{1}{2}F^{\mu\rho C}u^{\sigma B}u^\lambda_C H_{\rho\sigma\lambda} \\ - \frac{1}{2}a^{\rho CB}u^\sigma{}_C H^\mu{}_{\rho\sigma} - D^\mu f^B - \frac{1}{2}a^\mu f^B + a^{\mu BC}f_C + \frac{1}{2}a^{\mu CB}f_C + a^{\mu BC}f_C\,.\end{aligned} \tag{3.160}$$

Finally the time-time component (found directly from DFT) is given by

$$u^{\mu A}u^{\nu}{}_{A}\mathcal{R}_{\mu\nu}+2u^{\mu A}u^{\nu}{}_{A}D_{\mu}D_{\nu}\phi-\frac{1}{4}u^{\mu A}u^{\nu}{}_{A}H_{\mu\rho\sigma}H_{\nu}{}^{\rho\sigma}=\frac{3}{2}\mathcal{S}^{\rho AB}D_{\rho}\Phi_{AB}$$
$$-\frac{3}{4}a^{\mu}D_{\mu}\Phi-\frac{3}{4}a^{\mu}a_{\mu}\Phi+\mathcal{S}^{\mu AB}\mathcal{S}_{\mu AC}\Phi_{B}^{C}-2f^{A}f_{A}+\epsilon^{AB}\mathcal{Q}_{AB}\,, \tag{3.161}$$

where

$$\mathcal{Q}_{AB}=-\frac{1}{2}u^{\rho}{}_{A}u^{\sigma}{}_{B}D^{\lambda}H_{\rho\sigma\lambda}+u^{\mu}{}_{A}u^{\nu}{}_{B}H_{\mu\nu\rho}D^{\rho}\phi+u^{\mu}{}_{A}u^{\nu}{}_{B}D_{\mu}\mathcal{A}_{\nu}+2f_{A}u^{\mu}_{B}D_{\mu}\phi$$
$$+\frac{3}{2}u^{\mu}_{A}D_{\mu}f_{B}-u^{\mu}_{A}u^{\nu C}\mathcal{S}^{\rho}{}_{CB}H_{\mu\nu\rho}-a^{\rho}H_{AB\rho}+f_{A}\mathcal{K}_{B}+3\mathcal{A}^{\mu}\mathcal{S}_{\mu AC}\Phi_{B}^{C}$$
$$-\frac{1}{4}F^{\mu\nu}{}_{A}u^{\rho C}H_{\rho\mu\nu}\Phi_{BC}-\frac{1}{4}F^{\mu\nu}{}_{B}u^{\rho}{}_{A}H_{\rho\mu\nu}\Phi+\frac{1}{4}F^{\mu\nu C}u^{\rho}{}_{A}H_{\rho\mu\nu}\Phi_{BC}\,. \tag{3.162}$$

We remind the reader of the definitions of some of the fields appearing in these equations:

$$\begin{aligned}F_{\mu\nu}{}^{A}&=2\nabla_{[\mu}\tau_{\nu]}{}^{A}\equiv f^{A}\epsilon_{BC}\tau_{\mu}{}^{B}\tau_{\nu}{}^{C}+2a_{[\mu}{}^{BA}\tau_{\nu]B}+\widetilde{F}_{\mu\nu}{}^{A}\,,\\ a_{\mu}{}^{AB}&=u^{\rho A}F_{\rho\mu}{}^{B}\equiv\mathcal{S}_{\mu}{}^{AB}+\frac{1}{2}\eta^{AB}a_{\mu}+\epsilon^{AB}\mathcal{A}_{\mu}.\end{aligned} \tag{3.163}$$

These equations are clearly more complicated than the ones found for TNC or Carroll, as there is no obvious way to combine the time and space projections of an equation into a single simpler equation. Despite this technical difficulty, it is possible to show that these equations are invariant under the SNC algebra as expected. In the next section we will discuss these equations and their relations to previous results.

3.5 Comparison with known results

3.5.1 TNC beta functions

In the previous chapter we computed the beta functions for string theory describing a Type I TNC target space. Let us recall the results. The equations obtained by setting those beta functions to zero were the two twistlessness contraints

$$F_{\mu\nu} = a_\mu \tau_\nu - \tau_\mu a_\nu \,, \tag{3.164}$$
$$b_{\mu\nu} = \mathfrak{e}_\mu \tau_\nu - \tau_\mu \mathfrak{e}_\nu \,, \tag{3.165}$$

two scalar equations

$$D \cdot a + a^2 = 2\mathfrak{e}^2 + 2\left(a \cdot D\phi\right) \,, \tag{3.166}$$
$$D \cdot \mathfrak{e} = 2\left(\mathfrak{e} \cdot D\phi\right) \,, \tag{3.167}$$

and two tensor equations

$$\begin{aligned}\mathcal{R}_{(\mu\nu)} - \frac{1}{4} H^{\rho\sigma}{}_{(\mu} H_{\nu)\rho\sigma} + 2D_{(\mu} D_{\nu)}\phi = \frac{\mathfrak{e}^2\left(2\Phi\tau_\mu\tau_\nu - \bar{h}_{\mu\nu}\right) - \mathfrak{e}_\mu \mathfrak{e}_\nu}{2} \\ + \left(\Delta_S\right)^\rho_{(\mu} D_{\nu)} a_\rho - \mathfrak{e}^\sigma \left(\Delta_T\right)^\rho_{(\mu} H_{\nu)\rho\sigma} + \frac{a_\mu a_\nu - 2a^2\,\Phi\tau_\mu\tau_\nu}{2} \,,\end{aligned} \tag{3.168}$$

$$\begin{aligned}D^\rho H_{\rho\mu\nu} + a^\rho H_{\rho\mu\nu} - 2H^\rho{}_{\mu\nu} D_\rho \phi = 2\left(\Delta_S\right)^\rho_{[\mu} D_{\nu]}\mathfrak{e}_\rho - 2\left(\Delta_T\right)^\rho_{[\mu} D_{\nu]}\mathfrak{e}_\rho - 2a_{[\mu}\mathfrak{e}_{\nu]} \\ + \left(2\hat{\upsilon}^\rho D_\rho\phi - D_\rho \upsilon^\rho\right) b_{\mu\nu} \,,\end{aligned} \tag{3.169}$$

where we remind the reader of the definitions of the projectors $\left(\Delta_T\right)^\mu_\nu \equiv -\hat{\upsilon}^\mu \tau_\nu$ and $\left(\Delta_S\right)^\mu_\nu \equiv h^{\mu\rho}\bar{h}_{\rho\nu}$, which satisfy

$$\left(\Delta_S\right)^\mu_\nu + \left(\Delta_T\right)^\mu_\nu = \delta^\mu_\nu \,. \tag{3.170}$$

We now want to compare these equations with the ones obtained from DFT. First of all, notice that in the previous chapter twistless torsion was assumed from the start, which explains the difference between the equations (3.164)-(3.165) and (3.52). To compare the rest of the equations, we impose the twistlessness constraints in equations (3.53)-(3.56) and (3.64). We then find two scalar equations

$$D \cdot a + a^2 = \mathfrak{e}^2 + 2\left(a \cdot D\phi\right) \,, \tag{3.171}$$
$$D \cdot \mathfrak{e} = 2\left(\mathfrak{e} \cdot D\phi\right) \,, \tag{3.172}$$

and two tensor equations

$$\mathcal{R}_{(\mu\nu)} - \frac{1}{4}H^{\rho\sigma}{}_{(\mu}H_{\nu)\rho\sigma} + 2D_{(\mu}D_{\nu)}\phi = \frac{2\mathfrak{e}^2\Phi\tau_\mu\tau_\nu - \mathfrak{e}_\mu\mathfrak{e}_\nu}{2} - \frac{2a^2\,\Phi\tau_\mu\tau_\nu - a_\mu a_\nu}{2} + (\Delta_S)^{\rho}_{(\mu}\,D_{\nu)}a_\rho - \mathfrak{e}^\sigma\,(\Delta_T)^{\rho}_{(\mu}\,H_{\nu)\rho\sigma}\,, \tag{3.173}$$

$$D^\rho H_{\rho\mu\nu} + a^\rho H_{\rho\mu\nu} - 2H^\rho{}_{\mu\nu}D_\rho\phi = 2\,(\Delta_S)^{\rho}_{[\mu}\,D_{\nu]}\mathfrak{e}_\rho - 2\,(\Delta_T)^{\rho}_{[\mu}\,D_{\nu]}\mathfrak{e}_\rho - 2a_{[\mu}\mathfrak{e}_{\nu]} + (2\hat{\upsilon}^\rho D_\rho\phi - D_\rho\upsilon^\rho)\,b_{\mu\nu}\,, \tag{3.174}$$

where in the last equation we also used the Bianchi identity for $b_{\mu\nu}$. The main difference between these two sets of equations is the factor in front of $\mathfrak{e}^2$ in (3.166) and (3.171). The further difference in equations (3.168) and (3.173) arise precisely because of this factor of 2, since one needs to use the scalar equations to prove that the remaining equations are $U(1)_m$ invariant. In other words, the form of Einstein's equations depends on the factor in front of $\mathfrak{e}^2$ in the scalar equation, because of the requirement of $U(1)_m$ invariance, also see upcoming equations (3.175) and (3.176). Hence it looks like the DFT and beta functions computations produce almost the same result with the difference being a factor of two in (3.166) and (3.171). It is worth noting that this difference goes away when we consider torsionless geometries, since in that case we have $\mathfrak{e}_\mu = 0$ and worldsheet beta functions and Double Field Theory give rise to the same set of equations in the target space.

Despite this difference, both sets of equations describe spacetimes with the same symmetries. In fact, we could generalize them further by modifying equations (3.171) and (3.173) by introducing an arbitrary constant λ[15]:

$$D\cdot a + a^2 = (1+\lambda)\mathfrak{e}^2 + 2\,(a\cdot D\phi)\ , \tag{3.175}$$

$$\mathcal{R}_{(\mu\nu)} - \frac{1}{4}H^{\rho\sigma}{}_{(\mu}H_{\nu)\rho\sigma} + 2D_{(\mu}D_{\nu)}\phi = \frac{\mathfrak{e}^2\left(2\Phi\tau_\mu\tau_\nu - \lambda\,\bar{h}_{\mu\nu}\right) - \mathfrak{e}_\mu\mathfrak{e}_\nu}{2} - \frac{2a^2\,\Phi\tau_\mu\tau_\nu - a_\mu a_\nu}{2} + (\Delta_S)^{\rho}_{(\mu}\,D_{\nu)}a_\rho - \mathfrak{e}^\sigma\,(\Delta_T)^{\rho}_{(\mu}\,H_{\nu)\rho\sigma}\,, \tag{3.176}$$

while keeping the other equations unchanged. These equations are $U(1)_m$

[15]Note that this arbitraryness is a consequence of the fact that we are considering twistless torsion. For generic torsion the value of λ is fixed to be $\lambda = 0$ by symmetry.

invariant for any choice of the constant λ[16] (although recall that this is only true if we impose twistless torsion) and they are written in terms of manifestly boost-invariant quantities. Therefore, they describe theories with extended Galilean symmetry. The only difference between the theories with $\lambda = 0$ and $\lambda = 1$ is that the former can be found by a variational principle starting from the action (3.50), while the latter can be found by requiring vanishing of Weyl anomaly in the string embedding.

Finally, note that to arrive at the parametrization (3.46)-(3.47) one needs to perform field redefinitions involving both the B-field and $\aleph$. While such field redefinitions are allowed at the classical level, it is not clear whether this will produce the same actions at the quantum level as the path integral measure may, in principle, transform as well. This question is beyond the scope of this work and should be addressed separately in future.

3.5.2 SNC beta functions

In [9, 77] the beta functions for a worldsheet with SNC target space have been computed. They imposed the so called *foliation constraint* through their computation:

$$\nabla_{[\mu}\tau_{\nu]}{}^{A} = 0 = F_{\mu\nu}{}^{A}. \tag{3.177}$$

If we impose this geometrical constraint, the equations (3.152)-(3.162) become much simpler:

$$\begin{aligned}
D^{\rho}H_{\rho}{}^{\mu\nu} - 2D^{\rho}\phi\, H_{\rho}{}^{\mu\nu} &= 0,\\
D^{\mu}D_{\mu}\phi - 2D^{\mu}\phi\, D_{\mu}\phi &= -\frac{1}{12}H^{\rho\mu\nu}H_{\rho\mu\nu},\\
\mathcal{R}^{(\mu\nu)} + 2D^{(\mu}D^{\nu)}\phi - \frac{1}{4}H^{\mu}{}_{\rho\sigma}H^{\nu\rho\sigma} &= 0,\\
\mathcal{R}^{(\mu A)} + 2u^{\rho A}D^{\mu}D_{\rho}\phi - \frac{1}{4}H^{\mu}{}_{\rho\sigma}H_{\lambda}{}^{\rho\sigma}u^{\lambda A} &= \frac{1}{2}\epsilon^{A}{}_{B}u^{\sigma B}\left(D^{\rho}H^{\mu}{}_{\rho\sigma} - 2H^{\mu}{}_{\rho\sigma}D^{\rho}\phi\right),\\
\mathcal{R}^{A}_{A} + 2u^{\nu}{}_{A}D^{A}D_{\nu}\phi - \frac{1}{4}H_{A\rho\sigma}H^{A\rho\sigma} &= -\frac{1}{2}\epsilon^{AB}\left(D^{\lambda}H_{AB\lambda} - 2H_{AB\lambda}D^{\lambda}\phi\right),
\end{aligned} \tag{3.178}$$

[16] To see this one needs to use (B.10) and (B.11) to show that the variation of $\mathcal{R}_{(\mu\nu)}$ contains the term $\tau_{(\mu}D_{\nu)}\sigma\, D\cdot a = \lambda\mathfrak{e}^2\tau_{(\mu}D_{\nu)}\sigma + \dots$, where we are ignoring terms not proportional to λ. Notice that this cancels the variation $-\frac{1}{2}\lambda\mathfrak{e}^2\delta\bar{h}_{\mu\nu}$ coming from the r.h.s. of (3.176). It is also easy to see that the remaining terms proportional to λ are independently zero since the $U(1)_m$ variation of $\mathfrak{e}^2$ is zero in the twistless case.

where in the first equation we used the fact that the foliation constraint implies[17]

$$\Omega_{\mu\nu} = u^{\rho}_{A}\tau^{A}_{[\mu}\Omega_{\nu]\rho} \qquad \Longrightarrow \qquad \Omega^{\mu\nu} = 0\,. \tag{3.179}$$

Note that the connection used in the present work is in general different from the one used in the literature, however they are actually equal when the foliation constraint is imposed. The connection used in [9, 77] is

$$\bar{\Gamma}^{\rho}_{\mu\nu} = \frac{1}{2}h^{\rho\sigma}\left(\partial_{\mu}\bar{h}_{\nu\sigma} + \partial_{\nu}\bar{h}_{\mu\sigma} - \partial_{\sigma}\bar{h}_{\mu\nu}\right) + \frac{1}{2}u^{\rho A}u^{\sigma}_{A}\left(\partial_{\mu}\tau_{\nu\sigma} + \partial_{\nu}\tau_{\mu\sigma} - \partial_{\sigma}\tau_{\mu\nu}\right)\,, \tag{3.180}$$

where $\tau_{\mu\nu} \equiv \tau^{A}_{\mu}\tau_{\nu A}$. One can check that when $F^{A}_{\mu\nu} = 0$ we have

$$\Gamma^{\rho}_{\mu\nu} = \bar{\Gamma}^{\rho}_{\mu\nu}\,. \tag{3.181}$$

This means that the equations (3.178) are exactly the same as the ones that are obtained by requiring the cancellation of the Weyl anomaly in the worldsheet theory.

3.5.3 Comparison between TNC and SNC equations of motion

It was found in [10] that, under certain assumptions which we review below, the SNC worldsheet action reduces to that of TNC. It is then natural to ask whether the equations of motion of SNC also reduce to the ones of TNC. The basic condition under which SNC reduces to TNC is the presence of an isometry along a compact longitudinal direction. We can then split the SNC spacetime directions as $m = (\mu, u)$, where u is the compact direction and μ will describe the TNC directions. Then we impose the following gauge choice on the SNC fields:

$$m^{A}_{M} = 0\,, \qquad \tau^{0}_{\nu} = 0\,, \qquad \tau^{1}_{u} = 1\,, \qquad E^{A'}_{u} = 0\,. \tag{3.182}$$

Substituting this ansatz in the equations of motion would be quite tedious. Luckily, since the equations of motion are obtained from an action, which in turn is obtained from the generalized metric of DFT, it will suffice to compare the generalized metrics rather than the equations or the actions.

[17] Note that this is a sort of ”twistlelss condition” on the field strength of ω_{μ}.

Comparing (3.46)-(3.47) with (3.145) we see that the two parametrizations are indeed the same once we impose the following identifications:

$$\begin{aligned} \upsilon_1^\mu &= 0\,, & \upsilon_1^u &= 1\,, \\ \upsilon_0^\mu &= \upsilon^\mu\,, & \upsilon_0^u &= \upsilon^\mu \aleph_\mu\,, \\ h^{\mu\nu}_{(SNC)} &= h^{\mu\nu}_{(TNC)}, & h^{\mu u}_{(SNC)} &= h^{\mu\rho}\aleph_\rho\,, \\ h^{uu}_{(SNC)} &= h^{\rho\sigma}\aleph_\rho\aleph_\sigma\,, & B^{(SNC)}_{\mu\nu} &= B^{(TNC)}_{\mu\nu}\,, \\ B^{(SNC)}_{\mu u} &= -m_\mu\,. \end{aligned} \tag{3.183}$$

In addition, it is not hard to check that the invariant measure of SNC (3.148) correctly reduces to the one of TNC (3.51).

This identification implies that the SNC equations of motion will match the ones obtained for TNC, at least in the DFT formulation. Unfortunately, the SNC beta functions have only been computed under the assumption that the foliation constraint is satisfied[18], and similarly the TNC beta functions have been computed assuming twistless torsion, so we cannot compare the full SNC and TNC equations obtained from DFT with the ones obtained by setting the beta functions to zero. However we already showed that when ${F_{\mu\nu}}^A = 0$ the SNC equations (3.178) do indeed match the ones computed in [9, 77]. Using the identification between the SNC and TNC generalized metrics, this also implies that the SNC beta functions with ${F_{\mu\nu}}^A = 0$ match the TNC beta functions with $F_{\mu\nu} = 0$.

[18]The SNC beta functions without the requirement of the foliation constraint were also computed, but only at the linearized level.

Chapter 4

Non-relativistic membranes

Having studied non-relativistic particles and strings, we now turn to the last natural step of this thesis: non-relativistic membranes. To do this, we will use an approach similar to the one used for Stringy Newton Cartan, namely we will expand the basic fields appearing in the eleven-dimensional supergravity action for large c and then send $c \to \infty$. This will yield an action for eleven-dimensional *non-relativistic* supergravity, together with the corresponding equations of motion. We will also analyze this low-energy limit of M-theory through the framework of Exceptional Field Theory (ExFT), a generalization of Double Field Theory which we introduced in the previous chapter.

The eleven-dimensional theory we construct has a number of interesting features:

- *Membrane Newton-Cartan geometry (see section 4.1.1).* The geometry has three 'longitudinal' and eight 'transverse' directions, which we can describe in terms of an eleven-dimensional Newton-Cartan metric structure. This appears by taking the eleven-dimensional metric and its inverse to have the form

$$\begin{aligned} \hat{g}_{\mu\nu} &= c^2 \eta_{AB} \tau_\mu{}^A \tau_\nu{}^B + c^{-1} H_{\mu\nu} \,, \\ \hat{g}^{\mu\nu} &= c H^{\mu\nu} + c^{-2} \eta^{AB} \tau^\mu{}_A \tau^\nu{}_B \,, \end{aligned} \tag{4.1}$$

where $A = 0, 1, 2$ labels the *longitudinal* Newton-Cartan vielbeins, or clock forms, $\tau_\mu{}^A$, and $H^{\mu\nu}$ and $\tau^\mu{}_A$ are projective inverses obeying the Newton-Cartan completeness relations

$$H^{\mu\rho} H_{\rho\nu} + \tau^\mu{}_A \tau_\nu{}^A = \delta^\mu_\nu \,, \quad H^{\mu\nu} \tau_\nu{}^A = 0 = H_{\mu\nu} \tau^\nu{}_A \,, \quad \tau^\mu{}_A \tau_\mu{}^B = \delta^B_A \,. \tag{4.2}$$

We also expand the three-form as

$$\hat{C}_{\mu\nu\rho} = -c^3 \epsilon_{ABC} \tau_\mu{}^A \tau_\nu{}^B \tau_\rho{}^C + C_{\mu\nu\rho} + c^{-3} \widetilde{C}_{\mu\nu\rho} \,. \tag{4.3}$$

Here c is a dimensionless parameter whose $c \to \infty$ limit can be interpreted as a non-relativistic limit. It is the geometry that results from this limit that we refer to as membrane Newton-Cartan. The powers of c in (4.1), along with the leading order power in (4.3), follow the pattern of the powers of the harmonic function in the M2 brane supergravity solution (see (4.180)). Note that c is *not* the speed of light, although it is related to it, see discussion after (4.114). (The minus sign in the c^3 term in (4.3) is a choice of convention, and matches with e.g. expressions in the SNC literature on dimensional reduction [16].)

- *Transverse self-duality (see section 4.1.2).* Requiring singular terms to cancel in the $c \to \infty$ limit requires that the finite part $F_{\mu\nu\rho\sigma} = 4\partial_{[\mu} C_{\nu\rho\sigma]}$ of the four-form field strength obey a *self-duality constraint* in the eight-dimensional transverse space. This is a consequence of the presence of the Chern-Simons term in the eleven-dimensional action.

- *Dual degrees of freedom (see section 4.1.4).* The subleading part $\widetilde{C}_{\mu\nu\rho}$ of the three-form in the expansion appears in the dynamics with its equation of motion imposing the self-duality constraint. The anti-self-dual transverse projection of the field strength $\widetilde{F}_{\mu\nu\rho\sigma} = 4\partial_{[\mu} \widetilde{C}_{\nu\rho\sigma]}$ of this subleading part can be identified with the totally longitudinal part of the seven-form field strength dual to $F_{\mu\nu\rho\sigma}$. Hence the non-relativistic limit involves what would normally be physical and dual degrees of freedom, however, rather than being related to each other as would usually be the case, these degrees of freedom get reorganised into separately self- and anti-self-dual parts.

- *Dilatation invariance and a 'missing' equation of motion (see section 4.2.2).* The eleven-dimensional theory is invariant under a 'dilatation' symmetry which scales each field with a weight inherited from the power of c that accompanies them in the initial expansion. This is an 'emergent' local symmetry [16] and it has the effect of removing a variational degree of freedom when we vary the finite part of the action. Hence, at this order, there is a 'missing' equation of motion. As we have seen in the previous chapter, this is a familiar feature of non-relativistic theories, with the naively missing equation corresponding to the Poisson equation for the Newtonian gravitational potential. However, we can identify this missing equation by looking at the next order in the $1/c$ expansion [16, 25, 27, 103]. Indeed, here we identify this missing equation by extracting it from the dilatation variation of the action at the next subleading order. In parallel with the situation in the DFT description of the NSNS sector that we saw in the previous chapter, we also find it directly from the equations of motion of the exceptional field theory description.

- *Boost invariance (see section 4.2.3).* The eleven-dimensional theory is also invariant under Galilean boost transformations of the form

$$\begin{aligned} \delta H_{\mu\nu} &= 2\Lambda_{(\mu}{}^{A}\tau_{\nu)A}\,, \\ \delta \tau^{\mu}{}_{A} &= -H^{\mu\nu}\Lambda_{\nu A}\,, \\ \delta C_{\mu\nu\rho} &= -3\epsilon_{ABC}\Lambda_{[\mu}{}^{A}\tau_{\nu}{}^{B}\tau_{\rho]}{}^{C}\,, \end{aligned} \tag{4.4}$$

  where the (infinitesimal) boost parameter $\Lambda_{\mu}{}^{A}$ satisfies $\tau^{\mu}{}_{A}\Lambda_{\mu}{}^{B} = 0$. The slightly unusual feature here is the transformation of the three-form itself. This transformation (4.4) is to be expected based on similar observations in the case of stringy Newton-Cartan. There one can either introduce additional one-form gauge fields transforming under boosts, and treat the two-form gauge field as invariant, or else absorb the former into the latter via a sort of Stueckelberg gauge fixing [10,16]. We do not introduce additional one-forms and so generalise this second picture.

- *Reduction to type IIA SNC (see section 4.3.1).* Reduction of the theory on a longitudinal isometry direction produces the full type IIA SNC theory, coupling the known NSNS sector to RR fields. This is the same theory that we studied in the previous chapter (after setting the Z_A generator to zero).

- *Reduction to type IIA D2NC (see section 4.3.2)* Reduction of the theory on a transverse isometry directions produces a novel type IIA non-relativistic theory, that can be associated to D2 branes rather than strings.

- *Exceptional field theory embedding and U-duality (see section 4.4).* Finally, the eleven-dimensional MNC theory can be very naturally embedded within exceptional field theory (which also manifestly breaks

Lorentz invariance and treats original and dual degrees of freedom together), demonstrating that the same exceptional Lie algebraic structures that appear in the relativistic theory are preserved by the limit. Furthermore, we can easily use ExFT to study transformations between relativistic and non-relativistic geometries, and to obtain equations of motion which are otherwise missing from the action of the non-relativistic theory. The achievement of ExFT is to present a unified treatment of both eleven- and ten-dimensional supergravities in which $E_{d(d)}$ symmetry is manifest. The metric and gauge field degrees of freedom are reorganised into $E_{d(d)}$ multiplets. For instance, the wholly d-dimensional components of the metric and three-form (and possibly also of the dual six-form) appear in a *generalised metric*. For the cases $d = 3, 4$, this has an expression

$$\mathcal{M}_{MN} = |\hat{g}|^{1/(9-d)} \begin{pmatrix} \hat{g}_{ij} + \frac{1}{2}\hat{C}_i{}^{pq}\hat{C}_{jpq} & \hat{C}_i{}^{kl} \\ \hat{C}_k{}^{ij} & 2\hat{g}^{i[k}\hat{g}^{l]j} \end{pmatrix} . \tag{4.5}$$

If we adopt the same expansion as in equation (4.1), then in the limit $c \to \infty$, we obtain an alternative *non-relativistic* or *non-Riemannian* parametrisation[1]

$$\mathcal{M}_{MN} = \Omega^{\frac{2}{9-d}} \begin{pmatrix} \mathcal{M}_{ij} & \mathcal{M}_i{}^{kl} \\ \mathcal{M}_k{}^{ij} & \mathcal{M}^{ijkl} \end{pmatrix} , \tag{4.6}$$

with

$$\begin{aligned} \mathcal{M}_{ij} &= H_{ij} - \epsilon_{ABC}\tau_{(i}{}^A C_{j)kl}\tau^{kB}\tau^{lC} + C_{ikl}C_{jmn}H^{km}\tau^{ln} , \\ \mathcal{M}_i{}^{kl} &= -\epsilon_{ABC}\tau_i{}^A\tau^{kB}\tau^{lC} + 2C_{ipq}H^{p[k}\tau^{l]q} , \\ \mathcal{M}^{ijkl} &= 2H^{i[k}\tau^{l]j} + 2\tau^{i[k}H^{l]j} , \end{aligned} \tag{4.7}$$

and where Ω is a measure factor, and $\tau^{ij} \equiv \tau^i{}_A\tau^j{}_B\eta^{AB}$. This alternative parametrisation then changes the nature of the duality relationships encoded by the dynamics of the generalised gauge fields of exceptional field theory. This allows the exceptional field theory formulation to automatically capture the interesting reorganisation of degrees of

[1]The flat space limit of (4.1) was already studied in exceptional field theory in [76], and the general non-Riemannian parametrisation of the SL(5) generalised metric worked out - this can be shown to be equivalent to (4.6). However a full analysis of the Newton-Cartan interpretation and dynamics was not carried out.

freedom implied by the non-relativistic limit. In addition, the missing equation of motion is associated to variations which do not preserve the non-relativistic nature of the parametrisation (4.6) of the generalised metric.

The outline of this chapter is very simple. In section 4.1 we carry out the expansion at the level of the bosonic action. In section 4.2 we discuss the equations of motion and symmetries. In section 4.3, we carry out dimensional reductions to type IIA. In section 4.4, we discuss the embedding in ExFT.

4.1 Membrane Newton-Cartan limit and eleven-dimensional SUGRA

4.1.1 Setting up the expansion

Metric We start by writing the eleven-dimensional metric and its inverse as

$$\hat{g}_{\mu\nu} = c^2 \tau_{\mu\nu} + c^{-1} H_{\mu\nu} \,, \quad \hat{g}^{\mu\nu} = c H^{\mu\nu} + c^{-2} \tau^{\mu\nu} \,. \tag{4.8}$$

We can view this simply as a field redefinition which introduces the eleven-dimensional Newton-Cartan variables alongside the (dimensionless) parameter c. We will seek to send c to infinity and interpret the result as a non-relativistic limit. In principle, we can also think of this ansatz as containing the first terms in an infinite expansion in c^{-3}, and we will occasionally allow such a perspective to influence our presentation. However, we leave the development of the full non-relativistic expansion to future work. To see that the field redefinition (4.8) makes sense in Newton-Cartan terms we look at the condition $\delta_\mu^\nu = \hat{g}_{\mu\rho}\hat{g}^{\rho\nu}$, which gives at order c^3, c^0 and c^{-3} respectively the following three conditions[2]:

$$\tau_{\mu\rho} H^{\rho\nu} = 0 \,, \quad \tau_{\mu\rho}\tau^{\rho\nu} + H_{\mu\rho} H^{\rho\nu} = \delta_\mu^\nu \,, \quad H_{\mu\rho}\tau^{\rho\nu} = 0 \,. \tag{4.9}$$

We view these as the defining conditions for $\tau_{\mu\nu}$, viewed as a longitudinal Newton-Cartan metric (of Lorentzian signature), and $H^{\mu\nu}$, viewed as the corresponding orthogonal transverse Newton-Cartan metric (of Euclidean

[2]Note that the last of these identity heavily relies on the fact that we are truncating the expansion of the metric at order c^{-2}. If we included more subleading terms this identity would not hold anymore.

signature).[3] Letting $A = 0,1,2$ and $a = 1,\dots 8$ denote longitudinal and transverse flat indices, respectively, we can introduce projective vielbeins such that

$$\tau_{\mu\nu} \equiv \tau_\mu{}^A \tau_\nu{}^B \eta_{AB}\,, \quad \tau^{\mu\nu} \equiv \tau^\mu{}_A \tau^\nu{}_B \eta^{AB}\,, \quad \tau^\mu{}_A \tau_\mu{}^B = \delta_A^B\,, \tag{4.10}$$

$$H^{\mu\nu} \equiv h^\mu{}_a h^\nu{}_b \delta^{ab}\,, \quad H_{\mu\nu} \equiv h^a{}_\mu h^b{}_\nu \delta_{ab}\,, \quad h^\mu{}_a h^b{}_\mu = \delta_a^b\,, \tag{4.11}$$

and hence obeying the Newton-Cartan completeness relations following from (4.9). Here η_{AB} is the flat three-dimensional Minkowski metric and δ_{ab} is the flat Euclidean eight-dimensional metric. We can then compute the determinant of the eleven-dimensional metric:

$$\begin{aligned} \det \hat{g}_{\mu\nu} &= -c^{-2}\Omega^2\,, \\ \Omega^2 &\equiv -\tfrac{1}{3!8!}\epsilon^{\mu_1\dots\mu_{11}}\epsilon^{\nu_1\dots\nu_{11}}\tau_{\mu_1\nu_1}\tau_{\mu_2\nu_2}\tau_{\mu_3\nu_3}H_{\mu_4\nu_4}\dots H_{\mu_{11}\nu_{11}}\,, \end{aligned} \tag{4.12}$$

where $\epsilon^{\mu_1\dots\mu_{11}}$ denotes the eleven-dimensional Levi-Civita symbol. Hence $\sqrt{-\hat{g}} = c^{-1}\Omega$ and it is Ω which will be used as the measure factor in the non-relativistic action. In terms of the vielbeins, we can write

$$\Omega = \left| \tfrac{1}{3!8!}\epsilon^{\mu\nu\rho\sigma_1\dots\sigma_8}\epsilon_{ABC}\epsilon_{a_1\dots a_8}\tau_\mu{}^A\tau_\nu{}^B\tau_\rho{}^C h^{a_1}{}_{\sigma_1}\dots h^{a_8}{}_{\sigma_8} \right| \tag{4.13}$$

and note that

$$\partial_\mu \ln\Omega = \tau^\nu{}_A \partial_\mu \tau_\nu{}^A + h^\nu{}_a \partial_\mu h^a{}_\nu\,. \tag{4.14}$$

We can obtain further useful identities by substituting the expressions (4.8) into contractions of the Levi-Civita symbol and the metric. One that we will use later is

$$\begin{aligned} n! H^{[\mu_1|\nu_1}\dots H^{|\mu_n]\nu_n} = &-\frac{\epsilon^{\mu_1\dots\mu_n\lambda_1\dots\lambda_{11-n}}\epsilon^{\nu_1\dots\nu_n\sigma_1\dots\sigma_{11-n}}}{3!(8-n)!\Omega^2} \\ &\times \tau_{\lambda_1\sigma_1}\dots\tau_{\lambda_3\sigma_3}H_{\lambda_4\sigma_4}\dots H_{\lambda_{11-n}\sigma_{11-n}}\,. \end{aligned} \tag{4.15}$$

Three-form For the three-form, let

$$\hat{C}_3 = C_3 - \tfrac{1}{6}c^3\epsilon_{ABC}\tau^A\wedge\tau^B\wedge\tau^C + c^{-3}\widetilde{C}_3\,, \tag{4.16}$$

[3] As in the stringy Newton-Cartan case, we could choose to include additional one-forms in the expansion (4.8), however these can be eliminated by a Stueckelberg gauge fixing [10,16].

so that

$$\hat{F}_4 = F_4 - \tfrac{1}{2}c^3\epsilon_{ABC}d\tau^A \wedge \tau^B \wedge \tau^C + c^{-3}\widetilde{F}_4 \,, \tag{4.17}$$

where

$$F_4 \equiv dC_3 \,, \quad \widetilde{F}_4 \equiv d\widetilde{C}_3 \,. \tag{4.18}$$

Although $\widetilde{C}_3$ is subleading, it will explicitly appear in the action and dynamics of the non-relativistic limit. Its equation of motion will impose a self-duality constraint on F_4, and we will be able to identify a certain projection of its field strength with the totally longitudinal components of the dual seven-form field strength. We can therefore interpret the subleading part of $\hat{C}_3$ as being 'dual' to the finite part. This is clearly a general fact: the Hodge star itself has an expansion in powers of c and so inevitably mixes up the terms at different powers of c in any p-form it acts on. What is non-trivial is that the Chern-Simons term of the eleven-dimensional theory will lead to both C_3 and $\widetilde{C}_3$ playing a role in the non-relativistic limit.

4.1.2 Expanding the action

The action for the eleven-dimensional metric and three-form is

$$S = \int \mathrm{d}^{11}x \left(\sqrt{|\hat{g}|} \left[\hat{R}(\hat{g}) - \tfrac{1}{48}\hat{F}^{\mu\nu\rho\sigma}\hat{F}_{\mu\nu\rho\sigma} \right] + \tfrac{\epsilon^{\mu_1\cdots\mu_{11}}}{144^2}\hat{F}_{\mu_1\ldots}\hat{F}_{\mu_5\ldots}\hat{C}_{\mu_9\mu_{10}\mu_{11}} \right) . \tag{4.19}$$

Here $\hat{F}_4 = d\hat{C}_3$. In form notation the Chern-Simons term is $\frac{1}{6}\hat{F}_4 \wedge \hat{F}_4 \wedge \hat{C}_3$, the equation of motion of the three-form is $d\hat{\star}\hat{F}_4 = \frac{1}{2}\hat{F}_4 \wedge \hat{F}_4$ and its Bianchi identity is $d\hat{F}_4 = 0$. The Hodge dual field strength is $\hat{F}_7 = \hat{\star}\hat{F}_4$, which obeys the Bianchi identity $d\hat{F}_7 = \frac{1}{2}\hat{F}_4 \wedge \hat{F}_4$ and the equation of motion $d\hat{\star}\hat{F}_7 = 0$.

Chern-Simons term We start with the expansion of the Chern-Simons term. Leaving wedge products implicit, we can simply compute

$$\begin{aligned} \tfrac{1}{6}\hat{F}_4\hat{F}_4\hat{C}_3 &= \tfrac{1}{6}F_4F_4C_3 - \tfrac{1}{6}(3c^3F_4F_4 + 6F_4\widetilde{F}_4)\tfrac{1}{6}\epsilon_{ABC}\tau^A\tau^B\tau^C \\ &\quad - \tfrac{1}{3}d\left(c^3F_4C_3\tfrac{1}{6}\epsilon_{ABC}\tau^A\tau^B\tau^C + \tfrac{1}{6}\epsilon_{ABC}\tau^A\tau^B\tau^C(F_4\widetilde{C}_3 + C_3\widetilde{F}_4) \right) + \mathcal{O}(c^{-3}) \,. \end{aligned} \tag{4.20}$$

We drop the total derivative.

Kinetic term for three-form First, let's write the component expression

$$\hat{F}_{\mu_1\mu_2\mu_3\mu_4} = -6c^3 T_{[\mu_1\mu_2}{}^A \tau_{\mu_3}{}^B \tau_{\mu_4]}{}^C \epsilon_{ABC} + F_{\mu_1\mu_2\mu_3\mu_4} + c^{-3}\widetilde{F}_{\mu_1\mu_2\mu_3\mu_4}\,, \tag{4.21}$$

where we introduce the Newton-Cartan torsion

$$T_{\mu\nu}{}^A \equiv 2\partial_{[\mu}\tau_{\nu]}{}^A\,. \tag{4.22}$$

Any term involving three $H^{\mu\nu}$ contracting the first term in (4.21) vanishes as one $H^{\mu\nu}$ must necessarily contract a $\tau_\mu{}^A$. As a result,

$$\begin{aligned}
&\sqrt{|\hat{g}|}\hat{g}^{\mu_1\mu_4}\dots\hat{g}^{\mu_4\nu_4}\hat{F}_{\mu_1\dots\mu_4}\hat{F}_{\nu_1\dots\nu_4} \\
&= \Omega\Big[c^3\left(H^{\mu_1\nu_1}\dots H^{\mu_4\nu_4}F_{\mu_1\mu_2\mu_3\mu_4}F_{\nu_1\nu_2\nu_3\nu_4} - 12H^{\mu_1\nu_1}H^{\mu_2\nu_2}T_{\mu_1\mu_2}{}^A T_{\nu_1\nu_2 A}\right) \\
&\quad - 24H^{\mu\nu}T_{\mu\rho}{}^A T_{\nu\sigma}{}^B \tau^\rho{}_A \tau^\sigma{}_B - 12H^{\mu_1\nu_1}H^{\mu_2\nu_2}F_{\mu_1\mu_2\mu_3\mu_4}T_{\nu_1\nu_2}{}^A \tau^{\mu_3 B}\tau^{\mu_4 C}\epsilon_{ABC} \\
&\qquad + 4H^{\mu_1\nu_1}H^{\mu_2\nu_2}H^{\mu_3\nu_3}\tau^{\mu_4\nu_4}F_{\mu_1\mu_2\mu_3\mu_4}F_{\nu_1\nu_2\nu_3\nu_4} \\
&\qquad\quad + 2H^{\mu_1\nu_1}\dots H^{\mu_4\nu_4}F_{\mu_1\mu_2\mu_3\mu_4}\widetilde{F}_{\nu_1\nu_2\nu_3\nu_4}\Big] + \mathcal{O}(c^{-3})\,.
\end{aligned} \tag{4.23}$$

Kinetic term/Chern-Simons cancellations and self-duality We now examine the $\mathcal{O}(c^3)$ terms in (4.20) and (4.23) which involve a field strength F_4, as well as the $\mathcal{O}(c^0)$ terms involving the subleading $\widetilde{F}_4$. These cannot possibly be cancelled by a contribution from the expansion of the Ricci scalar. The relevant terms are:

$$\begin{aligned}
&-\tfrac{1}{2\cdot 4!}\Omega H^{\mu_1\nu_1}H^{\mu_2\nu_2}H^{\mu_3\nu_3}H^{\mu_4\nu_4}F_{\mu_1\mu_2\mu_3\mu_4}(c^3F_{\nu_1\nu_2\nu_3\nu_4} + 2\widetilde{F}_{\nu_1\nu_2\nu_3\nu_4}) \\
&-\tfrac{1}{2\cdot 4!4!3!}\epsilon^{\mu_1\dots\dots\mu_{11}}F_{\mu_1\mu_2\mu_3\mu_4}(c^3F_{\mu_5\mu_6\mu_7\mu_8} + 2\widetilde{F}_{\mu_5\mu_6\mu_7\mu_8})\epsilon_{ABC}\tau_{\mu_9}{}^A\tau_{\mu_{10}}{}^B\tau_{\mu_{11}}{}^C\,.
\end{aligned} \tag{4.24}$$

To cancel the terms at order c^3, we are led to require the following constraint:

$$\Omega H^{\mu_1\nu_1}H^{\mu_2\nu_2}H^{\mu_3\nu_3}H^{\mu_4\nu_4}F_{\nu_1\nu_2\nu_3\nu_4} = -\tfrac{\epsilon^{\mu_1\dots\dots\mu_{11}}}{4!3!}F_{\mu_5\mu_6\mu_7\mu_8}\epsilon_{ABC}\tau_{\mu_9}{}^A\tau_{\mu_{10}}{}^B\tau_{\mu_{11}}{}^C\,. \tag{4.25}$$

This says that the totally transverse part of $F_{\mu\nu\rho\sigma}$ is self-dual (or anti-self-dual). This is self-consistent thanks to (4.15). We will refer to this as the self-duality constraint.

Three-form equation of motion As a sanity check that requiring the constraint (4.25) is sensible and necessary, let us at this point also take the limit at the level of the equation of motion of the three-form gauge field. We will revisit the equations of motion, including that of the metric, in more detail in section 4.2. For the three-form, we have originally:

$$\partial_\sigma(\sqrt{|\hat{g}|}\hat{g}^{\mu\lambda_1}\hat{g}^{\nu\lambda_2}\hat{g}^{\rho\lambda_3}\hat{g}^{\sigma\lambda_4}\hat{F}_{\lambda_1\ldots\lambda_4}) = \tfrac{1}{2\cdot 4!4!}\epsilon^{\mu\nu\rho\sigma_1\ldots\sigma_8}\hat{F}_{\sigma_1\ldots\sigma_4}\hat{F}_{\sigma_5\ldots\sigma_8}\,. \tag{4.26}$$

Inserting the expansion, one has firstly at $\mathcal{O}(c^3)$ that

$$\begin{aligned}\partial_\sigma\left(\Omega H^{\mu\lambda_1}H^{\nu\lambda_2}H^{\rho\lambda_3}\ H^{\sigma\lambda_4}F_{\lambda_1\ldots\lambda_4}\right)&\\ = -\tfrac{1}{3!4!}\epsilon^{\mu\nu\rho\sigma\sigma_1\ldots\sigma_7}&\partial_\sigma(F_{\sigma_1\ldots\sigma_4}\epsilon_{ABC}\tau_{\sigma_5}{}^A\tau_{\sigma_6}{}^B\tau_{\sigma_7}{}^C)\,,\end{aligned} \tag{4.27}$$

which is the duality relation (4.25) under a derivative.

At $\mathcal{O}(c^0)$ we have the finite equation of motion

$$\begin{aligned}&\partial_\sigma\Big(\Omega\big(4H^{[\mu|\lambda_1}H^{|\nu|\lambda_2}H^{|\rho|\lambda_3}\tau^{|\sigma]\lambda_4}F_{\lambda_1\ldots\lambda_4} - 6H^{[\mu|\lambda_1}H^{|\nu|\lambda_2}\tau^{|\rho|B}\tau^{|\sigma]C}T_{\lambda_1\lambda_2}{}^A\epsilon_{ABC}\\
&\qquad + H^{\mu\lambda_1}H^{\nu\lambda_2}H^{\rho\lambda_3}H^{\sigma\lambda_4}\widetilde{F}_{\lambda_1\ldots\lambda_4}\big)\Big)\\
&= \tfrac{1}{2\cdot 4!4!}\epsilon^{\mu\nu\rho\sigma_1\ldots\sigma_8}(F_{\sigma_1\ldots\sigma_4}F_{\sigma_5\ldots\sigma_8} - 12\epsilon_{ABC}T_{\sigma_1\sigma_2}{}^A\tau_{\sigma_3}{}^B\tau_{\sigma_4}{}^C\widetilde{F}_{\sigma_5\ldots\sigma_8})\,.\end{aligned} \tag{4.28}$$

This will be reproduced from the action that we find below.

Ricci scalar Now we come to the Ricci scalar. A very quick way to take the limit is to start with the explicit expression for the Ricci scalar in terms of the metric and its derivatives:

$$\begin{aligned}\hat{R} = \frac{1}{4}\hat{g}^{\mu\nu}\partial_\mu\hat{g}_{\rho\sigma}\partial_\nu\hat{g}^{\rho\sigma} - \frac{1}{2}\hat{g}^{\mu\nu}\partial_\nu\hat{g}^{\rho\sigma}\partial_\rho\hat{g}_{\mu\sigma}&\\
- \frac{1}{4}\hat{g}^{\mu\nu}\partial_\mu\ln\hat{g}\,\partial_\nu\ln\hat{g} - \hat{g}^{\mu\nu}\partial_\mu\partial_\nu\ln\hat{g} - \partial_\mu\ln\hat{g}\,\partial_\nu\hat{g}^{\mu\nu} - \partial_\mu\partial_\nu\hat{g}^{\mu\nu}&\,.\end{aligned} \tag{4.29}$$

Calculating the expansion is trivial. One has $\hat{R} = c^4 R^{(4)} + cR^{(0)} + \mathcal{O}(c^{-2})$ with

$$\begin{aligned} R^{(4)} &= \frac{1}{4} H^{\mu\nu} \partial_\nu H^{\rho\sigma} \partial_\mu \tau_{\rho\sigma} - \frac{1}{2} H^{\mu\nu} \partial_\nu H^{\rho\sigma} \partial_\rho \tau_{\mu\sigma} \,, \\ R^{(0)} &= \frac{1}{4} H^{\mu\nu} (\partial_\mu \tau_{\rho\sigma} \partial_\nu \tau^{\rho\sigma} + \partial_\mu H_{\rho\sigma} \partial_\nu H^{\rho\sigma}) + \frac{1}{4} \tau^{\mu\nu} \partial_\mu \tau_{\rho\sigma} \partial_\nu H^{\rho\sigma} \\ &\quad - \frac{1}{2} H^{\mu\nu} \partial_\nu \tau^{\rho\sigma} \partial_\rho \tau_{\mu\sigma} - \frac{1}{2} H^{\mu\nu} \partial_\nu H^{\rho\sigma} \partial_\rho H_{\mu\sigma} - \frac{1}{2} \tau^{\mu\nu} \partial_\nu H^{\rho\sigma} \partial_\rho \tau_{\mu\sigma} \\ &\quad - H^{\mu\nu} \partial_\mu \ln\Omega \, \partial_\nu \ln\Omega - 2H^{\mu\nu} \partial_\mu \partial_\nu \ln\Omega - 2\partial_\mu \ln\Omega \, \partial_\nu H^{\mu\nu} - \partial_\mu \partial_\nu H^{\mu\nu} \,. \end{aligned} \tag{4.30}$$

Recall that the measure $\sqrt{-\hat{g}}$ introduces a further power of c^{-1}. The singular piece can be easily rewritten as

$$R^{(4)} = -\tfrac{1}{2} H^{\mu\nu} H^{\rho\sigma} (\partial_\mu \tau_\rho{}^A \partial_\nu \tau_\sigma{}^B - \partial_\rho \tau_\mu{}^A \partial_\nu \tau_\sigma{}^B) \eta_{AB} = -\tfrac{1}{4} H^{\mu\nu} H^{\rho\sigma} T_{\mu\rho}{}^A T_{\nu\sigma A} \,. \tag{4.31}$$

This cancels exactly the remaining singular term appearing in the expansion (4.23) of the kinetic term for the three-form. An entirely similar cancellation appeared in the NSNS sector expansion of [16], and as noted there is reminiscent of what happens when taking the Gomis-Ooguri limit on the string worldsheet.

4.1.3 Result of expansion and covariant formulation

Action and constraint Combining (4.20), (4.23) and (4.30) we obtain the expansion of the eleven-dimensional SUGRA action in the form $S = c^3 S^{(3)} + c^0 S^{(0)} + \dots$. The singular part is:

$$\begin{aligned} S^{(3)} = - \int \mathrm{d}^{11}x \, \tfrac{1}{2\cdot 4!} F_{\mu_1\dots\mu_4} F_{\nu_1\dots\nu_4} \\ \times \left(\Omega H^{\mu_1\nu_1} \dots H^{\mu_4\nu_4} + \tfrac{1}{4!3!} \epsilon^{\mu_1\dots\mu_4\nu_1\dots\nu_7} \epsilon_{ABC} \tau_{\nu_5}{}^A \tau_{\nu_6}{}^B \tau_{\nu_7}{}^C \right) , \end{aligned} \tag{4.32}$$

and in order to have a good $c \to \infty$ limit, we impose the constraint

$$\Omega H^{\mu_1\nu_1} H^{\mu_2\nu_2} H^{\mu_3\nu_3} H^{\mu_4\nu_4} F_{\nu_1\nu_2\nu_3\nu_4} = -\tfrac{\epsilon^{\mu_1\dots\dots\mu_{11}}}{4!3!} F_{\mu_5\mu_6\mu_7\mu_8} \epsilon_{ABC} \tau_{\mu_9}{}^A \tau_{\mu_{10}}{}^B \tau_{\mu_{11}}{}^C \,, \tag{4.33}$$

to ensure that $S^{(3)}$ vanishes[4]. The finite part of the action is:

$$\begin{aligned} S^{(0)} = \int \mathrm{d}^{11}x\,\Omega\,\big[R^{(0)} &+ \tfrac{1}{2}H^{\mu\nu}T_{\mu\rho}{}^{A}T_{\nu\sigma}{}^{B}\tau^{\rho}{}_{A}\tau^{\sigma}{}_{B} \\ &- \tfrac{1}{12}H^{\mu_1\nu_1}H^{\mu_2\nu_2}H^{\mu_3\nu_3}\tau^{\mu_4\nu_4}F_{\mu_1\mu_2\mu_3\mu_4}F_{\nu_1\nu_2\nu_3\nu_4} \\ &+ \tfrac{1}{4}H^{\mu_1\nu_1}H^{\mu_2\nu_2}F_{\mu_1\ldots\mu_4}\epsilon_{ABC}T_{\nu_1\nu_2}{}^{A}\tau^{\mu_3 B}\tau^{\mu_4 C} \\ &- \tfrac{1}{\cdot 4!}\widetilde{F}_{\nu_1\nu_2\nu_3\nu_4}\big(H^{\mu_1\nu_1}H^{\mu_2\nu_2}H^{\mu_3\nu_3}H^{\mu_4\nu_4}F_{\mu_1\mu_2\mu_3\mu_4} \\ &\quad + \tfrac{1}{4!3!\Omega}\epsilon^{\nu_1\nu_2\nu_3\nu_4\mu_1\ldots\mu_7}F_{\mu_1\mu_2\mu_3\mu_4}\epsilon_{ABC}\tau_{\mu_5}{}^{A}\tau_{\mu_6}{}^{B}\tau_{\mu_7}{}^{C}\big)\big] \\ + \tfrac{1}{6}F_4 \wedge F_4 \wedge C_3\,, \end{aligned} \tag{4.34}$$

where $R^{(0)}$ is defined in (4.30). The equation of motion of $C_{\mu\nu\rho}$ gives exactly (4.28), and we will discuss the equations of motion of the Newton-Cartan fields in detail in section 4.2. The equation of motion of $\widetilde{C}_{\mu\nu\rho}$ is (4.27), giving the constraint under a derivative. Alternatively if we were just to take the action (4.34) at face value, forgetting about its origin via an expansion of the three-form, we could make the choice to view $\widetilde{F}_{\mu\nu\rho\sigma}$ as an independent field, serving as a Lagrange multiplier imposing the constraint in its form (4.33).

Symmetries The action is diffeomorphism invariant (as follows from the covariant rewriting we carry out below), as well as gauge invariant under $\delta C_3 = d\lambda_2$, $\delta\widetilde{C}_3 = d\widetilde{\lambda}_2$. The vielbeins $h^a{}_\mu$ and $\tau^A{}_\mu$ transform under SO(8) and SO(1,2) rotational symmetries respectively, which are also symmetries of the action. The non-relativistic theory is also invariant under Galilean boosts and a dilatation symmetry.

The Galilean boosts mix the longitudinal and transverse degrees of freedom. The parameter for such a boost is denoted $\Lambda_\mu{}^A$ such that $\tau^\mu{}_A\Lambda_\mu{}^B = 0$. We can give the (infinitesimal) action of these symmetries as

$$\begin{aligned} \delta_\Lambda H_{\mu\nu} &= 2\Lambda_{(\mu}{}^{A}\tau_{\nu)A}\,, \\ \delta_\Lambda \tau^\mu{}_A &= -H^{\mu\nu}\Lambda_{\nu A}\,, \\ \delta_\Lambda C_{\mu\nu\rho} &= -3\epsilon_{ABC}\Lambda_{[\mu}{}^{A}\tau_{\nu}{}^{B}\tau_{\rho]}{}^{C}\,. \end{aligned} \tag{4.35}$$

[4]Strictly speaking this is a sufficient condition for the vanishing of $S^{(3)}$, as we could alternatively integrate by parts and use (4.27). However the full constraint (4.33) will follow from the expansion of the metric equations of motion that we discuss in section 4.2.1, as well as in the expansion of the dual field strength discussed in section 4.1.4, and also follows directly from the exceptional field theory formulation of section 4.4.

The action $S^{(0)}$ is invariant under these transformations on using the self-duality constraint. One way for the action to be exactly invariant would be to treat $\widetilde{F}_{\mu\nu\rho\sigma}$ as an independent field transforming as

$$\delta_\Lambda \widetilde{F}_{\mu\nu\rho\sigma} = -4\Lambda_{[\mu}{}^A F_{\nu\rho\sigma]\lambda}\tau^\lambda{}_A\,, \tag{4.36}$$

or to have $\widetilde{C}_{\mu\nu\rho}$ transform in a way leading to this transformation.

The dilatations are meanwhile induced by the expansion in powers of c, with the dilatation weight of each field equal to the power of c which accompanies it in the expansion. The (infinitesimal) action of dilatations is hence:

$$\begin{aligned} \delta_\lambda H^{\mu\nu} &= +\lambda H^{\mu\nu}\,,\\ \delta_\lambda H_{\mu\nu} &= -\lambda H_{\mu\nu}\,,\\ \delta_\lambda \tau^\mu{}_A &= -\lambda \tau^\mu{}_A\,,\\ \delta_\lambda \tau_\mu{}^A &= +\lambda \tau_\mu{}^A\,,\\ \delta_\lambda C_{\mu\nu\rho} &= 0\,. \end{aligned} \tag{4.37}$$

Note $\delta\Omega = -\lambda\Omega$. For λ coordinate dependent this is a symmetry of the action $S^{(0)}$ on using the self-duality constraint (4.33). If we treat $\widetilde{F}_{\mu\nu\rho\sigma}$ as an independent field transforming as $\delta_\lambda\widetilde{F}_{\mu\nu\rho\sigma} = -3\lambda\widetilde{F}_{\mu\nu\rho\sigma}$, then the action $S^{(0)}$ is exactly invariant. We will explicitly verify the invariance of the action and study these symmetries in more detail in section 4.2.

Newton-Cartan connections and covariant rewriting The way we obtained the action (4.34) was by a straightforward computation at the level of the metric and three-form. To better understand the result, we rewrite the action in a covariant way by introducing the following connection

$$\Gamma^\rho_{\mu\nu} = \tau^\rho{}_A\partial_\mu\tau_\nu{}^A + \frac{1}{2}H^{\rho\sigma}\left(\partial_\mu H_{\sigma\nu} + \partial_\nu H_{\mu\sigma} - \partial_\sigma H_{\mu\nu}\right)\,, \tag{4.38}$$

whose covariant derivative we denote by ∇_μ. This satisfies the following metric compatibility conditions:

$$\nabla_\rho H^{\mu\nu} = 0 = \nabla_\rho \tau_\mu{}^A\,, \tag{4.39}$$

though it is not the unique solution[5]. The antisymmetric component of (4.38) is the torsion (4.22):

$$\Gamma^{\rho}_{[\mu\nu]} = \frac{1}{2}\tau^{\rho}{}_{A}T_{\mu\nu}{}^{A}\,. \tag{4.40}$$

It is also useful to define the 'acceleration' and its trace

$$a_{\mu}{}^{AB} \equiv -\tau^{\rho A}T^{B}_{\rho\mu}\,, \qquad a_{\mu} \equiv a_{\mu}{}^{AB}\eta_{AB}\,, \tag{4.41}$$

as well as its symmetric traceless component

$$a_{\mu}{}^{\{AB\}} \equiv a_{\mu}{}^{(AB)} - \frac{1}{d_L}\eta^{AB}a_{\mu}\,, \tag{4.42}$$

where d_L is the dimension of the longitudinal space (which is $d_L = 3$ here, but we will will also use this notation in the reduction to the $d_L = 2$ case of SNC in section 4.3.1). The final tensor that will appear is the extrinsic curvature defined by

$$\mathcal{K}_{\mu\nu A} = \frac{1}{2}\mathcal{L}_{\tau^{\rho A}}H_{\mu\nu}\,, \qquad \mathcal{K}_{A} \equiv H^{\mu\nu}\mathcal{K}_{\mu\nu A}\,, \tag{4.43}$$

and obeying the following useful identities

$$\tau^{\mu(A}\mathcal{K}_{\mu\nu}{}^{B)} = 0, \qquad \nabla_{\mu}\tau^{\nu A} = H^{\nu\rho}\mathcal{K}_{\mu\rho}{}^{A}\,. \tag{4.44}$$

Finally, let's introduce some notation to make the expressions more compact. Given an arbitrary tensor $M_{\mu\nu}$ carrying lower indices, we will employ for convenience the following short-hand notation:

$$M^{\mu\nu} \equiv H^{\mu\rho}H^{\nu\sigma}M_{\rho\sigma}\,, \quad M_{AB} \equiv \tau^{\mu}{}_{A}\tau^{\nu}{}_{B}M_{\mu\nu}\,, \quad \nabla_{\rho}M_{AB} \equiv \nabla_{\rho}\left(\tau^{\mu}{}_{A}\tau^{\nu}{}_{B}M_{\mu\nu}\right)\,, \tag{4.45}$$

and similarly for tensors of arbitrary rank. The meaning of raised indices should then hopefully clear from context – note that e.g. the field strengths, Newton-Cartan torsion and covariant derivative are all naturally defined with lower curved indices so when they appear instead with raised curved or longitudinal flat indices this uses the above notation.

[5] Here ∇ acts only on the curved indices. It would also be possible to define a connection covariant under local $SO(1,2)$ transformations by replacing the partial derivative $\partial_{\mu}\tau_{\nu}{}^{A}$ term with a spin covariant derivative, as we did in the previous chapter.

The action can then be written in terms of these manifestly covariant quantities as

$$S = \int \mathrm{d}^{11}x\,\Omega\,\left(\mathcal{L} + \mathcal{L}_{\widetilde{F}} + \Omega^{-1}\mathcal{L}_{\text{top}}\right), \tag{4.46}$$

with

$$\begin{aligned}
\mathcal{L} &= \mathcal{R} - a^{\mu AB}a_{\mu(AB)} + \frac{3}{2}a^{\mu}a_{\mu} - \frac{1}{12}F^{\mu\nu\rho A}F_{\mu\nu\rho A} + \frac{1}{4}\epsilon_{ABC}F^{AB\mu\nu}T_{\mu\nu}{}^{C}\,,\\
\mathcal{L}_{\widetilde{F}} &= -\frac{1}{4!}\widetilde{F}_{\nu_1\ldots\nu_4}\left(F^{\nu_1\ldots\nu_4} + \frac{1}{4!3!\Omega}\epsilon^{\nu_1\ldots\nu_4\mu_1\ldots\mu_7}F_{\mu_1\ldots\mu_4}\epsilon_{ABC}\tau_{\mu_5}{}^{A}\tau_{\mu_6}{}^{B}\tau_{\mu_7}{}^{C}\right),\\
\mathcal{L}_{\text{top}} &= \frac{1}{6}F_4\wedge F_4\wedge C_3 = \frac{1}{6}\frac{1}{3!4!^2}\epsilon^{\mu_1\ldots\mu_{11}}F_{\mu_1\ldots\mu_4}F_{\mu_5\ldots\mu_8}C_{\mu_9\ldots\mu_{11}}\,,
\end{aligned} \tag{4.47}$$

where the Ricci scalar $\mathcal{R}$ is defined in terms of the usual Riemann curvature tensor of the connection (4.38) via

$$\mathcal{R}^{\rho}{}_{\sigma\mu\nu} = \partial_\mu\Gamma^{\rho}_{\nu\sigma} - \partial_\nu\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}\,,\quad \mathcal{R} = \mathcal{R}^{\rho}{}_{\mu\rho\nu}H^{\mu\nu}\,. \tag{4.48}$$

4.1.4 Dual field strength

The appearance of the two field strengths F_4 and $\widetilde{F}_4$ in the finite action (4.34) may seem rather exotic. In fact, we can relate the latter to components of the dual seven-form field strength, revealing that the non-relativistic action involves a partially democratic treatment of what are originally dual degrees of freedom. In eleven-dimensional SUGRA, we have

$$\hat{F}_7 = d\hat{C}_6 + \tfrac{1}{2}\hat{C}_3\wedge\hat{F}_4\,,\qquad \hat{F}_7 = \hat{\star}\hat{F}_4\,. \tag{4.49}$$

With our expansion, we can compute $\hat{\star}\hat{F}_4$ in components:

$$\begin{aligned}
(\hat{\star}\hat{F}_4)_{\mu_1\ldots\mu_7} = \Omega\epsilon_{\mu_1\ldots\mu_7\nu_1\ldots\nu_4}\big(&c^3H^{\nu_1\lambda_1}\ldots H^{\nu_4\lambda_4}F_{\rho_1\ldots\rho_4} + H^{\nu_1\lambda_1}\ldots H^{\nu_4\lambda_4}\widetilde{F}_{\lambda_1\ldots\lambda_4}\\
&+ 4H^{\nu_1\lambda_1}\ldots H^{\nu_3\lambda_3}\tau^{\nu_4\lambda_4}F_{\lambda_1\ldots\lambda_4}\\
&- 6H^{\nu_1\lambda_1}H^{\nu_2\lambda_2}T_{\lambda_1\lambda_2}{}^{A}\tau^{\nu_3B}\tau^{\nu_4C}\epsilon_{ABC}\big) + \mathcal{O}(c^{-3})\,.
\end{aligned} \tag{4.50}$$

We then search for an expansion of $\hat{C}_6$ that can reproduce the singular term and lead to a sensible definition of the dual six-form in the non-relativistic theory. This is provided by

$$\hat{C}_6 = -\tfrac{1}{2}c^3C_3\wedge\tfrac{1}{6}\epsilon_{ABC}\tau^A\wedge\tau^B\wedge\tau^C + C_6 - \tfrac{1}{2}\widetilde{C}_3\wedge\tfrac{1}{6}\epsilon_{ABC}\tau^A\wedge\tau^B\wedge\tau^C\,, \tag{4.51}$$

leading to

$$\hat{F}_7 = -\tfrac{\epsilon_{ABC}}{6}c^3\tau^A\wedge\tau^B\wedge\tau^C\wedge F_4+dC_6+\tfrac{1}{2}C_3\wedge F_4-\tfrac{\epsilon_{ABC}}{6}\tau^A\wedge\tau^B\wedge\tau^C\wedge\widetilde{F}_4\,, \quad (4.52)$$

where we are ignoring terms of order c^{-3}. The singular piece in (4.52) agrees with that in (4.50) on using the self-duality constraint (4.33) obeyed by F_4. From the finite terms, we can define in the non-relativistic limit the quantity $F_7 \equiv dC_6 + \frac{1}{2}C_3 \wedge F_4$ which obeys again $dF_7 = \frac{1}{2}F_4 \wedge F_4$. We could also define this quantity directly in the non-relativistic theory after taking the limit by starting with the equation of motion (4.28) of the gauge field. In that case, we would define the dual seven-form field strength to be the quantity appearing under the exterior derivative, including all terms on the left-hand side of (4.28) as well as that involving $d\tau$ on the right-hand side. In components, this means

$$\begin{aligned}F_{\mu_1\ldots\mu_7} = \tfrac{1}{4!}\Omega\epsilon_{\mu_1\ldots\mu_7\nu_1\ldots\nu_4}(&H^{\nu_1\lambda_1}\ldots H^{\nu_4\lambda_4}\widetilde{F}_{\lambda_1\ldots\lambda_4} + 4H^{\nu_1\lambda_1}\ldots H^{\nu_3\lambda_3}\tau^{\nu_4\lambda_4}F_{\lambda_1\ldots\lambda_4}\\ &- 6H^{\nu_1\lambda_1}H^{\nu_2\lambda_2}T_{\lambda_1\lambda_2}{}^{A}\tau^{\nu_3 B}\tau^{\nu_4 C}\epsilon_{ABC}\\ &+ \tfrac{1}{4!3!}\Omega^{-1}\epsilon^{\nu_1\ldots\nu_4\lambda_1\ldots\lambda_7}\epsilon_{ABC}\tau_{\lambda_1}{}^{A}\tau_{\lambda_2}{}^{B}\tau_{\lambda_3}{}^{C}\widetilde{F}_{\lambda_4\ldots\lambda_7})\,.\end{aligned} \quad (4.53)$$

Now, we can take the totally longitudinal contraction

$$\begin{aligned}F_{\mu_1\ldots\mu_4 ABC} = &\tfrac{1}{4!}\Omega\epsilon_{\mu_1\ldots\mu_4\nu_1\ldots\nu_4\sigma_1\sigma_2\sigma_3}\tau^{\sigma_1}{}_A\tau^{\sigma_2}{}_B\tau^{\sigma_3}{}_C H^{\nu_1\lambda_1}\ldots H^{\nu_4\lambda_4}\widetilde{F}_{\lambda_1\ldots\lambda_4}\\ &+ \epsilon_{ABC}\widetilde{F}_{\mu_1\ldots\mu_4}\,.\end{aligned} \quad (4.54)$$

Using (4.15), it can be shown that whereas the transverse part of $F_{\mu\nu\rho\sigma}$ obeys a self-duality constraint, the longitudinal part of $F_{\mu_1\ldots\mu_7}$ obeys an anti-self-duality constraint:

$$\begin{aligned}&\Omega H^{\mu_1\nu_1}\ldots H^{\mu_4\nu_4}F_{\mu_1\ldots\mu_4\sigma_1\sigma_2\sigma_3}\tau^{\sigma_1}{}_A\tau^{\sigma_2}{}_B\tau^{\sigma_3}{}_C\\ &\qquad = +\tfrac{1}{4!3!}\epsilon^{\mu_1\ldots\mu_4\nu_1\ldots\mu_4\lambda_1\ldots\lambda_3}\epsilon_{DEF}\tau_{\lambda_1}{}^{D}\tau_{\lambda_2}{}^{E}\tau_{\lambda_3}{}^{F}F_{\mu_1\ldots\mu_4 ABC}\,.\end{aligned} \quad (4.55)$$

The conclusion is that (4.54) shows that the totally longitudinal part of $F_{\mu_1\ldots\mu_7}$ can be identified with the anti-self-dual transverse part of $\widetilde{F}_{\mu\nu\rho\sigma}$. Notice that the longitudinal part of the latter projects trivially out of the action, and in fact it is exactly the projection as on the right-hand side of (4.54) which appears in (4.34). Hence we can re-express the terms in the Lagrangian involving $\widetilde{F}_{\mu\nu\rho\sigma}$ as

$$\begin{aligned}\mathcal{L}_{\widetilde{F}} = -\tfrac{1}{2}\tfrac{1}{4!}F_{\mu_1\ldots\mu_4\lambda_1\ldots\lambda_3}\tfrac{1}{6}\epsilon^{ABC}\tau^{\lambda_1}{}_A\tau^{\lambda_2}{}_B\tau^{\lambda_3}{}_C\times \\ \times\left(H^{\mu_1\nu_1}\ldots H^{\mu_4\nu_4} + \tfrac{1}{4!3!\Omega}\epsilon^{\mu_1\ldots\mu_4\nu_1\ldots\nu_7}\epsilon_{DEF}\tau_{\nu_5}{}^D\tau_{\nu_6}{}^E\tau_{\nu_7}{}^F\right)F_{\nu_1\nu_2\nu_3\nu_4}\,.\end{aligned} \tag{4.56}$$

This appearance of (components of) both the four-form and its dual together in the action is again reminiscent of exceptional field theory.

4.2 Equations of motion and symmetries

We have expanded the action, and now we turn our attention to the equations of motion, and the role played by the non-relativistic dilatation and boost symmetries.

4.2.1 Equations of motion from expansion

To keep track of the equations of motion at each order, we will consider the result of expanding the variation of the action. We will explicitly find that this gives the same results as varying the expansion of the action we considered previously. The reason we take this approach is that it will provide a useful way to keep track of which parts of the expansion of the eleven-dimensional equations of motion appear at which order. Recall that we view our non-relativistic limit as arising from a field redefinition, and we do not consider possible subleading terms which would occur in a true non-relativistic expansion. That said, we set up the expansion below in a way that would be reminiscent of such an expansion.

The relativistic equations of motion are obtained from the variation of the action (4.19):

$$\delta S = \int \mathrm{d}^{11}x(\sqrt{|\hat{g}|}\delta\hat{g}^{\mu\nu}\mathcal{G}_{\mu\nu} + \delta\hat{C}_{\mu\nu\rho}\mathcal{E}^{\mu\nu\rho})\,, \tag{4.57}$$

where

$$\begin{aligned}\mathcal{G}_{\mu\nu} &= \mathcal{R}_{\mu\nu} - \tfrac{1}{12}\hat{F}_{\mu\rho_1\ldots\rho_3}\hat{F}_\nu{}^{\rho_1\ldots\rho_3} - \tfrac{1}{2}\hat{g}_{\mu\nu}(\mathcal{R} - \tfrac{1}{48}\hat{F}^{\rho_1\ldots\rho_4}\hat{F}_{\rho_1\ldots\rho_4})\,,\\ \mathcal{E}^{\mu\nu\rho} &= -\tfrac{1}{6}\left(\partial_\sigma(\sqrt{|\hat{g}|}\hat{F}^{\mu\nu\rho\sigma}) - \tfrac{1}{2\cdot4!\cdot4!}\epsilon^{\mu\nu\rho\sigma_1\ldots\sigma_8}\hat{F}_{\sigma_1\ldots\sigma_4}\hat{F}_{\sigma_5\ldots\sigma_8}\right)\,.\end{aligned} \tag{4.58}$$

We consider the non-relativistic expansion of the fields, in the form

$$\begin{aligned} \hat{g}^{\mu\nu} &= cH^{\mu\nu} + c^{-2}\tau^{\mu\nu} \,, \\ \hat{g}_{\mu\nu} &= c^2\tau_{\mu\nu} + c^{-1}H_{\mu\nu} \,, \\ \hat{C}_{\mu\nu\rho} &= c^3\omega_{\mu\nu\rho} + C_{\mu\nu\rho} + c^{-3}\widetilde{C}_{\mu\nu\rho} \,, \end{aligned} \tag{4.59}$$

where $\omega_{\mu\nu\rho} = -\epsilon_{ABC}\tau_\mu{}^A\tau_\nu{}^B\tau_\rho{}^C$. Both $\mathcal{G}$ and $\mathcal{E}$ admit an expansion in powers of c^3, with

$$\begin{aligned} \mathcal{G} &= c^6\mathcal{G}^{(6)} + c^3\mathcal{G}^{(3)} + c^0\mathcal{G}^{(0)} + c^{-3}\mathcal{G}^{(-3)} + \dots \,, \\ \mathcal{E} &= c^3\mathcal{E}_{(3)} + c^0\mathcal{E}_{(0)} + c^{-3}\mathcal{E}_{(-3)} + \dots \,. \end{aligned} \tag{4.60}$$

We now re-organise the variation of the action that results from (4.59), by inserting the expressions (4.59) for the metric and three-form. We choose to consider the variations of $\tau^\mu{}_A$ and $H^{\mu\nu}$ as independent, in terms of which

$$\delta\omega_{\mu\nu\rho} = -\omega_{\mu\nu\rho}\tau^D_\lambda\delta\tau^\lambda{}_D - 3\omega_{\lambda[\mu\nu}H_{\rho]\kappa}\delta H^{\lambda\kappa} \,. \tag{4.61}$$

The general result at order c^{3n} following from (4.57) is that

$$\begin{aligned} \delta S^{(3n)} = \int \mathrm{d}^{11}x \Big[&\delta H^{\mu\nu}(\Omega\mathcal{G}^{(3n)}_{\mu\nu} - 3\omega_{\mu\rho\sigma}H_{\lambda\nu}\mathcal{E}^{\rho\sigma\lambda}_{(3n-3)}) \\ &+ \delta\tau^\mu{}_A(2\tau^{\nu A}\Omega\mathcal{G}^{(3n+3)}_{\mu\nu} - \tau^A{}_\mu\omega_{\rho\sigma\lambda}\mathcal{E}^{\rho\sigma\lambda}_{(3n-3)}) \\ &+ \delta C_{\mu\nu\rho}\mathcal{E}^{\mu\nu\rho}_{(3n)} + \delta\widetilde{C}_{\mu\nu\rho}\mathcal{E}^{\mu\nu\rho}_{(3n+3)} \Big] \,, \end{aligned} \tag{4.62}$$

using $\sqrt{|\hat{g}|} = \Omega c^{-1}$. Hence, in general, if we expand the theory up to order $3k$, for $k \leq n \leq 2$, the equations of motion will be

$$\begin{aligned} \mathcal{G}^{(3n)}_{\langle\mu\nu\rangle} &= 3H_{\lambda\langle\mu}\omega_{\nu\rangle\rho\sigma}\Omega^{-1}\mathcal{E}^{\lambda\rho\sigma}_{(3n-3)} \,, \\ 2\mathcal{G}^{(3n+3)}_{\mu A} &= \tau_{\mu A}\omega_{\rho\sigma\lambda}\Omega^{-1}\mathcal{E}^{\rho\sigma\lambda}_{(3n-3)} \,, \\ \mathcal{E}^{\mu\nu\rho}_{(3n)} &= 0 \,, \end{aligned} \tag{4.63}$$

with the understanding that $\mathcal{G}^{(9)} = \mathcal{E}^{(6)} = 0$. The angle bracket notation takes into account that the variation of $H^{\mu\nu}$ is constrained by $\delta H^{\mu\nu}\tau_\mu{}^A\tau_\nu{}^B = 0$. We can solve this constraint by letting $\delta H^{\mu\nu} = H^{\rho(\mu}H_{\rho\sigma}M^{\nu)\sigma}$ such that the naive variation $\delta H^{\mu\nu}T_{\mu\nu} = 0$ implies instead the equation of motion

$$T_{\langle\mu\nu\rangle} = \tfrac{1}{2}(H_{\mu\rho}H^{\rho\sigma}T_{(\sigma\nu)} + H_{\nu\rho}H^{\rho\sigma}T_{(\mu\sigma)}) = 0 \,, \tag{4.64}$$

which is symmetric and obeys $\tau^{\mu}{}_{A}\tau^{\nu}{}_{B}T_{\langle\mu\nu\rangle} = 0$. Note that the equation of motion for $\widetilde{C}$ at each order is exactly that of C at the previous order.

We should contrast the equations of motion (4.63) with the result of independently expanding $\mathcal{G}$ and $\mathcal{E}$. If we naively set each other of the expansion of the latter to zero, we would find the equations $\mathcal{G}^{(3n)} = 0 = \mathcal{E}^{(3n)}$ at any given order. However, in the non-relativistic expansion, treating $\tau^{\mu}{}_{A}$ and $H^{\mu\nu}$ as independent fields, then equation (4.63) says that we cannot simply expand the relativistic equations and set each order independently to zero unless we consider the full expansion (potentially infinite if treating subleading terms). A similar subtlety is the question of which equations of motion we are meant to expand. For instance, in the relativistic theory both $\mathcal{E}^{\mu\nu\rho} = 0$ and $g_{\mu\sigma}g_{\rho\kappa}g_{\sigma\lambda}\mathcal{E}^{\sigma\kappa\lambda} = 0$ are equivalent, but lead to different truncations to finite order in the $1/c$ expansion. Here we have made the choice to expand the equations of motion that appear conjugate to the variations $\delta g^{\mu\nu}$ and $\delta C_{\mu\nu\rho}$.

Let us look for example at the first two orders, c^6 and c^3. If we simply wanted to expand the theory up to order c^6 we would find the equation $\left(\mathcal{G}^{(6)} - 3\omega\mathcal{E}^{(3)}H\right)_{\langle\mu\nu\rangle} = 0$, however if we proceed with expanding up to order c^3 we find that the equation for the three-form tells us that $\mathcal{E}^{(3)} = 0$, so that we can safely impose the two equations $\mathcal{G}^{(6)}_{\langle\mu\nu\rangle} = \mathcal{E}^{(3)} = 0$ *independently*.

Matters are further complicated by a number of 'off-shell' identities obeyed by the terms appearing in the expansion of $\mathcal{G}$ and $\mathcal{E}$. These identities will feature heavily below, and in fact are crucial for the consistency and symmetries of the non-relativistic truncation.

To put all these ideas together, we now look in detail at the first orders of the expansion of (4.57).

Terms at $\mathcal{O}(c^6)$ Here we encounter the leading terms in the expansions of $\mathcal{G}$ and $\mathcal{E}$. First of all, we have

$$\mathcal{G}^{(6)}_{\mu\nu} = \tfrac{1}{2}\tau_{\mu\nu}\left(\tfrac{1}{2}T^{A}_{\rho_1\sigma_1}T^{B}_{\rho_2\sigma_2}\eta_{AB}H^{\rho_1\sigma_1}H^{\rho_2\sigma_2} + \tfrac{1}{48}H^{\rho_1\sigma_1}\dots H^{\rho_4\sigma_4}F_{\rho_1\dots\rho_4}F_{\sigma_1\dots\sigma_4}\right), \tag{4.65}$$

which obeys $\mathcal{G}_{\langle\mu\nu\rangle} = 0$ automatically. Hence the $\delta H^{\mu\nu}$ variation at order c^6 does not imply an actual equation of motion. One also has

$$\mathcal{E}^{\mu\nu\rho}_{(3)} = -\tfrac{1}{6}\partial_\sigma\left(\Omega F^{\mu\nu\rho\sigma} + \tfrac{1}{3!4!}\epsilon^{\mu\nu\rho\sigma\sigma_1\dots\sigma_7}F_{\sigma_1\dots\sigma_4}\epsilon_{ABC}\tau_{\sigma_5}{}^{A}\tau_{\sigma_6}{}^{B}\tau_{\sigma_7}{}^{C}\right). \tag{4.66}$$

This is the self-duality constraint under a derivative. It obeys $\tau_\mu{}^A\tau_\nu{}^B\mathcal{E}^{\mu\nu\rho}_{(3)} = 0$, and so also the $\delta\tau$ variation at order c^6 vanishes identically. This is however necessary for consistency: the expansion of the action itself started only at order c^3, i.e. $S^{(6)} \equiv 0$. Hence at this order we do not obtain any equations of motion.

Terms at $\mathcal{O}(c^3)$ At this order, there was a non-zero $S^{(3)}$ given by (4.32), for which we required the self-duality constraint (4.33) to set to zero. Let us see how this information is reproduced. First of all, the variation of C_3 coming from (4.62) at this order implies $\mathcal{E}_{(3)} = 0$. The variation of $\tau^\mu{}_A$ involves a contribution from $\mathcal{E}_{(0)}$, which can be read off from the finite part of the expansion of the three-form equation of motion, which was (4.28). For convenience, we repeat this here:

$$\begin{aligned}\mathcal{E}^{\mu\nu\rho}_{(0)} = -\tfrac{1}{6}\partial_\sigma\Big(\Omega\big(4H^{[\mu|\lambda_1}H^{|\nu|\lambda_2}H^{|\rho|\lambda_3}\tau^{|\sigma]\lambda_4}F_{\lambda_1\ldots\lambda_4} + \widetilde{F}^{\mu\nu\rho\sigma}\\ - 6H^{[\mu|\lambda_1}H^{|\nu|\lambda_2}\tau^{|\rho|B}\tau^{|\sigma]C}T_{\lambda_1\lambda_2}{}^A\epsilon_{ABC}\big)\Big)\\ + \tfrac{\epsilon^{\mu\nu\rho\sigma_1\ldots\sigma_8}}{2\cdot 3!4!4!}\big(F_{\sigma_1\ldots\sigma_4}F_{\sigma_5\ldots\sigma_8} - 12\epsilon_{ABC}T_{\sigma_1\sigma_2}{}^A\tau_{\sigma_3}{}^B\tau_{\sigma_4}{}^C\widetilde{F}_{\sigma_5\ldots\sigma_8}\big)\,.\end{aligned} \tag{4.67}$$

What one finds then is that

$$\begin{aligned}&2\tau^{\nu A}\Omega\mathcal{G}^{(6)}_{\mu\nu} - \tau_\mu{}^A\omega_{\rho\sigma\lambda}\mathcal{E}^{(0)\rho\sigma\lambda}\\ &= \tfrac{\Omega\tau_\mu{}^A}{2\cdot 4!}F_{\nu_1\ldots\nu_4}\left(F^{\nu_1\ldots\nu_4} + \tfrac{\epsilon^{\nu_1\ldots\nu_4\rho_1\ldots\rho_7}}{\Omega 3!4!}F_{\rho_1\ldots\rho_4}\epsilon_{ABC}\tau_{\rho_5}{}^A\tau_{\rho_6}{}^B\tau_{\rho_7}{}^C\right)\,,\end{aligned} \tag{4.68}$$

which is proportional to the self-duality constraint. For the terms accompanying the $\delta H^{\mu\nu}$ variation one finds

$$\begin{aligned}&\delta H^{\mu\nu}(\Omega\mathcal{G}^{(3)}_{\mu\nu} - 3\omega_{(\mu|\rho\sigma}H_{\lambda|\nu)}\mathcal{E}^{\rho\sigma\lambda}_{(0)})\\ &\quad= \delta H^{\mu\nu}\big(\tfrac{1}{4\cdot 4!^2}\epsilon_{ABC}H_{\lambda_1(\mu}\tau_{\nu)}{}^A\tau_{\lambda_2}{}^B\tau_{\lambda_3}{}^C F_{\sigma_1\ldots\sigma_4}F_{\sigma_5\ldots\sigma_8}\epsilon^{\lambda_1\ldots\lambda_3\sigma_1\ldots\sigma_8}\\ &\qquad - \tfrac{\Omega}{12}F_{\mu\rho_1\ldots\rho_3}F_\nu{}^{\rho_1\ldots\rho_3} + \tfrac{\Omega}{96}H_{\mu\nu}F^2\big)\,,\end{aligned} \tag{4.69}$$

such that after projecting using (4.64)

$$\begin{aligned}\Omega\mathcal{G}^{(3)}_{\langle\mu\nu\rangle} - 3\omega_{\langle\mu|\rho\sigma}H_{\lambda|\nu\rangle}\mathcal{E}^{\rho\sigma\lambda}_{(0)} = \tfrac{\epsilon^{\lambda_1\ldots\lambda_3\sigma_1\ldots\sigma_8}}{8\cdot 4!^2}\epsilon_{ABC}H_{\lambda_1(\mu}\tau_{\nu)}{}^A\tau_{\lambda_2}{}^B\tau_{\lambda_3}{}^C F_{\sigma_1\ldots\sigma_4}F_{\sigma_5\ldots\sigma_8}\\ + \tfrac{\Omega}{96}H_{\mu\nu}F^2 - \tfrac{\Omega}{12}H_{\kappa(\mu}F_{\nu)\rho\sigma\lambda}F^{\kappa\rho\sigma\lambda}\,,\end{aligned} \tag{4.70}$$

using the obvious shorthand for raised indices and F^2 instead of writing $H^{\mu\nu}$ multiple times. This exactly reproduces the variation $\delta S^{(3)}$ of the leading part of the expansion of the action (4.32). Then, after projecting and using the Schouten identity, (4.69) or (4.70) can be shown to again be proportional to the self-duality constraint (specifically: the time-space projection of the first term combines with the time-space projection of the third term, and the space-space projection of the second term combines with the space-space projection of the third term).

Hence the sole equation of motion we obtain at this order is the self-duality constraint. This is consistent with what we required from the expansion of the action.

Terms at $\mathcal{O}(c^0)$ We next consider (4.62) with $n = 0$. First of all, the equation of motion of C indeed gives $\mathcal{E}_{(0)}$, as in (4.67), while that of $\widetilde{C}$ gives the constraint in the form $\mathcal{E}_{(3)}$. This is exactly what we obtain from varying the finite action $S^{(0)}$ directly. Note that the longitudinal projection of $\mathcal{E}_{(0)}$ in conjunction with the self-duality constraint implies the equation

$$\tfrac{1}{2}\eta_{AB}H^{\mu\rho}H^{\nu\sigma}T_{\mu\nu}{}^{A}T_{\rho\sigma}{}^{B} = -\tfrac{1}{48}H^{\mu_1\nu_1}\dots H^{\mu_4\nu_4}F_{\mu_1\dots\mu_4}F_{\nu_1\dots\nu_4}\,, \tag{4.71}$$

thereby reproducing the equation we would get by setting $\mathcal{G}^{(6)} = 0$ (compare (4.65)). Hence although we could not set $\mathcal{G}^{(6)} = 0$ previously, the non-relativistic theory is not missing this equation. Note that for generic non-vanishing F_4, equation (4.71) is incompatible with imposing foliation-type constraints on the MNC torsion such that the left-hand side vanishes, however if F_4 is also restricted to vanish (for example) one could require such constraints (as is always possible in the NSNS sector case [16]).

Now we turn to the equations of motion following from the variations of τ and H. For simplicity, we present here the independent equations of motion after projecting onto longitudinal (time) and transverse (space) components. The temporal and spatial projectors are defined as

$$(\Delta_T)^{\mu}{}_{\nu} = \tau^{\mu}{}_{A}\tau_{\nu}{}^{A}\,, \qquad (\Delta_S)^{\mu}{}_{\nu} = H^{\mu\rho}H_{\rho\nu}\,, \qquad (\Delta_T)^{\mu}{}_{\nu} + (\Delta_S)^{\mu}{}_{\nu} = \delta^{\mu}_{\nu}\,. \tag{4.72}$$

We start with the equations of motion of τ. The trace of the time projection gives an equation involving the Ricci scalar:

$$\begin{aligned}\mathcal{R} = {}& \tfrac{7}{3}\nabla^{\mu}a_{\mu} + a^{\mu\{AB\}}a_{\mu AB} + \tfrac{7}{6}a^2 + \tfrac{1}{36}F_{A\nu\rho\sigma}F^{A\nu\rho\sigma} - \tfrac{1}{6}\epsilon_{ABC}F^{AB\rho\sigma}T_{\rho\sigma}{}^{C}\\ &+ \tfrac{1}{4!}\widetilde{F}_{\mu\nu\rho\sigma}\left(F^{\mu\nu\rho\sigma} + \tfrac{1}{\Omega 4!3!}\epsilon_{ABC}\epsilon^{\mu\nu\rho\sigma\lambda_1\dots\lambda_7}F_{\lambda_1\dots\lambda_4}\tau_{\lambda_5}{}^{A}\tau_{\lambda_6}{}^{B}\tau_{\lambda_7}{}^{C}\right)\,.\end{aligned} \tag{4.73}$$

The traceless part of the time-time projection is:

$$\begin{aligned}&\nabla^\mu a_{\mu\{AB\}} + a^\mu a_{\mu\{AB\}} + a_{\mu[C(A]}a^\mu{}_{\{B)D\}}\eta^{CD} + \tfrac{1}{12}F_A{}^{\mu\nu\rho}F_{B\mu\nu\rho}\\ &\quad= \epsilon_{(A|CD}F_{|B)}{}^{C\mu\nu}T_{\mu\nu}{}^D - \tfrac{\eta_{AB}}{3}\left(-\tfrac{1}{12}F^{C\mu\nu\rho}F_{C\mu\nu\rho} + \epsilon_{CDE}F^{\mu\nu CD}T_{\mu\nu}{}^E\right).\end{aligned} \tag{4.74}$$

The space projection is

$$\nabla_\rho T^{\mu\rho}{}_A + a_{\rho AC}T^{\mu\rho C} = \tfrac{1}{6}F^{\mu\nu\rho\sigma}F_{A\nu\rho\sigma} - \tfrac{1}{2}\epsilon_{ABC}F^{\mu\rho\sigma B}T_{\rho\sigma}{}^C\,. \tag{4.75}$$

Finally, we consider the equations of motion of H. The space-space projection is:

$$\begin{aligned}&\mathcal{R}^{(\mu\nu)} - a^{\mu AB}a^\nu{}_{\{AB\}} + \tfrac{1}{6}\left(a^\mu a^\nu - a^2 H^{\mu\nu}\right)\\ &\quad= \tau^{\rho A}\nabla^{(\mu}T^{\nu)}{}_{\rho A} + \tfrac{1}{6}H^{\mu\nu}\nabla^\rho a_\rho + \tfrac{1}{4}F^{\mu\rho\sigma A}F^\nu{}_{\rho\sigma A} - \tfrac{1}{36}H^{\mu\nu}F^{A\rho\sigma\lambda}F_{A\rho\sigma\lambda}\\ &\qquad - \tfrac{1}{2}\epsilon_{ABC}F^{(\mu|\rho AB}T^{|\nu)}{}_\rho{}^C + \tfrac{1}{24}H^{\mu\nu}\epsilon_{ABC}F^{\rho\sigma AB}T_{\rho\sigma}{}^C\\ &\qquad + \tfrac{1}{6}F^{(\mu|\rho\sigma\lambda}\widetilde{F}^{|\nu)}{}_{\rho\sigma\lambda} - \tfrac{1}{48}H^{\mu\nu}F^{\rho\sigma\lambda\kappa}\widetilde{F}_{\rho\sigma\lambda\kappa}\\ &\qquad + \tfrac{1}{2}H^{\mu\nu}\Big(-\mathcal{R} + \tfrac{7}{3}\nabla^\rho a_\rho + a^{\rho\{AB\}}a_{\rho AB} + \tfrac{7}{6}a^2\\ &\qquad\qquad + \tfrac{1}{36}F_{A\lambda\rho\sigma}F^{A\lambda\rho\sigma} - \tfrac{1}{6}\epsilon_{ABC}F^{AB\rho\sigma}T_{\rho\sigma}{}^C\Big)\,.\end{aligned} \tag{4.76}$$

Combining the trace of (4.76) with (4.73) we find that the self-duality constraint (4.33) appears (contracted with $\widetilde{F}_{\mu\nu\rho\sigma}$).

The time-space projection is (with $\epsilon^A{}_{BC} \equiv \eta^{AD}\epsilon_{DBC}$)

$$\begin{aligned}&\mathcal{R}^{(\mu A)} - a^\mu{}_{BC}a^{A(BC)} + \tfrac{1}{2}a_B a^{\mu BA} = \tfrac{1}{4}\epsilon^A{}_{BC}\nabla^\rho F^\mu{}_\rho{}^{BC} + \tfrac{1}{4}\epsilon^A{}_{BC}a_\rho F^{\mu\rho BC}\\ &\quad + \tfrac{1}{4}\epsilon_{BCD}a_\rho{}^{AB}F^{\mu\rho CD} + \tfrac{1}{4}F^{AB\rho\sigma}F^\mu{}_{B\rho\sigma} + \tfrac{1}{4}\epsilon_{BCD}F^{ABC\rho}T_\rho{}^{\mu D}\\ &\quad + \tfrac{1}{2}a^{\rho BA}\nabla_\rho\tau^\mu{}_B - \tfrac{1}{2}\nabla^2\tau^{\mu A} - a^\rho\nabla_\rho\tau^{\mu A} - \tfrac{1}{2}a^{\mu BA}\mathcal{K}_B + \tfrac{1}{2}a^\mu\mathcal{K}^A\\ &\quad - \tfrac{1}{2}\nabla_B a^{\mu BA} + \nabla^A a^\mu + \tfrac{1}{2}T^\mu{}_{\sigma B}\nabla^B\tau^{\sigma A} + \tfrac{1}{2}\nabla_\rho\nabla^\mu\tau^{\rho A} - \tfrac{1}{2}\tau^\rho{}_B\nabla^\mu a_\rho{}^{AB}\\ &\quad + \tfrac{1}{6}F^{(\mu}{}_{\nu\rho\sigma}\widetilde{F}^{A)\nu\rho\sigma} - \tfrac{\epsilon^{\lambda_1\ldots\lambda_3\sigma_1\ldots\sigma_8}}{4\cdot 4!^2\Omega}\epsilon^A{}_{BC}\tau_{\lambda_2}{}^B\tau_{\lambda_3}{}^C H^{\mu\kappa}H_{\kappa\lambda_1}F_{\sigma_1\ldots\sigma_4}\widetilde{F}_{\sigma_5\ldots\sigma_8}\,.\end{aligned} \tag{4.77}$$

One can verify that these are indeed exactly the equations of motions that one gets by varying the finite part of the action, $S^{(0)}$, given in (4.34).

4.2.2 Dilatations and a 'missing' equation of motion

We already mentioned the existence of a dilatation transformation given by (4.37), whose origin lay in the expansion in powers of c. There is evidently a freedom to rescale c by some constant while simultaneously rescaling the component fields such that the eleven-dimensional fields are unchanged. This *rigid dilatation* leaves the full action invariant. Hence for an infinitesimal dilatation, with $\delta_\lambda c = -\lambda c$, we have the transformations (4.37), and clearly order-by-order for the action we should have

$$\begin{aligned}\delta_\lambda S^{(6)} &= 6\lambda S^{(6)}\,,\\ \delta_\lambda S^{(3)} &= 3\lambda S^{(3)}\,,\\ \delta_\lambda S^{(0)} &= 0\cdot\lambda S^{(0)}\,,\\ \delta_\lambda S^{(-3)} &= -3\lambda S^{(-3)}\,,\end{aligned} \tag{4.78}$$

and so on. Recall that $S^{(6)}$ and $\delta S^{(6)}$ vanish identically, so the first two of these are just $0=0$.

A powerful consequence of the rigid dilatations is that if we know the equations of motion for the action $S^{(3k)}$ at a given order $k\neq 0$ we can immediately write down an action that produces them (which will agree up to total derivatives with that arising from the expansion). This works by applying the formula (4.62) for the variation and specialising to the dilatation variation. This is guaranteed to produce $3kS^{(3k)}$. This singles out the finite order action as being special, as here knowing the equations of motion and dilatation symmetry is not enough to determine its form. Furthermore, for this case we can promote the dilatation parameter to be coordinate dependent, and obtain a *local dilatation* symmetry.

Let's verify these statements. Under a rigid dilatation with parameter λ, the variation of the c^3 part of the action is

$$\delta_\lambda S^{(3)} = \int \mathrm{d}^{11}x\,\Omega\left(\lambda\mathcal{G}^{(3)}_{\mu\nu}H^{\mu\nu} - \lambda\left(2(\mathcal{G}^{(6)})^A{}_A + 3\epsilon_{ABC}\Omega^{-1}\mathcal{E}^{ABC}_{(0)}\right)\right)\,, \tag{4.79}$$

where $\mathcal{E}^{ABC}\equiv \tau_\mu{}^A\tau_\nu{}^B\tau_\rho{}^C\mathcal{E}^{\mu\nu\rho}$. It can be checked that $\mathcal{G}^{(3)}_{\mu\nu}H^{\mu\nu}=0$. Then, if we denote the self-duality constraint by

$$\Theta^{\mu_1\ldots\mu_4} \equiv H^{\mu_1\rho_1}\ldots H^{\mu_4\rho_4}F_{\rho_1\ldots\rho_4} + \tfrac{1}{\Omega 3!4!}\epsilon^{\mu_1\ldots\mu_4\rho_1\ldots\rho_7}F_{\rho_1\ldots\rho_4}\epsilon_{ABC}\tau_{\rho_5}{}^A\tau_{\rho_6}{}^B\tau_{\rho_7}{}^C\,, \tag{4.80}$$

we have

$$2(\mathcal{G}^{(6)})^A{}_A + 3\epsilon_{ABC}\Omega^{-1}\mathcal{E}^{ABC}_{(0)} = 3\tfrac{1}{2\cdot 4!}F_{\mu_1\dots\mu_4}\Theta^{\mu_1\dots\mu_4}\,, \tag{4.81}$$

hence indeed referring to (4.32) for $S^{(3)}$ we indeed have

$$\delta_\lambda S^{(3)} = 3\lambda S^{(3)}\,. \tag{4.82}$$

Next consider the finite part of the action:

$$\begin{aligned}\delta_\lambda S^{(0)} = \int \mathrm{d}^{11}x\,\Omega\,\Big[\lambda\mathcal{G}^{(0)}_{\mu\nu}H^{\mu\nu} - \lambda\left(2(\mathcal{G}^{(3)})^A{}_A + 3\epsilon_{ABC}\Omega^{-1}\mathcal{E}^{ABC}_{(-3)}\right)\\ + \Omega^{-1}\mathcal{E}^{(3)\mu\nu\rho}\delta_\lambda\widetilde{C}_{\mu\nu\rho}\Big]\,.\end{aligned} \tag{4.83}$$

Now we can show that

$$\mathcal{G}^{(0)}_{\mu\nu}H^{\mu\nu} - \left(2(\mathcal{G}^{(3)})^A{}_A + 3\epsilon_{ABC}\Omega^{-1}\mathcal{E}^{ABC}_{(-3)}\right) = -\tfrac{1}{8}\widetilde{F}_{\mu_1\dots\mu_4}\Theta^{\mu_1\dots\mu_4}\,, \tag{4.84}$$

such that using $\mathcal{E}^{\mu\nu\rho}_{(3)} = -\frac{1}{6}\partial_\sigma\Theta^{\mu\nu\rho\sigma}$ we have

$$\begin{aligned}\delta_\lambda S^{(0)} &= \int \mathrm{d}^{11}x\,(-\tfrac{1}{8}\lambda\widetilde{F}_{\mu\nu\rho\sigma}\Theta^{\mu\nu\rho\sigma} - \tfrac{1}{6}\partial_\sigma\Theta^{\mu\nu\rho\sigma}\delta_\lambda\widetilde{C}_{\mu\nu\rho})\,,\\ &= \int \mathrm{d}^{11}x\,(-\tfrac{1}{8}\lambda\widetilde{F}_{\mu\nu\rho\sigma}\Theta^{\mu\nu\rho\sigma} - \tfrac{1}{24}\Theta^{\mu\nu\rho\sigma}\delta_\lambda\widetilde{F}_{\mu\nu\rho\sigma})\,,\end{aligned} \tag{4.85}$$

after integrating by parts. For arbitrary local λ, we therefore have $\delta_\lambda S^{(0)} = 0$ *on imposing the self-duality constraint*, irrespective of the transformation of $\widetilde{C}_{\mu\nu\rho}$. Alternatively, if we require that

$$\delta_\lambda\widetilde{F}_{\mu\nu\rho\sigma} = -3\lambda\widetilde{F}_{\mu\nu\rho\sigma}\,, \tag{4.86}$$

then (4.85) vanishes identically without use of the constraint. This would mean accepting a non-local transformation for $\widetilde{C}_{\mu\nu\rho}$ itself, which is not completely outlandish given the discussion in section 4.1.4 suggests we may think of it as being a dual degree of freedom to C_3.

What this means in practice is that the action $S^{(0)}$ is invariant under variations of $H^{\mu\nu}$ and $\tau^\mu{}_A$ of the form (4.37). This implies that there is a 'direction' in the space of variations which leaves the action $S^{(0)}$ unchanged (or at best produces the self-duality constraint, which is not an independent equation of motion). Hence if we vary $S^{(0)}$ to obtain the equations of motion of $H^{\mu\nu}$ and $\tau^{\mu}{}_{A}$, we will find that we are 'missing' an equation of motion. This is exactly as in the NSNS sector case (see previous chapter and [16]) and reflects a known difficulty, even in the purely gravitational context, of obtaining the Poisson equation from an action principle for non-relativistic theories [25, 27], at least at first order.

Thus, in order to obtain an equation of motion for this missing variation, we go one step further in the expansion. The variation of $S^{(-3)}$, from (4.62), is:

$$\begin{aligned}\delta S^{(-3)} = \int \mathrm{d}^{11}x \Big[&\delta H^{\mu\nu}(\Omega \mathcal{G}^{(-3)}_{\mu\nu} - 3\omega_{\mu\rho\sigma}H_{\lambda\nu}\mathcal{E}^{\rho\sigma\lambda}_{(-6)}) + \delta\widetilde{C}_{\mu\nu\rho}\mathcal{E}^{\mu\nu\rho}_{(0)} \\ &+ \delta\tau^{\mu}{}_{A}(2\tau^{\nu A}\Omega\mathcal{G}^{(0)}_{\mu\nu} - \tau^{A}{}_{\mu}\omega_{\rho\sigma\lambda}\mathcal{E}^{\rho\sigma\lambda}_{(-6)}) + \delta C_{\mu\nu\rho}\mathcal{E}^{\mu\nu\rho}_{(-3)}\Big] .\end{aligned} \tag{4.87}$$

For dilatations we have

$$\begin{aligned}\delta_\lambda S^{(-3)} = \int \mathrm{d}^{11}x \Big[&\lambda\left(H^{\mu\nu}\Omega\mathcal{G}^{(-3)}_{\mu\nu} - 2\Omega(\mathcal{G}^{(0)})_{A}{}^{A} - 3\epsilon_{ABC}\mathcal{E}^{ABC}_{(-6)}\right) \\ &+ \delta_\lambda \widetilde{C}_{\mu\nu\rho}\mathcal{E}^{\mu\nu\rho}_{(0)}\Big] .\end{aligned} \tag{4.88}$$

With constant λ, equation (4.78) implies that

$$S^{(-3)} = \int \mathrm{d}^{11}x\,(\Omega\mathcal{N} + \widetilde{C}_{\mu\nu\rho}\mathcal{E}^{\mu\nu\rho}_{(0)}) , \tag{4.89}$$

where we defined the combination

$$\mathcal{N} \equiv \tfrac{1}{3}(-H^{\mu\nu}\mathcal{G}^{(-3)}_{\mu\nu} + 2(\mathcal{G}^{(0)})_{A}{}^{A}) + \epsilon_{ABC}\Omega^{-1}\mathcal{E}^{ABC}_{(-6)} . \tag{4.90}$$

Crucially, (4.90) does not vanish on applying the self-duality constraint, unlike the combination of terms (4.81) and (4.84) which appeared at the previous orders, and nor is it a combination of any other equations of motion resulting from the finite action. It can therefore be used as the equation of motion of the 'dilatation mode'. (We are not really interested in the $\widetilde{C}$ variation appearing in (4.88), which multiplies something we have already taken into account as an equation of motion.) It involves the fully longitudinal part of $\mathcal{G}^{(0)}$, which has not yet appeared in the equations of motion. Hence, we identify it with the 'Poisson equation', in which the longitudinal part of $C_{\mu\nu\rho}$ plays the role of the Newton potential (as did the longitudinal part of the B-field in the Stueckelberg gauge-fixed NSNS sector). This is because $\mathcal{E}_{(-6)}$

is the first equation of motion which contains two derivatives acting on the former. Explicitly,

$$\begin{aligned}
&\mathcal{E}^{\mu\nu\rho}_{(-6)} \\
&= -\tfrac{1}{6}\partial_\sigma\Big(\Omega\big(4H^{[\mu|\lambda_1}\tau^{|\nu|\lambda_2}\tau^{|\rho|\lambda_3}\tau^{|\sigma]\lambda_4}F_{\lambda_1\dots\lambda_4} + 6H^{[\mu|\lambda_1}H^{|\nu|\lambda_2}\tau^{|\rho|\lambda_3}\tau^{|\sigma]\lambda_4}\widetilde{F}_{\lambda_1\dots\lambda_4}\Big) \\
&\quad + \tfrac{1}{2\cdot 4!4!3!}\epsilon^{\mu\nu\rho\sigma_1\dots\sigma_8}\widetilde{F}_{\sigma_1\dots\sigma_4}\widetilde{F}_{\sigma_5\dots\sigma_8}\,.
\end{aligned} \tag{4.91}$$

Intriguingly, the combination of $\mathcal{G}^{(-3)}$ and $\mathcal{G}^{(0)}$ appearing in (4.90) has a somewhat murky relationship to the 'trace-reversed' version of the metric equation of motion. The equation $\mathcal{G}_{\mu\nu} = 0$ in the original eleven-dimensional theory can be simplified somewhat by taking its trace and solving that for the Ricci scalar. This trace is

$$\hat{g}^{\mu\nu}\mathcal{G}_{\mu\nu} = -\tfrac{9}{2}R + \tfrac{1}{32}\hat{F}^2 \tag{4.92}$$

and the equation of motion without the Ricci scalar is

$$\bar{\mathcal{G}}_{\mu\nu} \equiv \mathcal{G}_{\mu\nu} - \tfrac{1}{9}\hat{g}_{\mu\nu}\hat{g}^{\rho\sigma}\mathcal{G}_{\rho\sigma} = R_{\mu\nu} - \tfrac{1}{12}\hat{F}_\mu{}^{\rho\sigma\lambda}\hat{F}_{\nu\rho\sigma\lambda} + \tfrac{1}{144}\hat{g}_{\mu\nu}\hat{F}^2\,, \tag{4.93}$$

for which

$$\tau^{\mu\nu}\bar{\mathcal{G}}^{(0)}_{\mu\nu} = \tfrac{1}{3}(2\tau^{\mu\nu}\mathcal{G}^{(0)}_{\mu\nu} - H^{\mu\nu}\mathcal{G}^{(-3)}_{\mu\nu})\,, \tag{4.94}$$

which is exactly the combination appearing in (4.90). Note the relative numerical factors here are the same as the relative numerical factors in the powers of c in the expansion.

Now, what exactly is the equation (4.90)? Expanding the metric equation contributions and covariantising everything, one arrives at

$$\begin{aligned}
\tau^{\mu\nu}\bar{\mathcal{G}}^{(0)}_{\mu\nu} &= 2\tau^{\mu A}\nabla^\rho\mathcal{K}_{\mu\rho A} - \nabla^A\mathcal{K}_A - \tfrac{1}{4}a^{ABC}a_{ABC} - \tfrac{1}{2}a^{ABC}a_{ACB} - a^A a_A \\
&\quad - \epsilon_{ABC}F^{DAB\rho}a_{\rho D}{}^C - \tfrac{1}{8}F^{AB\mu\nu}F_{AB\mu\nu} + \tfrac{1}{48}\widetilde{F}^{\mu\nu\rho\sigma}\widetilde{F}_{\mu\nu\rho\sigma} + \tfrac{1}{4}\epsilon_{ABC}\widetilde{F}^{\mu\nu AB}T_{\mu\nu}{}^C \\
&\quad - a^A\mathcal{K}_A + \mathcal{K}^{\mu\nu A}\mathcal{K}_{\mu\nu A} - 2\tau^{\mu A}\tau^{\nu B}\nabla_\nu a_{\mu[AB]} - \tau^{\mu\nu}\nabla_\mu a_\nu\,,
\end{aligned} \tag{4.95}$$

$$\begin{aligned}
\epsilon_{ABC}\tau_\mu{}^A\tau_\nu{}^B\tau_\rho{}^C\Omega^{-1}\mathcal{E}^{\mu\nu\rho}_{(-6)} &= -\tfrac{1}{6}\epsilon_{ABC}\nabla^\mu F^{ABC}{}_\mu - \tfrac{1}{4}\epsilon_{ABC}\widetilde{F}^{AB\mu\nu}T_{\mu\nu}{}^C \\
&\quad + \tfrac{\epsilon^{\lambda_1\dots\sigma_1\dots\sigma_8}}{2\cdot 4!^2\Omega}\tfrac{1}{6}\widetilde{F}_{\sigma_1\dots\sigma_4}\widetilde{F}_{\sigma_5\dots\sigma_8}\epsilon_{ABC}\tau_{\lambda_1}{}^A\tau_{\lambda_2}{}^B\tau_{\lambda_3}{}^C\,,
\end{aligned} \tag{4.96}$$

hence the covariant Poisson equation is

$$\begin{aligned}\mathcal{N} = & -\tfrac{1}{6}\epsilon_{ABC}(\nabla^\mu F^{ABC}{}_\mu + a_\mu F^{ABC\mu} + 3a_{\mu D}{}^A F^{BCD\mu}) - \tfrac{1}{8}F^{AB\mu\nu}F_{AB\mu\nu} \\ & + \tfrac{1}{48}\widetilde{F}^{\mu\nu\rho\sigma}\widetilde{F}_{\mu\nu\rho\sigma} + \tfrac{\Omega^{-1}}{2\cdot 4!^2 3!}\epsilon^{\lambda_1\dots\lambda_3\sigma_1\dots\sigma_8}\widetilde{F}_{\sigma_1\dots\sigma_4}\widetilde{F}_{\sigma_5\dots\sigma_8}\epsilon_{ABC}\tau_{\lambda_1}{}^A\tau_{\lambda_2}{}^B\tau_{\lambda_3}{}^C \\ & - \nabla^A\mathcal{K}_A - a^A\mathcal{K}_A - \mathcal{K}^{\mu\nu A}\mathcal{K}_{\mu\nu A} - 2a^{\mu[AB]}\mathcal{K}_{\mu AB} - 2\tau^{\mu\nu}\nabla_\mu a_\nu \\ & - a^{ABC}(\tfrac{1}{4}a_{ABC} + \tfrac{1}{2}a_{ACB} + \eta_{BC}a_A) \\ = & \, 0\,. \end{aligned} \tag{4.97}$$

Note that this expression could equivalently be rewritten in terms of the Ricci tensor, using the following identity:

$$\mathcal{R}^A{}_A = \tau^{\mu\nu}\mathcal{R}_{\mu\nu} = -\nabla^A\mathcal{K}_A - \mathcal{K}^{\mu\nu A}\mathcal{K}_{\mu\nu A} - a^{\mu AB}\mathcal{K}_{\mu AB}\,. \tag{4.98}$$

Remarkably, equation (4.97) transforms covariantly under *local* dilatations. Exactly this equation will also be selected by the exceptional field theory description as an 'extra' equation of motion that one can not find from the variation of the finite part of the action. Furthermore, under Galilean boosts (discussed in next subsection), it transforms into the other equations of motions. All this is in keeping with the properties of the missing Poisson equation in the NSNS sector and supports including equation (4.97) as an equation of motion of the non-relativistic theory.

If we think in terms of the expansion it might seem strange to find the rest of the equations of motion from the expansion at order c^0 and this extra equation from order c^{-3}. Clearly, if we would vary the action $S^{(-3)}$ we would find additional $\mathcal{O}(c^{-3})$ contributions to the finite equations of motion, and if we would vary the action $S^{(-6)}$ we would find additional $\mathcal{O}(c^{-3})$ contributions to the equation of motion (4.97), i.e. it would become $\mathcal{N} = \mathcal{O}(c^{-3})$. The guiding philosophy is to find the lowest order non-zero equation of motion resulting from the variations of the action. For the Poisson equation associated to the degree of freedom that disappears into dilatations at the level of $S^{(0)}$, this happens to arise at lower order than the other equations of motion.

As a final remark, just as in the NSNS sector case [16], it is also possible to define a covariant derivative that is covariant with respect to dilatations. Letting b_μ denote this dilatation connection, and simultaneously introducing $\omega_\mu{}^{AB}$ as the longitudinal spin connection, we this new affine connection is defined by the following metric compatibility conditions

$$\begin{aligned}\widetilde{\nabla}_\mu \tau_\nu{}^A &= \partial_\mu \tau_\nu{}^A - \omega_\mu{}^{AB}\tau_{\nu B} - b_\mu \tau_\nu{}^A - \widetilde{\Gamma}^\rho_{\mu\nu}\tau_\rho{}^A = 0\,,\\ \widetilde{\nabla}_\mu H^{\rho\sigma} &= \partial_\mu H^{\rho\sigma} - b_\mu H^{\rho\sigma} + \widetilde{\Gamma}^\rho_{\mu\lambda}H^{\lambda\sigma} + \widetilde{\Gamma}^\sigma_{\mu\lambda}H^{\rho\lambda} = 0\,.\end{aligned} \tag{4.99}$$

The solution to these equations is

$$\widetilde{\Gamma}^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} - \tau^\rho{}_A\left(b_\mu \tau_\nu{}^A + \omega_\mu{}^{AB}\tau_{\nu B}\right) - \frac{1}{2}H^{\rho\sigma}\left(b_\mu H_{\nu\rho} + b_\nu H_{\mu\rho} - b_\rho H_{\mu\nu}\right)\,, \tag{4.100}$$

where the dilatation and spin connections are explicitly given by

$$b_\mu = \frac{1}{3}a_\mu + \frac{1}{6}\tau_\mu{}^A a_A\,, \quad \omega_\mu{}^{AB} = -a_\mu{}^{[AB]} + \frac{1}{2}\tau_\mu{}^C a^{AB}{}_C + \tau_\mu{}^{[A}a^{B]}\,. \tag{4.101}$$

4.2.3 Boost invariance

Now let's consider the boost transformations defined in (4.35). The calculations are very similar to those in the previous subsection. The variation of $S^{(3)}$ under (4.35) vanishes identically. The variation of the finite action gives

$$\begin{aligned}\delta S^{(0)} = \int \mathrm{d}^{11}x\Big[&-\Lambda_\rho{}^A\left(2H^{\mu\rho}\tau^\nu{}_A\Omega\mathcal{G}^{(3)}_{\mu\nu} + 3\epsilon_{ABC}\tau_\mu{}^B\tau_\nu{}^C\mathcal{E}^{\mu\nu\rho}_{(0)}\right)\\ &+ \delta_\Lambda \widetilde{C}_{\mu\nu\rho}\mathcal{E}^{\mu\nu\rho}_{(3)}\Big]\,,\end{aligned} \tag{4.102}$$

and the combination of $\mathcal{G}$ and $\mathcal{E}$ terms appearing here is

$$\begin{aligned}&-2\Omega\mathcal{G}^{(3)}_{A\mu}\Lambda^{\mu A} - 3\epsilon_{ABC}\mathcal{E}^{\mu AB}_{(0)}\Lambda_\mu{}^C\\ &\qquad = \tfrac{1}{6}F^{A\mu\nu\rho}\Lambda^\sigma{}_A F_{\sigma\mu\nu\rho} - \tfrac{\epsilon^{\lambda_1\ldots\lambda_3\sigma_1\ldots\sigma_8}}{4\cdot 4!^2\Omega}F_{\sigma_1\ldots\sigma_4}F_{\sigma_5\ldots\sigma_8}\Lambda_{\lambda_1}{}^A\tau_{\lambda_2}{}^B\tau_{\lambda_3}{}^C\epsilon_{ABC}\,.\end{aligned} \tag{4.103}$$

Using $\Lambda_{\mu A}\tau^\mu{}_B = 0$ and the Schouten identity this can be shown to be proportional to the self-duality constraint. Hence the finite action $S^{(0)}$ is invariant under boosts up to a total derivative and the self-duality constraint. To make the action boost-invariant off-shell we must improve the transformations (4.4) by requiring $\widetilde{F}$ to transform as well, similarly to (4.86). The improved boost transformations are

$$\begin{aligned}\delta_\Lambda H_{\mu\nu} &= 2\Lambda_{(\mu}{}^A\tau_{\nu)A}\,, & \delta_\Lambda \tau^\mu{}_A &= -H^{\mu\nu}\Lambda_{\nu A}\,,\\ \delta_\Lambda C_{\mu\nu\rho} &= -3\epsilon_{ABC}\Lambda_{[\mu}{}^A\tau_\nu{}^B\tau_{\rho]}{}^C\,, & \delta_\Lambda \widetilde{F}_{\mu\nu\rho\sigma} &= -4\tau^\lambda{}_A F_{\lambda[\mu\nu\rho}\Lambda_{\sigma]}{}^A\,.\end{aligned} \tag{4.104}$$

Furthermore, one can then check that the set of equations of motion presented in the previous sections is boost-invariant (i.e. closed under boosts) as expected. This includes the extra equation of motion (4.97), which under boosts transforms into the time-space projection of the equation of motion of $H^{\mu\nu}$, equation (4.77), as well as the self-duality constraint. This further implies that it is consistent to include it on the same footing as the remaining equations of motion that can be derived by varying $S^{(0)}$. Indeed, one can obtain the boost variation directly from that of $S^{(-3)}$, which is:

$$\delta S^{(-3)} = \int \mathrm{d}^{11}x \Big[-\Lambda_\rho{}^A \left(2H^{\mu\rho}\tau^\nu{}_A \Omega \mathcal{G}^{(0)}_{\mu\nu} + 3\epsilon_{ABC}\tau_\mu{}^B \tau_\nu{}^C \mathcal{E}^{\mu\nu\rho}_{(-3)} \right) + \delta_\Lambda \widetilde{C}_{\mu\nu\rho} \mathcal{E}^{\mu\nu\rho}_{(0)} \Big] . \tag{4.105}$$

The quantity in round brackets is exactly the time-space projection of the $H^{\mu\nu}$ equation of motion. (As a side-remark, note that this means that the boost variation of $S^{(-3)}$ is not identically zero, although it is zero on using the equations of motion following from the finite action.)

4.3 Dimensional reductions and type IIA Newton-Cartan

In this section we will propose reductions from the eleven-dimensional Newton-Cartan theory to ten-dimensional type IIA Newton-Cartan theories. We have a choice of whether to reduce on a longitudinal or a transverse direction. Reducing on a longitudinal direction will lead to type IIA stringy Newton-Cartan with RR fields. Reducing on a transverse direction will lead to a novel type IIA Newton-Cartan geometry which can be thought of as arising from a non-relativistic limit associated to D2 branes rather than strings. Similar reductions have been carried out in [104, 105] from the M2 worldvolume theory.

For comparison with the reduction ansatzes below, let us record here the usual decomposition of the eleven-dimensional metric and three-form into ten-dimensional fields:

$$d\hat{s}^2_{11} = e^{4\hat{\Phi}/3}(dy + \hat{A}_1)^2 + e^{-2\hat{\Phi}/3} d\hat{s}^2_{10} , \quad \hat{C}_3 = \hat{A}_3 + \hat{B}_2 \wedge dy , \tag{4.106}$$

where y denotes the direction on which we reduce.

Index book-keeping In this section, we denote the eleven-dimensional Newton-Cartan fields and curved spacetime indices with hats, thus $\hat{h}^a{}_{\hat{\mu}}$, $\hat{\tau}_{\hat{\mu}}{}^A$, $\hat{\Omega}$, and so on such that the eleven-dimensional coordinates are $x^{\hat{\mu}} = (x^\mu, y)$, with $\mu = 0, \dots, 9$. We assume that we have an isometry in the y direction. The eleven-dimensional three-forms are denoted $C_{\hat{\mu}\hat{\nu}\hat{\rho}}$, $\widetilde{C}_{\hat{\mu}\hat{\nu}\hat{\rho}}$.

4.3.1 Type IIA SNC

Here we present a reduction ansatz which produces the known Stueckelberg gauge-fixed form of the SNC NSNS sector action, supplemented with RR fields.

Reduction ansatz We want to reduce on a longitudinal direction. We therefore split the longitudinal index $A = (\mathsf{A}, 2)$ with $\mathsf{A} = 0, 1$. Then we single out

$$\hat{\tau}^2 \equiv e^{2\Phi/3}(dy + A_\mu dx^\mu)\,, \tag{4.107}$$

thereby defining the dilaton Φ and RR one-form A_μ that will appear in the reduced theory. If we take $\hat{\tau}_2 = e^{-2\Phi/3}\partial_y$ then the remaining pair of Newton-Cartan clock forms and vectors must have the form

$$\hat{\tau}^{\mathsf{A}} = e^{-\Phi/3}\tau_\mu{}^{\mathsf{A}} dx^\mu\,, \quad \hat{\tau}_{\mathsf{A}} = e^{+\Phi/3}(\tau^\mu{}_{\mathsf{A}}\partial_\mu, -\tau^\nu{}_{\mathsf{A}} A_\nu \partial_y)\,. \tag{4.108}$$

A compatible ansatz for the transverse vielbein is

$$\hat{h}^a{}_{\hat{\mu}} = (e^{-\Phi/3} h^a{}_\mu, 0)\,, \quad \hat{h}^{\hat{\mu}}{}_a = (e^{\Phi/3} h^\mu{}_a, -e^{\Phi/3} h^\nu{}_a A_\nu)\,. \tag{4.109}$$

These are such that $\tau_\mu{}^{\mathsf{A}}$, $\tau^\mu{}_{\mathsf{A}}$ and $h^\mu{}_a$, $h^a{}_\mu$ are ten-dimensional fields obeying the usual stringy Newton-Cartan completeness identities. We can define $\tau_{\mu\nu} \equiv \tau_\mu{}^{\mathsf{A}}\tau_\nu{}^{\mathsf{B}}\eta_{\mathsf{AB}}$, $H_{\mu\nu} \equiv h^a{}_\mu h^b{}_\nu \delta_{ab}$, and similarly for the projective inverses. We also have

$$\hat{\Omega} = e^{-8\Phi/3}\Omega\,, \quad \Omega \equiv \tfrac{1}{2!8!}\epsilon^{\mu\nu\sigma_1\dots\sigma_8}\epsilon_{\mathsf{AB}}\epsilon_{a_1\dots a_8}\tau_\mu{}^{\mathsf{A}}\tau_\nu{}^{\mathsf{B}} h^{a_1}{}_{\sigma_1}\dots h^{a_8}{}_{\sigma_8}\,. \tag{4.110}$$

Finally, we make the traditional decomposition of the three-form and its field strength:

$$\begin{aligned} C_3 &= A_3 + B_2 \wedge dy\,,\\ F_4 &= G_4 + H_3 \wedge (dy + A_1)\,,\\ G_4 &= dA_3 - A_1 \wedge \mathcal{H}_3\,,\\ \mathcal{H}_3 &= dB_2\,, \end{aligned} \tag{4.111}$$

where $A_1 \equiv A_\mu dx^\mu$, along with

$$\begin{aligned} \widetilde{C}_3 &= \widetilde{A}_3 + \widetilde{B}_2 \wedge dy\,, \\ \widetilde{F}_4 &= \widetilde{G}_4 + \widetilde{\mathcal{H}}_3 \wedge (dy + A_1)\,, \\ \widetilde{G}_4 &= d\widetilde{A}_3 - A_1 \wedge \widetilde{\mathcal{H}}_3\,, \\ \widetilde{\mathcal{H}}_3 &= d\widetilde{B}_2\,. \end{aligned} \tag{4.112}$$

Interpretation as an expansion Inserting the above ansatz into the original limit (4.1) gives

$$\begin{aligned} d\hat{s}^2_{11} &= c^2 e^{4\Phi/3}(dy + A_1)^2 + e^{-2\Phi/3}(c^2\tau_{\mu\nu} + c^{-1}H_{\mu\nu})dx^\mu dx^\nu\,, \\ \hat{C}_3 &= -c^3 \tfrac{1}{2}\epsilon_{\mathsf{AB}}\tau^{\mathsf{A}} \wedge \tau^{\mathsf{B}} \wedge dy + A_3 + B_2 \wedge dy + c^{-3}(\widetilde{A}_3 + \widetilde{B}_2 \wedge dy)\,. \end{aligned} \tag{4.113}$$

Hence according to (4.106) this translates into the following expansion of the ten-dimensional type IIA string frame metric $\hat{g}_{\mu\nu}$, NSNS two-form, $\hat{B}_2$, and dilaton $\hat{\Phi}$:

$$\begin{aligned} \hat{g}_{\mu\nu} &= c_s^2\tau_{\mu\nu} + H_{\mu\nu}\,, \\ \hat{B}_2 &= -c_s^2\epsilon_{\mathsf{AB}}\tau^{\mathsf{A}} \wedge \tau^{\mathsf{B}} + B_2 + c_s^{-2}\widetilde{B}_2\,, \\ e^{\hat{\Phi}} &= c_s e^{\Phi}\,, \end{aligned} \tag{4.114}$$

where $c_s \equiv c^{3/2}$. This is nothing but the limit leading to stringy Newton-Cartan [16]. Recall that in that case the true speed of light C was rescaled by a dimensionless parameter, $C \to c_s C$, and the non-relativistic limit was defined as $c_s \to \infty$, which also gives the relation between the expansion parameter c and the speed of light C.

In addition, we have an expansion of the RR fields:

$$\hat{A}_3 = A_3 + c_s^{-2}\widetilde{A}_3\,, \qquad \hat{A}_1 = A_1\,. \tag{4.115}$$

It is clear from these expressions that we can equivalently view this reduction as the result of the usual M-theory to type IIA reduction using (4.106) followed by the SNC field redefinitions of (4.114) and (4.115). At first glance, this is not completely general, given that the ansatz for the RR 1-form A_1 does not involve a subleading term while the other gauge fields do. A justification for the above ansatz is that it correctly produces the NSNS sector dynamics of SNC. Modifications to the ansatz would involve relaxing the implicit Stueckelberg gauge-fixing in 11-dimensions and comparing this to the possible ten-dimensional expansions. We do not consider this in this thesis.

Constraint The constraint (4.33) becomes

$$\Omega H^{\mu_1\nu_1}H^{\mu_2\nu_2}H^{\mu_3\nu_3}H^{\mu_4\nu_4}G_{\nu_1\nu_2\nu_3\nu_4} = -\tfrac{1}{4!2!}\epsilon^{\mu_1\dots\mu_{10}}G_{\mu_5\mu_6\mu_7\mu_8}\epsilon_{\mathsf{AB}}\tau_{\mu_9}{}^{\mathsf{A}}\tau_{\mu_{10}}{}^{\mathsf{B}} \tag{4.116}$$

and so only involves the RR 4-form field strength. The field strength of the NSNS two-form is not constrained. This is to be expected, as the limit of the NSNS sector alone makes sense without any constraint, and in the eleven-dimensional case the constraint arose as a consequence of the Chern-Simons term, which is not present in the truncation to the NSNS sector.

Type IIA SNC with RR fields The action obtained from the reduction ansatz (4.108) and (4.109) is

$$S_{\text{IIA SNC}} = \int \mathrm{d}^{10}x\,\Omega\,\left(e^{-2\Phi}\mathcal{L} + \mathcal{L}_{\widetilde{G}} + \Omega^{-1}\mathcal{L}_{\text{top}}\right)\,, \tag{4.117}$$

with

$$\begin{aligned}
\mathcal{L} &= \mathcal{R} - a^{\mu\mathsf{AB}}a_{\mu\{\mathsf{AB}\}} + (a^\mu - 2D^\mu\Phi)(a_\mu - 2D_\mu\Phi) - \tfrac{1}{12}\mathcal{H}^{\mu\nu\rho}\mathcal{H}_{\mu\nu\rho} \\
&\quad - \tfrac{\epsilon^{\mathsf{AB}}}{2}\mathcal{H}_{\mathsf{A}\mu\nu}T^{\mu\nu}{}_{\mathsf{B}} - \tfrac{1}{2}e^{2\Phi}G^{\mu\mathsf{A}}G_{\mu\mathsf{A}} - \tfrac{1}{12}e^{2\Phi}G^{\mu\nu\rho\mathsf{A}}G_{\mu\nu\rho\mathsf{A}} + \tfrac{\epsilon^{\mathsf{AB}}}{4}e^{2\Phi}G_{\mathsf{AB}\rho\sigma}G^{\rho\sigma}\,, \\
\mathcal{L}_{\widetilde{G}} &= -\tfrac{1}{4!}\widetilde{G}_{\nu_1\dots\nu_4}\left(G^{\nu_1\dots\nu_4} + \tfrac{1}{4!2!\Omega}\epsilon^{\nu_1\dots\nu_4\mu_1\dots\mu_6}G_{\mu_1\dots\mu_4}\epsilon_{\mathsf{AB}}\tau_{\mu_5}{}^{\mathsf{A}}\tau_{\mu_6}{}^{\mathsf{B}}\right)\,, \\
\mathcal{L}_{\text{top}} &= \tfrac{1}{2}dA_3\wedge dA_3\wedge B_2\,,
\end{aligned} \tag{4.118}$$

using the field strengths defined in (4.111) and (4.112) along with $G_{\mu\nu} \equiv 2\partial_{[\mu}A_{\nu]}$. As before, we write for convenience $G^{\mu\nu} \equiv H^{\mu\rho}H^{\nu\sigma}G_{\rho\sigma}$. The Ricci scalar and connection, torsion, acceleration and so on are defined in the same way as before but for the SNC geometry. If we ignore the RR fields, this is exactly the Stueckelberg gauge fixed action for NSNS SNC (note that the subleading component $\widetilde{B}_2$ only appears in the definition of $\widetilde{G}_4$). Furthermore, one can check that the reduction of the Poisson equation agrees with the Poisson equation for SNC, with of course additional contributions from the

RR sector. The reduced Poisson equation is found to be

$$\begin{aligned}
&-\tfrac{1}{2}\epsilon_{\mathsf{AB}}\nabla_\mu\mathcal{H}^{\mathsf{AB}\mu}+\nabla^{\mathsf{A}}\mathcal{K}_{\mathsf{A}}-2\tau^{\mu\nu}\nabla_\mu\nabla_\nu\Phi+2\tau^{\mu\nu}\nabla_\mu a_\nu+\epsilon_{\mathsf{AB}}\mathcal{H}^{\mathsf{AB}\mu}\nabla_\mu\Phi\\
&+\mathcal{K}^{\mu\nu\mathsf{A}}\mathcal{K}_{\mu\nu\mathsf{A}}+a^{\mathsf{A}}\mathcal{K}_{\mathsf{A}}+2a^{\mu[\mathsf{AB}]}\mathcal{K}_{\mu\mathsf{AB}}+a^{\mathsf{ABC}}\left(\tfrac{1}{4}a_{\mathsf{ABC}}+\tfrac{1}{2}a_{\mathsf{ACB}}+\eta_{\mathsf{BC}}a_{\mathsf{A}}\right)\\
&+\tfrac{1}{4}\mathcal{H}^{\mathsf{A}\mu\nu}\mathcal{H}_{\mathsf{A}\mu\nu}-\epsilon_{\mathsf{AB}}\mathcal{H}^{\mathsf{CB}\mu}\left(a_{\mu\mathsf{C}}{}^{\mathsf{A}}+\tfrac{1}{2}a_\mu\delta^{\mathsf{A}}_{\mathsf{C}}\right)+\tfrac{1}{4}e^{2\Phi}\left(G^{\mathsf{AB}}G_{\mathsf{AB}}+\tfrac{1}{2}G^{\mathsf{AB}\mu\nu}G_{\mathsf{AB}\mu\nu}\right)\\
&-2a^{\mathsf{A}}\nabla_{\mathsf{A}}\Phi-e^{2\Phi}\tfrac{1}{48}\left(\widetilde{G}^{\mu\nu\rho\sigma}\widetilde{G}_{\mu\nu\rho\sigma}+\tfrac{1}{48\Omega}\epsilon^{\lambda_1\lambda_2\mu_1\ldots\mu_8}\widetilde{G}_{\mu_1\ldots\mu_4}\widetilde{G}_{\mu_5\ldots\mu_8}\epsilon_{\mathsf{AB}}\tau_{\lambda_1}{}^{\mathsf{A}}\tau_{\lambda_2}{}^{\mathsf{B}}\right)\\
&=0\,.
\end{aligned}\tag{4.119}$$

In this case [16], it is the longitudinal components of the NSNS two-form playing the role of the Newton potential. It is also interesting to look at the reduction of the equation (4.71), which was the equation of motion of the longitudinal components of the three-form. This reduces to

$$\tfrac{1}{2}\eta_{\mathsf{AB}}H^{\mu\rho}H^{\nu\sigma}T_{\mu\nu}{}^{\mathsf{A}}T_{\rho\sigma}{}^{\mathsf{B}}=-\tfrac{1}{48}e^{2\Phi}H^{\mu_1\nu_1}\ldots H^{\mu_4\nu_4}G_{\mu_1\ldots\mu_4}G_{\nu_1\ldots\nu_4}\,,\tag{4.120}$$

and in particular in the truncation to the NSNS sector the right-hand side is zero. This allows imposing foliation constraints on the NSNS sector SNC torsion $T_{\mu\nu}{}^{\mathsf{A}}$, such as those discussed in [16].

4.3.2 Type IIA D2NC

General decompositions breaking local rotational invariance The next reduction we do involves reducing on a transverse reduction. This breaks part of the local SO(8) rotational invariance. Accordingly, write the flat index $a=(\mathsf{a},\bar{\imath})$, with $\mathsf{a}=1,\ldots,8-q$ and $\bar{\imath}=1\ldots q$. Simultaneously we can consider a *different* decomposition of the spacetime coordinate index $\hat{\mu}=(\mu,i)$ where μ is n-dimensional and i is $(11-n)$-dimensional. We then pick a lower triangular form for the vielbein $\hat{h}^a{}_{\hat{\mu}}$ such that

$$\hat{h}^a{}_{\hat{\mu}}=\begin{pmatrix} h^{\mathsf{a}}{}_\mu & 0\\ A_\mu{}^k h^{\bar{\imath}}{}_k & h^{\bar{\imath}}{}_i\end{pmatrix}\,.\tag{4.121}$$

The condition $\hat{h}^a{}_{\hat{\mu}}\hat{\tau}^{\hat{\mu}}{}_A=0$ implies

$$h^{\mathsf{a}}{}_\mu\hat{\tau}^\mu{}_A=0\,,\quad h^{\bar{\imath}}{}_i(\hat{\tau}^i{}_A+A_\mu{}^i\hat{\tau}^\mu{}_A)=0\,.\tag{4.122}$$

The diagonal blocks in (4.121) will in general not be square. Two interesting examples however are to take these blocks to be square and invertible. In this subsection, we will take the lower right block to be a non-zero 1×1 matrix, and perform a reduction to a novel type of type IIA Newton-Cartan geometry associated to D2 branes. In section 4.4, we will take the upper left block to be an invertible $(11-d) \times (11-d)$ matrix, and offer a description of the M-theory Newton-Cartan theory in terms of exceptional field theory.

Transverse reduction to type IIA The dimensional reduction to type IIA corresponds to taking $n = 10$, and $q = 1$ above. We again label the coordinates again as $x^{\hat{\mu}} = (x^\mu, y)$. In this case $h^{\bar{y}}{}_y$ is a scalar and we can identify it with the dilaton as $h^{\bar{y}}{}_y \equiv e^{2\Phi/3}$.[6] Using the conditions (4.122), the full Kaluza-Klein ansatz is:

$$\hat{h}^a{}_{\hat{\mu}} = \begin{pmatrix} e^{-\Phi/3} h^{\mathsf{a}}{}_\mu & 0 \\ e^{2\Phi/3} A_\mu & e^{2\Phi/3} \end{pmatrix}, \quad \hat{h}^{\hat{\mu}}{}_a = \begin{pmatrix} e^{\Phi/3} h^\mu{}_{\mathsf{a}} & 0 \\ -e^{\Phi/3} A_\nu h^\nu{}_{\mathsf{a}} & e^{-2\Phi/3} \end{pmatrix}, \tag{4.123}$$

$$\hat{\tau}_{\hat{\mu}}{}^A = e^{-\Phi/3}(\tau_\mu{}^A, 0), \quad \hat{\tau}^{\hat{\mu}}{}_A = e^{+\Phi/3}(\tau^\mu{}_A, -A_\nu \tau^\nu{}_A), \tag{4.124}$$

plus the same definitions (4.111) and (4.112) for the three-forms and field strengths. We also have

$$\hat{\Omega} = e^{-8\Phi/3}\Omega, \quad \Omega \equiv \tfrac{1}{3!7!}\epsilon^{\mu\nu\rho\sigma_1\dots\sigma_7}\epsilon_{ABC}\epsilon_{\mathsf{a}_1\dots\mathsf{a}_7}\tau_\mu{}^A\tau_\nu{}^B\tau_\rho{}^C h^{\mathsf{a}_1}{}_{\sigma_1}\dots h^{\mathsf{a}_7}{}_{\sigma_1}. \tag{4.125}$$

Interpretation as an expansion Inserting the above ansatz into the original limit (4.1) gives

$$\begin{aligned} d\hat{s}^2_{11} &= c^{-1}e^{4\Phi/3}(dy + A_1)^2 + e^{-2\Phi/3}(c^2\tau_{\mu\nu} + c^{-1}H_{\mu\nu}), \\ \hat{C}_3 &= -c^3 e^{-\Phi}\tfrac{1}{3!}\epsilon_{ABC}\tau^A \wedge \tau^B \wedge \tau^C + A_3 + B_2 \wedge dy + c^{-3}(\widetilde{A}_3 + \widetilde{B}_2 \wedge dy). \end{aligned} \tag{4.126}$$

Hence according to (4.106) this translates into the following expansion of the ten-dimensional type IIA string frame metric $\hat{g}_{\mu\nu}$, RR three-form, $\hat{C}_2$, and dilaton $\hat{\Phi}$:

$$\begin{aligned} \hat{g}_{\mu\nu} &= c_D^2\tau_{\mu\nu} + c_D^{-2}H_{\mu\nu}, \\ \hat{C}_3 &= -c_D^4\epsilon_{ABC}e^{-\Phi}\tau^A \wedge \tau^B \wedge \tau^C + C_3 + c_D^{-4}\widetilde{C}_3, \\ e^{\hat{\Phi}} &= c_D^{-1}e^{\Phi}, \end{aligned} \tag{4.127}$$

[6]Enthusiasts of non-relativistic geometries could also consider null reductions of the already non-relativistic theory.

along with expansions for the NSNS two-form, $\hat{B}_2$, and RR one-form, $\hat{A}_1$:

$$\hat{B}_2 = B_2 + c_D^{-4}\widetilde{B}_2, , \quad \hat{A}_1 = A_1\,, \tag{4.128}$$

where $c_D \equiv c^{3/4}$. This is an expansion and non-relativistic limit associated to the D2 brane (the powers of c_D appear in the same way as those of the harmonic function in the D2 brane SUGRA solution). We can refer to it as D2 Newton-Cartan (D2NC).

Constraint The constraint (4.33) becomes

$$\begin{aligned}\Omega H^{\mu_1\nu_1}H^{\mu_2\nu_2}H^{\mu_3\nu_3}H^{\mu_4\nu_4}G_{\nu_1\nu_2\nu_3\nu_4} &= \tfrac{\epsilon^{\mu_1\cdots\mu_{10}}}{3!3!}e^{-\Phi}H_{\mu_5\mu_6\mu_7}\epsilon_{ABC}\tau_{\mu_8}{}^{A}\tau_{\mu_9}{}^{B}\tau_{\mu_{10}}{}^{C}\,,\\ \Omega e^{-\Phi}H^{\mu_1\nu_1}H^{\mu_2\nu_2}H^{\mu_3\nu_3}\mathcal{H}_{\nu_1\nu_2\nu_3} &= \tfrac{\epsilon^{\mu_1\cdots\mu_{10}}}{4!3!}G_{\mu_4\mu_5\mu_6\mu_7}\epsilon_{ABC}\tau_{\mu_8}{}^{A}\tau_{\mu_9}{}^{B}\tau_{\mu_{10}}{}^{C}\,,\end{aligned} \tag{4.129}$$

which are equivalent. So now we have a duality relation between the RR three-form gauge field and the NSNS two-form.

Type IIA D2 Newton-Cartan theory The action obtained from the reduction ansatz (4.123) and (4.124) is

$$S_{\text{D2NC}} = \int \mathrm{d}^{10}x\,\Omega\,\left(e^{-2\Phi}\mathcal{L} + \mathcal{L}_{\widetilde{G}} + \Omega^{-1}\mathcal{L}_{\text{top}}\right)\,, \tag{4.130}$$

with

$$\begin{aligned}\mathcal{L} =& \mathcal{R} - a^{\mu AB}a_{\mu(AB)} + \tfrac{3}{2}a^{\mu}a_{\mu} - 5a^{\mu}D_{\mu}\Phi + \tfrac{9}{2}D^{\mu}\Phi D_{\mu}\Phi - \tfrac{1}{4}\mathcal{H}^{\mu\nu A}\mathcal{H}_{\mu\nu A}\\ &- \tfrac{1}{4}e^{2\Phi}G^{\mu\nu}G_{\mu\nu} - \tfrac{1}{12}e^{2\Phi}G^{\mu\nu\rho A}G_{\mu\nu\rho A} + \tfrac{1}{4}e^{\Phi}\epsilon^{ABC}G_{AB\rho\sigma}T^{\rho\sigma}{}_{C},\\ \mathcal{L}_{\widetilde{G}} =& -\tfrac{1}{4!}\widetilde{G}_{\nu_1\ldots\nu_4}\left(G^{\nu_1\ldots\nu_4} - \tfrac{1}{3!^2\Omega}e^{-\Phi}\epsilon^{\nu_1\ldots\nu_4\mu_1\ldots\mu_6}\mathcal{H}_{\mu_1\ldots\mu_3}\epsilon_{ABC}\tau_{\mu_4}{}^{A}\tau_{\mu_5}{}^{B}\tau_{\mu_6}{}^{C}\right)\\ &- \tfrac{1}{3!}e^{-2\Phi}\widetilde{\mathcal{H}}_{\nu_1\ldots\nu_3}\left(\mathcal{H}^{\nu_1\ldots\nu_3} - \tfrac{\epsilon^{\nu_1\cdots\nu_3\mu_1\cdots\mu_7}}{4!3!\Omega}e^{\Phi}G_{\mu_1\ldots\mu_4}\epsilon_{ABC}\tau_{\mu_5}{}^{A}\tau_{\mu_6}{}^{B}\tau_{\mu_7}{}^{C}\right),\\ =& -\tfrac{1}{4!}\left(\widetilde{G}_{\nu_1\ldots\nu_4} - e^{-\Phi}\widetilde{\mathcal{H}}_{\rho_1\ldots\rho_3}\tfrac{\epsilon^{\rho_1\cdots\rho_3\sigma_1\cdots\sigma_7}\epsilon_{ABC}}{3!^2\Omega}H_{\nu_1\sigma_1}\ldots H_{\nu_4\sigma_4}\tau_{\sigma_5}{}^{A}\tau_{\sigma_6}{}^{B}\tau_{\sigma_7}{}^{C}\right)\\ &\times\left(G^{\nu_1\ldots\nu_4} - \tfrac{1}{3!^2\Omega}e^{-\Phi}\epsilon^{\nu_1\ldots\nu_4\mu_1\ldots\mu_6}\mathcal{H}_{\mu_1\ldots\mu_3}\epsilon_{ABC}\tau_{\mu_4}{}^{A}\tau_{\mu_5}{}^{B}\tau_{\mu_6}{}^{C}\right)\,,\\ \mathcal{L}_{\text{top}} =& \tfrac{1}{2}dA_3\wedge dA_3\wedge B_2\,,\end{aligned} \tag{4.131}$$

where the field strengths are defined as in (4.111) and (4.112) with again $G_2 \equiv dA_1$. Note that we obtain what appears to be an extra contribution to the dilaton kinetic term due to the $e^{-\Phi}$ factor that in the expansion of $\hat{C}_3$ in (4.127). We could alter this by redefining the RR fields in the reduced theory. In addition, the reduction of the Poisson equation (4.97) gives

$$
\begin{aligned}
&\tfrac{1}{6}e^{\Phi}\epsilon_{ABC}\left(\nabla_\mu G^{ABC\mu} + a_\mu G^{ABC\mu} + 3a_{\mu D}{}^{A}G^{DBC\mu}\right) - \tfrac{1}{3}e^{\Phi}\epsilon_{ABC}G^{ABC\mu}\nabla_\mu\Phi \\
&+ \nabla^A\mathcal{K}_A - 3\tau^{\mu\nu}\nabla_\mu\nabla_\nu\Phi - 3a^A\nabla_A\Phi + 2\nabla^A\Phi\nabla_A\Phi - \mathcal{K}^A\nabla_A\Phi + 2\tau^{\mu\nu}\nabla_\mu a_\nu \\
&+ \mathcal{K}^{\mu\nu A}\mathcal{K}_{\mu\nu A} + a^A\mathcal{K}_A + 2a^{\mu[AB]}\mathcal{K}_{\mu AB} + a^{ABC}\left(\tfrac{1}{4}a_{ABC} + \tfrac{1}{2}a_{ACB} + \eta_{BC}a_A\right) \\
&+ \tfrac{1}{4}\mathcal{H}^{AB\mu}\mathcal{H}_{AC\mu} + \tfrac{1}{8}e^{2\Phi}\left(G^{AB\mu\nu}G_{AB\mu\nu} + 4G^{A\mu}G_{A\mu}\right) - e^{2\Phi}\tfrac{1}{48}\widetilde{G}^{\mu\nu\rho\sigma}\widetilde{G}_{\mu\nu\rho\sigma} \\
&- \tfrac{1}{12}\widetilde{\mathcal{H}}^{\mu\nu\rho}\widetilde{\mathcal{H}}_{\mu\nu\rho} + e^{-\Phi}\tfrac{1}{4!3!3!\Omega}\epsilon^{\lambda_1\lambda_2\lambda_3\mu_1\ldots\mu_7}\epsilon_{ABC}\tau_{\lambda_1}{}^{A}\tau_{\lambda_2}{}^{B}\tau_{\lambda_3}{}^{C}\widetilde{G}_{\mu_1\ldots\mu_4}\widetilde{\mathcal{H}}_{\mu_5\ldots\mu_7} = 0\,.
\end{aligned}
\tag{4.132}
$$

As in the MNC case, the longitudinal components of the three-form gauge field play the role of the Newton potential.

4.4 Dimensional decompositions and exceptional field theory description

4.4.1 Exceptional field theory

We will now discuss the exceptional field theory description of the eleven-dimensional MNC theory. ExFT automatically has a number of features in common with the non-relativistic theory: breaking of eleven-dimensional Lorentz symmetry, a geometry arising from mixing metric and form-field components, and the inclusion of dual degrees of freedom. We will see how it provides a unified framework treating the relativistic and non-relativistic theory on an equal footing, which demonstrates that the same exceptional Lie algebraic structures that underlie the relativistic theory are present in the non-relativistic one. In addition, the ExFT equations of motion include the additional missing Poisson equation.

We will focus particularly on the relatively unexceptional case of the SL(3) $\times$ SL(2) ExFT [106]. This makes use of an $(8+3)$-dimensional split of the eleven-dimensional spacetime. As such, it is a very natural fit for the $(8+3)$-dimensional split into transverse and longitudinal directions present in the MNC expansion. The SL(3)$\times$SL(2) ExFT includes a Riemannian metric for the eight-dimensional part of the spacetime, but the 3-dimensional part is described by an 'extended geometry' involving an SL(3) $\times$ SL(2) symmetric generalised metric. By decomposing the eleven-dimensional Newton-Cartan theory appropriately, we will replace the transverse Newton-Cartan metric with an invertible eight-dimensional metric, $\hat{H}^{\hat{\mu}\hat{\nu}} \to g^{\mu\nu}$, and the longitudinal metric with an invertible 3-dimensional metric, $\hat{\tau}_{\hat{\mu}\nu} \to \tau_{ij}$, which will be embedded into the generalised metric description. This drastic simplification of the geometry is nonetheless sufficient to highlight the key features of the theory.

It would also be interesting to consider for example the opposite $(3+8)$-dimensional split corresponding to the $E_{8(8)}$ ExFT, embedding the transverse metric into the $E_{8(8)}$ generalised metric. However as the known formulation of ExFT makes use of a Riemannian metric for the unextended part of the spacetime, this is not immediately available for our purposes. Evidently, for any given $E_{d(d)}$ ExFT, one can construct or imagine multiple other 'hybrid' formulations depending on how one chooses to separate or mix the longitudinal and transverse directions. More ambitiously, one could choose to work with the recently fully constructed 'master' E_{11} ExFT [107], for which no coordinate decomposition is necessary. Evidently this would eschew the technical difficulties of the latter in favour of the technicalities associated to working with an infinite-dimensional algebra. In this chapter, although many features that we will see are quite general, we describe the explicit details mainly for the $d \leq 4$ cases.

ExFT ingredients The basic idea behind ExFT is to replace d-dimensional vectors with *generalised vectors* V^M transforming in a specified representation of $E_{d(d)}$. This representation is such that we can decompose V^M under GL(d) as $V^M = (V^i, V_{ij}, V_{ijklm}, \dots)$ where V^i is a d-dimensional vector, V_{ij} and V_{ijklm} a two- and five-form, and the ellipsis corresponds to higher rank mixed symmetry tensors that appear for $d \geq 7$.[7] Generalised vectors are used to provide an $E_{d(d)}$-compatible local symmetry of *generalised diffeomorphisms*. These are defined in terms of a *generalised Lie derivative* which acts

[7] This decomposition is relevant to the description of eleven-dimensional SUGRA. There are also mutually inequivalent GL($d-1$) decompositions relevant to the description of type II SUGRA.

on a generalised vector V^M of weight λ_V as

$$\begin{aligned}\delta_U V^M &= \mathcal{L}_U V^M \\ &\equiv U^N \partial_N V^M - V^N \partial_N U^M + Y^{MN}{}_{PQ} \partial_N U^P V^Q + (\lambda_V - \tfrac{1}{9-d}) \partial_N U^N V^M \,. \end{aligned} \tag{4.133}$$

Here $Y^{MN}{}_{PQ}$ is constructed from invariant tensors of $E_{d(d)}$. This together with the weight term with coefficient $-1/(9-d)$ appear such that this generalised Lie derivative involves an infinitesimal $E_{d(d)}$, rather than GL(N) transformation. The partial derivatives written here formally involve an extended set of coordinates y^M. However, consistency requires the imposition of a constraint $Y^{MN}{}_{PQ}\partial_M \partial_N = 0$ where the derivatives can act on a single field or a product of fields. One solution to this constraint is to view the d-dimensional partial derivatives as being embedded such that $\partial_M = (\partial_i, 0, \dots, 0)$. We always assume we have made this choice below. (An alternative solution leads to a ten-dimensional type IIB description.)

Given this choice, for the $d \leq 4$ cases we will look at in detail, the action of $U^M = (u^i, \lambda_{ij})$ on $V^M = (V^i, V_{ij})$ (both having generalised diffeomorphism weight $1/(9-d)$) is $\mathcal{L}_U V^M = (L_u V^i, L_u V^{ij} - 3V^k \partial_{[k}\lambda_{ij]})$, where L_u denotes the usual d-dimensional Lie derivative. Identifying the two-form components λ_{ij} with the gauge transformation parameter of a three-form $\hat{\boldsymbol{C}}_{ijk}$, this means we can write $V^M = (V^i, \widetilde{V}_{ij} - \hat{\boldsymbol{C}}_{ijk} V^k)$, with $\widetilde{V}_{ij}$ gauge invariant. We use this to give explicit parametrisations for the ExFT fields.

The field content of ExFT is as follows. We now let $\mu, \nu, \dots$ be $(11-d)$-dimensional indices. We then have an $(11-d)$-dimensional metric, $g_{\mu\nu}$, which is a scalar of weight $-2/(9-d)$ under generalised diffeomorphisms. The $E_{d(d)}$ extended geometry is equipped with a generalised metric, $\mathcal{M}_{MN}$, transforming as a rank two symmetric tensor of weight zero under generalised diffeomorphisms. In addition, there is a 'tensor hierarchy' of gauge fields, starting with an $(11-d)$-dimensional one-form $\mathcal{A}_\mu{}^M$, and continuing with p-forms $\mathcal{B}_{\mu\nu}$, $\mathcal{C}_{\mu\nu\rho}$, ... in particular representations of $E_{d(d)}$. This set of fields mimics and extends what appears in a dimensional decomposition (or reduction) of the bosonic fields of supergravity.

Dimensional decomposition and field redefinitions We describe now the dimensional decomposition used to embed eleven-dimensional SUGRA in the ExFT framework. We split the eleven-dimensional coordinates $x^{\hat{\mu}} = (x^\mu, y^i)$, making an $(11-d)+d$ split. The supergravity degrees of freedom are then similarly decomposed under this split, classified according to their nature from the point of view of $(11-d)$-dimensional spacetime, and then rearranged into multiplets of the exceptional groups $E_{d(d)}$. We assume no restriction on the coordinate dependence. This can be viewed as a partial fixing of the local Lorentz symmetry in which we choose the eleven-dimensional vielbein $\hat{e}^{\hat{a}}{}_{\hat{\mu}}$ and hence metric $\hat{g}_{\hat{\mu}\hat{\nu}}$ to be

$$\hat{e}^{\hat{a}}{}_{\hat{\mu}} = \begin{pmatrix} |\phi|^{-\frac{1}{2(9-d)}} e^{\mathsf{a}}{}_{\mu} & 0 \\ A_{\mu}{}^{k}\phi^{\bar{\imath}}{}_{k} & \phi^{\bar{\imath}}{}_{i} \end{pmatrix} , \quad \hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} |\phi|^{-\frac{1}{9-d}} g_{\mu\nu} + \phi_{kl} A_{\mu}{}^{k} A_{\nu}{}^{l} & \phi_{ik} A_{\nu}{}^{l} \\ \phi_{jk} A_{\nu}{}^{k} & \phi_{ij} \end{pmatrix} , \tag{4.134}$$

where $e^{\mathsf{a}}{}_{\mu}$ is a vielbein for an $(11-d)$-dimensional (Einstein frame) metric $g_{\mu\nu}$ and $\phi^{\bar{\imath}}{}_{i}$ is a vielbein for a d-dimensional metric ϕ_{ij}, with $|\phi| \equiv |\det(\phi_{ij})|$. Normally one takes $g_{\mu\nu}$ to be Lorentzian, such that this corresponds to fixing the Lorentz symmetry as $\mathrm{SO}(1,10) \to \mathrm{SO}(1,10-d) \times \mathrm{SO}(d)$, however we can also take it to be Euclidean, such that $\mathrm{SO}(1,10) \to \mathrm{SO}(11-d) \times \mathrm{SO}(1,d-1)$. The latter choice is relevant for the version of ExFT applicable to the non-relativistic theory.

The 'Kaluza-Klein vector' $A_{\mu}{}^{i}$ has a field strength defined by

$$F_{\mu\nu}{}^{i} = 2\partial_{[\mu} A_{\nu]}{}^{i} - 2A_{[\mu|}{}^{j}\partial_{j} A_{|\nu]}{}^{i} \,. \tag{4.135}$$

Letting L denote the d-dimensional Lie derivative, the Kaluza-Klein vector also appears as the connection in the derivative $D_{\mu} = \partial_{\mu} - L_{A_{\mu}}$ which is covariant with respect to d-dimensional diffeomorphisms, using the transformation $\delta_{\Lambda} A_{\mu}{}^{i} = D_{\mu}\Lambda^{i}$ induced by the action of 11-dimensional diffeomorphisms on (4.134).

For the three-form and its field strength, we define a succession of gauge field components (denoted by bold font) via

$$\hat{C}_3 = \hat{\boldsymbol{C}}_3 + \hat{\boldsymbol{C}}_{2i} Dy^{i} + \tfrac{1}{2}\hat{\boldsymbol{C}}_{1ij} Dy^{i} Dy^{j} + \tfrac{1}{3!}\hat{\boldsymbol{C}}_{ijk} Dy^{i} Dy^{j} Dy^{k} \tag{4.136}$$

where $Dy^{i} \equiv dy^{i} + A_{\mu}{}^{i} dx^{\mu}$, the subscripts on the right-hand side denote the form degree in $(11-d)$ dimensions, and we omit the implicit wedge products. Similarly, for $\hat{F}_4 = d\hat{C}_3$ we let

$$\begin{aligned} \hat{F}_4 = \hat{\boldsymbol{F}}_4 + \hat{\boldsymbol{F}}_{3i} Dy^{i} + \tfrac{1}{2}\hat{\boldsymbol{F}}_{2ij} Dy^{i} Dy^{j} + \tfrac{1}{3!}\hat{\boldsymbol{F}}_{1ijk} Dy^{i} Dy^{j} Dy^{k} \,, \\ + \tfrac{1}{4!}\hat{\boldsymbol{F}}_{ijkl} Dy^{i} Dy^{j} Dy^{k} Dy^{l} \,, \end{aligned} \tag{4.137}$$

The persistence of hats reflects the fact that we still want to take the non-relativistic limit of all these quantities. Explicit component expressions can be found in appendix C.1. We can make similar redefinitions for the dual six-form and its field strength.

Metric and generalised metrics The metric $g_{\mu\nu}$ appearing in (4.134) is directly used as the $(11-d)$-dimensional ExFT metric (the generalised diffeomorphism weight $-2/(9-d)$ follows from the conformal factor in (4.134)).

The generalised metric $\mathcal{M}_{MN}$, or its generalised vielbein, may be defined as an $E_{d(d)}$ element valued in a coset $E_{d(d)}/H_d$ where H_d is the maximal compact subgroup (in the Euclidean case) or a non-compact version thereof (in the Lorentzian case). Under generalised diffeomorphisms it transforms as a rank two symmetric tensor of weight zero. It is normally parametrised in terms of the wholly d-dimensional components of the eleven-dimensional fields, ϕ_{ij} and $\hat{\boldsymbol{C}}_{ijk}$, in a manner consistent with its transformation under generalised diffeomorphisms. For $d \geq 6$, this parametrisation also includes internal components of the dual-six form. For simplicity, we will restrict to $d \leq 4$, in which case the conventional parametrisation of the generalised metric is given by

$$\mathcal{M}_{MN} = |\phi|^{1/(9-d)} \begin{pmatrix} \phi_{ij} + \frac{1}{2}\hat{\boldsymbol{C}}_i{}^{pq}\hat{\boldsymbol{C}}_{jpq} & \hat{\boldsymbol{C}}_i{}^{kl} \\ \hat{\boldsymbol{C}}_k{}^{ij} & 2\phi^{i[k}\phi^{l]j} \end{pmatrix} . \tag{4.138}$$

The conformal factor here ensures that $|\det \mathcal{M}| = 1$.

In specific cases, we can find factorisations of the generalised metric leading to simpler expressions. This includes the $\mathrm{SL}(3)\times\mathrm{SL}(2)$ ExFT. Here, generalised vectors $V^M = (V^i, V_{ij})$ transform in the $(\mathbf{3},\mathbf{2})$ of $\mathrm{SL}(3)\times\mathrm{SL}(2)$, with $i, j, \ldots$ three-dimensional. We can dualise V_{ij} using the three-dimensional epsilon symbol, and define $\widetilde{V}^i \equiv \frac{1}{2}\epsilon^{ijk}\widetilde{V}_{jk}$. Introduce an SL(2) fundamental index, $\alpha = 1, 2$, and let $V^M \equiv V^{i\alpha}$ with $V^{i1} \equiv V^i$ and $V^{i2} \equiv \widetilde{V}^i$. In terms of this basis we have a factorisation

$$\mathcal{M}_{MN} = \mathcal{M}_{i\alpha,j\beta} = \mathcal{M}_{ij}\mathcal{M}_{\alpha\beta} , \tag{4.139}$$

where $\mathcal{M}_{ij} = \mathcal{M}_{ji}$ with $|\det \mathcal{M}_{ij}| = 1$, and $\mathcal{M}_{\alpha\beta} = \mathcal{M}_{\beta\alpha}$ with $|\det \mathcal{M}_{\alpha\beta}| = 1$. When ϕ_{ij} has Lorentzian signature, the expressions which reproduce

(4.138) are

$$\begin{aligned} \mathcal{M}_{ij} &= |\phi|^{-1/3}\phi_{ij}\,, \\ \mathcal{M}_{\alpha\beta} &= \begin{pmatrix} |\phi|^{1/2} - |\phi|^{-1/2}\hat{\boldsymbol{C}}^2 & -|\phi|^{-1/2}\hat{\boldsymbol{C}} \\ -|\phi|^{-1/2}\hat{\boldsymbol{C}} & -|\phi|^{-1/2} \end{pmatrix}\,, \\ \hat{\boldsymbol{C}} &\equiv \tfrac{1}{3!}\epsilon^{ijk}\hat{\boldsymbol{C}}_{ijk}\,. \end{aligned} \tag{4.140}$$

Gauge fields and dual degrees of freedom Along with the Kaluza-Klein vector, $A_\mu{}^i$, coming from the metric decomposition (4.134), the p-forms obtained from the decomposition (4.136) of the three-form fit into $E_{d(d)}$-valued multiplets denoted $\mathcal{A}_\mu$, $\mathcal{B}_{\mu\nu}$, $\mathcal{C}_{\mu\nu\rho}$, ... Their field strengths are denoted $\mathcal{F}_{\mu\nu}$, $\mathcal{H}_{\mu\nu\rho}, \mathcal{J}_{\mu\nu\rho\sigma}$, ... To obtain full $E_{d(d)}$ representations, we have to include here the set of p-forms obtained by decomposing the dual six-form. This is unsurprising from the point of $E_{d(d)}$ U-duality transformations, which mix electric and magnetic degrees of freedom (e.g. M2 and M5 branes) coupling respectively to p-forms and their duals.

For $d = 3$, this works as follows [106]. The ExFT gauge fields $\mathcal{A}_\mu{}^{i\alpha}$, $\mathcal{B}_{\mu\nu i}$, $\mathcal{C}_{\mu\nu\rho}{}^\alpha$, $\mathcal{D}_{\mu\nu\rho\sigma}{}^i$ have weights $1/6, 2/6, 3/6, 4/6$ respectively, and their field strengths are denoted $\mathcal{F}_{\mu\nu}{}^{i\alpha}$, $\mathcal{H}_{\mu\nu\rho i}$, $\mathcal{J}_{\mu\nu\rho\sigma}{}^\alpha$ and $\mathcal{K}_{\mu\nu\rho\sigma\lambda}{}^i$ (the latter does not appear in the action). Under generalised diffeomorphisms, $\mathcal{F}^{i\alpha}$ transforms as a generalised vector of weight $1/6$, while $\mathcal{H}$ and $\mathcal{J}$ transform via the generalised Lie derivative acting as

$$\begin{aligned} \mathcal{L}_\Lambda \mathcal{H}_i &= \Lambda^{j\beta}\partial_{j\beta}\mathcal{H}_i + \partial_{i\beta}\Lambda^{j\beta}\mathcal{H}_j\,, \\ \mathcal{L}_\Lambda \mathcal{J}^\alpha &= \Lambda^{j\beta}\partial_{j\beta}\mathcal{J}^\alpha - \partial_{j\beta}\Lambda^{j\alpha}\mathcal{J}^\beta + \partial_{j\beta}\Lambda^{j\beta}\mathcal{J}^\alpha\,. \end{aligned} \tag{4.141}$$

These field strengths obey Bianchi identities:

$$3\mathcal{D}_{[\mu}\mathcal{F}_{\nu\rho]}{}^{i\alpha} = \epsilon^{ijk}\epsilon^{\alpha\beta}\partial_{j\beta}\mathcal{H}_{\mu\nu\rho k}\,, \tag{4.142}$$

$$4\mathcal{D}_{[\mu}\mathcal{H}_{\nu\rho\sigma]i} + 3\epsilon_{ijk}\epsilon_{\alpha\beta}\mathcal{F}_{[\mu\nu}{}^{j\alpha}\mathcal{F}_{\rho\sigma]}{}^{k\beta} = \partial_{i\alpha}\mathcal{J}_{\mu\nu\rho\sigma}{}^\alpha\,, \tag{4.143}$$

$$5\mathcal{D}_{[\mu}\mathcal{J}_{\nu\rho\sigma\lambda]}{}^\alpha + 10\mathcal{F}_{[\mu\nu}{}^{i\alpha}\mathcal{H}_{\rho\sigma\lambda]i} = \epsilon^{\alpha\beta}\partial_{i\beta}\mathcal{K}_{\mu\nu\rho\sigma\lambda}{}^i\,, \tag{4.144}$$

where $\mathcal{D}_\mu \equiv \partial_\mu - \mathcal{L}_{\mathcal{A}_\mu}$. The ExFT one-form can be simply identified as $\mathcal{A}_\mu{}^M = (A_\mu{}^i, \frac{1}{2}\epsilon^{ijk}\boldsymbol{C}_{\mu jk})$. The two-form $\mathcal{B}_{\mu\nu i}$ transforms in the $(\bar{\mathbf{3}}, \mathbf{1})$ of SL(3) × SL(2) and is identified (up to a further field redefinition) with $\hat{\boldsymbol{C}}_{\mu\nu i}$. However, rather than give the precise field redefinitions for the potentials, it is simpler to work at the level of the field strengths. These are all tensors under generalised diffeomorphisms, meaining in particular that they transform in a particular way under d-dimensional three-form gauge transformations. This allows us to decompose in terms of manifestly gauge invariant combinations

$$\mathcal{F}_{\mu\nu}{}^{i1} \equiv F_{\mu\nu}{}^{i}\,, \quad \mathcal{F}_{\mu\nu}{}^{i2} \equiv \frac{1}{2}\epsilon^{ijk}(\hat{\boldsymbol{F}}_{\mu\nu jk} - \hat{\boldsymbol{C}}_{jkl}\hat{\boldsymbol{F}}_{\mu\nu}{}^{l})\,, \quad \mathcal{H}_{\mu\nu\rho i} \equiv -\hat{\boldsymbol{F}}_{\mu\nu\rho i}\,, \tag{4.145}$$

where $F_{\mu\nu}{}^{i}$, $\hat{\boldsymbol{F}}_{\mu\nu\rho i}$ and $\hat{\boldsymbol{F}}_{\mu\nu jk}$ are gauge invariant and can be exactly identified with the quantities defined in (4.137) with $F_{\mu\nu}{}^{i}$ as in (4.135).[8]

The three-form situation is then where it gets interesting. There is a single 8-dimensional three-form $\hat{\boldsymbol{C}}_{\mu\nu\rho}$ obtained from the eleven-dimensional one. There is also a single three-form $\hat{\boldsymbol{C}}_{\mu\nu\rho ijk}$ coming from the eleven-dimensional six-form. Together these form an SL(3) singlet and SL(2) doublet, for which the field strength obeys a self-duality constraint reproducing (in the relativistic case!) the correct duality relationship between the field strengths $\hat{\boldsymbol{F}}_{\mu\nu\rho\sigma}$ and $\hat{\boldsymbol{F}}_{\mu\nu\rho\sigma ijk}$. This duality constraint, which has to be imposed by hand, involves the eight-dimensional Hodge star acting on the 8-dimensional indices and the SL(2) generalised metric acting on the SL(2) indices:

$$\sqrt{|g|}\mathcal{M}_{\alpha\beta}\mathcal{J}^{\mu\nu\rho\sigma\beta} = -48\kappa\epsilon_{\alpha\beta}\epsilon^{\mu\nu\rho\sigma\lambda_1\ldots\lambda_4}\mathcal{J}_{\lambda_1\ldots\lambda_4}{}^{\beta}\,. \tag{4.146}$$

The coefficient κ is fixed via the self-consistency of (4.146) (in both the cases where $g_{\mu\nu}$ has Lorentzian or Euclidean signature, with $\mathcal{M}_{\alpha\beta}$ having the opposite) to be $\kappa = \pm\frac{1}{2\cdot(24)^2}$, with the choice of sign being a matter of convention (equivalent to changing the sign of the three-form in eleven dimensions). This is consistent with decomposing the SL(2) doublet of four-form field strengths as

$$\mathcal{J}_{\mu\nu\rho\sigma}{}^{1} \equiv \hat{\boldsymbol{F}}_{\mu\nu\rho\sigma}\,, \quad \mathcal{J}_{\mu\nu\rho\sigma}{}^{2} \equiv \tfrac{1}{6}\epsilon^{ijk}(\hat{\boldsymbol{F}}_{\mu\nu\rho\sigma ijk} - \hat{\boldsymbol{C}}_{ijk}\hat{\boldsymbol{F}}_{\mu\nu\rho\sigma})\,. \tag{4.147}$$

Thus in general, ExFT treats simultaneously degrees of freedom coming from the three-form with dual degrees of freedom coming from the six-form, encoding the duality relations between them in its dynamics.

[8] The minus sign in $\mathcal{H}_{\mu\nu\rho i}$ ensures that the ExFT Bianchi identities (4.143) and (4.144) reproduce those coming from SUGRA in (C.17) and is otherwise simply a matter of convention in terms of what we call $\mathcal{B}_{\mu\nu i}$.

Dynamics: $\mathrm{SL}(3)\times\mathrm{SL}(2)$ **ExFT pseudo-action** The ExFT Lagrangian can be uniquely fixed by the requirement of invariance under the local symmetries (generalised diffeomorphisms, gauge transformations of the tensor hierarchy, and finally $(11-d)$-dimensional diffeomorphisms). When $11-d$ is even, this gives a pseudo-action which must be accompanied by a self-duality constraint such as (4.146). This includes the case $d=3$. The pseudo-action in this case can be written as $S=\int \mathrm{d}^8x\,\mathrm{d}^6y\sqrt{|g|}\mathcal{L}_{\text{ExFT}}$ where the Lagrangian has the (quite general) expression

$$\mathcal{L}_{\text{ExFT}} = R_{\text{ext}}(g) + \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{int}} + \sqrt{|g|}^{-1}\mathcal{L}_{\text{top}}\,. \tag{4.148}$$

Here, with $\mathcal{D}_\mu = \partial_\mu - \mathcal{L}_{\mathcal{A}_\mu}$, we have

$$\begin{aligned}
R_{\text{ext}}(g) &= \frac{1}{4}g^{\mu\nu}\mathcal{D}_\mu g_{\rho\sigma}\mathcal{D}_\nu g^{\rho\sigma} - \frac{1}{2}g^{\mu\nu}\mathcal{D}_\mu g^{\rho\sigma}\mathcal{D}_\rho g_{\nu\sigma} + \frac{1}{4}g^{\mu\nu}\mathcal{D}_\mu \ln g\mathcal{D}_\nu \ln g\\
&\quad + \frac{1}{2}\mathcal{D}_\mu \ln g\mathcal{D}_\nu g^{\mu\nu}\,, &(4.149)\\
\mathcal{L}_{\text{kin}} &= \frac{1}{4}\mathcal{D}_\mu\mathcal{M}^{ij}\mathcal{D}^\mu\mathcal{M}_{ij} + \frac{1}{4}\mathcal{D}_\mu\mathcal{M}_{\alpha\beta}\mathcal{D}^\mu\mathcal{M}^{\alpha\beta} - \frac{1}{4}\mathcal{M}_{ij}\mathcal{M}_{\alpha\beta}\mathcal{F}_{\mu\nu}{}^{i\alpha}\mathcal{F}^{\mu\nu j\beta}\\
&\quad - \frac{1}{12}\mathcal{M}^{ij}\mathcal{H}_{\mu\nu\rho i}\mathcal{H}^{\mu\nu\rho}{}_j - \frac{1}{96}\mathcal{M}_{\alpha\beta}\mathcal{J}_{\mu\nu\rho\sigma}{}^{\alpha}\mathcal{J}^{\mu\nu\rho\sigma\beta}\,, &(4.150)\\
\mathcal{L}_{\text{int}} &= \frac{1}{4}\mathcal{M}^{MN}\partial_M\mathcal{M}^{kl}\partial_N\mathcal{M}_{kl} + \frac{1}{4}\mathcal{M}^{MN}\partial_M\mathcal{M}^{\alpha\beta}\partial_N\mathcal{M}_{\alpha\beta}\\
&\quad - \frac{1}{2}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_K\mathcal{M}_{LN} + \frac{1}{2}\partial_M\mathcal{M}^{MN}\partial_N\ln|g|\\
&\quad + \frac{1}{4}\mathcal{M}^{MN}\left(\partial_M g_{\mu\nu}\partial_N g^{\mu\nu} + \partial_M\ln|g|\partial_N\ln|g|\right)\,. &(4.151)
\end{aligned}$$

The topological (Chern-Simons) term can be defined via its variation:

$$\begin{aligned}
\delta\mathcal{L}_{\text{top}} = \kappa\epsilon^{\mu_1\ldots\mu_8}\Big(&- \delta\mathcal{A}_{\mu_1}{}^{i\alpha}\epsilon_{\alpha\beta}\mathcal{J}_{\mu_2\ldots\mu_5}{}^{\beta}\mathcal{H}_{\mu_6\mu_7\mu_8 i}\\
&+ 6\Delta\mathcal{B}_{\mu_1\mu_2 i}\left(\epsilon_{\alpha\beta}\mathcal{F}_{\mu_3\mu_4}{}^{i\alpha}\mathcal{J}_{\mu_5\ldots\mu_8}{}^{\beta} - \tfrac{4}{9}\epsilon^{ijk}\mathcal{H}_{\mu_3\mu_4\mu_5 j}\mathcal{H}_{\mu_6\mu_7\mu_8 k}\right)\\
&+ 4\Delta\mathcal{C}_{\mu_1\mu_2\mu_3}{}^{\alpha}\epsilon_{\alpha\beta}\left(\mathcal{D}_{\mu_4}\mathcal{J}_{\mu_5\ldots\mu_8}{}^{\beta} + 4\mathcal{F}_{\mu_4\mu_5}{}^{i\beta}\mathcal{H}_{\mu_6\ldots\mu_8 i}\right)\\
&- \partial_{i\alpha}\Delta\mathcal{D}_{\mu_1\ldots\mu_4}{}^{i}\mathcal{J}_{\mu_5\ldots\mu_8}{}^{\alpha}\Big)\,,
\end{aligned} \tag{4.152}$$

where the 'improved' Δ variation includes by definition contributions of variations of lower rank gauge fields, for explicit expressions (which we do not require) see [106]. Finally, we must impose the constraint (4.146) after varying the above pseudo-action.

4.4.2 Obtaining the eleven-dimensional Newton-Cartan theory via ExFT

In this subsection, we perform a dimensional decomposition of the eleven-dimensional MNC variables, and use this to explain how exceptional field theory describes this theory.

Dimensional decomposition of eleven-dimensional Newton-Cartan theory We start with the eleven-dimensional coordinates labelled as $x^{\hat{\mu}} = (x^\mu, y^i)$ with $\mu = 1, \ldots, 11-d$ and $i = 1\ldots, d$. We keep all coordinate dependence on y^i throughout. Thus this is a decomposition rather than a reduction. In terms of the vielbein decomposition (4.121), we take $q = d-3$ and $n = 11-d$. The flat indices are $\mathsf{a} = 1, \ldots, 11-d$ and $\bar{\imath} = 1, \ldots, d-3$. Explicitly, we take the SO(8) vielbein to have the form

$$\hat{h}^a{}_{\hat{\mu}} = \begin{pmatrix} \Omega^{-\frac{1}{9-d}} e^{\mathsf{a}}{}_\mu & 0 \\ A_\mu{}^k h^{\bar{\imath}}{}_k & h^{\bar{\imath}}{}_i \end{pmatrix} , \quad \hat{h}^{\hat{\mu}}{}_a = \begin{pmatrix} \Omega^{\frac{1}{9-d}} e^\mu{}_{\mathsf{a}} & 0 \\ -\Omega^{\frac{1}{9-d}} e^\rho{}_{\mathsf{a}} A_\rho{}^k & h^i{}_{\bar{\imath}} \end{pmatrix} , \tag{4.153}$$

with $e^{\mathsf{a}}{}_\mu$ an invertible vielbein for an $(11-d)$-dimensional metric, $g_{\mu\nu} = e^{\mathsf{a}}{}_\mu e^{\mathsf{b}}{}_\nu \delta_{\mathsf{ab}}$. We also have to take

$$\hat{\tau}_{\hat{\mu}}{}^A = (A_\mu{}^i \tau_i{}^A, \tau_i{}^A) , \quad \hat{\tau}^{\hat{\mu}}{}_A = (0, \tau^i{}_A) . \tag{4.154}$$

where $\tau_{ij} = \tau_i{}^A \tau_j{}^B \eta_{AB}$, with $A = 0, 1, 2$ as before. The conformal factor Ω appearing in (4.153) is defined by

$$\Omega^2 = -\tfrac{1}{3!(d-3)!} \epsilon^{i_1\ldots i_d} \epsilon^{j_1\ldots j_d} \tau_{i_1 j_1} \tau_{i_2 j_2} \tau_{i_3 j_3} H_{i_4 j_4} \ldots H_{i_d j_d} , \tag{4.155}$$

and related to that of the eleven-dimensional theory by $\hat{\Omega} = (\det e) \Omega^{-\frac{2}{9-d}}$. It is useful to write down the full transverse and longitudinal metrics:

$$\begin{aligned}
\hat{H}_{\hat{\mu}\hat{\nu}} &= \begin{pmatrix} \Omega^{-\frac{2}{9-d}} g_{\mu\nu} + H_{kl} A_\mu{}^k A_\nu{}^l & H_{jk} A_\mu{}^k \\ H_{ik} A_\nu{}^k & H_{ij} \end{pmatrix} , \\
\hat{\tau}_{\hat{\mu}\hat{\nu}} &= \begin{pmatrix} A_\mu{}^k A_\nu{}^l \tau_{kl} & A_\mu{}^k \tau_{kj} \\ A_\nu{}^k \tau_{ki} & \tau_{ij} \end{pmatrix} , \\
\hat{H}^{\hat{\mu}\hat{\nu}} &= \begin{pmatrix} \Omega^{\frac{2}{9-d}} g^{\mu\nu} & -\Omega^{\frac{2}{9-d}} g^{\mu\rho} A_\rho{}^j \\ -\Omega^{\frac{2}{9-d}} g^{\nu\sigma} A_\sigma{}^i & H^{ij} + \Omega^{\frac{2}{9-d}} g^{\rho\sigma} A_\rho{}^i A_\sigma{}^j \end{pmatrix} , \\
\hat{\tau}^{\hat{\mu}\hat{\nu}} &= \begin{pmatrix} 0 & 0 \\ 0 & \tau^{ij} \end{pmatrix} .
\end{aligned} \tag{4.156}$$

In this way all the degenerate structure is encoded in the d-dimensional part of the spacetime, with a degenerate d-dimensional metric $H_{ij} \equiv h^{\bar{\imath}}{}_i h^{\bar{\jmath}}{}_j \delta_{\bar{\imath}\bar{\jmath}}$. This ensures that the metric $g_{\mu\nu}$ can be identified with the metric appearing in exceptional field theory, while the degenerate Newton-Cartan metric structure will appear in the generalised metric. In addition, we redefine the three-form and its field strength according to (4.136) and (4.137), now without hats:

$$C_3 = \boldsymbol{C}_3 + \boldsymbol{C}_{2i} Dy^i + \tfrac{1}{2}\boldsymbol{C}_{1ij} Dy^i Dy^j + \tfrac{1}{3!}\boldsymbol{C}_{ijk} Dy^i Dy^j Dy^k \,, \tag{4.157}$$

$$\begin{aligned} F_4 = \boldsymbol{F}_4 + \boldsymbol{F}_{3i} Dy^i + \tfrac{1}{2}\boldsymbol{F}_{2ij} Dy^i Dy^j + \tfrac{1}{3!}\boldsymbol{F}_{1ijk} Dy^i Dy^j Dy^k \\ + \tfrac{1}{4!}\boldsymbol{F}_{ijkl} Dy^i Dy^j Dy^k Dy^l \,, \end{aligned} \tag{4.158}$$

where again $Dy^i \equiv dy^i + A_\mu{}^i dx^\mu$. We carry out an analogous decomposition for $\widetilde{C}_3$ and $\widetilde{F}_4$, and for C_6 and F_7. Finally, we can consider the Newton-Cartan torsion: with $\hat{T}_{\hat{\mu}\hat{\nu}}{}^A \equiv 2\partial_{[\hat{\mu}}\hat{\tau}_{\hat{\nu}]}{}^A$ we have

$$\begin{aligned} &T_{ij}{}^A \equiv \hat{T}_{ij}{}^A = 2\partial_{[i}\tau_{j]}{}^A \,, \quad T_{\mu i}{}^A \equiv \hat{T}_{\mu i}{}^A - A_\mu{}^j \hat{T}_{ji} = D_\mu \tau_i{}^A \,, \\ &T_{\mu\nu}{}^A \equiv \hat{T}_{\mu\nu}{}^A - 2\hat{T}_{[\mu|i|}{}^A A_{\nu]}{}^i + A_\mu{}^i A_\nu{}^j \hat{T}_{ij} = F_{\mu\nu}{}^j \tau_j{}^A \,. \end{aligned} \tag{4.159}$$

Embedding the limit in ExFT Let's start by considering the expansions (4.1) and (4.3) of the original eleven-dimensional metric and three-form. We make use of the decompositions (4.156) and (4.157) for the Newton-Cartan variables and three-form appearing in the decomposition, and then use these to work out the decomposition (4.134) of the eleven-dimensional metric and that (4.136) of the three-form. The potentially singular terms as $c \to \infty$ then appear in the d-dimensional components of the metric and of the three-form, with

$$\phi_{ij} = c^2 \tau_{ij} + c^{-1} H_{ij} \,, \quad \hat{\boldsymbol{C}}_{ijk} = -c^3 \epsilon_{ABC} \tau_i{}^A \tau_j{}^B \tau_k{}^C + \boldsymbol{C}_{ijk} + c^{-3} \widetilde{\boldsymbol{C}}_{ijk} \,. \tag{4.160}$$

The metric $g_{\mu\nu}$ and Kaluza-Klein vector $A_\mu{}^i$ appearing in (4.134) are then exactly those appearing in $\hat{H}_{\mu\nu}$ in (4.156). The redefined form components carrying an $(11-d)$-dimensional index are all non-singular, so $\hat{\boldsymbol{C}}_{\mu ij} = \boldsymbol{C}_{\mu ij} + \mathcal{O}(c^{-3})$, and so on. One point of danger is that $\hat{\boldsymbol{C}}_{ijk}$ still appears in the field strengths (4.137) of these fields. However, consulting the more explicit expressions (C.16), one sees that the field strength $\mathcal{F}_{\mu\nu}{}^M$ appearing in ExFT in fact involves the combination $\mathcal{F}_{\mu\nu ij} = \hat{\boldsymbol{F}}_{\mu\nu ij} - \hat{\boldsymbol{C}}_{ijk}F_{\mu\nu}{}^k$, which is in fact independent of $\hat{\boldsymbol{C}}_{ijk}$, such that $\hat{\boldsymbol{F}}_{\mu\nu ij} - \hat{\boldsymbol{C}}_{ijk}F_{\mu\nu}{}^k = \boldsymbol{F}_{\mu\nu ij} - \boldsymbol{C}_{ijk}F_{\mu\nu}{}^k$.

For the generalised metric (4.138), inserting the expressions (4.160) one finds that all terms at leading order in c cancel, and sending $c \to \infty$ one has a manifestly finite and boost invariant expression[9]:

$$\mathcal{M}_{MN} = \Omega^{\frac{2}{9-d}} \begin{pmatrix} \mathcal{M}_{ij} & \mathcal{M}_i{}^{kl} \\ \mathcal{M}_k{}^{ij} & \mathcal{M}^{ijkl} \end{pmatrix} , \tag{4.161}$$

with

$$\begin{aligned} \mathcal{M}_{ij} &= H_{ij} - \epsilon_{ABC}\tau_{(i}{}^A\boldsymbol{C}_{j)kl}\tau^{kB}\tau^{lC} + \boldsymbol{C}_{ikl}\boldsymbol{C}_{jmn}H^{km}\tau^{ln} , \\ \mathcal{M}_i{}^{kl} &= -\epsilon_{ABC}\tau_i{}^A\tau^{kB}\tau^{lC} + 2\boldsymbol{C}_{ipq}H^{p[k}\tau^{l]q} , \\ \mathcal{M}^{ijkl} &= 2H^{i[k}\tau^{l]j} + 2\tau^{i[k}H^{l]j} . \end{aligned} \tag{4.162}$$

The parametrisation (4.161) can be viewed as a *non-Riemannian parametrisation* of the generalised metric, and viewed simply as an alternative possibility to taking the usual form (4.138). The reason why this is a *non-Riemannian parametrisation* is most clearly seen by looking at the inverse generalised metric $\mathcal{M}^{MN}$. In the Riemannian case, the parametrisation (4.138) implies that the $d \times d$ block $\mathcal{M}^{ij}$ is given by $\mathcal{M}^{ij} = |\hat{\phi}|^{-1/(9-d)}\hat{\phi}^{ij}$ and therefore corresponds to the inverse spacetime metric. Assuming this block is invertible then uniquely fixes (given the definition of the generalised metric as a particular coset element obeying certain properties) the rest of the parametrisation. In the non-Riemannian case, we instead have $\mathcal{M}^{ij} = \Omega^{-\frac{2}{9-d}}H^{ij}$, which is non-invertible. This leads instead to an alternative parametrisation. This is exactly as in the DFT case [66], which was generalised to ExFT in [76]. The expression (4.161) can be checked to be equivalent to the non-Riemannian SL(5) generalised metric worked out from first principles in [76]. In fact, from this point of view, one need not even go through the complications of taking the limit, but simply write down (4.161), insert it into the ExFT and study the resulting dynamics.

Returning to the embedding of the expansion in ExFT, we also need to worry about the singular pieces in the expansion of the dual gauge field

[9] Proving this requires the fact that $H^{i[k}H^{l]j} = 0$ when H^{ij} has rank 1. For $d > 4$ this would suggest we would have problems, however starting at $d = 5$ the representation on which $\mathcal{M}_{MN}$ acts enlarges and the structure of the generalised metric therefore changes. Note for $d > 5$ it will also explicitly contain components of the dual six-form.

$\hat{C}_6$. This inevitably appears in the tensor hierarchy for all exceptional field theories. From (4.51), we have $\hat{C}_6 \sim c^3 C_3 \wedge \tau \wedge \tau \wedge \tau + \dots$, and so given the decomposition according to (4.154) and (4.157), any component of $\hat{C}_6$ carrying three d-dimensional indices will be singular, i.e. $\hat{\boldsymbol{C}}_{\mu\nu\rho ijk}$, $\hat{\boldsymbol{C}}_{\mu\nu ijkl}$, $\hat{\boldsymbol{C}}_{\mu ijklm}$, $\hat{\boldsymbol{C}}_{ijklmn}$. The claim is that, remarkably, all such singularities cancel automatically thanks to the precise combinations of $\hat{C}_6$ and $\hat{C}_3$ that appear in the ExFT fields. For $d = 3, 4$, this is most straightforwardly checked at the level of the ExFT field strengths.[10]. One sees from (4.147) for $\mathrm{SL}(3) \times \mathrm{SL}(2)$ (and from (C.43) for SL(5)) that the components of $\hat{F}_7$ always appear in the combinations $\hat{\boldsymbol{F}}_{\mu\nu\rho\sigma ijk} - \hat{\boldsymbol{C}}_{ijk}\hat{\boldsymbol{F}}_{\mu\nu\rho\sigma}$ and $\hat{\boldsymbol{F}}_{\mu\nu\rho ijkl} + 4\hat{\boldsymbol{C}}_{[ijk}\hat{\boldsymbol{F}}_{|\mu\nu\rho\sigma|l]}$ exactly such that the singularity coming from $\hat{\boldsymbol{C}}_{ijk}$ cancels that coming from $\hat{F}_7$, which was written down in (4.52). That the ExFT gauge potentials themselves are non-singular can further be verified by hunting down the correct field redefinitions that relate the ExFT gauge fields to the eleven-dimensional ones. Note that for $d \geq 6$ the components $\hat{\boldsymbol{C}}_{ijklmn}$ are present and appear in the generalised metric itself: we have not verified explicitly but the expectation would be that it does so in a way that ensures the generalised metric remains finite.

Summary From the above we can conclude that the fields used in ExFT are manifestly non-singular in the non-relativistic limit (equivalently this shows that the fields which are U-duality covariant in a genuine dimensional reduction are non-singular). We can also view the distinction between the relativistic and the non-relativistic eleven-dimensional theory as being solely governed by the choice of parametrisation of the generalised metric. Having picked a generalised metric parametrisation, it is then consistent to directly identify the ExFT gauge fields and metric $g_{\mu\nu}$ with the gauge field components and metric of the decomposed relativistic *or* non-relativistic theory.

This is summarised in figure 4.1. The upper triangular half of this diagram corresponds to first embedding the relativistic fields in ExFT in the usual manner, with a Riemannian parametrisation of the generalised metric, and then taking the non-relativistic limit giving a non-Riemannian parametrisation. The lower triangular half corresponds to first taking the non-relativistic limit for the original eleven-dimensional fields, and then embedding these into ExFT, giving the same non-Riemannian parametrisation. In both cases, one

[10] Only the field strengths appear in the equations of motion, and the action can also be defined solely in terms of the field strengths by rewriting the Chern-Simons term in a standard way as an integral over a higher-dimensional spacetime.

needs to make the appropriate dimensional decomposition of the fields of the Newton-Cartan theory, corresponding to fixing the local tangent space (non-Lorentzian) symmetry.

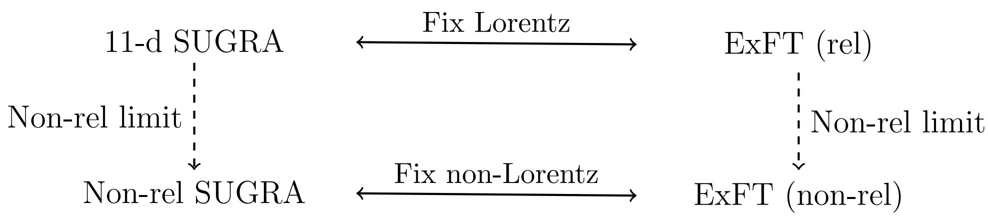

Figure 4.1: Relationship between non-relativistic limit and non-relativistic parametrisation of ExFT

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Inserting the non-Riemannian parametrisation into the ExFT action or equations of motion will then reproduce the finite action and equations of motion results from taking the limit, after decomposing. For the action, we calculate this decomposition in appendix C.1. What we will show next is that, remarkably, the ExFT equations of motion also automatically reproduce the Poisson equation (4.97).

4.4.3 Generalised metric and equations of motion

We now take a closer look at the consequences of using the non-relativistic parametrisation of the generalised metric. We focus on the $d = 3$ SL(3) $\times$ SL(2) ExFT. For the $d = 3$ Newton-Cartan geometry, H^{ij} and H_{ij} have rank zero and so are identically zero. The longitudinal metric τ_{ij} is a three-by-three matrix and in fact invertible, with $\Omega^2 = -\det\tau$. The resulting non-Riemannian parametrisation of the generalised metric (4.139) is

$$\mathcal{M}_{ij} = \Omega^{-2/3}\tau_{ij}\,, \quad \mathcal{M}_{\alpha\beta} = \begin{pmatrix} 2\varphi & 1 \\ 1 & 0 \end{pmatrix}\,, \quad \varphi \equiv \tfrac{1}{3!}\epsilon^{ijk}\boldsymbol{C}_{ijk}\,. \tag{4.163}$$

Comparing (4.163) and (4.140), we can note that (4.163) is the most general possible SL(2) non-Riemannian parametrisation (up to the sign of the off-diagonal components), as this is completely fixed by requiring $\mathcal{M}_{22} = 0$ which prevents us from interpreting that component as the determinant of a standard three-dimensional spacetime metric.

Normally, the generalised metric $\mathcal{M}_{\alpha\beta}$ encodes two degrees of freedom. It is clear that the non-Riemannian parametrisation given by (4.163) is restricted and is missing one degree of freedom. We may identify this missing degree of freedom with the overall scale of the longitudinal metric, as the latter only appears in the combination $|\det\tau|^{-1/3}\tau_{ij}$, which is conformally invariant. This makes the dilatation invariance trivial in this formulation.

If we insert this parametrisation into the SL(3) × SL(2) pseudo-action, with Lagrangian (4.148), we find that $\mathcal{L}_{\text{int}}$ as defined in (4.151) vanishes, while

$$\frac{1}{4}\mathcal{D}_\mu\mathcal{M}^{ij}\mathcal{D}^\mu\mathcal{M}_{ij} + \frac{1}{4}\mathcal{D}_\mu\mathcal{M}_{\alpha\beta}\mathcal{D}^\mu\mathcal{M}^{\alpha\beta} = \frac{1}{4}D_\mu(\Omega^{2/3}\tau^{ij})D^\mu(\Omega^{-2/3}\tau_{ij})\,. \quad (4.164)$$

This reproduces exactly the expected terms in the $d = 3$ case of (C.26) and (C.27).

Notice that the kinetic terms for $\mathcal{M}_{\alpha\beta}$ completely drop out. So if we insert the non-relativistic parametrisation into the action, and then vary with respect to φ, we will never find an equation involving $\mathcal{D}^\mu\mathcal{D}_\mu\varphi$, i.e the Poisson equation. However, instead we can consider the equations of motion of the generalised metric, which can be evaluated independently of its choice of parametrisation. These will provide the missing Poisson equation. This is exactly analogous to the situation in DFT, see the discussions in [83] and in the previous chapter. One has to make a choice about whether you allow the equations of motion that follow from variations of the generalised metric that do not preserve the non-Riemannian parametrisation. In both the DFT SNC case, and the present case, there is exactly one such independent variation, which provides an additional equation of motion beyond what is obtained by varying the fields of the parametrisation themselves.

Let's see how this works. Naively, the result of varying the generalised metric $\mathcal{M}_{\alpha\beta}$ in the action is

$$\delta S = \int \mathrm{d}^8x\,\mathrm{d}^6Y\,\sqrt{g}\delta\mathcal{M}^{\alpha\beta}\mathcal{K}_{\alpha\beta}\,, \quad (4.165)$$

with

$$\begin{aligned}
\mathcal{K}_{\alpha\beta} &= -\tfrac{1}{4}\tfrac{1}{\sqrt{g}}\left(\mathcal{D}_\mu(\sqrt{g}\mathcal{D}^\mu\mathcal{M}_{\alpha\beta}) - \mathcal{M}_{\alpha\gamma}\mathcal{M}_{\beta\delta}\mathcal{D}_\mu(\sqrt{g}\mathcal{D}^\mu\mathcal{M}^{\gamma\delta})\right)\\
&+ \frac{1}{4}\mathcal{M}_{\alpha\gamma}\mathcal{M}_{\beta\delta}\mathcal{M}_{ij}\mathcal{F}_{\mu\nu}{}^{i\gamma}\mathcal{F}^{\mu\nu j\delta} + \frac{1}{96}\mathcal{M}_{\alpha\gamma}\mathcal{M}_{\beta\delta}\mathcal{J}_{\mu\nu\rho\sigma}{}^{\gamma}\mathcal{J}^{\mu\nu\rho\sigma\delta}\\
&+ \tfrac{1}{4}\mathcal{M}^{ij}\left(\partial_{i(\alpha|}\mathcal{M}^{kl}\partial_{j|\beta)}\mathcal{M}_{kl} + \partial_{i(\alpha|}\mathcal{M}^{\gamma\delta}\partial_{j|\beta)}\mathcal{M}_{\gamma\delta} + \partial_{i(\alpha|}g_{\mu\nu}\partial_{j|\beta)}g^{\mu\nu}\right)\\
&- \tfrac{1}{2}\mathcal{M}^{ij}\partial_{i\alpha}\partial_{j\beta}\ln g + \tfrac{1}{\sqrt{g}}\partial_{i(\alpha|}(\sqrt{g}\partial_{j|\beta)}\mathcal{M}^{ij})\\
&- \tfrac{1}{2}\mathcal{M}^{ij}\left(\partial_{i(\alpha|}\mathcal{M}^{kl}\partial_{k|\beta)}\mathcal{M}_{lj} + \partial_{i(\alpha|}\mathcal{M}^{\gamma\delta}\partial_{j\gamma}\mathcal{M}_{|\beta)\delta}\right)\\
&+ \tfrac{1}{2\sqrt{g}}(\partial_{i\gamma}(\sqrt{g}\mathcal{M}^{ij}\mathcal{M}^{\gamma\delta}\partial_{j(\alpha}\mathcal{M}_{\beta)\delta}) - \mathcal{M}_{\gamma(\alpha}\mathcal{M}_{\beta)\delta}\partial_{j\kappa}(\sqrt{g}\mathcal{M}^{ij}\mathcal{M}^{\epsilon\gamma}\partial_{i\epsilon}\mathcal{M}^{\kappa\delta})\\
&- \tfrac{1}{4\sqrt{g}}(\partial_{i\gamma}(\sqrt{g}\mathcal{M}^{ij}\mathcal{M}^{\gamma\delta}\partial_{j\delta}\mathcal{M}_{\alpha\beta}) - \mathcal{M}_{\alpha\gamma}\mathcal{M}_{\beta\delta}\partial_{i\epsilon}(\sqrt{g}\mathcal{M}^{ij}\mathcal{M}^{\epsilon\kappa}\partial_{j\kappa}\mathcal{M}^{\gamma\delta})\,.
\end{aligned}\tag{4.166}$$

Now, the variation $\delta\mathcal{M}^{\alpha\beta}$ cannot be arbitrary but must preserve the condition $|\det\mathcal{M}| = 1$. This ensures that one gets two rather than three independent equations, corresponding to the usual two degrees of freedom encoded in $\mathcal{M}_{\alpha\beta}$. The true equation of motion taking this into account is:

$$\mathcal{R}_{\alpha\beta} \equiv \mathcal{K}_{\alpha\beta} - \tfrac{1}{2}\mathcal{M}_{\alpha\beta}\mathcal{M}^{\gamma\delta}\mathcal{K}_{\gamma\delta} = 0\,. \tag{4.167}$$

This can be thought of as the vanishing of a generalised Ricci tensor, $\mathcal{R}_{\alpha\beta}$. For the non-Riemannian parametrisation (4.163), the two independent equations are

$$\mathcal{R}_{22} = \mathcal{K}_{22} = 0\,, \quad \mathcal{R}_{11} - 2\varphi\mathcal{R}_{22} = \mathcal{K}_{11} - 2\varphi K_{12} = 0\,. \tag{4.168}$$

Setting $\partial_{i1} \equiv \partial_i$, $\partial_{i2} = 0$, we have explicitly that

$$\mathcal{K}_{22} = +\tfrac{1}{4}\mathcal{M}_{ij}F_{\mu\nu}{}^{i}F^{\mu\nu j} + \tfrac{1}{96}\boldsymbol{F}_{\mu\nu\rho\sigma}\boldsymbol{F}^{\mu\nu\rho\sigma} = 0\,. \tag{4.169}$$

This is the equation of motion (4.71) arising as the totally longitudinal part of the equation of motion of the three-form. This is consistent with its appearance here as the equation of motion of φ, which is indeed the totally longitudinal part of the three-form.

The other equation of motion is (after using (4.169))

$$\begin{aligned}
0 &= \mathcal{K}_{11} - 2\varphi K_{12}\\
&= -\tfrac{1}{\sqrt{g}}\tfrac{1}{6}\epsilon^{ijk}D_\mu(\sqrt{g}g^{\mu\nu}\boldsymbol{F}_{\nu ijk})\\
&\quad - \tfrac{1}{8}\mathcal{M}^{km}\mathcal{M}^{ln}\boldsymbol{F}_{\mu\nu kl}\boldsymbol{F}^{\mu\nu}{}_{mn} + \tfrac{1}{96}\boldsymbol{F}_{\mu\nu\rho\sigma ijk}\boldsymbol{F}^{\mu\nu\rho\sigma}{}_{lmn}\tfrac{1}{3!3!}\epsilon^{ijk}\epsilon^{lmn}\\
&\quad + \tfrac{1}{4}\mathcal{M}^{ij}\left(\partial_i\mathcal{M}^{kl}\partial_j\mathcal{M}_{kl} + \partial_i g_{\mu\nu}\partial_j g^{\mu\nu}\right) - \tfrac{1}{2}\mathcal{M}^{ij}\partial_i\mathcal{M}^{kl}\partial_k\mathcal{M}_{jl}\\
&\quad - \tfrac{1}{2}\mathcal{M}^{ij}\partial_i\partial_j\ln g - \tfrac{1}{\sqrt{g}}\partial_i(\sqrt{g}\partial_j\mathcal{M}^{ij})\,.
\end{aligned}\tag{4.170}$$

Here we have $\boldsymbol{F}_{\mu ijk} = D_\mu \boldsymbol{C}_{ijk} - 3\partial_{[i}\boldsymbol{C}_{|\mu|jk]}$, having used $\mathcal{D}_\mu \mathcal{M}_{11} = D_\mu \mathcal{M}_{11} - \epsilon^{ijk}\partial_i \mathcal{A}_{\mu jk}\mathcal{M}_{12}$. We can then identify (4.170) as the Poisson equation for $\varphi \equiv \frac{1}{6}\epsilon^{ijk}\boldsymbol{C}_{ijk}$, as it has the form $\frac{1}{\sqrt{g}}D_\mu(\sqrt{g}D^\mu\varphi) + \cdots = 0$. It is conjugate to the variation $\delta\mathcal{M}^{11}$. For the non-Riemannian parametrisation, $\mathcal{M}^{11} = 0$, so allowing this variation corresponds to allowing variations that do not respect the parametrisation. In terms of the expansion of $\mathcal{M}^{\alpha\beta}$ in powers of $1/c$, this variation is subleading in origin. Finally, one can precisely check that this equation (4.170) is indeed exactly the Poisson equation (4.97), which we found at subleading order in the expansion of the relativistic theory, and here is rewritten in terms of ExFT variables after making the dimensional decomposition of all the fields. (It can be easily checked that the gauge field terms match, using (C.29) to relate the seven-form components appearing here to those of $\widetilde{F}_4$, and a patient calculation shows that inserting the dimensional decomposition of the eleven-dimensional fields matches perfectly.)

Structure of generalised Ricci tensor Geometrically, $\mathcal{R}_{\alpha\beta}$ should be thought of as (the SL(2) part of) a generalised Ricci tensor. It is a symmetric generalised tensor of weight 0 and obeys $\mathcal{M}^{\alpha\beta}\mathcal{R}_{\alpha\beta} = 0$. When we take the relativistic parametrisation (4.140) of the generalised metric, it can therefore be parametrised as

$$\mathcal{R}_{\alpha\beta} = \frac{1}{2}\begin{pmatrix} 1 & \hat{\boldsymbol{C}} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} |\phi|^{1/2}\mathcal{R}_\phi & \mathcal{R}_C \\ \mathcal{R}_C & |\phi|^{-1/2}\mathcal{R}_\phi \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \hat{\boldsymbol{C}} & 1 \end{pmatrix}, \tag{4.171}$$

with $\mathcal{R}_\phi$ and $\mathcal{R}_C$ tensors of three-dimensional weight 0, such that the variation of the action leads to

$$\delta S \supset -\int \mathrm{d}^8x\,\mathrm{d}^6y\,\sqrt{g}\left(\frac{\delta|\phi|^{1/2}}{|\phi|^{1/2}}\mathcal{R}_\phi + |\phi|^{-1/2}\delta\hat{\boldsymbol{C}}\mathcal{R}_C\right). \tag{4.172}$$

Let's examine what happens to the components of $\mathcal{R}_{\alpha\beta}$ in the non-relativistic limit. We have $|\phi|^{1/2} = \Omega c^3$, $\hat{\boldsymbol{C}} = -c^3\Omega + C + c^{-3}\widetilde{C}$. This leads to the expression

$$\mathcal{R}_{\alpha\beta} = \frac{1}{2}\begin{pmatrix} 1 & \boldsymbol{C} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c^3\Omega(\mathcal{R}_\phi - \mathcal{R}_C) & \mathcal{R}_C - \mathcal{R}_\phi \\ \mathcal{R}_C - \mathcal{R}_\phi & c^{-3}\Omega^{-1}\mathcal{R}_\phi \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \boldsymbol{C} & 1 \end{pmatrix}. \tag{4.173}$$

So in principle the independent equations are still $\mathcal{R}_C$ and $\mathcal{R}_\phi$. However, we already know that this generalised Ricci tensor has no leading parts in

c when we take the limit (because none of the ExFT fields contain singular terms). If we expand

$$\mathcal{R}_\phi = c^3\mathcal{R}_\phi^{(3)} + c^0\mathcal{R}_\phi^{(0)} + c^{-3}\mathcal{R}_\phi^{(-3)}\,, \quad \mathcal{R}_C = c^3\mathcal{R}_C^{(3)} + c^0\mathcal{R}_C^{(0)} + c^{-3}\mathcal{R}_C^{(-3)}\,, \tag{4.174}$$

it must be that we have $\mathcal{R}_\phi^{(3)} = \mathcal{R}_C^{(3)}$, $\mathcal{R}_\phi^{(0)} = \mathcal{R}_C^{(0)}$, viewed as off-shell identities, and the independent equations of motion, i.e. those appearing as the actual finite entries of $\mathcal{R}_{\alpha\beta}$, are actually

$$\mathcal{R}_\phi^{(3)} = 0\,, \quad \mathcal{R}_\phi^{(-3)} - \mathcal{R}_C^{(-3)} = 0\,. \tag{4.175}$$

The former is conjugate to $\delta\mathcal{M}^{22}$ and the latter to the $\delta\mathcal{M}^{11}$ that is forbidden if we insist on keeping a non-Riemannian parametrisation. We can go back to the variation (4.172) and expand that:

$$\delta S = -\int \mathrm{d}^8x\,\mathrm{d}^6y\,\sqrt{g}\left(\delta\ln\Omega(\mathcal{R}_\phi - \mathcal{R}_C) + \Omega^{-1}c^{-3}\delta C\mathcal{R}_C\right)\,, \tag{4.176}$$

hence the first non-zero variations are

$$\delta S = -\int \mathrm{d}^8x\,\mathrm{d}^6y\,\sqrt{g}\left(c^{-3}\delta\ln\Omega(\mathcal{R}_\phi^{(-3)} - \mathcal{R}_C^{(-3)}) + \Omega^{-1}\delta C\mathcal{R}_C^{(3)}\right)\,. \tag{4.177}$$

We see again that we get the longitudinal equation of motion for the three-form at finite order, and the extra Poisson equation of motion comes from a subleading variation associated to the variation of the volume factor Ω, which otherwise has no dynamics associated to it in this formulation.

4.4.4 Generating non-relativistic generalised metrics via U-duality

Non-trivial U-duality transformations act as SL(2) transformations on the generalised metric $\mathcal{M}_{\alpha\beta}$, via $\mathcal{M} \to \mathcal{M}' = U^T\mathcal{M}U$ with $\det U = 1$. Parametrising $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the transformation of the non-relativistic parametrisation (4.163) gives

$$\mathcal{M}'_{\alpha\beta} = \begin{pmatrix} 2a(a\varphi + c) & 2ab\varphi + ad + bc \\ 2ab\varphi + ad + bc & 2b(b\varphi + d) \end{pmatrix}\,, \tag{4.178}$$

and this remains in the non-relativistic form only if $b = 0$, or else if φ is constant and $d = -b\varphi$. In the former case, the effect of the transformation is $\varphi \to a(a\varphi + c)$ and so amounts to a scaling and shift of the three-form.

The genuine non-geometric U-dualities correspond to the SL(2) inversion symmetry with $a = d = 0$, $bc = -1$. If $\varphi < 0$, this takes us from the non-relativistic parametrisation to a a relativistic one with

$$\phi_{ij} = (-\tfrac{1}{2\varphi})^{2/3}(\det\tau)^{-1/3}\tau_{ij}\,, \qquad \boldsymbol{C}_{ijk} = -\tfrac{1}{2\varphi}\epsilon_{ijk}\,. \tag{4.179}$$

These obey $|\det\phi| = \boldsymbol{C}^2$ which corresponds to a 'critical' three-form.

We can apply this to a real supergravity background along the lines of [76, 108], namely the M2 brane solution in the form

$$\mathrm{d}s^2 = f^{-2/3}\eta_{ij}dy^i dy^j + f^{1/3}\delta_{\mu\nu}dx^\mu dx^\nu\,, \qquad C_{ijk} = (f^{-1} + \gamma)\epsilon_{ijk}\,, \tag{4.180}$$

where the harmonic function f obeys $\partial_\mu\partial^\mu f = 0$ and γ is a constant. This has constant exceptional field theory eight-dimensional metric, $g_{\mu\nu} = \delta_{\mu\nu}$, while

$$\mathcal{M}_{ij} = \eta_{ij}\,, \quad \mathcal{M}_{\alpha\beta} = \begin{pmatrix} -\gamma(f+2\gamma) & -(1+\gamma f) \\ -(1+\gamma f) & -f \end{pmatrix}\,. \tag{4.181}$$

Sending $f \to 0$ corresponds exactly to the original limit (4.1). Alternatively, we can formally U-dualise along the y^i directions (including time) to obtain a solution with

$$\mathcal{M}_{\alpha\beta} = \begin{pmatrix} -f & 1+\gamma f \\ 1+\gamma f & -\gamma(f+2\gamma) \end{pmatrix}\,. \tag{4.182}$$

The standard M2 solution has $\gamma = -1$ and $f = 1 + \frac{q}{r^6}$, with $r^2 \equiv \delta_{\mu\nu}x^\mu x^\nu$. In this case, the generalised metric (4.182) corresponds to the *negative M2* solution [109]:

$$\begin{aligned} \mathrm{d}s^2 &= \widetilde{f}^{-2/3}\eta_{ij}dy^i dy^j + \widetilde{f}^{1/3}\delta_{\mu\nu}dx^\mu dx^\nu\,, \\ C_{ijk} &= (\widetilde{f}^{-1} - 1)\epsilon_{ijk}\,, \\ \widetilde{f} &= 1 - \tfrac{q}{r^6}\,. \end{aligned} \tag{4.183}$$

This solution has a naked singularity at $\widetilde{f} = 0 \Leftrightarrow f - 2 = 0$. Evidently the generalised metric (4.182) is non-singular everywhere and at $\widetilde{f} = 0$ becomes non-relativistic. This suggests [12] interpreting such backgrounds as containing a singular locus at which the geometry degenerates to a non-relativistic one.

If we alternatively take $\gamma = 0$ then the generalised metric (4.182) has the non-relativistic form everywhere, with $\varphi \equiv -\frac{1}{2}f$. If we now reconsider the equation of motion (4.170) which can only be found by varying the generalised metric before inserting the parametrisation, then this is exactly the equation $\nabla^2 f = 0$ obeyed by the harmonic function. Finally, we can reconstruct the full eleven-dimensional MNC geometry:

$$\hat{\tau}_{\hat{\mu}}{}^{A} = (0, \delta_i{}^A)\,, \quad \hat{H}^{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \delta^{\mu\nu} & 0 \\ 0 & 0 \end{pmatrix}\,, \quad C_{012} = -\tfrac{1}{2}f\,. \tag{4.184}$$

4.4.5 Gauge fields and self-duality in SL(3)×SL(2) ExFT

Now let's look at what happens in the gauge field sector of the SL(3) × SL(2) ExFT. Let's repeat the parametrisations (4.145) and (4.147) now for the field strength components of the non-relativistic theory:

$$\mathcal{F}_{\mu\nu}{}^{i1} \equiv F_{\mu\nu}{}^{i}\,, \quad \mathcal{F}_{\mu\nu}{}^{i2} \equiv \frac{1}{2}\epsilon^{ijk}(\boldsymbol{F}_{\mu\nu jk} - \boldsymbol{C}_{jkl}\boldsymbol{F}_{\mu\nu}{}^{l})\,, \quad \mathcal{H}_{\mu\nu\rho i} \equiv -\boldsymbol{F}_{\mu\nu\rho i}\,, \tag{4.185}$$

$$\mathcal{J}_{\mu\nu\rho\sigma}{}^{1} \equiv \boldsymbol{F}_{\mu\nu\rho\sigma}\,, \quad \mathcal{J}_{\mu\nu\rho\sigma}{}^{2} \equiv \frac{1}{6}\epsilon^{ijk}(\boldsymbol{F}_{\mu\nu\rho\sigma ijk} - \boldsymbol{C}_{ijk}\boldsymbol{F}_{\mu\nu\rho\sigma})\,. \tag{4.186}$$

Then the kinetic terms (4.150) in the SL(3) × SL(2) ExFT pseudo-action (4.148) are

$$\begin{aligned} &-\tfrac{1}{4}\mathcal{M}_{ij}\mathcal{M}_{\alpha\beta}\mathcal{F}_{\mu\nu}{}^{i\alpha}\mathcal{F}^{\mu\nu j\beta} - \tfrac{1}{12}\mathcal{M}^{ij}\mathcal{H}_{\mu\nu\rho i}\mathcal{H}^{\mu\nu\rho}{}_{j} \\ &\qquad = -\tfrac{1}{4}\Omega^{-2/3}\tau_{ij}F^{\mu\nu i}\epsilon^{jkl}\boldsymbol{F}_{\mu\nu kl} - \frac{1}{12}\Omega^{2/3}\tau^{ij}\boldsymbol{F}_{\mu\nu\rho i}\boldsymbol{F}^{\mu\nu\rho}{}_{j}\,, \end{aligned} \tag{4.187}$$

which matches the corresponding terms in the decomposition (C.26) of the non-relativistic action.

To discuss the three-form gauge field, consider the SL(3) × SL(2) ExFT equation of motion obtained from the pseudo-action by varying $\mathcal{C}_{\mu\nu\rho}{}^{\alpha}$:

$$\begin{aligned} &\mathcal{D}_{\sigma}(\sqrt{|g|}\mathcal{M}_{\alpha\beta}\mathcal{J}^{\mu\nu\rho\sigma\beta}) - 2\partial_{i\alpha}(\sqrt{|g|}\mathcal{M}^{ij}\mathcal{H}^{\mu\nu\rho}{}_{j}) \\ &- 48\kappa\epsilon_{\alpha\beta}\epsilon^{\mu\nu\rho\sigma_1\ldots\sigma_5}\left(\mathcal{D}_{\sigma_1}\mathcal{J}_{\sigma_2\ldots\sigma_5}{}^{\beta} + 4\mathcal{F}_{\sigma_1\sigma_2}{}^{i\beta}\mathcal{H}_{\sigma_3\sigma_4\sigma_5 i}\right) = 0\,. \end{aligned} \tag{4.188}$$

After varying, we must also impose the constraint (4.146). This constraint involves the generalised metric, and so it is sensitive to whether we are describing the relativistic or non-relativistic theory. However, in either case, using the constraint in the equation of motion of $\mathcal{C}_{\mu\nu\rho}{}^2$ in fact produces the Bianchi identity (4.144) for $\mathcal{J}_{\mu\nu\rho\sigma}{}^1 = \boldsymbol{F}_{\mu\nu\rho\sigma}$. In the relativistic case, with the Riemannian parametrisation (4.140) of the generalised metric (or its Euclidean version), we could go on to use the constraint to eliminate $\mathcal{J}_{\mu\nu\rho\sigma}{}^2$ from the equation of motion of $\mathcal{C}_{\mu\nu\rho}{}^2$. The result would be the equation of motion of the three-form $\boldsymbol{C}_{\mu\nu\rho}$ following from the decomposition of eleven-dimensional SUGRA.

Now let's consider the situation where the generalised metric admits the non-relativistic parametrisation (4.163). In this case, choosing the minus sign for κ, the constraint (4.146) implies that

$$\begin{aligned}\sqrt{g}\,\boldsymbol{F}^{\mu\nu\rho\sigma} &= -\tfrac{1}{4!}\epsilon^{\mu\nu\rho\sigma\lambda_1\ldots\lambda_4}\boldsymbol{F}_{\lambda_1\ldots\lambda_4}\,,\\ \sqrt{g}\,\boldsymbol{F}^{\mu\nu\rho\sigma}{}_{ijk} &= +\tfrac{1}{4!}\epsilon^{\mu\nu\rho\sigma\lambda_1\ldots\lambda_4}\boldsymbol{F}_{\lambda_1\ldots\lambda_4 ijk}\,.\end{aligned} \tag{4.189}$$

So we can no longer eliminate $\boldsymbol{F}_{\mu\nu\rho\sigma ijk}$ in favour of $\boldsymbol{F}_{\mu\nu\rho\sigma}$. This is clearly as expected for the MNC theory for which the former indeed appears explicitly in the action and equations of motion (note it is related to $\widetilde{\boldsymbol{F}}_{\mu\nu\rho\sigma}$ via (C.29)). We therefore see that the ExFT constraint gives not only the expected constraint (4.33) that the original four-form field strength becomes self-dual, but also the duality condition with opposite sign which is obeyed by the dual seven-form (4.55). Thus the SL(3) × SL(2) ExFT contains the expected degrees of freedom of the non-relativistic theory, and efficiently rearranges them into self-dual and anti-self-dual parts automatically on the non-Riemannian parametrisation.

Chapter 5

Discussion and Outlook

The main focus of this thesis was to derive actions and equations of motion describing non-relativistic physics. The results we obtained are the non-relativistic versions of (the bosonic sector of) the usual relativistic ten- and eleven-dimensional supergravity actions that have been known for many years. The study of non-relativistic physics can be considered to be still in its early stages. This means that several results one would like to obtain can be (hopefully) derived by closely following the same approach already known for their relativistic counterparts. However, non-relativistic theories offer a degree of novelty as can be seen for example in Chapter 2, with the non-relativistic Polyakov action involving several different fields and the quantum fields expansion being much more complicated than in the relativistic case. Nonetheless, we were able to derive the target-space equations of motion, altough after imposing the twistless torsion constraint and with the caveat of Section 3.5. It would be interesting to derive the beta functions when no constraint on torsion is imposed. However, this involves solving the non-relativistic geodesic equation in general, which turns out to not have a simple linear or quadratic dependance on the embedding fields. Alternatively, one could try to study a theory where the worldsheet itself is non-relativistic, which should be equivalent to Spin Matrix Theory [8]

In Chapter 3 we were able to bring this analysis one step further and find the equations of motion *and* actions for TNC, SNC and Carrollian gravity, while in Chapter 4 we obtained an action and equations of motion for eleven-dimensional non-relativistic supergravity. An obvious direction to expand this work would be to find solutions to the equations of motion we presented here. For example, the definition of a black hole in a non-Riemannian manifold is unclear [110–113], and we believe that the actions presented in this work would help clarify their physical interpretation. The fact that we have an action at our disposal allows us to derive interesting properties of a given solution. In particular, variations of on-shell actions (note that the DFT on-shell action is trivially zero up to boundary terms) would allow for investigation of the thermodynamical properties of such hypothetical black holes and possibly relate them to holographically dual theories in nonrelativistic quantum plasmas. Moreover, there has been recently a renewed interest in Carrollian gravity, as it has been suggested that it may help to better understand inflation and dark matter [114].

In our analysis of non-relativistic theories, we have mostly concerned ourselves with 'Type I' expansions, but it would be equally interesting to derive similar results for an expansion of the type given in (1.9), where more fields are included giving a very different algebra. For example, in the eleven-dimensional case of Chapter 4 we could expand the metrix as $g_{\mu\nu} = c^2\tau_{\mu\nu} + c^{-1}H_{\mu\nu} + c^{-4}X_{\mu\nu} + \dots$. It is possible to check that doing so does not affect the expansion of the action up to order c^0, and it would be expected on general grounds [115] that the first appearance of the first subleading terms simply re-imposes the equations of motion already encountered (as we saw with $\tilde{C}_3$ and the equations of motion of C_3). In addition, we could reformulate the expansion by introducing additional one-form gauge fields (as for this case in [105]), accompanied by a shift symmetry, such that the three-form $C_{\mu\nu\rho}$ does not transform under boosts. The resulting more general expansion could then be attacked order-by-order without necessarily sending $c \to \infty$ or truncating as we did. Here it would be interesting to compare with the approach of [27], inputting the eleven-dimensional three-form as matter. A complicating feature, relative to usual $1/c$ expansions of general relativity leading to Newton-Cartan [25, 103, 115] for example, is that the longitudinal vielbein appears in both the metric and three-form and does so at different orders in c.

Another possible approach would be to try and find an embedding of the Type II TNC in Double Field Theory (if possible). However, we expect this to be nontrivial. Recall that, upon BRST quantization, a critical bosonic string theory can only be anomaly-free when the trace of the generalized metric satisfies $\mathrm{Tr}\,\mathcal{H} = n - \bar{n} = 0$ [81]. On the other hand, we know that it is possible to derive all the equations of motion from an action principle for Type II TNC [25], which implies that the embedding of this theory in DFT should have $n \times \bar{n} = 0$. The only way to have $n = \bar{n}$ and $n \times \bar{n} = 0$

is if $n = \bar{n} = 0$, but this corresponds to a Riemannian geometry and so we cannot identify this theory with Type II TNC. This seems to indicate that Type II TNC cannot be embedded in DFT as a quantum consistent theory, or maybe it cannot be embedded in DFT at all. Either way, this direction should be explored in more detail.

A different route is the study of the spacetime actions for the non-relativistic duality web [28] in eleven and ten dimensions. This can proceed both by applying standard dimensional reduction and dualisation to our eleven-dimensional action, and by applying similar methods to individual supergravities by taking covariant non-relativistic limits associated to each p-brane present in the theory. In Chapter 4 we performed a dimensional reduction to type IIA, but we did not discuss the expected T-duality relationship to type IIB, for example. Similarly, there is presumably a heterotic SNC which could be obtained by reducing non-relativistic M-theory on a longitudinal interval, although it is not immediately obvious what the result of reducing on a transverse interval should be. Note that the appearance of the original and dual field strength together in the eleven-dimensional theory suggests that the appropriate formalism for describing generalisations of Newton-Cartan geometries in type II should be the formalism where the RR p-forms are treated 'democratically' [116], accompanied by a self-duality constraint.

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Samenvatting

De belangrijkste focus van dit proefschrift is de afleiding van niet-relativistische deeltjes-, snaar- en membraanacties en bewegingsvergelijkingen. In het bijzonder zijn de theorieën die we beschouwen gebaseerd op (generalisaties van) de Galilese algebra/Newton-Cartan zwaartekracht. Ons beginpunt is het berekenen van de bètafuncties van een niet-relativistische snaartheorie met Torsionele Newton Cartan-symmetrieën in de doelruimte. In analogie met de gebruikelijke relativistische snaartheorie worden de vergelijkingen, die worden verkregen door deze bètafuncties op nul te zetten, vervolgens geïnterpreteerd als de doelruimte-bewegingsvergelijkingen voor (Type I) Torsionele Newton Cartan-zwaartekracht. Vervolgens leiden we een doelruimte-actie af voor deze theorie, evenals voor andere niet-Riemanniaanse theorieën die er nauw mee verwant zijn: Carrolliaanse en String-achtige Newton Cartan-zwaartekracht. Deze acties komen overeen met verschillende niet-Riemanniaanse limieten van de bosonische sector van de gebruikelijke tiendimensionale superzwaartekrachtacties. Ten slotte bestuderen we een niet-relativistische limiet van M-theorie, waarvan de lage energielimiet een theorie geeft die we Membraan Newton Cartan-zwaartekracht noemen, die moet worden beschouwd als de niet-relativistische limiet van de bosonische sector van elfdimensionale superzwaartekracht. Twee conceptueel verschillende dimensionale reducties kunnen dan uitgevoerd worden op MNC-zwaartekracht: een ervan blijkt precies dezelfde SNC-zwaartekracht te zijn die hierboven is genoemd, terwijl de andere een nieuw type niet-relativistische theorie is die is gekoppeld aan $D2$-branen.

Appendix A

Type I TNC beta functions: computational details

A.1 Geodesic equation and normal coordinates in TNC geometry

The action of a particle moving in a TNC background is given by [1]

$$\mathcal{S}_{\text{part}} \propto \int d\lambda \frac{\bar{h}_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}{\tau_\sigma \dot{x}^\sigma} \,. \tag{A.1}$$

The geodesic equation can be obtained by minimizing such action, the corresponding equations of motion are found to be

$$\left[\frac{1}{2}\partial_s \bar{h}_{\mu\nu} - \partial_\mu \bar{h}_{\sigma\nu} - \frac{\left(\bar{h}_{\mu\nu}F_{\sigma\rho} - \tau_\sigma \partial_\rho \bar{h}_{\mu\nu}\right)\dot{x}^\rho}{2N} - \frac{\dot{N}\bar{h}_{\mu\nu}\tau_\sigma}{N^2}\right]\dot{x}^\mu \dot{x}^\nu + \frac{\dot{N}\bar{h}_{sn}\dot{x}^\nu - \tau_\sigma \bar{h}_{\mu\nu}\ddot{x}^\mu x^\nu}{N} = \bar{h}_{\sigma\nu}\ddot{x}^\nu \,, \tag{A.2}$$

where we have defined $N \equiv \tau_p \dot{x}^p$. Contracting (A.2) with $h^{\sigma\lambda}$ gives us the geodesic equation

$$\ddot{x}^\lambda + \Gamma^\lambda_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \frac{\dot{N}}{N}\dot{x}^\lambda - \frac{\bar{h}_{\mu\nu}F_{\sigma\rho}h^{\sigma\lambda}}{2N}\dot{x}^\mu \dot{x}^\nu \dot{x}^\rho \,. \tag{A.3}$$

We want to construct a solution of (A.3) such that $x^\mu(0) = X_0^\mu$ and $x^\mu(1) = X_0^\mu + l_s\bar{Y}^\mu$ where we can identify the vector $\dot{x}^\mu(0) = l_s Y^\mu$. The following expansion on l_s satisfying the previously mentioned conditions can be constructed

$$x^\mu = X_0^\mu + \lambda l_s Y^\mu + \frac{\lambda^2}{2} l_s^2 Y_2^\mu + \mathcal{O}(l_s^3)\,, \tag{A.4}$$

substituting (A.4) in (A.3) it follows that

$$\left(Y_2^\lambda + \Gamma^\lambda_{\mu\nu} Y^\mu Y^\nu\right) = \frac{\tau_\nu Y_2^\nu Y^\lambda + \partial_\mu \tau_\nu Y^\mu Y^\nu Y^\lambda - \frac{1}{2}\bar{h}_{\mu\nu} h^{\lambda\sigma} F_{\sigma\rho} Y^\rho Y^\mu Y^\nu}{\tau_\kappa Y^\kappa}\,, \tag{A.5}$$

where all the geometric background functions are evaluated at X_0^μ. Equation (A.5) has a solution of the form

$$Y_2^\lambda = -\Gamma^\lambda_{\mu\nu} Y^\mu Y^\nu - G^\lambda_{\mu\nu} Y^\mu Y^\nu\,, \tag{A.6}$$

with $G^\lambda_{\mu\nu}$ a tensor satisfying

$$\tau_{(\rho} G^\lambda_{\mu\nu)} = \tau_\sigma G^\sigma_{(\mu\nu}\delta^\lambda_{\rho)} - \frac{1}{2}\bar{h}_{(\mu\nu} F_{\rho)\sigma} h^{\sigma\lambda}\,. \tag{A.7}$$

For (A.7) to have a solution it is necessary to impose $F_{\rho\sigma} h^{\rho\mu} h^{\sigma\nu} = 0$, obtaining $G^\lambda_{\mu\nu} = \frac{1}{2}\bar{h}_{\mu\nu} a_\sigma h^{\sigma\lambda}$, meaning that the quantum field $\bar{Y}^\mu$ can be written in terms of the covariant vector Y^μ as

$$\bar{Y}^\mu = Y^\mu - \frac{l_s}{2}\left(\Gamma^\mu_{\rho\sigma} + \frac{1}{2}\bar{h}_{\rho\sigma} a_\nu h^{\mu\nu}\right) Y^\rho Y^\sigma + \mathcal{O}\left(l_s^2\right)\,. \tag{A.8}$$

A.2 Tree level contributions from the Dilaton

In this section we will compute the tree level contributions to the beta functions. To this end we will need the contribution to the energy-momentum tensor coming from (2.20) and then compute its (classical) trace. Notice that the energy-momentum tensor will receive a contribution from this term even when the worldsheet is flat. The result is given by

$$\left(-\frac{2\pi}{\alpha'}\right)\gamma^{\alpha\beta} T^{Dil}_{\alpha\beta} = -\hat{\Box}\phi = -\hat{\Box} X^\mu \partial_\mu \phi - \gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \partial_\mu \partial_\nu \phi\,, \tag{A.9}$$

where $\hat{\Box}$ is the d'Alembertian on the worldsheet, $\hat{\Box} = \gamma^{\alpha\beta}\partial_\alpha\partial_\beta$. To rewrite this in a useful way we need the equations of motion for the classical fields. These are found by varying the Lagrangian (2.19):

$$\begin{aligned} 0 =& -\gamma^{\alpha\beta}\left(\partial_\rho\bar{h}_{\mu\nu} - 2\partial_\mu\bar{h}_{\nu\rho}\right)\partial_\alpha X^\mu\partial_\beta X^\nu + 2\bar{h}_{\mu\rho}\hat{\Box}X^\mu + 2\Delta\lambda^\alpha\partial_\alpha X^\mu\partial_{[\mu}\tau_{\rho]} \\ &+ \tau_\rho\partial_\alpha\Delta\lambda^\alpha - 2\Sigma\lambda^\alpha\partial_\alpha X^\mu\partial_{[\mu}\aleph_{\rho]} - \aleph_\rho\partial_\alpha\Sigma\lambda^\alpha \\ &- \epsilon^{\alpha\beta}\left(\partial_\rho\bar{B}_{\mu\nu} - 2\partial_\mu\bar{B}_{\rho\nu}\right)\partial_\alpha X^\mu\partial_\beta X^\nu\,, \end{aligned} \tag{A.10}$$

$$0 = e_-^\alpha\partial_\alpha X^\mu\tau_\mu + e_-^\alpha\left(\partial_\alpha\eta + \partial_\alpha X^\mu\aleph_\mu\right)\,, \tag{A.11}$$

$$0 = e_+^\alpha\partial_\alpha X^\mu\tau_\mu - e_+^\alpha\left(\partial_\alpha\eta + \partial_\alpha X^\mu\aleph_\mu\right)\,, \tag{A.12}$$

$$0 = \partial_\alpha\Sigma\lambda^\alpha\,, \tag{A.13}$$

where

$$\Delta\lambda^\beta \equiv \lambda_- e_+^\beta - \lambda_+ e_-^\beta\,, \qquad \Sigma\lambda^\beta \equiv \lambda_- e_+^\beta + \lambda_+ e_-^\beta\,. \tag{A.14}$$

We now multiply the first equation by $\frac{1}{2}h^{\rho\rho'}$, the second equation by $e_+^\beta\partial_\beta$ and the third one by $e_-^\beta\partial_\beta$ to find

$$\begin{aligned} -\hat{\Box}X^\rho =& \left(\Gamma^\rho_{\mu\nu} + \hat{\upsilon}^\rho\partial_\mu\tau_\nu\right)\partial_\alpha X^\mu\partial_\beta X^\nu\gamma^{\alpha\beta} - \frac{1}{2}h^{\rho\sigma}H_{\sigma\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu\epsilon^{\alpha\beta} \\ &+ \hat{\upsilon}^\rho\tau_\mu\hat{\Box}X^\mu + h^{\rho\sigma}\Delta\lambda^\alpha\partial_\alpha X^\mu\partial_{[\mu}\tau_{\sigma]} - h^{\rho\sigma}\Sigma\lambda^\alpha\partial_\alpha X^\mu\partial_{[\mu}\aleph_{\sigma]}\,, \end{aligned} \tag{A.15}$$

$$\left(\tau_\mu + \aleph_\mu\right)\Box X^\mu = e_+^\alpha e_-^\beta\left(\partial_\mu\tau_\nu + \partial_\mu\aleph_\nu\right)\partial_\alpha X^\mu\partial_\beta X^\nu - \hat{\Box}\eta\,, \tag{A.16}$$

$$\left(\tau_\mu - \aleph_\mu\right)\Box X^\mu = e_+^\alpha e_-^\beta\left(\partial_\nu\tau_\mu - \partial_\nu\aleph_\mu\right)\partial_\alpha X^\mu\partial_\beta X^\nu + \hat{\Box}\eta\,, \tag{A.17}$$

where we have also used (A.13) to simplify (A.15). By adding and subtracting (A.16) and (A.17) we find

$$\begin{aligned} \tau_\mu\hat{\Box}X^\mu &= e_+^\alpha e_-^\beta\left(\partial_{(\mu}\tau_{\nu)} + \partial_{[\mu}\aleph_{\nu]}\right)\partial_\alpha X^\mu\partial_\beta X^\nu \\ &= -\left(\gamma^{\alpha\beta}\partial_\mu\tau_\nu + \epsilon^{\alpha\beta}\partial_\mu\aleph_\nu\right)\partial_\alpha X^\mu\partial_\beta X^\nu\,, \end{aligned} \tag{A.18}$$

$$\aleph_\mu\hat{\Box}X^\mu = -\left(\epsilon^{\alpha\beta}\partial_\mu\tau_\nu - \gamma^{\alpha\beta}\partial_\mu\aleph_\nu\right)\partial_\alpha X^\mu\partial_\beta X^\nu - \hat{\Box}\eta\,. \tag{A.19}$$

Substituting (A.18) in (A.15) we finally have

$$\begin{aligned} -\hat{\Box}X^\rho =& \left(\Gamma^\rho_{\mu\nu}\gamma^{\alpha\beta} - \hat{\upsilon}^\rho\partial_\mu\aleph_\nu\epsilon^{\alpha\beta} - \frac{1}{2}h^{\rho\sigma}H_{\rho\mu\nu}\epsilon^{\alpha\beta}\right)\partial_\alpha X^\mu\partial_\beta X^\nu \\ &+ h^{\rho\sigma}\Delta\lambda^\alpha\partial_\alpha X^\mu\partial_{[\mu}\tau_{\sigma]} - h^{\rho\sigma}\Sigma\lambda^\alpha\partial_\alpha X^\mu\partial_{[\mu}\aleph_{\sigma]}\,. \end{aligned} \tag{A.20}$$

Now that we have an expression for $\hat{\Box}X^\rho$ in terms of $\partial_\alpha X^\rho$, we can rewrite (A.9) as

$$\begin{aligned}\frac{2\pi}{l_s^2}\gamma^{\alpha\beta}T^{Dil}_{\alpha\beta} &= -h^{\rho\sigma}D_\rho\phi\,\partial_{[\mu}\tau_{\sigma]}\,\Delta\lambda^\alpha\partial_\alpha X^\mu + h^{\rho\sigma}D_\rho\phi\,\partial_{[\mu}\aleph_{\sigma]}\,\Sigma\lambda^\alpha\partial_\alpha X^\mu \\ &+\left[\gamma^{\alpha\beta}D_\mu D_\nu\phi + \epsilon^{\alpha\beta}\left(\hat{\upsilon}^\rho D_\rho\phi\,\partial_\mu\aleph_\nu + \frac{1}{2}h^{\rho\sigma}D_\rho\phi H_{\sigma\mu\nu}\right)\right]\partial_\alpha X^\mu\partial_\beta X^\nu \\ = -\frac{1}{2}&\left[\beta^\phi_{\rho\sigma}\partial_\alpha X_0^\rho\partial_\beta X_0^\sigma\gamma^{\alpha\beta} + \bar{\beta}^\phi_{\rho\sigma}\partial_\alpha X_0^\rho\partial_\beta X_0^\sigma\epsilon^{\alpha\beta}\right. \\ &\left.+\beta^\phi_\mu\Delta\lambda^\alpha\partial_\alpha X^\mu + \bar{\beta}^\phi_\mu\Sigma\lambda^\alpha\partial_\alpha X^\mu\right],\end{aligned} \tag{A.21}$$

from which one can easily read the dilaton contributions to the beta functions (2.81)-(2.86) :

$$\begin{aligned}\beta^\phi_{\mu\nu} &= -2\mathring{D}_{(\mu}\mathring{D}_{\nu)}\phi - 2G^\lambda_{\mu\nu}\mathring{D}_\lambda\phi\,, \\ \bar{\beta}^\phi_{\mu\nu} &= -b_{\mu\nu}\hat{\upsilon}^\rho\mathring{D}_\rho\phi - h^{\rho\sigma}H_{\sigma\mu\nu}\mathring{D}_\rho\phi\,, \\ \beta^\phi_\mu &= h^{\rho\sigma}F_{\mu\sigma}\mathring{D}_\rho\phi\,, \\ \bar{\beta}^\phi_\mu &= -h^{\rho\sigma}b_{\mu\sigma}\mathring{D}_\rho\phi\,.\end{aligned} \tag{A.22}$$

For completeness we mention that the time projection of (A.15) is given by

$$\begin{aligned}\partial_\alpha\Delta\lambda^\alpha &= \left[\hat{\upsilon}^\rho\left(D_\rho\bar{h}_{\mu\nu} - 2D_\mu\bar{h}_{\nu\rho}\right)\gamma^{\alpha\beta} + \left(\hat{\upsilon}^\rho H_{\rho\mu\nu} + 2\Phi b_{\mu\nu}\right)\gamma^{\alpha\beta}\right]\partial_\alpha X_0^\mu\partial_\beta X_0^\nu \\ &+ a_\mu\Delta\lambda^\alpha\partial_\alpha X_0^\mu - \mathfrak{e}_\mu\Sigma\lambda^\alpha\partial_\alpha X_0^\mu\,.\end{aligned} \tag{A.23}$$

A.3 Covariant expansion of the one loop effective action

We will make use of (2.39), (2.48) and (2.49) to write down the covariant expansion of the couplings appearing on the Polyakov action (2.11). Starting with the $\lambda\eta$ couplings

$$\int\frac{d^2\sigma}{l_s^2}e\left[\left(\lambda_+e^\beta_- + \lambda_-e^\beta_+\right)\partial_\beta\eta\right] = \int d^2\sigma e\left[\Sigma\bar{\Lambda}^\beta\mathring{\nabla}_\beta\bar{H}\right] + \mathcal{O}\left(l_s\right)\,, \tag{A.24}$$

where we have defined $\Sigma\bar{\Lambda}^\beta \equiv \left(\bar{\Lambda}_+ e^\beta_- + \bar{\Lambda}_- e^\beta_+\right)$. We can then look at the $\bar{h}_{\alpha\beta}$ coupling

$$\begin{aligned}\int \frac{d^2\sigma}{l_s^2} e\gamma^{\alpha\beta}\bar{h}_{\alpha\beta} = \int d^2\sigma e \Big[& \bar{h}_{\mu\nu}\mathring{\nabla}_\alpha Y^\mu \mathring{\nabla}^\alpha Y^\nu \\ & + 2\mathring{D}_\sigma \bar{h}_{\mu\nu}\mathring{\nabla}_\alpha Y^\mu Y^\sigma \partial^\alpha X_0^\nu \\ & + \left(\frac{1}{2}\mathring{D}_\rho \mathring{D}_\sigma \bar{h}_{\mu\nu} + \mathring{R}^\lambda{}_{\rho\sigma\mu}\bar{h}_{\nu\lambda}\right) Y^\rho Y^\sigma \partial_\alpha X_0^\mu \partial_\beta X_0^\nu \Big] \\ + \mathcal{O}(l_s)\,, &\end{aligned} \tag{A.25}$$

where we recall that $H = d\bar{B}$, $F = d\tau$, $b = d\aleph$, $\Delta\lambda^\beta \equiv \lambda^0_- e^\beta_+ - \lambda^0_+ e^\beta_-$, and $\Sigma\lambda^\beta \equiv \lambda^0_- e^\beta_+ + \lambda^0_+ e^\beta_-$.

Moving to the vector couplings we have

$$\begin{aligned}\int \frac{d^2\sigma}{l_s^2} e\left[\lambda_\pm e^\alpha_\mp \tau_\alpha\right] = \int d^2\sigma e \Big[& \bar{\Lambda}_\pm e^\alpha_\mp \left(\mathring{\nabla}_\alpha (\tau_\mu Y^\mu) - F_{\mu\rho} Y^\rho \partial_\alpha X_0^\mu\right) \\ & + \frac{1}{2} F_{\mu\nu}\lambda^0_\pm e^\alpha_\mp Y^\mu \mathring{\nabla}_\alpha Y^\nu \\ & + \frac{1}{2}\mathring{D}_\rho F_{\sigma\mu} Y^\rho Y^\sigma \lambda^0_\pm e^\alpha_\mp \partial_\alpha X_0^\mu \Big] \\ + \mathcal{O}(l_s)\,, &\end{aligned} \tag{A.26}$$

and

$$\begin{aligned}\int \frac{d^2\sigma}{l_s^2} e\left[\Sigma\lambda^\alpha \aleph_\alpha\right] = \int d^2\sigma e \Big[& \Sigma\bar{\Lambda}^\alpha \left(\aleph_\mu \mathring{\nabla}_\alpha Y^\mu + \mathring{D}_\rho \aleph_\mu Y^\rho \partial_\alpha X_0^\mu\right) \\ & + \frac{1}{2} b_{\mu\nu}\Sigma\lambda^\alpha Y^\mu \mathring{\nabla}_\alpha Y^\nu \\ & + \frac{1}{2}\mathring{D}_\rho b_{\sigma\mu} Y^\rho Y^\sigma \Sigma\lambda^\alpha \partial_\alpha X_0^\mu \Big] \\ + \mathcal{O}(l_s)\,, &\end{aligned} \tag{A.27}$$

where we have used (2.47) as well as the identity

$$\mathring{R}^\lambda{}_{\sigma\rho\mu}\aleph_\lambda = -\mathring{D}_\rho \mathring{D}_\mu \aleph_\sigma + \mathring{D}_\mu \mathring{D}_\rho \aleph_\sigma\,. \tag{A.28}$$

We can finally move to the last coupling:

$$\int \frac{d^2\sigma\, e}{l_s^2}\epsilon^{\alpha\beta}\bar{B}_{\alpha\beta}(X) = \int d^2\sigma e\epsilon^{\alpha\beta}\left[H_{\sigma\mu\nu}\mathring{\nabla}_\alpha Y^\mu Y^\sigma \partial_\beta X_0^\nu \right. \\ \left. + \frac{1}{2}\mathring{D}_\rho H_{\sigma\mu\nu} Y^\rho Y^\sigma \partial_\alpha X_0^\mu \partial_\beta X_0^\nu\right] \\ + \mathcal{O}\left(l_s\right) , \tag{A.29}$$

where we have used the identity

$$\int d^2\sigma e\epsilon^{\alpha\beta}\left[\bar{B}_{\mu\nu}\mathring{\nabla}_\alpha Y^\mu \mathring{\nabla}_\beta Y^\nu\right] = \int d^2\sigma e\epsilon^{\alpha\beta}\left[\left(\mathring{D}_\nu \bar{B}_{\sigma\mu}\right) Y^\sigma \mathring{\nabla}_\alpha Y^\mu \partial_\beta X_0^\nu \right. \\ \left. + \frac{1}{2}\left(\mathring{R}^\lambda{}_{\rho\mu\nu}\bar{B}_{\lambda\sigma}\right) Y^\rho Y^\sigma \partial_\alpha X^\mu \partial_\beta X^\nu\right] . \tag{A.30}$$

Before combining (A.24), (A.25), (A.26), (A.27) and (A.29) to write down the action $\bar{S}_0$ we will take a look at the transformation properties of $\bar{H}$ inherited from the Kalb-Ramond $U(1)$ transformation, namely if the transformation of the original fields are

$$\begin{aligned} \delta_\Lambda \aleph_\mu &= \partial_\mu \Lambda_u(X) , \\ \delta_\Lambda \eta &= -\Lambda_u(X) , \end{aligned} \tag{A.31}$$

then the quantum field $\bar{H}$ will transform as

$$\delta\bar{H} = -D_\rho \Lambda_u Y^\rho + \mathcal{O}\left(l_s\right) . \tag{A.32}$$

It is then convenient to define a new field $\hat{H}$ as

$$\hat{H} = \bar{H} + \aleph_\mu Y^\mu , \tag{A.33}$$

such that $\delta\hat{H} = \mathcal{O}\left(l_s\right)$, making it invariant under the Kalb-Ramond $U(1)$ transformation at the level of the action $\bar{S}_0$. By making use of this field redefinition, the action $\bar{S}_0$ is written as

$$\begin{aligned}\bar{S}_0 = -\int \frac{d^2\sigma e}{4\pi} \Big[& \bar{h}_{\mu\nu}\mathring{\nabla}_\alpha Y^\mu \nabla^\alpha Y^\nu - \bar{\Lambda}_+ e^\beta_- \left(\mathring{\nabla}_\beta \hat{H} + \mathring{\nabla}_\beta\left(\tau_\mu Y^\mu\right)\right) \\ & -\bar{\Lambda}_- e^\beta_+ \left(\mathring{\nabla}_\beta \hat{H} - \mathring{\nabla}_\beta\left(\tau_\mu Y^\mu\right)\right)\Big] \\ - \int \frac{d^2\sigma e}{4\pi} & \left[\bar{\Lambda}_+ Y^\rho \left(F_{\mu\rho} + b_{\mu\rho}\right) e^\beta_- \partial_\beta X^\mu_0 - \bar{\Lambda}_- Y^\rho \left(F_{\mu\rho} - b_{\mu\rho}\right) e^\beta_+ \partial_\beta X^\mu_0\right] \\ - \int \frac{d^2\sigma e}{4\pi} & \Big[\left(\gamma^{\alpha\beta} A_{\sigma\mu\nu} + \epsilon^{\alpha\beta}\bar{A}_{\sigma\mu\nu}\right) Y^\sigma \mathring{\nabla}_\alpha Y^\mu \partial_\beta X^\nu_0 \\ & + \frac{1}{2}\left(\Delta\lambda^\beta F_{\mu\nu} - \Sigma\lambda^\beta b_{\mu\nu}\right) Y^\mu \mathring{\nabla}_\alpha Y^\nu\Big] \\ - \int \frac{d^2\sigma e}{4\pi} & \Big[\left(\gamma^{\alpha\beta} C_{\rho\sigma\mu\nu} + \epsilon^{\alpha\beta}\bar{C}_{\rho\sigma\mu\nu}\right) Y^\rho Y^\sigma \partial_\alpha X^\mu_0 \partial_\beta X^\nu_0 \\ & + \left(\Delta\lambda^\alpha B_{\rho\sigma\mu} + \Sigma\lambda^\alpha \bar{B}_{\rho\sigma\mu}\right) Y^\rho Y^\sigma \partial_\alpha X^\mu_0\Big] ,\end{aligned} \tag{A.34}$$

where the coefficients $\{A, \bar{A}, C, \bar{C}, B, \bar{B}\}$ are given by

$$\begin{aligned} A_{\sigma\mu\nu} &= 2\mathring{D}_\sigma \bar{h}_{\mu\nu} , \\ \bar{A}_{\sigma\mu\nu} &= H_{\sigma\mu\nu} , \\ C_{\rho\sigma\mu\nu} &= \frac{1}{2}\mathring{D}_\rho \mathring{D}_\sigma \bar{h}_{\mu\nu} + \mathring{R}^\lambda{}_{(\rho\sigma)(\mu}\bar{h}_{\nu)\lambda} , \\ \bar{C}_{\rho\sigma\mu\nu} &= \frac{1}{2}\mathring{D}_\rho H_{\sigma\mu\nu} , \\ B_{\rho\sigma\mu} &= \frac{1}{2}\mathring{D}_\rho F_{\sigma\mu} , \\ \bar{B}_{\rho\sigma\mu} &= -\frac{1}{2}\mathring{D}_\rho b_{\sigma\mu} . \end{aligned} \tag{A.35}$$

A.4 Deriving the beta functions

Making use of decomposition (2.59) and assuming we are working on the critical spacetime dimension, the Weyl variation of the effective action can be written as (up to seond order in spacetime derivatives)

$$\delta_\psi \bar{\Gamma}[\Psi_0](0) = \delta_\psi \langle \mathcal{S}_1 + \tilde{\mathcal{S}}_1 + \mathcal{S}_2 + \tilde{\mathcal{S}}_2 \rangle_0 + \frac{i}{2}\delta_\psi \langle \left(\tilde{\mathcal{S}}_1 \tilde{\mathcal{S}}_1 + 2\tilde{\mathcal{S}}_1 \mathcal{S}_1 \right) + \mathcal{S}_1 \mathcal{S}_1 \rangle_0 \,, \tag{A.36}$$

where we have also made use of the Ward identity

$$\int d^2\sigma J^\alpha_{IJ} \langle Y^I(\sigma)\partial_\alpha Y^J(\sigma)\rangle = -\frac{1}{2}\int d^2\sigma \langle Y^I(\sigma) Y^J(\sigma)\rangle \partial_\alpha J^\alpha_{IJ} \,, \tag{A.37}$$

with J^α_{IJ} an arbitrary spacetime tensor introduced to move the disconnected part of the variation to one order higher in derivatives. We will start by computing the two-point correlations

$$\begin{aligned}
&\delta_\psi \langle \mathcal{S}_1 + \mathcal{S}_2 \rangle \\
&= -\int \frac{d^2\sigma\psi}{4\pi} \left\{ \left[-\mathring{R}_{\mu\nu} + \left(\frac{1}{2}\mathring{D}\cdot a + a^2 \left(\frac{d_c}{4} + \frac{3}{4} \right) \right) \bar{h}_{\mu\nu} \right] \gamma^{\alpha\beta}\partial_\alpha X^\mu_0 \partial_\beta X^\nu_0 \right. \\
&+ \left[\frac{1}{2} h^{\rho\sigma}\mathring{D}_\rho H_{\sigma\mu\nu} - \mathring{D}_\mu \mathfrak{e}_\nu + \frac{d_c+2}{4} a_\rho h^{\rho\sigma} H_{\sigma\mu\nu} + \frac{\mathring{D}_\lambda \hat{v}^\lambda}{2} b_{\mu\nu} \right] \epsilon^{\alpha\beta}\partial_\alpha X^\mu_0 \partial_\beta X^\nu_0 \\
&+ \left[\frac{1}{2}\mathring{D}\cdot a + \left(\frac{d_c}{4} + \frac{3}{4} \right) a^2 \right] \tau_\nu \Delta\lambda^\alpha \partial_\alpha X^\nu_0 \\
&+ \left. \left[-\frac{1}{2}\mathring{D}\cdot \mathfrak{e} - \left(\frac{d_c}{4} + \frac{3}{4} \right) a\cdot \mathfrak{e} \right] \tau_\nu \Sigma\lambda^\alpha \partial_\alpha X^\nu_0 \right\} + \dots
\end{aligned} \tag{A.38}$$

where we have neglected terms that will not contribute to the final result, " $\cdot$ " denotes an inner product with respect to $h^{\rho\sigma}$ and where we have used the identity

$$\delta_\psi \int d^2\sigma J_{IJ} \langle Y^I(\sigma) Y^J(\sigma) \rangle = \int d^2\sigma J_{ij}\delta^{ij}\psi \,, \tag{A.39}$$

with J_{IJ} an arbitrary tensor, this last identity follows from the renormalization of the propagators (2.58). In deriving (A.38) we have introduced a total derivative[1] $\int d^2\sigma \partial_\alpha \mathfrak{e}^\alpha$, made use of the Ward identity (A.37), the background equation (A.20), the Bianchi identity

[1]If this total derivative is not included then the $U(1)$ mass variation of the antisymmetric beta function will not be zero but rather a total derivative, leaving the effective action invariant but not the beta function itself.

$$\left(\mathring{D}_\mu \bar{h}_{\sigma\nu} + \mathring{D}_\nu \bar{h}_{\sigma\mu} - \mathring{D}_\sigma \bar{h}_{\mu\nu} + \bar{h}_{\mu\nu} a_\sigma\right) h^{\lambda\sigma} = 0\,, \tag{A.40}$$

and the TNC identities

$$\begin{aligned} h^{\rho\sigma}\left(\frac{1}{2}\mathring{D}_\rho \mathring{D}_\sigma \bar{h}_{\mu\nu} - \mathring{D}_\rho \mathring{D}_{(\mu}\bar{h}_{\nu)\sigma}\right) &= \frac{1}{2}\left(\mathring{D}\cdot a + a^2\right)\bar{h}_{\mu\nu} + \mathring{D}_{(\mu}\bar{h}_{\nu)\sigma} a_\rho h^{\rho\sigma}\,, \\ h^{\rho\sigma} a_\sigma \mathring{D}_\mu \bar{h}_{\nu\sigma} &= \left(\Delta_T\right)^\rho_\nu \mathring{D}_\mu a_\rho - \frac{1}{2} a_\mu a_\nu - \frac{1}{2} a^2 \bar{h}_{\mu\nu}\,, \\ h^{\rho\sigma} \mathring{R}^\lambda_{(\mu\nu)\rho} \bar{h}_{\sigma\lambda} &= -\mathring{R}_{\mu\nu} - \frac{1}{4} a_\mu a_\nu - \frac{1}{2}\left(\Delta_S\right)^\lambda_{(\mu}\mathring{D}_{\nu)} a_\lambda\,, \\ h^{\rho\sigma}\hat{\upsilon}^\lambda \mathring{D}_\rho \bar{h}_{\sigma\lambda} &= -\mathring{D}_\lambda \hat{\upsilon}^\lambda\,, \end{aligned} \tag{A.41}$$

where $\left(\Delta_T\right)^\rho_\mu$ and $\left(\Delta_S\right)^\rho_\mu$ are the usual TNC temporal and spatial projectors

$$\begin{aligned} \left(\Delta_T\right)^\rho_\mu &\equiv -\hat{\upsilon}^\rho \tau_\mu\,, \\ \left(\Delta_S\right)^\rho_\mu &\equiv h^{\rho\lambda}\bar{h}_{\lambda\mu}\,. \end{aligned} \tag{A.42}$$

To compute the four point functions arising from $\langle \mathcal{S}_1 \mathcal{S}_1 \rangle_0$ we will need the non-vanishing four-point identities

$$\begin{aligned} \delta_\psi \int \tfrac{d^2\sigma d^2\sigma'}{2\pi i} J_{IJKL}\langle Y^I \partial_\alpha Y^J(\sigma) Y^K \partial_\beta Y^L(\sigma')\rangle &= -\int d^2\sigma \psi J_{ijkl}\delta^{ik}\delta^{jl}\gamma_{\alpha\beta}\,, \\ \delta_\psi \int \tfrac{d^2\sigma d^2\sigma'}{4\pi i} J_{IJ}\langle \Lambda_+ Y^I(\sigma)\Lambda_- Y^J(\sigma')\rangle &= -\int d^2\sigma\psi J_{ij}\delta^{ij}\,, \\ \delta_\psi \int d^2\sigma d^2\sigma' J_{IJK}\langle Y^I e^\beta_\mp \partial_\beta Y^J(\sigma)\Lambda_\pm Y^K(\sigma')\rangle &= c_0 \int d^2\sigma \psi \frac{J_{i\lambda j}\hat{\upsilon}^\lambda \delta^{ij}}{\sqrt{2\Phi}}\,, \end{aligned} \tag{A.43}$$

with $\{J_{IJKL}, J_{IJ}, J_{IJK}\}$ arbitrary $\mathcal{O}\left(D^2\right)$ tensors, c_0 an arbitrary constant and where the last identity is only true *up to second order in spacetime derivatives*[2]. The presence of c_0 might seem like a problem to the uniqueness of the resulting beta functions, however by noting that the $U(1)$ mass symmetry is non-compatible with the derivative expansion[3] we find that constants

[2]This identity can be derived through integration by parts and making use of propagators (2.58).

[3]A $U(1)$ mass transformation changes the $\mathcal{O}\left(D\right)$ of the actions $S_0^{[a]}$.

of the c_0 type will be completely fixed by asking for $U(1)$ mass invariance at second order in covariant derivatives. The ambiguity in defining the $\mathcal{O}(D)$ can also be seen from the two point function Ward identity (A.37) as well as from the four point function identity

$$\int d^2\sigma d^2\sigma' \left(V^{\alpha}_{(IJ)}V^{\beta}_{(KL)}\right)\langle Y^I(\sigma)\partial_\alpha Y^J(\sigma)Y^K(\sigma')\partial_\beta Y^L(\sigma')\rangle = \mathcal{O}\left(D^3\right)\,, \tag{A.44}$$

with V^{α}_{IJ} an arbitrary tensor. Making use of (A.43), it is found that

$$\begin{aligned}\frac{i}{2}\delta_\psi\langle\mathcal{S}_1\mathcal{S}_1\rangle = &-\int\frac{d^2\sigma\psi}{4\pi}\Bigg[\tfrac{1}{4}H^{\rho\sigma}{}_\mu H_{\rho\sigma\nu} + c_1\left(\Delta_T\right)^\lambda_\mu \mathring{D}_\nu a_\rho - \bar{h}_{\mu\nu}\left(\mathfrak{e}^2+\frac{a^2}{4}\right)\\ &+\frac{\mathfrak{e}^2\left(2\Phi\tau_\mu\tau_\nu+\bar{h}_{\mu\nu}\right)-\mathfrak{e}_\mu\mathfrak{e}_\nu}{2} + c_4\mathfrak{e}_\rho h^{\rho\sigma}\left(\Delta_T\right)^\lambda_\mu H_{\nu\lambda\sigma}\\ &-a^2\Phi\tau_\mu\tau_\nu\left(c_1+\frac{5}{2}\right)\Bigg]\gamma^{\alpha\beta}\partial_\alpha X^\mu_0\partial_\beta X^\nu_0\\ &-\int\frac{d^2\sigma\psi}{4\pi}\Bigg[-\frac{a_\rho h^{\rho\sigma}H_{\sigma\mu\nu}}{2}+\left(c_3-\frac{1}{2}\right)a_\rho h^{\rho\sigma}\left(\Delta_T\right)^\lambda_\mu H_{\sigma\nu\lambda}\\ &+c_2\left(\Delta_T\right)^\rho_\mu\mathring{D}_\nu\mathfrak{e}_\rho + a_\mu\mathfrak{e}_\nu\Bigg]\epsilon^{\alpha\beta}\partial_\alpha X^\mu_0\partial_\beta X^\nu_0\\ &-\int\frac{d^2\sigma\psi}{4\pi}\left[\left(c_5a^2+c_6\mathfrak{e}^2\right)\Delta\lambda^\alpha+\left(c_7 a\cdot\mathfrak{e}\right)\Sigma\lambda^\alpha\right]\tau_\nu\partial_\alpha X^\nu_0\,.\end{aligned} \tag{A.45}$$

To derive (A.45) we made use of the projected Bianchi identity

$$\left(\mathring{D}_\rho\bar{h}_{\sigma\nu}-\mathring{D}_\sigma\bar{h}_{\rho\nu}-\frac{1}{2}a_\rho\bar{h}_{\sigma\nu}+\frac{1}{2}a_\sigma\bar{h}_{\rho\nu}\right)=0\,, \tag{A.46}$$

the Bianchi identity (A.40), the TNC identities

$$\begin{aligned}h^{\rho\sigma}a_\sigma\mathring{D}_\mu\bar{h}_{\nu\sigma} &= \left(\Delta_T\right)^\rho_\nu\mathring{D}_\mu a_\rho-\frac{1}{2}a_\mu a_\nu-\frac{1}{2}a^2\bar{h}_{\mu\nu}\,,\\ h^{\rho\sigma}\mathfrak{e}_\sigma\mathring{D}_\mu\bar{h}_{\nu\sigma} &= \left(\Delta_T\right)^\rho_\nu\mathring{D}_\mu\mathfrak{e}_\rho-\frac{1}{2}\mathfrak{e}_\mu a_\nu-\frac{1}{2}\left(\mathfrak{e}\cdot a\right)\bar{h}_{\mu\nu}\,,\end{aligned} \tag{A.47}$$

and we have used the Ward identity (A.43) to introduce the $\mathcal{O}\left(D^2\right)$ zeros

$$\begin{aligned}\int \frac{d^2\sigma}{4\pi}\left[\left(\frac{a_\mu a_\nu - \mathfrak{e}_\mu\mathfrak{e}_\nu + \left(a^2-\mathfrak{e}^2\right)\left(\bar{h}_{\mu\nu}+2\Phi\tau_\mu\tau_\nu\right)}{2}\right)\gamma^{\alpha\beta}\right]\partial_\alpha X_0^\mu\partial_\beta X_0^\nu \\ + \int \frac{d^2\sigma}{4\pi}\left[\left(a_\mu\mathfrak{e}_\nu\right)\epsilon^{\alpha\beta}\right]\partial_\alpha X_0^\mu\partial_\beta X_0^\nu = \mathcal{O}\left(D^3\right)\,,\end{aligned} \tag{A.48}$$

$$\int \frac{d^2\sigma}{4\pi}\left[a_\mu a_\nu + a^2\left(\bar{h}_{\mu\nu}+2\Phi\tau_\mu\tau_\nu\right)\right]\partial_\alpha X_0^\mu\partial^\alpha X_0^\nu = \mathcal{O}\left(D^3\right)\,. \tag{A.49}$$

Following an analogous procedure we can compute the contributions from $\tilde{\mathcal{S}}$, in particular we find that

$$\begin{aligned}\delta_\psi\left\langle\tilde{\mathcal{S}}_1+\tilde{\mathcal{S}}_2+\frac{i}{2}\tilde{\mathcal{S}}_1\left(\tilde{\mathcal{S}}_1+\mathcal{S}_1\right)\right\rangle \\ = -\int\frac{d^2\sigma\psi}{4\pi}\left[\frac{\left(\Delta_T\right)_\mu^{\ \lambda}\mathring{D}_\nu a_\lambda}{2}+\tilde{c}_0\Phi\left(a\cdot\mathfrak{e}\right)\tau_\mu\tau_\nu\right. \\ \left. + \left(c_0-1\right)a^2\Phi\tau_\mu\tau_\nu\right]\partial_\alpha X_0^\mu\partial^\alpha X_0^\nu \\ -\int\frac{d^2\sigma\psi}{4\pi}\left[-\frac{a_\rho h^{\rho\sigma}\left(\Delta_T\right)_\mu^{\ \lambda}H_{\sigma\nu\lambda}}{2}\right]\epsilon^{\alpha\beta}\partial_\alpha X_0^\mu\partial_\beta X_0^\nu \\ -\int\frac{d^2\sigma\psi}{4\pi}\left[\frac{a\cdot e}{4}\Sigma\lambda^\alpha-\frac{a^2}{4}\Delta\lambda^\alpha\right]\tau_\nu\partial_\alpha X_0^\nu\,,\end{aligned} \tag{A.50}$$

where we have used the identities

$$\begin{aligned}\left(\bar{h}_{\rho\sigma}-\bar{h}_{\rho\lambda}\bar{h}_{\sigma\kappa}h^{\lambda\kappa}\right)\mathring{D}_\mu e_i^\rho\mathring{D}_\nu e_j^\sigma\delta^{ij} &= -\frac{1}{2}a^2\tau_\mu\tau_\nu\Phi\,, \\ \left(\delta_\sigma^\lambda - h^{\lambda\kappa}\bar{h}_{\kappa\sigma}\right)e_i^\rho\mathring{D}_\mu e_j^\sigma\delta^{ij} &= \frac{1}{2}a_\sigma h^{\rho\sigma}\left(\Delta_T\right)_\mu^{\ \lambda}\,.\end{aligned} \tag{A.51}$$

We can note that the analogous $\tilde{\mathcal{S}}$ computation in the standard bosonic string will result in a vanishing result, however in our case this is no longer true as $h^{\rho\sigma}\bar{h}_{\sigma\lambda}\neq\delta_\lambda^\rho$ as well as due to the presence of a non-trivial coupling with the Lagrange multipliers. Combining (A.38), (A.45), (A.50), and the classical dilaton contribution (A.22) results in the beta functions

$$\beta_\mu = \left[\frac{1}{2}\mathring{D}\cdot a + \left(\frac{d_c}{4} + \frac{1}{2} + c_5\right) a^2 + c_6 \mathfrak{e}^2 - a\cdot \mathring{D}\phi\right]\tau_\mu\,, \tag{A.52}$$

$$\bar{\beta}_\mu = -\left[\frac{1}{2}\mathring{D}\cdot \mathfrak{e} + \left(\frac{d_c}{4} + \frac{1}{2} - c_7\right) a\cdot\mathfrak{e} - \mathfrak{e}\cdot \mathring{D}\phi\right]\tau_\mu\,, \tag{A.53}$$

$$\begin{aligned}\beta_{\mu\nu} = & -\mathring{R}_{\mu\nu} + \frac{1}{4}H^{\rho\sigma}{}_\mu H_{\rho\sigma\nu} + \left[\frac{1}{2}\mathring{D}\cdot a + \left(\frac{d_c}{4} + \frac{1}{2}\right) a^2 - \mathfrak{e}^2 - a\cdot\mathring{D}\phi\right]\bar{h}_{\mu\nu} \\ & + c_4 \mathfrak{e}_\rho h^{\rho\sigma}\left(\Delta_T\right)^\lambda_\mu H_{\nu\lambda\sigma} + \left[\tilde{c}_0 a\cdot\mathfrak{e} + \left(c_0 - c_1 - \frac{7}{2}\right)a^2\right]\Phi\tau_\mu\tau_\nu \\ & + \left[c_1 + \frac{1}{2}\right]\left(\Delta_T\right)^\lambda_\mu \mathring{D}_\nu a_\lambda + \frac{\mathfrak{e}^2\left(2\Phi\tau_\mu\tau_\nu + \bar{h}_{\mu\nu}\right) - \mathfrak{e}_\mu\mathfrak{e}_\nu}{2} - 2\mathring{D}_\mu\mathring{D}_\nu\phi\,,\end{aligned} \tag{A.54}$$

$$\begin{aligned}\bar{\beta}_{\mu\nu} = & \frac{1}{2}h^{\rho\sigma}\mathring{D}_\rho H_{\sigma\mu\nu} + \frac{d_c}{4}a_\rho h^{\rho\sigma}H_{\sigma\mu\nu} - \mathring{D}_\mu\mathfrak{e}_\nu + c_2\left(\Delta_T\right)^\rho_\mu \mathring{D}_\nu\mathfrak{e}_\rho + a_\mu\mathfrak{e}_\nu \\ & + \frac{\mathring{D}_\lambda \upsilon^\lambda}{2}b_{\mu\nu} + \left(c_3 - 1\right)a_\rho h^{\rho\sigma}\left(\Delta_T\right)^\lambda_\mu H_{\sigma\nu\lambda} - \left(\hat{\upsilon}^\rho b_{\mu\nu} + h^{\rho\sigma}H_{\sigma\mu\nu}\right)\mathring{D}_\rho\phi\,.\end{aligned} \tag{A.55}$$

The free coefficients in (A.52)-(A.55) can be fixed by asking for $\{\beta, \bar{\beta}_\mu, \beta_{\mu\nu}, \bar{\beta}_{\mu\nu}\}$ to be gauge invariant, this condition fixes the coefficients to $c_0 = 3, \tilde{c}_0 = 0, c_1 = -\frac{1}{2}, c_2 = 2, c_3 = 1,\ c_4 = -1, c_5 = 0, c_6 = -1, c_7 = \frac{1}{2}$ resulting in the beta functions (2.74)-(2.77) presented in the main text.

Appendix B

TNC, SNC and Carrollian identities

B.1 TNC identities

B.1.1 Geometric identities

The connection we use is

$$\Gamma^{\rho}_{\mu\nu} = -\hat{\upsilon}^{\rho}\partial_{\mu}\tau_{\nu} + \frac{1}{2}h^{\rho\sigma}\left(\partial_{\mu}\bar{h}_{\nu\sigma} + \partial_{\nu}\bar{h}_{\mu\sigma} - \partial_{\sigma}\bar{h}_{\mu\nu}\right) . \tag{B.1}$$

Integration by parts is not as straightforward as in usual general relativity. Instead we have

$$D_{\mu}A^{\mu} = e^{-1}\partial_{\mu}\left(e\,A^{\mu}\right) - a_{\mu}A^{\mu} + 2A^{\mu}D_{\mu}\phi\,, \tag{B.2}$$

where A^{μ} is an arbitrary vector.

Many geometric identities can be derived from the completeness relation

$$-\hat{\upsilon}^{\mu}\tau_{\nu} + h^{\mu\rho}\bar{h}_{\rho\nu} = \delta^{\mu}_{\nu}. \tag{B.3}$$

For example we can take the derivative of this relation and then multiply with $\bar{h}$. This gives

$$D_{\rho}\bar{h}_{\mu\nu} = 2\tau_{(\mu}\bar{h}_{\nu)\lambda}D_{\rho}\hat{\upsilon}^{\lambda} - 2\tau_{\mu}\tau_{\nu}D_{\rho}\Phi\,, \tag{B.4}$$

and

$$D_{\sigma}D_{\rho}\bar{h}_{\mu\nu} = 2\tau_{(\mu}\bar{h}_{\nu)\lambda}D_{\sigma}D_{\rho}\hat{\upsilon}^{\lambda} + 2\tau_{(\mu}\bar{D}_{\sigma}h_{\nu)\lambda}D_{\rho}\hat{\upsilon}^{\lambda} - 2\tau_{\mu}\tau_{\nu}D_{\sigma}D_{\rho}\Phi\,. \tag{B.5}$$

More useful identities can be derived by the definition of the connection:

$$\begin{aligned} h^{\rho\sigma} D_\sigma \bar{h}_{\mu\nu} &= 4h^{\rho\sigma} F_{\sigma(\mu}\tau_{\nu)}\Phi + 2\tau_{(\mu}D_{\nu)}\hat{\upsilon}^\rho \,, \\ \hat{\upsilon}^\rho D_\rho \upsilon^\mu &= h^{\rho\sigma}\left(D_\sigma\Phi + 2a_\sigma\Phi\right) \,, \\ h^{\mu\rho} D_\rho \hat{\upsilon}^\nu &= h^{\nu\rho} D_\rho \hat{\upsilon}^\mu + 2\Phi F_{\rho\sigma} h^{\mu\rho} h^{\nu\sigma} \,. \end{aligned} \tag{B.6}$$

B.1.2 Variational calculus

We choose the independent geometric fields to be $h^{\mu\nu}, \hat{\upsilon}^\mu$ and Φ. The variations of the dependant fields are given by

$$\begin{aligned} \delta\bar{h}_{\mu\nu} &= -2\tau_\mu\tau_\nu\delta\Phi + 2\tau_{(\mu}\bar{h}_{\nu)\rho}\delta\hat{\upsilon}^\rho - \bar{h}_{\mu\rho}\bar{h}_{\nu\sigma}\delta h^{\rho\sigma} \,, \\ \delta\tau_\mu &= \tau_\mu\tau_\rho\delta\hat{\upsilon}^\rho - \bar{h}_{\mu\rho}\tau_\sigma\delta \,. \end{aligned} \tag{B.7}$$

The variation of the measure is then

$$\delta e = \frac{e}{2}\left(\bar{h}^{\rho\sigma}\delta\bar{h}_{\rho\sigma} - \frac{\delta\Phi}{\Phi} - 4\delta\phi\right) = \frac{e}{2}\left(-\bar{h}_{\rho\sigma}\delta h^{\rho\sigma} + 2\tau_\rho\delta\hat{\upsilon}^\rho - 4\delta\phi\right) \,. \tag{B.8}$$

The variation of the acceleration is

$$\delta a_\mu = \delta\hat{\upsilon}^\rho F_{\rho\mu} + 2\hat{\upsilon}^\rho D_{[\rho}\delta\tau_{\mu]} - a_\mu\hat{\upsilon}^\rho\delta\tau_\rho \,. \tag{B.9}$$

The variation of the connection is

$$\begin{aligned} \delta\Gamma^\rho_{\mu\nu} &= -\frac{1}{2}\delta\hat{\upsilon}^\rho F_{\mu\nu} - \hat{\upsilon}^\rho D_\mu\delta\tau_\nu - \frac{1}{2}\hat{\upsilon}^\rho F_{\mu\nu}\tau_\sigma\delta\hat{\upsilon}^\sigma \\ &+ \frac{1}{2}h^{\rho\sigma}\left(D_\mu\delta\bar{h}_{\nu\sigma} + D_\nu\delta\bar{h}_{\mu\sigma} - D_\sigma\delta\bar{h}_{\mu\nu}\right) + \frac{1}{2}\delta h^{\rho\sigma}\left(D_\mu\bar{h}_{\nu\sigma} + D_\nu\bar{h}_{\mu\sigma} - D_\sigma\bar{h}_{\mu\nu}\right) \\ &- 2\Phi\tau_{(\mu}F_{\nu)\sigma}\delta h^{\rho\sigma} - 2\tau_{(\mu}F_{\nu)\sigma}h^{\rho\sigma}\delta\Phi - \delta\tau_{(\mu}F_{\nu)\sigma}h^{\rho\sigma}\Phi + h^{\rho\sigma}\bar{h}_{\lambda(\mu}F_{\nu)\sigma}\delta\hat{\upsilon}^\lambda \,. \end{aligned} \tag{B.10}$$

The Palatini identity in the presence of torsion is

$$\delta\mathcal{R}_{\mu\nu} = D_\rho\delta\Gamma^\rho_{\nu\mu} - D_\nu\delta\Gamma^\rho_{\rho\mu} - 2\Gamma^\sigma_{[\nu\rho]}\delta\Gamma^\rho_{\sigma\mu} = D_\rho\delta\Gamma^\rho_{\nu\mu} - D_\nu\delta\Gamma^\rho_{\rho\mu} + \hat{\upsilon}^\sigma F_{\nu\rho}\delta\Gamma^\rho_{\sigma\mu} \,. \tag{B.11}$$

The indipendent matter fields are $\aleph_\mu, B_{\mu\nu}$ and ϕ. The variation of $b_{\mu\nu}$ is

$$\delta b_{\mu\nu} = 2D_{[\mu}\delta\aleph_{\nu]} - 2\Gamma^\rho_{[\mu\nu]}\delta\aleph_\rho \,, \tag{B.12}$$

so that the variation of the electric field is

$$\delta\mathfrak{e}_\mu = \delta\hat{\upsilon}^\rho b_{\rho\mu} + 2D_{[\mu}\delta\aleph_{\nu]} - 2\hat{\upsilon}^\nu\Gamma^\rho_{[\mu\nu]}\delta\aleph_\rho \,, \tag{B.13}$$

while that of $H_{\mu\nu\rho}$ can be written as

$$\delta H_{\rho\mu\nu} = 3D_{[\rho}\delta B_{\mu\nu]} - 2\Gamma^{\sigma}_{[\rho\mu]}\delta B_{\nu\sigma} - 2\Gamma^{\sigma}_{[\nu\rho]}\delta B_{\mu\sigma} - 2\Gamma^{\sigma}_{[\mu\nu]}\delta B_{\rho\sigma} \,. \tag{B.14}$$

The Kalb-Ramond fields satisfy the usual Bianchi identities

$$dH = db = 0 \,. \tag{B.15}$$

In the *twistless* case we have $b_{\mu\nu} = \mathfrak{e}_\mu \tau_\nu - \mathfrak{e}_\nu \tau_\mu$, then we can express the Bianchi identity in terms of $\mathfrak{e}_\mu$ rather than $b_{\mu\nu}$. We find

$$\partial_{[\rho} b_{\mu\nu]} = 2D_{[\mu}\mathfrak{e}_\nu \tau_{\rho]} + 2a_{[\mu}\mathfrak{e}_\nu \tau_{\rho]} = 0 \,. \tag{B.16}$$

A similar identity can be derived for the acceleration, yielding

$$\partial_{[\rho} F_{\mu\nu]} = 2D_{[\mu} a_\nu \tau_{\rho]} = 0 \,. \tag{B.17}$$

B.2 Carroll identities

B.2.1 Geometric identities

The connection we utilize when varying the action is

$$\Gamma^{\rho}_{\mu\nu} = -\upsilon^\rho \partial_\mu \hat{\tau}_\nu + \frac{1}{2}\hat{h}^{\rho\sigma}\left(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}\right) \,, \tag{B.18}$$

from which we find the following identitiy

$$D_\mu A^\mu = e^{-1}\partial_\mu\left(e\, A^\mu\right) - a_\mu A^\mu + 2A^\mu D_\mu \phi \,, \tag{B.19}$$

valid for any arbitrary vector A^μ. From the completeness relation and the definition of the connection it is possible to derive the following identities:

$$\begin{aligned} D_\rho h_{\mu\nu} &= -\mathcal{K}_{\rho\mu}\hat{\tau}_\nu - \mathcal{K}_{\rho\nu}\hat{\tau}_\mu \,, \\ D_\mu \upsilon^\nu &= -\hat{h}^{\nu\rho}\mathcal{K}_{\rho\mu} \,, \\ \upsilon^\rho D_\rho \upsilon^\mu &= 0 \,, \end{aligned} \tag{B.20}$$

where the extrinsic curvature was defined in the main text as

$$\mathcal{K}_{\mu\nu} = -\frac{1}{2}\mathcal{L}_\upsilon h_{\mu\nu} = -\frac{1}{2}\left(\upsilon^\rho \partial_\rho h_{\mu\nu} + \left(\partial_\mu \upsilon^\rho\right) h_{\rho\nu} + \left(\partial_\nu \upsilon^\rho\right) h_{\mu\rho}\right) . \tag{B.21}$$

B.2.2 Variational calculus

We will use $\hat{h}^{\mu\nu}, \upsilon^\mu, \Phi$ as indipendent geometric fields. The other geometric fields are related to them via

$$
\begin{aligned}
\delta h_{\mu\nu} &= 2\tau_{(\mu}h_{\nu)\rho}\delta\upsilon^\rho - h_{\mu\rho}h_{\nu\sigma}\delta\hat{h}^{\rho\sigma} \,, \\
\delta\hat{\tau}_\mu &= \tau_\mu\tau_\rho\delta\upsilon^\rho - h_{\mu\rho}\tau_\sigma\delta\hat{h}^{\rho\sigma} \,, \\
\delta\bar{h}^{\mu\nu} &= \delta\hat{h}^{\mu\nu} - 4\Phi\upsilon^{(\mu}\delta\upsilon^{\nu)} - 2\delta\Phi\,\upsilon^\mu\upsilon^\nu \,.
\end{aligned}
\tag{B.22}
$$

The variation of the measure is then

$$
\delta e = \frac{e}{2}\left(\bar{h}^{\rho\sigma}\delta\bar{h}_{\rho\sigma} + \frac{\delta\Phi}{\Phi} - 4\delta\phi\right) = \frac{e}{2}\left(-h_{\mu\nu}\delta\hat{h}^{\mu\nu} + 2\hat{\tau}_\mu\delta\upsilon^\mu - 4\delta\phi\right) \,. \tag{B.23}
$$

The variation of the acceleration is

$$
\delta a_\mu = \delta\upsilon^\rho F_{\rho\mu} + 2\upsilon^\rho D_{[\rho}\delta\hat{\tau}_{\mu]} - a_\mu\upsilon^\rho\delta\hat{\tau}_\rho \,. \tag{B.24}
$$

The variation of the connection is

$$
\begin{aligned}
\delta\Gamma^\rho_{\mu\nu} =& -\frac{1}{2}\delta\upsilon^\rho F_{\mu\nu} - \upsilon D_\mu\delta\hat{\tau}_\nu - \frac{1}{2}\upsilon^\rho F_{\mu\nu}\hat{\tau}_\sigma\delta\upsilon^\sigma + \hat{h}^{\rho\sigma}h_{\lambda(\mu}F_{\nu)\sigma}\delta\upsilon^\lambda \\
&+ \frac{1}{2}\hat{h}^{\rho\sigma}\left(D_\mu\delta h_{\nu\sigma} + D_\nu\delta h_{\mu\sigma} - D_\sigma\delta h_{\mu\nu}\right) + \frac{1}{2}\delta\hat{h}^{\rho\sigma}\left(D_\mu h_{\nu\sigma} + D_\nu h_{\mu\sigma} - D_\sigma h_{\mu\nu}\right) \,.
\end{aligned}
\tag{B.25}
$$

The Palatini identity is

$$
\delta\mathcal{R}_{\mu\nu} = D_\rho\delta\Gamma^\rho_{\nu\mu} - D_\nu\delta\Gamma^\rho_{\rho\mu} - 2\Gamma^\sigma_{[\nu\rho]}\delta\Gamma^\rho_{\sigma\mu} = D_\rho\delta\Gamma^\rho_{\nu\mu} - D_\nu\delta\Gamma^\rho_{\rho\mu} + \upsilon^\sigma F_{\nu\rho}\delta\Gamma^\rho_{\sigma\mu} \,. \tag{B.26}
$$

The variations of the Kalb-Ramond matter fields are given by (B.12)-(B.14).

B.3 SNC identities

B.3.1 Geometric identities

The connection is

$$
\Gamma^\rho_{\mu\nu} = -u^\rho{}_A\nabla_\mu\tau_\nu{}^A + \frac{1}{2}h^{\rho\sigma}\left(D_\mu\bar{h}_{\nu\sigma} + D_\nu\bar{h}_{\mu\sigma} - D_\sigma\bar{h}_{\mu\nu}\right) \,, \tag{B.27}
$$

where we introduced the spin connection via

$$\nabla_\mu \tau_\nu{}^A = \partial_\mu \tau_\nu{}^A + \Omega_\mu{}^{AB} \tau_{\nu B} \equiv \partial_\mu \tau_\nu{}^A + \omega_\mu \epsilon^A{}_B \tau_\nu{}^B \,, \tag{B.28}$$

and our convention for the longitudinal epsilon symbol is $\epsilon_{01} = +1$. This connection is boost invariant and compatible with $h^{\mu\nu}$ and $\tau_\mu{}^A$,

$$D_\rho h^{\mu\nu} = D_\rho \tau_\mu^A = 0 \,, \tag{B.29}$$

where we are using the symbol D (not to be confused with the dimensionality of spacetime) to denote the full covariant derivative, i.e.

$$D_\mu \tau_\nu{}^A = \partial_\mu \tau_\nu{}^A - \Gamma^\rho_{\mu\nu} \tau_\rho{}^A + \omega_\mu \epsilon^A{}_B \tau_\nu{}^B = 0 \,. \tag{B.30}$$

Integration by parts is then performed with the use of the identity

$$D_\mu A^\mu = e^{-1} \partial_\mu \left(e\, A^\mu \right) - a_\mu A^\mu + 2 A^\mu D_\mu \phi \,, \tag{B.31}$$

where we recall $a_\mu = a_\mu{}^{AB} \eta_{AB}$.

From the completeness relation we find the following decomposition

$$D_\rho \bar{h}_{\mu\nu} = 2 \tau_{(\mu}{}^A \bar{h}_{\nu)\sigma} D_\rho u^\sigma{}_A - \tau_\mu{}^A \tau_\nu{}^B D_\rho \Phi_{AB} \,. \tag{B.32}$$

From the definition of the connection we find

$$h^{\rho\sigma} \left(D_\mu \bar{h}_{\nu\sigma} + D_\nu \bar{h}_{\mu\sigma} - D_\sigma \bar{h}_{\mu\nu} \right) = -2 h^{\rho\sigma} F_{\sigma(\mu}{}^A \tau_{\nu)}{}^B \Phi_{AB} \,. \tag{B.33}$$

From the projections of this identity it follows that

$$h^{\rho[\mu} D_\rho u^{\nu]}{}_A = \frac{1}{2} F^{\mu\nu B} \Phi_{BA} \tag{B.34}$$

and

$$u^\nu{}_{(A} D_\nu u^\mu{}_{B)} = h^{\mu\nu} \left(a_{\nu(A}{}^C \Phi_{B)C} + \frac{1}{2} D_\nu \Phi_{AB} \right) \,. \tag{B.35}$$

Since Φ_{AB} is a 2×2 symmetric matrix, we can write its inverse as

$$(\Phi^{-1})_{AB} = \frac{\Phi_{AB} - \eta_{AB} \Phi}{\det \Phi} \,, \tag{B.36}$$

where $\det \Phi = \frac{1}{2} \epsilon^{AB} \epsilon^{CD} \Phi_{AC} \Phi_{BD}$ and $\Phi = \Phi^A_A$. Moreover, since the longitudinal indices can only take two different values, we have that

$$T^{[\alpha\beta\gamma\dots]} = 0 \,, \tag{B.37}$$

for any tensor T with three or more antisymmetrized longitudinal indices.

The field strength of τ_μ^A can be decomposed as

$$F_{\mu\nu}{}^A = f^A \tau_\mu{}^B \tau_\nu{}^C \epsilon_{BC} + 2a_{[\mu}{}^{BA}\tau_{\nu]B} + \mathcal{F}_{\mu\nu}{}^A\,, \tag{B.38}$$

with

$$u^\mu{}_A \mathcal{F}_{\mu\rho B} = 0, \qquad\qquad u^{\mu A} a_\mu{}^{BC} = \epsilon^{AB} f^C\,. \tag{B.39}$$

B.3.2 Variational calculus

The independent fields are $u^\mu{}_A, h^{\mu\nu}, \Phi^{AB}$ and $B_{\mu\nu}$. They are related to the other SNC fields via

$$\begin{aligned} \delta\tau_\mu{}^A &= \tau_\mu{}^B \tau_\rho{}^A \delta u^\rho{}_B - \tau_\rho{}^A \bar{h}_{\mu\sigma}\delta h^{\rho\sigma}\,, \\ \delta\bar{h}_{\mu\nu} &= 2\tau_{(\mu}{}^A \bar{h}_{\nu)\rho} u^\rho{}_A - \bar{h}_{\mu\rho}\bar{h}_{\nu\sigma}\delta h^{\rho\sigma} - \tau_\mu{}^A \tau_\nu{}^B \delta\Phi_{AB}\,. \end{aligned} \tag{B.40}$$

The variation of the connection is

$$\begin{aligned} \delta\Gamma^\rho_{\mu\nu} = &-u^\rho{}_A\left(D_\mu\delta\tau_\nu{}^A + \Gamma^\sigma_{[\mu\nu]}\delta\tau_\sigma{}^A + \epsilon^A{}_B\tau_{\nu B}\delta\omega_\mu\right) - \delta u^\rho{}_A \Gamma^\sigma_{[\mu\nu]}\tau_\sigma{}^A \\ &+ \tfrac{1}{2}h^{\rho\sigma}\left(D_\mu\delta\bar{h}_{\nu\sigma} + D_\nu\delta\bar{h}_{\mu\sigma} - D_\sigma\delta\bar{h}_{\mu\nu}\right) + \tfrac{1}{2}\delta h^{\rho\sigma}\left(D_\mu\bar{h}_{\nu\sigma} + D_\nu\bar{h}_{\mu\sigma} - D_\sigma\bar{h}_{\mu\nu}\right) \\ &+ h^{\rho\sigma}\left(\Gamma^\lambda_{[\mu\sigma]}\delta\bar{h}_{\lambda\nu} + \Gamma^\lambda_{[\nu\sigma]}\delta\bar{h}_{\lambda\mu}\right) + \delta h^{\rho\sigma}\left(\Gamma^\lambda_{[\mu\sigma]}\bar{h}_{\lambda\nu} + \Gamma^\lambda_{[\nu\sigma]}\bar{h}_{\lambda\mu}\right)\,. \end{aligned} \tag{B.41}$$

The variation of the Ricci scalar can be found as usual through the Palatini identity:

$$\begin{aligned} \delta R_{\mu\nu} &= D_\rho\delta\Gamma^\rho_{\nu\mu} - D_\nu\delta\Gamma^\rho_{\rho\mu} - 2\Gamma^\rho_{[\nu\sigma]}\delta\Gamma^\sigma_{\rho\mu} \\ &= D_\rho\delta\Gamma^\rho_{\nu\mu} - D_\nu\delta\Gamma^\rho_{\rho\mu} - 2u^\rho{}_A\nabla_{[\nu}\tau^A_{\sigma]}\delta\Gamma^\sigma_{\rho\mu}\,. \end{aligned} \tag{B.42}$$

The variation of the field strength of the B-field can be read off from (B.14).

B.4 Actions in Einstein frame

In this section we will perform a conformal redefinition of the metric complex to rewrite the non-relativistic string frame actions (3.50), (3.89), (3.102) and (3.147) in Einstein frame.

B.4.1 TNC

The action and equations of motion can be written in terms of the basic fields $h^{\mu\nu}$, $\hat{v}^{\mu}$ and Φ. The aforementioned fields transform as

$$\begin{aligned} h^{\mu\nu} &\rightarrow e^{-\alpha\phi} h^{\mu\nu} \,, \\ \upsilon^{\mu} &\rightarrow e^{-\alpha\phi} \upsilon^{\mu} \,, \\ \Phi &\rightarrow e^{-\alpha\phi} \Phi \,, \end{aligned} \tag{B.43}$$

where $e^{-\alpha\phi}$ is the conformal factor and $\alpha \equiv \frac{4}{d-1}$. Then, in Einstein frame, the TNC action (3.50) can be rewritten as

$$\begin{aligned} S = \int d^d x \sqrt{\frac{\det \bar{h}_{\mu\nu}}{2\Phi}} \Big[&\mathcal{R} + \frac{1}{2} a^{\mu} a_{\mu} + \frac{1}{2} e^{-2\alpha\phi} \mathfrak{e}^{\mu} \mathfrak{e}_{\mu} - \alpha D^{\mu}\phi D_{\mu}\phi \\ &- \frac{1}{12} e^{-2\alpha\phi} H^{\mu\nu\rho} H_{\mu\nu\rho} - \frac{1}{2} e^{-2\alpha\phi} \hat{\upsilon}^{\rho} H_{\rho\mu\nu} b^{\mu\nu} - \frac{1}{2} \left(F^{\mu\nu} F_{\mu\nu} + e^{-2\alpha\phi} b^{\mu\nu} b_{\mu\nu} \right) \Phi \Big] \,. \end{aligned} \tag{B.44}$$

B.4.2 Carroll

The action and equations of motion are written in terms of the basic fields $\hat{h}^{\mu\nu}$, υ^{μ} and Φ. These fields transform as

$$\begin{aligned} \hat{h}^{\mu\nu} &\rightarrow e^{-\alpha\phi} \hat{h}^{\mu\nu} \,, \\ \upsilon^{\mu} &\rightarrow e^{-\alpha\phi} \upsilon^{\mu} \,, \\ \Phi &\rightarrow e^{+\alpha\phi} \Phi \,, \end{aligned} \tag{B.45}$$

where once again we have $\alpha \equiv \frac{4}{d-1}$. In Einstein frame, the action (3.89) is

$$\begin{aligned} S = \int d^d x \sqrt{2\Phi \det \bar{h}_{\mu\nu}} \Big[&\mathcal{R} + \frac{1}{2} a^{\mu} a_{\mu} + \frac{1}{2} e^{-2\alpha\phi} \mathfrak{e}^{\mu} \mathfrak{e}_{\mu} - \alpha D^{\mu}\phi D_{\mu}\phi + 2\mathcal{K} \upsilon^{\mu} D_{\mu}\Phi \\ &+ 2\Phi \left(\mathcal{K}^{\mu\nu} \mathcal{K}_{\mu\nu} - \mathcal{K}^2 + \alpha\, \upsilon^{\mu} D_{\mu}\phi\, \upsilon^{\nu} D_{\nu}\phi + \frac{1}{4} e^{-2\alpha\phi} \upsilon^{\rho} \upsilon^{\sigma} H^{\mu\nu}{}_{\rho} H_{\mu\nu\sigma} \right) \\ &- \frac{1}{12} e^{-2\alpha\phi} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{2} e^{-2\alpha\phi} b^{\mu\nu} H_{\mu\nu\rho} \upsilon^{\rho} \Big] \,. \end{aligned} \tag{B.46}$$

Similarily we can transform the action (3.102):

$$S = \int d^d x \sqrt{2\Phi \det \bar{h}_{\mu\nu}} \left[\bar{\mathcal{R}} + \frac{1}{2} a^\mu a_\mu + \frac{1}{2} e^{-2\alpha\phi} \mathfrak{e}^\mu \mathfrak{e}_\mu - \alpha \, \bar{D}^\mu \phi \bar{D}_\mu \phi \right.$$
$$\left. - \frac{1}{12} e^{-2\alpha\phi} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{2} e^{-2\alpha\phi} b^{\mu\nu} \upsilon^\rho H_{\mu\nu\rho} \right] . \tag{B.47}$$

B.4.3 SNC

The action and equations of motion can be written in terms of the basic fields $h^{\mu\nu}$, υ_A^μ and Φ_{AB}, which transform as

$$\begin{aligned} h^{\mu\nu} &\to e^{-\alpha\phi} h^{\mu\nu} \, , \\ u_A^\mu &\to e^{-\alpha\phi} u_A^\mu \, , \\ \Phi_{AB} &\to e^{-\alpha\phi} \Phi_{AB} \, , \end{aligned} \tag{B.48}$$

where now we have $\alpha \equiv \frac{4}{D}$. The action (3.147) can be written in Einstein frame as

$$S = \int d^D x \sqrt{\frac{\det \bar{h}_{\mu\nu}}{\det \Phi_{AB}}} \left[\mathcal{R} - a^{\mu AB} (a_{\mu(AB)} - \frac{1}{2} \eta_{AB} a_\mu) + a^\mu a_\mu + \alpha \, a^\mu D_\mu \phi \right.$$
$$+ \alpha(\alpha - 1) D^\mu \phi \, D_\mu \phi - \frac{1}{2} F^{\mu\nu A} F^B_{\mu\nu} (\Phi_{AB} - \frac{1}{2} \eta_{AB} \Phi)$$
$$\left. + \frac{1}{2} e^{-\alpha\phi} \epsilon_{AB} u^{\rho A} F^{\mu\nu B} H_{\rho\mu\nu} - \frac{1}{12} e^{-2\alpha\phi} H^{\rho\mu\nu} H_{\rho\mu\nu} \right] . \tag{B.49}$$

Appendix C

Non-relativistic parametrizations of ExFT

C.1 Dimensional decomposition of non-relativistic action for ExFT

Decomposition of $R^{(0)}$ Consider the part of the scalar curvature $R^{(0)}$ as defined in (4.30) not involving the longitudinal metric, but just the transverse metrics $\hat{H}_{\hat{\mu}\hat{\nu}}$ and $\hat{H}^{\hat{\mu}\hat{\nu}}$ and the measure factor $\hat{\Omega}$. In the dimensional decomposition used in exceptional field theory, the latter two factorise as

$$\hat{H}_{\hat{\mu}\hat{\nu}} = U_{\hat{\mu}}{}^{\hat{\rho}} U_{\hat{\nu}}{}^{\hat{\sigma}} \bar{H}_{\hat{\rho}\hat{\sigma}} \,, \quad \hat{H}^{\hat{\mu}\hat{\nu}} = (U^{-1})_{\hat{\rho}}{}^{\hat{\mu}} (U^{-1})_{\hat{\sigma}}{}^{\hat{\nu}} \bar{H}^{\hat{\rho}\hat{\sigma}} \,, \tag{C.1}$$

with

$$U_{\hat{\mu}}{}^{\hat{\nu}} = \begin{pmatrix} \delta_\mu{}^\nu & A_\mu{}^j \\ 0 & \delta_i{}^j \end{pmatrix} \,, \quad \bar{H}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} G_{\mu\nu} & 0 \\ 0 & H_{ij} \end{pmatrix} \,, \quad \bar{H}^{\hat{\mu}\hat{\nu}} = \begin{pmatrix} G^{\mu\nu} & 0 \\ 0 & H^{ij} \end{pmatrix} \,. \tag{C.2}$$

Here $G^{\mu\nu}$ is the inverse of $G_{\mu\nu}$, but H^{ij} and H_{ij} are not invertible. The idea is to completely factor out the matrix U from derivatives of $\hat{G}$. Defining

$$\partial_{\hat{\mu}} \hat{H}_{\hat{\nu}\hat{\rho}} = U_{\hat{\mu}}{}^{\hat{\sigma}} U_{\hat{\nu}}{}^{\hat{\lambda}} U_{\hat{\rho}}{}^{\hat{\kappa}} \overline{\partial H}_{\hat{\sigma}\hat{\lambda}\hat{\kappa}} \,, \quad \partial_{\hat{\mu}} \hat{H}^{\hat{\nu}\hat{\rho}} = U_{\hat{\mu}}{}^{\hat{\sigma}} (U^{-1})_{\hat{\lambda}}{}^{\hat{\nu}} (U^{-1})_{\hat{\kappa}}{}^{\hat{\rho}} \overline{\partial h}_{\hat{\sigma}}{}^{\hat{\lambda}\hat{\kappa}} \,, \tag{C.3}$$

we have the relatively simple expressions

$$\begin{aligned}
\overline{\partial H}_{\mu\hat\nu\hat\rho} &= \begin{pmatrix} \bar D_\mu G_{\nu\rho} & H_{kl}\bar D_\mu A_\nu{}^l \\ H_{jl}\bar D_\mu A_\rho{}^k & \bar D_\mu H_{jk} \end{pmatrix} , \\
\overline{\partial H}_{i\hat\nu\hat\rho} &= \begin{pmatrix} \partial_i G_{\nu\rho} & H_{kl}\partial_i A_\nu{}^l \\ H_{jl}\partial_i A_\rho{}^k & \partial_i H_{jk} \end{pmatrix} , \\
\overline{\partial H}_\mu{}^{\hat\nu\hat\rho} &= \begin{pmatrix} \bar D_\mu G^{\nu\rho} & -G^{\nu\sigma}\bar D_\mu A_\sigma{}^k \\ -G^{\rho\sigma}\bar D_\mu A_\sigma{}^j & \bar D_\mu H^{jk} \end{pmatrix} , \\
\overline{\partial H}_i{}^{\hat\nu\hat\rho} &= \begin{pmatrix} \partial_i G^{\nu\rho} & -G^{\nu\sigma}\partial_i A_\sigma{}^k \\ -G^{\rho\sigma}\partial_i A_\sigma{}^j & \partial_i H^{jk} \end{pmatrix} ,
\end{aligned} \tag{C.4}$$

where $\bar D_\mu \equiv \partial_\mu - A_\mu{}^i\partial_i$. For instance, consider the following terms in the scalar curvature:

$$\tfrac14 \bar H^{\hat\mu\hat\nu}\overline{\partial H}_{\hat\mu\hat\rho\hat\sigma}\overline{\partial H}_{\hat\nu}{}^{\hat\rho\hat\sigma} - \tfrac12 \bar H^{\mu\nu}\overline{\partial H}_\mu{}^{\rho\sigma}\overline{\partial H}_{\rho\nu\sigma} . \tag{C.5}$$

A fairly straightforward calculations shows that these equal

$$\begin{aligned}
&\tfrac14 G^{\mu\nu}D_\mu G_{\rho\sigma}D_\nu G^{\rho\sigma} - \tfrac12 G^{\mu\nu}D_\mu G^{\rho\sigma}D_\rho G_{\nu\sigma} - \tfrac14 G^{\mu\nu}G^{\rho\sigma}H_{ij}F_{\mu\rho}{}^iF_{\nu\sigma}{}^j \\
&+ \tfrac14 G^{\mu\nu}D_\mu H_{ij}D_\nu H^{ij} + \tfrac14 H^{ij}(\partial_i G_{\rho\sigma}\partial_j G^{\rho\sigma} + \partial_i H_{kl}\partial_j H^{kl}) - \tfrac12 H^{ij}\partial_i H^{kl}\partial_k H_{jl} \\
&- \tfrac12(\delta^i_k + H^{ij}H_{jk})\bar D_\mu A_\nu{}^k\partial_i G^{\mu\nu} + G^{\mu\nu}H^{ij}H_{jk}\partial_l A_\mu{}^k\partial_i A_\nu{}^l ,
\end{aligned} \tag{C.6}$$

where $F_{\mu\nu}{}^i \equiv 2\bar D_{[\mu}A_{\nu]}{}^i$, $D_\mu = \partial_\mu - L_{A_\mu}$, and acting on $G_{\mu\nu}$ and $G^{\mu\nu}$, we have $D_\mu = \bar D_\mu$.

Next, consider the part of $R^{(0)}$ that involves τ:

$$\tfrac14 \hat H^{\hat\mu\hat\nu}\partial_{\hat\mu}\hat\tau_{\hat\rho\hat\sigma}\partial_{\hat\nu}\hat\tau^{\hat\rho\hat\sigma} + \tfrac14 \hat\tau^{\hat\mu\hat\nu}\partial_{\hat\mu}\tau_{\hat\rho\hat\sigma}\partial_{\hat\nu}\hat H^{\hat\rho\hat\sigma} - \tfrac12 \hat\tau^{\hat\mu\hat\nu}\partial_{\hat\nu}H^{\hat\rho\hat\sigma}\partial_{\hat\rho}\hat\tau_{\hat\mu\hat\sigma} - \tfrac12 \hat H^{\hat\mu\hat\nu}\partial_{\hat\nu}\hat\tau^{\hat\rho\hat\sigma}\partial_{\hat\rho}\hat\tau_{\hat\mu\hat\sigma} . \tag{C.7}$$

Similar calculations to above give

$$\begin{aligned}
&\tfrac14 G^{\mu\nu}D_\mu\tau_{ij}D_\nu\tau^{ij} + g^{\mu\nu}\tau^{ik}\tau_{kj}\partial_i A_\mu{}^l\partial_l A_\nu{}^j - \tfrac12\tau^{ik}\tau_{kj}\bar D_\mu A_\nu{}^k\partial_i G^{\mu\nu} \\
&\qquad + \tfrac14 H^{ij}\partial_i\tau_{kl}\partial_j\tau^{kl} + \tfrac14\tau^{ij}\partial_i\tau_{kl}\partial_j H^{kl} - \tfrac12\tau^{ij}\partial_j H^{kl}\partial_k\tau_{il} - \tfrac12 H^{ij}\partial_j\tau^{kl}\partial_k\tau_{il} .
\end{aligned} \tag{C.8}$$

The terms involving $\tau^{ik}\tau_{kj}$ on the first line here combine with the terms involving $H^{ik}H_{kj}$ in the last line of (C.6) and sum up to give $\delta^i_j = H^{ik}H_{kj} + \tau^{ik}\tau_{kj}$, after which point the rest of the calculation proceeds identically to that normally used in exceptional field theory.

Finally one has the terms

$$- \bar{G}^{\hat{\mu}\hat{\nu}} \bar{\partial}_{\hat{\mu}} \ln \hat{\Omega}\, \bar{\partial}_{\hat{\nu}} \ln \hat{\Omega} + 2\bar{\partial}_{\hat{\mu}} \ln \hat{\Omega}\, \overline{\partial G}_{\hat{\nu}}{}^{\hat{\mu}\hat{\nu}} - \partial_{\hat{\mu}}\partial_{\hat{\nu}} \hat{G}^{\hat{\mu}\hat{\nu}} - \hat{G}^{\hat{\mu}\hat{\nu}} \partial_{\hat{\mu}}\partial_{\hat{\nu}} \ln \hat{\Omega}\,, \tag{C.9}$$

where $\hat{\Omega}$ has weight 1, and in the final two terms $\bar{\partial}_\mu \equiv \bar{D}_\mu$, $\bar{\partial}_i \equiv \partial_i$. Note $D_\mu \ln \hat{\Omega} = \bar{D}_\mu \ln \hat{\Omega} - \partial_i A_\mu{}^i$. We let $\hat{\Omega} = \Omega\sqrt{|G|}$, where Ω has weight 1 under internal diffeomorphisms. Straightforward manipulations allow one to rewrite (C.9) in the decomposition and combine with (C.6) and (C.8) After dropping a total derivative, the final result is:

$$\begin{aligned} R^{(0)}(\hat{H},\hat{\tau}) = R_{\text{ext}}(G) + R^{(0)}(H,\tau) - \tfrac{1}{4} F_{\mu\nu}{}^i F_{\rho\sigma}{}^j G^{\mu\rho} G^{\nu\sigma} H_{ij} \\ + \tfrac{1}{4} G^{\mu\nu} (D_\mu H_{ij} D_\nu H^{ij} + D_\mu \tau_{ij} D_\nu \tau^{ij} + D_\mu \ln \Omega^2 D_\nu \ln \Omega^2) \\ + \tfrac{1}{4} H^{ij} (\partial_i G_{\mu\nu} \partial_j G^{\mu\nu} + \partial_i \ln |G| \partial_j \ln |G|)\,, \end{aligned} \tag{C.10}$$

where

$$\begin{aligned} R_{\text{ext}}(g) = & \tfrac{1}{4} G^{\mu\nu} D_\mu G_{\rho\sigma} D_\nu G^{\rho\sigma} - \tfrac{1}{2} G^{\mu\nu} D_\mu G^{\rho\sigma} D_\rho G_{\nu\sigma} - \tfrac{1}{4} G^{\mu\nu} D_\mu \ln |G| D_\nu \ln |G| \\ & - D_\mu \ln |G| D_\nu G^{\mu\nu} - G^{\mu\nu} D_\mu D_\nu \ln |G| - D_\mu D_\nu G^{\mu\nu}\,, \end{aligned} \tag{C.11}$$

$$\begin{aligned} R^{(0)}(H,\tau) = & + \tfrac{1}{4} H^{ij} \partial_i \tau_{kl} \partial_j \tau^{kl} + \tfrac{1}{4} \tau^{ij} \partial_i \tau_{kl} \partial_j H^{kl} - \tfrac{1}{2} \tau^{ij} \partial_j H^{kl} \partial_k \tau_{il} \\ & + \tfrac{1}{4} H^{ij} \partial_i H_{kl} \partial_j H^{kl} - \tfrac{1}{2} H^{ij} \partial_j H^{kl} \partial_k H_{il} - \tfrac{1}{4} H^{ij} \partial_i \ln \Omega^2 \partial_j \ln \Omega^2 \\ & - \tfrac{1}{2} H^{ij} \partial_j \tau^{kl} \partial_k \tau_{il} - \partial_i \ln \Omega^2 \partial_j H^{ij} - \partial_i \partial_j H^{ij} - H^{ij} \partial_i \partial_j \ln \Omega^2\,. \end{aligned} \tag{C.12}$$

The measure factor is $\hat{\Omega} = \Omega\sqrt{|G|}$. To obtain an Einstein frame action, we let

$$G_{\mu\nu} = \Omega^{-\frac{2}{9-d}} g_{\mu\nu}\,. \tag{C.13}$$

Gauge fields The compact expressions (4.136) and (4.137) are equivalent to

$$\begin{aligned} \boldsymbol{C}_{\hat{\mu}\hat{\nu}\hat{\rho}} &= (U^{-1})^{\hat{\lambda}_1}{}_{\hat{\mu}} (U^{-1})^{\hat{\lambda}_2}{}_{\hat{\nu}} (U^{-1})^{\hat{\lambda}_3}{}_{\hat{\rho}} C_{\hat{\lambda}_1 \dots \hat{\lambda}_3}\,, \\ \boldsymbol{F}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} &= (U^{-1})^{\hat{\lambda}_1}{}_{\hat{\mu}} (U^{-1})^{\hat{\lambda}_2}{}_{\hat{\nu}} (U^{-1})^{\hat{\lambda}_3}{}_{\hat{\rho}} (U^{-1})^{\hat{\lambda}_4}{}_{\hat{\sigma}} F_{\hat{\lambda}_1 \dots \hat{\lambda}_4}\,, \end{aligned} \tag{C.14}$$

giving in components

$$\begin{aligned}
&\boldsymbol{C}_{ijk} \equiv C_{ijk}\,, \quad \boldsymbol{C}_{\mu ij} \equiv C_{\mu ij} - A_\mu{}^k C_{ijk}\,,\\
&\boldsymbol{C}_{\mu\nu i} \equiv C_{\mu\nu i} - 2A_{[\mu}{}^j C_{\nu]ij} + A_\mu{}^j A_\nu{}^k C_{ijk}\,,\\
&\boldsymbol{C}_{\mu\nu\rho} \equiv C_{\mu\nu\rho} - 3A_{[\mu}{}^i C_{\nu\rho]i} + 3A_{[\mu}{}^i A_\nu{}^j C_{\rho]ij} - A_\mu{}^i A_\nu{}^j A_\rho{}^k C_{ijk}\,,
\end{aligned} \tag{C.15}$$

$$\begin{aligned}
&\boldsymbol{F}_{mnpq} = 4\partial_{[m}\boldsymbol{C}_{npq]}\,, \qquad \boldsymbol{F}_{\mu mnp} = D_\mu \boldsymbol{C}_{mnp} - 3\partial_{[m}\boldsymbol{C}_{|\mu|np]}\\
&\boldsymbol{F}_{\mu\nu mn} = 2D_{[\mu}\boldsymbol{C}_{\nu]mn} + \boldsymbol{F}_{\mu\nu}{}^p \boldsymbol{C}_{pmn} + 2\partial_{[m}\boldsymbol{C}_{|\mu\nu|n]}\,,\\
&\boldsymbol{F}_{\mu\nu\rho m} = 3D_{[\mu}\boldsymbol{C}_{\nu\rho]m} + 3\boldsymbol{F}_{[\mu\nu}{}^n \boldsymbol{C}_{\rho]mn} - \partial_m \boldsymbol{C}_{\mu\nu\rho}\,,\\
&\boldsymbol{F}_{\mu\nu\rho\sigma} = 4D_{[\mu}\boldsymbol{C}_{\nu\rho\sigma]} + 6\boldsymbol{F}_{[\mu\nu}{}^m \boldsymbol{C}_{\rho\sigma]m}\,,
\end{aligned} \tag{C.16}$$

where $F_{\mu\nu}{}^i$ is as defined in (4.135). The original Bianchi identity $dF_4 = 0$ becomes a set of equations

$$\begin{aligned}
D_\mu \boldsymbol{F}_{mnpq} &= 4\partial_{[m}\boldsymbol{F}_{npq]}\,,\\
2D_{[\mu}\boldsymbol{F}_{\nu]mnp} &= -3\partial_{[m|}\boldsymbol{F}_{\mu\nu|np]} - F_{\mu\nu}{}^q \boldsymbol{F}_{qmnp}\,,\\
3D_{[\mu}\boldsymbol{F}_{\nu\rho]mn} &= 2\partial_{[m|}\boldsymbol{F}_{\mu\nu\rho|n]} + 3F_{[\mu\nu}{}^p \boldsymbol{F}_{\rho]pmn}\,,\\
4D_{[\mu}\boldsymbol{F}_{\nu\rho\sigma]m} &= -\partial_m \boldsymbol{F}_{\mu\nu\rho\sigma} + 6F_{[\mu\nu}{}^p \boldsymbol{F}_{\rho\sigma]mp}\,,\\
5D_{[\mu}\boldsymbol{F}_{\nu\rho\sigma\lambda]} &= 10F_{[\mu\nu}{}^m \boldsymbol{F}_{\rho\sigma\lambda]m}\,.
\end{aligned} \tag{C.17}$$

The above formulae are applicable to any dimensional reduction. In particular for the 11-dimensional MNC theory they allow us to easily decompose the terms in the action (4.34). For example, using the Einstein frame metric to raise indices, the kinetic terms for the field strength are:

$$\begin{aligned}
&-\tfrac{1}{12}\hat{H}^{\hat{\mu}_1\hat{\nu}_1}\hat{H}^{\hat{\mu}_2\hat{\nu}_2}\hat{H}^{\hat{\mu}_3\hat{\nu}_3}\hat{\tau}^{\hat{\mu}_4\hat{\nu}_4}F_{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3\hat{\mu}_4}F_{\hat{\nu}_1\hat{\nu}_2\hat{\nu}_3\hat{\nu}_4}\\
&\quad = -\tfrac{1}{12}\Omega^{6/(9-d)}\tau^{ij}\boldsymbol{F}^{\mu\nu\rho}{}_i \boldsymbol{F}_{\mu\nu\rho j} - \tfrac{1}{4}\Omega^{4/(9-d)}H^{ij}\tau^{kl}\boldsymbol{F}_{\mu\nu ik}\boldsymbol{F}^{\mu\nu}{}_{jl}\\
&\qquad - \tfrac{1}{4}\Omega^{2/(9-d)}H^{ij}H^{kl}\tau^{pq}\boldsymbol{F}_{\mu ikp}\boldsymbol{F}^{\nu}{}_{jlq} - \tfrac{1}{4}H^{ij}H^{kl}H^{mn}\tau^{pq}\boldsymbol{F}_{ikmp}\boldsymbol{F}_{jlnq}\,.
\end{aligned} \tag{C.18}$$

Similar manipulations apply to the rest of the action. Let us also indicate how the factorisation applies to an equation of the form $\partial_{\hat{\sigma}} X^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} = \Theta^{\hat{\mu}\hat{\nu}\hat{\rho}}$ where X has weight 1, and both X and Θ admit a factorisation via U^{-1} in terms of quantities $\bar{X}$ and $\bar{\Theta}$ independent of bare $A_\mu{}^i$. This is of course the form of the gauge field equation of motion (4.28). After decomposing, one has the simple expression

$$D_\sigma \bar{X}^{\hat{\mu}\hat{\nu}\hat{\rho}\sigma} + \partial_l \bar{X}^{\hat{\mu}\hat{\nu}\hat{\rho}l} + \tfrac{3}{2}F_{\kappa\lambda}{}^l \delta_l^{[\hat{\mu}}\bar{X}^{\hat{\nu}\hat{\rho}]\kappa\lambda} = \bar{\Theta}^{\hat{\mu}\hat{\nu}\hat{\rho}}\,. \tag{C.19}$$

Constraint The constraint (4.33) decomposes in terms of the redefined strengths:

$$\begin{aligned}
\Omega^{\frac{6}{9-d}} g^{\mu_1\nu_1}\dots g^{\mu_4\nu_4}\boldsymbol{F}_{\nu_1\dots\nu_4} &= -\frac{\epsilon^{\mu_1\dots\mu_4\hat{\nu}_1\dots\hat{\nu}_4 ijk}\epsilon_{ABC}}{4!\cdot 6\sqrt{g}}\tau_i{}^A\tau_j{}^B\tau_k{}^C\boldsymbol{F}_{\hat{\nu}_1\dots\hat{\nu}_4}\,,\\
\Omega^{\frac{4}{9-d}} g^{\mu_1\nu_1}\dots g^{\mu_3\nu_3}H^{ij}\boldsymbol{F}_{\nu_1\nu_2\nu_3 j} &= -\frac{\epsilon^{\mu_1\dots\mu_3 i\hat{\nu}_1\dots\hat{\nu}_4 pqr}\epsilon_{ABC}}{4!\cdot 6\sqrt{g}}\tau_p{}^A\tau_q{}^B\tau_r{}^C\boldsymbol{F}_{\hat{\nu}_1\dots\hat{\nu}_4}\,,\\
\Omega^{\frac{2}{9-d}} g^{\mu_1\nu_1}g^{\mu_2\nu_2}H^{i_1j_1}H^{i_2j_2}\boldsymbol{F}_{\nu_1\nu_2 j_1 j_2} &= -\frac{\epsilon^{\mu_1\mu_2 i_1 i_2\hat{\nu}_1\dots\hat{\nu}_4 pqr}\epsilon_{ABC}}{4!\cdot 6\sqrt{g}}\tau_p{}^A\tau_q{}^B\tau_r{}^C\boldsymbol{F}_{\hat{\nu}_1\dots\hat{\nu}_4}\,,\\
g^{\mu_1\nu_1}H^{i_1j_1}\dots H^{i_3j_3}\boldsymbol{F}_{\nu j_1 j_2 j_3} &= -\frac{\epsilon^{\mu i_1\dots i_3\hat{\nu}_1\dots\hat{\nu}_4 pqr}.\epsilon_{ABC}}{4!\cdot 6\sqrt{g}}\tau_p{}^A\tau_q{}^B\tau_r{}^C\boldsymbol{F}_{\hat{\nu}_1\dots\hat{\nu}_4}\,,\\
\Omega^{-\frac{2}{9-d}} H^{i_1j_1}\dots H^{i_4j_4}\boldsymbol{F}_{j_1j_2j_3j_4} &= -\frac{\epsilon^{i_1\dots i_4\hat{\nu}_1\dots\hat{\nu}_4 pqr}\epsilon_{ABC}}{4!\cdot 6\sqrt{g}}\tau_p{}^A\tau_q{}^B\tau_r{}^C\hat{\boldsymbol{F}}_{\hat{\nu}_1\dots\hat{\nu}_4}\,.
\end{aligned} \tag{C.20}$$

For instance, when $d=3$ only the first of these is non-zero, giving:

$$\begin{aligned}
\sqrt{g}\Omega g^{\mu_1\nu_1}\dots g^{\mu_4\nu_4}\boldsymbol{F}_{\nu_1\dots\nu_4} &= -\tfrac{1}{4!}\epsilon^{\mu_1\dots\mu_4\nu_1\dots\nu_4 ijk}\tfrac{1}{6}\epsilon_{ABC}\tau_i{}^A\tau_j{}^B\tau_k{}^C\boldsymbol{F}_{\nu_1\dots\nu_4}\,,\\
&= -\tfrac{1}{4!}\epsilon^{\mu_1\dots\mu_4\nu_1\dots\nu_4}\Omega\boldsymbol{F}_{\nu_1\dots\nu_4}\,.
\end{aligned} \tag{C.21}$$

When $d=4$ only the first two are non-zero:

$$\begin{aligned}
\sqrt{g}\Omega^{\frac{6}{5}} g^{\mu_1\nu_1}\dots g^{\mu_4\nu_4}\boldsymbol{F}_{\nu_1\dots\nu_4} &= -\tfrac{1}{3!}\epsilon^{\mu_1\dots\mu_4\nu_1\dots\nu_3 lijk}\tfrac{1}{6}\epsilon_{ABC}\tau_i{}^A\tau_j{}^B\tau_k{}^C\boldsymbol{F}_{\nu_1\nu_2\nu_3 l}\,,\\
\sqrt{g}\Omega^{\frac{4}{5}} g^{\mu_1\nu_1}\dots g^{\mu_3\nu_3}H^{ij}\boldsymbol{F}_{\nu_1\nu_2\nu_3 j} &= -\tfrac{1}{4!}\epsilon^{\mu_1\dots\mu_3 i\nu_1\dots\nu_4 pqr}\tfrac{1}{6}\epsilon_{ABC}\tau_p{}^A\tau_q{}^B\tau_r{}^C\boldsymbol{F}_{\nu_1\dots\nu_4}\,,
\end{aligned} \tag{C.22}$$

or if we take $\frac{1}{6}\epsilon^{ijkl}\epsilon_{ABC}\tau_i{}^A\tau_j{}^B\tau_k{}^C h_l = \Omega$ these are

$$\begin{aligned}
\sqrt{g}\Omega^{\frac{1}{5}} g^{\mu_1\nu_1}\dots g^{\mu_4\nu_4}\boldsymbol{F}_{\nu_1\dots\nu_4} &= \tfrac{1}{3!}\eta\epsilon^{\mu_1\dots\mu_4\nu_1\dots\nu_3}h^l\boldsymbol{F}_{\nu_1\nu_2\nu_3 l}\,,\\
\sqrt{g}g^{\mu_1\nu_1}\dots g^{\mu_3\nu_3}H^{ij}\boldsymbol{F}_{\nu_1\nu_2\nu_3 j} &= \tfrac{1}{4!}\eta\epsilon^{\mu_1\dots\mu_3\nu_1\dots\nu_4}h^i\Omega^{\frac{1}{5}}\boldsymbol{F}_{\nu_1\dots\nu_4}\,.
\end{aligned} \tag{C.23}$$

Here $H^{ij} = h^i h^j$ (as it has rank 1), and so both of these are equivalent.

Result Putting everything together, the dimensional decomposition of the finite action $S^{(0)}$ is:

$$S^{(0)} = \int \mathrm{d}^{11-d}x\,\mathrm{d}^d y\sqrt{g}(R_{\text{ext}}(g) + \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{int}} + \mathcal{L}_{\widetilde{F}} + \sqrt{g}^{-1}\mathcal{L}_{\text{CS}})\,. \tag{C.24}$$

Here, using $g^{\mu\nu}$ to raise $(11-d)$-dimensional indices, we have

$$\begin{aligned} R_{\text{ext}}(g) =& \frac{1}{4} g^{\mu\nu} D_\mu g_{\rho\sigma} D_\nu g^{\rho\sigma} - \frac{1}{2} g^{\mu\nu} D_\mu g^{\rho\sigma} D_\rho g_{\nu\sigma} \\ & + \frac{1}{4} g^{\mu\nu} D_\mu \ln g D_\nu \ln g + \frac{1}{2} D_\mu \ln g D_\nu g^{\mu\nu} , \end{aligned} \tag{C.25}$$

$$\begin{aligned} \mathcal{L}_{\text{kin}} =& \tfrac{1}{4}(D_\mu H^{ij} D^\mu H_{ij} + D_\mu \tau^{ij} D^\mu \tau_{ij} - \tfrac{1}{9-d} D_\mu \ln \Omega^2 D^\mu \ln \Omega^2) \\ & + \tfrac{1}{2} H^{ij} \boldsymbol{F}_{\mu ikl} \epsilon_{ABC} D^\mu \tau_j{}^A \tau^{kB} \tau^{lC} - \tfrac{1}{4} H^{ij} H^{kl} \tau^{pq} \boldsymbol{F}_{\mu ikp} \boldsymbol{F}^\mu{}_{jlq} \\ & + \tfrac{1}{4} \Omega^{\frac{2}{9-d}} \big(- F_{\mu\nu}{}^i F^{\mu\nu j} H_{ij} + \boldsymbol{F}_{\mu\nu kl} F^{\mu\nu m} \epsilon_{ABC} \tau_m^A \tau^{kB} \tau^{lC} - H^{ij} \tau^{kl} \boldsymbol{F}_{\mu\nu ik} \boldsymbol{F}^{\mu\nu}{}_{jl} \big) \\ & - \tfrac{1}{12} \Omega^{\frac{4}{9-d}} \tau^{ij} \boldsymbol{F}_{\mu\nu\rho i} \boldsymbol{F}^{\mu\nu\rho}{}_j + \tfrac{1}{2} D_\mu \tau_k{}^A \tau_A{}^k D^\mu \tau_l{}^B \tau_B{}^l \end{aligned} \tag{C.26}$$

and

$$\begin{aligned} \Omega^{\frac{2}{9-d}} \mathcal{L}_{\text{int}} =& \tfrac{1}{4} H^{ij} \left(\partial_i g^{\mu\nu} \partial_j g_{\mu\nu} + \partial_i \ln g \partial_j \ln g \right) + \tfrac{1}{2} \Omega^{\frac{2}{9-d}} \partial_i (H^{ij} \Omega^{-\frac{2}{9-d}}) \partial_j \ln g \\ & + \tfrac{1}{4} H^{ij} \partial_i \tau_{kl} \partial_j \tau^{kl} + \tfrac{1}{4} \tau^{ij} \partial_i \tau_{kl} \partial_j H^{kl} - \tfrac{1}{2} \tau^{ij} \partial_j H^{kl} \partial_k \tau_{il} \\ & - \tfrac{1}{2} H^{ij} \partial_j \tau^{kl} \partial_k \tau_{il} + \tfrac{1}{4} H^{ij} \partial_i H_{kl} \partial_j H^{kl} - \tfrac{1}{2} H^{ij} \partial_j H^{kl} \partial_k H_{il} \\ & + \tfrac{1}{4} \tfrac{d-7}{(9-d)^2} H^{ij} \partial_i \ln \Omega^2 \, \partial_j \ln \Omega^2 - \tfrac{1}{9-d} \partial_i \ln \Omega^2 \, \partial_j H^{ij} \\ & - \tfrac{1}{4} H^{ij} H^{kl} H^{mn} \tau^{pq} \boldsymbol{F}_{ikmp} \boldsymbol{F}_{jlnq} + \tfrac{1}{4} H^{im} H^{jn} \boldsymbol{F}_{ijkl} \epsilon_{ABC} T_{mn}{}^A \tau^{kB} \tau^{lC} \\ & + \tfrac{1}{2} H^{ij} T_{ik}{}^A \tau^k{}_A T_{jl}{}^B \tau^l{}_B \, . \end{aligned} \tag{C.27}$$

The term $\mathcal{L}_{\widetilde{F}}$ consists of a sum of contractions of $\widetilde{\boldsymbol{F}}_{\mu\nu\rho\sigma}$, $\widetilde{\boldsymbol{F}}_{\mu\nu\rho i}$, etc. (following analogous redefinition of the components) with the constraints as decomposed in (C.20). For instance, when $d=3$,

$$\mathcal{L}_{\widetilde{F}} = -\tfrac{1}{4!} \widetilde{\boldsymbol{F}}_{\mu_1\dots\mu_4} (\sqrt{g} \Omega g^{\mu_1\nu_1} \dots g^{\mu_4\nu_4} \boldsymbol{F}_{\nu_1\dots\nu_4} + \tfrac{1}{4!} \epsilon^{\mu_1\dots\mu_4\nu_1\dots\nu_4} \Omega \boldsymbol{F}_{\nu_1\dots\nu_4}) \, , \tag{C.28}$$

In this case the relationship between the dual seven-form field strength and $\widetilde{\boldsymbol{F}}_{\mu\nu\rho\sigma}$ gives

$$\tfrac{1}{6} \epsilon^{ijk} \boldsymbol{F}_{\mu_1\dots\mu_4 ijk} = \Omega (\widetilde{\boldsymbol{F}}_{\mu_1\dots\mu_4} + \tfrac{1}{4!} \sqrt{g} \epsilon_{\mu_1\dots\mu_4\nu_1\dots\nu_4} \widetilde{\boldsymbol{F}}^{\nu_1\dots\nu_4}) \, . \tag{C.29}$$

When $d=4$,

$$\begin{aligned} \mathcal{L}_{\widetilde{F}} =& -\tfrac{1}{3!} \left(\widetilde{\boldsymbol{F}}_{\mu_1\mu_2\mu_3 i} h^i - \Omega^{1/5} \epsilon^{\lambda_1\dots\lambda_4\sigma_1\dots\sigma_3} \tfrac{1}{4!} \tfrac{1}{\sqrt{g}} g_{\sigma_1\mu_1} \dots g_{\sigma_3\mu_3} \widetilde{\boldsymbol{F}}_{\lambda_1\dots\lambda_4} \right) \times \\ & \times \left(\sqrt{g} \Omega^{\frac{4}{5}} g^{\mu_1\nu_1} \dots g^{\mu_3\nu_3} h^j \boldsymbol{F}_{\nu_1\nu_2\nu_3 j} - \Omega \tfrac{1}{4!} \epsilon^{\mu_1\dots\mu_3\nu_1\dots\nu_4} \boldsymbol{F}_{\nu_1\dots\nu_4} \right) , \end{aligned} \tag{C.30}$$

Using (4.54) we can rewrite (C.30) in terms of the dual seven-form field strength directly as

$$\mathcal{L}_{\widetilde{F}} = \frac{\epsilon^{ijkl}}{3!4!}\boldsymbol{F}_{\mu_1\ldots\mu_3 ijkl}\left(\sqrt{g}\Omega^{-\frac{1}{5}}g^{\mu_1\nu_1}\ldots g^{\mu_3\nu_3}h^j\boldsymbol{F}_{\nu_1\nu_2\nu_3 j} - \frac{\epsilon^{\mu_1\ldots\mu_3\nu_1\ldots\nu_4}}{4!}\boldsymbol{F}_{\nu_1\ldots\nu_4}\right). \tag{C.31}$$

Finally, the Chern-Simons term can be worked out by taking wedge products of (4.137) and (4.136), we do not display this explicitly.

C.2 The SL(5) ExFT and its non-relativistic parametrisation

In the $d = 4$ case, more of the degenerate Newton-Cartan structure is preserved.

Elements of SL(5) **ExFT** For $d = 4$, generalised vectors $V^M = (V^i, V_{ij})$ transform in the **10** of SL(5), with $i, j, \ldots$ now four-dimensional. This representation is the antisymmetric representation, and we can see this more clearly as follows. Let $\mathcal{M}, \mathcal{N}, \ldots$ denote fundamental five-dimensional indices of SL(5). Then we can equivalently write a generalised vector as carrying an antisymmetric pair of such indices, $V^M \equiv V^{\mathcal{MN}} = -V^{\mathcal{NM}}$, and on writing $\mathcal{M} = (i, 5)$ we can identify $V^{i5} \equiv V^i$, and $V^{ij} \equiv \frac{1}{2}\epsilon^{ijkl}V_{kl}$. The generalised Lie derivative acting on vectors of weight λ_V is explicitly

$$\mathcal{L}_\Lambda V^{\mathcal{MN}} = \tfrac{1}{2}\Lambda^{\mathcal{PQ}}\partial_{\mathcal{PQ}}V^{\mathcal{MN}} + 2\partial_{\mathcal{PQ}}\Lambda^{\mathcal{P}[\mathcal{M}}V^{\mathcal{N}]\mathcal{Q}} + \tfrac{1}{2}(1 + \lambda_V + \omega)\partial_{\mathcal{PQ}}\Lambda^{\mathcal{PQ}}V^{\mathcal{MN}}. \tag{C.32}$$

The section condition is $\epsilon^{\mathcal{MNPQK}}\partial_{\mathcal{MN}}\partial_{\mathcal{PQ}} = 0$, and below we work with the M-theory solution, where splitting $\mathcal{M} = (i, 5)$ the derivatives ∂_{ij} are viewed as identically zero, and the derivatives ∂_{i5} are identified with the 4-dimensional partial derivatives.

In this case, the generalised metric admits a factorisation

$$\mathcal{M}_{\mathcal{MN},\mathcal{PQ}} = -(m_{\mathcal{MP}}m_{\mathcal{QN}} - m_{\mathcal{MQ}}m_{\mathcal{PN}}), \tag{C.33}$$

where the 'little metric' $m_{\mathcal{MN}}$ is symmetric and has unit determinant. The overall sign in this expression needed for the ExFT action to reproduce SUGRA correctly when we include timelike signatures in the generalised metric, according to the conventions of [76].

The gauge fields, $\mathcal{A}_\mu{}^M$, $\mathcal{B}_{\mu\nu\mathcal{M}}$, $\mathcal{C}_{\mu\nu\rho}{}^{\mathcal{M}}$ and $\mathcal{D}_{\mu\nu\rho\sigma M}$ have weights 1/5, 2/5, 3/5 and 4/5 respectively, with field strengths denoted $\mathcal{F}_{\mu\nu}{}^M$, $\mathcal{H}_{\mu\nu\rho\mathcal{M}}$, $\mathcal{J}_{\mu\nu\rho\sigma}{}^{\mathcal{N}}$ and $\mathcal{K}_{\mu\nu\rho\sigma\lambda M}$. Under generalised diffeomorphisms, $\mathcal{F}^M$ transforms as a generalised vector of weight 1/5, while $\mathcal{H}$ and $\mathcal{J}$ transform via the generalised Lie derivative acting as

$$\begin{aligned} \mathcal{L}_\Lambda \mathcal{H}_{\mathcal{M}} &= \tfrac{1}{2}\Lambda^{\mathcal{P}\mathcal{Q}}\partial_{\mathcal{P}\mathcal{Q}}\mathcal{H}_{\mathcal{M}} + \mathcal{H}_{\mathcal{P}}\partial_{\mathcal{M}\mathcal{Q}}\Lambda^{\mathcal{P}\mathcal{Q}}\,,\\ \mathcal{L}_\Lambda \mathcal{J}^{\mathcal{M}} &= \partial_{\mathcal{P}\mathcal{Q}}(\tfrac{1}{2}\Lambda^{\mathcal{P}\mathcal{Q}}\mathcal{J}^{\mathcal{M}}) - \partial_{\mathcal{P}\mathcal{Q}}\Lambda^{\mathcal{P}\mathcal{M}}\mathcal{J}^{\mathcal{Q}}\,. \end{aligned} \tag{C.34}$$

They obey Bianchi identities:

$$3\mathcal{D}_{[\mu}\mathcal{F}_{\nu\rho]}{}^{\mathcal{M}\mathcal{N}} = \tfrac{1}{2}\epsilon^{\mathcal{M}\mathcal{N}\mathcal{P}\mathcal{Q}\mathcal{K}}\partial_{\mathcal{P}\mathcal{Q}}\mathcal{H}_{\mu\nu\rho\mathcal{K}}\,, \tag{C.35}$$

$$4\mathcal{D}_{[\mu}\mathcal{H}_{\nu\rho\sigma]\mathcal{M}} + \tfrac{3}{4}\epsilon_{\mathcal{M}\mathcal{N}\mathcal{P}\mathcal{K}\mathcal{L}}\mathcal{F}_{[\mu\nu}{}^{\mathcal{N}\mathcal{P}}\mathcal{F}_{\rho\sigma]}{}^{\mathcal{K}\mathcal{L}} = \partial_{\mathcal{N}\mathcal{M}}\mathcal{J}_{\mu\nu\rho\sigma}{}^{\mathcal{N}}\,, \tag{C.36}$$

$$5\mathcal{D}_{[\mu}\mathcal{J}_{\nu\rho\sigma\lambda]}{}^{\mathcal{M}} + 10\mathcal{F}_{[\mu\nu}{}^{\mathcal{M}\mathcal{N}}\mathcal{H}_{\rho\sigma\lambda]\mathcal{N}} = \tfrac{1}{2}\epsilon^{\mathcal{M}\mathcal{N}\mathcal{P}\mathcal{Q}\mathcal{K}}\partial_{\mathcal{N}\mathcal{P}}\mathcal{K}_{\mu\nu\rho\sigma\lambda\mathcal{Q}\mathcal{K}}\,. \tag{C.37}$$

The dynamics follow from the variation of an action $S = \int \mathrm{d}^7x\,\mathrm{d}^{10}y\mathcal{L}_{\text{ExFT}}$ where $\mathcal{L}_{\text{ExFT}}$ has the same form as (4.148), with R_{ext} again as defined in (4.149), and [117]

$$\mathcal{L}_{\text{kin}} = \tfrac{1}{12}\mathcal{D}_\mu\mathcal{M}_{MN}\mathcal{D}^\mu\mathcal{M}^{MN} - \tfrac{1}{4}\mathcal{M}_{MN}\mathcal{F}_{\mu\nu}{}^M\mathcal{F}^{\mu\nu N} - \tfrac{1}{12}m^{\mathcal{M}\mathcal{N}}\mathcal{H}_{\mu\nu\rho\mathcal{M}}\mathcal{H}^{\mu\nu\rho}{}_{\mathcal{N}}\,, \tag{C.38}$$

$$\begin{aligned} \mathcal{L}_{\text{int}}(m,g) = {} & \tfrac{1}{12}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_N\mathcal{M}_{KL} - \tfrac{1}{2}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_K\mathcal{M}_{LN}\\ & + \tfrac{1}{2}\partial_M\mathcal{M}^{MN}\partial_N\ln|g| + \tfrac{1}{4}\mathcal{M}^{MN}\left(\partial_M g_{\mu\nu}\partial_N g^{\mu\nu} + \partial_M\ln|g|\partial_N\ln|g|\right)\,. \end{aligned} \tag{C.39}$$

The topological term can be defined via its variation (again up to a choice of sign equivalent to changing the sign of $\hat{C}_3$ in eleven-dimensional SUGRA):

$$\begin{aligned} \delta\mathcal{L}_{\text{top}} = \tfrac{-\epsilon^{\mu_1\cdots\mu_7}}{6\cdot 4!}\Big(& 2\delta\mathcal{A}_{\mu_1}{}^{\mathcal{M}\mathcal{N}}\mathcal{H}_{\mu_2\mu_3\mu_4\mathcal{M}}\mathcal{H}_{\mu_5\mu_6\mu_7\mathcal{N}} + 6\mathcal{F}_{\mu_1\mu_2}{}^{\mathcal{M}\mathcal{N}}\Delta\mathcal{B}_{\mu_3\mu_4\mathcal{M}}\mathcal{H}_{\mu_5\mu_6\mu_7\mathcal{N}}\\ & \partial_{\mathcal{N}\mathcal{M}}\Delta\mathcal{C}_{\mu_1\mu_2\mu_3}{}^{\mathcal{N}}\mathcal{J}_{\mu_4\ldots\mu_7}{}^{\mathcal{M}}\Big)\,. \end{aligned} \tag{C.40}$$

We refer to the original paper [117] or the review [118] for explicit details.

Review of 11-dimensional SUGRA embedding We start with the little metric, $m_{\mathcal{M}\mathcal{N}}$. The parametrisation reproducing (4.138) is

$$m_{\mathcal{M}\mathcal{N}} = |\phi|^{1/10} \begin{pmatrix} |\phi|^{-1/2}\phi_{ij} & -|\phi|^{-1/2}\phi_{ik}\hat{\boldsymbol{C}}^k \\ -|\phi|^{-1/2}\phi_{jk}\hat{\boldsymbol{C}}^k & |\phi|^{1/2}(-1)^t + |\phi|^{-1/2}\phi_{kl}\hat{\boldsymbol{C}}^k\hat{\boldsymbol{C}}^l \end{pmatrix} , \qquad \hat{\boldsymbol{C}}^i \equiv \tfrac{1}{3!}\epsilon^{ijkl}\hat{\boldsymbol{C}}_{jkl} . \tag{C.41}$$

For the gauge fields, we can again identify $\mathcal{A}_\mu^M = (A_\mu{}^i, \hat{\boldsymbol{C}}_{\mu ij})$. However, we already require dualisations when treating the two-forms. We get four 7-dimensional two-forms, $\hat{\boldsymbol{C}}_{\mu\nu i}$ and a single three-form $\hat{\boldsymbol{C}}_{\mu\nu\rho}$. The latter can be dualised into an extra two-form, $\widetilde{C}_{\mu\nu}$ (identifiable with the components $\hat{\boldsymbol{C}}_{\mu\nu ijkl}$ of the six-form in eleven-dimensions) such that $\mathcal{B}_{\mu\nu\mathcal{M}} \sim (\hat{\boldsymbol{C}}_{\mu\nu i}, \widetilde{C}_{\mu\nu})$ gives a five-dimensional representation of SL(5). Meanwhile, we can view $\hat{\boldsymbol{C}}_{\mu\nu\rho}$ together with the four four-forms $\hat{\boldsymbol{C}}_{\mu\nu\rho ijk}$ as comprising the conjugate five-dimensional representation. The equations of motion of the SL(5) ExFT then imply that the field strengths of these two- and three-forms are related by duality. This involves the seven-dimensional Hodge star acting on the seven-dimensional indices and the generalised metric acting on the SL(5) indices:

$$\sqrt{|g|}m^{\mathcal{M}\mathcal{P}}\mathcal{H}^{\mu\nu\rho}{}_{\mathcal{P}} = -\tfrac{1}{4!}\epsilon^{\mu\nu\rho\sigma_1\ldots\sigma_4}\mathcal{J}_{\sigma_1\ldots\sigma_4}{}^{\mathcal{M}} . \tag{C.42}$$

Again, the field strengths are all tensors under generalised diffeomorphisms, we may make the (usual) identifications consistent with the Bianchi identities [118]

$$\begin{aligned} \mathcal{F}_{\mu\nu}{}^{i5} &= F_{\mu\nu}{}^i , & \mathcal{F}_{\mu\nu}{}^{ij} &= \tfrac{1}{2}\epsilon^{ijkl}(\hat{\boldsymbol{F}}_{\mu\nu kl} - \hat{\boldsymbol{C}}_{klm}\hat{\boldsymbol{F}}_{\mu\nu}{}^m) , \\ \mathcal{H}_{\mu\nu\rho i} &= -\hat{\boldsymbol{F}}_{\mu\nu\rho i} , & \mathcal{H}_{\mu\nu\rho 5} &= -\tfrac{1}{4!}\epsilon^{ijkl}(\hat{\boldsymbol{F}}_{\mu\nu\rho ijkl} - 4\hat{\boldsymbol{F}}_{\mu\nu\rho i}\hat{\boldsymbol{C}}_{jkl}) , \\ \mathcal{J}_{\mu\nu\rho\sigma}{}^5 &= -\hat{\boldsymbol{F}}_{\mu\nu\rho\sigma} , & \mathcal{J}_{\mu\nu\rho\sigma}{}^i &= +\tfrac{1}{3!}\epsilon^{ijkl}(\hat{\boldsymbol{F}}_{\mu\nu\rho\sigma jkl} - \hat{\boldsymbol{C}}_{jkl}\hat{\boldsymbol{F}}_{\mu\nu\rho\sigma}) . \end{aligned} \tag{C.43}$$

Generalised metric The distinction between Riemannian and non-Riemannian parametrisations can be seen at the level of the unit-determinant five-by-five little generalised metric. For an M-theory parametrisation, this can be written as:

$$m_{\mathcal{M}\mathcal{N}} = \begin{pmatrix} m_{ij} & m_{i5} \\ m_{j5} & m_{55} \end{pmatrix} , \quad m_{55}\det(m_{ij}) - \tfrac{1}{6}m_{i5}m_{j5}\epsilon^{iklm}\epsilon^{jpqr}m_{kp}m_{lq}m_{mr} = 1 . \tag{C.44}$$

If $\det(m_{ij}) \neq 0$ this leads to the Riemannian parametrisation (C.41) encoding a four-dimensional metric, g_{ij}, and a three-form, $\hat{\boldsymbol{C}}_{ijk}$. However, we can also have $\det(m_{ij}) = 0$ with m_{ij} of rank 3 and this leads to a non-Riemannian parametrisation which was worked out in [76]. We can rediscover this parametrisation by taking the non-relativistic limit of (C.41) using (4.160). The resulting expression for $m_{\mathcal{M}\mathcal{N}}$ is

$$m_{\mathcal{M}\mathcal{N}} = \Omega^{-4/5}\times \begin{pmatrix} \tau_{ij} & \frac{\epsilon^{klmn}\epsilon_{ABC}}{6} H_{ik}\tau_l{}^A\tau_m{}^B\tau_n{}^C - \tau_{ik}\boldsymbol{C}^k \\ \frac{\epsilon^{klmn}\epsilon_{ABC}}{6} H_{jk}\tau_l{}^A\tau_m{}^B\tau_n{}^C - \tau_{jk}\boldsymbol{C}^k & \tau_{ij}\boldsymbol{C}^i\boldsymbol{C}^j - \frac{\epsilon^{jklm}\epsilon_{ABC}}{3} H_{ij}\tau_k{}^A\tau_l{}^B\tau_m{}^C\boldsymbol{C}^i \end{pmatrix}, \tag{C.45}$$

in terms of four-dimensional Newton-Cartan variables and $\boldsymbol{C}^i \equiv \frac{1}{3!}\epsilon^{ijkl}\boldsymbol{C}_{jkl}$. The unit determinant constraint implies that

$$-\tfrac{1}{3!}\epsilon^{i_1\ldots i_4}\epsilon^{j_1\ldots j_4}\tau_{i_1j_1}\tau_{i_2j_2}\tau_{i_3j_3}H_{i_4j_4} = \Omega^2\,, \tag{C.46}$$

which is the definition of Ω^2 in this case. As H_{ij} has rank 1, we can introduce a projective vielbein h_i such that $H_{ij} = h_ih_j$ and we take

$$\tfrac{1}{6}\epsilon^{ijkl}\epsilon_{ABC}\tau_i{}^A\tau_j{}^B\tau_k{}^C h_l = \Omega\,, \tag{C.47}$$

choosing to fix an arbitrary sign (by sending $\tau_i{}^A \to -\tau_i{}^A$ if necessary) which could appear here (Ω is assumed positive). Then (C.45) can be written as

$$m_{\mathcal{M}\mathcal{N}} = \Omega^{-4/5}\begin{pmatrix} \tau_{ij} & -\Omega h_i - \tau_{ik}\boldsymbol{C}^k \\ -\Omega h_j - \tau_{jk}\boldsymbol{C}^k & \tau_{ij}\boldsymbol{C}^i\boldsymbol{C}^j + 2\Omega h_i\boldsymbol{C}^i \end{pmatrix}, \tag{C.48}$$

which in this form can be checked to correspond to the parametrisation written down in [76] from first principles. Note that the boost invariance, acting as

$$\delta h_i = h^j\Lambda_j{}^A\tau_{iA}\,, \quad \delta\boldsymbol{C}^i = -\Omega\Lambda_j{}^A h^j\tau^i{}_A\,, \quad \tau^i{}_A\Lambda_i{}^B = 0\,, \tag{C.49}$$

corresponds to a shift symmetry of the parametrisation (C.48) pointed out in [76]. This generalises the Milne shift redundancy of the DFT non-Riemannian parametrisation [66]. Here we introduced the inverse vielbeins h^i and $\tau^i{}_A$ obeying the obvious relations

$$h_ih^i = 1\,, \quad \tau^i{}_A\tau_j{}^A + h^ih_j = \delta^i_j\,, \quad \tau^i{}_Ah_i = 0 \quad \tau_i{}^Ah^i = 0\,, \quad \tau^i{}_A\tau_i{}^B = \delta^B_A\,. \tag{C.50}$$

The generalised metric in the 10×10 representation followng from the little metric (C.45) can be seen to take the form (4.161), after rewriting in the basis where generalised indices run over vector and two-form indices, and using the identities

$$\begin{aligned} \epsilon^{i_1 \dots i_3 k} \epsilon^{j_1 \dots j_3 l} \tau_{kl} &= -3! \Omega^2 \big(\tau^{j_1 [i_1} \tau^{i_2 | j_2 |} H^{i_3] j_3} + \tau^{j_2 [i_1} \tau^{i_2 | j_3 |} H^{i_3] j_1} \\ &\qquad + \tau^{j_3 [i_1} \tau^{i_2 | j_1 |} H^{i_3] j_2} \big) \,, \\ \epsilon^{i_1 \dots i_3 k} \epsilon^{j_1 \dots j_3 l} H_{kl} &= -3! \Omega^2 \tau^{i_1 [j_1 |} \tau^{i_2 | j_2 |} \tau^{i_3 | j_3]} \,. \end{aligned} \tag{C.51}$$

It is useful to record the explicit expression for the inverse little metric:

$$m^{\mathcal{M}\mathcal{N}} = \Omega^{4/5} \begin{pmatrix} \tau^{ij} - 2\Omega^{-1} h^{(i} \boldsymbol{C}^{j)} & -\Omega^{-1} h^i \\ -\Omega^{-1} h^j & 0 \end{pmatrix} . \tag{C.52}$$

Clearly, variations $\delta m^{\mathcal{M}\mathcal{N}}$ with $\delta m^{55} \neq 0$ do not preserve this parametrisation. This means that if we look at the equations of motion $\mathcal{R}_{\mathcal{M}\mathcal{N}} = 0$ of the generalised metric, we expect that $\mathcal{R}_{55} = 0$ provides an additional equation of motion that we would not find by varying the action evaluated on the non-relativistic parametrisation.

Field strengths and self-duality in SL(5) **ExFT** Our field strengths (C.43) are now

$$\begin{aligned} \mathcal{F}_{\mu\nu}{}^{i5} &= F_{\mu\nu}{}^i \,, & \mathcal{F}_{\mu\nu}{}^{ij} &= \tfrac{1}{2} \epsilon^{ijkl} (\boldsymbol{F}_{\mu\nu kl} - \boldsymbol{C}_{klm} \boldsymbol{F}_{\mu\nu}{}^m) \,, \\ \mathcal{H}_{\mu\nu\rho i} &= -\boldsymbol{F}_{\mu\nu\rho i} \,, & \mathcal{H}_{\mu\nu\rho 5} &= -\tfrac{1}{4!} \epsilon^{ijkl} (\boldsymbol{F}_{\mu\nu\rho ijkl} - 4 \boldsymbol{F}_{\mu\nu\rho i} \boldsymbol{C}_{jkl}) \,, \\ \mathcal{J}_{\mu\nu\rho\sigma}{}^5 &= -\boldsymbol{F}_{\mu\nu\rho\sigma} \,, & \mathcal{J}_{\mu\nu\rho\sigma}{}^i &= +\tfrac{1}{3!} \epsilon^{ijkl} (\boldsymbol{F}_{\mu\nu\rho\sigma jkl} - \boldsymbol{C}_{jkl} \boldsymbol{F}_{\mu\nu\rho\sigma}) \,. \end{aligned} \tag{C.53}$$

The kinetic terms (C.38) in the SL(5) ExFT action are:

$$\begin{aligned} &- \tfrac{1}{4} \mathcal{M}_{MN} \mathcal{F}^{\mu\nu M} \mathcal{F}_{\mu\nu}{}^N - \tfrac{1}{12} m^{\mathcal{M}\mathcal{N}} \mathcal{H}^{\mu\nu\rho}{}_{\mathcal{M}} \mathcal{H}_{\mu\nu\rho\mathcal{N}} \\ &= -\tfrac{1}{4} \Omega^{2/5} \left(H_{ij} F^{\mu\nu i} F_{\mu\nu}{}^j - \epsilon^{ABC} \tau_{iA} \tau_B{}^j \tau_C{}^k F^{\mu\nu i} \boldsymbol{F}_{\mu\nu jk} + \tau^i_C \tau^{jC} H^{kl} \boldsymbol{F}^{\mu\nu}{}_{ik} \boldsymbol{F}_{\mu\nu jl} \right) \\ &\quad - \tfrac{1}{12} \Omega^{4/5} \tau^{ij} \boldsymbol{F}^{\mu\nu\rho}{}_i \boldsymbol{F}_{\mu\nu\rho j} + \tfrac{1}{6} \Omega^{-1/5} h^i \boldsymbol{F}^{\mu\nu\rho}{}_i \tfrac{1}{4!} \epsilon^{jklm} \boldsymbol{F}_{\mu\nu\rho jklm} \,, \end{aligned} \tag{C.54}$$

which match exactly the corresponding terms in (C.26) and (C.30), including the appearance of components of the dual seven-form field strength.

We see again that the ExFT description automatically contains the correct dual fields to reproduce the non-relativistic action immediately. It's worthwhile to go into some detail about the appearance of dual fields in the relativistic case. As mentioned above, the decomposition of the 11-dimensional three-form in the $(7+4)$-dimensional split produces four two-forms, $\hat{\boldsymbol{C}}_{\mu\nu i}$ and a single three-form, $\hat{\boldsymbol{C}}_{\mu\nu\rho}$. We exchange the latter for an additional two-form, $\hat{\boldsymbol{C}}_{\mu\nu}$, in order to obtain the five-dimensional SL(5) multiplet $\mathcal{B}_{\mu\nu\mathcal{M}} = (\hat{\boldsymbol{C}}_{\mu\nu i}, \hat{\boldsymbol{C}}_{\mu\nu})$. This is normally done by introducing the two-form into the action as a Lagrange multiplier enforcing the Bianchi identity for $\hat{\boldsymbol{F}}_{\mu\nu\rho\sigma}$. When this is done, the terms involving $\hat{\boldsymbol{F}}_4$ in the action are schematically $\hat{\boldsymbol{F}}_4 \wedge \star_7 \hat{\boldsymbol{F}}_4 - \hat{\boldsymbol{C}}_2 \wedge (d\hat{\boldsymbol{F}}_4 + \dots) + \hat{\boldsymbol{F}}_4 \wedge X_3$, where X_3 denotes whatever appears alongside $\hat{\boldsymbol{F}}_4$ in the decomposition of the Chern-Simons term. Integrating by parts one defines a field strength $H_3 \sim d\hat{\boldsymbol{C}}_2 + X_3$ and treating $\hat{\boldsymbol{F}}_4$ then as an independent field, one can integrate that out of the action to produce a kinetic term for H_3. The latter is then the $\mathcal{M}=5$ component of the ExFT field strength $\mathcal{H}_{\mu\nu\rho\mathcal{M}}$, and in this way the ExFT action matches the partially dualised SUGRA action.

In the non-relativistic theory, there is already no kinetic term for $\boldsymbol{F}_4$ in the decomposed action, as seen from (C.26). It only appears (linearly) in the constraint term (C.30), schematically in the form $\boldsymbol{F}_4 \wedge (\star_7 \widetilde{\boldsymbol{F}}_4 + \widetilde{\boldsymbol{F}}_{3i} h^i)$. So instead if we carry out the same procedure, we find that $\boldsymbol{F}_4$ equation of motion sets $H_3 = \star_7 \widetilde{\boldsymbol{F}}_4 + \widetilde{\boldsymbol{F}}_{3i} h^i$, which in this case exactly corresponds to the relationship between the dual seven-form and $\widetilde{F}_4$ as expressed by (4.54). Hence now it is this H_3 that we identify with $\mathcal{H}_{\mu\nu\rho ijkl}$ via the above arguments. All this exactly mirrors what happened for the SL(3) $\times$ SL(2) case.

We finish with a brief look at the equations of motion. The field strength $\mathcal{J}_{\mu\nu\rho\sigma}$ of the gauge field $\mathcal{C}_{\mu\nu\rho}$ only appears in the topological term. This gauge field also appears in the field strength $\mathcal{H}_{\mu\nu\rho}$. Its equation of motion has the form $\partial_{\mathcal{M}\mathcal{N}} \theta^{\mu\nu\rho\mathcal{N}} = 0$ where

$$\theta^{\mu\nu\rho\mathcal{M}} \equiv \sqrt{g} m^{\mathcal{M}\mathcal{P}} \mathcal{H}^{\mu\nu\rho}{}_{\mathcal{P}} + \tfrac{1}{4!} \epsilon^{\mu\nu\rho\sigma_1\dots\sigma_4} \mathcal{J}_{\sigma_1\dots\sigma_4}{}^{\mathcal{M}} \,. \tag{C.55}$$

Meanwhile the equation of motion of $\mathcal{B}_{\mu\nu\mathcal{M}}$ is

$$\begin{aligned} 0 = \mathcal{D}_\rho(\sqrt{g} m^{\mathcal{M}\mathcal{N}} \mathcal{H}^{\mu\nu\rho}{}_{\mathcal{N}}) + \tfrac{1}{8} \epsilon^{\mathcal{M}\mathcal{P}\mathcal{Q}\mathcal{K}\mathcal{L}} \partial_{\mathcal{P}\mathcal{Q}}(\sqrt{g} \mathcal{M}_{\mathcal{K}\mathcal{L},\mathcal{K}'\mathcal{L}'} \mathcal{F}^{\mu\nu\mathcal{K}'\mathcal{L}'}) \\ - \tfrac{2}{4!} \epsilon^{\mu\nu\lambda_1\dots\lambda_5} \mathcal{F}_{\lambda_1\lambda_2}{}^{\mathcal{M}\mathcal{N}} \mathcal{H}_{\lambda_3\dots\lambda_5\mathcal{N}} \,. \end{aligned} \tag{C.56}$$

The $\mathcal{M}=5$ component combines with the $\mathcal{M}=5$ component of the Bianchi identity (C.37) to give $\mathcal{D}_\rho \theta^{\mu\nu\rho 5} = 0$. Hence we integrate and set $\theta^{\mu\nu\rho\mathcal{M}} = 0$. Let's examine the content of this constraint. Firstly, the $\theta^{\mu\nu\rho 5}$ component implies

$$\Omega^{-1/5}\sqrt{g}h^j \boldsymbol{F}^{\mu\nu\rho}{}_j - \tfrac{1}{4!}\epsilon^{\mu\nu\rho\sigma_1\ldots\sigma_4}\boldsymbol{F}_{\sigma_1\ldots\sigma_4} = 0\,. \tag{C.57}$$

This is the 11-dimensional self-duality constraint (4.33) on the transverse part of the four-form field strength, here decomposed as in (C.22). Secondly, setting $\theta^{\mu\nu\rho i} - C^i\theta^{\mu\nu\rho 5} = 0$ and projecting gives

$$\begin{aligned}\sqrt{g}\Omega^{-1/5}\boldsymbol{F}^{\mu\nu\rho}{}_{ijkl} + \tfrac{1}{4!}\epsilon^{\mu\nu\rho\sigma_1\ldots\sigma_4}4h_{[i|}\boldsymbol{F}_{\sigma_1\ldots\sigma_4|jkl]} = 0\,,\\ \sqrt{g}\Omega^{4/5}\tau^{iA}\boldsymbol{F}^{\mu\nu\rho}{}_i - \tfrac{1}{4!}\epsilon^{\mu\nu\rho\sigma_1\ldots\sigma_4}\tau_i^A\tfrac{1}{3!}\epsilon^{ijkl}\boldsymbol{F}_{\sigma_1\ldots\sigma_4 jkl} = 0\,.\end{aligned} \tag{C.58}$$

The first of these is part of the self-duality condition (4.55) obeyed by the totally longitudinal part of the dual-seven form. The second is part of the duality between the partly longitudinal four-form and the rest of the seven-form. We see again that the ExFT rearrangement of degrees of freedom exactly captures the novel features of the eleven-dimensional non-relativistic limit.

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