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Accelerating Charge: Add-Ons to Rest Mass and Field Energy

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Abstract: We present—in the framework of classical theory—a self-consistent derivation scheme which produces equations for the calculation of add-ons to the full field energy and to the effective mass of a charge moving with acceleration, which may be practically used for analyses in various scenarios. The charge is treated as a quasi-point-like charge; this helps to resolve the complications of the "infinite" electromagnetic energy, which are avoided by the procedure of slightly "spreading" the charge. As a result, the concept of the size of the particle takes a straightforward physical interpretation. Indeed, it is within the charge spread, at scales smaller than Compton's length, where the quantum-field-mechanics approach to be applied. Beyond this region, no "infinite" tails of quantities accumulate. The seeming divergences of the integrals at the upper limits are not physical if one takes into account that the charge moves with acceleration only for a finite duration of time; every real physical process has its beginning and its end. The key focus of this paper is on the methodological aspects of the calculations.

Keywords: bremsstrahlung; radiation friction; field energy

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1. Introduction

A charged particle radiates when it moves (in a vacuum) with acceleration. When this acceleration is caused by a collision of charged particles, the radiation is called bremsstrahlung. When this acceleration is caused by an external magnetic field, the radiation is called the magnetic bremsstrahlung or, more commonly, the synchrotron radiation. Nowadays, the term "electromagnetic bremsstrahlung" is often used in the much wider context, regardless of the specific mechanism that causes the acceleration of the charged particle.

Quite a number of studies has been devoted to this effect; for some notable papers and reviews, see, e.g., [1–13]. Bremsstrahlung is used as a source of X-radiation production [14–17]. It appears in high-power laser–plasma interactions [18,19], is used for diagnostic purposes [20–29], and determines the dynamics of fast (relativistic) runaway electrons [30,31]. Bremsstrahlung is frequently observed and has been extensively studied in astrophysics (see, e.g., [4,32–37]). Bremsstrahlung plays a role in the physics of cosmic microwave background distortions [38,39], can be an essential electromagnetic radiation process for plasma around compact objects [40,41], and is responsible for high-energy gamma-radiation production in galaxy clusters [42–44] and stellar flares [29,45–47]. Within the realm of accelerating charges, electrons have received high attention. Accelerated electrons produce radio emission when they move in a magnetic field, especially near the magnetic poles of pulsars. These electrons are a source of synchrotron radiation when they are affected by external magnetic field (see, e.g., [48]). Accelerated electrons in high-energy environments, such as near the magnetic poles of pulsars or within the magnetic fields of magnetars, produce Roentgen radiation and are accelerated to extremely high energies (see, e.g., [49–51]). Bremsstrahlung is one of the primary mechanisms of radiative energy loss from plasma. The effect is mainly characterized by its differential cross section (DCS), whose relativistic expansion was derived in the seminal publications by Hans Bethe and Walter Heitler [52,53]. The inverse process, the so-called bremsstrahlung absorption, is one of the principal mechanisms of laser energy transfer in experiments of inertial confinement fusion (ICF) [54–56].

Given the unconditional importance of the issues related to this topic, in this paper, we address a significant methodological issue. In the process, the charged particle interacts with the field which it generates, and this interaction leads to the deceleration of the particle. Naturally, there should be no self-acceleration. Generally, one can immediately write the equations for the energy balance as follows:

$$\frac{dE_{\rm p}}{dt} = -P, \quad \frac{dW_{\rm f}}{dt} = P - \oint d\boldsymbol{\Sigma} \cdot \boldsymbol{S}\,, \tag{1}$$

where $E_{\rm p}$ is the full/kinetic energy of the charged particle (in the relativistic case, $E_{\rm p} = mc^2\gamma$, where m is the effective rest mass of the charged particle, c denotes the speed of light, and γ is the Lorentz factor); $W_{\rm f} = (8\pi)^{-1}\overline{(E^2 + H^2)}$ is the energy of the field surrounding the charge, with E and H the strengths of electric and magnetic fields, respectively, and the bar shows that integration is taken over the volume occupied by the field; the positively defined quantity P is the power of the force of radiation friction (in the first equation of the system (1), it leads to the particle's slowdown, whereas in the second equation, it leads to the increase in the magnitude of the electromagnetic field); and the surface integral on the right hand side (r.h.s.) of the second equation is an "outflow" (described by Poynting vector $\boldsymbol{S} = (c/4\pi)[\boldsymbol{E}, \boldsymbol{H}]$, where the square brackets denote the vector cross product) of the field energy through a surrounding surface $\boldsymbol{\Sigma}$ located at a large distance from the charge.

The sense of expressions (1) might seem trivial; however, as a result of the interaction of the charge with its accompanying field and with the radiated field, the charge's rest "effective mass" changes ($m \to m + \delta m$) and the field's energy density also changes ($W_{\rm f} \to W_{\rm f} + \delta W_{\rm f}$). The task of obtaining the explicit expressions for δm and $\delta W_{\rm f}$ is not trivial.

In this paper, we analytically give—in the framework of classical electrodynamics and relativity—the methodology and the derivations for δm and $\delta W_{\rm f}$ in the spirit of the founding paper by Paul Dirac [57] with details of the derivations; see [58–67] for a more in-depth examination.

This paper is organized as follows. Section 2 formulates the model. Section 3 shows how the energy–momentum tensors for the particle/charge and the field are constructed. Section 4 discusses the effect of the radiation friction force. Section 5 analyzes the structure of the field in the vicinity of the charge. Section 6 concludes with final remarks. Appendix A gives the details of the calculations.

2. Model

We consider a flat spacetime characterized by the Galilean metrics with the positive time-space signature $(+---)$ (in contrast to the often-used "negative" signature $(-+++)$; see, e.g., [61]). Each world-point is described by a vector pointing to a position/state with contra-variant components $q^i = (ct, q^\alpha)$, t is the time measured by a remote observer. The Latin letter indices take the values 0 (time), $1, 2$, and 3 (space), while the Greek letter indices take the values $1, 2$, and 3. With this definition of coordinates, s (see just below) and q^i have the length dimension, L. The spacetime metric is fixed by the interval ds: $ds^2 = g_{ik} dq^i dq^k$, where g_{ik} is the metric tensor. For a flat spacetime, $g_{ik} = (1, -\delta_{\mu\nu})$, where $\delta_{\mu\nu}$ is the Kronecker delta. In the latter case, $ds = cdt\sqrt{1-\beta^2} \equiv c\gamma^{-1}dt$. The contra-variant components of 4-velocity are $u^0 = \gamma$, $u^\alpha = \gamma\beta^\alpha$, where $\beta^\alpha = v^\alpha/c$, and the covariant ones are $u_0 = u^0$, $u_\alpha = -u^\alpha$. Similarly to the definition of 4-speed $u^i = dq^i/ds = (\gamma, \gamma\boldsymbol{\beta})$ (certainly, $u_i u^i = g_{ik} u^i u^k = u^0 u^0 - u^\alpha u^\alpha = \gamma^2 - \gamma^2\beta^2 = 1$), the quantity $w^i = du^i/ds = d^2q^i/ds^2$ can be called the contra-variant components of 4-acceleration (we follow here definitions of Ref. [59] (pp. 41 and 318). The dimensions of the quantities are $[u^i] = 1$, $[w^i] = L^{-1}$. The components of acceleration are

$$w^i = (w^0, \boldsymbol{w}) = \left(\frac{\gamma^4}{c^2} \boldsymbol{\beta} \cdot \boldsymbol{a},\ \frac{\gamma^2}{c^2} \boldsymbol{a} + \frac{\gamma^4}{c^2} \boldsymbol{\beta} (\boldsymbol{\beta} \cdot \boldsymbol{a}) \right). \tag{2}$$

Here, $\boldsymbol{a} = c(d\boldsymbol{\beta}/dt)$ is the common three-dimensional acceleration with the dimension of $L^1 T^{-2}$, where T denotes the time dimension.

3. Energy–Momentum Tensors for Particle and Field

For the electromagnetic field without particles, the Lagrangian density is the scalar:

$$\Lambda = -\frac{1}{16\pi} F_{ik} F^{ik} . \tag{3}$$

Let us remind that, symbolically, $F_{ij} = (\boldsymbol{E}, \boldsymbol{H})$, $F^{ij} = (-\boldsymbol{E}, \boldsymbol{H})$, i.e., $F_{0\alpha} = E_\alpha$, $F^{0\alpha} = -E_\alpha$, etc. Expressed in terms of $\boldsymbol{E}$ and $\boldsymbol{H}$, the convolution $F_{ik}F^{ik} = -2(E^2 - H^2)$. The sign of Equation (3) is explained in Ref. [59] (p. 69). The electromagnetic tensor F_{kl} is expressed via potentials A_l as

$$F_{kl} = \partial_k A_l - \partial_l A_k. \tag{4}$$

Hereafter, the operator $\partial_i \equiv \partial/\partial q^i$ where q^i denotes the contravariant 4-coordinate.

Let us stress that the energy–momentum tensor is determined according to the standard scheme: from action to Lagrangian to extremum of action to evolution equation to derivative of the Lagrangian (with respect to the spacetime coordinates) to consideration of the evolution equation to transfer of the terms to the law of conservation for the introduced tensor in the 4-divergence form.

The generalized coordinates for the field are the components of the 4-potential of the field, A_k. Thus, the mixed energy–momentum tensor $T_i^k[\mathrm{f}]$ is introduced as

$$T_i^k[\mathrm{f}] = (\partial_i A_l) \frac{\partial \Lambda}{\partial(\partial_k A_l)} - \Lambda\, \delta_i^k. \tag{5}$$

The symbolic argument f here indicates that the marked quantity refers to the electromagnetic field. After calculations—and keeping in mind that the variation of Λ (which gives $T_i^k[\mathrm{f}]$) is defined, with accuracy, up to the term that is the divergence of something—we find the following expressions for the energy–momentum tensor of the electromagnetic field:

$$T^{ik}[\mathrm{f}] = \frac{1}{4\pi} (-F^{il} F_l^k + \frac{1}{4} g^{ik} F_{lm} F^{lm}) \tag{6}$$

and for the mixed tensor:

$$T_i^k[\mathrm{f}] = \frac{1}{4\pi} (-F_i^l F_l^k + \frac{1}{4} \delta_i^k F_{lm} F^{lm}). \tag{7}$$

The components of the tensor in terms of the electric and magnetic field intensities are the quantity $T^{00}[\mathrm{f}]$, which coincides with the energy density $(1/8\pi)(E^2 + H^2)$, and the components $T^{0\alpha}[\mathrm{f}]$, which are the spatial components of the Poynting vector divided by c, i.e., S^α / c. Next, we differentiate Equation (6) and take into account the Maxwell equations:

$$\partial_k F^{kl} = \frac{4\pi}{c} j^l\ , \quad \partial_i F_{lm} = -\partial_l F_{mi} - \partial_m F_{il}\ , \tag{8}$$

where j^k is the current, and then permute the indices, cancel certain terms out, and obtain the result—the divergence of the mixed tensor:

$$\partial_k T_i^k[\mathrm{f}] = -\frac{1}{c} F_{ik} j^k. \tag{9}$$

The sense of the term on the r.h.s. of Equation (9) is straightforward: it is the "source" of the electromagnetic field—the field density changes if there is a "collision" of a moving charge ($j^k \neq 0$) with an electromagnetic field ($F_{ik} \neq 0$). Equation (9) shows that indeed no separate conservation of energy exists for only the electromagnetic field. The particles that create this field must also be included in the consideration.

The total energy–momentum for the closed system of particle plus field to be conserved: $\partial_k (T_i^k[\mathrm{p}] + T_i^k[\mathrm{f}]) = 0$. Then, for the particle, one must certainly have

$$\partial_k T_i^k[\mathrm{p}] = +\frac{1}{c} F_{ik} j^k, \tag{10}$$

with the plus sign in front of the source-term on the r.h.s. of Equation (10).

Now, let us turn to the question of the structure of the energy–momentum tensor for the particle, $T^{ik}[\mathrm{p}]$. First, one must move from the model of a point-like particle whose mass distribution is described by the Dirac delta-function as $\mu = m\delta(\boldsymbol{r} - \boldsymbol{r}_\mathrm{p})$, where $\boldsymbol{r}_\mathrm{p}$ is the radius vector for the particle, to a model with an arbitrary density distribution μ, which is normalized by the condition $\int dV\, \mu = m$.

However, expressing the energy–momentum tensor for the mass distribution in the form $T_i^k[\mathrm{p}] = \mu c^2 u_i u^k$ is a mistake. The correct expression is given by the formula

$$T^{ik}[\mathrm{p}] = \mu c \frac{dx^i}{ds}\frac{dx^k}{dt} = \mu c^2 u^i u^k \left(\frac{1}{c}\frac{ds}{dt}\right). \tag{11}$$

The rationale for this is given, e.g., in Ref. [59], p. 88, which states: "The 'four-momentum density' of the particles is given by $\mu c u_i$. We know that this density is the component $T^{0\alpha}/c$ of the energy–momentum tensor, i.e., $T^{0\alpha} = \mu c^2 u^\alpha$ ($\alpha = 1, 2, 3$). But the mass density is the time component of the four-vector $(\mu/c)(dq^i/dt)$ (in analogy to the charge density; see §28). Therefore the energy–momentum tensor of the system of non-interacting particles is $T^{ik} = \mu c u^i u^k (ds/dt)$ (33.5). As expected, this tensor is symmetric".

This result can also be obtained by direct calculation. We take into consideration the condition $u_k u^k = 1$, which holds along the particle's real world-line.

For a free non-point-like particle, the integral of action is written as

$$S[\mathrm{p}] = -mc\int ds \;\rightarrow\; S[\mathrm{p}] = -\int dt\, dV \mu c u_k u^k \frac{ds}{dt}, \tag{12}$$

where V is the volume bounded by the surface $\boldsymbol{\Sigma}$.

As soon as the integral $S[\mathrm{p}]$ is the relativistic-invariant quantity, and $dtdV$ and $u_k u^k$ are also invariants in Equation (12), then the invariant is $\mu^* = \mu(ds/cdt)$ and not μ. Thus, for the continuous distribution of mass, the Lagrangian density (per 4-volume unit) is

$$\Lambda[\mathrm{p}] = -\mu^* c^2\, u_k u^k. \tag{13}$$

Including the constraint of being on the real world-line, the quantity Λ has to be written as

$$\Lambda[\mathrm{p}] = -\mu^* c^2\, u_k u^k + \lambda_1 (u_l u^l - 1) + \lambda_2. \tag{14}$$

The second term in Equation (14) is zero on the real world-line but its derivative is not:

$$\frac{\partial \Lambda[\mathrm{p}]}{\partial u_k} = -\mu^* c^2\, u^k + \lambda_1 u^k. \tag{15}$$

The energy–momentum tensor is

$$u_i \frac{\partial \Lambda[\mathrm{p}]}{\partial u_k} - \Lambda \delta_i^k = -u_i \mu^* c^2 u^k + \lambda_1 u_i u^k + \delta_i^k \mu^* c^2 u_l u^l - \delta_i^k \lambda_1 (u_l u^l - 1) - \delta_i^k \lambda_2. \tag{16}$$

For indeterminate coefficients λ_1 and λ_2, we require that

$$-u_i \mu^* c^2 u^k + \lambda_1 u_i u^k + \delta_i^k \mu^* c^2 u_l u^l - \delta_i^k \lambda_1 (u_l u^l - 1) - \delta_i^k \lambda_2 = u_i \mu^* c^2 u^k, \tag{17}$$

i.e.,

$$-u_i \mu^* c^2 u^k - u_i \mu^* c^2 u^k + \lambda_1 u_i u^k + \delta_i^k \mu^* c^2 u_l u^l - \delta_i^k \lambda_1 (u_l u^l - 1) - \delta_i^k \lambda_2 = 0. \tag{18}$$

From Equation (18), one finds that the constant coefficients $\lambda_{1,2}$ may be fixed as

$$\lambda_1 = 2\mu^* c^2, \quad \lambda_2 = \mu^* c^2. \tag{19}$$

Thus, as soon as $u^l u_l = 1$, the mixed tensor for the particle, as formulated in Ref. [59], is

$$T_i^k[\mathrm{p}] = \mu^* c^2 u_i u^k = \mu\, c\, u_i \frac{dx^k}{dt}\,. \tag{20}$$

Now, let us calculate the derivative of the tensor $T_i^k[\mathrm{p}]$:

$$\partial_k T_i^k[\mathrm{p}] = c u_i \partial_k \left(\mu \frac{dx^k}{dt} \right) + \mu\, c \frac{dx^k}{dt} \partial_k u_i\,. \tag{21}$$

The first term in this expression is zero because of the conservation of mass $\partial_k(\mu dx^k/dt) = 0$, and therefore,

$$\partial_k T_i^k[\mathrm{p}] = \mu\, c \frac{dx^k}{dt} \frac{\partial u_i}{\partial x^k} = \mu\, c \frac{du_i}{dt} = \frac{ds}{dt} \mu\, c \frac{du_i}{ds}\,. \tag{22}$$

4. Effect of Radiation Friction Force

For the next step, we use the 4-dimensional equation of motion of the charge in the electromagnetic field, written in the form that includes the radiation friction force:

$$mc \frac{du_i}{ds} = \frac{e}{c} F_{ik} u^k + mc f_i\,. \tag{23}$$

Surely, the radiation friction force f_i has to satisfy the condition $u^i f_i = 0$ because of $u^i u_i = 1$ and $F_{ik} = -F_{ki}$. By transitioning to the continuous distributions of charge and mass and using the equality $\mu/m = \rho/e$, where e is the particle charge and ρ is the charge density, one obtains

$$\mu c \frac{du_i}{ds} = \frac{\rho}{c} F_{ik} u^k + \mu c f_i \quad \to \quad \mu c \frac{du_i}{dt} = \frac{1}{c} F_{ik} \rho u^k \frac{ds}{dt} + \mu c f_i \frac{ds}{dt}\,, \tag{24}$$

or, introducing the current j^k,

$$\mu c \frac{du_i}{dt} = \frac{1}{c} F_{ik} j^k + \mu c f_i \frac{ds}{dt}\,. \tag{25}$$

Thus, for the particle in the field, one finds the equation:

$$\partial_k T_i^k[\mathrm{p}] = \mu c \frac{du_i}{dt} = \frac{1}{c} F_{ik} j^k + \mu c f_i \frac{ds}{dt}\,. \tag{26}$$

In order for the conservation law to be satisfied for the particle–field system,

$$\partial_k (T_i^k[\mathrm{f}] + T_i^k[\mathrm{p}]) = 0, \tag{27}$$

it is necessary to add the compensating term $Q_i = -\mu c f_i (ds/dt)$ to the r.h.s. of Equation (9). Then, Equation (9) becomes

$$\partial_k T_i^k[\mathrm{f}] = -\frac{1}{c} F_{ik} j^k - \mu c f_i \frac{ds}{dt} \,. \tag{28}$$

Let us parameterize the effect of the radiation friction in the leading approximation by the following expression (see, e.g., [57–59]):

$$f^i = \frac{2e^2}{3mc^2} (\delta_k^i - u^i u_k) \frac{d^2 u^k}{ds^2} \quad \text{or} \quad f_i = \frac{2e^2}{3mc^2} (g_{ik} - u_i u_k) \frac{d^2 u^k}{ds^2} \,, \tag{29}$$

which certainly satisfies the mandatory requirement $u^i f_i = 0$. The quantity $2e^2/3mc^2$ here is nothing but the so-called "classical radius of the electron", r_e (with the accuracy within a numerical coefficient of the order of 1). Variants of Equation (29), in which additional nonlocal terms are added, are also presented in the literature, but discussion of this issue is beyond the scope of this paper. Equation (29) can be rewritten in an alternative form, taking into account the feature that $u_k u^k = 1$ and, consequently, for the derivatives $u_k (du^k/ds) = 0$ and $u_k (d^2 u^k/ds^2) = -(du_i/ds)(du^i/ds)$.

Thus, for the particle, Equation (23) written for the covariant component of 4-velocity becomes

$$\mu c \frac{du_i}{ds} = \frac{\rho}{c} F_{ik} u^k + \mu c r_e (\frac{d^2 u_i}{ds^2} + u_i \frac{du_k}{ds} \frac{du^k}{ds}) \,. \tag{30}$$

The analogous equation for the field is

$$\partial_k T_i^k[\mathrm{f}] = -\frac{\rho}{c} F_{il} u^l \frac{ds}{dt} - \mu c r_e (\frac{d^2 u_i}{ds^2} + u_i \frac{du_k}{ds} \frac{du^k}{ds}) \frac{ds}{dt}. \tag{31}$$

For the time component of Equation (31) with $u^0 = cdt/ds$ (which is equal to γ), one explicitly obtains:

$$\frac{1}{c} \partial_t T_0^0[\mathrm{f}] + \partial_\alpha T_0^\alpha[\mathrm{f}] = -\frac{\rho}{c} F_{0\alpha} u^\alpha \frac{ds}{dt} - \mu c r_e (\frac{dw_0}{ds} + u_0 w_k w^k) \frac{ds}{dt}, \tag{32}$$

where $T_0^0[\mathrm{f}] = (8\pi)^{-1}(E^2 + H^2) \equiv W_\mathrm{f}$ is the spatial density of the electromagnetic field energy, $T_0^\alpha[\mathrm{f}] = c^{-1} S^\alpha$. Here, $F_{0\alpha} = E_\alpha$, $F^{0\alpha} = -E^\alpha$ are the spatial components of the anti-symmetrical electromagnetic tensor; and $u^0 = \gamma$, $u^\alpha = \gamma \beta^\alpha$. One has

$$\frac{1}{c} \partial_t W_\mathrm{f} + \mu c r_e \frac{dw_0}{dt} = -\frac{\rho}{c} E_\alpha \gamma \beta^\alpha \frac{ds}{dt} + \mu c r_e \left(-w_k w^k u_0 \frac{ds}{dt} \right) - \frac{1}{c} \boldsymbol{\nabla} \cdot \mathbf{S} \,, \tag{33}$$

i.e.,

$$\frac{1}{c} \partial_t W_\mathrm{f} + \mu c r_e \frac{dw_0}{dt} = -\frac{\rho}{c} c E_\alpha \beta^\alpha + \mu c^2 r_e (-w_k w^k) - \frac{1}{c} \boldsymbol{\nabla} \cdot \mathbf{S} \,, \tag{34}$$

i.e.,

$$\partial_t W_\mathrm{f} + \mu c^2 r_e \frac{dw_0}{dt} = -\rho c (\boldsymbol{\beta} \cdot \boldsymbol{E}) + \mu c^3 r_e (-w_k w^k) - \boldsymbol{\nabla} \cdot \mathbf{S} \,. \tag{35}$$

After integration over a certain volume V, one obtains:

$$\frac{d}{dt}(\overline{W_{\rm f}} + mc^2 r_e w_0) = -\oint d\boldsymbol{\Sigma} \cdot \boldsymbol{S} + mc^3 r_e(-w_k w^k) - ec\overline{\boldsymbol{\beta} \cdot \boldsymbol{E}}\,, \quad \text{where} \quad \overline{W_{\rm f}} = \int dV \frac{E^2 + H^2}{8\pi}\,. \tag{36}$$

The first term in the r.h.s. of Equation (36) has a straightforward interpretation, namely it gives the electromagnetic energy flow "outflowing" from the volume V through the remote surface Σ. The contribution to this term is made only by the most slowly decreasing part of the field created by the accelerating charge.

The second term in the r.h.s. of Equation (36) represents the power of the radiation friction force: it increases the total energy of the electromagnetic field surrounding the moving charge. It is straightforward to verify using the definition of w^i in Equation (2) that

$$P = mc^3 r_e(-w^i w_i) = mc^3 r_e(-g_{ik} w^k w^i) \equiv mc^3 r_e(-w^0 w^0 + w^\mu w^\mu) = mc^{-1} r_e(1-\beta^2) a^2 \gamma^8 (1 - \beta^2 \sin^2\psi) = \frac{2}{3}\frac{e^2}{c^3}\gamma^6(a^2 - [\boldsymbol{\beta}, \boldsymbol{a}]^2), \tag{37}$$

where ψ is the angle between the directions of the spatial velocity, $\boldsymbol{v} = c\boldsymbol{\beta}$, and the spatial acceleration $\boldsymbol{a}$ of the particle.

Let us illustrate the elementary calculation so that to follow the details (here, in this calculation only, $\boldsymbol{v} \equiv \boldsymbol{\beta}$, in order to not be confused with the indices). The first term in the r.h.s. of Equation (37) gives $-\gamma^8 v^2 a^2 \cos^2\psi$ due to the scalar product feature. The next term is $\gamma^8(a^\beta\gamma^{-2} + v^\beta va\cos\psi)(a_\beta\gamma^{-2} + v_\beta va\cos\psi)$, i.e., $\gamma^8(a^2\gamma^{-4} + 2\gamma^{-2}a^2v^2\cos^2\psi + a^2v^4\cos^2\psi) = \gamma^8 a^2(1 - 2v^2 + v^4 + 2v^2\cos^2\psi - 2v^4\cos^2\psi + v^4\cos^2\psi) = \gamma^8 a^2(1 - 2v^2 + 2v^2\cos^2\psi + v^4\sin^2\psi) = \gamma^8 a^2(1 - 2v^2\sin^2\psi + v^4\sin^2\psi)$. Next, it is transformed into $\gamma^8 a^2(-v^2\cos^2\psi + 1 - 2v^2 + 2v^2\cos^2\psi + v^4\sin^2\psi) = \gamma^8 a^2(1 - 2v^2 + v^2\cos^2\psi + v^4\sin^2\psi) = \gamma^8 a^2(1 - v^2 - v^2\sin^2\psi + v^4\sin^2\psi) = \gamma^8 a^2(1-v^2)(1 - v^2\sin^2\psi)$. Finally, one finds the resulting expression:

$$-\frac{2}{3}e^2 c w^i w_i = \frac{2}{3}\frac{e^2}{c^3}\gamma^6(a^2 - a^2\beta^2\sin^2\psi). \tag{38}$$

The expression (38) can be rewritten as

$$P = \frac{2}{3}\frac{e^2}{c^3}\frac{(a^2 - [\boldsymbol{a}, \boldsymbol{\beta}]^2)}{(1-\beta^2)^3}. \tag{39}$$

It is necessary to remember the features of the cross product in Cartesian and curvilinear coordinates.

When the charged particle moves with much slower than the speed of light ($\beta \ll 1$) in an inertial frame of reference, Equation (37) gives

$$dE = \frac{2}{3}\frac{e^2}{c^3}a^2 dt'\,. \tag{40}$$

Equation (40) is the known Larmor's formula [68] for bremsstrachlung. Here, certainly, $a = a(t')$, $t' = t - R/c$ and $dt = dt'$.

In the r.h.s. of Equation (36), the third term, $\mathfrak{E} = -ec\,\overline{\boldsymbol{\beta} \cdot \boldsymbol{E}} = \mathfrak{E}_\beta + \mathfrak{E}_a + \mathfrak{E}_{\rm f}$, represents the work performed by the charge in a unit of time against the field $\boldsymbol{E}$ in order to generate an additional electromagnetic field. Section 5 discusses in detail how to evaluate the work $\mathfrak{E}$ and clarifies on each term.

5. Structure of Field in Vicinity of Charge

In Equation (36), the field $\boldsymbol{E} = \boldsymbol{E}_\beta + \boldsymbol{E}_a + \boldsymbol{E}_{\rm f}$ is measured at the location of the charge and is composed of three terms. The last term, $\boldsymbol{E}_{\rm f}$, is the free (external) field satisfying the homogenous-wave equation. The first two terms, $\boldsymbol{E}_\beta$ and $\boldsymbol{E}_a$, result from the Lienard–Wichert formulation.

Indeed, when a point-like charge e moves in vacuum along a certain trajectory, $\boldsymbol{r}(t)$, the electromagnetic fields, $\boldsymbol{E}$ and $\boldsymbol{H}$, are determined by the known expressions that follow from the Lienard–Wiechert potentials [59]. More precisely, since the field potential $\boldsymbol{A}$ satisfies an inhomogeneous-wave equation in which the source is the charge, then the potential is found using the Green function method [69]. For a point-like source, by rearranging the order of integration, the mentioned Lienard–Wiechert potentials are found, and then the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ are calculated. The resulting expressions are known (see, e.g., [59]):

$$\boldsymbol{E}_{\text{LW}} = \boldsymbol{E}_\beta + \boldsymbol{E}_a = \frac{e}{\gamma^2 R^2}\frac{\boldsymbol{n}-\boldsymbol{\beta}}{(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^3} + \frac{e}{c^2 R}\frac{[\boldsymbol{n},[(\boldsymbol{n}-\boldsymbol{\beta}),\boldsymbol{a}]]}{(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^3}, \quad \boldsymbol{H} = [\boldsymbol{n},\boldsymbol{E}]. \tag{41}$$

Here, $R = |\boldsymbol{R}|$ is the distance between the observation point described by $\boldsymbol{r}(t)$, and the charge location described by $\boldsymbol{r}_0(t')$. The vectors in the expressions (41) are, therefore, $\boldsymbol{R} = \boldsymbol{r}_0(t) - \boldsymbol{r}(t')$, $\boldsymbol{n} = \boldsymbol{R}/R$. The field $\boldsymbol{E}_\beta \sim R^{-2}$ describes the field accompanying the charge, and the field $\boldsymbol{E}_a \sim a/R$ describes the field radiated by the charge. The fields $\boldsymbol{E}$ and $\boldsymbol{H}$ in the left hand side (l.h.s.) of the expressions (41) are taken here at the point of observation $\boldsymbol{r}(t)$ at the instant t. The quantities in the r.h.s. of expressions (41) are taken at the instant t', which satisfies the equation $t = t' + c^{-1}|\boldsymbol{r}_0(t) - \boldsymbol{r}(t')|$. Vector $\boldsymbol{R} = \boldsymbol{r}_0(t) - \boldsymbol{r}(t')$ is drawn from the point where the charge is located, to the observation point. Also, the derivative $\boldsymbol{v}(t') \equiv \boldsymbol{v}' = d\boldsymbol{r}(t')/dt'$ is the velocity of the charge, and the acceleration is $\boldsymbol{a} = d\boldsymbol{v}'/dt'$. Let us also write $\boldsymbol{v}' = c\boldsymbol{\beta}$.

Hereafter, we focus on the terms $\boldsymbol{E}_\beta$ and $\boldsymbol{E}_a$ satisfying the inhomogeneous-wave equation, and not on $\boldsymbol{E}_{\text{f}}$, which satisfies the homogeneous-wave equation.

In the expression $\mathfrak{E} = -ec\,\overline{\boldsymbol{\beta}\cdot\boldsymbol{E}} = \mathfrak{E}_\beta + \mathfrak{E}_a + \mathfrak{E}_{\text{f}}$, the first term $\mathfrak{E}_\beta$ is zero. Indeed,

$$\mathfrak{E}_\beta \sim \int_0^\pi d\theta \sin\theta \frac{\boldsymbol{\beta}\cdot\boldsymbol{n} - \beta^2}{(1-\boldsymbol{\beta}\cdot\boldsymbol{n})^3} = 0, \tag{42}$$

where the relations for the components $\boldsymbol{\beta} = (0,0,\beta)$ and $\boldsymbol{n} = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$ have been used.

The term $\mathfrak{E}_a$ is not zero. Here, we are speaking about the interaction of the charge with the generated radiation field: just below in this Section, the charge density is parameterized by a "spread-out" three-dimensional delta-function in order to emphasize the point that the charge is localized in a relatively small space-region of the size ϵ. Then, in the third term in the r.h.s. of Equation (36), $\boldsymbol{E}$ is the field to be integrated over the domain (quasi-point-like) where this charge is located. To evaluate this term, we parameterize the delta-function as follows: $\delta_\epsilon(r) = (4\pi)^{-1}3\epsilon^2/(r^2+\epsilon^2)^{5/2}$. This function satisfies all the necessary requirements: it is symmetrical with respect to the angular variables, has a sharp maximum at $r = 0$, quickly falls to zero when $r \to \infty$, and the volume integral of $\delta_\epsilon(r)$ (at $dV = dr\, r^2 \sin\theta\, d\theta\, d\phi$) results in 1. Note that $r^{-2}\partial_r r^2 \partial_r(-(4\pi)^{-1}(r^2+\epsilon^2) = \delta_\epsilon(r)$. Therefore, in the denominator of the expression for the field $\boldsymbol{E}_a$, one should put $R^2 \to R^2 + \epsilon^2$; that is, the nonphysical singularity at $r = 0$ is avoided.

Generally, it can be assumed that the parameter $\epsilon \sim \hbar/mc$, with $\hbar$ the reduced Planck constant. Indeed, classical electrodynamics is limited to scales larger than the Compton length. At smaller scales, quantum effects come into play.

Calculation of the integral $\mathfrak{E}_a$ results into the following:

$$\mathfrak{E}_a = -ec\,\overline{\boldsymbol{\beta}\cdot\boldsymbol{E}_a} = -ec\int_0^\infty dR R^2 \left(\frac{1}{4\pi}\frac{3\epsilon^2}{(R^2+\epsilon^2)^{5/2}}\int d\Omega\left(\frac{e}{c^2}\right)\frac{1}{(R^2+\epsilon^2)^{1/2}}\frac{\boldsymbol{\beta}\cdot[\boldsymbol{n},[(\boldsymbol{n}-\boldsymbol{\beta}),\boldsymbol{a}]]}{(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^3}\right., \tag{43}$$

i.e.,

$$\mathfrak{E}_a = -\frac{3e^2}{4\pi c\epsilon}\int_0^\infty dy \frac{y^2}{(y^2+1)^3}\int d\Omega \frac{\boldsymbol{\beta}\cdot[\boldsymbol{n},[(\boldsymbol{n}-\boldsymbol{\beta}),\boldsymbol{a}]]}{(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^3} = -\frac{3}{4\pi}\frac{\pi}{16}\frac{e^2}{c\epsilon}\int d\Omega \frac{\boldsymbol{\beta}\cdot[\boldsymbol{n},[(\boldsymbol{n}-\boldsymbol{\beta}),\boldsymbol{a}]]}{(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^3}. \tag{44}$$

From the decomposition $[\boldsymbol{n},[(\boldsymbol{n}-\boldsymbol{\beta}),\boldsymbol{a}]] = (\boldsymbol{n}-\boldsymbol{\beta})(\boldsymbol{n}\cdot\boldsymbol{a}) - \boldsymbol{a}(\boldsymbol{n}\cdot(\boldsymbol{n}-\boldsymbol{\beta})) = \boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{a}) - \boldsymbol{\beta}(\boldsymbol{n}\cdot\boldsymbol{a}) - \boldsymbol{a}(\boldsymbol{n}\cdot\boldsymbol{n}) + \boldsymbol{a}(\boldsymbol{\beta}\cdot\boldsymbol{n})$, we rewrite the numerator in the angular integral of Equation (44) as $K(\theta,\phi) = \boldsymbol{\beta}\cdot[\boldsymbol{n},[(\boldsymbol{n}-\boldsymbol{\beta}),\boldsymbol{a}]] = (\boldsymbol{\beta}\cdot\boldsymbol{n})(\boldsymbol{n}\cdot\boldsymbol{a}) - (\boldsymbol{\beta}\cdot\boldsymbol{\beta})(\boldsymbol{n}\cdot\boldsymbol{a}) - (\boldsymbol{\beta}\cdot\boldsymbol{a})(\boldsymbol{n}\cdot\boldsymbol{n}) + (\boldsymbol{\beta}\cdot\boldsymbol{a})(\boldsymbol{\beta}\cdot\boldsymbol{n}) = -(\boldsymbol{\beta}\cdot\boldsymbol{a}) - \beta^2(\boldsymbol{n}\cdot\boldsymbol{a}) + (\boldsymbol{\beta}\cdot\boldsymbol{n})(\boldsymbol{n}\cdot\boldsymbol{a}) + (\boldsymbol{\beta}\cdot\boldsymbol{a})(\boldsymbol{\beta}\cdot\boldsymbol{n})$. Then, we express all vectors in the components $\boldsymbol{\beta} = (0,0,\beta)$, $\boldsymbol{a} = (a\sin\psi, 0, a\cos\psi)$, and $\boldsymbol{n} = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$, and calculate the integrals. Then, one finds that

$$\mathfrak{E}_a = \frac{3\pi}{16}\frac{e^2}{c\epsilon}\left[\frac{1}{\beta^2(1-\beta^2)}\left(1 - \frac{1-\beta^2}{\beta}\ln\sqrt{\frac{1+\beta}{1-\beta}}\right)\right](\boldsymbol{\beta}\cdot\boldsymbol{a}). \tag{45}$$

Taking into account that the time derivative of the modulus of the vector $\boldsymbol{\beta}(t)$ is written as $d\beta/dt = d\sqrt{\boldsymbol{\beta}\cdot\boldsymbol{\beta}}/dt = (\beta)^{-1}\boldsymbol{\beta}\cdot d\boldsymbol{\beta}/dt$, Equation (45) can be rewritten as

$$\mathfrak{E}_a = -\frac{3\pi}{16}\frac{e^2}{\epsilon}\left[1 - \frac{\operatorname{Arctanh}\beta}{\beta}\right]_{,\beta}\frac{d\beta}{dt} = \frac{d}{dt}\left[-\frac{3\pi}{16}\frac{e^2}{\epsilon}\left(1 - \frac{\operatorname{Arctanh}\beta}{\beta}\right)\right]. \tag{46}$$

This expression describes the work of the charged particle against the (radiated) electromagnetic field generated by the particle. Insert this expression into the equation of particle motion:

$$mc\frac{du_i}{ds} = \frac{e}{c}\overline{F_{il}u^l} + mcr_e\left(\frac{d^2u_i}{ds^2} + u_i\frac{du_k}{ds}\frac{du^k}{ds}\right). \tag{47}$$

For the time component of Equation (47) with $u^0 = cdt/ds$ (which is equal to γ), one explicitly obtains

$$\frac{d}{ds}mc\gamma = \frac{e}{c}\overline{F_{0\alpha}u^\alpha} + \mu cr_e\left(\frac{dw_0}{ds} + \gamma w_k w^k\right). \tag{48}$$

After some straightforward transformations, Equation (48) is reduced to

$$\frac{d}{dt}\left[\frac{mc^2}{\sqrt{1-\beta^2}} - \frac{2e^2}{3c^2}\frac{\boldsymbol{\beta}\cdot\boldsymbol{a}}{(1-\beta^2)^2} + \frac{3\pi}{16}\frac{e^2}{\epsilon}\left(\frac{\operatorname{Arctanh}\beta}{\beta} - 1\right)\right] = -P. \tag{49}$$

The expression (49) gives the link between the change of full/kinetic energy $E_\mathrm{p} \to E_\mathrm{p}^*$ (the expression in the square brackets, which includes the rest energy) of the charged particle and the power P of the radiation friction force acting onto the particle.

6. Results and Discussion

This analysis employed the framework of classical electrodynamics and relativity.

Equation (49) can be formally written as $(d/dt)(m^*c^2/\sqrt{1-\beta^2}) = -P$, where the notation m^* for the "effective mass" ($m^* = m + \delta m$) is introduced:

$$m^* = m + \frac{3\pi}{16}\frac{e^2}{c^2\epsilon}\sqrt{1-\beta^2}\left(\frac{\operatorname{Arctanh}\beta}{\beta} - 1\right) - \frac{2e^2}{3c^4}\frac{\boldsymbol{\beta}\cdot\boldsymbol{a}}{(1-\beta^2)^{3/2}}. \tag{50}$$

The expression within the square brackets in Equation (49)—denoted as E_p^*—just signifies that the link between the full/kinetic energy of a moving charged ($e^2 \neq 0$) particle and its velocity β, becomes more complicated than for a moving neutral ($e^2 = 0$) particle when the last two terms in E_p^* disappear. Equation (50), rewritten in the form

$$\frac{m^*}{m} = 1 + \frac{9\pi}{32}\frac{r_e}{\epsilon}\sqrt{1-\beta^2}\left(\frac{\text{Arctanh}\,\beta}{\beta} - 1\right) - \frac{r_e}{c\tau}\frac{\beta^2}{(1-\beta^2)^{3/2}}\left(\frac{\boldsymbol{\beta}}{\beta}\cdot\frac{\boldsymbol{a}\tau}{\beta c}\right) = 1 + S_1(\beta) - S_2(\beta)\left(\frac{\boldsymbol{\beta}}{\beta}\cdot\frac{\boldsymbol{a}\tau}{\beta c}\right), \tag{51}$$

allows us to estimate the field's contribution to the change: $E_{\text{p}0} = mc^2 \to E_\text{p} = mc^2\gamma \to E_\text{p}^*$. Here, τ is the characteristic time interval during which the particle velocity changes significantly. The second (always positive) term in the r.h.s. of Equation (51) shows that the "mass"—which is an inertial property—"increases" even more when the particle is moving ($\beta \neq 0$ or $\gamma \neq 1$) because of the presence of the field ($e^2 \neq 0$). If the space scale ϵ is assumed to be of the order of the Compton length $\lambda_\text{C} = \hbar/(mc)$, then the coefficient r_e/ϵ in the second term of Equation (51) is of the order of $e^2/(\hbar c) \simeq 1/137.04$ which is the fine-structure constant ($\simeq 7\cdot 10^{-3}$). Then, for all values of β, the mass add-on from the second term in Equation (51) is small ($\delta m < 0.003$).

A similar reasoning may be applied to the third term in Equation (51), where the scalar product $\boldsymbol{\beta}\cdot\boldsymbol{a}$ may be either positive or negative.

Let us estimate the order of the magnitude of the acceleration as $a \sim c\Delta\beta/\tau \sim c\beta/\tau$. Here, τ is the characteristic time of the meaningful change in the particle velocity. Then, the coefficient in front of the function $\beta^2/(1-\beta^2)^{3/2}$, resulting from the third term in the r.h.s. of Equation (51), namely $r_e/c\tau$, can be written as $r_e/c\tau \sim 10^{-23}/\tau$. Here, numerical values of $r_e \simeq 2.8\times 10^{-13}$ and $c \simeq 3\cdot 10^{10}$ in the CGS (cm–g–s) system are used. On the other hand, just as there exists the characteristic space-scale $\epsilon \sim \hbar/(mc)$, so does the characteristic time-scale $\tau_\epsilon \sim \epsilon/c = (\hbar/(mc))/c = \hbar/mc^2 \simeq 10^{-21}$ exist, where the numerical values $mc^2 \simeq 0.5\times 10^6\, ev$ and $\hbar \simeq 6.6\times 10^{-16}\, ev\cdot s$ are used. Thus, τ must be larger than 10^{-21} s in order for the classical electrodynamics to be applicable. Therefore, coefficient $r_e/c\tau$ has to be less than 10^{-2}. Then—in order to remain within the bounds of applicability of the classical theory—the acceleration $\boldsymbol{a}$ (which is of the order of the ratio of the velocity change to the time-interval over which this change occurs) must be less than the characteristic value $c\beta/\tau_\epsilon$. Thus, for the moderate values of velocity ($0 < \beta < 0.9$), the third term in Equation (51) is of the order of the fine-structure constant, i.e., the term is small relative to 1.

Thus, we may state that in most common cases, the add-on δm to the inertial rest mass m is small and the approximation $m^* \simeq m$ may be used. The add-on can be significant for the very fast processes (with very small $\tau \ll 10^{-21}$) occurring within the ultra-relativistic domain ($\beta \to 1$). But that is already stepping into the topic of nuclear collision.

The part of the parameter m^* that is linearly dependent on the acceleration—i.e., the third term in Equation (51)—can be interpreted as the so-called "added mass". It appears only when the charge is moving with acceleration. To make quite a rough analogy, a similar phenomenon occurs in classical hydrodynamics (see, e.g., [70,71]) when a material body moving with acceleration appears to "pull" into its motion some part of the fluid surrounding the body. More accurately, the "added mass" is a quantity which has the dimension of a mass and which is "added" to the mass of the body (which moves unevenly in a liquid medium) in order to take into account the influence of the medium on this body. For example, if an absolutely rigid body of mass m moves rectilinearly in an ideal fluid under the influence of force f, then the resistance of the medium is proportional to the body's acceleration a, i.e., $m\,a = f - m_a\,a$ or $(m + m_a)a = f$, where the coefficient of proportionality m_a is called the "added mass". Consequently, a body with mass m moves in a liquid in the same way as it would have moved in the emptiness and had the mass $m + m_a$. The value of the "added mass" depends on the shape of the body, its orientation relative to the direction of movement, the density of the medium, etc. Thus, one has to pay particular attention when applying such an interpretation to electrodynamic systems.

We return to Equation (36) and, after substituting Equation (46) into Equation (36), one obtains

$$\frac{dW}{dt} = -P_r + P, \tag{52}$$

where

$$W = \overline{W_{\rm f}} - (m^* - m)c^2\gamma \equiv \overline{W_{\rm f}} + \left[mc^2 r_e w_0 + \frac{3\pi}{16}\frac{e^2}{\epsilon}\left(1 - \frac{\text{Arctanh}\,\beta}{\beta}\right)\right],$$
$$P_r = \oint d\boldsymbol{\Sigma}\cdot\boldsymbol{S} \quad \text{and} \quad P = mc^3 r_e(-w_k w^k)\,,$$
$$w^i = (w^0, \boldsymbol{w}) = \left(\frac{\gamma^4}{c^2}\boldsymbol{\beta}\cdot\boldsymbol{a},\, \frac{\gamma^2}{c^2}\boldsymbol{a} + \frac{\gamma^4}{c^2}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{a})\right). \tag{53}$$

Equation (52) has a straightforward physical sense. It explains how the energy is balanced between three processes: radiation escaping through the remote spherical surface $\boldsymbol{\Sigma}$, excitation of the electromagnetic field by the particle decelerating by the action of the radiative friction force, and the change in the field's volume density. On the other hand, Equation (52) shows that the three quantities—the change in the field energy per unit time (dW/dt), the total energy flux of the outflowing field (P_r), and the power of the "radiative friction force" (P)—are connected by only one relationship. In a general case, they are far from being equal to each other in terms of the absolute magnitude. Only if a stationary motion with $dW/dt = 0$ is considered, then the equality $P = P_r$ takes place. This circumstance was noted in a methodological note [72].

To renormalize the energy of the electromagnetic field accompanying the moving charge, one has to replace $\overline{W_{\rm f}}$ with W in Equation (53).

The field energy W, as it follows from Equation (53), is composed of three components: the first component does not depend on the charge's acceleration a, the second component is linearly proportional to a, and the third component is proportional to a^2. To calculate the first component, the integration over the volume is carried out over the entire space; the singularity at the location of the charge is avoided via the use of a small spatial parameter ϵ. As for the components with a and a^2, one has to take into account that any physical process is the process with initial conditions and with a finite duration. In the case under study, the initial time-moment is the moment when the charge begins to move with acceleration. It is from this moment that the radiation field emerges and the terms, linear and quadratic with respect to the acceleration, appear in the expression for W. Furthermore, all field quantities calculated from the Lienard–Wiechert potentials are taken at time t', which is calculated from the equation $R(t') = c(t - t')$. This means that the size of the space-region, where the field generated by the accelerating charge exists, is finite, so the volume integral is taken not from zero to infinity, but from zero to a finite value calculated from the equation $R(t') = c(t - t')$. In such a case, there is no divergence of the volume integral at the upper limit.

If one associates the field energy with some effective mass (of the electromagnetic field) using the relation $W = m_{\rm f}c^2$, then Equations (51) and (53) actually just denotes a renormalization of the field mass due to the appearance of the add-on dependent on both the velocity of the charge and its acceleration.

Note that the first and the second terms in the r.h.s. of Equation (52), cannot be equal to each other because, for the term P, the force is applied to the charge along its trajectory, while for the term P_r, the energy flux of the field is integrated over the surface with a large characteristic radius. It is certain that, in full accord with the spirit of the field theory, the energy flux through the surface is determined by the field in the vicinity of this surface and not by the field along the charge's trajectory, which lays inside the volume bounded by the surface.

To conclude, this paper aimed to present—in the framework of classical theory—a self-consistent derivation scheme which produces equations that may be practically used for analyses of various scenarios. The charge is treated as a quasi-point-like charge; this helps to resolve the complications of the "infinite" electromagnetic energy, which are avoided by the procedure of slightly "spreading" the charge. As a result, the physical interpretation of the finite size of a particle is presented in a new light. Indeed, it is within the charge, at scales smaller than Compton's length of an order of $\hbar/mc$, where the quantum-mechanics approach to be applied. Beyond this region, no "infinite" tails of quantities accumulate. The seeming divergences of the integrals at the upper limits are not physical if one takes into account that the charge moves with acceleration only for a finite duration of time. In every real physical scenario, there is always a starting and an end moment.

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Appendix A. The Green Function: Interpretation and Application

When a neutral particle is point-like, its spatial mass density distribution is $\mu = m\delta^{(3)}(\mathbf{x} - \mathbf{x}_\mathrm{p}(t))$; its integral of action is $S_\mathrm{p} = -mc\int ds$ (see [59], §27).

When one considers a particle with an arbitrary mass density distribution μ, its integral of action can be written as

$$S_\mathrm{p} = -c\int d\Omega\,\mu\sqrt{1-\beta_\alpha\beta^\alpha} = -c\int d\Omega\,\mu\gamma^{-1}. \tag{A1}$$

Here, $d\Omega = d(ct)dV$, μ is normalized by the condition $m = \int dV\,\mu$. The interval in the 4-dimensional spacetime $ds = \sqrt{dq_i dq^i} = cdt\sqrt{\beta_i\beta^i} = cdt\sqrt{1-\beta_\alpha\beta^\alpha}$. We introduce $\beta^\alpha = c^{-1}(dq^\alpha/dt)$, $\beta^i = (1,\boldsymbol{\beta})$, $\beta_i = (1,-\boldsymbol{\beta})$. The particle 4-velocity is $u^i = \gamma\beta^i$. The quantity $\gamma = (1-\beta^2)^{-1/2}$ is the Lorentz factor.

For the next step, the action of the charged particle plus an electromagnetic field is

$$S_\mathrm{p} \to S = S_\mathrm{p} + S_\mathrm{int} + S_\mathrm{f} =$$
$$-c\int d\Omega\,\mu\sqrt{1-\beta_\alpha\beta^\alpha} - \frac{1}{c}\int d\Omega\,\rho A_i\beta^i - \frac{1}{16\pi c}\int d\Omega\,F^{ik}F_{ik}, \tag{A2}$$

where $F_{ik} = \partial_i A_k - \partial_k A_i$ is the electromagnetic field strength tensor. The particle charge density ρ is normalized by the condition $e = \int dV\,\rho$, similar to the condition for the mass $m = \int dV\,\mu$.

From Equation (A2), the Lagrangian 4-volume density is

$$\Lambda = -c\,\mu\sqrt{1-\beta_\alpha\beta^\alpha} - \frac{1}{c}\,\rho A_0 - \frac{1}{c}\,\rho A_\nu\beta^\nu - \frac{1}{16\pi c}\,F^{ik}F_{ik}. \tag{A3}$$

The first two terms in Equation (A2) depend on the 4-spacetime location of the particle q^i, on its velocity $\beta^i = c^{-1}(dq^i/dt)$, and on the field via the 4-potential A_i. The third term depends only on the field. When varying the action S along coordinates δq^k, with $\delta\beta^i = c^{-1}\delta(dq^i/dt) = c^{-1}(d(\delta q^i)/dt)$, it is necessary to take into account that the last term does not contribute to the variation, but the second term does because when the position of the particle changes virtually, the field value at this new point differs from the prior one. We also take into account that according to the procedure for calculating variations and variational derivatives, the quantities β^i and q^k are considered as independent variables.

We remind that, generally speaking, the Lagrangian density is not uniquely determined; for instance, $S \to S + \lambda\int d\Omega\,\rho + \mathrm{const}$.

Next, the functional derivative of the action

$$\frac{\delta S}{\delta q^i} = -\frac{1}{c}\frac{d}{dt}\frac{\partial \Lambda}{\partial \beta^i} + \frac{\partial \Lambda}{\partial q^i} \tag{A4}$$

turns out to be equal to

$$\frac{\delta S}{\delta q^i} = -\frac{1}{c}\frac{d}{dt}\left(-\frac{c\mu\beta_i}{\sqrt{1-\beta_\alpha\beta^\alpha}} - \frac{1}{c}\,\rho A_i\right) - \frac{1}{c}\,\partial_i(\rho A_k\beta^k) =$$
$$\frac{d}{dt}\left(\frac{\mu\beta_i}{\sqrt{1-\beta_\alpha\beta^\alpha}}\right) + \frac{1}{c}\,\beta^k\partial_k(\rho A_i) - \frac{1}{c}\,\beta^k\partial_i(\rho A_k) - \frac{1}{c}\,\rho A_k\partial_i\beta^k. \tag{A5}$$

The derivative (A5) has to be zero. Note that the last term in expression (A5) has to be taken as zero because the quantities β^i and q^k during the derivation of the equation are considered as independent quantities.

Thus, the equation for the charged particle motion, with $dt = \gamma ds/c$, becomes

$$\frac{d}{ds}(\mu c u_i) = \frac{1}{c}u^k(\partial_i(\rho A_k) - \partial_k(\rho A_i)). \tag{A6}$$

The l.h.s. of (A6) can be rewritten as $\mu\, c\, du_i/ds + c\, u_i d\mu/ds$. However, the second term here is equal to zero for the charged particle, which moves along the trajectory $\mathbf{x} = \mathbf{x}_\mathrm{p}(t)$. Indeed, in this case, one has $d\mu/ds = \gamma\beta^k\partial_k\mu(\mathbf{x} - \mathbf{x}_\mathrm{p}(t))$. Here, one just has to take into account that the action of the operator $\beta^k\partial_k$ on any function, including the delta-function whose argument is $\theta = q^\nu - q_0^\nu(t)$, yields zero. Indeed, from the definition of the dimensionless velocity of the particle, $\boldsymbol{\beta}$, it follows that $\beta^k\partial_k(q^\mu - q_0^\mu(t)) = \beta^k\partial_k q^\mu - \beta^k\partial_k q_0^\mu(t) = \beta^k\delta_k^\mu - \partial_0 q_0^\mu(t) = 0$. Thus, the equation of motion for a non-point--like particle is

$$\mu c\frac{du_i}{ds} = \frac{1}{c}u^k(\partial_i(\rho A_k) - \partial_k(\rho A_i)), \tag{A7}$$

which must be solved together with the electromagnetic field equations for potentials A_i and their derivatives.

As for the point-like particle, i.e., when $(\mu,\rho) = (m,e)\delta^{(3)}(\mathbf{x} - \mathbf{x}_\mathrm{p}(t))$, the situation here is quite nuanced and require higher attention. Let us note here that the use of the delta-function, or its spread-out analogue, is nothing more than a case-appropriate approximation of an object, made for studying processes in the spacetime region where classical electrodynamics still applies, i.e., with the scales exceeding the Compton length. For smaller scales, quantum-field physics must be used.

Let us rewrite Equation (A7) so that the terms dependent on (μ,ρ) and on their derivatives are arranged at opposing sides:

$$\mu c\frac{du_i}{ds} - \rho\frac{1}{c}u^k(\partial_i A_k - \partial_k A_i) = \frac{1}{c}(A_k u^k\partial_i\rho - u^k A_i\partial_k\rho). \tag{A8}$$

Now, on the r.h.s. of Equation (A8), only the terms proportional to the potential are present. One can then conclude that the r.h.s. must equal zero because the potentials—and not their derivatives, which determine the "force characteristics"—must not appear directly in the equations of motion for "measurable" dynamic quantities, such as the equation of motion for a particle.

The feature that the potential A_k is equal to zero at the location of the point-like charge can be understood from the following considerations.

First, the electromagnetic field equations are written for the electromagnetic stress–energy tensor as $\partial_k F^{ik} = -(4\pi/c)j^i$ (see also Equation (8)). Second, after imposing the Lorentz condition $\partial_k A^k = 0$ on the potentials A_k, one obtains

$$\partial_k \partial^k A^i = \frac{4\pi}{c} j^i. \tag{A9}$$

Here, $\partial_k \partial^k = c^{-2}\partial_t^2 - \Delta$; the symbol $\Delta = \partial_\alpha \partial^\alpha$ is the Laplace operator; the current components, j^k, are $j^0 = c\rho$, $\boldsymbol{j} = \rho c \boldsymbol{\beta}$; and the tensor $F_{ik} = \partial_i A_k - \partial_k A_i$. Raising and lowering the indices are carried out according to the standard procedure using the metric tensor $g_{ik} = (1, -\delta_{\alpha\beta})$. Then,

$$A^i = 4\pi \int d\Omega' G(t - t', \mathbf{x} - \mathbf{x}') j^i(t', \mathbf{x}') + A_0^i, \tag{A10}$$

where $d\Omega' = cdt'dx'dy'dz'$; A_0 is the solution of the homogenous-wave equation; and the Green function is introduced:

$$\partial_k \partial^k G(t - t', \mathbf{x} - \mathbf{x}') = \delta^{(1)}(c(t - t'))\delta^{(3)}(\mathbf{x} - \mathbf{x}'). \tag{A11}$$

Using the Fourier transform, the full Green function may be found quite elementarily:

$$G(t - t', \mathbf{x} - \mathbf{x}') = G^r + G^a = \frac{c}{4\pi|\mathbf{x} - \mathbf{x}'|}\delta^{(1)}(c(t - t') - |\mathbf{x} - \mathbf{x}'|) + \frac{c}{4\pi|\mathbf{x} - \mathbf{x}'|}\delta^{(1)}(c(t - t') + |\mathbf{x} - \mathbf{x}'|). \tag{A12}$$

Using the properties of the delta-function, the full Green function of the wave equation in the spacetime with Galilean metrics can be rewritten as

$$G(t - t', \mathbf{x} - \mathbf{x}') = \frac{c}{2\pi}\delta^{(1)}(c^2(t - t')^2 - |\mathbf{x} - \mathbf{x}'|^2). \tag{A13}$$

From Equation (A13), it follows that the Green function is not zero when its argument is zero. The argument of the first delta-function in Equation (A12) can become zero when $t' < t$, i.e., at the previous moments of time; the argument of the second term in the same equation can be zero only when $t' > t$. The first term on the r.h.s. of the expression (A12) represents the so-called retarded Green function, the second term represents the advanced Green function. Because for all physical processes, the principle of causality must hold, one should deal only with the first term; the field at the observation point at instant t is determined by the superposition of impacts from the states of the system/particle collected only along its world-line at the *preceding*, $t' < t$, moments of time. The unnecessary branch of the Green function is cut off by choosing such an integration contour along the time axis so that the singularity point is bypassed (i.e., by the deformation of the contour in the vicinity of the singularity in the complex time domain). To perform this, a Heaviside cutoff function $H(\xi)$ is introduced with the following properties: $H(\xi) = 1$ when $\xi > 0$; $H(\xi) = 0$ when $\xi < 0$; and notably, $H(0) = 0$. Then, the Green function in Equation (A10) is written as

$$G^r = \frac{c}{4\pi}\delta^{(1)}(c^2(t - t')^2 - |\mathbf{x} - \mathbf{x}'|^2)H(t - t'). \tag{A14}$$

Now, for the point-like charge, we substitute Equation (A14) into Equation (A10) (without any free term A_0^i) and integrate it: first over the volume, and then over time. Hence, one obtains the Lienard–Wichert potentials. One finds that $A^k = (eu^k/|R_i u^i|)H(t - t^*)$. Here, u^k is the 4-velocity of the charge at $t = t^*$, $R^k = (c(t - t^*), \boldsymbol{r} - \boldsymbol{r}_0(t^*))$ and parameter t^* satisfies equation $R_k R^k = 0$. Certainly, $R^k \to 0$ at the point where the charge is located, but at this point, the Heavyside function $H(0) = 0$ is strictly equal to zero. Therefore, potential A_i is zero at the point where the point-like charge is located. Thus, the r.h.s. of Equation (A8) is zero for the point-like charge.

The above then points that, for the point-like charged particle, Equation (A8) degenerates into the classical expression:

$$mc\frac{d}{ds}u_i = \frac{e}{c}u^k(\partial_i A_k - \partial_k A_i) = \frac{e}{c}F_{ik}u^k\,. \tag{A15}$$

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