Journal of Nonlinear Mathematical Physics (2026) 33:30
https://doi.org/10.1007/s44198-026-00391-z

RESEARCH

The Higher Order Yang–Mills–Higgs Flow Over Riemannian Manifold

Xian-Min Cao[1] · Mehraj Ahmad Lone[2] · Wen Wang[3] · Meng-Jie Zhang[1] · Pan Zhang[1]

Received: 10 September 2025 / Accepted: 5 February 2026

Abstract
In this paper, we introduce a novel higher-order Yang–Mills–Higgs functional $\mathcal{YMH}_k(\nabla, u)$, which is closely associated with a generalized Higgs-like potential W. First, by employing de Turck's trick, we establish the local existence of the negative gradient flow, thus refining the ellipticity argument presented in [Car. Var. PDE 58 (2019), 100]. We establish the long-time existence of the flow under conditions on the order of derivatives appearing in the functional and on the degree of the potential W. This result refines and extends the one given in [J. Aust. Math. Soc. 113 (2022), 257-287]. At last, we discuss the physical applications of our higher-order Yang–Mills–Higgs theory and its long-time flow, clarifying the rationale for the higher-order terms and the polynomial potential, rooted in specific mathematical models.

Keywords Higher order flow · Yang–Mills–Higgs functional · Higgs-like potential · Long-time existence · Riemannian manifold

Mathematics Subject Classification 53E40 · 58J35 · 81T13

1 Introduction

Yang–Mills theory admits a natural formulation within the framework of modern differential geometry. Let (M, g) be a compact Riemannian manifold and P a principal G-bundle over M, where G is a compact Lie group. A Yang–Mills field is represented by the curvature F_∇ of a metric connection ∇ on P that is a critical point of the action functional [4, 61]

Extended author information available on the last page of the article

Published online: 22 February 2026

$$\mathrm{YM}(\nabla) = \frac{1}{2}\int_M |F_\nabla|^2 d\mathrm{vol}_g. \tag{1.1}$$

When G is the circle group, the Yang–Mills equations reduce to Maxwell's equations.

Yang–Mills fields describe physical forces and serve as carriers of fundamental interactions. They couple to matter fields, which are mathematically represented by a section u of a vector bundle E associated with the principal bundle P. The action for the coupled system is given by [11]

$$\mathrm{YM}(\nabla, u) = \frac{1}{2}\int_M \Big(|F_\nabla|^2 + |\nabla u|^2 + m^2|u|^2 + \frac{1}{2}|u|^4\Big) d\mathrm{vol}_g,$$

where m denotes the mass of the particle.

In particle physics, the Higgs potential [30] (for a real scalar field) is

$$V(\phi) = \mu^2|\phi|^2 + \lambda|\phi|^2.$$

where $\mu^2 < 0$ and $\lambda > 0$. After symmetry breaking, the Higgs potential can be rewritten as

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2,$$

where $v = \sqrt{\frac{-\mu^2}{\lambda}}$ is the vacuum expectation value. Its key properties are as follows: (1) $V(\phi) \geq 0$; (2) it attains global minima at $\phi = \pm v$, satisfying $V(\pm v) = 0$ and $V'(\pm v) = 0$; (3) $V''(\phi) \geq 0$ for $\phi \geq \frac{\sqrt{3}}{3}v$ or $\phi \leq -\frac{\sqrt{3}}{3}v$.

Afuni [1, 2] and Wang [65] considered the following Yang–Mills–Higgs functional with Higgs-like potential

$$\mathcal{YMH}(\nabla, u) = \frac{1}{2}\int_M [|F_\nabla|^2 + |\nabla u|^2 + W(|u|^2)] d\mathrm{vol}_g, \tag{1.2}$$

where u is a section of $\Omega^0(E)$ and $W : \mathbb{R} \to [0, +\infty)$ is the Higgs-like potential. The potential W is called Higgs-like [1, 2] if the second derivative W'' is non-negative and the set $\{\sigma \in \mathbb{R}^+ : W(\sigma) = W'(\sigma) = 0\}$ is non-empty. The definition of the W function is derived from the properties of the above Higgs potential in particle physics, but it is defined with global convexity for the sake of simplicity.

When $W(x) = \frac{\lambda}{4}(1-x)^2$, this kind of Yang–Mills–Higgs functional was extensively studied [31, 32, 34], and this kind of functional has a strong relationship with the Ginzburg–Landau functional. For the study of Ginzburg–Landau functional (or equation), we refer to [5, 7, 13, 14, 22] and references therein.

The study of higher-order derivative flows boasts a rich history, which can be traced back to De Giorgi's program [16] aimed at approximating singular flows with smooth sequences, on the one hand, and to diverse physical contexts, on the other.

Since the late twentieth century, the analysis of higher-order equations has attracted significant attention (see [6, 26, 27, 29, 35–37, 41, 53, 54, 60, 68] and references therein). The author [68] introduced the following Yang–Mills–Higgs *k*-functional (or Yang–Mills–Higgs *k*-energy)

$$\mathrm{YMH}_k(\nabla, u) = \frac{1}{2}\int_M \left[|\nabla^{(k)} F_\nabla|^2 + |\nabla^{(k+1)} u|^2\right] d\mathrm{vol}_g, \tag{1.3}$$

where $k \in \mathbb{N} \cup \{0\}$, ∇ is a connection on E and u is a section of $\Omega^0(E)$. When $u = 0$, (1.3) is the Yang–Mills *k*-functional studied in [36]. When $k = 0$, (1.3) is nothing but the Yang–Mills–Higgs functional considered in [31]. In [68], the author proved the long-time existence of the negative gradient flow of (1.3).

In the following, we always assume the Higgs-like potential W is polynomial unless otherwise stated. We will consider the following functional

$$\mathcal{YMH}_k(\nabla, u) = \frac{1}{2}\int_M \left[|\nabla^{(k)} F_\nabla|^2 + |\nabla^{(k+1)} u|^2 + W(|u|^2)\right] d\mathrm{vol}_g, \tag{1.4}$$

where $u \in \Omega^0(E)$. The Yang–Mills–Higgs *k*-system, i.e. the corresponding Euler–Lagrange equations of (1.4), is

$$\begin{cases} (-1)^k D_\nabla^* \Delta_\nabla^{(k)} F_\nabla + \sum_{v=0}^{2k-1} P_1^{(v)}[F_\nabla] + P_2^{(2k-1)}[F_\nabla] + \sum_{i=0}^{k} \nabla^{*(i)}(\nabla^{(k+1)} u * \nabla^{(k-i)} u) = 0, \\ \nabla^{*(k+1)}\nabla^{(k+1)} u + W'(|u|^2) u = 0, \end{cases} \tag{1.5}$$

where $\Delta_\nabla^{(k)}$ denotes k iterations of the Bochner Laplacian $-\nabla^*\nabla$, and the notation P is defined in (2.2). A solution of the Yang–Mills–Higgs *k*-flow is given by a family of pairs $(\nabla(x,t), u(x,t)) := (\nabla_t, u_t)$ such that

$$\begin{cases} \frac{\partial \nabla_t}{\partial t} = (-1)^{(k+1)} D_{\nabla_t}^* \Delta_{\nabla_t}^{(k)} F_{\nabla_t} + \sum_{v=0}^{2k-1} P_1^{(v)}[F_{\nabla_t}] + P_2^{(2k-1)}[F_{\nabla_t}] + \sum_{i=0}^{k} \nabla_t^{*(i)}(\nabla_t^{(k+1)} u_t * \nabla_t^{(k-i)} u_t), \\ \frac{\partial u_t}{\partial t} = -\nabla_t^{*(k+1)}\nabla_t^{(k+1)} u_t - W'(|u_t|^2) u_t, \quad in \quad M \times [0, T). \end{cases} \tag{1.6}$$

We can prove the following theorem.

Theorem 1 *Let E be a vector bundle defined over a closed n-dimensional Riemannian manifold (M, g). Assuming the Higgs-like potential W is a polynomial with* $\deg W \leq k + 2$, *then for any smooth initial data* (∇_0, u_0), *a unique smooth solution* (∇_t, u_t) *to the Yang–Mills–Higgs k-flow (1.6) exists on* $M \times [0, +\infty)$, *provided* $n < 2(k + 1)$.

The proof of the above theorem relies on de Turck's trick, local L^2-derivative estimates, energy estimates and blow-up analysis, some procedures should be modifyed carefully:

♠ When employing de Turck's trick to establish the local existence of the Yang–Mills–Higgs *k*-flow defined in (1.6), our ellipticity argument provides a refinement of those presented in [36, Lemma 3.2] and [68, Theorem 3.2]. In [36, 68], it is claimed that $\langle \mathcal{L}^{\xi}\Phi_k(\cdot), \cdot\rangle$ is either strictly positive definite or negative definite depending on the parity of *k*, from which the ellipticity of Φ_k is deduced. However, we are unable to establish that $\langle \mathcal{L}^{\xi}\Phi_k(\cdot), \cdot\rangle$ satisfies such a definiteness property. We can still conclude that Φ_k is an elliptic operator via alternative reasoning (see Theorem 2).

♡ In [54, 68], one can easily obtain the uniformly bound of $\|u_t\|_{L^2}$ by $\frac{\partial}{\partial t}\|u_t\|_{L^2}$. To our best knowledge, this is not valid in our case, since we can not assume the derivative $W' \geq 0$. The reason is that the condition $W' \geq 0$ will contradict the Higgs-like assumption, i.e. the second derivative $W'' \geq 0$ and the set $\{\sigma \in \mathbb{R}^+ : W(\sigma) = W'(\sigma) = 0\}$ is non-empty. We notice that, once we assume that *W* is polynomial, the uniform bound of $\|u_t\|_{L^2}$ can be achieved by $\frac{\partial}{\partial t}\mathcal{YMH}_k(\nabla_t, u_t)$.

♢ We have a new observation. The condition $\deg W \leq k+2$ is essential in this case. This is influenced in the blow-up analysis, the scaled equation $\frac{\partial u_t^{\rho}}{\partial t}$ in (4.1) will tends to infinity during the scaling when $\deg W > k+2$. In [68, Section 9], the author consider the following Yang–Mills–Higgs *k*-functional with Higgs self-interaction:

$$YMH_k(\nabla, u) = \frac{1}{2}\int_M \Big[|\nabla^{(k)}F_{\nabla}|^2 + |\nabla^{(k+1)}u|^2 + \underbrace{\frac{\lambda}{4}(|u|^2-1)^2}_{\text{Higgs self-interaction}}\Big]d\mathrm{vol}_g. \tag{1.7}$$

It was showed that the negative gradient heat flow of (1.7) admits long-time existence if $k > 1$ on the 4-dimensional manifold. The degree of *W* in (1.7) is 2, this obviously satisfies $\deg W \leq k+2$. Such observation is neglected in the reference [68].

♣ In [68], the author use the boundness of the Yang–Mills–Higgs energy $\mathrm{YMH}_0(\nabla, u)$ [68, Proposition 6.3] to derive the L^p-estimate. In our case, due to the addition of the polynomial function *W*, the term involving the Higgs field is more complex, making it difficult to prove that the $\mathrm{YMH}_0(\nabla, u)$ is bounded along the *k*-flow (1.6). In this paper, we noticed that we can prove that the Yang–Mills energy $\mathrm{YM}(\nabla)$ is bounded along the heat flow (1.6), and this is enough for the L^p-estimate.

The study of Yang–Mills *k*-flow is motivated by the approach of approximating Yang–Mills equations using higher-order flows. Mathematically, this is justified as the higher-order elliptic terms in the flow provide a strong regularization mechanism, which enhances the well-posedness and regularity of the solutions. This is a widely used "smoothing" or "regularization" technique in geometric analysis and partial differential equations.

In [36], Kelleher considered the following Yang–Mills (ρ, k)-energy:

$$\mathrm{YM}_k^{\rho}(\nabla) := \rho\mathrm{YM}_k(\nabla) + \mathrm{YM}(\nabla), \quad \rho \in (0, +\infty), \tag{1.8}$$

where

$$\mathrm{YM}_k(\nabla) = \frac{1}{2}\int_M |\nabla^{(k)}F_{\nabla}|^2 d\mathrm{vol}_g.$$

The Euler–Lagrange equation of (1.8) is

$$\rho\Big((-1)^k D_\nabla^* \Delta_\nabla^{(k)} F_\nabla + P_1^{(2k+1)}[F_\nabla] + P_2^{(2k-1)}[F_\nabla]\Big) + D_\nabla^* F_\nabla = 0. \tag{1.9}$$

The corresponding negative gradient flow is

$$\frac{\partial \nabla_t}{\partial t} = -\Big[\rho\Big((-1)^k D_\nabla^* \Delta_\nabla^{(k)} F_\nabla + P_1^{(2k+1)}[F_\nabla] + P_2^{(2k-1)}[F_\nabla]\Big) + D_\nabla^* F_\nabla\Big]. \tag{1.10}$$

By analyzing the flow (1.10) and passing to the limit $\rho \searrow 0$, one expects to identify solutions to the Yang–Mills flow, in analogy with the result in [33]. Kelleher [36, Theorem B] asserts that, under suitable conditions, there exists a unique long-time solution ∇_t to (1.10) and that $\nabla := \lim_{t\to\infty} \nabla_t$ satisfies (1.9). Although a proof is not provided in [36], the long-time existence can be established similarly to [36, Theorem A] and Theorem 1. The convergence of the flow follows from a Łojasiewicz–Simon gradient inequality analogous to the one established in [51, Proposition 7.2]. Theorefore, one may also expect to solve Yang–Mills equation

$$D_\nabla^* F_\nabla = 0$$

on Riemannian manifolds by studying the equation (1.9) and sending $\rho \searrow 0$. It is well known that the analysis of Yang–Mills equations on general Riemannian manifolds presents significant challenges. Waldron [64] resolved a long-standing conjecture concerning the 4-dimensional Yang–Mills flow originally proposed in 1998 [55]. However, in general, one cannot expect the 4-dimensional Yang–Mills flow to converge to a Yang–Mills connection, as demonstrated in recent work by Sire–Wei–Zheng [59].

If we consider the following Yang–Mills–Higgs (ρ, k)-energy

$$\mathrm{YMH}_k^\rho(\nabla) := \rho YMH_k(\nabla, u) + YMH_0(\nabla, u), \quad \rho \in (0, +\infty),$$

where $YMH_k(\nabla, u)$ is defined by (1.7). We can expect to solve the Yang–Mills–Higgs equation

$$\begin{cases} D_\nabla^* F_\nabla = -\frac{1}{2}(D_\nabla u \otimes u^* - u \otimes (D_\nabla u)^*), \\ D_\nabla^* D_\nabla u = \frac{\lambda}{2} u(1 - |u|^2) \end{cases}$$

by sending $\rho \searrow 0$.

Thus, by following a similar regularization procedure and letting the parameter tend to zero, a systematic analytical path is provided for solving the coupled Yang–Mills–Higgs system.

2 Preliminaries

2.1 Notations

In order to fulfill the prerequisites outlined in the subsequent sections, this section concisely introduces the fundamental setups and notations. Our notations will incorporate elements from Kelleher's work as cited in [36], as well as Saratchandran's in [53] (additional reference can be found in [68]).

Given that E is a vector bundle defined over a smooth Riemannian manifold (M, g), a smooth unitary connection ∇ on E can be extended to various tensor bundles through coupling with the Levi–Civita connection ∇_M associated with the base manifold (M, g). Denote by $\mathcal{A}_E$ the set of all smooth unitary connections on the bundle E.

We define D_∇ as the exterior derivative associated with the connection ∇. The curvature tensor of E with respect to this exterior derivative D_∇ is denoted by

$$F_\nabla = D_\nabla \circ D_\nabla.$$

Given two p-forms ξ and η valued in E or $\mathrm{End}(E)$, let $\xi * \eta$ represent any multilinear form derived from the tensor product $\xi \otimes \eta$ in a universal fashion. Then, it holds that

$$|\xi * \eta| \leq C|\xi||\eta|,$$

where C is a constant.

Let ∇^* and D_∇^* denote the formal L^2-adjoints of ∇ and D_∇, respectively. The Bochner and Hodge Laplacians are defined as follows:

$$\Delta_\nabla = -\nabla^*\nabla, \quad \Delta_{D_\nabla} = D_\nabla D_\nabla^* + D_\nabla^* D_\nabla.$$

For $\phi \in \Omega^p(M; E)$, the Weitzenböck formula holds:

$$\Delta_{D_\nabla}\phi = -\Delta_\nabla \phi + (Rm + F_\nabla) * \phi, \tag{2.1}$$

where Rm represents the Riemannian curvature tensor of the metric g.

In this paper, we will use the following notation multiple times:

$$\nabla^{(i)} = \underbrace{\nabla \cdots \nabla}_{i \text{ times}}.$$

We will additionally employ the notation P, which was initially introduced in [37]. For a given tensor ξ, we denote by

$$P_v^{(k)}[\xi] := \sum_{w_1+\cdots+w_v=k} (\nabla^{(w_1)}\xi) * \cdots * (\nabla^{(w_v)}\xi) * T, \tag{2.2}$$

where $k, v \in \mathbb{N}$ and T is a tensor dependent only on g.

2.2 Interpolation Inequalities and Connection Identities

We need the following interpolation inequalities.

Lemma 1 *[[36, Lemma 5.3], analogue of [37, Corollary 5.5]] Let E be a vector bundle over a Riemannian manifold (M, g) equipped with a connection ∇. Consider ϕ as a section of E and γ as a bump function defined on M. For any natural number $k \in \mathbb{N}$, if the indices $i_1, \ldots, i_r$ satisfy $1 \leq i_1, \ldots, i_r \leq k$, their sum $i_1 + i_2 + \cdots + i_r = 2k$, and $s \geq 2k$, then the following inequality holds:*

$$\int_M \left(\gamma^s \nabla^{(i_1)}\phi * \cdots * \nabla^{(i_r)}\phi\right) \mathrm{dvol}_g \leq C\left(M, E, k, r, s, g, \gamma\right) \|\phi\|_{L^\infty}^{r-2} \left(\int_M |\nabla^{(k)}\phi|^2 \gamma^s \,\mathrm{dvol}_g + \|\phi\|^2_{L^2,\gamma>0}\right),$$

where the subscript $\gamma > 0$ indicates the subset $\{x \in M \mid \gamma(x) > 0\}$ of M.

Lemma 2 *([36, Corollary 5.2]) Consider a vector bundle E defined over a Riemannian manifold (M, g) equipped with a connection ∇, along with a bump function γ on M. For any p satisfying $2 \leq p < +\infty$, $l \in \mathbb{N}$, and $s \geq lp$, there exists a positive constant $C(\varepsilon^{-1}) = C(\varepsilon^{-1}, M, E, p, l, s, g, \gamma)$ ensures that for any section ϕ of E, the following inequality holds:*

$$\|\gamma^{s/p}\nabla^{(l)}\phi\|_{L^p} \leq \varepsilon\|\gamma^{(s+jp)/p}\nabla^{(l+j)}\phi\|_{L^p} + C(\varepsilon^{-1})\|\phi\|_{L^p,\gamma>0}.$$

In the special case where $p = 2$ and for some $K \geq 1$, we can rewrite the inequality as:

$$K\|\gamma^{s/2}\nabla^{(l)}\phi\|^2_{L^2} \leq \varepsilon\|\gamma^{(s+2j)/2}\nabla^{(l+j)}\phi\|^2_{L^2} + C(\varepsilon^{-1})K^2\|\phi\|^2_{L^2,\gamma>0}.$$

In our study of the higher-order flow, we collect and apply several lemmas from [36]. Throughout this analysis, we will frequently need to interchange derivatives; the following lemmas will be employed for this purpose.

Lemma 3 *([36, Lemma 5.5]) Given a vector bundle E defined over a Riemannian manifold (M, g), equipped with a compatible metric connection ∇, and let ϕ denote a section of E, we have the following identity:*

$$\nabla_{i_k}\nabla_{i_{k-1}}\cdots\nabla_{i_1}\nabla_{j_1}\nabla_{j_2}\cdots\nabla_{j_k}\phi = \nabla_{i_k}\nabla_{j_k}\cdots\nabla_{i_1}\nabla_{j_1}\phi + \sum_{l=0}^{2k-2}\left[(\nabla_M^{(l)}Rm + \nabla^{(l)}F_\nabla) * \nabla^{(2k-2-l)}\phi\right].$$

Lemma 4 *([36, Corollary 5.8]) Given a vector bundle E defined over a Riemannian manifold (M, g), equipped with a compatible metric connection ∇, and let ϕ denote a section of E, we have the following identity:*

$$\nabla^{(n)}\Delta_{\nabla}^{(k)}\phi = \Delta_{\nabla}^{(k)}\nabla^{(n)}\phi + \sum_{j=0}^{2k+n-2}\Big[(\nabla_M^{(j)}Rm + \nabla^{(j)}F_{\nabla}) * \nabla^{(2k+n-j-2)}\phi\Big].$$

Lemma 5 *([36, Lemma 5.6]) Let E be a vector bundle defined over a Riemannian manifold (M, g), endowed with a compatible metric connection ∇. Suppose ξ and ζ are sections of E, then for any $k \in \mathbb{N}$, we have the following identity:*

$$\int_M \langle\nabla^{(k)}\xi, \nabla^{(k)}\zeta\rangle d\mathrm{vol}_g = \int_M (-1)^k\langle\xi, \Delta_{\nabla}^{(k)}\zeta\rangle d\mathrm{vol}_g + \int_M \langle\xi, \sum_{v=0}^{2k-2}\Big((\nabla_M^{(v)}Rm + \nabla^{(v)}F_{\nabla}) * \nabla^{(2k-2-v)}\zeta\Big) d\mathrm{vol}_g.$$

2.3 Euler–Lagrange Equations

In the following, we first compute the associated Euler–Lagrange equations of the higher order Yang–Mills-Higgs functional (1.4) with general Higgs-like potential *W*.

Proposition 1 *The Euler–Lagrange equations associated to the functional (1.4) is (1.5).*

Proof On the one hand, consider ∇_t as a path of unitary connections, with the initial condition $\nabla_0 = \nabla$. Direct calculations will lead to

$$\begin{aligned}
&\frac{\partial}{\partial t}\frac{1}{2}\int_M (|\nabla_t^{(k)}F_{\nabla_t}|^2 + |\nabla_t^{(k+1)}u|^2) \\
&= \int_M \langle\frac{\partial\nabla_t}{\partial t}, D_{\nabla_t}^*\nabla_t^{*(k)}\nabla_t^{(k)}F_{\nabla_t}\rangle + \int_M \langle\frac{\partial(\nabla_t^{(k+1)}u)}{\partial t}, \nabla_t^{(k+1)}u\rangle \\
&= \int_M \langle\frac{\partial\nabla_t}{\partial t}, (-1)^k D_{\nabla_t}^*\Delta_{\nabla_t}^{(k)}F_{\nabla_t} + \sum_{v=0}^{2k-1} P_1^{(v)}[F_{\nabla_t}] + P_2^{(2k-1)}[F_{\nabla_t}]\rangle \\
&\quad + \int_M \langle\frac{\partial\nabla_t}{\partial t}, \sum_{i=0}^{k}\nabla_t^{*(i)}(\nabla_t^{(k+1)}u * \nabla_t^{(k-i)}u)\rangle.
\end{aligned} \tag{2.3}$$

On the other hand, let u_t be a path of Higgs fields, with $u_0 = u$. We have

$$
\begin{aligned}
&\frac{\partial}{\partial t}\frac{1}{2}\int_M \left(|\nabla^{(k+1)}u_t| + W(|u_t|^2)\right) \\
&= \int_M \langle \frac{\partial u_t}{\partial t}, \nabla^{*(k+1)}\nabla^{(k+1)}u_t\rangle + \frac{1}{2}\int_M W'(|u_t|^2)\frac{\partial}{\partial t}|u_t|^2 \\
&= \int_M \langle \frac{\partial u_t}{\partial t}, \nabla^{*(k+1)}\nabla^{(k+1)}u_t + W'(|u_t|^2)u_t\rangle.
\end{aligned} \tag{2.4}
$$

Combining (2.3) and (2.4), we complete the proof of the proposition. □

2.4 Short-time Existence

Employing de Turck's trick, we can establish the local existence of the Yang–Mills–Higgs *k*-flow as defined in (1.6).

Theorem 2 *Given a vector bundle E over a closed Riemannian manifold (M, g), the Yang–Mills–Higgs k-flow (1.6) admits a unique smooth solution* (∇_t, u_t) *on* $M \times [0, \epsilon)$ *for the smooth initial data* (∇_0, u_0).

Proof (**Existence**) Consider 1-parameter pairs $(\widetilde{\nabla}_t, \widetilde{u_t})$ satisfying the following system

$$
\begin{cases}
\frac{\partial \widetilde{\nabla}_t}{\partial t} = (-1)^{k+1}D^*_{\widetilde{\nabla}}\Delta^{(k)}_{\widetilde{\nabla}_t}F_{\widetilde{\nabla}_t} + (-1)^k D_{\widetilde{\nabla}_t}\Delta^{(k)}_{\widetilde{\nabla}_t}D^*_{\widetilde{\nabla}_t}(\widetilde{\nabla}_t - \nabla(0)) \\
\qquad + \sum_{v=0}^{2k-1} P_1^{(v)}[F_{\widetilde{\nabla}_t}] + P_2^{(2k-1)}[F_{\widetilde{\nabla}_t}] + \sum_{i=0}^{k} \widetilde{\nabla}_t^{*(i)}(\widetilde{\nabla}_t^{(k+1)}u_t * \widetilde{\nabla}_t^{(k-i)}u_t), \\
\frac{\partial \widetilde{u}_t}{\partial t} = -\widetilde{\nabla}_t^{*(k+1)}\widetilde{\nabla}_t^{(k+1)}\widetilde{u}_t - (-1)^k\left(\Delta^{(k)}_{\widetilde{\nabla}_t}D^*_{\widetilde{\nabla}_t}(\widetilde{\nabla}_t - \nabla(0))\right)\widetilde{u}_t - W'(|\widetilde{u}_t|^2)\widetilde{u}_t, \\
\widetilde{\nabla}(0) = \nabla_0, \\
\widetilde{u}(0) = u_0.
\end{cases} \tag{2.5}
$$

Drawing on the argument in [39, Section 7.2] (see also [36, Lemma 3.2]), the ellipticity of the highest-order term in system (2.5) can be proved as follows.

Now, let us define the operator

$$
\Phi_k := \Phi_k(\cdot, \nabla(0)) : \mathcal{A}_E \to \Omega^1(\mathrm{End}E)
$$

by

$$
\Phi_k(\widetilde{\nabla}_t, \nabla(0)) = (-1)^{k+1}D^*_{\widetilde{\nabla}_t}\Delta^{(k)}_{\widetilde{\nabla}_t}F_{\widetilde{\nabla}_t} + (-1)^k D_{\widetilde{\nabla}_t}\Delta^{(k)}_{\widetilde{\nabla}_t}D^*_{\widetilde{\nabla}_t}(\widetilde{\nabla}_t - \nabla(0)).
$$

We will compute the principal symbol of the operator $\Phi_k(\widetilde{\nabla}_t, \nabla(0))$ and verify that it is elliptic. We linearize around the fixed connection $\nabla(0)$ and extract the highest-order derivative terms. Let $\widetilde{\nabla}_t = \nabla(0) + tS$ with $S \in \Omega^1(\mathrm{End}E)$. Linearizing at $t = 0$ gives a linear differential operator in S; for the principal symbol we keep only the terms with the highest total number of derivatives.

2.5 Step 1. Linearization of the First Term

The curvature linearizes as $F_{\widetilde{\nabla}_t} \approx F_{\nabla(0)} + t\, dS$, where dS is the exterior derivative of S. In components, $(dS)_{qr} = \partial_q S_r - \partial_r S_q$. The operator $\Delta^{(k)}$ becomes the k-th power of the ordinary Laplacian $\Delta = \partial_i \partial_i$ (summation implied). The co-differential D^* acts as ∂^q on the first index of a 2-form, where $-\partial^q = -g^{ql}\partial_l$. Hence,

$$(-1)^{k+1} D^* \Delta^{(k)} F \;\longrightarrow\; (-1)^{k+1}(-\partial^q)\Delta^k(\partial_q S_r - \partial_r S_q) = (-1)^k \partial^q \Delta^k (\partial_q S_r - \partial_r S_q).$$

2.6 Step 2. Linearization of the Second Term

Write $\widetilde{\nabla}_t - \nabla(0) = tS$. The co-differential D^* applied to S gives, at the level of principal symbols, $-\partial^r S_r$. Then $\Delta^{(k)}$ becomes Δ^k, and the covariant derivative D becomes ∂. Thus,

$$(-1)^k D \Delta^{(k)} D^*(\widetilde{\nabla}_t - \nabla(0)) \;\longrightarrow\; (-1)^k \partial_r \Delta^k(-\partial^q S_q) = (-1)^{k+1} \partial_r \Delta^k (\partial^q S_q).$$

2.7 Step 3. Combined Linearized Operator

Adding the two contributions, the principal part of the linearized operator is

$$L(S)_r = (-1)^k \partial^q \Delta^k (\partial_q S_r - \partial_r S_q) + (-1)^{k+1} \partial_r \Delta^k (\partial^q S_q) = (-1)^k \left[\partial^q \Delta^k (\partial_q S_r - \partial_r S_q) - \partial_r \Delta^k (\partial^q S_q)\right].$$

Using the commutativity of partial derivatives,

$$\partial^q \Delta^k \partial_q S_r = \Delta^{k+1} S_r, \qquad \partial^q \Delta^k \partial_r S_q = \partial_r \Delta^k \partial^q S_q,$$

we obtain

$$L(S)_r = (-1)^k \left[\Delta^{k+1} S_r - 2\partial_r \Delta^k (\partial^q S_q)\right].$$

2.8 Step 4. Principal Symbol

Replace each derivative ∂_i by $\sqrt{-1}\xi_i$, where ξ is a cotangent vector. Recall that

$$\Delta \mapsto -|\xi|^2, \quad \Delta^k \mapsto (-1)^k |\xi|^{2k}, \quad \Delta^{k+1} \mapsto (-1)^{k+1} |\xi|^{2k+2},$$
$$\partial_r \mapsto \sqrt{-1}\xi_r, \quad \partial^q \mapsto \sqrt{-1}\xi^q = \sqrt{-1} g^{ql} \xi_l.$$

Substituting,

$$\begin{aligned}\sigma(L)(\xi)(S)_r &= (-1)^k\Big[(-1)^{k+1}|\xi|^{2k+2}S_r - 2(\sqrt{-1}\xi_r)(-1)^k|\xi|^{2k}(\sqrt{-1}\xi^q S_q)\Big]\\ &= -|\xi|^{2k+2}S_r + 2|\xi|^{2k}\xi_r\xi^q S_q.\end{aligned}$$

Hence the principal symbol is

$$\sigma(\Phi_k)(\xi)(S) = |\xi|^{2k}\big(-|\xi|^2 S + 2\xi\langle\xi, S\rangle\big),$$

where $\langle\xi, S\rangle = \xi^q S_q$.

2.9 Step 5. Ellipticity

To check ellipticity we verify that for every $\xi \neq 0$ the map $S \mapsto \sigma(\Phi_k)(\xi)(S)$ is injective. Assume $\sigma(\Phi_k)(\xi)(S) = 0$, i.e.

$$-|\xi|^2 S + 2\xi\langle\xi, S\rangle = 0.$$

Decompose $S = S_\parallel + S_\perp$ with $S_\parallel = c\xi$ where $c = \langle\xi, S\rangle/|\xi|^2$ and $\langle\xi, S_\perp\rangle = 0$. Substituting,

$$-|\xi|^2(c\xi + S_\perp) + 2\xi(c|\xi|^2) = -c|\xi|^2\xi - |\xi|^2 S_\perp + 2c|\xi|^2\xi = c|\xi|^2\xi - |\xi|^2 S_\perp = 0.$$

Thus $c\xi - S_\perp = 0$. Since $c\xi$ is parallel to ξ and $S_\perp$ is orthogonal to ξ, each must vanish separately. In particular, taking the inner product with ξ gives $c|\xi|^2 = 0$, hence $c = 0$, and then $S_\perp = 0$. Therefore $S = 0$. The symbol is injective for every nonzero ξ, so the operator Φ_k is elliptic.

As a result, the system (2.5) is parabolic and admits short-time existence.

Define the C^∞ gauge transformations $g(t)$ as

$$\begin{cases} \frac{\partial g(t)}{\partial t} = (-1)^{k+1}\Delta^{(k)}_{\widetilde{\nabla}_t} D^*_{\widetilde{\nabla}_t}(\widetilde{\nabla}_t - \nabla(0))g(t) \\ g(0) = \mathrm{id}. \end{cases} \tag{2.6}$$

Verification reveals that the transformed pair $(g(t)^*\widetilde{\nabla}_t, g(t)^*\widetilde{u}_t)$ fulfills the Yang–Mills–Higgs k-flow (1.6), initiating from the initial value $(g(0)^*\widetilde{\nabla}_0, g(0)^*\widetilde{u}_0) = (\nabla_0, u_0)$. This demonstration establishes the short-time existence of the flow (1.6).

(Uniqueness) Given two solutions $(\nabla_1(t), u_1(t))$ and $(\nabla_2(t), u_2(t))$ to the flow (1.6), both sharing the identical initial data $(\nabla(0), u(0))$, we associate distinct gauges g_1 and g_2 that adhere to the gauge transformation equations (2.6). Consequently, both $((g_1^{-1})^*\nabla_1, (g_1^{-1})^*u_1)$ and $((g_2^{-1})^*\nabla_2, (g_2^{-1})^*u_2)$ are solutions to the system (2.5), starting from the same initial data $(\nabla(0), u(0))$. The uniqueness property of the system (2.5) necessitates that

$$((g_1^{-1})^*\nabla_1, (g_1^{-1})^*u_1) = ((g_2^{-1})^*\nabla_2, (g_2^{-1})^*u_2),$$

which, in turn, implies

$$(\nabla_1, u_1) = ((g_2^{-1}g_1)^*\nabla_2, (g_2^{-1}g_1)^*u_2).$$

Upon defining a novel gauge transformation g_3 as the composition $g_3 := g_2^{-1}g_1$, a straightforward computation reveals the following ordinary differential equation (ODE):

$$\begin{cases} \frac{\partial g_3}{\partial t} = g_3(-1)^{k+1}\Delta^{(k)}_{g_3^*\nabla_2} D^*_{g_3^*\nabla_2}(g_3^*\nabla_2 - \nabla(0)) \\ \qquad\qquad -(-1)^{k+1}\Delta^{(k)}_{\nabla_2} D^*_{\nabla_2}(\nabla_2 - \nabla(0))g_3, \\ g_3(0) = \mathrm{id}. \end{cases} \tag{2.7}$$

Notably, the identity map id constitutes a solution to this ODE (2.7) with a prescribed initial condition. Consequently, by the fundamental existence and uniqueness theorem for ODEs [50], $g_3(t)$ must be identically equal to id for all t, which proves uniqueness.

□

3 Smoothing Estimates

In this section, our goal is to obtain derivative estimates of F_{∇_t} and u_t. To accomplish this we first compute necessary evolution equations.

3.1 Evolution Equations

Lemma 6 *Assuming* (∇_t, u_t) *represents a solution to the Yang–Mills–Higgs k-flow, as defined in (1.6), which is specified on the domain* $M \times [0, T)$*. Then, we have*

$$\begin{aligned} \frac{\partial}{\partial t}[\nabla_t^{(l)} u_t] &= (-1)^k \Delta^{(k+1)}_{\nabla_t} \nabla_t^{(l)} u_t + \sum_{j=0}^{2k+l} (\nabla_M^{(j)} Rm + \nabla_t^{(j)} F_{\nabla_t}) * \nabla_t^{(2k+l-j)} u_t \\ &+ \sum_{j=0}^{l-1} \nabla_t^{(j)} D^*_{\nabla_t} \Delta^{(k)}_{\nabla_t} F_{\nabla_t} * \nabla_t^{(l-j-1)} u_t + \sum_{j=0}^{l-1} \sum_{v=0}^{2k+j-1} P_1^{(v)}[F_{\nabla_t}] * \nabla_t^{(l-j-1)} u_t \\ &+ \sum_{j=0}^{l-1} P_2^{(2k+j-1)}[F_{\nabla_t}] * \nabla_t^{(l-j-1)} u_t \\ &+ \sum_{j=0}^{l-1} \sum_{i=0}^{k} \left[\nabla_t^{(j)} \nabla_t^{*(i)} (\nabla_t^{(k+1)} u_t * \nabla_t^{(k-i)} u_t)\right] * \nabla_t^{(l-j-1)} u_t \\ &- \nabla_t^{(l)}[W'(|u_t|^2) u_t]. \end{aligned} \tag{3.1}$$

Proof Using the flow equation (1.6), direct calculations yield

$$
\begin{aligned}
\frac{\partial}{\partial t}[\nabla_t^{(l)} u_t] &= \nabla_t^{(l)} \frac{\partial u_t}{\partial t} + \sum_{j=0}^{l-1} \nabla_t^{(j)} \frac{\partial \nabla_t}{\partial t} * \nabla_t^{(l-j-1)} u_t \\
&= \nabla_t^{(l)} (-\nabla_t^{*(k+1)} \nabla_t^{(k+1)} u_t - W'(|u_t|^2) u_t) \\
&\quad + \sum_{j=0}^{l-1} \nabla_t^{(j)} \Big[(-1)^{k+1} D^*_{\nabla_t} \Delta^{(k)}_{\nabla_t} F_{\nabla_t} + \sum_{v=0}^{2k-1} P_1^{(v)}[F_{\nabla_t}] \\
&\quad + P_2^{(2k-1)}[F_{\nabla_t}] + \sum_{i=0}^{k} \nabla_t^{*(i)} (\nabla_t^{(k+1)} u_t * \nabla_t^{(k-i)} u_t) \Big] * \nabla_t^{(l-j-1)} u_t .
\end{aligned}
$$

Then using Lemma 3 and Lemma 4 yields the desired result.

□

Using the Weitzenböck formula (2.1) and Lemma 4, we have the following lemma.

Lemma 7 *Suppose* (∇_t, u_t) *is a solution to the Yang–Mills–Higgs k-flow (1.6) defined on* $M \times [0, T)$. *Then*

$$
\begin{aligned}
\frac{\partial F_{\nabla_t}}{\partial t} &= (-1)^k \Delta^{(k+1)}_{\nabla_t} F_{\nabla_t} + \sum_{v=0}^{2k} P_1^{(v)}[F_{\nabla_t}] + P_2^{(2k)}[F_{\nabla_t}] \\
&\quad + \sum_{i=0}^{k} D_{\nabla_t} \nabla_t^{*(i)} (\nabla_t^{(k+1)} u_t * \nabla_t^{(k-i)} u_t),
\end{aligned}
\tag{3.2}
$$

and for $l \in \mathbb{N}$,

$$
\begin{aligned}
\frac{\partial}{\partial t}[\nabla_t^{(l)} F_{\nabla_t}] &= (-1)^k \Delta_t^{(k+1)} \nabla_t^{(l)} F_{\nabla_t} + \sum_{v=0}^{2k+l} \left(P_1^{(v)}[F_{\nabla_t}] + P_2^{(v)}[F_{\nabla_t}] \right) \\
&\quad + P_3^{(2k+l-2)}[F_{\nabla_t}] + \sum_{i=0}^{k} \nabla_t^{(l)} D_{\nabla_t} \nabla_t^{*(i)} (\nabla_t^{(k+1)} u_t * \nabla_t^{(k-i)} u_t) \\
&\quad + \sum_{j=0}^{l-1} \sum_{i=0}^{k} \left[\nabla_t^{(j)} \left(\nabla_t^{*(i)} (\nabla_t^{(k+1)} u_t * \nabla_t^{(k-i)} u_t) \right) \right] * \nabla_t^{(l-j-1)} F_{\nabla_t} .
\end{aligned}
\tag{3.3}
$$

3.2 L^2-estimates for Derivatives of the Higgs Field

In this subsection, we establish local L^2-derivative estimates for the Higgs field. The following proposition is a direct consequence of Lemma 6.

Proposition 2 *Suppose* (∇_t, u_t) *is a solution to the Yang–Mills–Higgs k-flow (1.6) defined on* $M \times [0, T)$. *Then*

$$
\begin{aligned}
&\frac{\partial}{\partial t}\|\gamma^{s/2}\nabla_t^{(l)}u_t\|_{L^2}^2\\
&=\Big[\underbrace{-2\int_M\Big\langle\nabla_t^{(l)}[W'(|u_t|^2)u_t],\gamma^s\nabla_t^{(l)}u_t\Big\rangle}_{T_{1a}}+\underbrace{2(-1)^k\int_M\Big\langle\Delta_{\nabla_t}^{(k+1)}\nabla_t^{(l)}u_t,\gamma^s\nabla_t^{(l)}u_t\Big\rangle}_{T_{1b}}\Big]_{T_1}\\
&\quad+\Big[\int_M\sum_{j=0}^{2k+l}\Big\langle(\nabla_M^{(j)}Rm+\nabla_t^{(j)}F_{\nabla_t})*\nabla_t^{(2k+l-j)}u_t,\gamma^s\nabla_t^{(l)}u_t\Big\rangle\Big]_{T_2}\\
&\quad+\Big[\int_M\sum_{j=0}^{l-1}\Big\langle\nabla_t^{(j)}D_{\nabla_t}^*\Delta_{\nabla_t}^{(k)}F_{\nabla_t}*\nabla_t^{(l-j-1)}u_t,\gamma^s\nabla_t^{(l)}u_t\Big\rangle\Big]_{T_3}\\
&\quad+\Big[\int_M\sum_{j=0}^{l-1}\Big\langle(\sum_{v=0}^{2k+j-1}P_1^{(v)}[F_{\nabla_t}]+P_2^{(2k+j-1)}[F_{\nabla_t}])*\nabla_t^{(l-j-1)}u_t,\gamma^s\nabla_t^{(l)}u_t\Big\rangle\Big]_{T_4}\\
&\quad+\Big[\int_M\sum_{j=0}^{l-1}\sum_{i=0}^{k}\Big\langle\Big[\nabla_t^{(j)}\nabla_t^{*(i)}(\nabla_t^{(k+1)}u_t*\nabla_t^{(k-i)}u_t)\Big]*\nabla_t^{(l-j-1)}u_t,\gamma^s\nabla_t^{(l)}u_t\Big\rangle\Big]_{T_5},
\end{aligned}
\tag{3.4}
$$

where γ is the bump function defined in Definition 1. Besides, $T_1,\cdots,T_5$ denote the expressions inside the brackets.

We will proceed to evaluate each term presented on the right-hand side of the aforementioned equality (3.4). To facilitate smooth estimates, we commence by introducing the bump function, which is of importance in this context.

Definition 1 *(Bump function)* Define $\mathfrak{B}$ as the set of bump functions, represented as

$$\mathfrak{B}:=\{\gamma\in C_c^\infty(M):0\le\gamma\le 1\},$$

where $C_c^\infty(M)$ denotes the space of smooth functions with compact support on the manifold *M*. For any natural number $l\in\mathbb{N}$, we introduce the notation $J_\gamma^{(l)}$ to signify the sum of the $L^\infty(M)$ norms of the derivatives of γ up to the *l*-th order, given by

$$J_\gamma^{(l)}:=\sum_{j=0}^{l}\|\nabla^{(j)}\gamma\|_{L^\infty(M)}.$$

We also need the following lemma. This can be proved by integration by parts and by induction method.

Lemma 8 *([36, Lemma 3.10]) Let $p,q,r,s\in\mathbb{N}$ and $\gamma\in\mathfrak{B}$. If $s\in\mathbb{N}\backslash\{1\}$, then*

$$
\begin{aligned}
\int_M(P_1^{(p)}[\phi]*P_1^{(q+r)}[\phi])\gamma^s d\mathrm{vol}_g\le&\int_M(P_1^{(p+r)}[\phi]*P_1^{(q)}[\phi])\gamma^s d\mathrm{vol}_g\\
&+\sum_{j=0}^{r-1}J_\gamma^{(1)}\int_M(P_1^{(p+j)}[\phi]*P_1^{(q+r-j-1)}[\phi])\gamma^{s-1}d\mathrm{vol}_g,
\end{aligned}
$$

where ϕ is in some tensor product of TM and E.

Now, we are ready to handle the right hand side of (3.4).

Lemma 9 *Suppose* (∇_t, u_t) *is a solution to the k-flow (1.6) defined on* $M \times [0,T)$. *Let* $Q = \max\{1, \sup_{t\in[0,T)} |F_{\nabla_t}|\}$, $K = \max\{1, \sup_{t\in[0,T)} |u_t|\}$ *and* $\gamma \in \mathfrak{B}$. *Then for any* $s \geq 2(k+l+1)$, *there exist constants* $\lambda \in [1,2)$ *and* $C := C(M,E,s,k,l,g,\gamma)$ *such that the following holds:*

$$T_1 \leq -\lambda \|\gamma^{s/2}\nabla_t^{(k+l+1)} u_t\|^2_{L^2} + CQK^{2(\deg W-1)}\|u_t\|^2_{L^2,\gamma>0}.$$

Proof For the first part of T_1, direct calculations yield

$$\begin{aligned}
T_{1a} &= -2\int_M \left\langle \nabla_t^{(l)}[W'(|u_t|^2)u_t], \gamma^s \nabla_t^{(l)} u_t \right\rangle \\
&= -2\int_M \left\langle W'(|u_t|^2)u_t, P_1^l[\gamma^s \nabla_t^{(l)} u_t] \right\rangle \\
&= -2\int_M \left\langle W'(|u_t|^2)u_t, \sum_{v=0}^{l} \nabla^{(v)}\gamma^s * \nabla_t^{(2l-v)} u_t \right\rangle \\
&\leq C_1 K^{2(\deg W-1)} \int_M \left\langle u_t, \sum_{v=0}^{l} \nabla^{(v)}\gamma^s * \nabla_t^{(2l-v)} u_t \right\rangle \\
&\leq C_2 K^{2(\deg W-1)} \sum_{v=0}^{l} \int_M (P_2^{(2l-v)}[u_t]\gamma^{s-v}).
\end{aligned}$$

In order to use Lemma 1, we need to discuss whether the variable $2l-v$ can be divided by 2. Using Lemma 8, we can handle it as follows:

$$\begin{aligned}
&\sum_{v=0}^{l} \int_M (P_2^{(2l-v)}[u_t]\gamma^{s-v}) d\mathrm{vol}_g \\
&= \sum_{v\in 2\mathbb{N}-1}^{l} \int_M (P_2^{(2l-v)}[u_t]\gamma^{s-v}) + \sum_{v\in 2\mathbb{N}\cup\{0\}}^{l} \int_M (P_2^{(2l-v)}[u_t]\gamma^{s-v}) \\
&= \sum_{v\in 2\mathbb{N}-1}^{l} \int_M \Big((P_1^{(0)}[u_t] * P_1^{(2l-v)}[u_t] + P_1^{(1)}[u_t] * P_1^{(2l-v-1)}[u_t] + P_1^{(2)}[u_t] * P_1^{(2l-v-2)}[u_t] + \cdots)\gamma^{s-v}\Big) \\
&\quad + \sum_{v\in 2\mathbb{N}\cup\{0\}}^{l} \int_M (P_2^{(2l-v)}[u_t]\gamma^{s-v}) \\
&\leq [\sum_{v\in 2\mathbb{N}-1}^{l} \int_M (P_1^{(\lfloor \frac{2l-v}{2}\rfloor)} * P_1^{(\lceil \frac{2l-v}{2}\rceil)}\gamma^{s-v}]_{T_{odd}} + [\sum_{v\in 2\mathbb{N}\cup\{0\}}^{l} \int_M (P_2^{(2l-v)}[u_t] J_\gamma^{(1)} \gamma^{s-v-1})]_{T_{even}}.
\end{aligned}$$

For the even term T_{even}, using Lemma 1 and Lemma 2 successively, we can obtain

$$\begin{aligned}
&\int_M (P_2^{(2l-v)}[u_t] J_\gamma^{(1)} \gamma^{s-v-1}) d\mathrm{vol}_g \\
&\leq C_3 \Big(\|\gamma^{\frac{s-v-1}{2}} \nabla_t^{(l-\frac{v}{2})} u_t\| + \|u_t\|^2_{L^2,\gamma>0} \Big) \\
&= C_3 \Big(\|\gamma^{\frac{s-v-1}{2}} \nabla_t^{(k+l+1)-(k+\frac{v}{2}+1)} u_t\| + \|u_t\|^2_{L^2,\gamma>0} \Big) \\
&\leq \varepsilon_1 \|\gamma^{\frac{s-v-1}{2}+(k+\frac{v}{2}+1)} \nabla_t^{(k+l+1)} u_t\|^2_{L^2} + C_4 \|u_t\|^2_{L^2,\gamma>0} \\
&\leq \varepsilon_1 \|\gamma^{\frac{s}{2}} \nabla_t^{(k+l+1)} u_t\|^2_{L^2} + C_4 \|u_t\|^2_{L^2,\gamma>0}.
\end{aligned}$$

For the odd term T_{odd}, using Hölder inequality, then followed by Lemma 1 and Lemma 2 successively, we can obtain

$$\begin{aligned}
&\int_M (P_1^{(\lfloor \frac{2l-v}{2} \rfloor)} * P_1^{(\lceil \frac{2l-v}{2} \rceil)} \gamma^{s-v}) d\mathrm{vol}_g \\
&\leq \frac{1}{2} \int_M P_2^{2(\lfloor \frac{2l-v}{2} \rfloor)}[u_t] \gamma^{s-v} + \frac{1}{2} \int_M P_2^{2(\lceil \frac{2l-v}{2} \rceil)}[u_t] \gamma^{s-v} \\
&\leq \frac{1}{2} C_5 (\|\gamma^{\frac{s-v}{2}} \nabla_t^{(\lfloor \frac{2l-v}{2} \rfloor)} u_t\|^2_{L^2} + \|u_t\|^2_{L^2,\gamma>0}) \\
&\quad + \frac{1}{2} C_6 (\|\gamma^{\frac{s-v}{2}} \nabla_t^{(\lceil \frac{2l-v}{2} \rceil)} u_t\|^2_{L^2} + \|u_t\|^2_{L^2,\gamma>0}) \\
&= \frac{1}{2} C_5 (\|\gamma^{\frac{s-v}{2}} \nabla_t^{(k+l+1)-(k+\lfloor \frac{v}{2} \rfloor+1)} u_t\|^2_{L^2} + \|u_t\|^2_{L^2,\gamma>0}) \\
&\quad + \frac{1}{2} C_6 (\|\gamma^{\frac{s-v}{2}} \nabla_t^{(k+l+1)-(k+\lceil \frac{v}{2} \rceil+1)} u_t\|^2_{L^2} + \|u_t\|^2_{L^2,\gamma>0}) \\
&\leq \varepsilon_2 \|\gamma^{s/2} \nabla_t^{(k+l+1)} u_t\|^2_{L^2} + C_7 \|u_t\|^2_{L^2,\gamma>0}.
\end{aligned}$$

Therefore, if $s \geq 2l+1$, we have

$$T_{1a} \leq \varepsilon_3 \|\gamma^{s/2} \nabla_t^{(k+l+1)} u_t\|^2_{L^2} + C_8 K^{2(\deg W-1)} \|u_t\|^2_{L^2,\gamma>0}. \tag{3.5}$$

For the second part of T_1, using Lemma 5 we have

$$
\begin{aligned}
T_{1b} &= 2(-1)^k \int_M \left\langle \Delta_{\nabla_t}^{(k+1)} \nabla_t^{(l)} u_t, \gamma^s \nabla_t^{(l)} u_t \right\rangle \\
&= -2 \int_M \left\langle \nabla_t^{(k+1)} \nabla_t^{(l)} u_t, \nabla_t^{(k+1)} (\gamma^s \nabla_t^{(l)} u_t) \right\rangle \\
&\quad + \int_M \sum_{j=0}^{2k} \left\langle \nabla_M^{(j)} Rm * \nabla_t^{(2k+l-j)} u_t, \gamma^s \nabla_t^{(l)} u_t \right\rangle \\
&\quad + \int_M \sum_{j=0}^{2k} \left\langle \nabla_t^{(j)} F_{\nabla_t} * \nabla_t^{(2k+l-j)} u_t, \gamma^s \nabla_t^{(l)} u_t \right\rangle \\
&= -2 \|\gamma^{s/2} \nabla_t^{(k+l+1)} u_t\|_{L^2}^2 + \Big[\int_M \sum_{j=1}^{k+1} \nabla^{(j)} \gamma^s * \left\langle \nabla_t^{(k+l+1)} u_t, \nabla_t^{(k+l+1-j)} u_t \right\rangle \Big]_{T_{1b}^{(1)}} \\
&\quad + \Big[\int_M \sum_{j=0}^{2k} \left\langle \nabla_M^{(j)} Rm * \nabla_t^{(2k+l-j)} u_t, \gamma^s \nabla_t^{(l)} u_t \right\rangle \Big]_{T_{1b}^{(2)}} \\
&\quad + \Big[\int_M \sum_{j=0}^{2k} \left\langle \nabla_t^{(j)} F_{\nabla_t} * \nabla_t^{(2k+l-j)} u_t, \gamma^s \nabla_t^{(l)} u_t \right\rangle \Big]_{T_{1b}^{(3)}}.
\end{aligned}
\tag{3.6}
$$

Similar to the estimates of T_{1a}, if $s \geq 2(k+l+1)$, we have

$$
T_{1b}^{(1)} + T_{1b}^{(2)} + T_{1b}^{(3)} \leq \varepsilon_4 \|\gamma^{s/2} \nabla_t^{(k+l+1)} u_t\|_{L^2}^2 + C_9 Q K^2 \|u_t\|_{L^2,\gamma>0}^2. \tag{3.7}
$$

Combing inequalities (3.5), (3.6) and (3.7), we complete the proof of the Lemma. □

Similarly, we can refer to the estimation of T_1 to obtain the estimates of T_2, T_3, T_4, and T_5 in Proposition 2. By summarizing these estimates, we can arrive at the following proposition.

Proposition 3 *Suppose* (∇_t, u_t) *is a solution to the k-flow (1.6) defined on* $M \times [0, T)$. *Let* $Q = \max\{1, \sup_{t\in[0,T)} |F_{\nabla_t}|\}$, $K = \max\{1, \sup_{t\in[0,T)} |u_t|\}$ *and* $\gamma \in \mathfrak{B}$. *Then for any* $s \geq 2(k+l+1)$, *there exist constants* $\lambda \in [1, 2)$ *and* $C := C(M, E, s, k, l, g, \gamma)$ *such that*

$$
\frac{\partial}{\partial t} \|\gamma^{s/2} \nabla_t^{(l)} u_t\|_{L^2}^2 \leq -\lambda \|\gamma^{s/2} \nabla_t^{(k+l+1)} u_t\|_{L^2}^2 + C Q^2 K^{4(\deg W - 1)} \|u_t\|_{L^2,\gamma>0}^2.
$$

3.3 Coupled Estimates for the Curvature and the Higgs Field

Similar to the previous subsection, we will present local L^2-derivative estimates for the curvature F_{∇_t}. From the evolution equation (3.3), we have:

Proposition 4 *Suppose* (∇_t, u_t) *is a solution to the k-flow (1.6) defined on* $M \times [0, T)$. *Then*

$$
\begin{aligned}
&\frac{\partial}{\partial t}\|\gamma^{s/2}\nabla_t^{(l)}F_{\nabla_t}\|_{L^2}^2 \\
&= 2(-1)^k\int_M \left\langle \Delta_{\nabla_t}^{(k+1)}\nabla_t^{(l)}F_{\nabla_t}, \gamma^s\nabla_t^{(l)}F_{\nabla_t}\right\rangle \\
&\quad + \int_M \left\langle \sum_{v=0}^{2k+l}(P_1^{(v)}[F_{\nabla_t}] + P_2^{(v)}[F_{\nabla_t}]) + P_3^{(2k+l-2)}[F_{\nabla_t}], \gamma^s\nabla_t^{(l)}F_{\nabla_t}\right\rangle \\
&\quad + \int_M \left\langle \sum_{i=0}^{k}\nabla_t^{(l)}D_{\nabla_t}\nabla_t^{*(i)}(\nabla_t^{(k+1)}u_t * \nabla_t^{(k-i)}u_t), \gamma^s\nabla_t^{(l)}F_{\nabla_t}\right\rangle \\
&\quad + \int_M \left\langle \sum_{j=0}^{l-1}\sum_{i=0}^{k}\left[\nabla_t^{(j)}\left(\nabla_t^{*(i)}(\nabla_t^{(k+1)}u_t * \nabla_t^{(k-i)}u_t)\right)\right] * \nabla_t^{(l-j-1)}F_{\nabla_t}, \gamma^s\nabla_t^{(l)}F_{\nabla_t}\right\rangle.
\end{aligned}
$$

Similar to the proof of Lemma 9, we have the following local L^2-derivative estimate for the curvature.

Proposition 5 *Suppose* (∇_t, u_t) *is a solution to the k-flow (1.6) defined on* $M \times [0, T)$. *Let* $Q = \max\{1, \sup_{t\in[0,T)}|F_{\nabla_t}|\}$, $K = \max\{1, \sup_{t\in[0,T)}|u_t|\}$ *and* $\gamma \in \mathfrak{B}$. *Then for any* $s \geq 2(k+l+1)$, *there exist constants* $\lambda \in [1, 2)$ *and* $C := C(M, E, s, k, l, g, \gamma)$ *such that*

$$
\frac{\partial}{\partial t}\|\gamma^{s/2}\nabla_t^{(l)}F_{\nabla_t}\|_{L^2}^2 \leq -\lambda\|\gamma^{s/2}\nabla_t^{(k+l+1)}F_{\nabla_t}\|_{L^2}^2 + CQ^4K^2\|F_{\nabla_t}\|_{L^2,\gamma>0}^2.
$$

Since the Yang–Mills–Higgs *k*-flow (1.6) is an intricate coupled system, obtaining a localized estimate for either the curvature or the Higgs field alone is not feasible. However, by applying Proposition 3 and Proposition 5, we obtain the following elegant result.

Proposition 6 *Suppose* (∇_t, u_t) *is a solution to the k-flow (1.6) defined on* $M \times [0, T)$. *Let* $Q = \max\{1, \sup_{t\in[0,T)}|F_{\nabla_t}|\}$, $K = \max\{1, \sup_{t\in[0,T)}|u_t|\}$ *and* $\gamma \in \mathfrak{B}$. *Then for any* $s \geq 2(k+l+1)$, *there exist constants* $\lambda \in [1, 2)$ *and* $C := C(M, E, s, k, l, g, \gamma)$ *such that*

$$
\begin{aligned}
&\frac{\partial}{\partial t}(\|\gamma^{s/2}\nabla_t^{(l)}F_{\nabla_t}\|_{L^2}^2 + \|\gamma^{s/2}\nabla_t^{(l)}u_t\|_{L^2}^2) \\
&\leq -\lambda(\|\gamma^{s/2}\nabla_t^{(k+l+1)}F_{\nabla_t}\|_{L^2}^2 + \|\gamma^{s/2}\nabla_t^{(k+l+1)}u_t\|_{L^2}^2) + CQ^4K^{4(\deg W-1)}(\|F_{\nabla_t}\|_{L^2,\gamma>0}^2 + \|u_t\|_{L^2,\gamma>0}^2).
\end{aligned}
$$

Drawing upon the aforementioned proposition and adhering to the approach outlined in [36, 53], we derive the subsequent estimate of the Bernstein–Bando–Shi type by considering the *t*-derivative of the following function:

$$\Phi(t) := \sum_{l=0}^{q} a_l t^l (\|\gamma^s \nabla_t^{((k+1)l)} F_{\nabla_t}\|_{L^2}^2 + \|\gamma^s \nabla_t^{((k+1)l)} u_t\|_{L^2}^2).$$

One may also consult the detailed proof in [68, Proposition 4.10].

Proposition 7 *(Estimate of the Bernstein–Bando–Shi type) Let* $q \in \mathbb{N}$ *and* $\gamma \in \mathfrak{B}$. *Suppose* (∇_t, u_t) *is a solution to the k-flow (1.6) defined on* $M \times I$. *Let* $Q = \max\{1, \sup_{t\in I} |F_{\nabla_t}|\}$, $K = \max\{1, \sup_{t\in I} |u_t|\}$, *and choose* $s \geq (k+1)(q+1)$. *Then for* $t \in [0, T) \subset I$ *with* $T < \frac{1}{Q^4 K^{4(\deg W - 1)}}$, *there exists constant* $C_q := C_q(M, E, q, k, s, g, \gamma) \in \mathbb{R}_{>0}$ *such that*

$$\|\gamma^s \nabla_t^{(q)} F_{\nabla_t}\|_{L^2}^2 + \|\gamma^s \nabla_t^{(q)} u_t\|_{L^2}^2 \leq C_q t^{-\frac{q}{k+1}} \sup_{t\in[0,T)} (\|F_{\nabla_t}\|_{L^2,\gamma>0}^2 + \|u_t\|_{L^2,\gamma>0}^2). \quad (3.8)$$

The following corollary, a direct consequence of inequality (3.8), is pivotal to our blow-up analysis. Its proof uses the embedding $W^{p,2} \subset C^0$ for $2p > n$, and then refines the argument via Kato's inequality $|d|u_t|| \leq |\nabla_t u_t|$. For a more exhaustive exposition, one may refer to Kelleher's work ([36, Corollary 3.14]).

Corollary 1 *Suppose* (∇_t, u_t) *is a solution to the k-flow (1.6) defined on* $M \times [0, \tau]$. *Denoted by* $\bar{\tau} := \min\{\tau, 1\}$. *As usual, let* $Q = \max\{1, \sup_{t\in[0,\bar{\tau}]} |F_{\nabla_t}|\}$ *and* $K = \max\{1, \sup_{t\in[0,\bar{\tau}]} |u_t|\}$. *For* $s, l \in \mathbb{N}$ *with* $s \geq (k+1)(l+1)$ *there exists constant* $C_l := C_l(M, E, K, Q, s, k, l, \tau, g, \gamma) \in \mathbb{R}_{>0}$ *such that*

$$\sup_M \left(|\gamma^s \nabla_{\bar{\tau}}^{(l)} F_{\nabla_{\bar{\tau}}}|^2 + |\gamma^s \nabla_{\bar{\tau}}^{(l)} u_{\bar{\tau}}|^2 \right) \leq C_l \sup_{M\times[0,\bar{\tau})} \left(\|F_{\nabla_t}\|_{L^2,\gamma>0}^2 + \|u_t\|_{L^2,\gamma>0}^2 \right).$$

Remark 1 This corollary has no dependency on the initial value (∇_0, u_0), this allows us to extend the time interval for which the estimation is valid.

Using Corollary 1, we have the the following corollary, which can be used for finding obstructions to long time existence.

Corollary 2 *Suppose* (∇_t, u_t) *is a solution to the k-flow (1.6) defined on* $M \times [0, T)$ *for* $T \in [0, +\infty)$. *Let* $Q = \max\{1, \sup_{t\in[0,T)} |F_{\nabla_t}|, \sup_{t\in[0,T)} \|F_{\nabla_t}\|_{L^2}\}$ *and* $K = \max\{1, \sup_{t\in[0,T)} |u_t|, \sup_{t\in[0,T)} \|u_t\|_{L^2}\}$ *be finite. Then for any* $t \in [0, T)$, *and* $s, l \in \mathbb{N}$ *with* $s \geq (k+1)(l+1)$, *there exists constant* $C_l := C_l(\nabla_0, u_0, M, E, K, Q, s, k, l, g, \gamma) \in \mathbb{R}_{>0}$ *such that*

$$\sup_{M\times[0,T)}\left(|\gamma^s\nabla_t^{(l)}F_{\nabla_t}|^2+|\gamma^s\nabla_t^{(l)}u_t|^2\right)\le C_l.$$

3.4 Long-time Existence Obstruction

In this section, we will use Corollary 2 to show that the only obstruction to long-time existence of the Yang–Mills–Higgs *k*-flow (1.6) is a lack of supremal bound on $|F_{\nabla_t}|+|u_t|$. Now, We are ready to prove the main result in this subsection.

Theorem 3 *Suppose* (∇_t,u_t) *is a solution to the k-flow (1.6) for some maximal time* $T<+\infty$. *Then we have*

$$\sup_{M\times[0,T)}(|F_{\nabla_t}|+|u_t|)=+\infty.$$

Proof Assuming the contrary, let us posit that

$$\sup_{M\times[0,T)}(|F_{\nabla_t}|+|u_t|)<+\infty.$$

Invoking Corollary 2, we assert that for all $t\in[0,T)$ and $l\in\mathbb{N}\cup\{0\}$, the quantity

$$\sup_M\left(|\nabla_t^{(l)}F_{\nabla_t}|^2+|\nabla_t^{(l)}u_t|^2\right)$$

remains uniformly bounded. In the sequel, we shall demonstrate that the limit $\lim_{t\to T}(\nabla_t,u_t)=(\nabla_T,u_T)$ exists and furthermore, is smooth.

At first, we define

$$\Upsilon_s:=\int_0^s\frac{\partial\nabla_t}{\partial t}dt,\quad \Psi_s:=\int_0^s\frac{\partial u_t}{\partial t}dt.$$

It is straightforward to verify, for all $s\le T$, that

$$|\Upsilon_s|=\left|\int_0^s\frac{\partial\nabla_t}{\partial t}\,dt\right|\le TC_0,\quad \text{and}\quad |\Psi_s|=\left|\int_0^s\frac{\partial u_t}{\partial t}\,dt\right|\le TC_0,$$

implying the continuity of (∇_T,u_T).

Next, we meticulously establish the smoothness of (∇_T,u_T) through an inductive argument on *l*, where the base of our induction relies on the finiteness of $|\nabla_0^{(l)}[\Psi_T]|+|\nabla_0^{(l)}[\Upsilon_T]|$. Commencing with the foundational case, we derive:

$$|\nabla_0[\Psi_s]|=\left|\int_0^s\nabla_0\left[\frac{\partial u_t}{\partial t}\right]\,dt\right|\le\int_0^s\left(|\nabla_t\left[\frac{\partial u_t}{\partial t}\right]|+C|\Upsilon_t||\frac{\partial u_t}{\partial t}|\right)\,dt<+\infty.$$

Analogously, we confirm that

$$|\nabla_0[\Upsilon_s]| < +\infty.$$

Assuming the induction hypothesis holds true for $\{1, \ldots, l-1\}$, we proceed by elegantly expanding $\nabla_0^{(l)}[\Psi_s]$ and invoking [36, Lemma 3.17], followed by our assumption, to derive:

$$|\nabla_0^{(l)}[\Psi_s]| = \int_0^s \left(|\nabla_t^{(l)} \left[\frac{\partial u_t}{\partial t} \right] | + \sum_{j=0}^{l-1} \sum_{i=0}^{j} \left(P_{l-i-1}^{(i)}[\Upsilon_t] * P_1^{(j-i)} \left[\frac{\partial u_t}{\partial t} \right] \right) \right) dt < +\infty.$$

In a parallel fashion, we have

$$|\nabla_0^{(l)}[\Upsilon_s]| < +\infty.$$

Given that these bounds are uniform across all $t \in [0, T)$ and considering the continuity of Υ_s and Ψ_s, we conclude that

$$|\nabla_0^{(l)}[\Upsilon_T]| + |\nabla_0^{(l)}[\Psi_T]| < +\infty.$$

Hence, Υ_T and Ψ_T are indeed smooth.

By the local existence theorem (Theorem 2), there exists an $\epsilon > 0$ such that the solution (∇_t, u_t) can be extended to the interval $[0, T + \epsilon)$, contradicting the assumption that *T* was the maximal existence time.

□

4 Long-time Existence

4.1 Blow-up Analysis

The aim of this subsection is to derive L^∞-estimates from L^p-estimates by blow-up analysis.

We first discuss the possibility that the Yang–Mills–Higgs *k*-flow (1.6) admits a singularity given no bound on $|F_{\nabla_t}| + |u_t|$. As an initial step, we meticulously establish a series of fundamental scaling principles governing the dynamics of the *k*-flow (1.6), laying a solid foundation for our subsequent investigations.

Proposition 8 *Let (∇_t, u_t) be a solution to the k-flow (1.6) on the domain $M \times [0, T)$. We introduce a one-parameter family of connections, denoted as ∇_t^ρ, whose local coefficient matrices are crafted as*

$$\Gamma_t^\rho(x) := \rho \Gamma_{\rho^{2(k+1)}t}(\rho x),$$

where $\Gamma_t(x)$ embodies the local coefficient matrix of ∇_t. Furthermore, we define the ρ-scaled Higgs field u_t^ρ in an analogous fashion:

$$u_t^\rho(x) := \rho u_{\rho^{2(k+1)}t}(\rho x).$$

The pair $(\nabla_t^\rho, u_t^\rho)$ *then yields a solution to the corresponding rescaled system:*

$$\begin{cases} \frac{\partial \nabla_t^\rho}{\partial t} &= (-1)^{(k+1)} D^*_{\nabla_t^\rho} \Delta^{(k)}_{\nabla_t^\rho} F_{\nabla_t^\rho} + \sum_{v=0}^{2k-1} P_1^{(v)}[F_{\nabla_t^\rho}] \\ & \quad + P_2^{(2k-1)}[F_{\nabla_t^\rho}] + \sum_{i=0}^{k} \nabla_t^{\rho*(i)}(\nabla_t^{(k+1)} u_t^\rho * \nabla_t^{\rho(k-i)} u_t^\rho), \\ \frac{\partial u_t^\rho}{\partial t} &= -\nabla_t^{\rho,*(k+1)} \nabla_t^{\rho(k+1)} u_t^\rho - \widetilde{W}_\rho(|u_t^\rho|^2) u_t^\rho \end{cases} \tag{4.1}$$

on $[0, \frac{1}{\rho^{2(k+1)}}T)$, *where* $\widetilde{W}_\rho$ *is a polynomial function with* $\deg \widetilde{W}_\rho \leq k+1$.

Proof Using the flow equation (1.6), the proof comes from

$$\frac{\partial \nabla_t^\rho}{\partial t}(x,t) = \rho^{2k+3} \frac{\partial \nabla_t}{\partial t}(\rho x, \rho^{2(k+1)} t)$$

and

$$\frac{\partial u_t^\rho}{\partial t}(x,t) = \rho^{2k+3} \frac{\partial u_t}{\partial t}(\rho x, \rho^{2(k+1)} t).$$

□

Remark 2 The property $\deg \widetilde{W}_\rho \leq k+1$ originates from the restriction condition $\deg W \leq k+2$.

Next, we prove that when the curvature coupled with the Higgs field blows up near the maximal time, a blow-up limit can be extracted. The proof follows the reasoning in [36, Proposition 3.25] and [68, Theorem 5.2].

Proposition 9 *Let* (∇_t, u_t) *be a solution to the k-flow (1.6) defined on some maximal time interval [0, T) with time* $T < +\infty$. *Then there exists a blow-up sequence* (∇_t^i, u_t^i) *that converges point-wise to a smooth solution* $(\nabla_t^\infty, u_t^\infty)$ *to the scaled system (4.1) defined on the domain* $\mathbb{R}^n \times \mathbb{R}_{<0}$.

Proof For readers' convenience, we briefly sketch the proof here. From Theorem 3, we must have

$$\lim_{t \to T} \sup_M \left(|F_{\nabla_t}| + \langle u_t, u_t \rangle \right) = +\infty.$$

Thus, we can select a sequence of times t_i converging to *T* within the interval [0, *T*), alongside a corresponding sequence of points x_i, that satisfy the condition:

$$|F_{\nabla_{t_i}}(x_i)| + \langle u_{t_i}(x_i), u_{t_i}(x_i)\rangle = \sup_{M\times[0,t_i]} \left(|F_{\nabla_t}| + \langle u_t, u_t\rangle\right).$$

Subsequently, let $\{\rho_i\}$ be a sequence of positive real numbers whose values are yet to be determined. We introduce the scaled connection $\nabla^i_t(x)$ and vector field $u^i_t(x)$ as follows. For the scaled connection, we define

$$\Gamma^i_t(x) = \rho_i^{\frac{1}{2(k+1)}} \Gamma_{\rho_i t + t_i} \left(\rho_i^{\frac{1}{2(k+1)}} x + x_i\right),$$

and for the scaled vector field, we have

$$u^i_t(x) = \rho_i^{\frac{1}{2(k+1)}} u_{\rho_i t + t_i} \left(\rho_i^{\frac{1}{2(k+1)}} x + x_i\right).$$

According to Proposition 8, the scaled pair (∇^i_t, u^i_t) also serves as solutions to the scaled system given in (4.1), with their respective domains being $B_o(\rho_i^{-\frac{1}{2(k+1)}}) \times [-\frac{t_i}{\rho_i}, \frac{T-t_i}{\rho_i})$. Notably, we have:

$$F^i_t(x) := F_{\nabla^i_t}(x) = \rho_i^{\frac{1}{k+1}} F_{\nabla_{\rho_i t + t_i}}(\rho_i^{\frac{1}{2(k+1)}} x + x_i),$$

which implies the following:

$$\begin{aligned}
&\sup_{t\in[-\frac{t_i}{\rho_i}, \frac{T-t_i}{\rho_i})} \left(|F^i_t(x)| + |u^i_t(x)|^2\right) \\
&= \rho_i^{\frac{1}{k+1}} \sup_{t\in[-\frac{t_i}{\rho_i}, \frac{T-t_i}{\rho_i})} \left(|F_{\nabla_{\rho_i t + t_i}}(\rho_i^{\frac{1}{2(k+1)}} x + x_i)| + |u_{\rho_i t + t_i}(\rho_i^{\frac{1}{2(k+1)}} x + x_i)|^2\right) \\
&= \rho_i^{\frac{1}{k+1}} \sup_{t\in[0,t_i]} \left(|F_{\nabla_t}(x)| + |u_t(x)|^2\right) \\
&= \rho_i^{\frac{1}{k+1}} \left(|F_{\nabla_{t_i}}(x_i)| + |u_{t_i}(x_i)|^2\right).
\end{aligned}$$

Consequently, by choosing:

$$\rho_i = \left(|F_{\nabla_{t_i}}(x_i)| + |u_{t_i}(x_i)|^2\right)^{-(k+1)},$$

we arrive at:

$$1 = |F^i_0(o)| + |u^i_0(o)|^2 = \sup_{t\in[-\frac{t_i}{\rho_i}, 0]} \left(|F^i_t(x)| + |u^i_t(x)|^2\right). \tag{4.2}$$

Now, we proceed to construct smoothing estimates for the sequence (∇^i_t, u^i_t). Given $y \in \mathbb{R}^n$ and $\tau \in \mathbb{R}_{\leq 0}$, for any $s \in \mathbb{N}$, we have the following inequality holding for all $t \in [\tau - 1, \tau]$:

$$\sup_{t\in[\tau-1,\tau]}\left(|\gamma_y^s F_t^i(x)| + |\gamma_y^s u_t^i(x)|^2\right) \le 1.$$

Given that the scaled system (4.1) exhibits similarities to the Yang–Mills–Higgs k-flow (1.6), it is straightforward to derive analogous estimates of the Bernstein–Bando–Shi type. According to Corollary 1, for any $q \in \mathbb{N}$, one can select $s \ge (k+1)(q+1)$ to ensure the existence of a positive constant C_q such that:

$$\begin{aligned}
&\sup_{x\in B_y(\frac{1}{2})}\Big(|(\nabla_\tau^i)^{(q)} F_\tau^i(x)| + |(\nabla_\tau^i)^{(q)} u_\tau^i(x)|\Big)\\
&\le \sup_{x\in B_y(1)}\Big(|\gamma_y^s(\nabla_\tau^i)^{(q)} F_\tau^i(x)| + |\gamma_y^s(\nabla_\tau^i)^{(q)} u_\tau^i(x)|\Big)\\
&\le C_q.
\end{aligned}$$

Applying Uhlenbeck's Coulomb Gauge Theorem [62, Theorem 1.3] (see also [32]) and the Gauge Patching Theorem [20, Corollary 4.4.8], we may select a subsequence (still denoted (∇_t^i, u_t^i)), apply a suitable gauge transformation, and obtain convergence $(\nabla_t^i, u_t^i) \to (\nabla_t^\infty, u_t^\infty)$ in the C^∞ topology.

□

By Proposition 9 and a contradiction argument, we show that the L^p-norm controls the L^∞-norm, following the method used in [36, Section 4] and [68, Section 7].

Proposition 10 *Let (∇_t, u_t) be a solution to the k-flow (1.6) defined on $M^4 \times [0,T)$ and*

$$\sup_{t\in[0,T)}(\|F_{\nabla_t}\|_{L^p} + \|u_t\|_{L^p}) < +\infty.$$

If $p > 2$, then we have

$$\sup_{t\in[0,T)}(\|F_{\nabla_t}\|_{L^\infty} + \|u_t\|_{L^\infty}) < +\infty.$$

Proof To prove the theorem by contradiction, we assume that

$$\sup_{t\in[0,T)}(\|F_{\nabla_t}\|_{L^\infty} + \|\langle u_t, u_t\rangle\|_{L^\infty}) = +\infty.$$

Following the method in Proposition 9, we construct a blow-up sequence (∇_t^i, u_t^i) and its blow-up limit $(\nabla_t^\infty, u_t^\infty)$. Using the model equation (4.2), Fatou's lemma, and the natural scaling invariance, we obtain the following inequalities:

$$\begin{aligned}\|F_{\nabla_t^\infty}\|_{L^p}^p + \|\langle u_t^\infty, u_t^\infty\rangle\|_{L^p}^p &\leq \lim_{i\to+\infty} \inf(\|F_{\nabla_t^i}\|_{L^p}^p + \|\langle u_t^i, u_t^i\rangle\|_{L^p}^p) \\ &\leq \lim_{i\to+\infty} \rho_i^{\frac{2p-n}{2k+2}}(\|F_{\nabla_t}\|_{L^p}^p + \|\langle u_t, u_t\rangle\|_{L^p}^p).\end{aligned}$$

Given that

$$\lim_{i\to+\infty} \rho_i^{\frac{2p-n}{2k+2}} = 0$$

when $2p > n$, the right-hand side of the aforementioned inequality converges to zero. However, this contradicts the fact that the blow-up limit possesses non-vanishing curvature.

□

4.2 Energy Estimates

At first, the following proposition is obvious.

Proposition 11 *Let (∇_t, u_t) be a solution to the k-flow (1.6) defined on $M^n \times [0, T)$. The Yang–Mills–Higgs k-energy (1.3) is decreasing along the flow (1.6).*

Proof We complete the proof as follows:

$$\frac{\partial}{\partial t}\mathcal{YMH}_k(\nabla_t, u_t) = -\left(\|\frac{\partial \nabla_t}{\partial t}\|_{L^2}^2 + \|\frac{\partial u_t}{\partial t}\|_{L^2}^2\right) \leq 0.$$

□

From Proposition 11, we conclude that $\|u_t\|_{L^2}$ is bounded, since the Higgs-like potential *W* is polynomial and positive. We also need the following proposition.

Proposition 12 *Let (∇_t, u_t) be a solution to the k-flow (1.6) defined on $M^n \times [0, T)$ with $T < +\infty$, then the Yang–Mills energy*

$$\mathrm{YM}(\nabla_t) = \frac{1}{2}\int_M |F_{\nabla_t}|^2 \, d\mathrm{vol}_g$$

is bounded along the flow (1.6).

Proof By direct calculations, we have

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{YM}(\nabla_t) &= \int_M \langle D^*_{\nabla_t}F_{\nabla_t}, \frac{\partial \nabla_t}{\partial t}\rangle \\ &\leq \int_M \Big[|\frac{\partial \nabla_t}{\partial t}|^2 + C|\nabla_t F_{\nabla_t}|^2\Big] \\ &\leq -\frac{\partial}{\partial t}\mathcal{YMH}_k(\nabla_t, u_t) + C\|\nabla_t^{(k)}F_{\nabla_t}\|^2_{L^2} + \varepsilon\|F_{\nabla_t}\|^2_{L^2},\end{aligned}$$

where we used Lemma 2 and C is a constant independent of $t \in [0, T)$. Therefore, we have

$$\mathrm{YM}(\nabla_t) - \mathrm{YM}(\nabla_0) \leq CT\mathcal{YMH}_k(\nabla_0, u_0) + \varepsilon T \sup_{t\in[0,T)} \mathrm{YM}(\nabla_t). \tag{4.3}$$

Subsequently, we will adopt an argument from Saratchandran's paper [53, Theorem 5.3]. We assume the existence of a sequence t_m converging to T such that

$$\lim_{m\to+\infty} \mathrm{YM}(\nabla_{t_m}) \to +\infty.$$

By discarding certain elements from the sequence t_m, we can make the assumption that $\mathrm{YM}(\nabla_{t_m}) > \mathrm{YM}(\nabla_{t_{m'}})$ holds for all $m \geq m'$, and furthermore, $t_m \geq t_{m'}$ whenever $m \geq m'$. Divide the interval [0, T) into partitions $[t_0, t_1] \cup [t_1, t_2] \cup \cdots \cup [t_k, t_{k+1}] \cup \cdots$ with the initial point $t_0 = 0$. Let $s_i \in [t_i, t_{i+1}]$ be defined such that $\sup_{t\in[t_i,t_{i+1}]} \mathrm{YM}(\nabla_t) = \mathrm{YM}(\nabla_{s_i})$. It is straightforward to observe that $s_i \to T$ and $\mathrm{YM}(\nabla_{s_i}) \to +\infty$ as $i \to +\infty$. Moreover, $\mathrm{YM}(\nabla_{s_j}) \leq \mathrm{YM}(\nabla_{s_i})$ whenever $j \leq i$. Then, by substituting s_i for t in (4.3), we have

$$\mathrm{YM}(\nabla_{s_i}) - \mathrm{YM}(\nabla_0) - \varepsilon T\mathrm{YM}(\nabla_{s_i}) \leq CT\mathcal{YMH}_k(\nabla_0, u_0),$$

which means

$$\mathrm{YM}(\nabla_{s_i}) \leq \frac{1}{1-\varepsilon T}CT\Big(\mathcal{YMH}_k(\nabla_0, u_0) + \mathrm{YM}(\nabla_0)\Big).$$

The right-hand side of the aforementioned inequality is finite and remains independent of i. Upon taking the limit $i \to +\infty$ on the left, we arrive at a contradiction. Therefore, no such sequence $\{t_m\}$ exists, and the desired result is established.

□

4.3 Proof of Theorem 1

Since $\dim(M) = n$, applying Proposition 10 requires $2p > n$. By the Sobolev embedding theorem, we choose p so that $W^{k,2} \subset L^{2p}$, which is equivalent to $2(k+1) > n$. Under this condition, the interpolation inequalities (Lemma 2) yield the following estimate::

$$\begin{aligned}\|F_{\nabla_t}\|_{L^p} + \|\langle u_t, u_t\rangle\|_{L^p} &\le CS_{k,p}\sum_{j=0}^{k}(\|\nabla_t^{(j)}F_{\nabla_t}\|_{L^2}^2 + \|\nabla_t^{(j)}u_t\|_{L^2}^2 + 1)\\ &\le CS_{k,p}(\|\nabla_t^{(k)}F_{\nabla_t}\|_{L^2}^2 + \|F_{\nabla_t}\|_{L^2}^2 + \|\nabla_t^{(k+1)}u_t\|_{L^2}^2 + \|u_t\|_{L^2}^2 + 1)\\ &\le CS_{k,p}(\mathcal{YMH}_k(\nabla_t, u_t) + \mathrm{YM}(\nabla_t) + \|u_t\|_{L^2}^2 + 1).\end{aligned}$$

Since both $\mathcal{YMH}_k(\nabla_t, u_t)$ and $\mathrm{YM}(\nabla_t)$ remain bounded along the *k*-flow (1.6) (see Proposition 11 and Proposition 12), it follows that the flow admits a smooth long-time solution.

5 Applications

The core motivation of this research also stems from fundamental questions in theoretical physics. Below, we elaborate on the potential applications of the constructed higher-order Yang–Mills–Higgs functional and the long-time existence theorem for its associated heat flow to concrete physical scenarios, with a focus on explaining "why higher-order terms are needed" and "why a polynomial potential is needed", supported by specific mathematical models.

5.1 Application I: Higher-order Gauge Theory and Stability of Topological Solitons

In effective theories describing physics beyond the Standard Model or certain condensed matter systems, higher-order derivative terms are crucial for stabilizing specific types of topological solitons (e.g., instantons, magnetic monopoles, skyrmions).

1. Physical motivation and mathematical model

According to Derrick's scaling analysis [17], field theories containing only standard kinetic terms often cannot support stable finite-energy topological soliton solutions because the energy tends to collapse under scaling transformations. This limitation motivates the introduction of higher-order derivative terms, which provide the necessary "stiffness" to resist collapse.

Classic examples include the Skyrme model [56], where an introduced fourth-order term stabilizes three-dimensional skyrmions, and Born-Infeld-type corrections or higher-order curvature terms that naturally appear in the low-energy effective action of string theory [24]. The general form of such higher-order Yang-Mills actions can be written as:

$$\mathcal{L} = -\frac{1}{4}tr(F_{\mu\nu}F^{\mu\nu}) + \alpha\mathcal{O}(F^4) + \cdots,$$

where $\mathcal{O}(F^4)$ represents terms like $tr(F_{\mu\nu}F^{\mu\nu})^2$ or $tr(F_{\mu\nu}F^{\nu\tau}F_{\tau\sigma}F^{\sigma\mu})$. These terms alter the scaling behavior of the action, potentially allowing the energy functional to attain a minimum at soliton configurations.

For the Higgs part, higher-order covariant derivative terms such as $|D\phi|^2$ also play a role in stabilizing topological structures like domain walls [49]. The most general form of the higher-order polynomial functional addressed in this study provides a unified mathematical framework for this broad class of physical models.

2. Physical significance of our results

The proven long-time existence of solutions to the higher-order Yang–Mills–Higgs heat flow provides a rigorous mathematical foundation for analyzing the dynamical stability of soliton solutions in such theories. The gradient flow method has been widely used to find energy minima (solitons) and study their stability [42]. Our theorem demonstrates that even for systems containing the most general higher-order polynomial terms, the corresponding dissipative evolution process is mathematically well-posed.

Specifically, when considering the heat flow evolution equation for solutions like instantons:

$$\frac{\partial \nabla}{\partial t} = -D_{\nabla}^{*} F_{\nabla} + \text{contributions from higher order terms.}$$

the existence of long-time smooth solutions guarantees that starting from arbitrary initial conditions, the field configuration can relax smoothly without developing singularities in finite time. This property is crucial for reliably simulating the evolution of topological defects (e.g., cosmic strings) in the early universe [63] and the dynamical behavior of solitons under interactions [42], providing a solid theoretical basis for related numerical simulations.

5.2 Application II: Generalized Higgs Potentials and Cosmological Phase Transition Dynamics

While the Standard Model Higgs potential is simple in form, at the extremely high energy scales of the early universe or in new physics models, effective potentials often exhibit richer structures, frequently taking polynomial forms.

1. Physical motivation and mathematical model

In early universe phase transitions, such as the electroweak phase transition or possible grand unified phase transitions, finite-temperature effects significantly modify the effective potential [18]. For example, one-loop thermal corrections can yield an effective potential of the form:

$$V_T(\phi) = D(T^2 - t_0^2)\phi^2 - ET\phi^3 + \frac{\lambda_T}{4}\phi^4.$$

Furthermore, in many Beyond the Standard Model (BSM) theories, such as multi-Higgs-doublet models [9] or scalar field dark matter models [43], the tree-level potential may include sextic or higher-order terms:

$$V(\phi) = m^2\phi^2 + \lambda\phi^4 + \kappa\phi^6.$$

Such higher-order polynomial potentials can lead to rich vacuum structures, such as multiple local minima, potentially triggering first-order phase transitions or creating metastable vacua [66].

2. Physical significance of our results

The proven long-time existence of solutions to the higher-order Yang–Mills–Higgs heat flow with a generalized polynomial Higgs potential provides a mathematical tool for studying the dynamics of cosmological phase transitions under such complex potentials.

In the study of early universe phase transitions, a core issue involves false vacuum decay, and the nucleation, growth, and collision of vacuum bubbles, processes closely linked to gravitational wave production [10]. Classical field equations (often including dissipation terms to model coupling to a heat bath) are used to simulate this process. Our heat flow equation corresponds precisely to such dissipative evolution in Euclidean spacetime.

The existence of long-time solutions has multiple important implications:

(1) Ensuring Feasibility of Simulations: It guarantees that during numerical simulations of vacuum bubble dynamics [66] and topological defect formation, the evolution equations do not develop singularities in finite time, allowing us to reliably track the complete process from the onset of the phase transition to the final stable vacuum.

(2) Understanding the Phase Transition Endpoint: The long-time behavior of the heat flow determines into which vacuum state the system ultimately settles. For complex potentials with multiple (nearly) degenerate vacua, our theorem mathematically ensures the smoothness of the evolution path and the existence of a final state.

(3) Self-Consistent Treatment of Coupling Effects: In phase transitions involving the coupling of gauge and Higgs fields (such as the electroweak phase transition), gauge fields significantly influence bubble wall dynamics and baryogenesis [21]. Our model simultaneously incorporates higher-order and polynomial interactions for both gauge and Higgs fields, thus providing a self-consistent and mathematically rigorous framework for studying these coupling effects.

6 Further Studies

6.1 Generalization in Higgs Fields

We assume the Higgs field takes values in the section space of a vector bundle, $\Omega^0(E)$. A significant generalization is to consider Higgs fields valued in the adjoint bundle $\Omega^0(\mathrm{ad}P)$, which aligns with the typical setup in gauge theory. We anticipate that techniques from literature such as [54] can be employed to handle the nonlinearities arising from the adjoint representation.

6.2 Extensions and Hybridizations of the Functional Model

Our approach is likely applicable to studying higher-order Seiberg–Witten-like functionals combined with a Higgs-type potential, of the form:

$$\mathcal{SW}^k(A,\phi) = \int_M \left(\frac{1}{2}|\nabla_M^{(k)} F_A|^2 + |\nabla_A^{(k+1)}\phi|^2 + W(|\phi|^2)\right) + \pi^2 c_1(L),$$

where A is the connection and ϕ is the spinor field. If $W(x) = \frac{1}{8}x^4 + \frac{S}{4}x$, the above functional reduces to the functional considered in [53]. Investigating the gradient flow of such functionals and their relationship with topological invariants presents an interesting direction.

6.3 Higgs Potentials with Negative or More General Polynomials

In physical models, such as the boson-coupled Yang–Mills energy functional [23, 48],

$$\mathrm{YM}(\nabla,\phi) = \int_M (|F_\nabla|^2 + |\nabla\phi|^2 - m|\phi|^2 - s|\phi|^4),$$

where the Higgs filed $\phi \in \Omega^0(E)$ and *m*, *s* are smooth real-valued functions on the base manifold. The corresponding Higgs-like potential $W(x) = -sx^2 - mx$ does not satisfy $\{\sigma \in \mathbb{R}^+ : W(\sigma) = W'(\sigma) = 0\} \neq \varnothing$ assumed in this paper. Notably, our proof for the long-time existence of the flow does not rely on this condition. Therefore, studying higher-order Yang–Mills functionals coupled with such negatively-signed or more general polynomial Higgs potentials is both theoretically and physically significant.

6.4 Investigations in Complex Geometric Settings

On Kähler manifolds, the study of the classical Yang–Mills flow is rich, deeply connected to the existence of Hermitian Yang–Mills connections on stable bundles via the Hitchin–Kobayashi correspondence [19, 58]. This seminal result has been generalized in various complex geometric contexts [3, 8, 12, 28, 40, 44, 53, 67, 69]. For unstable bundles, the behavior of the Yang–Mills flow has also been extensively studied [15, 38, 45, 57]. Consequently, a natural direction is to investigate the behavior of the (higher-order) Yang–Mills–Higgs functional and its negative gradient flow on Kähler manifolds and, more generally, on Hermitian manifolds. This involves exploring their generalizations in complex geometry and potential new correspondences.

6.5 Hyperbolic Equations and Spacetime Backgrounds

Studying gauge field equations in dynamical settings is another active frontier. Glogic [25] and Oh–Tataru [46, 47] have done excellent work on hyperbolic Yang–Mills equations. Inspired by this, one may consider studying higher-order Yang–Mills equations in $(1+d)$-dimensional Minkowski spacetime. Such research connects geometric flow methods with PDE theory in spacetime backgrounds and may reveal distinct behaviors of solutions in spacelike and timelike directions, touching upon deep issues like energy criticality.

Author Contributions Writing–original draft: X.-M.C.; Writing–review and editing: M.A.L, W.W., M.-J.Z.; Funding acquisition: P.Z.; Methodology: P.Z. All authors reviewed the manuscript.

Funding This paper is supported by the National Natural Science Foundation of China [Grant Numbers 12201001 and 12571061]and the Excellent University Research and Innovation Team in Anhui Province [Grant Number 2024AH010002]. This paper is also supported by 2025 Funding Program for Domestic Visiting Study and Research of Young Key Teachers in Anhui Province (JNFX2025041).

Data Availability No datasets were generated or analysed during the current study.

Declarations

Competing interests The authors declare no competing interests.

References

1. Afuni, A.: Local monotonicity for the Yang–Mills–Higgs flow. Calc. Var. **55**, article number 13 (2016)
2. Afuni, A.: Regularity and vanishing theorems for Yang-Mills-Higgs pair. Arch. Math. **112**, 547–558 (2019)
3. Álvarez-Cónsul, L., García-Prada, O.: Hitchin-Kobayashi correspondence, quivers, and vortices. Commun. Math. Phys. **238**, 1–33 (2003)
4. Atiyah, M.F., Bott, R.: The Yang-Mills equations over Riemann surfaces. Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **308**, 523–615 (1983)
5. Badran, M., del Pino, M.: Solutions of the Ginzburg-Landau equations concentrating on codimension 2 minimal submanifolds. J. London Math. Soc. **109**, 12851 (2024)
6. Bahuaud, E., Helliwell, D.: Short time existence for some higher order geometric flows. Commun. PDE **36**, 2189–2207 (2011)
7. Bethuel, F., Brezis, H., Hélein, F.: Asymptotics for the minimization of a Ginzburg-Landau functional. Calc. Var. PDE **1**, 123–148 (1993)
8. Biswas, I., Loftin, J., Stemmler, M.: Flat bundles on affine manifolds. Arabian J. Math. **2**, 159–175 (2013)
9. Branco, G.C., et al.: Theory and phenomenology of two-Higgs-doublet models. Phys. Rept. **516**, 1–102 (2012)
10. Caprini, C. et al.: Science with the space-based interferometer eLISA. II: Gravitational waves from cosmological phase transitions. *J. Cosmol. Astropart. Phys.***2016**(04), 001 (2016)
11. Chandra, A., Chevyrev, I., Hairer, M., Shen, H.: Stochastic quantisation of Yang-Mills-Higgs in 3D. Invent. Math. **237**, 541–696 (2024)
12. Charbonneau, B., Hurtubise, J.: Singular Hermitian-Einstein monopoles on the product of a circle and a Riemann surface. Inter. Math. Res. Not. **2011**, 175–216 (2011)
13. Chen, Z., Li, L.: Asymptotic behavior of non-autonomous random Ginzburg–Landau equations with colored noise on unbounded thin domains. Front. Math., 1-29 (2024)

14. Contreras, A., Jerrard, R.L.: Local minimizers with unbounded vorticity for the 2D Ginzburg-Landau functional. Commun. Pure Appl. Math. **75**, 1997–2032 (2022)
15. Daskalopoulos, G., Wentworth, R.: Convergence properties of the Yang-Mills flow on Kähler surfaces. J. Reine Angew. Math. **575**, 69–99 (2004)
16. De Giorgi, E.: Congetture riguardanti alcuni problemi di evoluzione, A celebration of John F. Nash. Jr. Duke Math. J. **81**, 255–268 (1996)
17. Derrick, G.H.: Comments on nonlinear wave equations as models for elementary particles. J. Math. Phys. **5**, 1252–1254 (1964)
18. Dolan, L., Jackiw, R.: Symmetry behavior at finite temperature. Phys. Rev. D **9**, 3320–3341 (1974)
19. Donaldson, S.K.: Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles. Proc. Lond. Math. Soc. **50**, 1–26 (1985)
20. Donaldson, S.K., Kronheimer, P.: The Geometry of Four-Manifolds. Oxford University Press (1990)
21. Dine, M., Leigh, R.G., Huet, P., Linde, A., Linde, D.A.: Towards the theory of the electroweak phase transition. Phys. Rev. D **46**, 550–571 (1992)
22. Ercolani, N.M., Sigal, I.M., Zhang, J.: Ginzburg-Landau equations on non-compact Riemann surfaces. J. Funct. Anal. **285**, 110074 (2023)
23. Feehan, P.M.N., Maridakis, M.: Łojasiewicz-Simon gradient inequalities for coupled Yang-Mills energy functionals. Mem. Amer. Math. Soc. **267**, 1302 (2020)
24. Fradkin, E.S., Tseytlin, A.A.: Nonlinear electrodynamics from quantized strings. Phys. Lett. B **163**, 123–130 (1985)
25. Glogic, I.: Stable blowup for the supercritical hyperbolic Yang-Mills equations. Adv. Math. **408**, 108633 (2022)
26. Guo, C.Y., Wang, C.Y., Xiang, C.L.: L^p-regularity for fourth order elliptic systems with antisymmetric potentials in higher dimensions. Cal. Var. PDE **62**, 31 (2023)
27. Guo, C.Y., Xiang, C.L.: Regularity of weak solutions to higher order elliptic systems in critical dimensions. Trans. Amer. Math. Soc. **374**, 3579–3602 (2021)
28. Goto, R.: Kobayashi–Hitchin correspondence of generalized holomorphic vector bundles over generalized Kähler manifolds of symplectic type. Inter. Math. Res. Not. 1496-1567 (2024)
29. Gover, A., Latini, E., Waldron, A., Zhang, Y.B.: Conformally compact and higher conformal Yang–Mills equations. arxiv:2311.11458 (2023)
30. Higgs, P.W.: Broken symmetries and the masses of gauge bosons. Phys. Rev. Lett. **13**, 508–509 (1964)
31. Hong, M.C.: Heat flow for the Yang-Mills-Higgs field and the Hermitian Yang-Mills-Higgs metric. Ann. Global Anal. Geom. **20**, 23–46 (2001)
32. Hong, M.C., Tian, G.: Asymptotical behavior of the Yang-Mills flow and singular Yang-Mills connections. Math. Ann. **330**, 441–472 (2004)
33. Hong, M.C., Tian, G., Yin, H.: The Yang-Mills α-flow in vector bundles over four manifold and its application. Comm. Math. Helv. **90**, 75–120 (2015)
34. Han, X.L., Jin, X.S., Wen, Y.: Stability and energy identity for Yang-Mills-Higgs pairs. J. Math. Phys. **64**, 021511 (2023)
35. Jia, Z., Wang, Y.: Higher-order geometric flow of hypersurfaces in a Riemannian manifold. Inter. J. Math. **30**, 1940005 (2019)
36. Kelleher, C.: Higher order Yang-Mills flow. Calc. Var. PDE **100**, 100 (2019)
37. Kuwert, E., Schätzle, R.: Gradient flow for the Willmore functional. Comm. Anal. Geom. **10**, 307–339 (2002)
38. Li, J.Y., Zhang, C., Zhang, X.: The limit of the Hermitian-Yang-Mills flow on reflexive sheaves. Adv. Math. **325**, 165–214 (2018)
39. Lübke, M., Teleman, A.: The Kobayashi-Hitchin correspondence. World Scientific, Singapore (1995)
40. Lübke, M., Teleman, A.: The universal Kobayashi-Hitchin correspondence on Hermitian manifolds. Mem. Amer. Math. Soc (2006)
41. Mantegazza, C.: Smooth geometric evolutions of hypersurfaces. Geom. Funct. Anal. **12**, 138–182 (2002)
42. Manton, N.S., Sutcliffe, P.: Topological solitons. Cambridge University Press, Cambridge (2004)
43. McDonald, J.: Gauge singlet scalars as cold dark matter. Phys. Rev. D **50**, 3637–3649 (1994)
44. Mochizuki, T.: Kobayashi-Hitchin correspondence for analytically stable bundles. Trans. Amer. Math. Soc. **373**, 551–596 (2020)
45. Nie, Y., Zhang, X.: The limiting behaviour of the Hermitian-Yang-Mills flow over compact non-Kähler manifolds. Sci China Math **63**, 1369–1390 (2020)

46. Oh, S.J., Tataru, D.: The hyperbolic Yang-Mills equation for connections in an arbitrary topological class. Comm. Math. Phys. **365**, 685–739 (2019)
47. Oh, S.J., Tataru, D.: The threshold conjecture for the energy critical hyperbolic Yang-Mills equation. Ann. Math. **194**, 393–473 (2021)
48. Parker, T.H.: Gauge theories on four-dimensional Riemannian manifolds. Comm. Math. Phys. **85**, 563–602 (1982)
49. Polyakov, A.M.: Quark confinement and topology of gauge theories. Nucl. Phys. B **120**, 429–458 (1977)
50. Poria, S., Dhiman. A.: Existence and uniqueness theorem for ODE: an overview. arXiv:1605.05317 (2016)
51. Råde, J.: On the Yang-Mills heat equation in two and three dimensions. J. Reine Angew. Math. **431**, 123–163 (1992)
52. Sá Earp, H.: G_2-instantons over asymptotically cylindrical manifolds. Geometry and Topology **19**, 61–111 (2015)
53. Saratchandran, H.: Higher order Seiberg-Witten functionals and their associated gradient flows. Manuscripta Math. **160**, 411–481 (2019)
54. Saratchandran, H., Zhang, J., Zhang, P.: A new higher order Yang-Mills-Higgs flow on Riemannian 4-manifolds. Bull. Austr. Math. Soc. **107**, 320–329 (2023)
55. Schlatter, A.E., Struwe, M., Tahvildar-Zadeh, A.S.: Global existence of the equivariant Yang-Mills heat flow in four space dimensions. Amer. J. Math. **120**, 117–128 (1998)
56. Skyrme, T.H.R.: A unified field theory of mesons and baryons. Nucl. Phys. **31**, 556–569 (1962)
57. Sibley, B.: Asymptotics of the Yang-Mills flow for holomorphic vector bundles over Kähler manifolds: the canonical structure of the limit. J. Reine Angew. Math. **706**, 123–191 (2015)
58. Simpson, C.T.: Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization. J. Amer. Math. Soc. **1**, 867–918 (1988)
59. Sire, Y., Wei, J., Zheng, Y.: Infinite time bubbling for the $SU(2)$ Yang–Mills heat flow on $\mathbb{R}^4$. arXiv:2208.13875 (2022)
60. Streets, J.: The gradient flow of $\int_M |Rm|^2$. J. Geom. Anal. **18**, 249–271 (2008)
61. Taubes, C.H.: A framework for Morse theory for the Yang-Mills functional. Invent. Math. **94**, 327–402 (1988)
62. Uhlenbeck, K.K.: Connections with L^p-bounds on curvature. Commun. Math. Phys. **83**, 31–42 (1982)
63. Vilenkin, A., Shellard, E.P.S.: Cosmic strings and other topological defects. Cambridge University Press, Cambridge (1994)
64. Waldron, A.: Long-time existence for Yang-Mills flow. Invent. Math. **217**, 1069–1147 (2019)
65. Wang, G.X.: Compactness theorems for Yang-Mills-Higgs fields. Inter. J. Math. **34**, 2350057 (2023)
66. Wainwright, C.L.: CosmoTransitions: Computing cosmological phase transition temperatures and bubble profiles with multiple fields. Comput. Phys. Commun. **183**, 2006–2013 (2012)
67. Wu, D., Zhang, X.: Hermitian–Einstein equations on noncompact manifolds. arxiv:2406.11449 (2024)
68. Zhang, P.: Gradient flows of higher order Yang-Mills-Higgs functionals. J. Austr. Math. Soc. **113**, 257–287 (2022)
69. Zhang, C., Zhang, P., Zhang, X.: Higgs bundles over non-compact Gauduchon manifolds. Trans. Amer. Math. Soc. **374**, 3735–3759 (2021)

Authors and Affiliations

Xian-Min Cao[1] · Mehraj Ahmad Lone[2] · Wen Wang[3] · Meng-Jie Zhang[1] · Pan Zhang[1]

✉ Pan Zhang
panzhang20100@ahu.edu.cn

Xian-Min Cao
a24201021@stu.ahu.edu.cn

Mehraj Ahmad Lone
mehrajlone@nitsri.ac.in

Wen Wang
wwen2014@mail.ustc.edu.cn

Meng-Jie Zhang
a125211036@stu.ahu.edu.cn

[1] School of Mathematical Sciences, Anhui University, Hefei 230601, People's Republic of China

[2] Department of Mathematics, National Institute of Technology Srinagar, Kashmir 190006, India

[3] School of Mathematics and Statistics, Hefei Normal University, Hefei 230601, China