Motivic coaction on generalized hypergeometric functions

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Statement of Originality

I hereby declare that this thesis contains no material that has been accepted or is currently being submitted for any other degree, diploma, certificate, or other qualifications at the University of Oxford or elsewhere. This thesis and the research it refers to are the product of my work and were carried out individually or sometimes with collaborators, in which case they are mentioned specifically in each instance in the text. The work of other people, published or otherwise, is fully acknowledged by the standard referencing practices of the discipline.

Abstract

This thesis deals with the motivic coaction and single-valued map for hypergeometric functions. It is known that hypergeometric functions have series expansions whose coefficients can be expressed in terms of multiple zeta values and multiple polylogarithms, the classical Beta function being a motivating example. Multiple zeta values are examples of periods, complex numbers that can be expressed as algebraically defined integrals, and exhibit an action under the motivic Galois group, leading to a transcendental extension of the classical Galois theory. This suggests a natural question as to whether such group actions can be carried over to the hypergeometric functions in a way that is still compatible with the term-by-term action on the series coefficients. This question becomes even more pertinent when considering integrals relevant to physics, such as Feynman integrals in dimensional regularization, which can be evaluated in terms of hypergeometric functions. Feynman integrals are fundamental to computations in precision collider physics, making them an active area of current research. Thus, the appearance of periods and motivic Galois structures in physics are a rich source of investigation in fundamental physics and have played a vital role in the last few decades.

In this thesis, I focus on one aspect of this, which I will refer to as the coaction conjecture for hypergeometric functions. Concretely, a conjecture was proposed, and numerical evidence was provided by Abreu-Britto-Duhr-Gardi-Matthew for a coaction formula for hypergeometric functions that is faithful to the term-by-term action on their Taylor series coefficients. This conjecture was proved in the case of one-dimensional integrals by the work of Brown-Dupont. The key novelty in their work was the use of a generalized Ihara formula that allows for the computation of the coaction at all orders. The conjecture was further explored in the general setting by the work of Britto-Mizera-Rodriguez-Schlotterer, where the strategy was to use the Knizhnik–Zamolodchikov equation to derive series expansion for the hypergeometric functions in terms of multiple zeta values and multiple polylogarithms and recast the coaction conjecture to an equivalent one formulated in terms of the series coefficients. There has also been recent progress, of which I am a part, that was made in collaboration with Hadleigh Frost, Martijn Hidding, Carlos Rodriguez, Oliver Schlotterer and Bram Verbeek, where we bridge the two approaches mentioned above to give a Lie algebraic reformulation of the coaction conjecture as well as applications of the new approach to computing the single-

valued (trivial monodromy) map for hypergeometric functions. I will cover this development in this thesis. I also give an alternative proof of the coaction conjecture in a one-dimensional case that extends to a proof of the coaction conjecture in a two-dimensional case, which is a new result. This result in dimension two also proves an old conjecture on open superstring amplitudes due to Schlotterer-Stieberger and its reformulation in terms of the coaction of periods by Drummond-Ragoucy.

Contents

Chapter 1

Introduction

The classical Beta function has an exponential series expansion near the origin in terms of the single zeta values $\zeta(k) := \sum_{n\geq 1} \frac{1}{n^k}, k \in \mathbb{N}, k \geq 2$,

$$\beta(x,y) = \int_0^1 t^x (1-t)^y \frac{dt}{t(1-t)}\,,\; x,y \in \mathbb{C},\; Re(x) > 0\,,\, Re(y) > 0\,,$$
$$\frac{x\,y}{x+y}\beta(x,y) = \exp\big(\sum_{n\geq 2} \frac{(-1)^{n-1}\zeta(n)}{n}((x+y)^n - x^n - y^n)\big)\,. \tag{1.0.1}$$

This is interesting from a number-theoretic perspective because of the appearance of zeta values, which are, in fact, examples of periods. A period [49] is a complex number whose real and imaginary parts can be expressed as an integral of a rational function (over $\mathbb{Q}$) over a domain defined by polynomial inequalities with coefficients in $\mathbb{Q}$. Examples include many important numbers in mathematics and physics such as π, $\log(\alpha)$ for $\alpha \in \mathbb{N}$, multiple zeta values, special values of L-functions, perturbative string amplitudes, scattering amplitudes etc. The ring of periods is important because of their rich algebraic structure, including the existence of a group action by a group scheme $\mathcal{G}_{MT}$ called the motivic Galois group as envisioned by Grothendieck. This action has only been computed in some cases, such as for logarithms, multiple zeta values [21], iterated integrals on the upper half plane [22] etc. and is often written down in terms of the dual coaction of the Hopf algebra $\mathcal{O}(\mathcal{G}_{MT})$, which is the ring of functions of the group scheme $\mathcal{G}_{MT}$.

Going back to the example of the Beta function in equation 1.0.1, this means that the RHS has a term-by-term coaction, and it is thus natural to ask whether this structure can be imported to the LHS in a compatible way. For the Beta function

$B(x,y) = \frac{xy}{x+y}\beta(x,y)$ this is given simply by

$$\Delta(B) = B \otimes B\,. \tag{1.0.2}$$

This encapsulates the coaction on an infinite number of periods, the zeta values and explains the uniformity in their coaction[1] formula

$$\Delta(\zeta(n)) = \zeta(n) \otimes 1 + 1 \otimes \zeta(n)\,,\ n \in \mathbb{N}, n \geq 2\,. \tag{1.0.3}$$

Such algebraic structure for integrals become particularly important if one imagines physically relevant integrals on the left of the equation 1.0.1 such as Feynman integrals, amplitudes in quantum field theory, string theory etc. Since Feynman integrals are crucial in making precision predictions for collider experiments [10], studying their mathematical structure is necessary to come up with efficient methods for their computation. Hence, the search for coaction properties of Feynman amplitudes has been an active area of research since Cartier's vision [27] of a cosmic Galois group that encodes symmetries of physical quantities. Francis Brown arrived at the first instance of this phenomenon [25] for a family of convergent Feynman graphs following the work of Bloch-Esnault-Kreimer [9].

A further level of complexity is added, however, when one deals with Feynman integrals in dimensional regularization, which evaluates to hypergeometric functions. In this case, the Feynman integrals are not periods in the usual sense, but their Laurent coefficients are periods, such as the example of Beta function 1.0.1. Nevertheless, as we saw, it is possible to extend a coaction formula 1.0.2 to the Beta function itself, which is compatible with the term by term coaction. This observation was extended to the family of hypergeometric functions in a series of papers [1], [2], [4], [5] by Samuel Abreu, Ruth Britto, Claude Duhr, Einan Gardi and James Matthew. Specifically, they proposed a general coaction formula for hypergeometric functions. Also, they verified for several examples, at low orders or in some cases to all orders, that the proposed formula is compatible with the term-by-term coaction formula on the series expansion of the corresponding hypergeometric function. The surprising phenomenon here is that the coaction defined on the two sides comes from quite different structures with no apparent relation and yet is still compatible.

Brown-Dupont explored this phenomenon further in [26], where they put the coaction conjecture above in a rigorous framework and proved it in the case of one-dimensional hypergeometric integrals. The key new development was the

[1] For simplicity, we drop the reference to motivic and deRham periods for now, which would be required for a more precise formulation. Their definitions are needed and will be given in Chapter 2.

application of a generalized version of the Ihara formula that works at the level of generating series of periods and thus allows for computation of the coaction at all orders.

Another point of view was taken in [12] where the authors made use of the Knizhnik–Zamolodchikov (KZ) differential equation for hypergeometric functions to derive series representation in terms of multiple polylogarithms (MPLs) and multiple zeta values (MZVs) and gave equivalent criteria for the coaction conjecture in terms of an 'adjoint' operation that they computed explicitly in low orders. Another recent progress was announced in a collaboration [35] involving myself, Hadleigh Frost, Martijn Hidding, Carlos Rodriguez, Oliver Schlotterer and Bram Verbeek where we bridge the two approaches mentioned above to give a Lie algebraic reformulation of the coaction conjecture as well as applications of the new approach to compute the single-valued (trivial monodromy) map for multiple polylogarithms and hypergeometric functions. A companion paper [36] will give detailed proofs of the results announced in [35]. I cover a part of this development in this thesis. In particular, I will cover the above-mentioned reformulation of the coaction conjecture including proofs. I will also discuss some of the results on the construction of the single-valued map though we will be concise in this thesis and follow a slightly different approach[2] than what is expected to appear in [36]. Additionally, based on my independent work, I provide alternative proofs of some of the results in [26] and prove a couple of cases of the coaction conjecture in dimension two which is a new result. The origin of the conjectural coaction property in a two-dimensional case also goes back to a 2012 paper of Schlotterer-Stieberger [51] where the α' expansion of open superstring amplitudes was studied. In the cases considered, the α' expansion has an expression purely in terms of multiple zeta values, which satisfies an elegant property. It was observed that the coefficients of higher depth multiple zeta values can be determined in terms of the coefficients of single zeta values at least to low orders and that this holds generally was left as a conjecture. The fact that this phenomenon can be explained in terms of the coaction was elucidated in a 2013 paper by Drummond-Ragoucy [34]. This conjecture also follows from the results of this thesis.

We should also note that the work in [1], [2] goes beyond coaction formulae for integrals. They give a combinatorial coaction formula for generic one-loop Feynman diagrams in terms of operations called cuts and contractions of a graph. These results were further extended by the work of Matija Tapušković in [53, 54]. Similarly, the coaction formula for Feynman integrals has also been looked at in cases where they evaluate to higher genus or higher dimensional integrals, which is the next

[2]The changes relative to [36] are highlighted in section 5.6 via remark 5.6.6 and 5.6.7.

frontier for collider computations[3]. However, in this thesis, we will be limited in scope to focusing only on the coaction conjecture for hypergeometric functions.

Outline of the report - This thesis is organized as follows. Chapters 2 and 3 cover some preliminary background. Chapter 2 briefly discusses definitions and results related to periods, multiple polylogarithms, multiple zeta values, and the motivic coaction of periods. We also introduce the generalized Ihara coaction formula worked out in [26].

Hypergeometric functions are multi-valued functions and such integrals fit naturally under the realm of twisted deRham theory. Chapter 3 focuses on the notion of twisted periods and the interpretation of hypergeometric functions as twisted periods of the punctured Riemann Sphere. The description of the coaction conjecture in this setup following the work of [12] is also included in this chapter.

In Chapter 4, we cover many results, with proofs, about the generating series of multiple zeta values, also known as the Drinfeld associator. These will be crucial in proving our main result in Chapter 5. However, they only appear in section 5.5, so this chapter may be skipped in the first instance until needed later on. It should be noted that some of the results in this chapter may be new and of independent interest in its own right.

Chapter 5 is the very core of this text. In this chapter, we recall the connection between hypergeometric functions and multiple polylogarithms by using the KZ equation. Then, we derive equivalent criteria for the coaction conjecture in terms of certain identities for generating series of multiple polylogarithms and multiple zeta values. At the end of this chapter, we also include an application to compute the single-valued map for hypergeometric functions by using a purely algebraic approach described in [28]. The results in this section are also meant to lay the groundwork for analogous results in higher genus / dimensional settings [10] and this hope has already started to bear fruit. We make a brief remark on this later in the text.

In Chapter 6, we finally give a proof for some explicit instances of the coaction conjecture in dimensions one and two by using the equivalent criteria from Chapter 5. Chronologically speaking, the results in this chapter were initially obtained independently of the ones in chapter 5. However, the proofs in this report have been slightly modified to be in sync with the current progress. We also explain how the coaction formulae we prove in dimension two lead to the solution of an old conjecture about the series expansion of open superstring tree amplitudes [51], [34].

[3]Please refer to the survey article [10] and the extensive references therein for further information.

Finally, there are some commutator identities that are required in a long proof in section 5.5 and these are covered in the appendix 7.

One final point to note before we proceed is that coaction formulae exist not at the level of periods but at the level of motivic periods [24], which are certain disembodied avatars of the period integrals. The ring of motivic periods is conjectured to be isomorphic to the ring of periods, but this is not yet known. We only know that there is a surjective map from motivic periods to complex periods. So whenever we work with periods, choices are involved when we take a lift to motivic periods, which can not be avoided at the moment.

Chapter 2

Periods, Motivic Coaction and Multiple Zeta Values

2.1 Periods

The elementary definition of periods [49] states that "a period is a complex number whose real and imaginary parts can be represented as an absolutely convergent integral of a rational function with rational coefficients over an integration domain which can described in terms of polynomial inequalities with rational coefficients". Examples of periods include algebraic numbers, pi, logarithm of rational numbers, etc.

For our purposes, however, we need to work with the cohomological interpretation of periods, which is briefly discussed in the next section. Further details are available in references [21], [24] and [39].

2.1.1 Cohomological interpretation of periods

Let X be an algebraic variety defined over $\mathbb{Q}$ and $Y \subset X$ a sub-variety.

Definition 2.1.1. We denote by $\mathrm{H}^*_{\mathrm{dR}}(\mathrm{X}, \mathrm{Y})$ the relative algebraic deRham cohomology[1] of X relative to Y. It is a finite-dimensional vector space over $\mathbb{Q}$.

Definition 2.1.2. We denote by $H^*_B(X,Y) \coloneqq H^*(X(\mathbb{C}), Y(\mathbb{C}); \mathbb{Q})$ the relative Betti cohomology of the pair of spaces (X,Y). It is a finite-dimensional vector space over $\mathbb{Q}$. We also denote the corresponding Betti homology group by $H^B_*(X,Y)$.

[1]The relative algebraic deRham cohomology is defined in the textbook [44].

With the prescribed notation[2], we have Grothendieck's comparison theorem [43], which is an algebraic analogue of the classical deRham's theorem.

Theorem 2.1.3 (Comparison Isomorphism)**.** *There is a canonical isomorphism of the relative cohomology groups,*

$$comp_{B,dR} : H^i_{dR}(X,Y) \otimes_{\mathbb{Q}} \mathbb{C} \to H^i_B(X,Y) \otimes_{\mathbb{Q}} \mathbb{C}\,. \tag{2.1.1}$$

Remark 2.1.4. Note that the comparison isomorphism does not preserve the rational structure.

Definition 2.1.5 (Periods)**.** The ring of periods is defined as the subset of complex numbers that appear as a coefficient of a matrix of the comparison isomorphism 2.1.1.

Example 2.1.6. Let us consider $X = \mathbb{G}_m = \mathrm{Spec}(\mathbb{Q}[\mathrm{t}, \mathrm{t}^{-1}])$ and $Y = \emptyset$. Then we have

$$H^1_{dR}(X) = \mathbb{Q}\,[\frac{dt}{t}]\,,\quad H_1(X(\mathbb{C}), \mathbb{Q}) = \mathbb{Q}\,[\gamma]\,, \tag{2.1.2}$$

where γ is the counterclockwise oriented unit circle and the period pairing gives

$$\int_\gamma \frac{dt}{t} = 2\pi i\,. \tag{2.1.3}$$

2.1.2 Motivic periods and coaction

A period can have multiple integral representations. Motivic periods are disembodied integrals that keep track of the different representations of a period.

Notation 2.1.1. We write $H^\bullet(X,Y)$ to denote the triple $(H^\bullet_{\mathrm{dR}}(X,Y), H^\bullet_{\mathrm{B}}(X,Y), \mathrm{comp}_{\mathrm{B,dR}})$. We will denote elements of $H^\bullet_{dR}(X,Y)$, $H^B_\bullet(X,Y)$ and $H^\bullet_{dR}(X,Y)^\vee$ by $[\omega]$, $[\sigma]$ and $[\nu]$ respectively.

Definition 2.1.7 (Motivic periods)**.** The ring $\mathcal{P}^{\mathrm{m}}$ of motivic periods is defined as the $\mathbb{Q}$-vector space generated by the symbols $[H^\bullet(X,Y), [\sigma], [\omega]]^{\mathrm{m}}$, with $[\sigma] \in H^B_\bullet(X,Y)$, $[\omega] \in H^\bullet_{\mathrm{dR}}(X,Y)$, after factorisation modulo the following equivalence relations. The multiplicative structure is given by tensor product on the components.

(1) *Bilinearity* : $[H^\bullet(X,Y), [\sigma], [\omega]]^{\mathrm{m}}$ is bilinear in $[\omega]$ and $[\sigma]$.

(2) *Change of variables* : If $f\colon (X_1,Y_1) \to (X_2,Y_2)$ is a $\mathbb{Q}$-morphism of pairs of algebraic varieties, $[\sigma_1] \in H^{\mathrm{B}}_\bullet(X_1,Y_1)$ and $[\omega_2] \in H^\bullet_{\mathrm{dR}}(X_2,Y_2)$, then

$$[H^\bullet(X_1,Y_1), [\sigma_1], f^*[\omega_2]]^{\mathrm{m}} = [H^\bullet(X_2,Y_2), f_*[\sigma_1], [\omega_2]]^{\mathrm{m}},$$

[2] We will also allow Y to be the empty set $\emptyset$, in which case we will mean the usual non-relative cohomology.

where f^* and f_* are the pull-back and the push-forward of f, respectively.

(3) *Stoke's formula* : For every triple $Z \subset Y \subset X$,

$$[H^\bullet(Y,Z),[\partial\,\sigma],[\omega]]^{\mathrm{m}} = [H^\bullet(X,Y),[\sigma],[\delta\,\omega]]^{\mathrm{m}}\,.$$

where ∂ is the connecting morphism for relative singular homology and δ is the connecting morphism for relative deRham cohomology.

Definition 2.1.8 (Period map)**.** Note that the ring of motivic periods comes equipped with a homomorphism called the period map, which is given by the evaluation of integrals,

$$\begin{aligned} \mathrm{per}\colon\ \mathcal{P}^{\mathrm{m}} &\longrightarrow \mathcal{P}, \\ \left[H^k(X,Y),[\sigma],[\omega]\right]^{\mathrm{m}} &\longmapsto \int_\sigma \omega\,. \end{aligned} \tag{2.1.4}$$

The theory of motivic periods is related to the theory of another ring, the de Rham periods.

Definition 2.1.9 (deRham periods)**.** The ring of deRham periods denoted $\mathcal{P}^{\mathrm{dR}}$, is defined [24] analogously to motivic periods except that it is generated by symbols of the form $[H^\bullet(X,Y),[\nu],[\omega]]^{\mathrm{dR}}$ with $[\nu] \in H^\bullet_{dR}(X,Y)^\vee$ and $[\omega] \in H^\bullet_{dR}(X,Y)$.

Definition 2.1.10 (Motivic coaction)**.** The space of deRham periods forms a Hopf algebra and the space of motivic periods is a Hopf comodule over it [24]. Let $\{e_i\}_{i=1,\ldots,n}$ be a $\mathbb{Q}$-basis of $H^k_{dR}(X,Y)$ and $\left\{\check{e}_i\right\}_{i=1,\ldots,n}$ the associated vector dual basis of $H^k_{dR}(X,Y)^\vee$. Then the coaction is given by

$$\begin{aligned} \Delta\colon\ \mathcal{P}^{\mathrm{m}} &\longrightarrow \mathcal{P}^{\mathrm{m}} \otimes \mathcal{P}^{\mathrm{dR}}, \\ [H^k(X,Y),\sigma,\omega]^{\mathrm{m}} &\longmapsto \sum_{i=1}^{n}[H^k(X,Y),\sigma,e_i]^{\mathrm{m}} \otimes [H^k(X,Y),\check{e}_i,\omega]^{\mathrm{dR}}\,. \end{aligned} \tag{2.1.5}$$

2.2 Multiple zeta values and Multiple polylogarithms

Definition 2.2.1 (Multiple zeta values)**.** Multiple zeta values are a multi-parameter generalization of the classical zeta values and are defined as the infinite sum below

$$\zeta(n_1,\ldots,n_r) = \sum_{0<k_1<\ldots,k_r} \frac{1}{k_1^{n_1}\ldots k_r^{n_r}} \tag{2.2.1}$$

where the n_i's are positive integers with $n_r \geq 2$. We refer to $\zeta(n_1,\ldots,n_r)$ as a multiple zeta value with weight $n_1+\ldots+n_r$ and depth r.

Similarly, we have functions defined similarly to the above called multiple polylogarithms.

Definition 2.2.2 (Multiple polylogarithms in one variable)**.** The infinite sum

$$Li_{n_1,\ldots,n_r}(z) := \sum_{0<k_1<\ldots,k_r} \frac{z^{k_r}}{k_1^{n_1}\ldots k_r^{n_r}} \tag{2.2.2}$$

defined for positive integers n_i with $n_r \geq 2$ is called a multiple polylogarithm in one variable with weight $n_1 + \ldots + n_r$ and depth r. It converges absolutely on the open unit disk and extends continuously to the closed unit disc.

Multiple zeta values are an important class of periods. In fact, they exhibit multiple integral representations. By the work of Goncharov-Manin [42], for instance, it is known that they are periods of the moduli space of curves of genus zero at n points. Additionally, they also exhibit a representation as iterated integrals which makes them periods of the motivic fundamental groupoid of the thrice punctured projective line [21].

Definition 2.2.3 (Iterated integrals)**.** Let $k = \mathbb{R}$ or $\mathbb{C}$ and M be a smooth manifold over k. For a piecewise smooth path $\gamma : [0,1] \longrightarrow M$ and smooth k-valued 1-forms $\omega_1, \ldots, \omega_n$ on M we define the iterated integral as

$$\int_\gamma \omega_1, \ldots, \omega_n := \int_{0\leq t_1\leq\ldots\leq t_n\leq 1} f_1(t_1)dt_1 \ldots f_n(t_n)dt_n\ .$$

Here $\gamma^*(\omega_i) = f_i(t)dt$ are the pull-back of the forms ω_i to the unit interval for $i \in \{1, \ldots, n\}$.

The iterated integrals are closed under multiplication.

Lemma 2.2.4. *The iterated integrals satisfy the shuffle product formula*

$$\int_\gamma \omega_1 \ldots \omega_r \int_\gamma \omega_{r+1} \ldots \omega_{r+s} = \sum_{\sigma\in\Sigma(r,s)} \int_\gamma \omega_{\sigma(1)} \ldots \omega_{\sigma(r+s)} \tag{2.2.3}$$

where $\Sigma(r,s)$ *is the set of* $(r+s)$*-shuffles:*
$\Sigma(r,s) = \{\sigma \in \Sigma(r+s) : \sigma^{-1}(1) < \ldots < \sigma^{-1}(r)\ \cap \sigma^{-1}(r+1) < \ldots < \sigma^{-1}(r+s)\}$.

Multiple zeta values can be represented as an iterated integral in the following way.

Example 2.2.5. Let $M = \mathbb{C} \setminus \{0,1\}$. Let $\omega_0 = \frac{dz}{z}$, $\omega_1 = \frac{dz}{1-z}$ be complex valued $1-$ forms on M and $\gamma(t) = t$ be the inclusion of the unit interval on M, then we have

$$\zeta(n_1, \ldots, n_r) = \int_\gamma \omega_1\omega_0^{n_1-1}\omega_1\omega_0^{n_r-1}\ldots\omega_1\omega_0^{n_r-1}. \tag{2.2.4}$$

Example 2.2.6. Similarly, if we don't fix the endpoint and take $\gamma : [0,1] \longrightarrow \mathbb{C}$ to be a smooth path in M such that $\gamma(0) = 0$ and $\gamma(1) = z$ then we get the multiple polylogarithms in one variable

$$Li_{n_1,\ldots,n_r}(z) = \int_\gamma \omega_1 \omega_0^{n_1-1} \omega_1 \omega_0^{n_r-1} \ldots \omega_1 \omega_0^{n_r-1} . \tag{2.2.5}$$

The motivic coproduct for iterated integrals was first computed by Goncharov [41]. To write down this formula, it would be convenient to introduce another notation that accounts for varying endpoints of integration.

Notation 2.2.1. Let $a_i \in \mathbb{C}$ for $i = 0, 1, \ldots, n+1$. Then we define

$$I(a_0; a_1, \ldots, a_n; a_{n+1}) \coloneqq \int_{a_0}^{a_{n+1}} \frac{dt}{t - a_n} I(a_0; a_1, \ldots, a_{n-1}; t) . \tag{2.2.6}$$

In the new notation, the coproduct on the Hopf algebra of polylogarithms (taken modulo $i\pi$) is given by

$$\Delta(I^{\mathfrak{m}}(a_0; a_1, \ldots, a_n; a_{n+1})) \\ = \sum_{0=i_1<i_2<\ldots<i_k<i_{k+1}=n} I^{\mathfrak{m}}(a_0; a_{i_1}, \ldots, a_{i_k}; a_{n+1}) \otimes \left[\prod_{p=0}^{k} I^{\omega}(a_{i_p}; a_{i_p+1}, \ldots, a_{i_{p+1}-1}; a_{i_{p+1}}) \right] . \tag{2.2.7}$$

This can be upgraded into a coaction by defining

$$\Delta(i\pi) = \pi \otimes 1 . \tag{2.2.8}$$

Note that the coaction on motivic multiple zeta values is obtained by setting $z = 1$.

2.3 Multiple polylogarithms in more than one variable

In what follows, we will need to work with iterated integrals over logarithmic forms beyond just $\omega_0 = \frac{dz}{z}$, $\omega_1 = \frac{dz}{1-z}$. We may also encounter iterated integrals with singularities in contrast to what we have seen before. So we will modify our notation a bit and work with the one common in physics literature so as to be in sync with the work in [12], [35] and [36]. This is described below.

Notation 2.3.1. Let $A = \{a_1, \dots, a_n\}$ be a generic alphabet of complex numbers. Then, we define,

$$G(a_1, a_2, \dots, a_n; z) \coloneqq \int_0^z \frac{\mathrm{d}t}{t - a_1} G(a_2, \dots, a_n; t)\,. \tag{2.3.1}$$

We also impose $G(\emptyset; z) = 1$.

Remark 2.3.1. The iterated integrals above may be divergent at the endpoints. This is resolved via shuffle-regularization, that is by using the shuffle property 2.2.4 of iterated integrals along with the initial condition, $G(0; z) = \log(z)$ and $G(z; z) = -\log(z)$.

In order to keep track of the dependence on any number of variables, we introduce the generating series

$$\mathbb{G}\left[\begin{smallmatrix} e_{a_1} & e_{a_2} & \cdots & e_{a_n} \\ a_1 & a_2 & \cdots & a_n \end{smallmatrix}; z\right] = \sum_{r=0}^{\infty} \sum_{a_1, a_2, \dots, a_r \in A} e_{a_1} e_{a_2} \dots e_{a_r} G(a_r, \dots, a_2, a_1; z) \tag{2.3.2}$$

$$= 1 + \sum_{a_1 \in A} e_{a_1} G(a_1; z) + \sum_{a_1, a_2 \in A} e_{a_1} e_{a_2} G(a_2, a_1; z) + \dots \tag{2.3.3}$$

The generating series above satisfies the Knizhnik–Zamolodchikov (KZ) equation [47]

$$\frac{\partial}{\partial z} \mathbb{G}\left[\begin{smallmatrix} e_{a_1} & e_{a_2} & \cdots & e_{a_n} \\ a_1 & a_2 & \cdots & a_n \end{smallmatrix}; z\right] = \mathbb{G}\left[\begin{smallmatrix} e_{a_1} & e_{a_2} & \cdots & e_{a_n} \\ a_1 & a_2 & \cdots & a_n \end{smallmatrix}; z\right] \sum_{i=1}^{n} \frac{e_{a_i}}{z - a_i}\,. \tag{2.3.4}$$

Remark 2.3.2. For $A = \{0, 1\}$ we get the generating series of multiple polylogarithms in one variable.

$$\mathbb{G}(e_0, e_1; z) \coloneqq \sum_{w \in \{e_0, e_1\}^*} G(w)\, w\,. \tag{2.3.5}$$

Further, setting $z = 1$ in the above we get the generating series of regularized multiple zeta values [15], also known as the Drinfeld associator,

$$\mathbb{G}(e_0, e_1; 1) = \Phi(e_0, e_1) = \sum_{w \in \{e_0, e_1\}^*} (-1)^{d(w)} \zeta(w)\, w\,. \tag{2.3.6}$$

Here, $d(w)$, called the depth of $\zeta(w)$, is defined as the number of occurrences of e_1 in the word w.

2.3.1 Ihara coaction formula

We have already seen the coaction formula for iterated integrals due to Goncharov. However, it will be much more convenient to work instead with the (generalized) Ihara formula [23, 26], which compactly packages Goncharov's formula at the level of generating series. For convenience we will drop the tensor notation and write $a^{\mathfrak{m}}$ instead of $a^{\mathfrak{m}} \otimes 1 \in \mathcal{P}^{\mathfrak{m}} \otimes \mathcal{P}^{\omega}$ and a^{ω} instead of $1 \otimes a^{\omega} \in \mathcal{P}^{\mathfrak{m}} \otimes \mathcal{P}^{\omega}$.

Theorem 2.3.3 (Ihara formula)**.** *The motivic coaction on the generating series of multiple polylogarithms is given by*

$$\Delta\mathbb{G}^{\mathfrak{m}}\left[\begin{smallmatrix} e_{a_1} & e_{a_2} & \dots & e_{a_n} \\ a_1 & a_2 & \dots & a_n \end{smallmatrix}; z\right] = \mathbb{G}^{\mathfrak{m}}\left[\begin{smallmatrix} e'_{a_1} & e'_{a_2} & \dots & e'_{a_n} \\ a_1 & a_2 & \dots & a_n \end{smallmatrix}; z\right] \mathbb{G}^{\mathfrak{dr}}\left[\begin{smallmatrix} e_{a_1} & e_{a_2} & \dots & e_{a_n} \\ a_1 & a_2 & \dots & a_n \end{smallmatrix}; z\right] \tag{2.3.7}$$

where the e'_{a_j} for $j = 1, \dots, n$ are conjugates of the e_{a_j} given by

$$e'_{a_j} = \mathbb{G}^{\mathfrak{dr}}\left[\begin{smallmatrix} e_{a_1} & e_{a_2} & \dots & e_{a_n} \\ a_1 & a_2 & \dots & a_n \end{smallmatrix}; z = a_j\right] e_{a_j} \, \mathbb{G}^{\mathfrak{dr}}\left[\begin{smallmatrix} e_{a_1} & e_{a_2} & \dots & e_{a_n} \\ a_1 & a_2 & \dots & a_n \end{smallmatrix}; z = a_j\right]^{-1}. \tag{2.3.8}$$

Remark 2.3.4. We will often set $a_1 = 0$ in which case $e'_0 = e_0$ itself.

For $A = \{0, 1\}$ we get the coaction of the generating series of MPLs in one variable.

$$\Delta\, \mathbb{G}^{\mathfrak{m}}(e_0, e_1; z) = \mathbb{G}^{\mathfrak{m}}(e_0, e'_1; z)\, \mathbb{G}^{\mathfrak{dr}}(e_0, e_1; z) \tag{2.3.9}$$

where the conjugate e'_1 is equal to

$$e'_1 = \Phi^{\mathfrak{dr}}(e_0, e_1)\, e_1\, (\Phi^{\mathfrak{dr}}(e_0, e_1))^{-1}. \tag{2.3.10}$$

The coaction of the Drinfeld associator is obtained by setting $z = 1$ in 2.3.9.

In analogy to the Ihara formula in the one variable case 2.3.9, we will often refer to the terms

$$\mathbb{G}^{\mathfrak{dr}}\left[\begin{smallmatrix} e_{a_1} & e_{a_2} & \dots & e_{a_n} \\ a_1 & a_2 & \dots & a_n \end{smallmatrix}; z = a_j\right] \tag{2.3.11}$$

in 2.3.7 as generalized Drinfeld associators.

Chapter 3

Configuration space integrals, twisted periods and coaction

Hypergeometric integrals are multi-valued functions. Twisted deRham theory is the right framework for dealing with them. In this chapter, we will recall the interpretation of hypergeometric functions as twisted periods of the configuration space of the punctured Riemann sphere and express the coaction conjecture in this setup. We start by setting up notation on configuration spaces of the punctured Riemann sphere following [12].

3.1 Configuration spaces

We denote a genus-zero Riemann surface by

$$\mathbb{CP}^1 \coloneqq \mathbb{C} \cup \{\infty\} \,. \tag{3.1.1}$$

Let $n, p \in \mathbb{N}$.

Definition 3.1.1 (Configuration space)**.** For a a topological space X, the configuration space of p distinct points on X is defined as

$$\mathrm{Conf}_p(X) \coloneqq X^p \backslash \{(x_1, x_2, \ldots, x_p) \,|\, x_i = x_j \text{ for some } i \neq j\} \,. \tag{3.1.2}$$

Let $X \coloneqq \mathbb{CP}^1 \backslash \{(n-p)\,\text{points}\}$ be a Riemann Sphere with $(n-p)$ punctures. We denote by $\mathcal{C}^{(n,p)} \coloneqq \mathrm{Conf}_p(X)$, the configuration space of p distinct points on X. We also impose the following condition: $1 \leq p \leq n-3$. Note that the configuration space $\mathcal{C}^{(n,p)}$ is p dimensional.

Let us denote the inhomogeneous coordinates of the configuration space by z_i for $i = 2, 3, \ldots, p+1$ and the inhomogeneous coordinates of the $(n-p)$ fixed punctures by z_1 and z_j for $j = p+2, p+3, \ldots, n$.

We use the Möbius transformation to fix three coordinates and set

$$(z_1, z_{n-1}, z_n) = (0, 1, \infty) \ . \tag{3.1.3}$$

Further, we assume the fixed punctures $z_1, z_{p+2}, z_{p+3}, \ldots, z_{n-1}$ to be real and ordered as below.

$$0 = z_1 < z_{p+2} < z_{p+3} < \cdots < z_{n-2} < z_{n-1} = 1 \, . \tag{3.1.4}$$

In what follows, the hypergeometric integrals that we will consider will only involve integrals over coordinates $z_2, \ldots, z_{p+1}$ of the configuration space and so the fixed punctures $z_1, z_{p+2}, z_{p+3}, \ldots, z_n$ will be referred to as the unintegrated variables.

3.2 Twisted periods and hypergeometric functions

Hypergeometric functions can be interpreted as twisted periods on the configuration space of punctured Riemann spheres where the twist is defined in terms of the Koba-Nielsen factor.

The Koba–Nielsen factor $\mathrm{KN}^{(n,p)}$ is defined as

$$\mathrm{KN}^{(n,p)} = \prod_{2 \leq i \leq p+1} \left(z_{1i}^{s_{1i}} \prod_{i<j \leq n-1} z_{ij}^{s_{ij}} \right) \tag{3.2.1}$$

where the difference between punctures is denoted by

$$z_{ij} = z_i - z_j \tag{3.2.2}$$

and s_{ij} will be thought of either as 'generic' real numbers or formal variables.

We also extend this notation by defining

$$s_{ij} = s_{ji} \ \text{ and } \ s_{ii} = 0 \tag{3.2.3}$$

for $1 \leq i, j \leq n-1$.

The genericity condition imposes that we must have

$$s_{ij} \notin \mathbb{Z} \ \text{ and } \ \sum_{(i,j)} s_{ij} \notin \mathbb{Z} \tag{3.2.4}$$

where the last sum is over all distinct unordered pairs (i, j).

Finally, the twist is defined to be single-valued form $\omega_{(n,p)} = \mathrm{d}\log \mathrm{KN}^{(n,p)}$.

Definition 3.2.1 (Koba-Nielsen connection)**.** Let $O_{\mathcal{C}^{(n,p)}}$ denote the structure sheaf on $\mathcal{C}^{(n,p)}$ and $\Omega^1_{\mathcal{C}^{(n,p)}}$ the sheaf of differential 1-forms on $\mathcal{C}^{(n,p)}$. Then the twist 1-form $\omega_{(n,p)}$ defines an integrable connection on $\mathcal{C}^{(n,p)}$ called the Koba-Nielsen connection [48].

$$\nabla_{\omega_{(n,p)}} = d + \omega_{(n,p)} \wedge : O_{\mathcal{C}^{(n,p)}} \to \Omega^1_{\mathcal{C}^{(n,p)}} \,. \tag{3.2.5}$$

3.2.1 Twisted (co)-homology groups and periods

Definition 3.2.2. The twisted deRham cohomology group denoted $H^k_{dR}(X, \nabla_{\omega(n,p)})$ is defined as the cohomology of $\mathcal{C}^{(n,p)}$ with coefficients in the integrable connection $(O_{\mathcal{C}^{(n,p)}}, \nabla_{\omega_{(n,p)}})$

$$H^k_{dR}(X, \nabla_{\omega(n,p)}) \coloneqq H^k_{dR}(X, (O_{\mathcal{C}^{(n,p)}}, \nabla_{\omega_{(n,p)}}))\,. \tag{3.2.6}$$

It is known that the cohomology groups vanish except in the top dimension [6, 50] and that the dimension of the p^{th} twisted cohomology group $H^p_{dR}(X, \nabla_{\omega(n,p)})$ is equal to $d^{(n,p)}$ where

$$d^{(n,p)} = \frac{(n{-}3)!}{(n{-}3{-}p)!}\,. \tag{3.2.7}$$

Concretely, the p^{th} twisted deRham cohomology group $H^p_{dR}(X, \nabla_{\omega(n,p)})$ is a $\mathbb{Q}(s_{ij})$ vector space generated by equivalence classes of closed p-forms up to exact p-forms with respect to the differential $\nabla_{\omega_{(n,p)}}$.

The sheaf of horizontal sections of the connection $\nabla_{\omega_{(n,p)}}$, denoted $\mathcal{L}_{\omega_{(n,p)}} \coloneqq \ker(\nabla_{\omega_{(n,p)}})$ is a local system (locally constant sheaf) on $\mathcal{C}^{(n,p)}$. We also have the dual local system $\mathcal{L}^\vee_{\omega_{(n,p)}}$ defined as the sheaf of horizontal sections of the dual connection $\nabla^\vee_{\omega_{(n,p)}} = \nabla_{-\omega_{(n,p)}}$. The dual local system $\mathcal{L}^\vee_{\omega_{(n,p)}}$ is generated over $\mathbb{Q}(\exp(2\pi i s_{ij}))$ by the Koba-Nielsen factor.

$$\prod_{2\le i\le p+1} \left(z_{1i}^{s_{1i}} \prod_{i<j\le n-1} z_{ij}^{s_{ij}} \right)\,. \tag{3.2.8}$$

Definition 3.2.3. The twisted singular homology group denoted $H^B_k(X, \mathcal{L}_{\omega_{(n,p)}})$ is defined as the homology of $\mathcal{C}^{(n,p)}$ with coefficients in the local system $\mathcal{L}_{\omega_{(n,p)}}$. The dual singular homology group $H^B_k(X, \mathcal{L}^\vee_{\omega_{(n,p)}})$ and the singular cohomology group $H_B^k(X, \mathcal{L}_{\omega_{(n,p)}})$ are defined similarly.

Just like the deRham cohomology groups, the dual singular homology groups $H_k^B(X, \mathcal{L}^\vee_{\omega_{(n,p)}})$ vanish everywhere except the top dimension. And the p^{th} twisted homology group $H_p^B(X, \mathcal{L}^\vee_{\omega_{(n,p)}})$ is a $\mathbb{Q}(\exp(i\pi s_{ij})$ vector space generated by equivalence classes of 'loaded' cycles $\gamma^{(n,p)} \otimes \mathrm{KN}_{\gamma^{(\mathrm{n,p})}}$ where $\gamma^{(n,p)}$ is a closed p-cycle on $\mathcal{C}^{(n,p)}$and $\mathrm{KN}_{\gamma^{(\mathrm{n,p})}}$ is a section of the Koba-Nielsen factor on $\gamma^{(n,p)}$.

Note that going forward we work with the complex vector spaces of cohomology groups obtained by extending the scalars to $\mathbb{C}$. This will be implicit in the notation.

To define hypergeometric function as twisted periods we need to work with integrals over open bounded domains of $\mathcal{C}^{(n,p)}$. These are not closed cycles on $\mathcal{C}^{(n,p)}$but they are locally finite cycles on $\mathcal{C}^{(n,p)}$. So we also consider the locally finite homology[1] groups $H_k^{B,lf}(X, \mathcal{L}^\vee_{\omega_{(n,p)}})$.

There are relations between the various singular homology and cohomology groups. Assuming the genericity condition 3.2.4, we have for instance, the isomorphism of homology groups [7],

$$H_k^{B,lf}(X, \mathcal{L}^\vee_{\omega_{(n,p)}}) \cong H_k^B(X, \mathcal{L}^\vee_{\omega_{(n,p)}})\,. \tag{3.2.9}$$

Theorem 3.2.4. *The singular cohomology group is related to the dual homology group by the universal coefficient theorem*

$$H^k_B(X, \mathcal{L}_{\omega(n,p)}) \cong H_k^B(X, \mathcal{L}^\vee_{\omega(n,p)})^\vee\,. \tag{3.2.10}$$

Finally, we have the twisted version of the comparison isomorphism [29].

Theorem 3.2.5. *There is an isomorphism between the twisted cohomology groups*

$$H^k_{dR}(X, \nabla_{\omega(n,p)}) \cong H^k_B(X, \mathcal{L}_{\omega(n,p)})\,. \tag{3.2.11}$$

Using the universal coefficient theorem 3.2.10 we can rewrite the RHS above in terms of the dual homology group

$$H^k_{dR}(X, \nabla_{\omega(n,p)}) \cong H_k^B(X, \mathcal{L}^\vee_{\omega(n,p)})^\vee\,. \tag{3.2.12}$$

Assuming the genericity condition we can work with the locally finite homology group

$$H^k_{dR}(X, \nabla_{\omega(n,p)}) \cong H_k^{B,lf}(X, \mathcal{L}^\vee_{\omega(n,p)})^\vee\,. \tag{3.2.13}$$

[1]The locally finite homology group of a space is defined using complexes of formal infinite sums of singular chains in contrast to the classical singular homology which works with finite chains. For a formal definition see [11].

Finally, this isomorphism can be rephrased in terms of a bilinear pairing

$$H_k^{B,lf}(X, \mathcal{L}^\vee_{\omega(n,p)}) \otimes H^k_{dR}(X, \nabla_{\omega(n,p)}) \to \mathbb{C}\,. \tag{3.2.14}$$

Once a basis $\gamma_a^{(n,p)} \otimes \mathrm{KN}_{\gamma_a^{(n,p)}}$, $a \in \{1, \ldots, d^{(n,p)}\}$ and $\omega_b^{(n,p)}$, $b \in \{1, \ldots, d^{(n,p)}\}$ of the p^{th} locally finite dual homology group and p^{th} deRham cohomology group respectively is fixed in the above pairing, the matrix of twisted periods is given by

$$F_{ab}^{(n,p)} = \langle \gamma_a^{(n,p)} \otimes \mathrm{KN}_{\gamma_\mathrm{a}^{(\mathrm{n,p})}} \,|\, \omega_\mathrm{b}^{(\mathrm{n,p})} \rangle = \int_{\gamma_\mathrm{a}^{(\mathrm{n,p})}} \mathrm{KN}_{\gamma_\mathrm{a}^{(\mathrm{n,p})}}\, \omega_\mathrm{b}^{(\mathrm{n,p})}\,. \tag{3.2.15}$$

3.2.2 Twisted basis of cycles and forms

A basis of twisted forms and cycles for the p^{th} (co)homology groups was written down in [12] and we will introduce it in this section.

In the proposed basis, the twisted 'locally finite' cycles $\gamma_a^{(n,p)}$ correspond to bounded regions of the real section of $\mathcal{C}^{(n,p)}$, with boundaries contained in the union of hyperplanes $\{z_{ij} = 0\}$ appearing in the Koba–Nielsen factor $\mathrm{KN}^{(n,p)}$. And as for $\mathrm{KN}^{(n,p)}_{\gamma_a^{(n,p)}}$ we choose the section of $\mathrm{KN}^{(n,p)}$ on $\gamma_a^{(n,p)}$ so that each factor z_{ij} in the expression for $\mathrm{KN}^{n,p}_{\gamma_a^{(n,p)}}$ is positive. We will work with this choice implicitly and often omit the Koba-Nielsen factor in the basis of cycles. So one can think of the factor $\mathrm{KN}^{(n,p)}$ appearing in the period integral $F_{ab}^{(n,p)}$ as

$$\mathrm{KN}^{(n,p)} = \prod_{2 \le i \le p+1} \left(|z_{1i}|^{s_{1i}} \prod_{i<j \le n-1} |z_{ij}|^{s_{ij}} \right) \tag{3.2.16}$$

to reflect our choice of section.

As mentioned in section 2.3 of [12], the basis of bounded cycles corresponds to regions labeled by distinct real orderings of the p integrated variables $z_{i_1}, z_{i_2}, \ldots, z_{i_p}$ among the $(n{-}p)$ unintegrated variables in their fixed order and thus can be expressed combinatorially via the following recipe.

Let us write $\vec{A} = (A_1, A_2, \ldots, A_{p+1})$ to denote a partition of the ordered list of unintegrated variables $z_{p+2}, \ldots, z_{n-2}$ into possibly empty parts A_j.

Then we write

$$\gamma^{(n,p)}_{\vec{A},\vec{i}} = (1, A_1, i_1, A_2, i_2, A_3, \ldots, A_p, i_p, A_{p+1}, n{-}1, n) \tag{3.2.17}$$

to denote the interspersing of the variables to be integrated among the unintegrated ones.

Writing $A_k = (a_{k1}, a_{k2}, \dots, a_{k\ell_k})$ we can interpret the sequence $\dots, A_k, i_k, A_{k+1}, \dots$ as the range $z_{a_{k\ell_k}} < z_{i_k} < z_{a_{k+1,1}}$ for the associated integration variable z_{i_k}. If however we have $A_k = \emptyset$ or $A_{k+1} = \emptyset$ then we replace the above with $z_{i_{k-1}} < z_{i_k}$ and $z_{i_k} < z_{i_{k+1}}$ respectively.

Corresponding to the above choice of basis of cycles, an associated basis of forms is proposed to eliminate any poles with respect to s_{ij}'s in the series expansion of $F^{(n,p)}$.

We strip off the total differential and write

$$\omega^{(n,p)}_{\vec{A},\vec{i}} = \hat{\omega}^{(n,p)}_{\vec{A},\vec{i}} \prod_{k=2}^{p+1} dz_k \,, \text{ where} \tag{3.2.18}$$

$$\hat{\omega}^{(n,p)}_{\vec{A},\vec{i}} = \sum_{j_1 \in \{1, A_1\}} \frac{s_{i_1,j_1}}{z_{i_1,j_1}} \sum_{j_2 \in \{1, A_1, i_1, A_2\}} \frac{s_{i_2,j_2}}{z_{i_2,j_2}} \cdots \sum_{\substack{j_p \in \{1, A_1, i_1, A_2, \dots \\ \dots, A_{p-1}, i_{p-1}, A_p\}}} \frac{s_{i_p,j_p}}{z_{i_p,j_p}} \,. \tag{3.2.19}$$

We also state a basis $\nu^{(n,p)}_{\vec{A},\vec{i}}$ for the dual deRham cohomology group $H^k_{dR}(X, \nabla_{\omega(n,p)})^\vee$. They are derived from the basis of cycles $\gamma^{(n,p)}_{\vec{A},\vec{i}}$ in the sense that each $\nu^{(n,p)}_{\vec{A},\vec{i}}$ has logarithmic singularities with unit residues along the boundaries of $\gamma^{(n,p)}_{\vec{A},\vec{i}}$.

To describe the dual forms we first write a basis cycle as a product of intervals[2] in the form below.

$$\gamma^{(n,p)}_{\vec{A},\vec{i}} = \{z_{b_{i_1}} < z_{i_1} < z_{c_{i_1}}\} \times \{z_{b_{i_2}} < z_{i_2} < z_{c_{i_2}}\} \times \cdots \times \{z_{b_{i_p}} < z_{i_p} < z_{c_{i_p}}\}$$

so that the integral on the top form can be written as

$$\int_{\gamma^{(n,p)}_{\vec{A},\vec{i}}} \left(\prod_{k=2}^{p+1} \mathrm{d}z_k \right) = \int_{z_{b_{i_1}}}^{z_{c_{i_1}}} \mathrm{d}z_{i_1} \int_{z_{b_{i_2}}}^{z_{c_{i_2}}} \mathrm{d}z_{i_2} \dots \int_{z_{b_{i_p}}}^{z_{c_{i_p}}} \mathrm{d}z_{i_p}$$

i.e. for each integrated puncture z_{i_k}, the indices b_{i_k} and c_{i_k} label the variables adjacent to it in the ordering (3.2.17).

Finally, this allows us to write the dual forms as

$$\nu^{(n,p)}_{\vec{A},\vec{i}} = \hat{\nu}^{(n,p)}_{\vec{A},\vec{i}} \prod_{k=2}^{p+1} \mathrm{d}z_k \tag{3.2.20}$$

$$\hat{\nu}^{(n,p)}_{\vec{A},\vec{i}} = \left(\frac{1}{z_{i_1,b_{i_1}}} - \frac{1}{z_{i_1,c_{i_1}}} \right) \left(\frac{1}{z_{i_2,b_{i_2}}} - \frac{1}{z_{i_2,c_{i_2}}} \right) \cdots \left(\frac{1}{z_{i_p,b_{i_p}}} - \frac{1}{z_{i_p,c_{i_p}}} \right) .$$

[2] This choice is in general not unique but leads to the same cohomology class for dual differential forms in any case.

3.2.3 Coaction conjecture for hypergeometric functions

The following coaction formula is conjectured [3, 5, 12]

$$\Delta F_{ab}^{(n,p)} = \sum_{c=1}^{d^{(n,p)}} F_{ac}^{(n,p)} \otimes F_{cb}^{(n,p)}\,, \tag{3.2.21}$$

to be consistent with the coaction of Taylor coefficients in the series expansion of $F^{(n,p)}$.

We must be a bit careful here since the coaction is defined only at the motivic level and so the above formula should be adjusted. The motivic and deRham lift of the twisted periods can be defined analogously to how the motivic and deRham periods were defined in the previous section just by changing the cohomology groups with cohomology with coefficients [26]. So the correct form of the conjecture is to replace the LHS with its motivic version, whereas on the RHS, we replace the left factor with its motivic version and the right factor with its deRham counterpart.

$$\Delta F_{ab}^{(n,p),\,\mathfrak{m}} = \sum_{c=1}^{d^{(n,p)}} F_{ac}^{(n,p),\,\mathfrak{m}} \otimes F_{cb}^{(n,p),\,\omega}\,. \tag{3.2.22}$$

Equivalently, in terms of the period matrix we have

$$\Delta F^{(n,p),\,\mathfrak{m}} = F^{(n,p),\,\mathfrak{m}}\, F^{(n,p),\,\omega} \tag{3.2.23}$$

where we drop the tensor notation going forward by writing $a^{\mathfrak{m}}$ for $a^{\mathfrak{m}} \otimes 1 \in \mathcal{P}^{\mathfrak{m}} \otimes \mathcal{P}^{\omega}$ and a^{ω} for $1 \otimes a^{\omega} \in \mathcal{P}^{\mathfrak{m}} \otimes \mathcal{P}^{\omega}$.

In the final chapter of this text, we will prove the above coaction formula when $p = 1$ and $n = 4, 5$ as well as for $p = 2$ and $n = 5, 6$. The case $(n, p) = (4, 1)$ corresponds to the classical Beta function for which the above coaction is already known while the case $(n, p) = (5, 1)$ corresponds to Gauss's ${}_2F_1$ hypergeometric function for which the coaction conjecture was proved in [26]. However, we will give alternative proofs in this report by making use of certain equivalent criteria for the coaction conjecture which is discussed in the chapter 5. This alternative approach will also allow us to prove the coaction conjecture for $(n, p) = (5, 2)$, $(6, 2)$ which are new results. This result will also lead to a proof of the conjecture due to Schlotterer-Stieberger [51] and its reformulation due to Drummond-Ragoucy [34] on open superstring amplitudes that was alluded to in the introduction.

Chapter 4

Drinfeld associator and the initial value series

The focus of this chapter is to derive a series expansion for the Drinfeld associator in terms of a certain operator defined in terms of a derivation on the free Lie algebra in two variables. This allows us to write down explicitly the coefficients of conjugate Drinfeld associators which is required to prove some conjugate identities in section 5.5. We will work solely over the f-alphabet 5.1.2.

The results in this chapter will only be needed in section 5.5 so the reader may skip them until required later on.

4.1 Derivation and the circle operator

Let $g = Lie_K[e_0, e_1]$ be the free Lie algebra generated by letters e_0 and e_1 over a field K. Let $U(g)$ denote the universal enveloping algebra of g. Note that g embeds into $U(g)$.

To each element y of the Lie algebra g we can associate a derivation D_y on $U(g)$ as follows. For $y \in g$, define

$$D_y e_0 = 0,\ D_y e_1 = [e_1, y] \tag{4.1.1}$$

and extend the derivation to all of $U(g)$ by using the product rule of differentiation.

For $y \in g$ and $x \in U(g)$, we define the circle operator, a right action of g on $U(g)$ as

$$x \circ y := xy - D_y x\,. \tag{4.1.2}$$

In what follows, we will also need to work with the derivation $\widehat{D}$ which complements D. For $y \in g$, define

$$\widehat{D_y} e_0 = [e_0,\, y],\;\; D_y e_1 = 0 \tag{4.1.3}$$

and extend as a derivation to all of $U(g)$.

Note that for all $y \in g$ and $x \in U(g)$ we have

$$D_y x + \widehat{D_y} x = [x,\, y]\,. \tag{4.1.4}$$

We also have the complementary circle operation $\widehat{\circ}$ defined as follows. For $y \in g$ and $x \in U(g)$ we define

$$x \,\widehat{\circ}\, y = xy - \widehat{D}_y x\,. \tag{4.1.5}$$

From equation 4.1.4 it follows that we have

$$x \,\widehat{\circ}\, y = xy - \widehat{D}_y x = xy - ([x, y] - D_y x) = yx + D_y x\,. \tag{4.1.6}$$

4.2 Circle operator expansion

The following theorem is due to Francis Brown [18] though we include an independent proof below.

Theorem 4.2.1 (Circle Operator Expansion formula)**.** *For $j \in \mathbb{N}$, there exist words $p_{2j}, w_{2j+1} \in \mathbb{Q}\langle e_0, e_1\rangle$ of length $2j$ and $2j+1$ respectively such that the motivic and deRham Drinfeld associators have the following series expansion using circle operators,*

$$\Phi^{\mathfrak{m}}(e_0,\, e_1) = \Big(\sum_{j=0}^{\infty} (f_2^j)^{\mathfrak{m}} p_{2j}\Big) \circ \sum_{i_1, i_2, \ldots, i_r \in 2\mathbb{N}+1} (f_{i_1} f_{i_2} \ldots f_{i_r})^{\mathfrak{m}} w_{i_1} \circ \ldots \circ w_{i_r}, \tag{4.2.1}$$

$$\Phi^{\mathfrak{dr}}(e_0,\, e_1) = \sum_{r=0}^{\infty} \sum_{i_1, i_2, \ldots, i_r \in 2\mathbb{N}+1} (f_{i_1} f_{i_2} \ldots f_{i_r})^{\mathfrak{dr}} w_{i_1} \circ \ldots \circ w_{i_r} \tag{4.2.2}$$

where the circle operator is to be computed from left to right. Moreover, $p_2, w_{2j+1} \in Lie_{\mathbb{Q}}[e_0, e_1]$, that is, p_2, w_{2j+1} are in fact commutators in e_0, e_1 for $j \in \mathbb{N}$.

We fix $K = \mathbb{Q}$ and $g = Lie_{\mathbb{Q}}[e_0, e_1]$ in this section. Note that $U(g) = \mathbb{Q}\,\langle e_0, e_1\rangle$.

Recall that the $\mathbb{Q}$ algebra of motivic multiple zeta values in the f-alphabet is generated by non-commutative letters f_k for $k \geq 3$ odd and f_2 where f_2 commutes with all f_k. Therefore, we can express the motivic Drinfeld associator in the form below

$$\Phi^{\mathfrak{m}}(e_0,\, e_1) = \sum_{j\geq 0}\sum_{r\geq 0}\sum_{i_1,i_2,\dots,i_r\in 2\mathbb{N}+1} (f_2^j)^{\mathfrak{m}}\,(f_{i_1}f_{i_2}\dots f_{i_r})^{\mathfrak{m}}\,W_{2j,i_1,\dots,i_r} \tag{4.2.3}$$

for some words $W_{2j,i_1,\dots,i_r} \in U(g)$. Further, observe that due to the weight grading on motivic multiple zeta values, we have that $W_{2j,i_1,\dots,i_r}$ is a word of length $2j + i_1 + \dots + i_r$. Note that we will just write $W_{i_1,\dots,i_r}$ in case $r = 0$ and consider also the deRham Drinfeld associator,

$$\Phi^{\mathfrak{dr}}(e_0,\, e_1) = \sum_{r\geq 0}\sum_{i_1,i_2,\dots,i_r\in 2\mathbb{N}+1} (f_{i_1}f_{i_2}\dots f_{i_r})^{\mathfrak{dr}}\,W_{i_1,\dots,i_r}\,. \tag{4.2.4}$$

Note that there is no canonical isomorphism between the $\mathbb{Q}$ algebra of motivic MZVs and the algebra in the f-alphabet; however, the form of the results that follow in this section won't depend on the choice of isomorphism itself.

Lemma 4.2.2. *For $k = 2$ or $k \geq 3$ odd, the word $W_k \in U(g)$ that appears in the ansatz 4.2.3, 4.2.4 is actually a commutator in e_0, e_1, that is, $W_k \in Lie[e_0, e_1]$.*

Before we prove this result let us recall some prerequisites.

Let R be a commutative unital ring of characteristic zero. Let $R\langle e_0,\, e_1\rangle$, $R\langle\langle e_0,\, e_1\rangle\rangle$, denote the ring of polynomials and the ring of power series in non-commutative letters e_0, e_1 with coefficients in R respectively. Let Δ_S denote the coproduct on $R\langle\langle e_0,\, e_1\rangle\rangle$ which is defined on the generators by

$$\Delta_S(e_j) = e_j \otimes 1 + 1 \otimes e_j, \quad j = 0, 1, \tag{4.2.5}$$

and extended to the power series ring as a ring homomorphism.

Then, note that a polynomial word $W(e_0, e_1) \in R\langle e_0,\, e_1\rangle$ satisfies

$$\Delta_S(W(e_0, e_1)) = W(e_0, e_1) \otimes 1 + 1 \otimes W(e_0, e_1) \tag{4.2.6}$$

if and only if $W(e_0, e_1)$ is a commutator in e_0, e_1, that is, $W(e_0, e_1) \in Lie[e_0, e_1]$, by Friedrichs' theorem [46].

Let us fix R to be the algebra of motivic multiple zeta values. Also, recall that the motivic Drinfeld associator satisfies

$$\Delta_S(\Phi^{\mathfrak{m}}(e_0,\, e_1)) = \Phi^{\mathfrak{m}}(e_0,\, e_1) \otimes \Phi^{\mathfrak{m}}(e_0,\, e_1)\,. \tag{4.2.7}$$

Now, we can give a proof of lemma 4.2.2.

Proof. Making use of our ansatz 4.2.3 in the series coproduct 4.2.7 we get that

$$\sum_{j\geq 0}\sum_{r\geq 0}\sum_{i_1,i_2,\ldots,i_r\in 2\mathbb{N}+1}(f_2^j)^{\mathfrak{m}}\,(f_{i_1}f_{i_2}\ldots f_{i_r})^{\mathfrak{m}}\,\Delta_S(W_{2j,i_1,\ldots,i_r}) =$$
$$\sum_{j\geq 0}\sum_{r\geq 0}\sum_{i_1,i_2,\ldots,i_r\in 2\mathbb{N}+1}(f_2^j)^{\mathfrak{m}}\,(f_{i_1}f_{i_2}\ldots f_{i_r})^{\mathfrak{m}}\,W_{2j,i_1,\ldots,i_r}$$
$$\otimes\sum_{j\geq 0}\sum_{r\geq 0}\sum_{i_1,i_2,\ldots,i_r\in 2\mathbb{N}+1}(f_2^j)^{\mathfrak{m}}\,(f_{i_1}f_{i_2}\ldots f_{i_r})^{\mathfrak{m}}\,W_{2j,i_1,\ldots,i_r}\,. \tag{4.2.8}$$

For $k=2$ or $k\geq 3$ odd, let us compare the coefficient of f_k on both sides of the equation above. On the left side we have $\Delta_S(W_k)$, whereas on the right side we have $W_k(e_0,e_1)\otimes 1+1\otimes W_k(e_0,e_1)$. Therefore, $W_k(e_0,e_1)\in Lie[e_0,e_1]$.

The same proof holds for $\Phi^{\mathfrak{dr}}(e_0,\,e_1)$ except there is no W_2 term since $f_2^{\mathfrak{dr}}=0$. □

Next, we have a result that gives the structure of the coefficient words $W_{2j,i_1,\ldots,i_r}$ in the ansatz 4.2.3.

Lemma 4.2.3. *For $j,r\geq 0$ and $i_1,\ldots,i_r\in 2\mathbb{N}+1$ we have*

$$W_{2j,i_1,\ldots,i_r}=W_{2j}\circ W_{i_1}\circ\ldots\circ W_{i_r} \tag{4.2.9}$$

where the circle operator is to be evaluated from left to right.

In this section, let $\Delta_{\mathfrak{m}}$ denote the motivic coaction on the algebra of motivic multiple zeta values. Then, Ihara's coaction formula tells us that

$$\Delta_{\mathfrak{m}}(\Phi^{\mathfrak{m}}(e_0,\,e_1))=\Phi^{\mathfrak{m}}(e_0,\,e_1^{'})\,\Phi^{\mathfrak{dr}}(e_0,\,e_1) \tag{4.2.10}$$

where

$$e_1^{'}=\Phi^{\mathfrak{dr}}(e_0,\,e_1)\,e_1\,(\Phi^{\mathfrak{dr}}(e_0,\,e_1))^{-1}\,. \tag{4.2.11}$$

Recall also that the motivic coaction on the f-alphabet is given by deconcatenation,

$$\Delta_{\mathfrak{m}}(f_2^j f_{i_1}f_{i_2}\ldots f_{i_r})^{\mathfrak{m}}=(f_2^j)^{\mathfrak{m}}\sum_{q=0}^{r}(f_{i_1}f_{i_2}\ldots f_{i_q})^{\mathfrak{m}}(f_{i_{q+1}}\ldots f_{i_r})^{\mathfrak{dr}} \tag{4.2.12}$$

for $j\geq 0, r\geq 0$ and $i_1,\ldots,i_r\geq 3$ and odd.

Finally, we have the proof of lemma 4.2.10.

Proof. We prove this result by induction. First, note that for $r = 0$ and $j \neq 0$ or for $j = 0$ and $r = 1$, there is nothing to prove. So, the base case is covered, and the induction step follows from the following argument. Observe that

$$\Phi^{\mathfrak{dr}}(e_0, e_1) = 1 + \sum_{i_1 \in 2\mathbb{N}+1} f_{i_1}^{\mathfrak{dr}} W_{i_1} + \dots \tag{4.2.13}$$

and therefore

$$(\Phi^{\mathfrak{dr}}(e_0, e_1))^{-1} = 1 - \sum_{i_1 \in 2\mathbb{N}+1} f_{i_1}^{\mathfrak{dr}} W_{i_1} + \dots . \tag{4.2.14}$$

Therefore, we can write the deRham conjugate as

$$e_1^{'} = \Phi^{\mathfrak{dr}}(e_0, e_1)\, e_1\, (\Phi^{\mathfrak{dr}}(e_0, e_1))^{-1} = e_1 - \sum_{i_1 \in 2\mathbb{N}+1} f_{i_1}^{\mathfrak{dr}} [e_1, W_{i_1}] + \dots . \tag{4.2.15}$$

Now, making use of our ansatz 4.2.3 in Ihara's coaction formula 4.2.10 we get that the left side is equal to

$$\begin{aligned}\sum_{j\geq 0}\sum_{r\geq 0}\sum_{i_1,i_2,\dots,i_r \in 2\mathbb{N}+1} \Delta_{\mathfrak{m}}\big((f_2^j)^{\mathfrak{m}}\, f_{i_1}^{\mathfrak{m}} f_{i_2}^{\mathfrak{m}} \dots f_{i_r}^{\mathfrak{m}}\big)\, W_{2j,i_1,\dots,i_r}(e_0, e_1)\\ = \sum_{j\geq 0}\sum_{r\geq 0}\sum_{i_1,i_2,\dots,i_r \in 2\mathbb{N}+1} (f_2^j)^{\mathfrak{m}} \sum_{q=0}^{r} f_{i_1}^{\mathfrak{m}} f_{i_2}^{\mathfrak{m}} \dots f_{i_q}^{\mathfrak{m}}\, f_{i_{q+1}}^{\mathfrak{dr}} \dots f_{i_r}^{\mathfrak{dr}}\, W_{2j,i_1,\dots,i_r}(e_0, e_1)\,,\end{aligned} \tag{4.2.16}$$

whereas the right side is equal to

$$\begin{aligned}&\sum_{j\geq 0}\sum_{r\geq 0}\sum_{i_1,i_2,\dots,i_r \in 2\mathbb{N}+1} (f_2^j)^{\mathfrak{m}}\, f_{i_1}^{\mathfrak{m}} f_{i_2}^{\mathfrak{m}} \dots f_{i_r}^{\mathfrak{m}}\, W_{2j,i_1,\dots,i_r}(e_0, e_1^{'})\\ &\qquad\times \sum_{r\geq 0}\sum_{i_1,i_2,\dots,i_r \in 2\mathbb{N}+1} f_{i_1}^{\mathfrak{dr}} f_{i_2}^{\mathfrak{dr}} \dots f_{i_r}^{\mathfrak{dr}}\, W_{i_1,\dots,i_r}(e_0, e_1)\\ &= \sum_{j\geq 0}\sum_{r\geq 0}\sum_{i_1,i_2,\dots,i_r \in 2\mathbb{N}+1} (f_2^j)^{\mathfrak{m}}\, f_{i_1}^{\mathfrak{m}} f_{i_2}^{\mathfrak{m}} \dots f_{i_r}^{\mathfrak{m}}\, W_{2j,i_1,\dots,i_r}\Big(e_0,\ e_1 - \sum_{i_1 \in 2\mathbb{N}+1} f_{i_1}^{\mathfrak{dr}} [e_1, W_{i_1}] + \dots\Big)\\ &\qquad\times \sum_{r\geq 0}\sum_{i_1,i_2,\dots,i_r \in 2\mathbb{N}+1} f_{i_1}^{\mathfrak{dr}} f_{i_2}^{\mathfrak{dr}} \dots f_{i_r}^{\mathfrak{dr}}\, W_{i_1,\dots,i_r}(e_0, e_1)\,.\end{aligned} \tag{4.2.17}$$

From the left side of the Ihara coaction, we see that $W_{2j,i_1,\dots,i_r}(e_0, e_1)$ is the coefficient of

$$\Delta_{\mathfrak{m}}\big((f_2^j)^{\mathfrak{m}}\, f_{i_1}^{\mathfrak{m}} f_{i_2}^{\mathfrak{m}} \dots f_{i_r}^{\mathfrak{m}}\big) = (f_2^j)^{\mathfrak{m}} \sum_{q=0}^{r} f_{i_1}^{\mathfrak{m}} f_{i_2}^{\mathfrak{m}} \dots f_{i_q}^{\mathfrak{m}}\, f_{i_{q+1}}^{\mathfrak{dr}} \dots f_{i_r}^{\mathfrak{dr}} \tag{4.2.18}$$

and thus in particular the coefficient of

$$(f_2^j)^{\mathfrak{m}}\, f_{i_1}^{\mathfrak{m}}\, f_{i_2}^{\mathfrak{m}} \dots f_{i_{r-2}}^{\mathfrak{m}}\, f_{i_{r-1}}^{\mathfrak{m}}\, f_{i_r}^{\mathfrak{dr}}\,. \tag{4.2.19}$$

Consider also the coefficient of the above f-word on the right side. The motivic part of the word, $(f_2^j)^{\mathfrak{m}}\, f_{i_1}^{\mathfrak{m}}\, f_{i_2}^{\mathfrak{m}} \dots f_{i_{r-2}}^{\mathfrak{m}}\, f_{i_{r-1}}^{\mathfrak{m}}$, appears as a contribution from

$$\Phi^{\mathfrak{m}}(e_0,\, e_1^{'}) = 1 + \ldots + (f_2^j)^{\mathfrak{m}}\, f_{i_1}^{\mathfrak{m}}\, f_{i_2}^{\mathfrak{m}} \dots f_{i_{r-2}}^{\mathfrak{m}}\, f_{i_{r-1}}^{\mathfrak{m}}\, W_{2j,i_1,\ldots,i_{r-1}}(e_0, \mathbf{e_1^{'}}) + \ldots \tag{4.2.20}$$

with the accompanying word

$$W_{2j,i_1,\ldots,i_{r-1}}(e_0, \mathbf{e_1^{'}})\,, \tag{4.2.21}$$

whereas, the deRham part of the word, $f_{i_r}^{\mathfrak{dr}}$, appears either as a contribution from

$$e_1^{'} = \Phi^{\mathfrak{dr}}(e_0, e_1)\, e_1\, (\Phi^{\mathfrak{dr}}(e_0, e_1))^{-1} = e_1 - f_{i_r}^{\mathfrak{dr}}\, [e_1, W_{i_r}(e_0, e_1)] + \ldots \tag{4.2.22}$$

in $W_{2j,i_1,\ldots,i_{r-1}}(e_0, \mathbf{e_1^{'}})$ or as a contribution from the second factor

$$\Phi^{\mathfrak{dr}}(e_0,\, e_1) = 1 + f_{i_r}^{\mathfrak{dr}}\, W_{i_r}(e_0, e_1) + \ldots \tag{4.2.23}$$

in the Ihara coaction. In the latter case, we can ignore the deRham contribution from $e_1^{'}$ and get

$$W_{2j,i_1,\ldots,i_{r-1}}(e_0, e_1)\, W_{i_r}(e_0, e_1)\,. \tag{4.2.24}$$

as part of the coefficient of

$$(f_2^j)^{\mathfrak{m}}\, f_{i_1}^{\mathfrak{m}}\, f_{i_2}^{\mathfrak{m}} \dots f_{i_{r-2}}^{\mathfrak{m}}\, f_{i_{r-1}}^{\mathfrak{m}}\, f_{i_r}^{\mathfrak{dr}}\,. \tag{4.2.25}$$

For the former case, consider the word below

$$W_{2j,i_1,\ldots,i_{r-1}}(e_0, \mathbf{e_1^{'}}) = W_{2j,i_1,\ldots,i_{r-1}}(e_0, \mathbf{e_1} - \mathbf{f_{i_r}^{\mathfrak{dr}}}\, [\mathbf{e_1}, \mathbf{W_{i_r}}(\mathbf{e_0}, \mathbf{e_1})] + \ldots)\,. \tag{4.2.26}$$

and the coefficient of $f_{i_r}^{\mathfrak{dr}}$ in it. Since

$$e_1^{'} = e_1 - f_{i_r}^{\mathfrak{dr}}\, [e_1, W_{i_r}(e_0, e_1)] + \ldots \tag{4.2.27}$$

for every appearance of e_1 in the word $W_{2j,i_1,\ldots,i_{r-1}}(e_0, e_1)$, considering only one e_1 at a time, we get the term $-f_{i_r}\, [e_1, W_{i_r}(e_0, e_1)]$. Thus, taking them all together we get the coefficient of f_{i_r} as

$$-D_{W_{i_r}(e_0,e_1)} W_{2j,i_1,\ldots,i_{r-1}}(e_0, e_1)\,. \tag{4.2.28}$$

Finally, adding the two contributions we get the coefficient of

$$(f_2^j)^{\mathfrak{m}}\, f_{i_1}^{\mathfrak{m}}\, f_{i_2}^{\mathfrak{m}} \dots f_{i_{r-2}}^{\mathfrak{m}}\, f_{i_{r-1}}^{\mathfrak{m}}\, f_{i_r}^{\mathfrak{dr}} \;\text{ as} \tag{4.2.29}$$

$$W_{2j,i_1,\dots,i_r} = W_{2j,i_1,\dots,i_{r-1}}(e_0,e_1)\,W_{i_r}(e_0,e_1) - D_{W_{i_r}(e_0,e_1)}W_{2j,i_1,\dots,i_{r-1}}(e_0,e_1) \quad (4.2.30)$$
$$= W_{2j,i_1,\dots,i_{r-1}}(e_0,e_1) \circ W_{i_r}(e_0,e_1). \quad (4.2.31)$$

Hence, by induction, we get that

$$W_{2j,i_1,\dots,i_r} = W_{2j} \circ W_{i_1} \circ \dots \circ W_{i_r}\,. \quad (4.2.32)$$

□

Finally, we have the proof of Theorem 4.2.1.

Proof. Follows from the discussion in this section along with lemma 4.2.3 and 4.2.2 after taking $p_{2j} = W_{2j}$ and $w_{2j+1} = W_{2j+1}$. □

Our main goal in the following two sections is to write down the coefficients of the Drinfeld conjugate $\Phi^{\mathfrak{dr}}e_1(\Phi^{\mathfrak{dr}})^{-1}$ in terms of derivations.

Also note that going forward, we will only be working with deRham periods, so we will drop the $\mathfrak{dr}$ superscript from the f-alphabet for ease of notation.

4.3 Coefficients of the inverse series

We first need to understand the coefficients of the inverse $\Phi^{\mathfrak{dr}}(e_0,\,e_1)^{-1}$.

We know that

$$\Phi^{\mathfrak{dr}}(e_0,\,e_1)^{-1} = \Phi^{\mathfrak{dr}}(e_1,\,e_0)\,. \quad (4.3.1)$$

In this chapter, for a word $x(e_0,\,e_1) \in U(g)$, we write $\overleftrightarrow{x}(e_0,\,e_1)$ to denote $x(e_1,\,e_0)$, that is the word obtained by swapping e_0 and e_1 in x.

From 4.3.1 it is clear that we have for all $i \in 2\mathbb{N}+1$,

$$\overleftrightarrow{w_i}(e_0,\,e_1) = -w_i(e_0,\,e_1)\,. \quad (4.3.2)$$

Next, we need to understand how the swapping operation affects the derivations and the circle operator.

Lemma 4.3.1. *For all $x \in U(g)$ and $y \in g$, we have*

$$\overleftrightarrow{D_y x} = \widehat{D}_{\overleftrightarrow{y}}\overleftrightarrow{x}\,. \quad (4.3.3)$$

Proof. Since the derivation is linear it is enough to assume that x is a monomial word in $U(g)$. Then, by the product rule, D_y acts on x as follows. If only the letter e_0 occurs in x, then $D_y x = 0$ and the result is clear. Otherwise, for each occurrence of the letter e_1 in x, D_y replaces that e_1 by $[e_1, y]$ and adds it to the output. Finally the swapping operator in $\overleftrightarrow{D_y x}$ changes x by $\overleftrightarrow{x}$ and y by $\overleftrightarrow{y}$ which causes all the $[e_1, y]$'s to be replaced by $[e_0, \overleftrightarrow{y}]$. And thus we get $\widehat{D}_{\overleftrightarrow{y}}\overleftrightarrow{x}$ as the result. □

Lemma 4.3.2. *For all $x \in U(g)$ and $y \in g$, we have*

$$\overleftrightarrow{x \circ y} = \overleftrightarrow{x} \,\hat{\circ}\, \overleftrightarrow{y} \,. \tag{4.3.4}$$

Proof. From lemma 4.3.1 we have

$$\overleftrightarrow{x \circ y} = \overleftrightarrow{xy - D_y x} = \overleftrightarrow{x}\,\overleftrightarrow{y} - \overleftrightarrow{D_y x} = \overleftrightarrow{x}\,\overleftrightarrow{y} - \widehat{D}_{\overleftrightarrow{y}}\overleftrightarrow{x} = \overleftrightarrow{x} \,\hat{\circ}\, \overleftrightarrow{y} = \overleftrightarrow{y}\,\overleftrightarrow{x} + D_{\overleftrightarrow{y}}\overleftrightarrow{x} \tag{4.3.5}$$

and the last two equations follow from 4.1.5 and 4.1.6 respectively. □

Corollary 4.3.3. *The coefficient of $f_{i_1} \dots f_{i_r}$ in the series $\Phi^{\mathfrak{dr}}(e_0, e_1)^{-1}$ is equal to*

$$\overleftrightarrow{w_{i_1}} \,\hat{\circ}\, \dots \,\hat{\circ}\, \overleftrightarrow{w_{i_r}} \tag{4.3.6}$$

Proof. Follows from equation 4.3.1 and lemma 4.3.2. □

4.4 Coefficients of the conjugate series

Our next result is about the coefficients of the conjugate series $\Phi^{\mathfrak{dr}}(e_0, e_1)e_1\Phi^{\mathfrak{dr}}(e_0, e_1)^{-1}$. To be clear, we are looking at the coefficients of the words in the f-alphabet in terms of words in e_0, e_1. To motivate this result we describe a few computations.

1. The coefficient of 1 is e_1. This is clear since the constant term in both Φ and Φ^{-1} is 1.

2. The coefficient of f_i for $i \geq 3$ odd is

$$w_i\, e_1 + e_1 \overleftrightarrow{w_i} \,. \tag{4.4.1}$$

   Since the term f_i comes either from Φ with a coefficient of w_i or from Φ^{-1} with a coefficient of $\overleftrightarrow{w_i}$.

3. To figure out the coefficient of $f_{i_1}f_{i_2}$ we have to look at all the shuffle products that could lead to the appearance of the term $f_{i_1}f_{i_2}$ in the conjugate product series and we get

$$(w_{i_1} \circ w_{i_2})e_1 + w_{i_1}\, e_1\, \overleftrightarrow{w_{i_2}} + w_{i_2}\, e_1\, \overleftrightarrow{w_{i_1}} + e_1(\overleftrightarrow{w_{i_1}}\, \widehat{\circ}\, \overleftrightarrow{w_{i_2}}) \tag{4.4.2}$$

corresponding to the shuffle terms

$$f_{i_1}f_{i_2} ⧢ 1,\ f_{i_1} ⧢ f_{i_2},\ f_{i_2} ⧢ f_{i_1},\ 1 ⧢ f_{i_1}f_{i_2}\,. \tag{4.4.3}$$

From the above examples, the pattern is now clear. To be precise we introduce the following notation. For *monomial* words u, v and z in the f-alphabet we define

$$C(u,\, v;\, z) \coloneqq \text{coefficient of}\, z\, \text{in the shuffle product}\, u ⧢ v\,. \tag{4.4.4}$$

Also, for a word $f_{i_1}\dots f_{i_r}$ in the f-alphabet, we define

$$w(f_{i_1}\dots f_{i_r}) \coloneqq w_{i_1} \circ \dots \circ w_{i_r},\ \text{and} \tag{4.4.5}$$

$$\widehat{w}(f_{i_1}\dots f_{i_r}) \coloneqq \overleftrightarrow{w_{i_1}}\, \widehat{\circ} \dots \widehat{\circ}\, \overleftrightarrow{w_{i_r}}\,. \tag{4.4.6}$$

With the prescribed notation above we can write down the coefficient of the Drinfeld conjugate series.

Lemma 4.4.1. *For $r \in \mathbb{N}$, the coefficient of $f_{i_1}\dots f_{i_r}$ in the series $\Phi^{\mathfrak{dr}}(e_0,\, e_1)e_1\Phi^{\mathfrak{dr}}(e_0,\, e_1)^{-1}$ is equal to*

$$\sum_{\{(u,v)\colon f_{i_1}\cdots f_{i_r} \in u ⧢ v\}} C(u, v; f_{i_1}\dots f_{i_r})w(u)e_1\widehat{w(v)}\,. \tag{4.4.7}$$

Proof. Clear from looking at all deshuffles of the word $f_{i_1}\dots f_{i_r}$ and the coefficients of the corresponding components in the series Φ and Φ^{-1}. □

We are at a stage where we can write down the above coefficients purely in terms of derivations. To motivate this result we again describe a few examples.

1. The coefficient of 1 is e_1.
2. Using equation 4.3.2 we get that the coefficient of f_i for $i \geq 3$ odd is

$$w_i\, e_1 + e_1\, \overleftrightarrow{w_i} = w_i\, e_1 - e_1\, w_i = [w_i, e_1] = -D_{w_i}e_1\,. \tag{4.4.8}$$

3. The coefficient of $f_{i_1}f_{i_2}$ for $i_1,\, i_2 \geq 3$ odd and distinct is

$$(w_{i_1} \circ w_{i_2})e_1 + w_{i_1}\, e_1\, \overleftrightarrow{w_{i_2}} + w_{i_2}\, e_1\, \overleftrightarrow{w_{i_1}} + e_1(\overleftrightarrow{w_{i_1}}\, \widehat{\circ}\, \overleftrightarrow{w_{i_2}}) = D_{w_{i_2}}D_{w_{i_1}}e_1 \tag{4.4.9}$$

as can be easily verified. Similarly, for $i_1 = i_2 \geq 3$ odd we get

$$(w_{i_1} \circ w_{i_1})e_1 + 2w_{i_1}\, e_1\, \overleftrightarrow{w_{i_1}} e_1(\overleftrightarrow{w_{i_1}}\, \hat{\circ} \overleftrightarrow{w_{i_1}}) = D_{w_{i_1}} D_{w_{i_1}} e_1 \tag{4.4.10}$$

which is identical to the previous case when i_1 and i_2 are distinct.

This leads us to the following theorem.

Theorem 4.4.2 (D. Kamlesh). *For $i_1, \ldots, i_r \in 2\mathbb{N}+1$, the coefficient of $f_{i_1} \ldots f_{i_r}$ in the series $\Phi^{\mathfrak{dr}}(e_0,\, e_1)e_1\Phi^{\mathfrak{dr}}(e_0,\, e_1)^{-1}$ is equal to*

$$(-1)^l D_{w_{i_r}} \ldots D_{w_{i_1}} e_1 \,. \tag{4.4.11}$$

Convention: Before we proceed with the proof, we want to set a convention for this chapter. For the sake of simplicity and clarity, going forward, we treat all the indices i_j as distinct. This way of representation does not cause any loss of information and serves as a unifying notation. To illustrate with an example let us discuss the example 3 again when $i_1 = i_2$. So when we consider the deshuffles of $f_{i_1}^2$, we write

$$f_{i_1} f_{i_2} ⧢ 1,\; f_{i_1} ⧢ f_{i_2},\; f_{i_2} ⧢ f_{i_1},\; 1 ⧢ f_{i_1} f_{i_2} \,. \tag{4.4.12}$$

instead of

$$f_{i_1}^2 ⧢ 1,\; f_{i_1} ⧢ f_{i_1} = 2f_{i_1}^2,\; f_{i_1}^2 ⧢ 1 \tag{4.4.13}$$

since we are treating i_1 and i_2 as distinct indices. Also, following the first formulation, the coefficient of $f_{i_1}^2$ in the conjugate series $\Phi^{\mathfrak{dr}}(e_0,\, e_1)e_1\Phi^{\mathfrak{dr}}(e_0,\, e_1)^{-1}$ will be written as

$$(w_{i_1} \circ w_{i_2})e_1 + w_{i_1}\, e_1\, \overleftrightarrow{w_{i_2}} + w_{i_2}\, e_1\, \overleftrightarrow{w_{i_1}} + e_1(\overleftrightarrow{w_{i_1}}\, \hat{\circ} \overleftrightarrow{w_{i_2}}) = D_{w_{i_2}} D_{w_{i_1}} e_1 \,. \tag{4.4.14}$$

This observation allows us to rewrite lemma 4.4.1 as follows by setting all the shuffle coefficients $C(u, v; w)$ to 1.

Lemma 4.4.3. *For $r \in \mathbb{N}$, the coefficient of $f_{i_1} \ldots f_{i_r}$ in the series $\Phi^{\mathfrak{dr}}(e_0,\, e_1)e_1\Phi^{\mathfrak{dr}}(e_0,\, e_1)^{-1}$ is equal to*

$$\sum_{\{(u,v):\, f_{i_1} \cdots f_{i_r} \in u ⧢ v\}} C(u, v; f_{i_1} \ldots f_{i_r}) w(u) e_1 \widehat{w(v)} = \sum_{\{(u,v):\, f_{i_1} \cdots f_{i_r} \in u ⧢ v\}} w(u) e_1 \widehat{w(v)} \,. \tag{4.4.15}$$

Remark 4.4.4. Next, we add a small observation that will be used in the proof of theorem 4.4.2 below. For every pair (u, v) such that $f_{i_1} \ldots f_{i_r} \in u ⧢ v$, we clearly have that

$$f_{i_1} \ldots f_{i_r} f_{i_{r+1}} \in u f_{i_{r+1}} ⧢ v \text{ and } f_{i_1} \ldots f_{i_r} f_{i_{r+1}} \in u ⧢ v f_{i_{r+1}} \,. \tag{4.4.16}$$

Conversely, for any pair $(\tilde{u}, \tilde{v})$ such that $f_{i_1} \dots f_{i_r} f_{i_{r+1}} \in \tilde{u} \sqcup \tilde{v}$, either $\tilde{u}$ or $\tilde{v}$ must have the letter $f_{i_{r+1}}$ at its end. So we must be able to write either

$$(\tilde{u}, \tilde{v}) = (u f_{i_{r+1}}, v) \text{ or } (\tilde{u}, \tilde{v}) = (u, v f_{i_{r+1}})\,. \tag{4.4.17}$$

Proof. For arbitrary r, consider the derivation of the coefficient of $f_{i_1} \dots f_{i_r}$. We have

$$-D_{w_{i_{r+1}}}\Big(\sum_{\{(u,v)\colon f_{i_1} \dots f_{i_r} \in u \sqcup v\}} w(u) e_1 \widehat{w(v)}\Big) = \sum_{\{(u,v)\colon f_{i_1} \dots f_{i_r} \in u \sqcup v\}} \Big(-D_{w_{i_{r+1}}}(w(u) e_1 \widehat{w(v)})\Big)\,. \tag{4.4.18}$$

Focusing on the bracketed term on the right-hand side we get

$$-D_{w_{i_{r+1}}}(w(u) e_1 \widehat{w(v)}) = -D_{w_{i_{r+1}}}(w(u)) e_1 \widehat{w(v)} + w(u)(-D_{w_{i_{r+1}}} e_1)\widehat{w(v)} + w(u) e_1 (-D_{w_{i_{r+1}}}(\widehat{w(v)}))\,. \tag{4.4.19}$$

In particular

$$-D_{w_{i_{r+1}}} e_1 = [w_{i_{r+1}},\, e_1] = w_{i_{r+1}} e_1 - e_1 w_{i_{r+1}} = w_{i_{r+1}} e_1 + e_1 \overleftrightarrow{w_{i_{r+1}}} \tag{4.4.20}$$

and so the middle term can be rewritten as

$$w(u)(-D_{w_{i_{r+1}}} e_1)\widehat{w(v)} = w(u)(w_{i_{r+1}} e_1 + e_1 \overleftrightarrow{w_{i_{r+1}}})\widehat{w(v)}\,. \tag{4.4.21}$$

Putting above in the original bracketed term we get

$$-D_{w_{i_{r+1}}}(w(u) e_1 \widehat{w(v)}) = \Big(-D_{w_{i_{r+1}}}(w(u)) e_1 \widehat{w(v)} + w(u) w_{i_{r+1}} e_1 \widehat{w(v)}\Big) \tag{4.4.22}$$

$$+ \Big(w(u) e_1 \overleftrightarrow{w_{i_{r+1}}} \widehat{w(v)} + w(u) e_1 (-D_{w_{i_{r+1}}}(\widehat{w(v)}))\Big)\,. \tag{4.4.23}$$

The last term can be modified using

$$-D_{w_{i_{r+1}}} = D_{-w_{i_{r+1}}} = D_{\overleftrightarrow{w_{i_{r+1}}}} \tag{4.4.24}$$

to get

$$-D_{w_{i_{r+1}}}(w(u) e_1 \widehat{w(v)}) = (w(u) w_{i_{r+1}} - D_{w_{i_{r+1}}} w(u)) e_1 \widehat{w(v)} + w(u) e_1 (\overleftrightarrow{w_{i_{r+1}}} \widehat{w(v)} + D_{\overleftrightarrow{w_{i_{r+1}}}} \widehat{w(v)})\,. \tag{4.4.25}$$

Using the definition of the circle operators along with equation 4.1.2 and 4.1.6, we have

$$-D_{w_{i_{r+1}}}(w(u) e_1 \widehat{w(v)}) = (w(u) \circ w_{i_{r+1}}) e_1 \widehat{w(v)} + w(u) e_1 (\widehat{w(v)} \mathbin{\widehat{\circ}} \overleftrightarrow{w_{i_{r+1}}}) \tag{4.4.26}$$

$$= w(u f_{i_{r+1}}) e_1 \widehat{w(v)} + w(u) e_1 \widehat{w(v f_{i_{r+1}})}\,. \tag{4.4.27}$$

Therefore, the derivation of the coefficient of $f_{i_1}\dots f_{i_r}$ is

$$-D_{w_{i_{r+1}}}\Big(\sum_{\{(u,v):\, f_{i_1}\cdots f_{i_r}\in u\sqcup v\}} w(u)e_1\widehat{w(v)}\Big) \tag{4.4.28}$$

$$=\sum_{\{(u,v):\, f_{i_1}\cdots f_{i_r}\in u\sqcup v\}} \Big(w(uf_{i_{r+1}})e_1\widehat{w(v)}+w(u)e_1\widehat{w(vf_{i_{r+1}})}\Big) \tag{4.4.29}$$

$$=\sum_{\{(u,v):\, f_{i_1}\cdots f_{i_r}\in u\sqcup v\}} \Big(w(uf_{i_{r+1}})e_1\widehat{w(v)}\Big)+\sum_{\{(u,v):\, f_{i_1}\cdots f_{i_r}\in u\sqcup v\}} \Big(w(u)e_1\widehat{w(vf_{i_{r+1}})}\Big) \tag{4.4.30}$$

$$=\sum_{\{(u,v):\, f_{i_1}\cdots f_{i_r}f_{i_{r+1}}\in uf_{i_{r+1}}\sqcup v\}} w(uf_{i_{r+1}})e_1\widehat{w(v)}+\sum_{\{(u,v):\, f_{i_1}\cdots f_{i_r}f_{i_{r+1}}\in u\sqcup vf_{i_{r+1}}\}} w(u)e_1\widehat{w(vf_{i_{r+1}})}) \tag{4.4.31}$$

$$=\sum_{\{(\tilde u,\tilde v):\, f_{i_1}\cdots f_{i_r}f_{i_{r+1}}\in \tilde u\sqcup \tilde v\}} w(\tilde u)e_1\widehat{w(\tilde v)}\,. \tag{4.4.32}$$

This is the coefficient of $f_{i_1}\dots f_{i_r}f_{i_{r+1}}$ by proposition 4.4.3 and the last two steps use remark 4.4.4 discussed before the proof. Since the coefficient of 1 in $\Phi^{\mathfrak{dr}}(e_0,\, e_1)e_1\Phi^{\mathfrak{dr}}(e_0,\, e_1)^{-1}$ is e_1, we get the required result by induction. □

Next, we study conjugates when there are two Drinfeld factors involved.

Let $e(1,0), e(1,1), e(2,0), e(2,1)$ be non-commutative letters and consider the conjugate below.

$$\Phi(e(1,0),e(1,1))\,\Phi(e(2,0),e(2,1))\,e(2,1)\,\Phi(e(2,0),e(2,1))^{-1}\,\Phi(e(1,0),e(1,1))^{-1}\,. \tag{4.4.33}$$

Let us denote by $X := \Phi(e(2,0),e(2,1))\,e(2,1)\,\Phi(e(2,0),e(2,1))^{-1}$, the inner conjugate. Let $w\in Lie[e_0,e_1]$ be a commutator in letters $e(1,0)$, $e(1,1)$. For $j=1,2$ we write $w(j) := w(e(j,0),e(j,1))$. Further, let us define

$$D_{w(1)}(X)=[X,w(1)] \tag{4.4.34}$$

for all $w\in Lie[e_0,e_1]$. Then, X does the work of e_1 in theorem 4.4.2 and we get that

$$\begin{aligned}&\Phi(e(1,0),e(1,1))\,\Phi(e(2,0),e(2,1))\,e(2,1)\,\Phi(e(2,0),e(2,1))^{-1}\,\Phi(e(1,0),e(1,1))^{-1}\\&=\Phi(e(1,0),e(1,1))\,X\,\Phi(e(1,0),e(1,1))^{-1}=\sum_{l\ge 0}\sum_{i_1,\dots,i_l}(-1)^l f_{i_1}\dots f_{i_l}D_{w_{i_l}(1)}\dots D_{w_{i_1}(1)}\,X\end{aligned}$$

$$= \sum_{l\geq 0} \sum_{i_1,\dots,i_l} (-1)^l f_{i_1} \dots f_{i_l} D_{w_{i_l}(1)} \dots D_{w_{i_1}(1)} \Big(\sum_{m\geq 0} \sum_{j_1,\dots,j_m} (-1)^m f_{j_1} \dots f_{j_m} D_{w_{j_m}(2)} \dots D_{w_{j_1}(2)} (X) \Big). \tag{4.4.35}$$

To simplify the above expression, let u, v be words in the f-alphabet such that $f_{i_1}, \dots f_{i_r}$ occurs in the sum $u ⧢ v$. Then, we can write $u = f_{i_{a_1}} \dots f_{i_{a_l}}$ and $v = f_{i_{b_1}} \dots f_{i_{b_m}}$ where $l + m = r$ and a_j, b_k are distinct letters from the list $i_1, \dots, i_r$. Note that we must have $a_1 < a_2 < \dots < a_l$ and $b_1 < b_2 < \dots < b_m$ since a_j, b_k must preserve the internal ordering of $i_1, \dots, i_r$ for $f_{i_1}, \dots f_{i_r} \in f_{i_{a_1}} \dots f_{i_{a_l}} ⧢ f_{i_{b_1}} \dots f_{i_{b_m}}$ to hold.

Then, the coefficient of $f_{i_1} \dots f_{i_r}$ in the conjugate series 4.4.33 can be written as

$$\sum_{(u,v): f_{i_1} \dots f_{i_r} \in u ⧢ v} (-1)^r \, D_{w_{i_{a_l}}(1)} \dots D_{w_{i_{a_1}}(1)} \, D_{w_{i_{b_m}}(2)} \dots D_{w_{i_{b_1}}(2)} \, X \, . \tag{4.4.36}$$

We can simplify the above expression further by extending the definition of $D_{w(2)}$ to $Lie[e(1,0), e(1,1), e(2,0), e(2,1)]$ by setting

$$D_{w(2)}(w'(1)) = 0 \tag{4.4.37}$$

for all $w' \in Lie[e_0, e_1]$.

Theorem 4.4.5 (D. Kamlesh)**.** *The coefficient of $f_{i_1} \dots f_{i_r}$ in the conjugate series 4.4.33 is equal to*

$$(-1)^r D_{w_{i_r}(1)+w_{i_r}(2)} \dots D_{w_{i_1}(1)+w_{i_1}(2)} (e(2,1)) \, . \tag{4.4.38}$$

Proof. From 4.4.36 we know that the coefficient of $f_{i_1} \dots f_{i_r}$ is equal to

$$\sum_{(u,v): f_{i_1} \dots f_{i_r} \in u ⧢ v} (-1)^r \, D_{w_{i_{a_l}}(1)} \dots D_{w_{i_{a_1}}(1)} \, D_{w_{i_{b_m}}(2)} \dots D_{w_{i_{b_1}}(2)} \, e(2,1) \, . \tag{4.4.39}$$

where $u = f_{i_{a_1}} \dots f_{i_{a_l}}$ and $v = f_{i_{b_1}} \dots f_{i_{b_m}}$.

First, we focus on the expression below,

$$D_{w_{i_{a_l}}(1)} \dots D_{w_{i_{a_1}}(1)} \, D_{w_{i_{b_m}}(2)} \dots D_{w_{i_{b_1}}(2)} \, e(2,1) \, . \tag{4.4.40}$$

By definition 4.4.37, $D_{w_{i_{b_m}}(2)}$ acts on any $D_{w_{i_{a_j}}(1)}$ by zero and thus $D_{w_{i_{b_m}}(2)}$ commutes with $D_{w_{i_{a_j}}(1)}$. Therefore, we can push $D_{w_{i_{b_m}}(2)}$ to the left in the above equation. If $b_m > a_l$ then we push $D_{w_{i_{b_m}}(2)}$ all the way to the left or if $a_1 > b_m$ then we leave $D_{w_{i_{b_m}}(2)}$ inert. If not, there exists $k \in 1, \dots l$ such that $a_k > b_m > a_{k-1}$

and in this case we push $D_{w_{i_{b_m}}(2)}$ to the left until it sits between $D_{w_{i_{a_k}}(1)}$ and $D_{w_{i_{a_{k-1}}}(1)}$. We repeat the same procedure for $D_{w_{i_{b_{m-1}}}(2)}$. Note that by definition of the above procedure, $D_{w_{i_{b_{m-1}}}(2)}$ never gets pushed beyond $D_{w_{i_{b_m}}(2)}$ and so the process is still valid. We repeat this procedure until $D_{w_{i_{b_1}}(2)}$ is also exhausted and in the end, we are left with a rewriting of 4.4.40 as follows

$$D_{w_{i_r}(\epsilon_r^{(u,v)})} \dots D_{w_{i_1}(\epsilon_1^{(u,v)})}\, e(2,1) \tag{4.4.41}$$

where $\epsilon_j^{(u,v)}$ takes values in $1, 2$ depending on the pair (u, v). As we sum over the pairs (u, v) such that $f_{i_1} \dots f_{i_r} \in u ⧢ v$ then the sequence $\epsilon_r^{(u,v)}, \dots, \epsilon_1^{(u,v)}$ covers all binary sequences in letters 1 and 2 and thus we can rewrite equation 4.4.39 as

$$(-1)^r \big(D_{w_{i_r}(1)} + D_{w_{i_r}(2)}\big) \dots \big(D_{w_{i_1}(1)} + D_{w_{i_1}(2)}\big)\big(e(2,1)\big)\,. \tag{4.4.42}$$

Since D_w is linear in w we can write above as

$$(-1)^r D_{w_{i_r}(1)+w_{i_r}(2)} \dots D_{w_{i_1}(1)+w_{i_1}(2)}\big(e(2,1)\big)\,. \tag{4.4.43}$$

□

Remark 4.4.6. The above result can be extended when working with conjugates with more than two factors by appropriate extensions of the definition of derivations. This fact will be used in section 5.5.

4.5 Initial value series

For $i \in 2\mathbb{N}+1$, let M_i denote the abstract generators of a Lie algebra. In this section, we are interested in series of the form

$$\mathbb{M}^{\mathfrak{dr}} = \sum_{r=0}^{\infty} \sum_{i_1, i_2, \dots, i_r \in 2\mathbb{N}+1} (f_{i_1} \dots f_{i_r})^{\mathfrak{dr}} M_{i_1} \dots M_{i_r}\,. \tag{4.5.1}$$

In particular, we want to prove conjugate identities for the series $\mathbb{M}^{\mathfrak{dr}}$, analogous to what we saw earlier for the Drinfeld associator. Such series appear in section 5.2 where they will be referred to as an 'initial value series'. The results in this section will only be needed in section 5.5 so the reader may skip them until required later on.

Our first goal is to find the inverse of the series $\mathbb{M}^{\mathfrak{dr}}$. So we write down the ansatz

$$(\mathbb{M}^{\mathfrak{dr}})^{-1} = \sum_{r=0}^{\infty} \sum_{i_1, i_2, \dots, i_r \in 2\mathbb{N}+1} f_{i_1} \dots f_{i_r} W_{i_1, \dots, i_r} \tag{4.5.2}$$

and we want to compute the terms $W_{i_1,\dots,i_r}$.

Using the relation

$$\mathbb{M}^{\mathfrak{dr}}(\mathbb{M}^{\mathfrak{dr}})^{-1} = 1 \tag{4.5.3}$$

and comparing coefficients of the words $f_{i_1}\dots f_{i_r}$ in both sides we make the following observations.

1. The constant term of $(\mathbb{M}^{\mathfrak{dr}})^{-1}$ is 1.
2. Looking at the coefficient of f_{i_1} we get

$$M_{i_1} + W_{i_1} = 0\,. \tag{4.5.4}$$

   So we have $W_{i_1} = -M_{i_1}$ for all positive odd integers $i_1 \geq 3$.
3. The coefficient of $f_{i_1}f_{i_2}$ can be found by looking at words whose shuffle product has the term $f_{i_1}f_{i_2}$ in it. We have four possibilities -

$$f_{i_1}f_{i_2}⧢1,\ f_{i_1}⧢f_{i_2},\ f_{i_2}⧢f_{i_1},\ 1⧢f_{i_1}f_{i_2}\,. \tag{4.5.5}$$

   Looking at the coefficient contribution from each term we get the relation

$$M_{i_1}M_{i_2} + M_{i_1}W_{i_2} + M_{i_2}W_{i_1} + W_{i_1,i_2} = 0\,. \tag{4.5.6}$$

   Using 4.5.4 and simplifying we get that $W_{i_1,i_2} = M_{i_2}M_{i_1}$.

The general pattern of the coefficient equation is now clear. To state it precisely we define some notation. For positive odd indices $i_1,\dots,i_r$ and words $f_{i_1}\dots f_{i_r}$ we define

$$M(f_{i_1},\dots,f_{i_r}) = M_{i_1}\dots M_{i_r} \text{ and} \tag{4.5.7}$$

$$W(f_{i_1},\dots,f_{i_r}) = W_{i_1,\dots,i_r}\,. \tag{4.5.8}$$

Then we get the following relation after comparing coefficients in 4.5.3.

$$\sum_{\{(u,v)\colon f_{i_1}\cdots f_{i_r}\in u⧢v\}} C(u,v;f_{i_1}\dots f_{i_r})M(u)W(v) = 0\,. \tag{4.5.9}$$

Recall that we treat all the letters f_{k_i}'s as distinct for ease of computation, and thus, we can simplify the above as follows.

$$\sum_{\{(u,v)\colon f_{i_1}\cdots f_{i_r}\in u⧢v\}} M(u)W(v) = 0\,. \tag{4.5.10}$$

We use this relation to prove the following lemma.

Lemma 4.5.1. *The coefficient of $f_{i_1}\dots f_{i_r}$ in the series $(\mathbb{M}^{\mathfrak{dr}})^{-1}$, $W_{i_1,\dots,i_r}$ is equal to*

$$(-1)^r M_{i_r}\dots M_{i_1}\ . \tag{4.5.11}$$

Proof. We prove this result using induction. We have already checked a couple of base cases. So let r be a positive integer and assume that the lemma is true for all positive integers less than or equal to r, that is,

$$W_{i_1,\dots,i_j} = (-1)^j M_{i_j}\dots M_{i_1} \tag{4.5.12}$$

for positive integers $j \le r$. We will prove the result for $r+1$.

We split the pairs (u, v) such that $C(u, v; f_{i_1}\dots f_{i_{r+1}}) = 1$ into two types. First, we have the pairs where the word u ends in the letter $f_{i_{r+1}}$ and similarly we have the second type of pairs where the word v ends in the letter $f_{i_{r+1}}$.

Note that any pair $(uf_{i_{r+1}}, v)$ from the first type can be changed into a pair $(u, vf_{i_{r+1}})$ from the second type and vice versa. Further, the transfer of the letter $f_{i_{r+1}}$ along with our induction hypothesis 4.5.12 leads to the relation

$$M(uf_{i_{r+1}})W(v) + M(u)W(vf_{i_{r+1}}) = 0\,. \tag{4.5.13}$$

except when $u = f_{i_{r+1}}$ since in that case $v = f_{i_1}\dots f_{i_r}$ and the term $W_{i_1,\dots,i_r}$ is unknown.

However, the above equation 4.5.13 when used with the relation 4.5.10 cancels all the term except the one associated with $u = f_{i_{r+1}}$ and so we do indeed get

$$M(f_{i_{r+1}})W(f_{i_1}\dots f_{i_r}) + M(1)W(f_{i_1}\dots f_{i_r}f_{i_{r+1}}) = 0\,. \tag{4.5.14}$$

Again, by making use of the induction hypothesis 4.5.12 in the above relation we get

$$(-1)^r M_{i_{r+1}}M_{i_r}\dots M_{i_1} + W_{i_1,\dots,i_{r+1}} = 0\,. \tag{4.5.15}$$

This proves the lemma. □

Our next goal is to work out an analogous result to 4.4.2 for the initial value series $\mathbb{M}$.

Let e_0, e_1 be non-commutative variables such that $[e_0, M_i]$, $[e_1, M_i] \in g = Lie[e_0, e_1]$ for $i \ge 3$ odd. Then, for every positive odd integer $i \ge 3$, we define a derivation D_{M_i} on $U(g)$ by setting

$$D_{M_i}(e_0) = [e_0, M_i],\quad D_{M_i}(e_1) = [e_1, M_i] \tag{4.5.16}$$

and then extending the derivation by the Leibniz rule.

Note that the above definition implies

$$D_{M_i}(w) = [w, M_i] \tag{4.5.17}$$

for any word $w(e_0, e_1) \in U(g)$.

Let $X \in Lie[e_0, e_1]$. We want to compute the coefficients of the conjugate series $(\mathbb{M}^{\mathfrak{dr}})^{-1} X \mathbb{M}^{\mathfrak{dr}}$ and we start by looking at some examples.

1. The coefficient of 1 is X.
2. The coefficient of f_{i_1} is
$$-M_{i_1}X + XM_{i_1} = [X, M_{i_1}] = D_{M_{i_1}} X\,. \tag{4.5.18}$$
3. The coefficient of $f_{i_1} f_{i_2}$ can be worked out by looking at the deshuffles again and we get
$$\begin{aligned} M_{i_2}M_{i_1}X - M_{i_2}XM_{i_1} - M_{i_1}XM_{i_2} + XM_{i_1}M_{i_2} \\ = [[X, M_{i_1}], M_{i_2}] = D_{M_{i_2}} D_{M_{i_1}} X\,. \end{aligned} \tag{4.5.19}$$

The general pattern is clear and we reintroduce some notation to state it precisely. For positive odd indices $i_1, \ldots, i_r$ and words $f_{i_1}, \ldots, f_{i_r}$ we define

$$M(f_{i_1}, \ldots, f_{i_r}) = M_{i_1} \ldots M_{i_r} \text{ and} \tag{4.5.20}$$

$$W(f_{i_1}, \ldots, f_{i_r}) = W_{i_1,\ldots,i_r} = (-1)^r M_{i_r} \ldots M_{i_1}\,. \tag{4.5.21}$$

The below result is then clear.

Lemma 4.5.2. *The coefficient of $f_{i_1}, \ldots, f_{i_r}$ in the conjugate series $(\mathbb{M}^{\mathfrak{dr}})^{-1} X \mathbb{M}^{\mathfrak{dr}}$ is equal to*

$$\sum_{\{(u,v):\, f_{i_1} \cdots f_{i_r} \in u \sqcup\!\sqcup v\}} W(u) X M(v)\,. \tag{4.5.22}$$

However, we want to express the above coefficient in terms of derivations and the corresponding result is below.

Theorem 4.5.3 (D. Kamlesh)**.** *The coefficient of $f_{i_1} \ldots f_{i_r}$ in the conjugate series $(\mathbb{M}^{\mathfrak{dr}})^{-1} X \mathbb{M}^{\mathfrak{dr}}$ is equal to*

$$D_{M_{i_r}} \ldots D_{M_{i_1}}(X)\,. \tag{4.5.23}$$

Proof. We prove this result by induction. The first few cases have already been checked. So assume that the result is true for positive integers less than equal to r and we will prove the result for $r+1$.

From proposition 4.5.2 we know that the coefficient of $f_{i_1}, \ldots, f_{i_{r+1}}$ in the conjugate series $(\mathbb{M}^{\mathfrak{dr}})^{-1} X \mathbb{M}^{\mathfrak{dr}}$ is equal to

$$\sum_{\{(\tilde{u},\tilde{v}):\, f_{i_1}\cdots f_{i_{r+1}} \in \tilde{u} \sqcup \tilde{v}\}} W(\tilde{u}) X M(\tilde{v})\,. \tag{4.5.24}$$

By remark 4.4.4 we can split it into a sum

$$\sum_{\{(u,v):\, f_{i_1}\cdots f_{i_r} f_{i_{r+1}} \in u f_{i_{r+1}} \sqcup v\}} W(u f_{i_{r+1}}) X M(v) + \sum_{\{(u,v):\, f_{i_1}\cdots f_{i_r} f_{i_{r+1}} \in u \sqcup v f_{i_{r+1}}\}} W(u) X M(v f_{i_{r+1}})\,. \tag{4.5.25}$$

Again, using remark 4.4.4, we can also drop the factor of $f_{i_{r+1}}$ from the sum pairs to get

$$\sum_{\{(u,v):\, f_{i_1}\cdots f_{i_r} \in u \sqcup v\}} W(u f_{i_{r+1}}) X M(v) + \sum_{\{(u,v):\, f_{i_1}\cdots f_{i_r} \in u \sqcup v\}} W(u) X M(v f_{i_{r+1}}) \tag{4.5.26}$$

Then making use of the M and W notation 4.5.20 we can rewrite above as

$$\sum_{\{(u,v):\, f_{i_1}\cdots f_{i_r} \in u \sqcup v\}} [(-M_{i_{r+1}}) W(u) X M(v) + W(u) X M(v)(M_{i_{r+1}})] \tag{4.5.27}$$

$$= -M_{i_{r+1}} D_{M_{i_r}} \ldots D_{M_{i_1}}(X) + D_{M_{i_r}} \ldots D_{M_{i_1}}(X) M_{i_{r+1}} \tag{4.5.28}$$

$$= [D_{M_{i_r}} \ldots D_{M_{i_1}}(X), M_{i_{r+1}}] = D_{M_{i_{r+1}}} D_{M_{i_r}} \ldots D_{M_{i_1}}(X) \tag{4.5.29}$$

where the last two lines make use of the induction hypothesis and proposition 4.5.2. ☐

Chapter 5

Equivalent criteria for the coaction conjecture

The results in this chapter were achieved as part of a collaboration with Hadleigh Frost, Martijn Hidding, Carlos Rodriguez, Oliver Schlotterer and Bram Verbeek. This progress was first announced in [35] and a more detailed work is meant to appear in a future article [36].

In this chapter we explore the connection between period matrices of hypergeometric functions and series expansion in multiple zeta values and multiple polylogarithms. Our strategy is as follows. The matrix of twisted periods is known to satisfy the KZ differential equation. Since the generating series of multiple polylogarithms solves the KZ equation, this allows us to express the period matrix as a series expansion in polylogarithms up to a factor of initial values comprising of multiple zeta values computed by taking boundary limits. This allows us to derive equivalent criteria for the coaction conjecture in terms of the series coefficients by making use of the Ihara formula. To be completely general we will work abstractly in this chapter and not deal with any explicit period matrices. We will come back to the coaction conjecture in explicit cases in the next chapter. For reasons of convenience and aesthetics, we will also work with a slightly different notation in this chapter, which is discussed below.

5.1 Setup

Let $n \in \mathbb{N}$. We work with a holomorphic function $F = F^{(n)}$ of n-variables

$$F^{(n)} = F^{(n)}(z_1, \dots, z_n) \tag{5.1.1}$$

defined on the coordinates

$$\{(z_1, \dots, z_n \in \mathbb{C}^n) |\, z_i \neq 0, 1 \text{ and } z_i \neq z_j, \text{ for } 1 \leq i \neq j \leq n\}\,. \tag{5.1.2}$$

We also set

$$z_0 = 0,\ z_{n+1} = 1\,. \tag{5.1.3}$$

We should think of the function $F^{(n)}$ as a stand-in for the period matrix of hypergeometric functions. It is already known that[50, 6, 55] such period matrices satisfy matrix-type KZ equations[1]. Hence, we may suppose that the function F satisfies the following partial differential equations for $k = 1, \dots, n$,

$$\frac{\partial}{\partial z_k} F(z_1, \dots, z_n) = F(z_1, \dots, z_n)\Big(\sum_{j=0, j\neq k}^{n+1} \frac{e_{k,j}}{z_k - z_j} \Big) \tag{5.1.4}$$

where we think of $e_{k,j}$'s as the abstract generators of a Lie algebra satisfying the infinitesimal braid relations[2] :

$$e_{i,j} = e_{j,i} \ \text{ if } \ i \neq j\,, \tag{5.1.5}$$

$$[e_{i,j},\, e_{k,l}] = 0 \ \text{ if } \ i \neq j \neq k \neq l \ \text{ and} \tag{5.1.6}$$

$$[e_{i,j} + e_{j,k},\, e_{i,k}] = 0 \ \text{ if } \ i \neq j \neq k \tag{5.1.7}$$

for $i, j, k, l = 0, 1, \dots, n+1$.

We denote the differential factor on the right by

$$X^{(n)}_{k,F} := \sum_{j=0,\, j\neq k}^{n+1} \frac{e_{k,j}}{z_k - z_j} \tag{5.1.8}$$

so that we may write

$$\frac{\partial}{\partial z_k} F(z_1, \dots, z_n) = F(z_1, \dots,\, z_n)\, X^{(n)}_{k,F}\,. \tag{5.1.9}$$

Since $F^{(n)}$ satisfies linear differential equations we can solve for it in terms of iterated integrals. Concretely, we get a solution to 5.1.4 on the branch

$$0 = z_0 < z_1 \dots z_n < z_{n+1} = 1 \tag{5.1.10}$$

[1]A proof of this using the notion of S-bracket will appear in [36].

[2]The existence of braid relations for the $e_{i,j}$'s is also known and a new proof will appear in [36].

by integrating along the path

$$(0,0,\dots,0)\to(0,\dots,0,z_n)\to(0,\dots,0,z_{n-1},z_n)\to\dots\to(z_1,\dots,z_{n-1},z_n)\,. \tag{5.1.11}$$

This gives a series expression for F as

$$F(z_1,\dots,z_n)=$$
$$\mathbb{I}\,\mathbb{G}\left[\begin{smallmatrix} e^*_{n,0} & e_{n,n+1} \\ z_0 & z_{n+1}\end{smallmatrix};z_n\right]\dots\mathbb{G}\left[\begin{smallmatrix} e^*_{k,0} & e_{k,k+1} & \cdots & e_{k,n+1} \\ z_0 & z_{k+1} & \cdots & z_{n+1}\end{smallmatrix};z_k\right]\dots\mathbb{G}\left[\begin{smallmatrix} e^*_{1,0} & e_{1,2} & \cdots & e_{1,n+1} \\ z_0 & z_2 & \cdots & z_{n+1}\end{smallmatrix};z_1\right] \tag{5.1.12}$$

where

$$\mathbb{I}:=\lim_{z_n\to 0}\dots\lim_{z_1\to 0}F\,z_1^{-e^*_{1,0}}\dots z_n^{-e^*_{n,0}} \tag{5.1.13}$$

is the initial value factor and the starred entries for $k=1,\dots,n$ are given by

$$e^*_{k,0}=e_{k,0}+\sum_{j=1}^{k-1}e_{k,j}\,. \tag{5.1.14}$$

Note that we have $e^*_{1,0}=e_{1,0}$. Also, observe that the series of polylogarithms $\mathbb{G}(-;z_k)$ in 5.1.12 depends only on z_k and the variables with higher indices, that is, $z_{k+1},\dots,z_n$, but not on $z_1,\dots,z_{k-1}$.

For simplicity, we will sometimes abbreviate the series of polylogarithms in 5.1.12 as

$$\mathbb{G}_k=\mathbb{G}(z_k;\overrightarrow{e^*_{k,j}}):=\mathbb{G}\left[\begin{smallmatrix} e^*_{k,0} & e_{k,k+1} & \cdots & e_{k,n+1} \\ z_0 & z_{k+1} & \cdots & z_{n+1}\end{smallmatrix};z_k\right] \tag{5.1.15}$$

and write

$$F(z_1,\dots,z_n)=\mathbb{I}\,\mathbb{G}_n\dots\mathbb{G}_1\,. \tag{5.1.16}$$

Remark 5.1.1. It is known that the initial value factor $\mathbb{I}$ can be expressed in terms of multiple zeta values[3] and we usually write it in terms of the f-alphabet of multiple zeta values [19, 20] which is described below.

Definition 5.1.2 (f-alphabet)**.** The $\mathbb{Q}$-algebra of motivic multiple zeta values is non-canonically isomorphic to the $\mathbb{Q}$-algebra of polynomials in the commutative letter f_2 and non-commutative letters f_k, for each odd weight $k\in 2\mathbb{N}+1$, with multiplication given by shuffle product.

The coaction of a word in the f-alphabet is given by deconcatenation,

$$\Delta(f_2^j f_{i_1}f_{i_2}\dots f_{i_r})^{\mathfrak{m}}=(f_2^j)^{\mathfrak{m}}\sum_{q=0}^{r}(f_{i_1}f_{i_2}\dots f_{i_q})^{\mathfrak{m}}(f_{i_{q+1}}\dots f_{i_r})^{\mathfrak{dr}} \tag{5.1.17}$$

[3]This is known by the work of Terasoma on Selberg integrals [55].

where $j \geq 0, r \geq 0$ and $i_1, \ldots, i_r \in 2\mathbb{N}+1$.

In the rest of this text, we fix an isomorphism to the f-alphabet. It is known that the coefficients of $\zeta(2)$ and the odd zeta values $\zeta(2\mathbb{N}+1)$ in the Drinfeld associator can be written as a commutator which we denote by w_k. So we have an expression of the form

$$\Phi(e_0, e_1) = 1 + f_2\, w_2 + f_3\, w_3 + f_5\, w_5 + \ldots \tag{5.1.18}$$

The above mentioned features and more properties of the Drinfeld associator are covered in section 4. Also note that the choice of the isomorphism to the f-alphabet may change the commutators w_k but the form of the results in this text is independent of this choice and will stay the same.

Before proceeding to the next section we also recall the Ihara coaction formula for ease of reference.

The motivic coaction on the generating series $\mathbb{G}_k$ is given by

$$\Delta\mathbb{G}^{\mathfrak{m}}\left[\begin{smallmatrix} e^*_{k,0} & e_{k,k+1} & \cdots & e_{k,n+1} \\ z_0 & z_{k+1} & \cdots & z_{n+1} \end{smallmatrix}; z_k\right] = \\ \mathbb{G}^{\mathfrak{m}}\left[\begin{smallmatrix} e^*_{k,0} & e'_{k,k+1} & \cdots & e'_{k,n+1} \\ z_0 & z_{k+1} & \cdots & z_{n+1} \end{smallmatrix}; z_k\right] \mathbb{G}^{\mathfrak{dr}}\left[\begin{smallmatrix} e^*_{k,0} & e_{k,k+1} & \cdots & e_{k,n+1} \\ z_0 & z_{k+1} & \cdots & z_{n+1} \end{smallmatrix}; z_k\right] \tag{5.1.19}$$

where the conjugates $e'_{k,j}$ are defined by

$$e'_{k,j} = \mathbb{G}^{\mathfrak{dr}}\left[\begin{smallmatrix} e^*_{k,0} & e_{k,k+1} & \cdots & e_{k,n+1} \\ z_0 & z_{k+1} & \cdots & z_{n+1} \end{smallmatrix}; z_k \to z_j\right] e_{k,j} \left(\mathbb{G}^{\mathfrak{dr}}\left[\begin{smallmatrix} e^*_{k,0} & e_{k,k+1} & \cdots & e_{k,n+1} \\ z_0 & z_{k+1} & \cdots & z_{n+1} \end{smallmatrix}; z_k \to z_j\right]\right)^{-1}. \tag{5.1.20}$$

Here, the notation $z_k \to z_j$ means that we set $z_k = z_j$ in the generating series. We note again that the terms in the generating series are shuffle-regularized to remove any endpoint singularities.

5.2 Equivalent criteria

Theorem 5.2.1 (Frost-Hidding-Kamlesh-Rodriguez-Schlotterer-Verbeek)**.** *The following statements are equivalent -*

1.

$$\Delta(F^{\mathfrak{m}}) = F^{\mathfrak{m}}\, F^{\mathfrak{dr}}\,. \tag{5.2.1}$$

2.

$$\Delta(\mathbb{I}^{\mathfrak{m}}) = \mathbb{I}^{\mathfrak{m}}\, \mathbb{I}^{\mathfrak{dr}} \tag{5.2.2}$$

and for $1 \leq k \leq n$, we have

$$\Delta(\mathbb{G}^{\mathfrak{m}}_k) = (\mathbb{I}^{\mathfrak{dr}}\mathbb{G}^{\mathfrak{dr}}_n \dots \mathbb{G}^{\mathfrak{dr}}_{k+1})^{-1}\, \mathbb{G}^{\mathfrak{m}}_k\, (\mathbb{I}^{\mathfrak{dr}}\mathbb{G}^{\mathfrak{dr}}_n \dots \mathbb{G}^{\mathfrak{dr}}_{k+1})\, \mathbb{G}^{\mathfrak{dr}}_k. \tag{5.2.3}$$

3. $\Delta(\mathbb{I}^{\mathfrak{m}}) = \mathbb{I}^{\mathfrak{m}}\, \mathbb{I}^{\mathfrak{dr}}$ *and for $1 \leq k \leq n$, $k+1 \leq j \leq n+1$, we have*

$$(\mathbb{I}^{\mathfrak{dr}}\mathbb{G}^{\mathfrak{dr}}_n \dots \mathbb{G}^{\mathfrak{dr}}_{k+1})^{-1}\, e^*_{k,0}\, (\mathbb{I}^{\mathfrak{dr}}\mathbb{G}^{\mathfrak{dr}}_n \dots \mathbb{G}^{\mathfrak{dr}}_{k+1}) = e^*_{k,0} \quad \text{and} \tag{5.2.4}$$

$$(\mathbb{I}^{\mathfrak{dr}}\mathbb{G}^{\mathfrak{dr}}_n \dots \mathbb{G}^{\mathfrak{dr}}_{k+1})^{-1}\, e_{k,j}\, (\mathbb{I}^{\mathfrak{dr}}\mathbb{G}^{\mathfrak{dr}}_n \dots \mathbb{G}^{\mathfrak{dr}}_{k+1}) = \mathbb{G}^{\mathfrak{dr}}(z_k = z_j;\, \overrightarrow{e^*_{k,j}})\, e_{k,j}\, \mathbb{G}^{\mathfrak{dr}}(z_k = z_j;\, \overrightarrow{e^*_{k,j}})^{-1}. \tag{5.2.5}$$

4. *For $k \geq 1$, there exists P_{2k}, M_{2k+1} such that*

$$\mathbb{I}^{\mathfrak{m}} = \mathbb{P}^{\mathfrak{m}}\, \mathbb{M}^{\mathfrak{m}} \quad \text{where} \tag{5.2.6}$$

$$\mathbb{P}^{\mathfrak{m}} = \sum_{j=0}^{\infty} (f_2^j)^{\mathfrak{m}}\, P_{2j} \quad \text{and} \tag{5.2.7}$$

$$\mathbb{M}^{\mathfrak{m}} = \sum_{r=0}^{\infty} \sum_{i_1, i_2, \dots, i_r \in 2\mathbb{N}+1} (f_{i_1} f_{i_2} \dots f_{i_r})^{\mathfrak{m}} M_{i_1} M_{i_2} \dots M_{i_r}\,, \tag{5.2.8}$$

and the terms P_{2j}, M_i are the abstract generators of a free Lie algebra satisfying the following commutator relations.

(a) *For $i = 1, \dots, n$,*

$$[e^*_{i,0},\, M_{2k+1}] = [\sum_{j=0}^{i-1} e_{i,j}\,,\, M_{2k+1}] = 0\,. \tag{5.2.9}$$

(b) *For $i = 2, \dots, n+1$*

$$[e_{1,i}, M_{2k+1}] = -\sum_{j=1}^{2i-3} [e_{1,i}\,,\, w_{2k+1}(j)] \tag{5.2.10}$$

where $w_k(j) \coloneqq w_k(e(j,\, 0),\, e(j,\, 1))$ with

$$e(2a-1,\, 0) = e_{1,0} + \sum_{b=2}^{a} e_{1,b}\,, \tag{5.2.11}$$

$$e(2a-1,\, 1) = e(2a, 0) = e_{1,a+1}\,, \tag{5.2.12}$$

$$e(2a,\, 1) = e_{a+1,0} + \sum_{b=2}^{a} e_{a+1,b} \tag{5.2.13}$$

defined for $a = 1, \dots, n$.

(c) More generally, for $i = 1, \ldots, n$, $j = i+1, \ldots, n+1$ and $k \in 2\mathbb{N}+1$,

$$[e_{i,j}, M_k] = [\sum_{t=i}^{j-1} w_k(\sum_{r=0, r\neq i}^{t} e_{r,i},\, e_{i,t+1}) + \sum_{t=i}^{j-2} w_k(e_{i,t+1}, \sum_{r=0,r\neq i}^{t} e_{r,t+1}),\, e_{i,j}]. \tag{5.2.14}$$

Note that the w_k's above refer to the commutator coefficients 5.1.18 of the zeta values in the Drinfeld associator.

Remark 5.2.2. Before proceeding to the proof we briefly discuss the origin of the definitions and relations introduced in the criteria *4* above.

1. The factorization of the initial value factor $\mathbb{I}$ into the series factors $\mathbb{P}$ and $\mathbb{M}$ is intimately tied to the coaction property of $\mathbb{I}$. Recall that the Beta function satisfies the same coaction property and has a series expansion in terms of single zeta values 1.0.1 which can be split into two factors corresponding to only even and odd zeta values. This phenomenon can be generalized to the factorization for $\mathbb{I}$ mentioned in the theorem above when higher-depth zeta values also appear and was first observed in [51], [34] in the context of tree-level open superstring amplitudes. We will discuss this further at the end of section 6.2.
2. The identities 5.2.4, 5.2.5 in criteria *3* above involve conjugation by the factor $\mathbb{I}$ (with associated variables P_{2j}, M_i) on the left of the equation, whereas the right side of the equation consists only of series in the variables e_{ij}. To transform the left side of the equation into a series involving only e_{ij}'s additional relations are required between the Lie variables P_{2j}, M_i and e_{ij}. This is incorporated via the commutator relations listed in criteria *4*. These relations were already observed to hold numerically in low depths for specific matrix representations in (4.23) of [12].

Remark 5.2.3. The conjugate terms in the criteria *3* above also appear in [30], [31] in the context of single-valued iterated Eisenstein integrals, which is an elliptic analogue.

Proof. **I.** $1 \iff 2:$ Suppose that 1 holds, that is, $\Delta(F^{\mathfrak{m}}) = F^{\mathfrak{m}}\, F^{\mathfrak{dr}}$. Then, on setting $z_i = 0$ for $i = 1, \ldots, n$, we get that

$$\Delta(F^{\mathfrak{m}}(z_j = 0)) = F^{\mathfrak{m}}(z_j = 0)\, F^{\mathfrak{dr}}(z_j = 0) \tag{5.2.15}$$

and hence

$$\Delta(\mathbb{I}^{\mathfrak{m}}) = \mathbb{I}^{\mathfrak{m}}\, \mathbb{I}^{\mathfrak{dr}}\,. \tag{5.2.16}$$

Similarly, setting $z_i = 0$ for $i = 1, \ldots, n-1$ but not $i = n$ we have that

$$\Delta(\mathbb{I}^{\mathfrak{m}})\,\Delta(\mathbb{G}^{\mathfrak{m}}_n) = \Delta(\mathbb{I}^{\mathfrak{m}}\,\mathbb{G}^{\mathfrak{m}}_n) = (\mathbb{I}^{\mathfrak{m}}\mathbb{G}^{\mathfrak{m}}_n)\,(\mathbb{I}^{\mathfrak{dr}}\,\mathbb{G}^{\mathfrak{dr}}_n). \tag{5.2.17}$$

Therefore, applying 5.2.16 to the above equation we get

$$\Delta(\mathbb{G}^{\mathfrak{m}}_n) = (\mathbb{I}^{\mathfrak{dr}})^{-1}\,\mathbb{G}^{\mathfrak{m}}_n\,\mathbb{I}^{\mathfrak{dr}}\,\mathbb{G}^{\mathfrak{dr}}_n\,. \tag{5.2.18}$$

Repeating the steps above we get 5.2.3 and hence 2 holds.

Now, suppose that 2 holds. Then, multiplying equation 5.2.2 and 5.2.3 for $k = n$ to 1 we get 5.2.1 and therefore 1 holds.

II. 2 $\iff$ 3: Suppose that 2 holds. That is, for $1 \le k \le n$, we have

$$\begin{aligned}\Delta\mathbb{G}^{\mathfrak{m}}\left[\begin{smallmatrix} e^*_{k,0} & e_{k,k+1} & \cdots & e_{k,n+1} \\ z_0 & z_{k+1} & \cdots & z_{n+1}\end{smallmatrix}\,;z_k\right] = \\ (\mathbb{I}^{\mathfrak{dr}}\mathbb{G}^{\mathfrak{dr}}_n\ldots\mathbb{G}^{\mathfrak{dr}}_{k+1})^{-1}\,\mathbb{G}^{\mathfrak{m}}\left[\begin{smallmatrix} e^*_{k,0} & e_{k,k+1} & \cdots & e_{k,n+1} \\ z_0 & z_{k+1} & \cdots & z_{n+1}\end{smallmatrix}\,;z_k\right] \\ \times\,(\mathbb{I}^{\mathfrak{dr}}\mathbb{G}^{\mathfrak{dr}}_n\ldots\mathbb{G}^{\mathfrak{dr}}_{k+1})\,\mathbb{G}^{\mathfrak{dr}}\left[\begin{smallmatrix} e^*_{k,0} & e_{k,k+1} & \cdots & e_{k,n+1} \\ z_0 & z_{k+1} & \cdots & z_{n+1}\end{smallmatrix}\,;z_k\right].\end{aligned} \tag{5.2.19}$$

By Ihara formula 2.3.7, we also know that,

$$\begin{aligned}\Delta\mathbb{G}^{\mathfrak{m}}\left[\begin{smallmatrix} e^*_{k,0} & e_{k,k+1} & \cdots & e_{k,n+1} \\ z_0 & z_{k+1} & \cdots & z_{n+1}\end{smallmatrix}\,;z_k\right] = \\ \mathbb{G}^{\mathfrak{m}}\left[\begin{smallmatrix} e^*_{k,0} & e'_{k,k+1} & \cdots & e'_{k,n+1} \\ z_0 & z_{k+1} & \cdots & z_{n+1}\end{smallmatrix}\,;z_k\right]\,\mathbb{G}^{\mathfrak{dr}}\left[\begin{smallmatrix} e^*_{k,0} & e_{k,k+1} & \cdots & e_{k,n+1} \\ z_0 & z_{k+1} & \cdots & z_{n+1}\end{smallmatrix}\,;z_k\right]\end{aligned} \tag{5.2.20}$$

with $e'_{k,j}$ as defined in 5.1.20. Equating the two expressions, we can cancel the rightmost factor and compare the coefficient of weight 1 motivic logarithm in z_k on both sides. Observe that up to weight one we have

$$\mathbb{G}^{\mathfrak{m}}\left[\begin{smallmatrix} e^*_{k,0} & e_{k,k+1} & \cdots & e_{k,n+1} \\ z_0 & z_{k+1} & \cdots & z_{n+1}\end{smallmatrix}\,;z_k\right] = 1+\log^{\mathfrak{m}}(z_k)e^*_{k,0}+\sum_{j=k+1}^{n+1}\big(\log^{\mathfrak{m}}(z_j-z_k)-\log^{\mathfrak{m}}(z_j)\big)e_{k,j}+\ldots \tag{5.2.21}$$

Thus, the coefficient of $\log^{\mathfrak{m}}(z_k)$ in 5.2.19 is equal to

$$(\mathbb{I}^{\mathfrak{dr}}\mathbb{G}^{\mathfrak{dr}}_n\ldots\mathbb{G}^{\mathfrak{dr}}_{k+1})^{-1}\,e^*_{k,0}\,(\mathbb{I}^{\mathfrak{dr}}\mathbb{G}^{\mathfrak{dr}}_n\ldots\mathbb{G}^{\mathfrak{dr}}_{k+1}) \tag{5.2.22}$$

and in 5.2.20 it is equal to $e^*_{k,0}$. Therefore, we get

$$(\mathbb{I}^{\mathfrak{dr}}\mathbb{G}^{\mathfrak{dr}}_n\ldots\mathbb{G}^{\mathfrak{dr}}_{k+1})^{-1}\,e^*_{k,0}\,(\mathbb{I}^{\mathfrak{dr}}\mathbb{G}^{\mathfrak{dr}}_n\ldots\mathbb{G}^{\mathfrak{dr}}_{k+1}) = e^*_{k,0}. \tag{5.2.23}$$

Similarly, the coefficient of $\log^{\mathfrak{m}}(z_j - z_k)$ in 5.2.19 is equal to

$$(\mathbb{I}^{\mathfrak{dr}}\mathbb{G}^{\mathfrak{dr}}_n \dots \mathbb{G}^{\mathfrak{dr}}_{k+1})^{-1} e_{k,j} \, (\mathbb{I}^{\mathfrak{dr}}\mathbb{G}^{\mathfrak{dr}}_n \dots \mathbb{G}^{\mathfrak{dr}}_{k+1}) \,, \tag{5.2.24}$$

whereas, in 5.2.20 it is equal to

$$e^{'}_{k,j} = \mathbb{G}^{\mathfrak{dr}} \left[\begin{smallmatrix} e^*_{k,0} & e_{k,k+1} & \cdots & e_{k,n+1} \\ z_0 & z_{k+1} & \cdots & z_{n+1} \end{smallmatrix} ; z_j \right] e_{k,j} \, \mathbb{G}^{\mathfrak{dr}} \left[\begin{smallmatrix} e^*_{k,0} & e_{k,k+1} & \cdots & e_{k,n+1} \\ z_0 & z_{k+1} & \cdots & z_{n+1} \end{smallmatrix} ; z_j \right]^{-1} . \tag{5.2.25}$$

Thus, we also get the equation

$$\begin{aligned} &(\mathbb{I}^{\mathfrak{dr}}\mathbb{G}^{\mathfrak{dr}}_n \dots \mathbb{G}^{\mathfrak{dr}}_{k+1})^{-1} \, e_{k,j} \, (\mathbb{I}^{\mathfrak{dr}}\mathbb{G}^{\mathfrak{dr}}_n \dots \mathbb{G}^{\mathfrak{dr}}_{k+1}) = \\ &\qquad \mathbb{G}^{\mathfrak{dr}}(\overrightarrow{e^*_{k,j}}; z_k \to z_j) \, e_{k,j} \, \mathbb{G}^{\mathfrak{dr}}(\overrightarrow{e^*_{k,j}}; z_k \to z_j)^{-1}. \end{aligned} \tag{5.2.26}$$

Conversely, suppose that 3 holds. Then, in the Ihara formula,

$$\begin{aligned} &\Delta \mathbb{G}^{\mathfrak{m}} \left[\begin{smallmatrix} e^*_{k,0} & e_{k,k+1} & \cdots & e_{k,n+1} \\ z_0 & z_{k+1} & \cdots & z_{n+1} \end{smallmatrix} ; z_k \right] = \\ &\qquad \mathbb{G}^{\mathfrak{m}} \left[\begin{smallmatrix} e^*_{k,0} & e^{'}_{k,k+1} & \cdots & e^{'}_{k,n+1} \\ z_0 & z_{k+1} & \cdots & z_{n+1} \end{smallmatrix} ; z_k \right] \, \mathbb{G}^{\mathfrak{dr}} \left[\begin{smallmatrix} e^*_{k,0} & e_{k,k+1} & \cdots & e_{k,n+1} \\ z_0 & z_{k+1} & \cdots & z_{n+1} \end{smallmatrix} ; z_k \right] \end{aligned} \tag{5.2.27}$$

we can substitute for $e^*_{k,0}$ and $e^{'}_{k,j}$ with equations 5.2.23, 5.2.26 and take the common conjugating factor out to get

$$\begin{aligned} &\Delta \mathbb{G}^{\mathfrak{m}} \left[\begin{smallmatrix} e^*_{k,0} & e_{k,k+1} & \cdots & e_{k,n+1} \\ z_0 & z_{k+1} & \cdots & z_{n+1} \end{smallmatrix} ; z_k \right] = \\ &\qquad (\mathbb{I}^{\mathfrak{dr}}\mathbb{G}^{\mathfrak{dr}}_n \dots \mathbb{G}^{\mathfrak{dr}}_{k+1})^{-1} \, \mathbb{G}^{\mathfrak{m}} \left[\begin{smallmatrix} e^*_{k,0} & e_{k,k+1} & \cdots & e_{k,n+1} \\ z_0 & z_{k+1} & \cdots & z_{n+1} \end{smallmatrix} ; z_k \right] \\ &\qquad\qquad \times (\mathbb{I}^{\mathfrak{dr}}\mathbb{G}^{\mathfrak{dr}}_n \dots \mathbb{G}^{\mathfrak{dr}}_{k+1}) \, \mathbb{G}^{\mathfrak{dr}} \left[\begin{smallmatrix} e^*_{k,0} & e_{k,k+1} & \cdots & e_{k,n+1} \\ z_0 & z_{k+1} & \cdots & z_{n+1} \end{smallmatrix} ; z_k \right] . \end{aligned} \tag{5.2.28}$$

III. $3 \implies 4$ **:** Recall from remark 5.1.1 that the initial value series $\mathbb{I}$ can be expressed in terms of the multiple zeta values. Since the $\mathbb{Q}$ algebra of motivic multiple zeta values in the f-alphabet is generated by the non-commutative letters f_k for $k \geq 3$ odd and f_2 where f_2 commutes with all f_k, we can write the series $\mathbb{I}$ in the form below,

$$\mathbb{I}^{\mathfrak{m}} = \sum_{j \geq 0} \sum_{r \geq 0} \sum_{i_1, i_2, \dots, i_r \in 2\mathbb{N}+1} (f_2^j)^{\mathfrak{m}} \, (f_{i_1} f_{i_2} \dots f_{i_r})^{\mathfrak{m}} \, W_{2j, i_1, \dots, i_r} \tag{5.2.29}$$

for some Lie algebra generators $W_{2j,i_1,\dots,i_r}$. Now, suppose that 3 holds. Then we have $\Delta \mathbb{I}^{\mathfrak{m}} = \mathbb{I}^{\mathfrak{m}} \mathbb{I}^{\mathfrak{dr}}$. Therefore, we get that,

$$\Delta\mathbb{I}^{\mathfrak{m}} = \sum_{j\geq 0}\sum_{r\geq 0}\sum_{i_1,i_2,\ldots,i_r\in 2\mathbb{N}+1} (f_2^j)^{\mathfrak{m}}\,(f_{i_1}f_{i_2}\ldots f_{i_r})^{\mathfrak{m}}\,W_{2j,i_1,\ldots,i_r} \\ \times\sum_{j\geq 0}\sum_{r\geq 0}\sum_{i_1,i_2,\ldots,i_r\in 2\mathbb{N}+1} (f_{i_1}f_{i_2}\ldots f_{i_r})^{\mathfrak{dr}}\,W_{2j,i_1,\ldots,i_r} \tag{5.2.30}$$

since $f_2^{\mathfrak{dr}} = 0$. Recall also that the coaction on the f-alphabet is given by deconcatenation,

$$\Delta(f_2^j f_{i_1}f_{i_2}\ldots f_{i_r})^{\mathfrak{m}} = (f_2^j)^{\mathfrak{m}}\sum_{q=0}^{r}(f_{i_1}f_{i_2}\ldots f_{i_q})^{\mathfrak{m}}(f_{i_{q+1}}\ldots f_{i_r})^{\mathfrak{dr}} \tag{5.2.31}$$

for $j\geq 0, r\geq 0$ and $i_1,\ldots,i_r\in 2\mathbb{N}+1$. Therefore, the coaction of $\mathbb{I}^{\mathfrak{m}}$ is also given by

$$\Delta\mathbb{I}^{\mathfrak{m}} = \sum_{j\geq 0}\sum_{r\geq 0}\sum_{i_1,i_2,\ldots,i_r\in 2\mathbb{N}+1} (f_2^j)^{\mathfrak{m}}\sum_{q=0}^{r}(f_{i_1}f_{i_2}\ldots f_{i_q})^{\mathfrak{m}}(f_{i_{q+1}}\ldots f_{i_r})^{\mathfrak{dr}}\,W_{2j,i_1,\ldots,i_r} \tag{5.2.32}$$

So let us compare the coefficients on both sides. For $j\in\mathbb{N}\cup\{0\}$, $r\in\mathbb{N}$, the coefficient of $(f_2^j)^{\mathfrak{m}}f_{i_1}\ldots f_{i_{r-1}}\otimes f_{i_r}$ in 5.2.30 is equal to $W_{2j,i_1,\ldots,i_{r-1}}W_{i_r}$, whereas, in 5.2.32 it is equal to $W_{2j,i_1,\ldots,i_r}$. Therefore, by induction, we have that

$$W_{2j,i_1,\ldots,i_r} = W_{2j}W_{i_1}\ldots W_{i_r} \tag{5.2.33}$$

for all $j\in\mathbb{N}\cup\{0\}$ and $r\in\mathbb{N}$. Finally, relabelling the W_{2j} as P_{2j} and W_{i_r} as M_{i_r} we get that

$$\mathbb{I}^{\mathfrak{m}} = \sum_{j\geq 0}\sum_{r\geq 0}\sum_{i_1,i_2,\ldots,i_r\in 2\mathbb{N}+1} (f_2^j)^{\mathfrak{m}}\,(f_{i_1}f_{i_2}\ldots f_{i_r})^{\mathfrak{m}}\,P_{2j}M_{i_1}\ldots M_{i_r} \tag{5.2.34}$$

which can be simplified to

$$\mathbb{I}^{\mathfrak{m}} = \mathbb{P}^{\mathfrak{m}}\mathbb{M}^{\mathfrak{m}} \tag{5.2.35}$$

with the notation

$$\mathbb{P}^{\mathfrak{m}} = \sum_{j=0}^{\infty}(f_2^j)^{\mathfrak{m}}\,P_{2j} \quad\text{and} \tag{5.2.36}$$

$$\mathbb{M}^{\mathfrak{m}} = \sum_{r=0}^{\infty}\sum_{i_1,i_2,\ldots,i_r\in 2\mathbb{N}+1}(f_{i_1}f_{i_2}\ldots f_{i_r})^{\mathfrak{m}}M_{i_1}M_{i_2}\ldots M_{i_r}\,. \tag{5.2.37}$$

Next, we need to derive the commutator relations. In that direction, let us consider equation 5.2.4 which is valid for $k = 1,\ldots,n$,

$$(\mathbb{I}^{\mathfrak{dr}}\mathbb{G}_n^{\mathfrak{dr}}\ldots\mathbb{G}_{k+1}^{\mathfrak{dr}})^{-1}\,e_{k,0}^*\,(\mathbb{I}^{\mathfrak{dr}}\mathbb{G}_n^{\mathfrak{dr}}\ldots\mathbb{G}_{k+1}^{\mathfrak{dr}}) = e_{k,0}^*\,. \tag{5.2.38}$$

First, let us fix k. Then, on setting $z_j = 0$ for $j = k+1, \dots, n$, the above equation reduces to

$$(\mathbb{I}^{\mathfrak{dr}})^{-1}\, e^*_{k,0}\, \mathbb{I}^{\mathfrak{dr}} \;= e^*_{k,0}\,. \tag{5.2.39}$$

Since $\pi^{\mathfrak{dr}} = 0$, we have $\mathbb{P}^{\mathfrak{dr}} = 1$, $\mathbb{I}^{\mathfrak{dr}} = \mathbb{M}^{\mathfrak{dr}}$ and we get

$$(\mathbb{M}^{\mathfrak{dr}})^{-1}\, e^*_{k,0}\, \mathbb{M}^{\mathfrak{dr}} \;= e^*_{k,0}\,. \tag{5.2.40}$$

Observe that we have

$$(\mathbb{M}^{\mathfrak{dr}})^{-1} = 1 - \sum_{i_1 \in 2\mathbb{N}+1} f_{i_1} M_{i_1} + \dots \tag{5.2.41}$$

and thus the conjugate is equal to

$$(\mathbb{M}^{\mathfrak{dr}})^{-1}\, e^*_{k,0}\, \mathbb{M}^{\mathfrak{dr}} = e^*_{k,0} + \sum_{i_1 \in 2\mathbb{N}+1} [e^*_{k,0}, M_{i_1}] + \dots \tag{5.2.42}$$

Now, let $i \in \mathbb{N}$. Then, comparing the coefficient of f_{2i+1} on both sides of 5.2.40 we get that

$$[e^*_{k,0},\, M_{2i+1}] = 0\,. \tag{5.2.43}$$

Next, consider the equation 5.2.5 which is valid for $k = 1, \dots, n$ and $j = k+1, \dots, n$,

$$(\mathbb{I}^{\mathfrak{dr}}\mathbb{G}^{\mathfrak{dr}}_n \dots \mathbb{G}^{\mathfrak{dr}}_{k+1})^{-1}\, e_{k,j}\, (\mathbb{I}^{\mathfrak{dr}}\mathbb{G}^{\mathfrak{dr}}_n \dots \mathbb{G}^{\mathfrak{dr}}_{k+1}) = \\ \mathbb{G}^{\mathfrak{dr}}(z_k \to z_j;\, \overrightarrow{e^*_{k,j}})\, e_{k,j}\, \mathbb{G}^{\mathfrak{dr}}(z_k \to z_j;\, \overrightarrow{e^*_{k,j}})^{-1}. \tag{5.2.44}$$

Without loss of generality, let us fix $k = 1$ and $k+1 \le j \le n$ and rewrite the above as follows.

$$(\mathbb{I}^{\mathfrak{dr}})^{-1}\, e_{1,j}\, \mathbb{I}^{\mathfrak{dr}} = \\ \mathbb{G}^{\mathfrak{dr}}_n \dots \mathbb{G}^{\mathfrak{dr}}_2\, \mathbb{G}^{\mathfrak{dr}}(z_1 = z_j;\, \overrightarrow{e^*_{1,j}})\, e_{1,j}\, \big(\mathbb{G}^{\mathfrak{dr}}_n \dots \mathbb{G}^{\mathfrak{dr}}_2 \mathbb{G}^{\mathfrak{dr}}(z_1 = z_j;\, \overrightarrow{e^*_{1,j}})\big)^{-1}. \tag{5.2.45}$$

As before, the strategy is to look at the coefficients of single zeta values. However, we need to get rid of the polylogarithms first. We do this by taking the limits $z_i \to 0$ for $i = 2, \dots, n$ in the increasing order of i. This limit has been computed in the section 5.4 on initial values. It is the result 5.4.3 which we recall here.

$$\lim_{z_i \to 0,\, i \ge 2}\ \lim_{z_1 \to z_j}\ \mathbb{G}(z_n; \overrightarrow{e^*_{n,j}}) \dots \mathbb{G}(z_1; \overrightarrow{e^*_{1,j}})\, e_{1,l}\, \big(\mathbb{G}(z_n; \overrightarrow{e^*_{n,j}}) \dots \mathbb{G}(z_1; \overrightarrow{e^*_{1,j}})\big)^{-1} \\ = \prod_{a=1}^{2j-3} \Phi(e(a,\, 0),\, e(a,\, 1))\, e_{1,j} \prod_{a=1}^{2j-3} \big(\Phi(e(a,\, 0),\, e(a,\, 1))\big)^{-1}. \tag{5.2.46}$$

where the terms $e(a,0), e(a,1)$ are the same as the ones defined in 5.2.10.

Note that the inverse of the Drinfeld associator looks like

$$\Phi(e_0, e_1)^{-1} = 1 - f_2\, w_2 - f_3\, w_2 - f_5\, w_5 - \dots \tag{5.2.47}$$

Thus, the coefficient of f_{2i+1} in the above limit is equal to

$$\sum_{a=1}^{2j-3} [w_{2i+1}(e(a,0),\, e(a,1))\,,\, e_{1,j}]\,. \tag{5.2.48}$$

Similarly, the coefficient of f_{2i+1} in the conjugate $(\mathbb{I}^{\mathfrak{dr}})^{-1}\, e_{1,j}\, \mathbb{I}^{\mathfrak{dr}}$ is equal to $[e_{1,j},\, M_{2i+1}]$ and thus the equality 5.2.45 gives us the commutator relation 5.2.10 as required.

The proof of the relation 5.2.14 follows from similar limit computations in the case $k > 1$, so we will skip it here.

IV. $4 \implies 3$ **:** First, let us fix some notation. We write

$$Z_l = Z(z_l\,;\, e_{1,0} \dots e_{1,n+1}) := \mathbb{G}\left[\begin{smallmatrix} e_{1,0} & e_{1,2} & \dots & e_{1,n+1} \\ z_0 & z_2 & \dots & z_{n+1} \end{smallmatrix}\,;\, z_1 = z_l\right] \tag{5.2.49}$$

for $l = 2, \dots, n$. We often refer to the Z_l's as generalized Drinfeld associators.

We can consider the Drinfeld conjugate defined for $l = 2, \dots, n$ as

$$e^{'}_{1,l} = Z^{\mathfrak{dr}}(z_l\,;\, e_{1,0} \dots e_{1,n+1})\, e_{1,l}\, Z^{dR}(z_l\,;\, e_{1,0} \dots e_{1,n+1})^{-1}\,. \tag{5.2.50}$$

And we want to prove that for $l = 2, \dots, n$ we have

$$e^{'}_{1,l} = (\mathbb{M}^{\mathfrak{dr}}\mathbb{G}^{\mathfrak{dr}}_n \dots \mathbb{G}^{\mathfrak{dr}}_2)^{-1}\, e_{1,l}\, (\mathbb{M}^{\mathfrak{dr}}\mathbb{G}^{\mathfrak{dr}}_n \dots \mathbb{G}^{\mathfrak{dr}}_2)\,. \tag{5.2.51}$$

There are other conjugates to consider as well but the proof of those identities is similar so the above case is enough for the purpose of illustration.

Going forward we will drop the deRham superscript in the next few sections and assume it implicitly unless mentioned otherwise. Further, keeping the initial value constant factors on one side we can rephrase the conjugate identity as

$$(\mathbb{G}_n \dots \mathbb{G}_2\, Z_l)\, e_{1,l}\, (\mathbb{G}_n \dots \mathbb{G}_2\, Z_l)^{-1} = \mathbb{M}^{-1}\, e_{1,l}\, \mathbb{M}\,. \tag{5.2.52}$$

We will prove the above version provided the variables $e_{k,j}$ and M_k satisfy the commutator relations mentioned in the theorem. The steps are a bit involved and will be covered in the next few sections.

Briefly, our strategy is to show that both sides of the equation 5.2.52 have the same partial derivatives in z_k and the same initial limiting values. To this end, we first compute all the partial derivatives of the factors $\mathbb{G}_j$ and Z_l in section 5.3. Next, we compute the initial value limits of the left-hand side in section 5.4. This requires knowledge of analytic continuation of F beyond the standard simplex. This information is encoded in certain representations of the braid group which is recalled in section 5.4.1. In section 5.5 we deal with conjugate series of generalized Drinfeld associators. An explicit computation of the coefficients of conjugate Drinfeld associators is covered in chapter 4 which will be useful in section 5.5. Finally, we put the results together to get the conjugate identity in section 5.5.

□

5.3 Partial derivatives

We recall the derivatives of the series of polylogarithms $\mathbb{G}_k$,

$$\frac{\partial}{\partial z_k}\mathbb{G}_k = \mathbb{G}_k\,\Big(\frac{e^*_{k,0}}{z_k - z_0} + \sum_{j=k+1}^{n+1} \frac{e_{k,j}}{z_k - z_j}\Big)\,. \tag{5.3.1}$$

We also denote the differential factors by

$$X^{(n)}_{k,\mathbb{G}} := \frac{e^*_{k,0}}{z_k - z_0} + \sum_{j=k+1}^{n+1} \frac{e_{k,j}}{z_k - z_j} \tag{5.3.2}$$

so that we may write

$$\frac{\partial}{\partial z_k}\mathbb{G}_k = \mathbb{G}_k\, X^{(n)}_{k,\mathbb{G}}\,. \tag{5.3.3}$$

Lemma 5.3.1. *For $k = 2, \ldots, n$ and $l = 2, \ldots, n+1$, we have*

$$\frac{\partial}{\partial z_k}\,(\mathbb{M}^{-1}\, e_{1,l}\, \mathbb{M}) = 0\,. \tag{5.3.4}$$

Proof. This is clear since $\mathbb{M}$ is a constant series independent of the variables $z_2, \ldots, z_n$. □

Let us introduce some notation before we compute the partial derivatives of the polylogarithmic factor $\mathbb{G}_1$.

Notation 5.3.1. For $k = 2, \ldots, n$, we write $\tau_{(2k)} \in S_n$ to denote the transposition that swaps 2 and k. Note that we impose $\tau_{(2k)}(0) = 0$, $\tau_{(2k)}(n+1) = n+1$. We also write $\partial_k = \frac{\partial}{\partial z_k}$ to denote the partial derivative operator with respect to z_k.

Lemma 5.3.2. *For $k = 2, \dots, n$, we have*

$$\partial_k \mathbb{G}_1 = -\tau_{(2k)}(X^{(n)}_{2,G}) \mathbb{G}_1 + \mathbb{G}_1 X^{(n)}_{k,F} . \tag{5.3.5}$$

where $\tau_{(2k)}$ acts on the indices of z_j and e_{ij} in $X^{(n)}_{2,G}$.

Proof. First, we prove the result for $k = 2$. Taking the derivative with respect to z_2 in the following equation

$$F^{(n)} = \mathbb{G}_n \dots \mathbb{G}_1 \tag{5.3.6}$$

we get that

$$\partial_2 F^{(n)} = \partial_2(\mathbb{G}_n \dots \mathbb{G}_1), \text{ that is} \tag{5.3.7}$$

$$F^{(n)} X^{(n)}_{2,F} = \partial_2(\mathbb{G}_n \dots \mathbb{G}_1) \text{ by 5.1.9 and thus} \tag{5.3.8}$$

$$\mathbb{G}_n \dots \mathbb{G}_1 X^{(n)}_{2,F} = \mathbb{G}_n \dots \mathbb{G}_3 \partial_2(\mathbb{G}_2 \mathbb{G}_1) \tag{5.3.9}$$

since $\mathbb{G}_k$ does not depend on z_2 for $k \geq 3$. Since the factors $\mathbb{G}_k$ are invertible we can cancel factors on both sides to get

$$\mathbb{G}_2 \mathbb{G}_1 X^{(n)}_{2,F} = \partial_2(\mathbb{G}_2) \mathbb{G}_1 + \mathbb{G}_2 \partial_2(\mathbb{G}_1) = \mathbb{G}_2 X^{(n)}_{2,\mathbb{G}} \mathbb{G}_1 + \mathbb{G}_2 \partial_2(\mathbb{G}_1) \text{ by 5.3.3.} \tag{5.3.10}$$

Again, cancelling the factor $\mathbb{G}_2$ from both sides we get

$$\partial_2(\mathbb{G}_1) = -X^{(n)}_{2,\mathbb{G}} \mathbb{G}_1 + \mathbb{G}_1 X^{(n)}_{2,F} . \tag{5.3.11}$$

Finally, we can compute the partial derivative with respect to any $k = 2, \dots, n$ by applying the transposition $\tau_{(2k)}$. Precisely,

$$\partial_k \mathbb{G}_1 = \tau_{(2k)}(\partial_2 \mathbb{G}_1) = \tau_{(2k)}(-X^{(n)}_{2,\mathbb{G}} \mathbb{G}_1 + \mathbb{G}_1 X^{(n)}_{2,F}) = -\tau_{(2k)}(X^{(n)}_{2,\mathbb{G}}) \mathbb{G}_1 + \mathbb{G}_1 X^{(n)}_{k,F} \tag{5.3.12}$$

since $\tau_{(2k)}(X^{(n)}_{2,F}) = X^{(n)}_{k,F}$ and $\tau_{(2k)} \mathbb{G}_1 = \mathbb{G}_1$. □

Corollary 5.3.3. *For $k = 2, \dots, n$ we have*

$$\partial_k(\mathbb{G}_n \dots \mathbb{G}_2) = \mathbb{G}_n \dots \mathbb{G}_2(\tau_{(2k)} X^{(n)}_{2,\mathbb{G}}) . \tag{5.3.13}$$

Proof. We have

$$F^{(n)} = \mathbb{G}_n \dots \mathbb{G}_1 . \tag{5.3.14}$$

Taking partial derivatives on both sides we get

$$\partial_k F^{(n)} = \partial_k(\mathbb{G}_n \dots \mathbb{G}_1), \text{ that is} \tag{5.3.15}$$

$$\mathbb{G}_n \dots \mathbb{G}_1\, X^{(n)}_{k,\,F} = \partial_k(\mathbb{G}_n \dots \mathbb{G}_n)\, \mathbb{G}_1 + \mathbb{G}_n \dots \mathbb{G}_n \partial_k(\mathbb{G}_1)\,. \tag{5.3.16}$$

Applying lemma 5.3.2 we get

$$\mathbb{G}_n \dots \mathbb{G}_1\, X^{(n)}_{k,\,F} = \partial_k(\mathbb{G}_n \dots \mathbb{G}_2)\, \mathbb{G}_1 + \mathbb{G}_n \dots \mathbb{G}_2(-\tau_{(2k)}(X^{(n)}_{2,\mathbb{G}})\, \mathbb{G}_1 + \mathbb{G}_1\, X^{(n)}_{k,F}) \tag{5.3.17}$$

$$\text{and thus } \; \partial_k(\mathbb{G}_n \dots \mathbb{G}_2)\, \mathbb{G}_1 - \mathbb{G}_n \dots \mathbb{G}_2(\tau_{(2k)} X^{(n)}_{2,\mathbb{G}})\, \mathbb{G}_1 = 0\,. \tag{5.3.18}$$

Finally, cancelling the factor of $\mathbb{G}_1$ from both sides we are left with

$$\partial_k(\mathbb{G}_n \dots \mathbb{G}_2) = \mathbb{G}_n \dots \mathbb{G}_2(\tau_{(2k)} X^{(n)}_{2,\mathbb{G}})\,. \tag{5.3.19}$$

□

Corollary 5.3.4. *For $k = 2, \dots, n$ we have*

$$\partial_k\, (\mathbb{G}_n \dots \mathbb{G}_2)^{-1} = -(\tau_{(2k)} X^{(n)}_{2,\mathbb{G}})\, (\mathbb{G}_n \dots \mathbb{G}_2)^{-1}\,. \tag{5.3.20}$$

Proof. We have

$$\partial_k\big((\mathbb{G}_n \dots \mathbb{G}_2)\, (\mathbb{G}_n \dots \mathbb{G}_2)^{-1}\big) = \partial_k(1) = 0, \text{ therefore,} \tag{5.3.21}$$

$$\partial_k(\mathbb{G}_n \dots \mathbb{G}_2)\, (\mathbb{G}_n \dots \mathbb{G}_2)^{-1} + (\mathbb{G}_n \dots \mathbb{G}_2)\, \partial_k\, (\mathbb{G}_n \dots \mathbb{G}_2)^{-1} = 0\,. \tag{5.3.22}$$

Applying corollary 5.3.4 we get

$$\mathbb{G}_n \dots \mathbb{G}_2(\tau_{(2k)} X^{(n)}_{2,\mathbb{G}})\, (\mathbb{G}_n \dots \mathbb{G}_2)^{-1} + \mathbb{G}_n \dots \mathbb{G}_2\, \partial_k\, (\mathbb{G}_n \dots \mathbb{G}_2)^{-1} = 0 \tag{5.3.23}$$

and after cancelling factors, we are left with

$$\partial_k\, (\mathbb{G}_n \dots \mathbb{G}_2)^{-1} = -(\tau_{(2k)} X^{(n)}_{2,\mathbb{G}})\, (\mathbb{G}_n \dots \mathbb{G}_2)^{-1}\,. \tag{5.3.24}$$

□

Next, we will compute the partial derivatives of Z_l.

Proposition 5.3.5. *For For $k = 2, \dots, n$ and $l = 2, \dots, n, n+1$ we have*

$$\partial_k\, Z_l = -\tau_{(2k)}(X^{(n)}_{2,\mathbb{G}})\, Z_l + Z_l\, (X^{(n)}_{k,\,F} - \frac{e_{k,1}}{z_k - z_1} + \frac{e_{k,1}}{z_k - z_l}) \;\; \textit{if } k \neq l \textit{ and} \tag{5.3.25}$$

$$\partial_k\, Z_k = -\tau_{(2k)}(X^{(n)}_{2,\mathbb{G}})\, Z_k + Z_k\, (X^{(n)}_{k,\,F} - \frac{e_{k,1}}{z_k - z_1} + (X^{(n)}_{1,\,F} - \frac{e_{1,k}}{z_1 - z_k})|_{z_1 \to z_k})\,. \tag{5.3.26}$$

Proof. Case I: $k \neq l$.

We have

$$Z_l = \mathbb{G}(z_l;\, \overrightarrow{e_{1,j}}) = 1 + (\mathbb{G}(z_l;\, \overrightarrow{e_{1,j}}) - 1) = 1 + \int_0^{z_l} \big(\partial_1 \mathbb{G}(z_1;\, \overrightarrow{e_{1,j}})\big)\, dz_1\,. \tag{5.3.27}$$

From equation 5.1.9 we know that

$$\partial_1 \mathbb{G}_1 = \mathbb{G}_1\, X^{(n)}_{1,\,F} \tag{5.3.28}$$

and so we get

$$Z_l = 1 + \int_0^{z_l} \mathbb{G}_1\, X^{(n)}_{1,\,F}\, dz_1\,. \tag{5.3.29}$$

Taking the partial derivative with respect to z_k on both sides in the last equation we get

$$\partial_k\, Z_l = \partial_k \Big(\int_0^{z_l} \mathbb{G}_1\, X^{(n)}_{1,F}\, dz_1 \Big)\,. \tag{5.3.30}$$

Since $l \neq k$ we can use Leibniz's rule to push the partial derivative inside the integral and apply the product rule to get

$$\partial_k Z_l = \int_0^{z_l} \partial_k\, (\mathbb{G}_1)\, X^{(n)}_{1,F}\, dz_1 + \int_0^{z_l} \mathbb{G}_1\, \partial_k (X^{(n)}_{1,F})\, dz_1\,. \tag{5.3.31}$$

Denote the two terms on the right-hand side above by I and II, respectively. Applying lemma 5.3.2 we get

$$I = \int_0^{z_l} \partial_k\, (\mathbb{G}_1)\, X^{(n)}_{1,F}\, dz_1 = -\tau_{(2k)}(X^{(n)}_{2,\mathbb{G}}) \int_0^{z_l} \mathbb{G}_1\, X^{(n)}_{1,F} dz_1 + \int_0^{z_l} \mathbb{G}_1\, X^{(n)}_{k,\,F}\, X^{(n)}_{1,F} dz_1\,. \tag{5.3.32}$$

and we call the two terms on the right $I(a)$ and $I(b)$ respectively.

By equation 5.3.27 we can rewrite $I(a)$ as

$$I(a) = -\tau_{(2k)}(X^{(n)}_{2,\mathbb{G}}) \int_0^{z_l} \mathbb{G}_1\, X^{(n)}_{1,F} dz_1 = -\tau_{(2k)}(X^{(n)}_{2,\mathbb{G}})(Z_l - 1)\,. \tag{5.3.33}$$

Further, applying lemma 7.0.1 we can rewrite $I(b)$ as

$$I(b) = \int_0^{z_l} \mathbb{G}_1\, X^{(n)}_{k,\,F}\, X^{(n)}_{1,F} dz_1 = \int_0^{z_l} \mathbb{G}_1\, X^{(n)}_{1,\,F}\, X^{(n)}_{k,F} dz_1\,. \tag{5.3.34}$$

Next, recall equation 5.1.8 and note that we have

$$\partial_k X_{1,F}^{(n)} = \partial_k (\sum_{j=0, j\neq k}^{n+1} \frac{e_{k,j}}{z_k - z_j}) = \frac{e_{k,1}}{(z_1 - z_k)^2} = \partial_1 \frac{e_{k,1}}{z_k - z_1} \,. \tag{5.3.35}$$

Therefore, term II becomes

$$II = \int_0^{z_l} \mathbb{G}_1 \, \partial_k (X_{1,F}^{(n)}) \, dz_1 = \int_0^{z_l} \mathbb{G}_1 \, \partial_1 \frac{e_{k,1}}{z_k - z_1} dz_1 \,. \tag{5.3.36}$$

Using integration by parts we get

$$II = \big[\mathbb{G}_1 \, \frac{e_{k,1}}{z_k - z_1}\big]_0^{z_l} - \int_0^{z_l} \partial_1 \mathbb{G}_1 \frac{e_{k,1}}{z_k - z_1} dz_1 \,. \tag{5.3.37}$$

Call the two terms on the right in the above equation by II(a) and II(b) respectively. Then, we have

$$II(a) = \big[\mathbb{G}_1 \, \frac{e_{k,1}}{z_k - z_1}\big]_0^{z_l} = Z_l \frac{e_{k,1}}{z_k - z_l} - \frac{e_{k,1}}{z_k} \,, \tag{5.3.38}$$

$$II(b) = -\int_0^{z_l} \partial_1 \mathbb{G}_1 \frac{e_{k,1}}{z_k - z_1} dz_1 = -\int_0^{z_l} \mathbb{G}_1 \, X_{1,F}^{(n)} \frac{e_{k,1}}{z_k - z_1} dz_1 \,. \tag{5.3.39}$$

Adding terms $I(b)$ and $II(b)$ we get

$$I(b) + II(b) = \int_0^{z_l} \mathbb{G}_l \, X_{1,\,F}^{(n)} \, X_{k,F}^{(n)} \, dz_1 - \int_0^{z_l} \mathbb{G}_1 \, X_{1,F}^{(n)} \frac{e_{k,1}}{z_k - z_1} dz_1 \tag{5.3.40}$$

$$= \int_0^{z_l} \mathbb{G}_l \, X_{1,\,F}^{(n)} \big(X_{k,F}^{(n)} - \frac{e_{k,1}}{z_k - z_1} \big) \mathrm{d}z_1 \tag{5.3.41}$$

$$= \int_0^{z_l} \mathbb{G}_l \, X_{1,\,F}^{(n)} \, dz_1 \big(X_{k,F}^{(n)} - \frac{e_{k,1}}{z_k - z_1} \big) \tag{5.3.42}$$

since $\big(X_{k,F}^{(n)} - \frac{e_{k,1}}{z_k - z_1}\big)$ is independent of z_1. Following equation 5.3.27 we can rewrite above as

$$I(b) + II(b) = (Z_l - 1) \big(X_{k,F}^{(n)} - \frac{e_{k,1}}{z_k - z_1} \big) \,. \tag{5.3.43}$$

Finally, we add all the terms together to get

$$\partial_k Z_l = I + II = I(a) + I(b) + II(b) + II(a), \text{ that is} \tag{5.3.44}$$

$$\partial_k Z_l = -\tau_{(2k)} (X_{2,\mathbb{G}}^{(n)}) (Z_l - 1) + (Z_l - 1) \big(X_{k,F}^{(n)} - \frac{e_{k,1}}{z_k - z_1} \big) + Z_l \frac{e_{k,1}}{z_k - z_l} - \frac{e_{k,1}}{z_k} \,. \tag{5.3.45}$$

Expanding the brackets and pairing some terms together we are left with

$$-\tau_{(2k)}(X^{(n)}_{2,\mathbb{G}})\, Z_l + Z_l\big(X^{(n)}_{k,F} - \frac{e_{k,1}}{z_k - z_1} + \frac{e_{k,1}}{z_k - z_l}\big) + \big(\tau_{(2k)}(X^{(n)}_{2,\mathbb{G}}) - \frac{e_{k,1}}{z_k} + \frac{e_{k,1}}{z_k - z_1} - X^{(n)}_{k,F}\big)\,. \tag{5.3.46}$$

Now, note that the last bracketed term above is equal to

$$\tau_{(2k)}(X^{(n)}_{2,\mathbb{G}}) - \frac{e_{k,1}}{z_k} + \frac{e_{k,1}}{z_k - z_1} - X^{(n)}_{k,F} \tag{5.3.47}$$

$$= \frac{e_{k,0} + e_{k,1}}{z_k} + \sum_{j=2,\, j\neq k}^{n+1} \frac{e_{k,j}}{z_k - z_j} - \frac{e_{k,1}}{z_k} + \frac{e_{k,1}}{z_k - z_1} - X^{(n)}_{k,F} \tag{5.3.48}$$

$$= \sum_{j=0,\, j\neq k}^{n+1} \frac{e_{k,j}}{z_k - z_j} - X^{(n)}_{k,F} = 0 \tag{5.3.49}$$

by definition of $X^{(n)}_{k,F}$. Therefore, we are left with

$$\partial_k Z_l = -\tau_{(2k)}(X^{(n)}_{2,\mathbb{G}})\, Z_l + Z_l\big(X^{(n)}_{k,F} - \frac{e_{k,1}}{z_k - z_1} + \frac{e_{k,1}}{z_k - z_l}\big) \tag{5.3.50}$$

as required to show.

Case II: $k = l$.

Repeating the steps from the previous case we recall that

$$Z_k = 1 + \int_0^{z_k} (\partial_1\, \mathbb{G}_1)\, dz_1 = 1 + \int_0^{z_k} \mathbb{G}_1\, X^{(n)}_{1,\,F}\, dz_1 \tag{5.3.51}$$

and thus

$$\partial_k\, Z_k = \partial_k\big(\int_0^{z_k} \mathbb{G}_1\, X^{(n)}_{1,F}\, dz_1\big)\,. \tag{5.3.52}$$

Applying Leibniz's rule of differentiation under the integral sign we get

$$\partial_k\, Z_k = (\mathbb{G}_1\, X^{(n)}_{1,F})|_{z_1\to z_k} + \int_0^{z_k} \partial_k(\mathbb{G}_1\, X^{(n)}_{1,F})\, dz_1 \tag{5.3.53}$$

and we denote the latter terms by I and II respectively. Precisely, we have

$$I = (\mathbb{G}_1\, X^{(n)}_{1,F})|_{z_1\to z_k} = Z_k\, (X^{(n)}_{1,F})|_{z_1\to z_k} \tag{5.3.54}$$

and we already know how to evaluate II from the calculation in the previous case.

$$II = \int_0^{z_k} \partial_k(\mathbb{G}_1\, X^{(n)}_{1,F})\, dz_1 = -\tau_{(2k)}(X^{(n)}_{2,\mathbb{G}})\, Z_k + Z_k\big(X^{(n)}_{k,F} - \frac{e_{k,1}}{z_k - z_1} - \frac{e_{1,k}}{z_1 - z_k}|_{z_1\to z_k}\big)\,. \tag{5.3.55}$$

Adding terms I and II we get

$$\partial_k Z_k = -\tau_{(2k)}(X^{(n)}_{2,\mathbb{G}})\, Z_k + Z_k\, (X^{(n)}_{k,\,F} - \frac{e_{k,1}}{z_k - z_1} + (X^{(n)}_{1,\,F} - \frac{e_{1,k}}{z_1 - z_k})|_{z_1\to z_k})\,. \quad (5.3.56)$$

Since $X^{(n)}_{1,\,F} - \frac{e_{1,k}}{z_1-z_k}$ is convergent as $z_1 \to z_k$ we get a convergent expression in the equation above. This proves the required result. □

Corollary 5.3.6. *For $k = 2,\dots,n$ and $l = 2,\dots,n,n+1$ we have*

$$\partial_k Z_l^{-1} = Z_l^{-1}\,\tau_{(2k)}(X^{(n)}_{2,\mathbb{G}}) - (X^{(n)}_{k,\,F} - \frac{e_{k,1}}{z_k - z_1} + \frac{e_{k,1}}{z_k - z_l})\, Z_l^{-1} \quad \textit{if } k \neq l \textit{ and} \quad (5.3.57)$$

$$\partial_k Z_k^{-1} = Z_k^{-1}\,\tau_{(2k)}(X^{(n)}_{2,\mathbb{G}}) - (X^{(n)}_{k,\,F} - \frac{e_{k,1}}{z_k - z_1} + (X^{(n)}_{1,\,F} - \frac{e_{1,k}}{z_1 - z_k})|_{z_1 \to z_k})\, Z_k^{-1}. \quad (5.3.58)$$

Proof. The proof is the same in both cases, so we just consider the case when k is not equal to l. Then, we have

$$\partial_k(Z_l)\, Z_l^{-1} + Z_l \partial_k(Z_l^{-1}) = \partial_k(Z_l\, Z_l^{-1}) = \partial_k(1) = 0\,. \quad (5.3.59)$$

Therefore, applying proposition 5.3.5 we get

$$\big(-\tau_{(2k)}(X^{(n)}_{2,\mathbb{G}})\, Z_l + Z_l\, (X^{(n)}_{k,\,F} - \frac{e_{k,1}}{z_k - z_1} + \frac{e_{k,1}}{z_k - z_l})\big)\, Z_l^{-1} + Z_l \partial_k(Z_l^{-1}) = 0\,. \quad (5.3.60)$$

After left multiplication with Z_l^{-1} and shuffling terms we get

$$\partial_k Z_l^{-1} = Z_l^{-1}\,\tau_{(2k)}(X^{(n)}_{2,\mathbb{G}}) - (X^{(n)}_{k,\,F} - \frac{e_{k,1}}{z_k - z_1} + \frac{e_{k,1}}{z_k - z_l})\, Z_l^{-1}\,. \quad (5.3.61)$$

□

Theorem 5.3.7 (Frost-Hidding-Kamlesh-Rodriguez-Schlotterer-Verbeek)**.** *For $k = 2,\dots,n$ and $l = 2,\dots,n+1$, we have*

$$\partial_k\big(\mathbb{G}_n \dots \mathbb{G}_2\, Z_l\, e_{1,l}\, Z_l^{-1}\, (\mathbb{G}_n \dots \mathbb{G}_2)^{-1}\big) = 0\,. \quad (5.3.62)$$

Proof. Write $S = \mathbb{G}_n \dots \mathbb{G}_2$ and $Y = Z_l\, e_{1,l}\, Z_l^{-1}$. Then, we have

$$\partial_k(S\,Y\,S^{-1}) = \partial_k(S)\,Y\,S^{-1} + S\,\partial_k(Y)\,S^{-1} + S\,Y\,\partial_k(S^{-1}) \;\text{ with} \quad (5.3.63)$$

$$\partial_k(S) = S\,(\tau_{(2k)}X^{(n)}_{2,\mathbb{G}}) \;\text{ and }\; \partial_k(S^{-1}) = -(\tau_{(2k)}X^{(n)}_{2,\mathbb{G}})\,S^{-1} \quad (5.3.64)$$

by proposition 5.3.3 and corollary 5.3.4 respectively. Therefore,

$$\partial_k(S)\,Y\,S^{-1} + S\,Y\,\partial_k(S^{-1}) = S\,(\tau_{(2k)}X^{(n)}_{2,\mathbb{G}})\,Y\,S^{-1} - S\,Y\,(\tau_{(2k)}X^{(n)}_{2,\mathbb{G}})\,S^{-1} \quad (5.3.65)$$

$$\text{and thus,}\ \ \partial_k(S)\,Y\,S^{-1} + S\,Y\,\partial_k(S^{-1}) = S\,[\tau_{(2k)}X^{(n)}_{2,\mathbb{G}},\, Y]\,S^{-1}\,. \tag{5.3.66}$$

Further, we have

$$\partial_k(Y) = \partial_k(Z_l)\,e_{1,l}\,Z_l^{-1} + Z_l\,e_{1,l}\,\partial_k(Z_l^{-1})\,. \tag{5.3.67}$$

Case 1: $k \neq l$.

$$\begin{aligned}
\partial_k(Y) &= \big(-\tau_{(2k)}(X^{(n)}_{2,\mathbb{G}})\,Z_l + Z_l\,(X^{(n)}_{k,\,F} - \frac{e_{k,1}}{z_k - z_1} + \frac{e_{k,1}}{z_k - z_l})\big)\,e_{1,l}\,Z_l^{-1} \\
&\quad + Z_l\,e_{1,l}\big(Z_l^{-1}\,\tau_{(2k)}(X^{(n)}_{2,\mathbb{G}}) - (X^{(n)}_{k,\,F} - \frac{e_{k,1}}{z_k - z_1} + \frac{e_{k,1}}{z_k - z_l})\,Z_l^{-1}\big) \\
&= -[\tau_{(2k)}X^{(n)}_{2,\mathbb{G}},\, Z_l\,e_{1,l}\,Z_l^{-1}] + Z_l\,[X^{(n)}_{k,\,F} - \frac{e_{k,1}}{z_k - z_1} + \frac{e_{k,1}}{z_k - z_l},\, e_{1,l}]\,Z_l^{-1} \\
&= -[\tau_{(2k)}X^{(n)}_{2,\mathbb{G}},\, Y] + Z_l\,[X^{(n)}_{k,\,F} - \frac{e_{k,1}}{z_k - z_1} + \frac{e_{k,1}}{z_k - z_l},\, e_{1,l}]\,Z_l^{-1}\,.
\end{aligned} \tag{5.3.68}$$

Conjugating the above equation with S and adding it to equation 5.3.66 we get

$$\begin{aligned}
\partial_k(S)\,Y\,S^{-1} + S\,Y\,\partial_k(S^{-1}) + S\,\partial_k(Y)\,S^{-1} &= S\,[\tau_{(2k)}X^{(n)}_{2,\mathbb{G}},\, Y]\,S^{-1} \\
- S\,[\tau_{(2k)}X^{(n)}_{2,\mathbb{G}},\, Y]\,S^{-1} &+ S\,Z_l\,[X^{(n)}_{k,\,F} - \frac{e_{k,1}}{z_k - z_1} + \frac{e_{k,1}}{z_k - z_l},\, e_{1,l}]\,Z_l^{-1}\,S^{-1} \\
&= S\,Z_l\,[X^{(n)}_{k,\,F} - \frac{e_{k,1}}{z_k - z_1} + \frac{e_{k,1}}{z_k - z_l},\, e_{1,l}]\,Z_l^{-1}\,S^{-1}\,.
\end{aligned} \tag{5.3.69}$$

Now, recall that

$$X^{(n)}_{k,F} = \sum_{j=0}^{n+1} \frac{e_{k,j}}{z_k - z_j} \tag{5.3.70}$$

and consider the commutator

$$[X^{(n)}_{k,\,F} - \frac{e_{k,1}}{z_k - z_1} + \frac{e_{k,1}}{z_k - z_l},\, e_{1,l}] = [\sum_{j=0,\,j\neq k}^{n+1} \frac{e_{k,j}}{z_k - z_j} - \frac{e_{k,1}}{z_k - z_1} + \frac{e_{k,1}}{z_k - z_l},\, e_{1,l}] \tag{5.3.71}$$

$$= [\sum_{j=0,\,j\neq 1,k,l}^{n+1} \frac{e_{k,j}}{z_k - z_j} + \frac{e_{k,l} + e_{k,1}}{z_k - z_l},\, e_{1,l}]\,. \tag{5.3.72}$$

For $1 \neq j \neq k \neq l$ we have $[e_{k,j},\, e_{1,l}] = 0$ and $[e_{k,l} + e_{k,1},\, e_{1,l}] = 0$. Therefore, the commutator above is zero and we get that

$$\partial_k\big(\mathbb{G}_n \ldots \mathbb{G}_2\,Z_l\,e_{1,l}\,Z_l^{-1}\,(\mathbb{G}_n \ldots \mathbb{G}_2)^{-1}\big) = \partial_k(S\,Y\,S^{-1}) = 0\,. \tag{5.3.73}$$

Case II: $k = l$.

$$
\begin{aligned}
\partial_k(Y) &= \partial_k(Z_k)\, e_{1,k}\, Z_k^{-1} + Z_k\, e_{1,k}\, \partial_k(Z_k^{-1}) \\
&= \big(-\tau_{(2k)}(X^{(n)}_{2,\mathbb{G}})\, Z_k + Z_k\, (X^{(n)}_{k,\,F} - \frac{e_{k,1}}{z_k - z_1} + (X^{(n)}_{1,\,F} - \frac{e_{1,k}}{z_1 - z_k})|_{z_1 \to z_k})\big)\, e_{1,k}\, Z_k^{-1} \\
&+ Z_k\, e_{1,k}\, \big(Z_k^{-1}\, \tau_{(2k)}(X^{(n)}_{2,\mathbb{G}}) - (X^{(n)}_{k,\,F} - \frac{e_{k,1}}{z_k - z_1} + (X^{(n)}_{1,\,F} - \frac{e_{1,k}}{z_1 - z_k})|_{z_1 \to z_k}\big)\, Z_k^{-1}) \\
&= -[\tau_{(2k)} X^{(n)}_{2,\mathbb{G}},\, Z_k\, e_{1,k}\, Z_k^{-1}] + Z_k\, [X^{(n)}_{k,\,F} - \frac{e_{k,1}}{z_k - z_1} + (X^{(n)}_{1,\,F} - \frac{e_{1,k}}{z_1 - z_k})|_{z_1 \to z_k},\, e_{1,k}]\, Z_k^{-1} \\
&= -[\tau_{(2k)} X^{(n)}_{2,\mathbb{G}},\, Y] + Z_k\, [X^{(n)}_{k,\,F} - \frac{e_{k,1}}{z_k - z_1} + (X^{(n)}_{1,\,F} - \frac{e_{1,k}}{z_1 - z_k})|_{z_1 \to z_k},\, e_{1,k}]\, Z_k^{-1}\,.
\end{aligned}
\tag{5.3.74}
$$

Conjugating the above equation with S and adding it to equation 5.3.66 we get

$$
\begin{aligned}
&\partial_k(S)\, Y\, S^{-1} + S\, Y\, \partial_k(S^{-1}) + S\, \partial_k(Y)\, S^{-1} = S\, [\tau_{(2k)} X^{(n)}_{2,\mathbb{G}},\, Y]\, S^{-1} \\
&- S\, [\tau_{(2k)} X^{(n)}_{2,\mathbb{G}},\, Y]\, S^{-1} + S\, Z_k\, [X^{(n)}_{k,\,F} - \frac{e_{k,1}}{z_k - z_1} + (X^{(n)}_{1,\,F} - \frac{e_{1,k}}{z_1 - z_k})|_{z_1 \to z_k},\, e_{1,k}]\, Z_k^{-1}\, S^{-1} \\
&= S\, Z_k\, [X^{(n)}_{k,\,F} - \frac{e_{k,1}}{z_k - z_1} + (X^{(n)}_{1,\,F} - \frac{e_{1,k}}{z_1 - z_k})|_{z_1 \to z_k},\, e_{1,k}]\, Z_k^{-1}\, S^{-1}\,.
\end{aligned}
\tag{5.3.75}
$$

Recall that

$$
X^{(n)}_{k,\,F} - \frac{e_{k,1}}{z_k - z_1} = \sum_{j=0,\, j\neq k}^{n+1} \frac{e_{k,j}}{z_k - z_j} - \frac{e_{k,1}}{z_k - z_1} = \sum_{j=0,\, j\neq 1,k}^{n+1} \frac{e_{k,j}}{z_k - z_j} \quad \text{and} \tag{5.3.76}
$$

$$
(X^{(n)}_{1,\,F} - \frac{e_{1,k}}{z_1 - z_k})|_{z_1 \to z_k} = \Big(\sum_{j=0,\, j\neq 1}^{n+1} \frac{e_{1,j}}{z_1 - z_j} - \frac{e_{1,k}}{z_1 - z_k}\Big)|_{z_1 \to z_k} \tag{5.3.77}
$$

$$
= \Big(\sum_{j=0,\, j\neq 1,k}^{n+1} \frac{e_{1,j}}{z_1 - z_j}\Big)|_{z_1 \to z_k} = \sum_{j=0,\, j\neq 1,k}^{n+1} \frac{e_{1,j}}{z_k - z_j}\,. \tag{5.3.78}
$$

Therefore,

$$
\partial_k(S\, Y\, S^{-1}) = S\, Z_k\, [\sum_{j=0,\, j\neq 1,k}^{n+1} \frac{e_{k,j}}{z_k - z_j} + \sum_{j=0,\, j\neq 1,k}^{n+1} \frac{e_{1,j}}{z_k - z_j},\, e_{1,k}]\, Z_k^{-1}\, S^{-1} \tag{5.3.79}
$$

$$
= S\, Z_k\, [\sum_{j=0,\, j\neq 1,k}^{n+1} \frac{e_{k,j} + e_{1,j}}{z_k - z_j},\, e_{1,k}]\, Z_k^{-1}\, S^{-1}\,. \tag{5.3.80}
$$

Since $[e_{k,j} + e_{1,j},\, e_{1,k}] = 0$ for $1 \neq j \neq k$ we get that

$$
\partial_k\, (S\, Y\, S^{-1}) = 0\,. \tag{5.3.81}
$$

□

5.4 Initial Value Limits

Our aim in this section is to compute the initial value limit

$$\lim_{z_j \to 0,\, j \geq 2} \mathbb{G}_n \ldots \mathbb{G}_2 \, Z_l \tag{5.4.1}$$

for $l = 2, \ldots, n+1$ where we take the limit in order of increasing j from 2 to n.

Observe that $Z_l = \lim_{z_1 \to z_l} \mathbb{G}_1$ and therefore

$$\lim_{z_j \to 0,\, j \geq 2} \mathbb{G}_n \ldots \mathbb{G}_1 \, Z_l = \lim_{z_j \to 0,\, j \geq 2} \mathbb{G}_n \ldots \mathbb{G}_2 \, \lim_{z_1 \to z_l} \mathbb{G}_1 = \lim_{z_j \to 0,\, j \geq 2} \, \lim_{z_1 \to z_l} \mathbb{G}_n \ldots \mathbb{G}_2 \, \mathbb{G}_1 \tag{5.4.2}$$

since the factors $\mathbb{G}_j$ are independent of z_1 for $j \geq 2$.

To compute the above limit we will need to work with the analytic continuation of hypergeometric functions. This is given in terms of a representation of the braid group which is discussed in the next section.

5.4.1 Representation of the braid group

The following discussion is adapted from Section 5 and Appendix B of [12]. We will skip the use of motivic superscripts in this section for ease of notation.

For $n \in \mathbb{N}$, $n \geq 2$ we write B_n to denote the Braid group on n strands. Recall that B_n is a non-commutative group generated by the symbols $\sigma_i := \sigma_{i,i+1}, 1 \leq i \leq n-1$ modulo the relations

$$\sigma_i \, \sigma_j = \sigma_j \, \sigma_i \quad \text{for} \quad |i-j| \geq 2 \quad \text{and} \tag{5.4.3}$$

$$\sigma_i \, \sigma_{i+1} \, \sigma_i = \sigma_{i+1} \, \sigma_i \, \sigma_{i+1} \quad \text{for} \quad 1 \leq 1 \leq n-2 \, . \tag{5.4.4}$$

Further, let S_n denote the symmetric group on n letters. There exists a canonical projection map

$$pr : B_n \to S_n \; := \; \sigma_{i,\, i+1} \mapsto \sigma^{pr}_{i,i+1} = (i,\, i+1) \tag{5.4.5}$$

that maps $\sigma_{i,i+1}$ to the transposition that swaps letters i and $i+1$.

It is known that the braid group B_n acts on the solutions to the differential equations 5.1.4 satisfied by F.

To introduce this group action, we introduce a new notation for the product of polylogarithmic factors,

$$\mathcal{G}(z_1, \ldots, z_n) = \mathbb{G}(z_n; \overrightarrow{e^*_{n,j}}) \ldots \mathbb{G}(z_1; \overrightarrow{e^*_{1,j}}) \, . \tag{5.4.6}$$

Note that $\mathcal{G}(z_1, \ldots, z_n)$ is a solution to the set of differential equations 5.1.4. It is obtained by integrating along the path

$$(0, 0, \ldots, 0) \to (0, \ldots, 0, z_n) \to (0, \ldots, 0, z_{n-1}, z_n) \to \ldots \to (z_1, \ldots, z_{n-1}, z_n) \tag{5.4.7}$$

adapted to the choice of branch

$$0 = z_0 < z_1 < z_2 < \ldots < z_{n-1} < z_n < z_{n+1} = 1\,. \tag{5.4.8}$$

The action of $g \in B_n$ on $\mathcal{G}(z_1, \ldots, z_n)$ is given by

$$g\big(\mathcal{G}(z_1, \ldots, z_n)\big) = \mathbb{X}(g)\, \mathcal{G}(z_1, \ldots, z_n) \ \text{ where } \ \mathbb{X}(g) \ \text{ satisfies} \tag{5.4.9}$$

$$X(g_1\, g_2) = g_1^{pr}(X(g_2))\, X(g_1) \tag{5.4.10}$$

and is given on the generators $\sigma_{i,\, i+1}$ by

$$\mathbb{X}(\sigma_{i,i+1}) = \Phi(\sum_{j=0}^{i-1} e_{j,\, i+1},\ e_{i,i+1})\exp(i\pi e_{i,i+1})\, \Phi(e_{i,i+1},\ \sum_{j=0}^{i-1} e_{j,\, i}) \tag{5.4.11}$$

for $i = 1, \ldots, n-1$.

In fact, we know that $g\big(\mathcal{G}(z_1, \ldots, z_n)\big)$ is a solution to 5.1.4 adapted to the branch ordering

$$0 = z_0 < z_{g^{pr}(1)} < z_{g^{pr}(2)} < \ldots < z_{g^{pr}(n-1)} < z_{g^{pr}(n)} < z_{n+1} = 1 \tag{5.4.12}$$

and is obtained by integrating along the path

$$\begin{aligned}&(0, 0, \ldots, 0) \to (0, \ldots, 0, z_{g^{pr}(n)}) \to (0, \ldots, 0, z_{g^{pr}(n-1)}, z_{g^{pr}(n)}) \to \ldots \\ &\quad \ldots \to (0, z_{g^{pr}(2)}, \ldots, z_{g^{pr}(n-1)}, z_{g^{pr}(n)}) \to (z_{g^{pr}(1)}, \ldots, z_{g^{pr}(n-1)}, z_{g^{pr}(n)})\,.\end{aligned} \tag{5.4.13}$$

Explicitly, we have

$$g\big(\mathcal{G}(z_1, \ldots, z_n)\big) = \mathbb{G}(z_{g^{pr}(n)};\ \overrightarrow{e_{n,\,j}^{g^{pr}*}}) \ldots \mathbb{G}(z_{g^{pr}(1)};\ \overrightarrow{e_{1,\,j}^{g^{pr}*}}) \ \text{ where} \tag{5.4.14}$$

$$\mathbb{G}(z_{g^{pr}(k)};\ \overrightarrow{e_{k,\,j}^{g^{pr}*}}) = \mathbb{G}\left[\begin{smallmatrix} e_{k,\,0}^{g^{pr}*} & e_{k,\,k+1}^{g^{pr}} & \cdots & e_{k,\,n}^{g^{pr}} & e_{k,\,n+1}^{g^{pr}} \\ z_0 & z_{g^{pr}(k+1)} & \cdots & z_{g^{pr}(n)} & z_{n+1} \end{smallmatrix}; z_{g^{pr}(k)}\right] \tag{5.4.15}$$

$$\text{with } \ e_{k,\,0}^{g^{pr}*} = e_{g^{pr}(k),\,0} + \sum_{j=1}^{k-1} e_{g^{pr}(k),\, g^{pr}(j)} \ \text{ for } \ k = 1, \ldots, n \tag{5.4.16}$$

$$\text{and } \ e_{k,\,j}^{g^{pr}} = e_{g^{pr}(k),\, g^{pr}(j)} \ \text{ for } \ k = 1, \ldots, n,\ j = k+1, \ldots, n+1\,. \tag{5.4.17}$$

So we can rewrite equation 5.4.9 as

$$\mathbb{G}(z_{g^{pr}(n)};\, \overrightarrow{e^{g^{pr}*}_{n,j}}) \dots \mathbb{G}(z_{g^{pr}(1)};\, \overrightarrow{e^{g^{pr}*}_{1,j}}) = \mathbb{X}(g)\, \mathbb{G}(z_n; \overrightarrow{e^{*}_{n,j}}) \dots \mathbb{G}(z_1; \overrightarrow{e^{*}_{1,j}})\,. \tag{5.4.18}$$

Finally, we include a small result that will be helpful later on.

Lemma 5.4.1. *Let $k \in \mathbb{N}$ and $g_1, \dots, g_k \in B_N$. Then, we have*

$$\mathbb{X}(g_1 \dots, g_k) = \prod_{j=k}^{1} g_1^{pr} \dots\, g_{j-1}^{pr}\, \mathbb{X}(g_j) \tag{5.4.19}$$

where the product on the right uses a descending order of indices.

Proof. We prove this result by induction. The base case $k = 1$ is clear so suppose that k is greater than 1 and that the result is true for all integers less than k. Then, equation 5.4.10 implies that

$$\mathbb{X}(g_1 \dots, g_k) = g_1^{pr} \dots g_{k-1}^{pr}(\mathbb{X}(g_k))\, \mathbb{X}(g_1 \dots, g_{k-1}))\,. \tag{5.4.20}$$

By the induction hypothesis, we have

$$\mathbb{X}(g_1 \dots, g_{k-1}) = \prod_{j=k-1}^{1} g_1^{pr} \dots\, g_{j-1}^{pr}\, \mathbb{X}(g_j) \tag{5.4.21}$$

and putting this in equation 5.4.20 we get

$$\mathbb{X}(g_1 \dots, g_k) = \prod_{j=k}^{1} g_1^{pr} \dots\, g_{j-1}^{pr}\, \mathbb{X}(g_j)\,. \tag{5.4.22}$$

□

5.4.2 Computation of the initial value limit

Proposition 5.4.2 (Frost-Hidding-Kamlesh-Rodriguez-Schlotterer-Verbeek)**.** *For $l = 2, \dots, n+1$, the initial value limit is equal to*

$$\lim_{z_j \to 0, j \geq 2}\ \lim_{z_1 \to z_l}\ \mathbb{G}(z_n; \overrightarrow{e^{*}_{n,j}}) \dots \mathbb{G}(z_1; \overrightarrow{e^{*}_{1,j}}) = \prod_{j=1}^{2l-3} \Phi(e(j,0),\, e(j,1)) \tag{5.4.23}$$

where the terms $e(j,0), e(j,1)$ are those defined in 5.2.11, 5.2.12, 5.2.13 in section 5.1.

Proof. We split the proof into two cases.

Case I : $\mathbf{l} = \mathbf{2}$.

Observe that

$$\lim_{z_j \to 0,\, j \geq 2} \lim_{z_1 \to z_2} \mathbb{G}(z_n; \overrightarrow{e^*_{n,j}}) \ldots \mathbb{G}(z_1; \overrightarrow{e^*_{1,j}})$$
$$= \lim_{z_j \to 0,\, j \geq 2} \lim_{z_1 \to z_l} \mathbb{G}(z_n; \overrightarrow{e^*_{n,j}}) \ldots \mathbb{G}(z_2; \overrightarrow{e^*_{2,j}}) \lim_{z_j \to 0,\, j \geq 2} \lim_{z_1 \to z_2} \mathbb{G}(z_1; \overrightarrow{e^*_{1,j}})$$
$$= \lim_{z_j \to 0,\, j \geq 2} \lim_{z_1 \to z_2} \mathbb{G}(z_1; \overrightarrow{e^*_{1,j}}) \quad (5.4.24)$$

since $\mathbb{G}(z_j; \overrightarrow{e^*_{2,j}})$ is independent of z_1 for $j \geq 2$ and is equal to 1 for $z_j = 0$.

So it is enough to focus on the limit below,

$$\lim_{z_j \to 0,\, j \geq 2} Z_2 = \lim_{z_j \to 0,\, j \geq 2} \lim_{z_1 \to z_2} \mathbb{G}_1 \,. \quad (5.4.25)$$

Consider the differential forms that make up the generating series of iterated integrals in $Z_2 = \mathbb{G}(z_2;\, \overrightarrow{e^*_{1,j}})$ under the transformation $t = z_2\, u$,

$$\frac{dt}{t} = \frac{z_2 du}{z_2 u} = \frac{du}{u} \,, \quad (5.4.26)$$
$$\frac{dt}{t - z_2} = \frac{z_2\, du}{z_2 u - z_2} = \frac{du}{u - 1} \text{ and} \quad (5.4.27)$$
$$\frac{dt}{t - z_j} = \frac{z_2 du}{z_2 u - z_j} \to 0 \text{ as } z_2 \to 0 \text{ for } j \neq 0, 2 \,. \quad (5.4.28)$$

Therefore, in Z_2, only the words with letters $e_{1,0}$ and $e_{1,2}$ survive in the limit 5.4.25 and we get that

$$\lim_{z_j \to 0,\, j \geq 2} Z_2 = \lim_{z_j \to 0,\, j \geq 2} \lim_{z_1 \to z_2} \mathbb{G}(z_1;\, \overrightarrow{e_{1,j}}) = \Phi(e_{1,0},\, e_{1,2}) = \Phi(e(1,0),\, e(1,1)) \quad (5.4.29)$$

by definition 5.2.11.

Case II : $\mathbf{l} > \mathbf{2}$.

Observe that the $z_2 \to 0$ limit does not commute with the shuffle regularized integration over $0 < t < z_l$ in Z_l as seen below.

$$\int_0^{z_l} \lim_{z_2 \to 0} \frac{dt}{t - z_2} = \int_0^{z_l} \frac{dt}{t} = \ln(z_l), \quad (5.4.30)$$

$$\lim_{z_2\to 0}\int_0^{z_l}\frac{dt}{t-z_2}=\lim_{z_2\to 0}\big(\ln(z_2-z_l)-\ln(z_2)\big)\neq\ \ln(z_l)\,. \tag{5.4.31}$$

This hinders our effort to compute the limit of Z_l via the same method as we did for $l=2$. To correct for this we bring z_1 closer to z_l via an action of the braid group.

Let $g=\sigma_{1,2}\,\sigma_{2,3}\ldots\sigma_{l-2,l-1}\in B_n$. Then $g^{pr}=(1,2,\ldots,\,l-2)\in S_n$ is the cycle of length $l-1$ that maps a to $a+1$ for $a=1,\ldots,l-2$, maps $l-1$ to 1 and a to a for $a=l,\ldots,n$. Therefore, according to 5.4.9 and 5.4.18,

$$g\big(\mathcal{G}(z_1,\ldots,z_n)\big)=\mathbb{G}(z_{g^{pr}(n)};\,\overrightarrow{e^{g^{pr}*}_{n,j}})\ldots\,\mathbb{G}(z_{g^{pr}(1)};\,\overrightarrow{e^{g^{pr}*}_{1,j}}) \tag{5.4.32}$$

gives a solution to the KZ-equations 5.1.4 adapted to the branch ordering

$$0=z_0<z_2<\ldots<z_{l-2}<\mathbf{z_{l-1}}<\mathbf{z_1}<\mathbf{z_l}<z_{l+1}<\ldots<z_n<z_{n+1}=1 \tag{5.4.33}$$

and we have

$$\mathbb{G}(z_{g^{pr}(n)};\,\overrightarrow{e^{g^{pr}*}_{n,j}})\ldots\,\mathbb{G}(z_{g^{pr}(1)};\,\overrightarrow{e^{g^{pr}*}_{1,j}})=\mathbb{X}(g)\,\mathbb{G}(z_n;\overrightarrow{e^{*}_{n,j}})\ldots\,\mathbb{G}(z_1;\overrightarrow{e^{*}_{1,j}})\,. \tag{5.4.34}$$

This allows us to express the required limit as

$$\begin{aligned}&\lim_{z_j\to 0,\,j\geq 2}\ \lim_{z_1\to z_l}\ \mathbb{G}(z_n;\overrightarrow{e^{*}_{n,j}})\ldots\,\mathbb{G}(z_1;\overrightarrow{e^{*}_{1,j}})\\ &\qquad=\mathbb{X}(g)^{-1}\lim_{z_j\to 0,\,j\geq 2}\ \lim_{z_1\to z_l}\ \mathbb{G}(z_{g^{pr}(n)};\,\overrightarrow{e^{g^{pr}*}_{n,j}})\ldots\,\mathbb{G}(z_{g^{pr}(1)};\,\overrightarrow{e^{g^{pr}*}_{1,j}})\end{aligned} \tag{5.4.35}$$

and we focus on the term

$$\mathbb{G}(z_{g^{pr}(n)};\,\overrightarrow{e^{g^{pr}*}_{n,j}})\ldots\,\mathbb{G}(z_{g^{pr}(1)};\,\overrightarrow{e^{g^{pr}*}_{1,j}})\,. \tag{5.4.36}$$

For $a=1,\ldots n$, we have

$$\mathbb{G}(z_{g^{pr}(a)};\,\overrightarrow{e^{g^{pr}*}_{a,j}})=\mathbb{G}\left[\begin{smallmatrix}e^{g^{pr}*}_{a,0} & e^{g^{pr}}_{a,a+1} & \cdots & e^{g^{pr}}_{a,n} & e^{g^{pr}}_{a,n+1}\\ z_0 & z_{g^{pr}(a+1)} & \cdots & z_{g^{pr}(n)} & z_{n+1}\end{smallmatrix};z_{g^{pr}(a)}\right]\,. \tag{5.4.37}$$

If $1\leq a\leq l-2$ then $2\leq g^{pr}(a)\leq l-1$ and thus $g^{pr}(a)\neq 1$. In this case the limit $\lim_{z_1\to z_l}\mathbb{G}(z_{g^{pr}(a)};-)$ is convergent. The limit below is equal to one,

$$\lim_{z_j\to 0,\,2\leq j\leq l-1}\ \lim_{z_1\to z_l}\ \mathbb{G}\left[\begin{smallmatrix}e^{g^{pr}*}_{a,0} & e^{g^{pr}}_{a,a+1} & \cdots & e^{g^{pr}}_{a,n} & e^{g^{pr}}_{a,n+1}\\ z_0 & z_{g^{pr}(a+1)} & \cdots & z_{g^{pr}(n)} & z_{n+1}\end{smallmatrix};z_{g^{pr}(a)}\right]=1 \tag{5.4.38}$$

since the argument $z_{g^{pr}(a)}$ of the polylogarithm goes to zero.

Again, for $l \leq a \leq n$, we have $l \leq g^{pr}(a) = a \leq n$, so $\mathbb{G}(z_{g^{pr}(a)}; -)$ is independent of z_1 and the limit below is equal to one by the same argument as before.

$$\lim_{z_j \to 0,\, l \leq j \leq n} \lim_{z_1 \to z_l} \mathbb{G}\left[\begin{smallmatrix} e^{g^{pr}*}_{a,\,0} & e^{g^{pr}}_{a,\,a+1} & \cdots & e^{g^{pr}}_{a,\,n} & e^{g^{pr}}_{a,\,n+1} \\ z_0 & z_{g^{pr}(a+1)} & \cdots & z_{g^{pr}(n)} & z_{n+1} \end{smallmatrix}; z_{g^{pr}(a)}\right] = 1\,. \tag{5.4.39}$$

On the other hand for $a = l - 1$, we have $g^{pr}(a) = 1$ and $\mathbb{G}(z_{g^{pr}(a)}; -)$ is dependent only on the variables $z_l, z_{l+1}, \ldots, z_n$. Thus we can compute the limit $z_1 \to z_l$ via the same method we used in the case $l = 2$ to get,

$$\lim_{z_j \to 0,\, j \geq 2} \lim_{z_1 \to z_l} \mathbb{G}\left[\begin{smallmatrix} e^{g^{pr}*}_{a,\,0} & e^{g^{pr}}_{a,\,a+1} & \cdots & e^{g^{pr}}_{a,\,n} & e^{g^{pr}}_{a,\,n+1} \\ z_0 & z_{g^{pr}(a+1)} & \cdots & z_{g^{pr}(n)} & z_{n+1} \end{smallmatrix}; z_1\right] = \Phi(e^{g^{pr}*}_{l-1,\,0},\, e^{g^{pr}}_{l-1,\,l})\,. \tag{5.4.40}$$

Recall that the equation 5.4.16, 5.4.17 and definition 5.2.11 give

$$e^{g^{pr}*}_{l-1,\,0} = e_{1,0} + \sum_{j=2}^{l-1} e_{1,\,j} = e(2l-3, 0) \;\text{ and }\; e^{g_{pr}}_{l-1,\,l} = e_{1,\,l} = e(2l-3, 1) \tag{5.4.41}$$

on setting $a = l - 1$.

Putting it all together, equation 5.4.35 gives

$$\lim_{z_j \to 0,\, j \geq 2} \lim_{z_1 \to z_l} \mathbb{G}(z_n; \overrightarrow{e^*_{n,j}}) \ldots \mathbb{G}(z_1; \overrightarrow{e^*_{1,j}}) = \mathbb{X}(g)^{-1}\, \Phi(e(2l-3, 0),\, e(2l-3, 1)) \tag{5.4.42}$$

and we will compute the factor $\mathbb{X}(g)^{-1}$ explicitly.

From equations 5.4.10 and 5.4.11 we know that[4]

$$\mathbb{X}(g) = \mathbb{X}(\sigma_{1,\,2}\, \sigma_{2,\,3} \ldots \sigma_{l-2,\,l-1}) = \prod_{i=l-2}^{1} \sigma^{pr}_{1,\,2} \ldots \sigma^{pr}_{i-1,\,i}\, \mathbb{X}(\sigma_{i,\,i+1}) \;\text{ and } \tag{5.4.43}$$

$$\mathbb{X}(\sigma_{i,i+1})^{\mathfrak{dr}} = \Phi(\sum_{j=0}^{i-1} e_{j,\,i+1},\, e_{i,i+1})^{\mathfrak{dr}} \exp(i\pi e_{i,i+1})^{\mathfrak{dr}}\, \Phi(e_{i,i+1},\, \sum_{j=0}^{i-1} e_{j,\,i})^{\mathfrak{dr}} \tag{5.4.44}$$

$$= \Phi(\sum_{j=0}^{i-1} e_{j,\,i+1},\, e_{i,i+1})^{\mathfrak{dr}}\, \Phi(e_{i,i+1},\, \sum_{j=0}^{i-1} e_{j,\,i})^{\mathfrak{dr}}\,. \tag{5.4.45}$$

Since $\sigma^{pr}_{1,\,2} \ldots \sigma^{pr}_{i-1,\,i} = (1, 2, \ldots, i) \in S_n$ we get that

$$\sigma^{pr}_{1,\,2} \ldots \sigma^{pr}_{i-1,\,i}\, \mathbb{X}(\sigma_{i,i+1}) = \Phi(e_{0,\,i+1} + \sum_{j=2}^{i} e_{j,\,i+1},\, e_{1,\,i+1})\, \Phi(e_{1,\,i+1},\, e_{1,\,0} + \sum_{j=2}^{i} e_{1,\,j}) \tag{5.4.46}$$

[4]Note that this uses a descending product

and therefore

$$\mathbb{X}(g) = \prod_{i=l-2}^{1} \Phi(e_{0,\,i+1} + \sum_{j=2}^{i} e_{j,\,i+1},\, e_{1,\,i+1})\, \Phi(e_{1,\,i+1},\, e_{1,\,0} + \sum_{j=2}^{i} e_{1,\,j}) \tag{5.4.47}$$

and finally by definition 5.2.11, 5.2.12, 5.2.13,

$$\mathbb{X}(g)^{-1} = \prod_{i=1}^{l-2} \Phi(e_{1,\,0} + \sum_{j=2}^{i} e_{1,\,j},\, e_{1,\,i+1})\, \Phi(e_{1,\,i+1},\, e_{0,\,i+1} + \sum_{j=2}^{i} e_{j,\,i+1}) \tag{5.4.48}$$

$$= \prod_{j=1}^{2l-4} \Phi(e(j,0), e(j,1))\,. \tag{5.4.49}$$

Hence, we can rewrite equation 5.4.42 to get

$$\lim_{z_j \to 0,\, j \geq 2} \lim_{z_1 \to z_l} \mathbb{G}(z_n; \overrightarrow{e^*_{n,j}}) \dots \mathbb{G}(z_1; \overrightarrow{e^*_{1,j}}) = \prod_{j=1}^{2l-3} \Phi(e(j,0),\, e(j,1)) \tag{5.4.50}$$

□

Corollary 5.4.3. *For $l = 2, \dots, n+1$, the conjugate limit is equal to*

$$\lim_{z_j \to 0,\, j \geq 2} \lim_{z_1 \to z_l} \mathbb{G}(z_n; \overrightarrow{e^*_{n,j}}) \dots \mathbb{G}(z_1; \overrightarrow{e^*_{1,j}})\, e_{1,\,l} \left(\mathbb{G}(z_n; \overrightarrow{e^*_{n,j}}) \dots \mathbb{G}(z_1; \overrightarrow{e^*_{1,j}})\right)^{-1}$$

$$= \prod_{j=1}^{2l-3} \Phi(e(j,\,0),\, e(j,\,1))\, e_{1,l} \prod_{j=1}^{2l-3} \left(\Phi(e(j,\,0),\, e(j,\,1))\right)^{-1}. \tag{5.4.51}$$

5.5 Conjugate identity

We recommend reading chapter 4 before continuing with this one, especially Theorem 4.4.2 and onward up to 4.4.5 and finally 4.5.3. The appendix 7 also contains some results on commutator identities that will be used in a proof in this section but these results can be referred to as and when needed.

Our aim in this section is to prove the conjugate identity

$$\lim_{z_j \to 0,\, j \geq 2} \lim_{z_1 \to z_l} \mathbb{G}(z_n; \overrightarrow{e^*_{n,j}}) \dots \mathbb{G}(z_1; \overrightarrow{e^*_{1,j}})\, e_{1,\,l} \left(\mathbb{G}(z_n; \overrightarrow{e^*_{n,j}}) \dots \mathbb{G}(z_1; \overrightarrow{e^*_{1,j}})\right)^{-1}$$

$$= \mathbb{M}^{-1} e_{1,\,l}\, \mathbb{M} \tag{5.5.1}$$

for $l = 2, \dots, n+1$.

So we need to understand the structure of both sides of the equation. Towards this direction, we first make some definitions in the spirit of theorem 4.4.5.

Let $l = 2, \dots, n+1$. Let $w(e_0, e_1) \in g$ be a commutator in e_0, e_1. We write $w(j)$ to denote $w(e(j,\, 0),\, e(j,\, 1))$ for $1 \leq j \leq 2l-3$.

For $j = 1, \dots, 2l-3$, we extend the derivation $D_{w(j)}$ defined on $Lie[e(j,\, 0),\, e(j,\, 1)]$ to
$Lie[e(1,\, 0),\, e(1,\, 1), \dots, e(2l-3,\, 0),\, e(2l-3,\, 1)]$ as follows. On the generators $e(j',\, \epsilon)$ with $j' = 1, \dots, 2l-1$, $\epsilon = 0, 1$, we define,

$$D_{w(j)}\, e(j', \epsilon) = [e(j', \epsilon), w(j)] \;\text{ if }\; j < j' \,, \tag{5.5.2}$$
$$D_{w(j)}\, e(j',\, 0) = 0 \;\text{ if }\; j = j' \,, \tag{5.5.3}$$
$$D_{w(j)}\, e(j',\, 1) = [e(j',\, 1),\, w(j)] \;\text{ if }\; j = j' \,, \tag{5.5.4}$$
$$D_{w(j)}\, e(j', \epsilon) = 0 \;\text{ if }\; j > j' \tag{5.5.5}$$

and extend it to $Lie[e(1,\, 0),\, e(1,\, 1), \dots, e(2l-3,\, 0),\, e(2l-3,\, 1)]$ via linearity and the product rule for derivations.

Note that when $j = j'$ then we just have the usual derivation 4.1.1 in the definition above.

We can also rephrase the above derivation in the following way. Let $w'(e_0, e_1) \in g$ be a commutator in e_0, e_1 and $j' = 1, \dots, 2n-1$, then we have

$$D_{w(j)}\, w'(j') = [w'(j'), w(j)] \;\text{ if }\; j < j' \,, \tag{5.5.6}$$
$$D_{w(j)}\, w'(j') = (D_{w(e_0,e_1)} w'(e_0, e_1))|_{e_0 \to e(j,0),\, e_1 \to e(j,1)} \;\text{ if }\; j = j' \,, \tag{5.5.7}$$
$$D_{w(j)}\, w'(j') = 0 \;\text{ if }\; j > j' \,. \tag{5.5.8}$$

Armed with this definition, a repeated application of theorem 4.4.5 gives the following result.

Proposition 5.5.1. *For $i_1, \dots, i_r \in 2\mathbb{N}+1$, the coefficient of $f_{i_1} \dots f_{i_r}$ in the conjugate limit*

$$\lim_{z_j \to 0,\, j \geq 2} \lim_{z_1 \to z_l} \mathbb{G}(z_n; \overrightarrow{e^*_{n,j}}) \dots \mathbb{G}(z_1; \overrightarrow{e^*_{1,j}})\, e_{1,l} \left(\mathbb{G}(z_n; \overrightarrow{e^*_{n,j}}) \dots \mathbb{G}(z_1; \overrightarrow{e^*_{1,j}})\right)^{-1}$$
$$= \prod_{j=1}^{2l-3} \Phi(e(j,\, 0),\, e(j,\, 1))\, e_{1,l} \prod_{j=1}^{2l-3} \left(\Phi(e(j,\, 0),\, e(j,\, 1))\right)^{-1} . \tag{5.5.9}$$

is equal to

$$(-1)^r\, D_{\sum_{j=1}^{2l-3} w_{i_r}(j)} \dots D_{\sum_{j=1}^{2l-3} w_{i_1}(j)}(e_{1,l}) \,. \tag{5.5.10}$$

Next, we recall an analogous result for the $\mathbb{M}$ series.

For $k \geq 3$ odd, we define a derivation D_{M_k} on $Lie[e(1,\,0),\,e(1,\,1),\ldots,e(2n-1,\,0),\,e(2n-1,\,1)]$ by

$$D_{M_k}\,e(j,\epsilon) = [e(j,\epsilon), M_k] \tag{5.5.11}$$

for $1 \leq j \leq 2n-1$, $\epsilon = 0,1$ and extend it to the Lie algebra by the product rule. The definition above makes sense due to 5.2.14. Further, note that the above definition implies

$$D_{M_k}\,w = [w, M_k] \tag{5.5.12}$$

for any $w \in Lie[e(1,\,0),\,e(1,\,1),\ldots,e(2n-1,\,0),\,e(2n-1,\,1)]$.

Proposition 5.5.2. *For $i_1,\ldots,i_r \in 2\mathbb{N}+1$, the coefficient of $f_{i_1}\ldots f_{i_r}$ in the conjugate series $(\mathbb{M}^{\mathfrak{dr}})^{-1}e_{1,l}\mathbb{M}^{\mathfrak{dr}}$ is equal to*

$$D_{M_{i_r}}\ldots D_{M_{i_1}}(e_{1,l})\,. \tag{5.5.13}$$

Proof. Follows from result 4.5.3. □

Next, we have a comparison result for the derivations.

Theorem 5.5.3 (Frost-Hidding-Kamlesh-Rodriguez-Schlotterer-Verbeek)**.** *For $k \geq 3$ odd and $2 \leq l \leq n+1$, we have an equality of derivations*

$$D_{M_k} = -D_{\sum_{j=1}^{2l-3} w_k(j)}\,. \tag{5.5.14}$$

on the Lie algebra $Lie[e(1,0),e(1,1),e(2,0),e(2,1),\ldots,e(2l-3,0),e(2l-3,1)]$.

Proof. We prove this result by induction on n.

Base case: $l = 2$.

We want to show that

$$D_{M_k} = -D_{w_k(1)} \tag{5.5.15}$$

on the Lie algebra $Lie[e(1,0),e(1,1)]$ for $k \geq 3$ odd where

$$e(1,0) = e_{1,0}\,,\quad e(1,1) = e_{1,2}\,. \tag{5.5.16}$$

It is enough to check that the equality holds on the generators $e(1,0), e(1,1)$, so we split the proof into two cases.

Case 1 : $e(1,0)$.

$$LHS = D_{M_k}(e(1,0)) = D_{M_k}(e_{1,0}) = [e_{1,0}, M_k] = 0 \tag{5.5.17}$$

by 5.2.9 applied to $i = 1$, whereas

$$RHS = -D_{w_k(1)}(e(1,0)) = 0 \tag{5.5.18}$$

by 5.5.3 applied to $j = 1 = j'$.

Case 2 : $e(1,1)$.

$$LHS = D_{M_k}(e(1,1)) = D_{M_k}(e_{1,2}) = [e_{1,2}, M_k] = -[e_{1,2}, w_k(1)] \tag{5.5.19}$$

by 5.2.10 applied to $i = 2$, whereas

$$RHS = -D_{w_k(1)}(e(1,1)) = -[e(1,1), w_k(1)] = -[e_{1,2}, w_k(1)] = LHS \tag{5.5.20}$$

by 5.5.4 applied to $j = 1 = j'$.

Induction step : Let $l > 2$. Suppose the result is true for all integers greater than 1 and up to l.

We want to show that the result is true for $l+1$. That is, for $k \geq 3$ odd, we have

$$D_{M_k} = -D_{\sum_{j=1}^{2l-1} w_k(j)} \tag{5.5.21}$$

on the Lie algebra $Lie[e(1,0), e(1,1), \ldots, e(2l-1,0), e(2l-1,1)]$.

It is enough to show that the equality holds on the generators $e(1,0)$, $e(1,1)$, $\ldots$, $e(2l-1,0)$, $e(2l-1,1)$. Now, observe that for $1 \leq j' \leq 2l-3$ and $\epsilon = 0,1$ we have

$$RHS = -D_{\sum_{j=1}^{2l-1} w_k(j)}(e(j',\epsilon)) = -D_{\sum_{j=1}^{2l-3} w_k(j)}(e(j',\epsilon)) \tag{5.5.22}$$

by 5.5.5. By the induction hypothesis, we know that

$$-D_{\sum_{j=1}^{2l-3} w_k(j)}(e(j',\epsilon)) = D_{M_k}(e(j',\epsilon)) = LHS \tag{5.5.23}$$

for $1 \leq j' \leq 2l-3$ and $\epsilon = 0,1$.

So it is enough to check the equality of derivations on $e(2l-2,0)$, $e(2l-2,1)$, $e(2l-1,0)$, $e(2l-1,1)$.

Case 1 : $e(2l-2,\, 0)$.

Note that $e(2l-2,\, 0) = e_{1,l}$ and

$$LHS = D_{M_k}\left(e(2l-2,\, 0)\right) = D_{M_k}\left(e_{1,l}\right) = \left[e_{1,l},\, M_k\right]. \tag{5.5.24}$$

On the other hand

$$RHS = -D_{\sum_{j=1}^{2l-1} w_k(j)}\, e(2l-2,\, 0) = -\sum_{j=1}^{2l-1} D_{w_k(j)}\, e(2l-2,\, 0) \tag{5.5.25}$$

$$= -\sum_{j=1}^{2l-3} D_{w_k(j)}\, e(2l-2,\, 0) - D_{w_k(2l-2)}\, e(2l-2,\, 0) - D_{w_k(2l-1)}\, e(2l-2,\, 0) \tag{5.5.26}$$

$$= -\sum_{j=1}^{2l-3} [e(2l-2,\, 0),\, w_k(j)] \tag{5.5.27}$$

by 5.5.2, 5.5.3, and 5.5.5 respectively.

Further, we know by 5.2.10 applied to $i = l$ that

$$[e_{1,\,l},\, M_k] = -\sum_{j=1}^{2l-3} [e_{1,\,l}\,,\, w_k(j)] = -\sum_{j=1}^{2l-3} [e(2l-2,\, 0),\, w_k(j)] \tag{5.5.28}$$

and therefore $LHS = RHS$.

Case 2 : $e(2l-2,\, 1)$.

Note that $e(2l-2,\, 1) = \sum_{a=0,a\neq 1}^{l-1} e_{l,\,a}$ and

$$LHS = D_{M_k}\,(e(2l-2,\, 1)) = [e(2l-2,\, 1), M_k] = [\sum_{a=0,a\neq 1}^{l-1} e_{l,\,a},\, M_k]\,. \tag{5.5.29}$$

From the commutator relation 5.2.9 applied to $i = l$ we know that

$$[\sum_{a=0}^{l-1} e_{l,\,a},\, M_k] = 0\,. \tag{5.5.30}$$

Therefore, the LHS is equal to

$$[\sum_{a=0,a\neq 1}^{l-1} e_{l,\,a},\, M_k] = -[e_{1,\,l},\, M_k] = [e_{1,l} \sum_{j=1}^{2l-3} w_k(j)] \tag{5.5.31}$$

by the commutator relation 5.2.10 applied to $i = l$.

Consider also the RHS,

$$RHS = -D_{\sum_{j=1}^{2l-2} w_k(j)}(e(2l-2,\, 1)) = -\sum_{j=1}^{2l-2} D_{w_k(j)}(e(2l-2,\, 1)) \tag{5.5.32}$$

$$= -\sum_{j=1}^{2l-3} D_{w_k(j)}(e(2l-2,\, 1)) - D_{w_k(2l-2)}(e(2l-2,\, 1)) \tag{5.5.33}$$

$$= -\sum_{j=1}^{2l-3}[e(2l-2,\,1),\,w_k(j)] - [e(2l-2,\,1),\,w_k(2l-2)] \text{ by } 5.5.2, 5.5.4 \tag{5.5.34}$$

$$= -\sum_{j=1}^{2l-2}[e(2l-2,\,1),\,w_k(j)] = -[\sum_{a=0,a\neq 1}^{l-1} e_{l,\,a},\sum_{j=1}^{2l-2} w_k(j)]\,. \tag{5.5.35}$$

We want to express the LHS in terms of factors on the RHS so we need to transform $e_{1,l}$. By lemma 7.0.2 we know that

$$[\sum_{a=0}^{l}\sum_{b=a+1}^{l} e_{a\,b},\,e_{c,\,d}] = 0 \tag{5.5.36}$$

for $0 \leq c \neq d \leq l$.

Further, recall the definition 5.2.11, 5.2.12, 5.2.13 and observe that, for $1 \leq j \leq 2l-3$, the maximum sub-script index in $w_k(j) = w_k(e(j,0), e(j,1))$ is equal to l. Therefore, lemma 7.0.2 applies and we get that

$$[\sum_{a=0}^{l}\sum_{b=a+1}^{l} e_{a\,b},\,\sum_{j=1}^{2l-3} w_k(j)] = 0\,. \tag{5.5.37}$$

Therefore,

$$[\sum_{a=0}^{l-1}\sum_{b=a+1}^{l-1} e_{a\,b} + e_{1,l} + \sum_{a=0,a\neq 1}^{l-1} e_{a,l},\,\sum_{j=1}^{2l-3} w_k(j)] = 0\,. \tag{5.5.38}$$

Thus the LHS is equal to

$$[e_{1,l}\sum_{j=1}^{2l-3} w_k(j)] = -[\sum_{a=0}^{l-1}\sum_{b=a+1}^{l-1} e_{a\,b} + \sum_{a=0,a\neq 1}^{l-1} e_{a,l},\,\sum_{j=1}^{2l-3} w_k(j)] \tag{5.5.39}$$

$$= -[\sum_{a=0}^{l-1}\sum_{b=a+1}^{l-1} e_{a\,b},\,\sum_{j=1}^{2l-3} w_k(j)] - [\sum_{a=0,a\neq 1}^{l-1} e_{a,l},\,\sum_{j=1}^{2l-3} w_k(j)]\,. \tag{5.5.40}$$

Since the RHS is equal to

$$-[\sum_{a=0,a\neq 1}^{l-1} e_{l,\,a},\sum_{j=1}^{2l-2} w_k(j)] = -[\sum_{a=0,a\neq 1}^{l-1} e_{l,\,a},\,w_k(2l-2)] - [\sum_{a=0,a\neq 1}^{l-1} e_{l,\,a},\sum_{j=1}^{2l-3} w_k(j)] \tag{5.5.41}$$

it is enough to show that

$$-[\sum_{a=0}^{l-1}\sum_{b=a+1}^{l-1} e_{a\,b},\, \sum_{j=1}^{2l-3} w_k(j)] = -[\sum_{a=0,a\neq 1}^{l-1} e_{l,\,a},\; w_k(2l-2)]\,. \tag{5.5.42}$$

Let us denote the two terms above by I and II, respectively.

Also, recall that

$$w_k(2l-2) = w_k(e(2l-2,0), e(2l-2,1)) = w_k(e_{1,l}, \sum_{j=0,j\neq 1}^{l-1} e_{j,l})\,. \tag{5.5.43}$$

Note that for $j = 1,\ldots,2l-4$, the maximum sub-script index in $w_k(j) = w_k(e(j,0), e(j,1))$ is equal to $l-1$. Therefore, lemma 7.0.2 applies and we get that

$$[\sum_{a=0}^{l-1}\sum_{b=a+1}^{l-1} e_{a\,b},\, \sum_{j=1}^{2l-4} w_k(j)] = 0. \tag{5.5.44}$$

Therefore,

$$I = -[\sum_{a=0}^{l-1}\sum_{b=a+1}^{l-1} e_{a\,b},\, \sum_{j=1}^{2l-3} w_k(j)] = -[\sum_{a=0}^{l-1}\sum_{b=a+1}^{l-1} e_{a\,b},\, w_k(2l-3)]\,. \tag{5.5.45}$$

Now, recall that

$$w_k(2l-3) = w_k(e(2l-3,0), e(2l-3,1)) = w_k(\sum_{j=0,j\neq 1}^{l-1} e_{1,j}, e_{1,l}) \tag{5.5.46}$$

and therefore

$$\begin{aligned} I &= -[\sum_{a=0}^{l-1}\sum_{b=a+1}^{l-1} e_{a\,b},\, w_k(2l-3)] \\ &= -[(\sum_{b=0,b\neq 1}^{l-1} e_{1,b}) + (\sum_{a=0,a\neq 1}^{l-1}\sum_{b=a+1,b\neq 1}^{l-1} e_{a,b}), w_k(\sum_{j=0,j\neq 1}^{l-1} e_{1,j}, e_{1,l})] \\ &= -[\sum_{b=0,b\neq 1}^{l-1} e_{1,b}, w_k(\sum_{j=0,j\neq 1}^{l-1} e_{1,j}, e_{1,l})] - [\sum_{a=0,a\neq 1}^{l-1}\sum_{b=a+1,b\neq 1}^{l-1} e_{a,b}, w_k(\sum_{j=0,j\neq 1}^{l-1} e_{1,j}, e_{1,l})]. \end{aligned} \tag{5.5.47}$$

Next, observe that

$$[\sum_{a=0,a\neq 1}^{l-1}\sum_{b=a+1,b\neq 1}^{l-1} e_{a,b},\sum_{j=0,j\neq 1}^{l-1} e_{1,j}]=0 \tag{5.5.48}$$

by lemma 7.0.3 and

$$[\sum_{a=0,a\neq 1}^{l-1}\sum_{b=a+1,b\neq 1}^{l-1} e_{a,b},e_{1,l}]=0 \tag{5.5.49}$$

because of distinct indices, that is, relation 5.1.6. Therefore,

$$[\sum_{a=0,a\neq 1}^{l-1}\sum_{b=a+1,b\neq 1}^{l-1} e_{a,b},w_k(\sum_{j=0,j\neq 1}^{l-1} e_{1,j},e_{1,l})]=0 \tag{5.5.50}$$

and we get that

$$I=-[\sum_{a=0}^{l-1}\sum_{b=a+1}^{l-1} e_{a\,b},\,w_k(2l-3)]=-[\sum_{b=0,b\neq 1}^{l-1} e_{1,b},w_k(\sum_{j=0,j\neq 1}^{l-1} e_{1,j},e_{1,l})]\,. \tag{5.5.51}$$

Since II is equal to

$$II=-[\sum_{a=0,a\neq 1}^{l-1} e_{l,\,a},\,w_k(2l-2)]=-[\sum_{a=0,a\neq 1}^{l-1} e_{l,\,a},\,w_k(e_{1,l},\sum_{j=0,j\neq 1}^{l-1} e_{j,l})] \tag{5.5.52}$$

it is enough to show that

$$-[\sum_{b=0,b\neq 1}^{l-1} e_{1,b},w_k(\sum_{j=0,j\neq 1}^{l-1} e_{1,j},e_{1,l})]=-[\sum_{a=0,a\neq 1}^{l-1} e_{l,\,a},\,w_k(e_{1,l},\sum_{j=0,j\neq 1}^{l-1} e_{j,l})]\,. \tag{5.5.53}$$

Equivalently, it is enough to show that

$$[\sum_{b=0,b\neq 1}^{l-1} e_{1,b},w_k(e_{1,l},\sum_{b=0,b\neq 1}^{l-1} e_{1,b}]+[\sum_{a=0,a\neq 1}^{l-1} e_{l,\,a},\,w_k(e_{1,l},\sum_{a=0,a\neq 1}^{l-1} e_{a,l})]=0. \tag{5.5.54}$$

Therefore we can apply lemma 7.0.4 with

$$x=e_{1,l}\,,\ y=\sum_{b=0,b\neq 1}^{l-1} e_{1,b}\,,\ z=\sum_{a=0,a\neq 1}^{l-1} e_{a,l} \tag{5.5.55}$$

provided we show that

$$w_k(e_{1,l},\sum_{b=0,b\neq 1}^{l-1} e_{1,b})=w_k(-\sum_{b=0,b\neq 1}^{l-1} e_{1,b}-\sum_{a=0,a\neq 1}^{l-1} e_{a,l},\sum_{b=0,b\neq 1}^{l-1} e_{1,b} \tag{5.5.56}$$

and

$$w_k(e_{1,l}, \sum_{a=0,a\neq 1}^{l-1} e_{a,l}) = w_k(-\sum_{b=0,b\neq 1}^{l-1} e_{1,b} - \sum_{a=0,a\neq 1}^{l-1} e_{a,l}, \sum_{a=0,a\neq 1}^{l-1} e_{a,l}\,. \qquad (5.5.57)$$

For this, it is enough to show that the following two commutator relations hold

$$[e_{1,l}, \sum_{b=0,b\neq 1}^{l-1} e_{1,b})] = [-\sum_{b=0,b\neq 1}^{l-1} e_{1,b} - \sum_{a=0,a\neq 1}^{l-1} e_{a,l}, \sum_{b=0,b\neq 1}^{l-1} e_{1,b})] \qquad (5.5.58)$$

$$[e_{1,l}, \sum_{a=0,a\neq 1}^{l-1} e_{a,l})] = [-\sum_{b=0,b\neq 1}^{l-1} e_{1,b} - \sum_{a=0,a\neq 1}^{l-1} e_{a,l}, \sum_{a=0,a\neq 1}^{l-1} e_{a,l})]\,. \qquad (5.5.59)$$

since in that case, we can replace every occurrence of $e_{1,l}$ by

$$e_{1,l} = -\sum_{b=0,b\neq 1}^{l-1} e_{1,b} - \sum_{a=0,a\neq 1}^{l-1} e_{a,l} \qquad (5.5.60)$$

$$\text{in } \; w_k(e_{1,l}, \sum_{b=0,b\neq 1}^{l-1} e_{1,b}), \quad w_k(e_{1,l}, \sum_{a=0,a\neq 1}^{l-1} e_{a,l})\,. \qquad (5.5.61)$$

Towards that end, recall as we have seen earlier that

$$[\sum_{a=0}^{l} \sum_{b=a+1}^{l} e_{a\,b}, e_{c,d}] = 0 \qquad (5.5.62)$$

for $0 \leq c \neq d \leq l$. Therefore,

$$[\sum_{a=0}^{l} \sum_{b=a+1}^{l} e_{a\,b}, \sum_{b=0,b\neq 1}^{l-1} e_{1,b})] = 0 \qquad (5.5.63)$$

and equivalently

$$[(e_{1,0}+\sum_{b=2}^{l-1})+(\sum_{i=2}^{l-1}(e_{i,0}+\sum_{j=i+1}^{l} e_{i,j}))+e_{1,l}+(e_{l,0}+\sum_{a=2}^{l-1} e_{a,l}), e_{1,0}+\sum_{b=2}^{l-1} e_{1,b})] = 0\,. \qquad (5.5.64)$$

Now, by lemma 7.0.3 we know that

$$[\sum_{i=2}^{l-1}(e_{i,0} + \sum_{j=i+1}^{l} e_{i,j}), e_{1,0} + \sum_{b=2}^{l-1} e_{1,b})] = 0\,. \qquad (5.5.65)$$

Therefore, we can simplify the earlier equation as

$$[(e_{1,0}+\sum_{b=2}^{l-1}+e_{1,l})+(e_{l,0}+\sum_{a=2}^{l-1}e_{a,l}),e_{1,0}+\sum_{b=2}^{l-1}e_{1,b}]=0 \tag{5.5.66}$$

and equivalently

$$[e_{1,l},e_{1,0}+\sum_{b=2}^{l-1}e_{1,b}]]=[-e_{1,0}+\sum_{b=2}^{l-1}e_{1,b}-e_{l,0}+\sum_{a=2}^{l-1}e_{a,l},\sum_{b=2}^{l-1}e_{1,b}] \tag{5.5.67}$$

The same proof works for the other commutator relation with 1 replaced by l and so we are done.

Case 3 : $e(2l-1,0)$.

Note that $e(2l-1,0)=e_{1,0}+\sum_{a=2}^{l}e_{1,a}$ and

$$LHS=D_{M_k}\left(e(2l-1,\,0)\right)=[e(2l-1,\,0),M_k] \tag{5.5.68}$$

$$=[e_{1,0}+\sum_{a=2}^{l}e_{1,a},\;M_k]=\sum_{a=2}^{l}[e_{1,a},\;M_k] \tag{5.5.69}$$

since $[e_{1,0},M_k]=0$ by 5.2.9. Again, by 5.2.9 we know that for $a=2,\dots,l$,

$$[\sum_{b=0}^{a-1}e_{a,b},M_k]=0 \tag{5.5.70}$$

and thus

$$[e_{1,a},M_k]=-[e_{a,0}+\sum_{b=2}^{a-1}e_{a,b},M_k]=-[e(2(a-1),1),M_k]\,. \tag{5.5.71}$$

Further, by the induction hypothesis and case 2, we know that for $a=2,\dots,l$,

$$-[e(2(a-1),1),M_k]=[e(2(a-1),1),\;\sum_{b=1}^{2(a-1)}w_k(b)]\,. \tag{5.5.72}$$

Therefore,

$$LHS=\sum_{a=2}^{l}[e_{1,a},\;M_k]=-\sum_{a=2}^{l}[e(2(a-1),1),M_k] \tag{5.5.73}$$

$$=\sum_{a=2}^{l}[e(2(a-1),1),\;\sum_{b=1}^{2(a-1)}w_k(b)]=\sum_{a=2}^{l}[e_{a,0}+\sum_{b=2}^{a-1}e_{a,b},\;\sum_{b=1}^{2(a-1)}w_k(b)]\,. \tag{5.5.74}$$

On the other hand,

$$RHS = -D_{\sum_{j=1}^{2l-1} w_k(j)}\, e(2l-1,\,0) \tag{5.5.75}$$
$$= -D_{\sum_{j=1}^{2l-2} w_k(j)}\, e(2l-1,\,0) - D_{w_k(2l-1)}\, e(2l-1,\,0) \tag{5.5.76}$$
$$= -[e(2l-1,\,0), \sum_{j=1}^{2l-2} w_k(j)] \text{ by definitions 5.5.2, 5.5.3,} \tag{5.5.77}$$
$$= -[e_{1,0} + \sum_{a=2}^{l} e_{1,a}, \sum_{j=1}^{2l-2} w_k(j)]\,. \tag{5.5.78}$$

Now, observe that the maximum subscript index in $w_k(j) = w_k(e(j,0), e(j,1))$ for $1 \leq j \leq 2l-2$ is equal to l. Therefore, by lemma 7.0.2 we get that

$$[\sum_{b=0}^{l} \sum_{a=b+1}^{l} e_{a,b}, \sum_{j=1}^{2l-2} w_k(j)] = 0 \tag{5.5.79}$$

and thus

$$[(e_{1,0} + \sum_{a=2}^{l} e_{1,a}) + \sum_{b=2}^{l}(e_{b,0} + \sum_{a=b+1}^{l} e_{a,b}), \sum_{j=1}^{2l-2} w_k(j)] = 0\,. \tag{5.5.80}$$

Therefore, we can rewrite the RHS as

$$[\sum_{b=2}^{l}(e_{b,0} + \sum_{a=b+1}^{l} e_{a,b}), \sum_{j=1}^{2l-2} w_k(j)]\,. \tag{5.5.81}$$

We claim that

$$\sum_{b=2}^{l}(e_{b,0} + \sum_{a=b+1}^{l} e_{a,b}) = \sum_{d=2}^{l}(e_{d,0} + \sum_{c=2}^{d-1} e_{c,d}) = \sum_{d=2}^{l} e(2(d-1),1)\,. \tag{5.5.82}$$

Proof :

$$LHS(claim) = \sum_{b=2}^{l}(e_{b,0} + \sum_{a=b+1}^{l} e_{a,b}) = \sum_{b=2}^{l} e_{b,0} + \sum_{b=2}^{l} \sum_{a=b+1}^{l} e_{a,b}. \tag{5.5.83}$$

Consider the following transformation of indices

$$d = l + 2 - b, \quad c = l + 2 - a \tag{5.5.84}$$

and observe that as b goes from 2 to l, d goes from l to 2. Similarly, as a goes from $b+1$ to l, then c goes from $l+1-b$ to 2 and hence from $l+1-(l+2-d) = d-1$ to 2. Therefore, $LHS(claim)$ is equal to

$$LHS(claim) = \sum_{d=l}^{2} e_{d,0} + \sum_{d=l}^{2} \sum_{c=d-1}^{2} e_{l+2-c,l+2-d} \,. \tag{5.5.85}$$

Now, applying the transformation $c \to l+2-c$ and $d \to l+2-d$ we get

$$LHS(claim) = \sum_{d=2}^{l} (e_{d,0} + \sum_{c=2}^{d-1} e_{c,d}) = \sum_{d=2}^{l} e(2(d-1), 1) \,. \tag{5.5.86}$$

We can apply the result from our claim to the original RHS to get

$$RHS = [\sum_{b=2}^{l} (e_{b,0} + \sum_{a=b+1}^{l} e_{a,b}), \sum_{j=1}^{2l-2} w_k(j)] \tag{5.5.87}$$

$$= [\sum_{d=2}^{l} e(2(d-1), 1), \sum_{j=1}^{2l-2} w_k(j)] \tag{5.5.88}$$

$$= [\sum_{a=2}^{l} e(2(a-1), 1), \sum_{j=1}^{2l-2} w_k(j)]. \tag{5.5.89}$$

Recall that the LHS is equal to

$$LHS = \sum_{a=2}^{l} [e(2(a-1), 1), \sum_{b=1}^{2(a-1)} w_k(b)] \,. \tag{5.5.90}$$

So it is enough to show that for $a = 2, \ldots, l$, $j = 1, \ldots, 2l-2$ and $j > 2(a-1)$,

$$[e(2(a-1), 1), w_k(j)] = [e(2(a-1), 1), w_k(e(j, 0), e(j, 1))] = 0 \,. \tag{5.5.91}$$

Or simply put it is sufficient to show that for $j > 2(a-1)$

$$[e(2(a-1), 1), e(j, 0)] = [e(2(a-1), 1), e(j, 1)] = 0 \,. \tag{5.5.92}$$

We split this into four cases.

Case I(0) : $(j, 0) = (2c-1, 0)$, $2 \le a \le l$ with $a+1 \le c \le l$.

Note that $j = 2c-1 > 2(a-1)$.

$$[e(2(a-1), 1), e(2c-1, 0)] = [e_{a,0} + \sum_{b=2}^{a-1} e_{a,b}, e_{1,0} + \sum_{d=2}^{c} e_{1,d}] \tag{5.5.93}$$

$$= [e_{a,0}, e_{1,0} + \sum_{d=2}^{c} e_{1,d}] + [\sum_{b=2}^{a-1} e_{a,b}, e_{1,0} + \sum_{d=2}^{c} e_{1,d}] \,. \tag{5.5.94}$$

For $d \neq a$ we have $[e_{a,0}, e_{1,d}] = 0$. Further, $[e_{a,0}, e_{1,0} + e_{1,a}] = 0$ and since $c > a$ there does exist a term $e_{1,a}$ in the sum $\sum_{d=2}^{c} e_{1,d}$. Therefore,

$$[e_{a,0}, e_{1,0} + \sum_{d=2}^{c} e_{1,d}] = 0 \,. \tag{5.5.95}$$

Next, we have

$$\sum_{b=2}^{a-1} [e_{a,b}, e_{1,0} + \sum_{d=2}^{c} e_{1,d}] = \sum_{b=2}^{a-1} [e_{a,b}, e_{1,0}] + \sum_{b=2}^{a-1} [e_{a,b}, \sum_{d=2}^{c} e_{1,d}] \tag{5.5.96}$$

$$= \sum_{b=2}^{a-1} [e_{a,b}, \, (e_{1,b} + e_{1,a}) + \sum_{d=2, d\neq b,a}^{c} e_{1,d}] = 0. \tag{5.5.97}$$

Case I(1) : $(j, 1) = (2c - 1, 1)$, $2 \leq a \leq l$, with $a + 1 \leq c \leq l$.

$$[e(2(a-1), 1), e(2c-1, 1)] = [e_{a,0} + \sum_{b=2}^{a-1} e_{a,b}, e_{1,c+1}] = 0 \tag{5.5.98}$$

since $1 \neq a \neq b \neq c + 1$.

Case II(0) : $(j, 0) = (2c, 0)$, $2 \leq a \leq l$ with $a + 1 \leq c \leq l$. Note that $j = 2c > 2(a - 1)$.

Since $e(2c, 0) = e(2c - 1, 1)$, the proof is the same as in the last case.

Case II(1) : $(j, 1) = (2c, 1)$, $a = 2, \ldots, l$ with $a + 1 \leq c \leq l$.

$$[e(2(a-1), 1), e(2c, 1)] = [e_{a,0} + \sum_{b=2}^{a-1} e_{a,b}, e_{c+1,0} + \sum_{d=2}^{c} e_{c+1,d}] = 0 \tag{5.5.99}$$

since the same proof as in I(0) with 1 replaced by $c + 1$ works.

Case 4 : $e(2l - 1, \, 1)$.

Note that $e(2l - 1, \, 1) = e_{1,\, l+1}$ and

$$LHS = D_{M_k}\, (e(2l-1, \, 1)) = D_{M_k}\, (e_{1,\, l+1}) = [e_{1,\, l+1}, \, M_k] \tag{5.5.100}$$

$$= -\sum_{j=1}^{2l-1}[e_{1,\,l+1}\,,\,w_k(j)] \tag{5.5.101}$$

$$= -\sum_{j=1}^{2l-1}[e(2l-1,\,1)\,,\,w_k(j)] \tag{5.5.102}$$

by 5.2.10 applied to $i = l+1$, whereas

$$RHS = -D_{\sum_{j=1}^{2l-1} w_k(j)}\,(e(2l-1,\,1)) \tag{5.5.103}$$

$$= -\sum_{j=1}^{2l-2} D_{w_k(j)}\,(e(2l-1,\,1)) - D_{w_k(2l-1)}\,(e(2l-1,\,1)) \tag{5.5.104}$$

$$= -\sum_{j=1}^{2l-2}[e(2l-1,\,1),\,w_k(j)] - [e(2l-1,\,1),\,w_k(2l-1)] \tag{5.5.105}$$

$$= -\sum_{j=1}^{2l-1}[e(2l-1,\,1),\,w_k(j)] = LHS \tag{5.5.106}$$

by 5.5.2 and 5.5.4 respectively.

□

Proposition 5.5.4 (Frost-Hidding-Kamlesh-Rodriguez-Schlotterer-Verbeek)**.** *For $l = 2, \ldots, n+1$, we have*

$$\lim_{z_j \to 0,\, j \geq 2}\ \lim_{z_1 \to z_l} \mathbb{G}(z_n; \overrightarrow{e^*_{n,j}}) \ldots \mathbb{G}(z_1; \overrightarrow{e^*_{1,j}})\, e_{1,\,l}\, \left(\mathbb{G}(z_n; \overrightarrow{e^*_{n,j}}) \ldots \mathbb{G}(z_1; \overrightarrow{e^*_{1,j}})\right)^{-1} = \mathbb{M}^{-1} e_{1,\,l}\, \mathbb{M} \tag{5.5.107}$$

Proof. For $r \geq 0$ and $i_1, \ldots, i_r \in 2\mathbb{N}+1$, the coefficient of $f_{i_1} \ldots f_{i_r}$ in the LHS is equal to

$$(-1)^r\, D_{\sum_{j=1}^{2l-3} w_{i_r}(j)} \ldots D_{\sum_{j=1}^{2l-3} w_{i_1}(j)}(e_{1,l})\,, \tag{5.5.108}$$

whereas in the RHS it is equal to

$$D_{M_{i_r}} \ldots D_{M_{i_1}}(e_{1,l})\,. \tag{5.5.109}$$

Applying theorem 5.5.3 to the last equation we get

$$D_{M_{i_r}} \ldots D_{M_{i_1}}(e_{1,l}) = (-1)^r\, D_{\sum_{j=1}^{2l-3} w_{i_r}(j)} \ldots D_{\sum_{j=1}^{2l-3} w_{i_1}(j)}(e_{1,l}). \tag{5.5.110}$$

□

Finally, the next result completes the proof of theorem 5.2.1 started in this chapter.

Theorem 5.5.5 (Frost-Hidding-Kamlesh-Rodriguez-Schlotterer-Verbeek)**.** *For $l = 2, \ldots, n$, we have*

$$(\mathbb{G}_n \ldots \mathbb{G}_2 Z_l)\, e_{1,l}\, (\mathbb{G}_n \ldots \mathbb{G}_2 Z_l)^{-1} = \mathbb{M}^{-1} e_{1,l}\, \mathbb{M}\,. \tag{5.5.111}$$

Further, in terms of commutators, this is given by

$$[e_{1,l}, M_r] = [\sum_{t=1}^{l-1} w_r(\sum_{r=0, r\neq 1}^{t} e_{1,r},\, e_{1,t+1}) + \sum_{t=1}^{l-2} w_r(e_{1,t+1}, \sum_{r=0,r\neq 1}^{t} e_{r,t+1}), e_{1,l}] \tag{5.5.112}$$

for $r \in 2\mathbb{N}+1$.

Proof. For the first relation, we know by 5.3.7 and 5.3.1 that for $k = 2, \ldots, n$ we have

$$\partial_k(LHS) = 0 = \partial_k(RHS)\,. \tag{5.5.113}$$

Further, by 5.5.4 we know that

$$\lim_{z_j \to 0,\, j\geq 2} (LHS) = RHS = \lim_{z_j \to 0,\, j \geq 2} RHS\,. \tag{5.5.114}$$

Therefore, LHS = RHS.

Finally, the commutator relation follows from the following application of theorem 5.5.3 and notations 5.2.11 5.2.12, 5.2.13.

$$D_{M_r}(e_{1,l}) = -D_{\sum_{j=1}^{2l-3} w_k(j)}(e_{1,l})\,. \tag{5.5.115}$$

□

5.6 Application : Single-valued periods

The multiple polylogarithms are multi-valued functions. However, it is possible to construct their single-valued versions, that is, meromorphic analogues with trivial monodromy but the same holomorphic differential. This was first worked out by Brown in the one-variable case $a_j \in \{0, 1\}$ in [17], in the multivariable case[5] in [16] and from a motivic point of view in [23]. Following the work of Brown [23], a combinatorial construction of the single-valued map was given in [28] by relying purely on the coproduct and the antipode map of the Hopf algebra of multiple

[5]See [14] for an independent approach to single-valued polylogarithms in two variables.

polylogarithms [41, 40]. Our goal in this section is to apply our results on the motivic coaction to the formulation in [28] to compute the single-valued image of the function F. We start by first computing this for the initial value series $\mathbb{I}$. We write sv to denote the single-valued map of multiple polylogarithms and multiple zeta values.

5.6.1 Initial value series

The values at 1 of the single-valued multiple-polylogarithm in one variable are referred to as the single-valued multiple zeta values, and these form a $\mathbb{Q}$ sub-algebra of the algebra of multiple zeta values. From [23] (section *7.2*) we know that the map sv takes the following simple form for the f-alphabet.

$$\mathrm{sv}\, f_{i_1} f_{i_2} \dots f_{i_r} = \sum_{j=0}^{r} f_{i_j} \dots f_{i_2} f_{i_1} ш (f_{i_{j+1}} \dots f_{i_r}) , \qquad \mathrm{sv}\, f_2 = 0 \tag{5.6.1}$$

This allows us to compute the sv map on the initial value series $\mathbb{I}$. To describe this, we introduce the transpose symbol ^{t}, which reverses the order of the generators M_i in the series $\mathbb{M}$,

$$(M_{i_1} \dots M_{i_r})^t = M_{i_r} \dots M_{i_1} . \tag{5.6.2}$$

Lemma 5.6.1. *The single valued map acts on the initial value series* $\mathbb{I} = \mathbb{P}\,\mathbb{M}$ *as follows.*

$$\mathrm{sv}\, \mathbb{I} = \mathrm{sv}\, \mathbb{M} = \mathbb{M}^t\, \mathbb{M} . \tag{5.6.3}$$

Proof. Since sv $f_2 = 0$ we have that

$$\mathrm{sv}\, \mathbb{P} = \mathrm{sv}\big(\sum_{j=0}^{\infty} f_2^j\, P_{2j}\big) = 1 . \tag{5.6.4}$$

Next, considering the f-alphabet with odd letters, we get

$$\mathrm{sv}\, \mathbb{M} = \mathrm{sv}\big(\sum_{r=0}^{\infty} \sum_{i_1,i_2,\dots,i_r \in 2\mathbb{N}+1} f_{i_1} f_{i_2} \dots f_{i_r}\, M_{i_1} M_{i_2} \dots M_{i_r}\big) \tag{5.6.5}$$

$$= \sum_{r=0}^{\infty} \sum_{i_1,i_2,\dots,i_r \in 2\mathbb{N}+1} \mathrm{sv}(f_{i_1} f_{i_2} \dots f_{i_r})\, M_{i_1} M_{i_2} \dots M_{i_r} \tag{5.6.6}$$

$$= \sum_{r=0}^{\infty} \sum_{i_1,i_2,\dots,i_r \in 2\mathbb{N}+1} \big(\sum_{j=0}^{r} f_{i_j} \dots f_{i_2} f_{i_1} ш f_{i_{j+1}} \dots f_{i_r}\big)\, M_{i_1} M_{i_2} \dots M_{i_r} \tag{5.6.7}$$

$$= \sum_{r=0}^{\infty} \sum_{i_1,i_2,\ldots,i_r \in 2\mathbb{N}+1} \sum_{j=0}^{r} \big(f_{i_j} \ldots f_{i_1} \, (M_{i_j} \ldots M_{i_1})^t ш f_{i_{j+1}} \ldots f_{i_r} \big) \, M_{i_{j+1}} \ldots M_{i_r} \big) \tag{5.6.8}$$

$$= \mathbb{M}^t \, \mathbb{M} \, . \tag{5.6.9}$$

Finally, we have

$$\mathrm{sv}\, \mathbb{I} = \mathrm{sv}(\mathbb{P}\, \mathbb{M}) = \mathrm{sv}\, \mathbb{M} = \mathbb{M}^t \, \mathbb{M}. \tag{5.6.10}$$

□

5.6.2 Multiple Polylogarithms

Next, let us focus on the single-valued multiple polylogarithms. As explained in section *3.4.3* of [28], one can construct single-valued polylogarithms in any number of variables from the coproduct Δ and the antipode S in the Hopf algebra of polylogarithms. This method works as follows.

One first computes the antipode of meromorphic polylogarithms via

$$S \, \Delta G(a_1, \ldots, a_w; z) = 0 \, , \qquad w \geq 1 \, . \tag{5.6.11}$$

Note that the antipode map acts only on the left side of the coproduct and not on the right side.

Given the antipode of meromorphic polylogarithms, the single-valued map for any number of variables is given by [28]

$$\mathrm{sv}\, G(a_1, \ldots, a_w; z) = \tilde{S} \Delta G(a_1, \ldots, a_w; z) \tag{5.6.12}$$

where the operation $\tilde{S}$ acts by taking the complex conjugate of the antipode map S and multiplying weight-w contributions by $(-1)^w$.

The above construction was used at the level of generating series to write down the single-valued image for multiple polylogarithms. For purposes of illustration, let us first state this result in the one variable case, that is, for the generating series $\mathbb{G}_n$. We use the bar notation $\overline{z}$ to denote complex conjugation. We also use the transpose notation ^{t} that reverses the order of letters e_a,

$$(e_{a_1} \ldots e_{a_r})^t = e_{a_r} \ldots e_{a_1} \, . \tag{5.6.13}$$

Lemma 5.6.2. *The single-valued image of the generating series $\mathbb{G}_n$ is given by*

$$\mathrm{sv}\, \mathbb{G}\left[\begin{smallmatrix} e^*_{n,0} & e_{n,n+1} \\ 0 & 1 \end{smallmatrix} ; z_n \right] = \overline{\mathbb{G}\left[\begin{smallmatrix} e^*_{n,0} & \widehat{e_{n,n+1}} \\ 0 & 1 \end{smallmatrix} ; z_n \right]}^{\,t} \, \mathbb{G}\left[\begin{smallmatrix} e^*_{n,0} & e_{n,n+1} \\ 0 & 1 \end{smallmatrix} ; z_n \right] \, . \tag{5.6.14}$$

where $\widehat{e_{n,n+1}}$ *is given by*

$$\widehat{e_{n,n+1}} = \mathrm{sv}\, e^{'}_{n,n+1} = \mathrm{sv}\, \Phi(e^*_{n,0}, e_{n,n+1})\; e_{n,n+1}\; \mathrm{sv}\, \Phi(e^*_{n,0}, e_{n,n+1})^{-1}\,. \tag{5.6.15}$$

Assuming the equivalent criteria of theorem 5.2.1 hold, we have the conjugate identity 5.2.5

$$\Phi(e^*_{n,0}, e_{n,n+1})\; e_{n,n+1}\; \Phi(e^*_{n,0}, e_{n,n+1})^{-1} = \mathbb{I}^{-1}\, e_{n,n+1}\, \mathbb{I}\,. \tag{5.6.16}$$

Thus, applying the single-valued map to the above equation we get

$$\mathrm{sv}\, \Phi(e^*_{n,0}, e_{n,n+1})\; e_{n,n+1}\; \mathrm{sv}\, \Phi(e^*_{n,0}, e_{n,n+1})^{-1} = \mathrm{sv}\mathbb{M}^{-1}\, e_{n,n+1}\, \mathrm{sv}\mathbb{M}\,. \tag{5.6.17}$$

We can use this relation along with the identity $[e^*_{n,0}, \mathbb{M}] = 0$ to reformulate lemma 5.6.2 as follows.

Lemma 5.6.3. *The single-valued image of the generating series* $\mathbb{G}_n$ *is given by*

$$\mathrm{sv}\, \mathbb{G}\begin{bmatrix} e^*_{n,0} & e_{n,n+1} \\ 0 & 1 \end{bmatrix}; z_n\Big] = \mathrm{sv}\, \mathbb{M}^{-1}\, \overline{\mathbb{G}\Big[\begin{smallmatrix} e^*_{n,0} & e_{n,n+1} \\ 0 & 1 \end{smallmatrix}; z_n\Big]}^{t}\, \mathrm{sv}\, \mathbb{M}\; \mathbb{G}\Big[\begin{smallmatrix} e^*_{n,0} & e_{n,n+1} \\ 0 & 1 \end{smallmatrix}; z_n\Big]\,. \tag{5.6.18}$$

Let us verify that this indeed gives us a single-valued series. First of all, it is clear that $\mathrm{sv}\, \mathbb{G}_n$ has the same holomorphic differential in z_n as $\mathbb{G}_n$. So it is enough to check that $\mathrm{sv}\, \mathbb{G}_n$ has trivial monodromy. We write $Disc_0$, $Disc_1$ to denote the monodromy operator for polylogarithms at 0 and 1 respectively. Then, we have

$$Disc_0\, \mathbb{G}\Big[\begin{smallmatrix} e^*_{n,0} & e_{n,n+1} \\ 0 & 1 \end{smallmatrix}; z_n\Big] = \exp(2\pi i e^*_{n,0})\, \mathbb{G}\Big[\begin{smallmatrix} e^*_{n,0} & e_{n,n+1} \\ 0 & 1 \end{smallmatrix}; z_n\Big] \quad \text{and} \tag{5.6.19}$$

$$Disc_0\, \overline{\mathbb{G}\Big[\begin{smallmatrix} e^*_{n,0} & e_{n,n+1} \\ 0 & 1 \end{smallmatrix}; z_n\Big]}^{t} = \overline{\mathbb{G}\Big[\begin{smallmatrix} e^*_{n,0} & e_{n,n+1} \\ 0 & 1 \end{smallmatrix}; z_n\Big]}^{t}\, \exp(-2\pi i e^*_{n,0})\,. \tag{5.6.20}$$

Therefore, on applying $Disc_0$ to 5.6.18 we get

$$\begin{aligned}
Disc_0\, \mathrm{sv}\, \mathbb{G}\Big[\begin{smallmatrix} e^*_{n,0} & e_{n,n+1} \\ 0 & 1 \end{smallmatrix}; z_n\Big] &= \mathrm{sv}\, \mathbb{M}^{-1}\; Disc_0\, \overline{\mathbb{G}\Big[\begin{smallmatrix} e^*_{n,0} & e_{n,n+1} \\ 0 & 1 \end{smallmatrix}; z_n\Big]}^{t} \\
&\quad \times \mathrm{sv}\, \mathbb{M}\; Disc_0\, \mathbb{G}\Big[\begin{smallmatrix} e^*_{n,0} & e_{n,n+1} \\ 0 & 1 \end{smallmatrix}; z_n\Big] \\
&= \mathrm{sv}\, \mathbb{M}^{-1}\; \overline{\mathbb{G}\Big[\begin{smallmatrix} e^*_{n,0} & e_{n,n+1} \\ 0 & 1 \end{smallmatrix}; z_n\Big]}^{t}\; \exp(-2\pi i e^*_{n,0}) \\
&\quad \times \mathrm{sv}\, \mathbb{M}\; \exp(2\pi i e^*_{n,0})\; \mathbb{G}\Big[\begin{smallmatrix} e^*_{n,0} & e_{n,n+1} \\ 0 & 1 \end{smallmatrix}; z_n\Big] \\
= \mathrm{sv}\, \mathbb{M}^{-1}\, \overline{\mathbb{G}\Big[\begin{smallmatrix} e^*_{n,0} & e_{n,n+1} \\ 0 & 1 \end{smallmatrix}; z_n\Big]}^{t}\, \mathrm{sv}\, \mathbb{M}\; \mathbb{G}\Big[\begin{smallmatrix} e^*_{n,0} & e_{n,n+1} \\ 0 & 1 \end{smallmatrix}; z_n\Big] &= \mathrm{sv}\, \mathbb{G}\Big[\begin{smallmatrix} e^*_{n,0} & e_{n,n+1} \\ 0 & 1 \end{smallmatrix}; z_n\Big]
\end{aligned} \tag{5.6.21}$$

since $[e^*_{n,0}, \mathbb{M}] = 0$ and thus we can put the exponential terms together in the second equation which then cancel out.

The monodromy at 1 is slightly more complicated.

$$Disc_1\, \mathbb{G}\left[\begin{smallmatrix} e^*_{n,0} & e_{n,n+1} \\ 0 & 1 \end{smallmatrix}; z_n\right] = \Phi(e^*_{n,0}, e_{n,n+1})\, \exp(2\pi i e_{n,n+1})\, \Phi(e^*_{n,0}, e_{n,n+1})^{-1}\, \mathbb{G}\left[\begin{smallmatrix} e^*_{n,0} & e_{n,n+1} \\ 0 & 1 \end{smallmatrix}; z_n\right]. \tag{5.6.22}$$

This leads to

$$Disc_1\, \overline{\mathbb{G}\left[\begin{smallmatrix} e^*_{n,0} & e_{n,n+1} \\ 0 & 1 \end{smallmatrix}; z_n\right]}^t = \overline{\mathbb{G}\left[\begin{smallmatrix} e^*_{n,0} & e_{n,n+1} \\ 0 & 1 \end{smallmatrix}; z_n\right]}^t (\Phi(e^*_{n,0}, e_{n,n+1})^{-1})^t\, \exp(-2\pi i e_{n,n+1})\, \Phi(e^*_{n,0}, e_{n,n+1})^t. \tag{5.6.23}$$

Now, we apply the operator $Disc_1$ to 5.6.18 to get

$$\begin{aligned} Disc_1\, \mathrm{sv}\, \mathbb{G}\left[\begin{smallmatrix} e^*_{n,0} & e_{n,n+1} \\ 0 & 1 \end{smallmatrix}; z_n\right] &= \mathrm{sv}\, \mathbb{M}^{-1}\, Disc_1\, \overline{\mathbb{G}\left[\begin{smallmatrix} e^*_{n,0} & e_{n,n+1} \\ 0 & 1 \end{smallmatrix}; z_n\right]}^t \\ &\quad \times \mathrm{sv}\, \mathbb{M}\, Disc_1\, \mathbb{G}\left[\begin{smallmatrix} e^*_{n,0} & e_{n,n+1} \\ 0 & 1 \end{smallmatrix}; z_n\right] \\ = \mathrm{sv}\, \mathbb{M}^{-1}\, \overline{\mathbb{G}\left[\begin{smallmatrix} e^*_{n,0} & e_{n,n+1} \\ 0 & 1 \end{smallmatrix}; z_n\right]}^t &(\Phi(e^*_{n,0}, e_{n,n+1})^{-1})^t\, \exp(-2\pi i e_{n,n+1})\, \Phi(e^*_{n,0}, e_{n,n+1})^t \\ \times\, \mathrm{sv}\, \mathbb{M}\, \Phi(e^*_{n,0}, e_{n,n+1})\, \exp(2\pi i e_{n,n+1})\, &\Phi(e^*_{n,0}, e_{n,n+1})^{-1}\, \mathbb{G}\left[\begin{smallmatrix} e^*_{n,0} & e_{n,n+1} \\ 0 & 1 \end{smallmatrix}; z_n\right]. \end{aligned} \tag{5.6.24}$$

Thus, for $Disc_1$ to act trivially above, it is enough to show that

$$\begin{aligned} &(\Phi(e^*_{n,0}, e_{n,n+1})^{-1})^t\, \exp(-2\pi i e_{n,n+1})\, \Phi(e^*_{n,0}, e_{n,n+1})^t \\ &\qquad \times \mathrm{sv}\, \mathbb{M}\, \Phi(e^*_{n,0}, e_{n,n+1})\, \exp(2\pi i e_{n,n+1})\, \Phi(e^*_{n,0}, e_{n,n+1})^{-1} = \mathrm{sv}\, \mathbb{M}. \end{aligned} \tag{5.6.25}$$

Recall that we have $\mathrm{sv}\, \mathbb{M} = \mathbb{M}^t\, \mathbb{M}$ and

$$\Phi(e^*_{n,0}, e_{n,n+1})\, e_{n,n+1}\, \Phi(e^*_{n,0}, e_{n,n+1})^{-1} = \mathbb{M}^{-1}\, e_{n,n+1}\, \mathbb{M}. \tag{5.6.26}$$

Therefore, we get

$$\mathbb{M}\, \Phi(e^*_{n,0}, e_{n,n+1})\, \exp(2\pi i e_{n,n+1})\, \Phi(e^*_{n,0}, e_{n,n+1})^{-1} = \exp(2\pi i e_{n,n+1})\, \mathbb{M}. \tag{5.6.27}$$

Similarly, with the transverse operation, we also get

$$(\Phi(e^*_{n,0}, e_{n,n+1})^{-1})^t\, \exp(-2\pi i e_{n,n+1})\, \Phi(e^*_{n,0}, e_{n,n+1})^t\, \mathbb{M}^t = \mathbb{M}^t\, \exp(-2\pi i e_{n,n+1}). \tag{5.6.28}$$

Finally, multiplying equation 5.6.28 and 5.6.27 gives 5.6.25 as required. Thus, we have verified that the formulation in theorem 5.6.2 and 5.6.3 lead to single-valued polylogarithms in one variable. We state the result in the general case below, starting with the extension to lemma 5.6.2.

Proposition 5.6.4. *The single-valued image of the generating series of polylogarithms $\mathbb{G}_k$ is given by*

$$\begin{aligned}\mathrm{sv}\,\mathbb{G}\left[\begin{smallmatrix} e^*_{k,0} & e_{k,k+1} & \cdots & e_{k,n+1} \\ z_0 & z_{k+1} & \cdots & z_{n+1}\end{smallmatrix}; z_k\right] \\ = \overline{\mathbb{G}\left[\begin{smallmatrix} e^*_{k,0} & \widehat{e_{k,k+1}} & \cdots & \widehat{e_{k,n+1}} \\ z_0 & z_{k+1} & \cdots & z_{n+1}\end{smallmatrix}; z_k\right]}^t \,\mathbb{G}\left[\begin{smallmatrix} e^*_{k,0} & e_{k,k+1} & \cdots & e_{k,n+1} \\ z_0 & z_{k+1} & \cdots & z_{n+1}\end{smallmatrix}; z_k\right]\end{aligned} \tag{5.6.29}$$

where we have for $j = k+1, k+2, \ldots, n+1$,

$$\widehat{e_{k,j}} = \mathrm{sv}\, e'_{k,j} = \mathrm{sv}\,\mathbb{G}(z_k \to z_j;\, \overrightarrow{e^*_{k,j}})\, e_{k,j}\, \mathrm{sv}\,\mathbb{G}(z_k \to z_j;\, \overrightarrow{e^*_{k,j}})^{-1}\,. \tag{5.6.30}$$

This result can be reformulated as follows and the same argument as before verifies the single-valuedness of the expression.

Proposition 5.6.5 (Frost-Hidding-Kamlesh-Rodriguez-Schlotterer-Verbeek)**.** *Suppose that the equivalent criteria in theorem 5.2.1 hold. Then, the single-valued image of the generating series of polylogarithms $\mathbb{G}_k$ is given by*

$$\begin{aligned}\mathrm{sv}(\mathbb{G}(z_k; \overrightarrow{e^*_{k,0}})) = \mathrm{sv}\left(\mathbb{M}\,\mathbb{G}(z_n; \overrightarrow{e^*_{n,0}})) \ldots \,\mathbb{G}(z_{k+1}; \overrightarrow{e^*_{k+1,0}})\right)^{-1} \overline{\mathbb{G}(z_k; \overrightarrow{e^*_{k,0}})^t} \\ \times\, \mathrm{sv}\left(\mathbb{M}\,\mathbb{G}(z_n; \overrightarrow{e^*_{n,0}})) \ldots \,\mathbb{G}(z_{k+1}; \overrightarrow{e^*_{k+1,0}})\right) \mathbb{G}(z_k; \overrightarrow{e^*_{k,0}})\,.\end{aligned} \tag{5.6.31}$$

Proof. The theorem 5.2.1 holds. Therefore, we can make use of the following conjugate identity 5.2.5 along with proposition 5.6.4.

$$\begin{aligned}(\mathbb{I}\mathbb{G}_n \ldots \mathbb{G}_{k+1})^{-1}\, e_{k,j}\, (\mathbb{I}\mathbb{G}_n \ldots \mathbb{G}_{k+1}) = \\ \mathbb{G}(z_k = z_j;\, \overrightarrow{e^*_{k,j}})\, e_{k,j}\, \mathbb{G}(z_k = z_j;\, \overrightarrow{e^*_{k,j}})^{-1}\,.\end{aligned} \tag{5.6.32}$$

Applying the sv map to the above equation and using 5.6.1 we get

$$\begin{aligned}\widehat{e_{k,j}} = \mathrm{sv}\, e'_{k,j} = \mathrm{sv}\,\mathbb{G}(z_k \to z_j;\, \overrightarrow{e^*_{k,j}})\, e_{k,j}\, \mathrm{sv}\,\mathbb{G}(z_k \to z_j;\, \overrightarrow{e^*_{k,j}})^{-1} \\ = \mathrm{sv}\,(\mathbb{M}\mathbb{G}_n \ldots \mathbb{G}_{k+1})^{-1}\, e_{k,j}\, \mathrm{sv}\,(\mathbb{M}\mathbb{G}_n \ldots \mathbb{G}_{k+1})\end{aligned} \tag{5.6.33}$$

We apply this to equation 5.6.29 along with the relation

$$[e^*_{k,0}, \mathbb{M}] = 0 \quad \text{and} \quad [e^*_{k,0}, \mathbb{G}_j] = 0 \tag{5.6.34}$$

and take conjugating factors out to get

$$\begin{aligned}\mathrm{sv}(\mathbb{G}(z_k; \overrightarrow{e^*_{k,0}})) = \mathrm{sv}\left(\mathbb{M}\,\mathbb{G}(z_n; \overrightarrow{e^*_{n,0}})) \ldots \,\mathbb{G}(z_{k+1}; \overrightarrow{e^*_{k+1,0}})\right)^{-1} \overline{\mathbb{G}(z_k; \overrightarrow{e^*_{k,0}})^t} \\ \times\, \mathrm{sv}\left(\mathbb{M}\,\mathbb{G}(z_n; \overrightarrow{e^*_{n,0}})) \ldots \,\mathbb{G}(z_{k+1}; \overrightarrow{e^*_{k+1,0}})\right) \mathbb{G}(z_k; \overrightarrow{e^*_{k,0}})\,.\end{aligned} \tag{5.6.35}$$

where we have cancelled out the two instances of complex conjugation. □

Remark 5.6.6. It is not necessary to use proposition 5.6.4 to prove this result. One can directly use the algebraic construction [28] of the single-valued map discussed earlier in the section to get the above result. This approach will appear in [36].

Remark 5.6.7. Note that the generating series $\mathrm{sv}\,\mathbb{G}_k$ has the same holomorphic differential in z_k as $\mathbb{G}_k$. However, the anti-holomorphic differential is clearly non-zero. The computation of this anti-holomorphic differential will also appear in [36].

5.6.3 Hypergeometric functions

We can use the above results to give a compact formulation for the single-valued image of F.

Theorem 5.6.8 (Frost-Hidding-Kamlesh-Rodriguez-Schlotterer-Verbeek)**.** *Suppose that the equivalent conditions of Theorem 5.2.1 hold. Then, the single-valued image of F is given by*

$$\mathrm{sv}\, F = \overline{F^t}\, F\,. \tag{5.6.36}$$

Proof. By proposition 5.6.5 we have

$$\begin{aligned}\mathrm{sv}F = \mathrm{sv}(\mathbb{M}\,\mathbb{G}_n \dots \mathbb{G}_1) &= \mathrm{sv}(\mathbb{M}\,\mathbb{G}_n \dots \mathbb{G}_2)\,(\mathrm{sv}\,\mathbb{G}_1) \\ &= \mathrm{sv}(\mathbb{M}\,\mathbb{G}_n \dots \mathbb{G}_2)\,\big(\mathrm{sv}(\mathbb{M}\,\mathbb{G}_n \dots \mathbb{G}_2)\big)^{-1}\,\overline{\mathbb{G}_1^t}\,\mathrm{sv}(\mathbb{M}\,\mathbb{G}_n \dots \mathbb{G}_2)\,\mathbb{G}_1 \\ &= \overline{\mathbb{G}_1^t}\,\mathrm{sv}(\mathbb{M}\,\mathbb{G}_n \dots \mathbb{G}_2)\,\mathbb{G}_1\,. \end{aligned} \tag{5.6.37}$$

Repeating the above argument for the factor $\mathrm{sv}(\mathbb{M}\,\mathbb{G}_n \dots \mathbb{G}_2)$ we can rewrite the above equation as

$$\mathrm{sv}\, F^{\mathfrak{m}} = \overline{\mathbb{G}_1^t} \dots \overline{\mathbb{G}_n^t}\,(\mathrm{sv}\,\mathbb{M})\,\mathbb{G}_n \dots \mathbb{G}_1\,. \tag{5.6.38}$$

Finally, we use lemma 5.6.1 to rewrite 5.6.37 as

$$\mathrm{sv}F = \overline{\mathbb{G}_1^t} \dots \overline{\mathbb{G}_n^t}\,\mathbb{M}^t\,\mathbb{M}\,\mathbb{G}_n \dots \mathbb{G}_1 \tag{5.6.39}$$

$$= \overline{(\mathbb{M}\,\mathbb{G}_n \dots \mathbb{G}_1)^t}\,(\mathbb{M}\,\mathbb{G}_n \dots \mathbb{G}_1) \tag{5.6.40}$$

$$= \overline{(F)^t}\,F\,. \tag{5.6.41}$$

□

Chapter 6

Application: Proof of the coaction conjecture

In this chapter, we will prove the coaction conjecture for the period matrix $F^{(n,p)}$ in the case $p = 1$ and $n = 4, 5$ as well as $p = 2$ and $n = 5, 6$.

6.1 Dimension 1

6.1.1 (n,p)=(4,1)

In this section, we set $n = 4$ and $p = 1$. Then $\mathcal{C}^{(n,p)}$is one-dimensional.

In terms of coordinates we have

$$\mathcal{C}^{(n,p)} = \{z_2 \in \mathbb{C} \,|\, z_2 \neq 0, z_2 \neq 1\}. \tag{6.1.1}$$

Further the (co)-homology groups are one-dimensional since $d^{(n,p)} = \frac{(4-3)!}{(4-3-1)!} = 1$. So the homology basis consists of just one element

$$\gamma_1^{(4,1)} = \{0 < z_2 < 1\} \tag{6.1.2}$$

and the cohomology basis is the class of

$$\omega_1^{(4,1)} = \frac{s_{12}}{z_2} dz_2\,. \tag{6.1.3}$$

Also, the Koba-Nielsen factor in this case is

$$KN^{(4,1)} = z_2^{s_{12}}(1 - z_2)^{s_{23}}\,. \tag{6.1.4}$$

Therefore, the period matrix is given by

$$F^{(4,1)} = (F^{(4,1)}_{1,1}) \tag{6.1.5}$$

where

$$F^{(4,1)}_{11} = \int_0^1 z_2^{s_{12}}(1-z_2)^{s_{23}}\, \frac{s_{12}}{z_2}\, dz_2\,,\ \mathrm{Re}(s_{ij}) > 0\,. \tag{6.1.6}$$

The poles in the integral above can only occur at the end points of integration. Clearly there is no pole at 1. On the other hand, near zero, the integrand behaves like $s_{12}z_2^{s_{12}-1}$ which integrates out to $z_2^{s_{12}}$ and evaluates to 1 after putting in the boundary values, so the integral is indeed well-defined.

Note that the above period integral can be expressed in terms of the well-known Beta and Gamma function as follows

$$F_{11} = \frac{s_{12}s_{23}}{s_{12}+s_{23}}B(s_{12},s_{23}) = \frac{\Gamma(1+s_{12})\Gamma(1+s_{23})}{\Gamma(1+s_{12}+s_{23})}\ . \tag{6.1.7}$$

This is a particularly special situation where we can write down explicitly an exponential series expansion formula for the period integral $F^{(4,1)}_{11}$ and its 'motivic' lift.

$$F^{(4,1),\mathfrak{m}}_{11} = \exp(\sum_{n\geq 2} \frac{(-1)^{n-1}\zeta^{\mathfrak{m}}(n)}{n}((s_{12}+s_{23})^n - s_{12}^n - s_{23}^n))\,. \tag{6.1.8}$$

Since the single zeta values have a particularly simple coaction formula

$$\Delta(\zeta^{\mathfrak{m}}(2n)) = \zeta^{\mathfrak{m}}(2n)\otimes 1\,, \tag{6.1.9}$$

$$\Delta(\zeta^{\mathfrak{m}}(2n+1)) = \zeta^{\mathfrak{m}}(2n+1)\otimes 1 + 1\otimes \zeta^{\mathfrak{dr}}(2n+1)\,, \tag{6.1.10}$$

we can compute the coaction directly as

$$\begin{aligned}\Delta(F^{(4,1),\mathfrak{m}}_{11}) &= \exp(\sum_{n\geq 2} \frac{(-1)^{n-1}\Delta(\zeta^{\mathfrak{m}}(n))}{n}((s_{12}+s_{23})^n - s_{12}^n - s_{23}^n)) \\ &= F^{(4,1),\mathfrak{m}}_{11}\, F^{(4,1),\mathfrak{dr}}_{11}\end{aligned} \tag{6.1.11}$$

in accordance with the coaction conjecture.

In the next section, we will provide another proof of the above result. Continuing on we may drop the superscripts (n,p) if it is clear from the context.

6.1.2 (n,p)=(5,1)

In this section we set $n = 5$ and $p = 1$. Then we have

$$\mathcal{C}^{(n,p)} = \{z_2 \in \mathbb{C} \,|\, z_2 \neq 0\,, z_2 \neq z_3\,, z_2 \neq 1\}.$$

So $\mathcal{C}^{(n,p)}$is still one dimensional but the (co)homology groups are now $d^{(5,1)} = 2$ dimensional.

We have as our homology basis the classes of following cycles

$$\gamma_1^{(5,1)} = \{0 < z_2 < z_3\}\,, \qquad \gamma_2^{(5,1)} = \{z_3 < z_2 < 1\}\,, \tag{6.1.12}$$

and as our cohomology basis the classes of following forms

$$\omega_1^{(5,1)} = \frac{s_{12}}{z_2}\, dz_2 \cong \big(\frac{s_{32}}{z_3 - z_2} + \frac{s_{42}}{1 - z_2}\big)\, dz_2\,, \tag{6.1.13}$$

$$\omega_2^{(5,1)} = \big(\frac{s_{12}}{z_2} + \frac{s_{23}}{z_2 - z_3}\big)\, dz_2 \cong \frac{s_{42}}{1 - z_2} dz_2\,. \tag{6.1.14}$$

And the KZ-factor is

$$\mathrm{KZ}^{(5,1)} = |z_2|^{s_{12}}\, |z_2 - z_3|^{s_{23}}\, |z_2 - 1|^{s_{24}}\,. \tag{6.1.15}$$

Next we can compute using integration by parts and partial fraction identities that the period matrix $F^{(5,1)}(z_3) = \big(\int_{\gamma_a^{(5,1)}} \mathrm{KN}^{(5,1)}\, \omega_{\mathrm{b}}^{(5,1)}\big)$ satisfies the KZ equation

$$\frac{d}{dz_3} F^{(5,1)}(z_3) = F^{(5,1)}(z_3)\big(\frac{E_{31}^{(5,1)}}{z_{31}} + \frac{(E_{34}^{(5,1)})}{z_{34}}\big) \tag{6.1.16}$$

for the matrices

$$E_{31}^{(5,1)} = \begin{pmatrix} s_{12}{+}s_{23} & 0 \\ -s_{12} & 0 \end{pmatrix}, \qquad E_{34}^{(5,1)} = \begin{pmatrix} 0 & -s_{24} \\ 0 & s_{23}{+}s_{24} \end{pmatrix}. \tag{6.1.17}$$

For convenience of notation we will write $E_{0,z_3} = E_{31}$ and $E_{1,z_3} = E_{34}$ and we may even drop the subscript z_3 in this section.

In this notation a solution to the KZ equation above is given in terms of multiple polylogarithms as follows.

$$\mathbb{G}(E_0, E_1; z_3) = \sum_{r=0}^{\infty} \sum_{\substack{a_1,a_2,\dots \\ \dots,a_r \in \{0,1\}}} \mathrm{G}(a_r, \dots, a_2, a_1; z_3) E^{(5,1)}_{a_1,z_3} E^{(5,1)}_{a_2,z_3} \dots E^{(5,1)}_{a_r,z_3} . \tag{6.1.18}$$

Since the KZ equation is linear we can solve for the period matrix $F^{(5,1)}(z_3)$ in terms of the above generating series by multiplying a prefactor matrix of initial values. Specifically, we have

$$F^{(5,1)}(z_3) = C_0^{(5,1)} \, \mathbb{G}(E_0, E_1; z_3) \tag{6.1.19}$$

where $C_0^{(5,1)} \coloneqq \lim_{z_3 \to 0} F^{(5,1)}(z_3) \, z_3^{-E_0}$.

To be able to compute the coaction of $F^{(5,1)}(z_3)$, we first need to compute the coaction of $C_0^{(5,1)}$. However, this requires us to first compute the initial value matrix. In that direction, the z_3 limit for the first row of the period matrix, that is, for the integral over $\gamma_1 = 0 < z_2 < z_3$, can be computed by first making a change of variables $z_2 = z_3 t_2$ to eliminate the dependence of z_3 from the integration path and then taking the limit.

A straightforward change of variables doesn't work for the second row but we can make use of the monodromy relations [8, 52] to write the integral over γ_2 in terms of integrals over paths which have no z_3 dependence or whose z_3 dependence can be eliminated with a change of variables.

In our case, the monodromy relation comes down to the vanishing of the following integral

$$\oint_{\mathcal{C}} (-z_2)^{s_{12}} (z_3 - z_2)^{s_{23}} (1 - z_2)^{s_{24}} \omega_a^{(5,1)} \tag{6.1.20}$$

valid for both $a = 1$ and 2, obtained by applying Cauchy's theorem to the contour depicted on the next page.

So we simply have the following relation

$$\int_{-\infty}^{\infty} (-z_2)^{s_{12}} (z_3 - z_2)^{s_{23}} (1 - z_2)^{s_{24}} \omega_a^{(5,1)} = 0 \, . \tag{6.1.21}$$

The above integral can be split into four intervals for z_2 which we correct with monodromy factors to get relations among period integrals.

1. $\gamma_{neg} = (-\infty, 0)$: All the factors in brackets are positive so we do not make any changes.

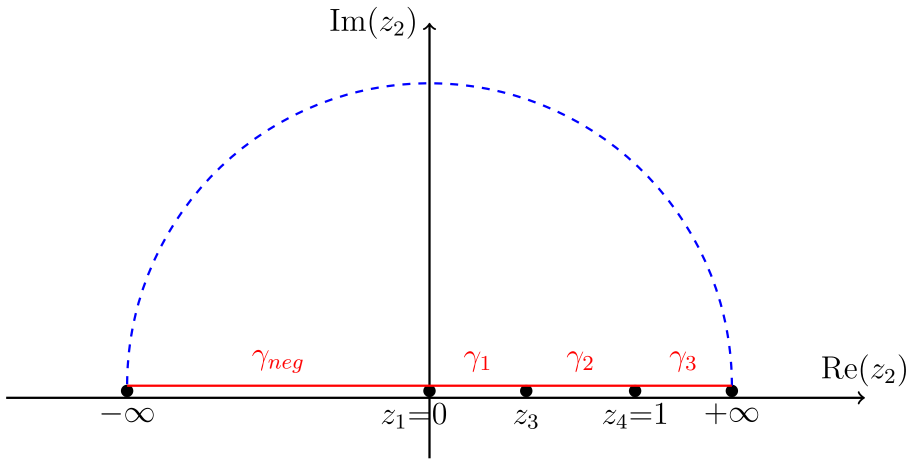

FIGURE 6.1: The closed contour $\mathcal{C}$ consists of the real line drawn in red and split into paths γ_{neg}, γ_1, γ_2, γ_3, whereas the blue semicircle simply indicates that $z_5 \to \pm\infty$ are identified on the Riemann sphere and does not contribute to the integral.

2. $\gamma_1 = (0, z_3)$: The first factor in brackets is negative so we can express the integral in this range as $\exp(\pm\pi s_{12}))F^{(5,1)}_{1a}$ with the signs depending on how we move around zero.
3. $\gamma_2 = (z_3, 1)$: Both the first and second factor are negative so we write $\exp(\pm\pi(s_{12} + s_{23}))F^{(5,1)}_{2a}$.
4. $\gamma_3 = (1, \infty)$: All the factors are negative so we write $\exp(\pm\pi(s_{12} + s_{23} + s_{24}))F^{(5,1)}_{3a}$ where we set $\gamma_3 = (1, \infty)$ and $F^{(5,1)}_{3a} = \int_{\gamma_3} KN^{(5,1)}\omega_a$.

Above gives us two relations, one each for a choice of sign on the exponential factors. Taking the difference of the two we can eliminate the integral over γ_{neg} and get that

$$\sin(\pi s_{12}))F^{(5,1)}_{1a} + \sin(\pi(s_{12}+s_{23}))F^{(5,1)}_{2a} + \sin(\pi(s_{12}+s_{23}+s_{24}))F^{(5,1)}_{3a} = 0\,. \quad (6.1.22)$$

We already know how to make a change of variables for the integral over γ_2 and γ_3 has no dependence on z_3. So we can just take the limit as $z_3 \to 0$ which gives us the second row of C_0.

$$C_0 = \begin{pmatrix} \frac{\Gamma(1+s_{12})\Gamma(1+s_{23})}{\Gamma(1+s_{12}+s_{23})} & 0 \\ \frac{s_{12}}{s_{12}+s_{23}}\frac{\Gamma(1+s_{24})\Gamma(1+s_{12}+s_{23})}{\Gamma(1+s_{12}+s_{23}+s_{24})} - \frac{\sin(\pi s_{12})}{\sin(\pi(s_{12}+s_{23}))}\frac{\Gamma(1+s_{12})\Gamma(1+s_{23})}{\Gamma(1+s_{12}+s_{23})} & \frac{\Gamma(1+s_{12}+s_{23})\Gamma(1+s_{24})}{\Gamma(1+s_{12}+s_{23}+s_{24})} \end{pmatrix}. \quad (6.1.23)$$

Note that we could also have looked at the limit of $F^{(5,1)}(z_3)$ as z_3 tends to 1. So we define $C_1^{(5,1)} := \lim_{z_3\to 1} F^{(5,1)}(z_3)(1-z_3)^{-E_1}$. Now, we know that the boundary values are related via the Drinfeld associator as follows.

$$C_1^{(5,1)} = C_0^{(5,1)}\,\Phi(E_0, E_1)\,. \tag{6.1.24}$$

To give a preview of what is to come, we note that the above will lead to some relations involving Beta function that will help us compute its coaction. This is to be expected because the factor $\Phi(E_0, E_1)$ is a meta-abelian quotient [33, 26] of the Drinfeld associator and its entries can be written down explicitly in terms of the Beta function and this suggests that the entries of C_1 should also consists of Beta functions.

Now we can compute the entries of C_1 directly by calculating the $z_3 \to 1$ limit. However we give here another recipe for deriving C_1 in terms of C_0.

Recall that we have

$$\frac{d}{dz_3}F^{(5,1)}(z_3) = F^{(5,1)}(z_3)\,\Big(\frac{E_0}{z_3} + \frac{E_1}{z_3-1}\Big)\,. \tag{6.1.25}$$

Define a change of variables $y_3 = 1 - z_3$ and $y_2 = 1 - z_2$. Then we can rewrite the above KZ equation as

$$-\frac{d}{dy_3}(F^{(5,1)}(1-y_3)) = -F^{(5,1)}(1-y_3)\Big(\frac{E_0}{y_3-1} + \frac{E_1}{y_3}\Big)\,. \tag{6.1.26}$$

Now, let's analyze $F^{(5,1)}(1-y_3)$. First, we have the change in basis of cycles

$$\gamma_1 = 0 < z_2 < z_3 \Leftrightarrow 1 > y_2 > y_3 = -\,\gamma_2(y_2, y_3)\,, \tag{6.1.27}$$

$$\gamma_2 = z_3 < z_2 < 1 \Leftrightarrow y_3 > y_2 > 0 = -\,\gamma_1(y_2, y_3)\,. \tag{6.1.28}$$

Secondly, we have the modification in the basis for forms

$$\omega_1 = \frac{s_{12}}{z_2}dz_2 = -\frac{s_{12}}{1-y_2}dy_2 \;\cong\; -\Big(\frac{s_{24}}{y_2} + \frac{s_{23}}{y_2-y_3}\Big)dy_2\,, \tag{6.1.29}$$

$$\omega_2 = \Big(\frac{s_{12}}{z_2} + \frac{s_{23}}{z_2-z_3}\Big)dz_2 \;\cong\; \frac{s_{24}}{1-z_2}dz_2 = -\frac{s_{24}}{y_2}dy_2\,. \tag{6.1.30}$$

Therefore, in the y-variables we have

$$\omega_1 = -\omega_2(y_2, y_3, 1\leftrightarrow 4)\,,\; \omega_2 = -\omega_1(y_2, y_3, 1\leftrightarrow 4) \tag{6.1.31}$$

where we keep track that we have to swap 1 and 4 in the subscripts of s_{ij}'s.

The KZ factor also modifies in alignment with our choice of sections since $z_i - z_j = y_j - y_i$ have the same sign for $1 \leq i, j \leq 2$ and thus we only have positive terms inside brackets in the expression for $\mathrm{KZ}^{(5,1)}$.

Therefore, in terms of matrix entries we have

$$F_{ij}^{(5,1)}(z_3) = F_{ij}^{(5,1)}(1-y_3) = F_{(3-i)(3-j),1\leftrightarrow 4}^{(5,1)}(y_3) \tag{6.1.32}$$

for $1 \leq i, j \leq 2$, since the signs from the basis of cycles and forms cancel out.

These kind of relations will be a general feature so we introduce a new notation $\check{A}$ for 2×2 matrices that interchanges entries (i, j) and $(3-i, 3-j)$ as well as swaps 1 and 4 in the indices for s_{ij}. Therefore, we have in the new notation

$$F^{(5,1)}(z_3) = F^{(5,1)}(1-y_3) = \check{F}^{(5,1)}(y_3) \tag{6.1.33}$$

and the updated differential equation

$$\frac{d}{dy_3}(\check{F}^{(5,1)}(y_3)) = (\check{F}^{(5,1)}(y_3))\Big(\frac{E_1}{y_3} + \frac{E_0}{y_3 - 1}\Big)\,. \tag{6.1.34}$$

Comparing with the original equation 6.1.25 we can observe two facts of interest. First we have the matrix equality

$$E_1 = \check{E}_0 \tag{6.1.35}$$

which can also be verified by checking the matrix entries. Secondly,

Lemma 6.1.1. *The limit as $z_3 \to 1$ satisfies $C_1^{(5,1)} = \check{C_0}^{(5,1)}$.*

Proof.

$$\begin{aligned} C_1^{(5,1)} &= \lim_{z_3 \to 1} F^{(5,1)}(z_3)(1-z_3)^{-E_1} \\ &= \lim_{y_3 \to 0} F^{(5,1)}(1-y_3)(y_3)^{-E_1} \\ &= \lim_{y_3 \to 0} (\check{F}^{(5,1)}(y_3))(y_3)^{-E_1} \\ &= \lim_{y_3 \to 0} (\check{F}^{(5,1)}(y_3))(y_3)^{-\check{E}_0} \\ &= \check{C_0}^{(5,1)} \end{aligned} \tag{6.1.36}$$

□

Therefore we can write down the entries of C_1 without having to compute the $z_3 \to 1$ limit of $F^{(5,1)}$ and this is stated below.

$$C_1 = \begin{pmatrix} \frac{\Gamma(1+s_{24}+s_{23})\Gamma(1+s_{12})}{\Gamma(1+s_{12}+s_{23}+s_{24})} & \frac{s_{24}}{s_{24}+s_{23}} \frac{\Gamma(1+s_{12})\Gamma(1+s_{24}+s_{23})}{\Gamma(1+s_{12}+s_{23}+s_{24})} - \frac{\sin(\pi s_{24})}{\sin(\pi(s_{24}+s_{23}))} \frac{\Gamma(1+s_{24})\Gamma(1+s_{23})}{\Gamma(1+s_{24}+s_{23})} \\ 0 & \frac{\Gamma(1+s_{24})\Gamma(1+s_{23})}{\Gamma(1+s_{23}+s_{24})} \end{pmatrix} . \tag{6.1.37}$$

Now let's focus again on the coaction! We just need two more ingredients.

The first thing is to observe that if we replace the sine factors in the entries of C_0 with its argument then the resulting matrix commutes with E_0. Concretely, let's take

$$C_0^\pi = \begin{pmatrix} \frac{\Gamma(1+s_{12})\Gamma(1+s_{23})}{\Gamma(1+s_{12}+s_{23})} & 0 \\ \frac{s_{12}}{s_{12}+s_{23}} \frac{\Gamma(1+s_{24})\Gamma(1+s_{12}+s_{23})}{\Gamma(1+s_{12}+s_{23}+s_{24})} - \frac{s_{12}}{s_{12}+s_{23}} \frac{\Gamma(1+s_{12})\Gamma(1+s_{23})}{\Gamma(1+s_{12}+s_{23})} & \frac{\Gamma(1+s_{12}+s_{23})\Gamma(1+s_{24})}{\Gamma(1+s_{12}+s_{23}+s_{24})} \end{pmatrix} . \tag{6.1.38}$$

Then, we have $[E_0, C_0^\pi] = 0$ and as a consequence $[E_0, C_0^{\mathfrak{dr}}] = 0$. Similarly, we also have $[E_1, C_1^\pi] = 0$ and $[E_1, C_1^{\mathfrak{dr}}] = 0$ where C_1^π is defined in an analogous way by getting rid of the factors of π coming from the monodromy relations.

The second ingredient that we need is the matrices C_1 and C_0 when we set $s_{23} = 0$. Let $\tilde{C}_1 = C_1(s_{23} \to 0)$, $\tilde{C}_0 = C_0(s_{23} \to 0)$, $\tilde{E}_0 = E_0(s_{23} \to 0)$ and $\tilde{E}_1 = E_1(s_{23} \to 0)$. Then, we have

$$\tilde{C}_1 = \begin{pmatrix} \frac{\Gamma(1+s_{12})\Gamma(1+s_{24})}{\Gamma(1+s_{12}+s_{24})} & \frac{\Gamma(1+s_{12})\Gamma(1+s_{24})}{\Gamma(1+s_{12}+s_{24})} - 1 \\ 0 & 1 \end{pmatrix} , \tag{6.1.39}$$

$$\tilde{C}_0 = \begin{pmatrix} 1 & 0 \\ \frac{\Gamma(1+s_{12})\Gamma(1+s_{24})}{\Gamma(1+s_{12}+s_{24})} - 1 & \frac{\Gamma(1+s_{12})\Gamma(1+s_{24})}{\Gamma(1+s_{12}+s_{24})} \end{pmatrix} \tag{6.1.40}$$

and

$$\tilde{C}_1 = \tilde{C}_0 \, \Phi(\tilde{E}_0, \tilde{E}_1) \, . \tag{6.1.41}$$

Theorem 6.1.2. *The motivic coaction of the initial value matrix is given by*

$$\Delta(C_0^{(5,1),\mathfrak{m}}) = C_0^{(5,1),\mathfrak{m}} C_0^{(5,1),\mathfrak{dr}} \, . \tag{6.1.42}$$

Proof (D. Kamlesh). Recall that by Ihara's coaction formula we have

$$\Delta(\Phi^{\mathfrak{m}}(\tilde{E}_0, \tilde{E}_1)) = \Phi^{\mathfrak{m}}(\tilde{E}_0, \tilde{E}_1{}') \, \Phi^{\mathfrak{dr}}(\tilde{E}_0, \tilde{E}_1) \, , \tag{6.1.43}$$

$$\tilde{E_1}' = \Phi^{\mathfrak{dr}}(\tilde{E_0}, \tilde{E_1})\, \tilde{E_1}\, (\Phi^{\mathfrak{dr}}(\tilde{E_0}, \tilde{E_1}))^{-1}\,. \tag{6.1.44}$$

Replacing the deRham Drinfeld associator with initial value matrices from 6.1.24 we get

$$\Phi^{\mathfrak{dr}}(\tilde{E_0}, \tilde{E_1}) = \tilde{C_0}^{-\mathfrak{dr}} \tilde{C_1}^{\mathfrak{dr}} \tag{6.1.45}$$

and so 6.1.44 gives

$$\tilde{E_1}' = \tilde{C_0}^{-\mathfrak{dr}}\, \tilde{C_1}^{\mathfrak{dr}}\, \tilde{E_1}\, \tilde{C_1}^{-\mathfrak{dr}}\, \tilde{C_0}^{\mathfrak{dr}} = \tilde{C_0}^{-\mathfrak{dr}}\, \tilde{E_1}\, \tilde{C_0}^{\mathfrak{dr}} \tag{6.1.46}$$

since we have $[E_1, C_1^{\mathfrak{dr}}] = 0$ and therefore $[\tilde{E_1}, \tilde{C}_1^{\mathfrak{dr}}] = 0$.

Using above in the coaction we get

$$\begin{aligned} \Delta(\Phi^{\mathfrak{m}}(\tilde{E_0}, \tilde{E_1})) &= \Phi^{\mathfrak{m}}(\tilde{E_0}, \tilde{E_1}')\, \Phi^{\mathfrak{dr}}(\tilde{E_0}, \tilde{E_1}) \\ &= \Phi^{\mathfrak{m}}(\tilde{C_0}^{-\mathfrak{dr}} \tilde{E_0} \tilde{C_0}^{\mathfrak{dr}}, \tilde{C_0}^{-\mathfrak{dr}} \tilde{E_1} \tilde{C_0}^{\mathfrak{dr}})\, \Phi^{\mathfrak{dr}}(\tilde{E_0}, \tilde{E_1}) \\ &= \tilde{C_0}^{-\mathfrak{dr}}\, \Phi^{\mathfrak{m}}(\tilde{E_0}, \tilde{E_1})\, \tilde{C_0}^{\mathfrak{dr}}\, \Phi^{\mathfrak{dr}}(\tilde{E_0}, \tilde{E_1}) \\ &= \tilde{C_0}^{-\mathfrak{dr}}\, \Phi^{\mathfrak{m}}(\tilde{E_0}, \tilde{E_1})\, \tilde{C_1}^{\mathfrak{dr}}\,. \end{aligned} \tag{6.1.47}$$

where the last step uses $\tilde{C_1}^{\mathfrak{dr}} = \tilde{C_0}^{\mathfrak{dr}}\, \Phi^{\mathfrak{dr}}(\tilde{E_0}, \tilde{E_1})$.

Since the coaction preserves products we can use 6.1.47 to get

$$\begin{aligned} \Delta(\tilde{C_1}^{\mathfrak{m}}) &= \Delta(\tilde{C_0}^{\mathfrak{m}})\, \Delta(\Phi^{\mathfrak{m}}(\tilde{E_0}, \tilde{E_1})) \\ &= \Delta(\tilde{C_0}^{\mathfrak{m}})\, \tilde{C_0}^{-\mathfrak{dr}} \Phi^{\mathfrak{m}}\, (\tilde{E_0}, \tilde{E_1})\, \tilde{C_1}^{\mathfrak{dr}}\,. \end{aligned} \tag{6.1.48}$$

Recall that the first row of $\tilde{C_0}^{\mathfrak{m}}$ is just $(1, 0)$ by 6.1.40 and $\Delta((1, 0)) = (1, 0)$. Thus $\Delta(\tilde{C_0}^{\mathfrak{m}}) = \tilde{C_0}^{\mathfrak{m}}\, \tilde{C_0}^{w}$ is verified up to the first row.

Applying this to 6.1.48 we get

$$\begin{aligned} \Delta(\tilde{C_1}^{\mathfrak{m}}) &= \tilde{C_0}^{\mathfrak{m}}\, \Phi^{\mathfrak{m}}(\tilde{E_0}, \tilde{E_1}), \tilde{C_1}^{\mathfrak{dr}} \\ &= \tilde{C_1}^{\mathfrak{m}}\, \tilde{C_1}^{\mathfrak{dr}} \end{aligned} \tag{6.1.49}$$

which is verified up to the first row. But this is enough to determine the coaction of the first row of $\tilde{C_1}^{\mathfrak{m}}$, see 6.1.39, and in particular the first entry of the row and so we get

$$\Delta\Big(\frac{\Gamma^{\mathfrak{m}}(1{+}s_{12})\Gamma^{\mathfrak{m}}(1{+}s_{24})}{\Gamma^{\mathfrak{m}}(1{+}s_{12}{+}s_{24})}\Big) = \frac{\Gamma^{\mathfrak{m}}(1{+}s_{12})\Gamma^{\mathfrak{m}}(1{+}s_{24})}{\Gamma^{\mathfrak{m}}(1{+}s_{12}{+}s_{24})}\, \frac{\Gamma^{\mathfrak{dr}}(1{+}s_{12})\Gamma^{\mathfrak{dr}}(1{+}s_{24})}{\Gamma^{\mathfrak{dr}}(1{+}s_{12}{+}s_{24})} \tag{6.1.50}$$

that is

$$\Delta F^{(4,1),\mathfrak{m}}(s_{12}, s_{24}) = F^{(4,1),\mathfrak{m}}(s_{12}, s_{24})\, F^{(4,1),\mathfrak{dr}}(s_{12}, s_{24}) \tag{6.1.51}$$

as already observed in the previous section.

Now, since the entries of $C_1^{(5,1),\mathfrak{m}}$ are composed entirely in terms of Beta functions and sine factors one can use the above result to check that we also have

$$\Delta(C_1^{(5,1),\mathfrak{m}}) = C_1^{(5,1),\mathfrak{m}}\, C_1^{(5,1),\mathfrak{dr}} \text{ and } \Delta(C_0^{(5,1),\mathfrak{m}}) = C_0^{(5,1),\mathfrak{m}}\, C_0^{(5,1),\mathfrak{dr}}\,. \tag{6.1.52}$$

□

And finally, we have the coaction conjecture for Gauss's ${}_2F_1$ hypergeometric function.

Theorem 6.1.3. *The motivic coaction of $F^{(5,1),\mathfrak{m}}$ is given by*

$$\Delta(F^{(5,1),\mathfrak{m}}) = F^{(5,1),\mathfrak{m}}\, F^{(5,1),\mathfrak{dr}}\,. \tag{6.1.53}$$

Proof (D. Kamlesh). By theorem 5.2.1, criteria 3, it is enough to know that the following conditions are satisfied.

1. $\Delta(C_0^{\mathfrak{m}}) = C_0^{\mathfrak{m}}\, C_0^{\mathfrak{dr}}\,.$
2. $(C_0^{\mathfrak{dr}})^{-1}\, E_0\, (C_0^{\mathfrak{dr}}) = 0\,.$
3. $(C_0^{\mathfrak{dr}})^{-1}\, E_1\, C_0^{\mathfrak{dr}} = \Phi^{\mathfrak{dr}}(E_0, E_1)\, E_1\, \Phi^{\mathfrak{dr}}(E_0, E_1)^{-1}\,.$

The first condition is just theorem 6.1.2. And we have already noted earlier in the section that

$$[E_0, C_0^{\mathfrak{dr}}] = 0 = [E_1, C_1^{\mathfrak{dr}}]\,. \tag{6.1.54}$$

Hence, the second condition is also satisfied. Finally, making use of the above relation and equation 6.1.24, we have

$$\begin{aligned} \Phi^{\mathfrak{dr}}(E_0, E_1)\, E_1\, \Phi^{\mathfrak{dr}}(E_0, E_1)^{-1} &= (C_0^{\mathfrak{dr}})^{-1}\, C_1^{\mathfrak{dr}}\, E_1\, (C_1^{\mathfrak{dr}})^{-1}\, C_0^{\mathfrak{dr}} && (6.1.55)\\ &= (C_0^{\mathfrak{dr}})^{-1}\, E_1\, C_0^{\mathfrak{dr}}\,. && (6.1.56) \end{aligned}$$

□

Remark 6.1.4. One can also use equivalent criteria 4 to prove the above theorem. Briefly, the property $\Delta(C_0^{\mathfrak{m}}) = C_0^{\mathfrak{m}}\, C_0^{\mathfrak{dr}}$ implies that we have

$$C_0^{\mathfrak{dr}} = \mathbb{M}^{\mathfrak{dr}} = \sum_{r=0}^{\infty} \sum_{i_1, i_2, \ldots, i_r \in 2\mathbb{N}+1} (f_{i_1} f_{i_2} \ldots f_{i_r})^{\mathfrak{m}} M_{i_1} M_{i_2} \ldots M_{i_r} \tag{6.1.57}$$

for some matrices M_i, $i \in \mathbb{N}$. But then $[E_0, C_0^{\mathfrak{dr}}] = 0$ implies that $[E_0, M_i] = 0$ for all $i \in \mathbb{N}$. Similarly, the relation $C_1^{\mathfrak{dr}} = C_0^{\mathfrak{dr}}\,\Phi(E_0, E_1)$ and $[E_1, C_1^{\mathfrak{dr}}] = 0$ implies that $[E_1, M_i] = [w_i, E_1]$ on comparing the coefficients of zeta values.

Corollary 6.1.5. *The single-valued image of the period matrix $F^{(4,1)}$ and $F^{(5,1)}$ is given by*

$$\mathrm{sv}F = \overline{F^t}\, F \tag{6.1.58}$$

Proof. This follows from theorem 5.6.8 and theorem 6.1.3. □

Remark 6.1.6. Summary of the steps above - Before proceeding to the next chapter which uses the same argument as this one, we summarize our steps as follows. To get the coaction of the period matrix $F^{(n,p)}$ for $(n,p) = (4,1)$ with $n - p = 3$ we first identified $F^{(n,p)}$ as a sub-matrix of the initial value matrices $C_0^{(5,1)}$ and $C_1^{(5,1)}$ for the $(n+1,p) = (5,1)$ period matrix. The initial value matrices are related via the Drinfeld associator $C_1^{(n+1,p)} = C_0^{(n+1,p)}\Phi(E_0, E_1)$ and thus as a consequence we also get a relation for the periods of (n,p). We set some of the s_{ij} entries to zero to simplify the matrix entries and the above relation and applied the coaction formula. The only other feature we needed is the coaction on the Drinfeld associator itself which follows from the Ihara's formula as well as the commutator relations $[E_0, C_0^{(n+1,p),\mathfrak{dr}}] = [E_1, C_1^{(n+1,p),\mathfrak{dr}}] = 0$. From the work of [13] it is known that we always get the period relation as observed here and only the commutator relations need to be known. As long as one can derive this the rest follows and we will this use approach in the next section.

6.2 Dimension 2

6.2.1 (n,p) = (5,2)

In this section we set $n = 5$ and $p = 2$.

So $\mathcal{C}^{(n,p)} = \{(z_2, z_3) \in \mathbb{C} \times \mathbb{C} \,|\, z_2 \neq 0,\ z_2 \neq z_3,\ z_2 \neq 1,\ z_3 \neq 0,\ z_3 \neq 1\}$ is two dimensional and the (co)homology groups are $d^{(5,2)} = 2$ dimensional as well.

The homology and cohomology basis are listed below.

$$\gamma_1^{(5,2)} = \{0 < z_2 < z_3 < 1\}, \qquad \gamma_2^{(5,2)} = \{0 < z_3 < z_2 < 1\}, \tag{6.2.1}$$

$$\hat{\omega}_1^{(5,2)} = \frac{s_{12}}{z_{12}}\left(\frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}}\right), \qquad \hat{\omega}_1^{(5,2)} = \frac{s_{13}}{z_{13}}\left(\frac{s_{12}}{z_{12}} + \frac{s_{32}}{z_{32}}\right). \tag{6.2.2}$$

The Koba-Nielsen factor is

$$KN^{(5,2)} = |z_2|^{s_{12}}|z_3|^{s_{12}}|z_3 - z_2|^{s_{23}}|1 - z_2|^{s_{24}}|1 - z_3|^{s_{34}} \tag{6.2.3}$$

and we have a 2×2 period matrix

$$F^{(5,2)}_{ab} = \int_{\gamma_a} \mathrm{KN}^{(5,2)}\, \omega_b\,,\ 1 \leq a,\, b \leq 2\,. \tag{6.2.4}$$

One way to study this period matrix is to follow the approach in [26] and integrate one variable at a time which allows us to express it as a linear combination of depth 2 Beta quotient of the Drinfeld associator whose coaction can be computed via Ihara's formula. However this gets combinatorially complex quite fast and is a bit unwieldy for higher values of n and p. So we will follow the strategy from the previous chapter instead.

6.2.2 (n,p) = (6,2)

In this section we set $n = 6$ and $p = 2$.

So $\mathcal{C}^{(n,p)} = \{(z_2, z_3) \in \mathbb{C}\times\mathbb{C} \,|\, z_2 \neq 0,\ z_2 \neq z_3,\ z_2 \neq z_4,\ z_2 \neq 1,\ z_3 \neq 0,\ z_3 \neq z_4,\ z_3 \neq 1\}$ is again two dimensional with coordinates z_2, z_3 and one unintegrated puncture z_4. The (co)homology groups are now $d^{(6,2)} = 6$ dimensional.

The homology and cohomology basis are listed below

$$\begin{aligned}
\gamma_1^{(6,2)} &= \{0 < z_2 < z_3 < z_4 < 1\}\,, &\quad \gamma_3^{(6,2)} &= \{0 < z_2 < z_4 < z_3 < 1\}\\
\gamma_5^{(6,2)} &= \{z_4 < z_2 < z_3 < 1\}\,, & \gamma_{2k}^{(6,2)} &= \gamma_{2k-1}^{(6,2)}\Big|_{2\leftrightarrow 3}\,,\quad k = 1,2,3 \text{ and}
\end{aligned} \tag{6.2.5}$$

$$\begin{aligned}
\hat{\omega}_1^{(6,2)} &= \frac{s_{12}}{z_{12}}\left(\frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}}\right), & \hat{\omega}_3^{(6,2)} &= \frac{s_{12}}{z_{12}}\left(\frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}} + \frac{s_{43}}{z_{43}}\right)\\
\hat{\omega}_5^{(6,2)} &= \left(\frac{s_{12}}{z_{12}} + \frac{s_{42}}{z_{42}}\right)\left(\frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}} + \frac{s_{43}}{z_{43}}\right), & \hat{\omega}_{2k}^{(6,2)} &= \hat{\omega}_{2k-1}^{(6,2)}\Big|_{2\leftrightarrow 3}\,,\quad k = 1,2,3
\end{aligned} \tag{6.2.6}$$

where the notation $2 \leftrightarrow 3$ means an interchange of indices.

We have the Koba-Nielsen factor

$$\mathrm{KN}^{(6,2)} = |z_2|^{s_{12}}|z_3|^{s_{12}}|z_3 - z_2|^{s_{23}}|z_4 - z_2|^{s_{24}}||z_4 - z_3|^{s_{34}}|1 - z_2|^{s_{24}}|1 - z_3|^{s_{35}} \tag{6.2.7}$$

and the period matrix $F^{(6,2)}(z_4)$ satisfies the KZ-equation

$$\frac{d}{dz_4}F^{(6,2)}(z_4) = F^{(6,2)}(z_4)\,\Big(\frac{E_{41}^{(6,2)}}{z_{41}} + \frac{(E_{45}^{(6,2)})}{z_{45}}\Big) \tag{6.2.8}$$

for the matrices

$$E_{41}^{(6,2)} = \begin{pmatrix} s_{123}{+}s_{24}{+}s_{34} & 0 & 0 & 0 & 0 & 0 \\ 0 & s_{123}{+}s_{24}{+}s_{34} & 0 & 0 & 0 & 0 \\ -s_{13}{-}s_{23} & -s_{13} & s_{12}{+}s_{24} & 0 & 0 & 0 \\ -s_{12} & -s_{12}{-}s_{23} & 0 & s_{13}{+}s_{34} & 0 & 0 \\ -s_{12} & s_{13} & -s_{12} & 0 & 0 & 0 \\ s_{12} & -s_{13} & 0 & -s_{12} & 0 & 0 \end{pmatrix},$$

$$E_{45}^{(6,2)} = \begin{pmatrix} 0 & 0 & -s_{35} & 0 & -s_{35} & s_{25} \\ 0 & 0 & 0 & -s_{25} & s_{35} & -s_{25} \\ 0 & 0 & s_{35}{+}s_{34} & 0 & -s_{23}{-}s_{25} & -s_{25} \\ 0 & 0 & 0 & s_{25}{+}s_{24} & -s_{35} & -s_{23}{-}s_{35} \\ 0 & 0 & 0 & 0 & s_{235}{+}s_{24}{+}s_{34}{-}s_{35} & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{235}{+}s_{24}{+}s_{34} \end{pmatrix} \tag{6.2.9}$$

and shorthand $s_{123} = s_{12} + s_{13} + s_{23}$, $s_{235} = s_{23} + s_{25} + s_{35}$.

Analogous to the previous chapter we write $E_0 = E_{0,z_4} = E_{41}$ and $E_1 = E_{1,z_4} = E_{45}$ and then a solution to the above differential equation can be expressed as

$$\mathbb{G}(E_0, E_1; z_4) = \sum_{r=0}^{\infty} \sum_{\substack{a_1,a_2,\dots \\ \dots,a_r\in\{0,1\}}} \mathrm{G}(a_r, \dots, a_2, a_1; z_4) E^{(6,2)}_{a_1,z_3} E^{(6,2)}_{a_2,z_3} \dots E^{(6,2)}_{a_r,z_3} \,. \tag{6.2.10}$$

We define

$$C_0^{(6,2)} := \lim_{z_4\to 0} F^{(6,2)}(z_4) z_4^{-E_0} \tag{6.2.11}$$

which allows us to express the period matrix as

$$F^{(6,2)}(z_4) = C_0\, \mathbb{G}(E_0, E_1; z_4)\,. \tag{6.2.12}$$

Therefore, it is imperative that we understand the initial values matrix C_0. The z_4 dependence on the basis cycles γ_1 and γ_2 can be eliminated by a change of variables $z_2 = t_2 z_4$ and $z_3 = t_3 z_4$ but this doesn't work for the other cycles $\gamma_j^{(6,2)}$ for $j = 3, 4, 5, 6$.

However, once again monodromy relations between the basis of bounded cycles and unbounded ones stated below and pictured on the next page come to the rescue.

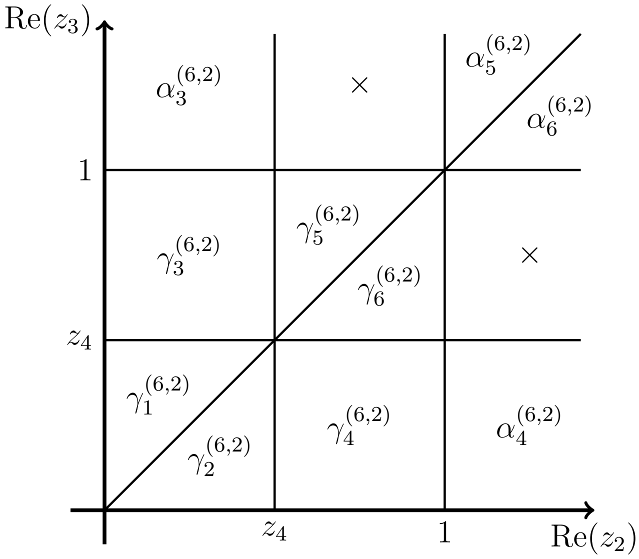

FIGURE 6.2: Integration contour γ's for $(6,2)$ period integrals and α's for monodromy relations. The contours marked $\times$ are not needed.

$$\begin{aligned}
\alpha_3^{(6,2)} &= \{(z_2,z_3)\in\mathbb{R}^2 \mid 0<z_2<z_4<1 \text{ and } 1<z_3<\infty\}\\
\alpha_5^{(6,2)} &= \{(z_2,z_3)\in\mathbb{R}^2 \mid 1<z_2<z_3<\infty\}\\
\alpha_4^{(6,2)} &= \alpha_3^{(6,2)}\Big|_{2\leftrightarrow 3}, \qquad \alpha_6^{(6,2)} = \alpha_5^{(6,2)}\Big|_{2\leftrightarrow 3}
\end{aligned} \tag{6.2.13}$$

The period integrals over the cycles above work better for taking the $z_4 \to 0$ limit since we can eliminate the z_4 dependence from the integration limits.

We illustrate one instance of the monodromy relations obtained from the vanishing of the following integral

$$\int_{z_3=-\infty}^{z_3=\infty}\int_{z_2=0}^{z_2=z_4} (z_2)^{s_{12}}(-z_3)^{s_{13}}(z_2-z_3)^{s_{23}}(z_4-z_2)^{s_{24}}(z_4-z_3)^{s_{34}}(1-z_2)^{s_{24}}(1-z_3)^{s_{35}}\omega_a^{(6,2)}$$

valid for $a\in\{1,\ldots,6\}$.

The integral above can be split into five regions which we correct with monodromy factors to get relations among period integrals. Note that we keep the range of z_2 fixed and we only vary z_3.

1. $(-\infty,0)$: All the factors in brackets are positive so we do not make any changes.
2. $(0,z_2)$: Then $-z_3<0$ so we can express the integral in this range as $\exp(\pm\pi s_{13}))F_{2a}^{(6,2)}$.

3. (z_2, z_4) : $z_2 - z_3 < 0$ so we write $\exp(\pm\pi(s_{13} + s_{23}))F^{(6,2)}_{1a}$.

4. $(z_4, 1)$: $(z_4 - z_3) < 0$ so we write $\exp(\pm\pi(s_{13} + s_{23} + s_{34}))F^{(6,2)}_{3a}$.

5. $(1, \infty)$: $(1 - z_3) < 0$ so we write $\exp(\pm\pi(s_{13} + s_{23} + s_{34} + s_{35}))F^{(6,2)}_{\alpha_3 a}$.

Eliminating the integral over the range $z_3 < 0$ we get

$$\sin(\pi s_{13}))F^{(6,2)}_{2a} + \sin(\pi(s_{13} + s_{23}))F^{(6,2)}_{1a} + \sin(\pi(s_{13} + s_{23} + s_{34}))F^{(6,2)}_{3a} + \sin(\pi(s_{13} + s_{23} + s_{34} + s_{35}))F^{(6,2)}_{\alpha_3\, a} = 0\,. \quad (6.2.14)$$

This helps us express the integral over γ_3 in terms of integrals over more well behaved cycles. Similarly, one can compute the rest of the monodromy relations and use it to compute the initial value $C_0^{6,2}$ as worked out in [12] which we state below verbatim and have also verified independently.

$$\begin{pmatrix} F^{(5,2)}_{11} & F^{(5,2)}_{12} & 0 & 0 & 0 & 0 \\ F^{(5,2)}_{21} & F^{(5,2)}_{22} & 0 & 0 & 0 & 0 \\ H^{(6,2)}_{11} & H^{(6,2)}_{12} & F^{(4,1)}(s_{12}, s_{24})F^{(4,1)}(s_{35}, s_{13}{+}s_{23}{+}s_{34}) & 0 & 0 & 0 \\ H^{(6,2)}_{21} & H^{(6,2)}_{22} & 0 & F^{(4,1)}(s_{13}, s_{34})F^{(4,1)}(s_{25}, s_{12}{+}s_{23}{+}s_{24}) & 0 & 0 \\ J^{(6,2)}_{11} & J^{(6,2)}_{12} & K^{(6,2)}_{11} & \frac{s_{13}}{s_{13}+s_{34}}\hat{F}^{(5,2)}_{12} & \hat{F}^{(5,2)}_{11} & \hat{F}^{(5,2)}_{12} \\ J^{(6,2)}_{21} & J^{(6,2)}_{22} & \frac{s_{12}}{s_{12}+s_{24}}\hat{F}^{(5,2)}_{21} & K^{(6,2)}_{22} & \hat{F}^{(5,2)}_{21} & \hat{F}^{(5,2)}_{22} \end{pmatrix}.$$

The entries $H^{(6,2)}_{1j}$ are given by

$$\begin{aligned} H^{(6,2)}_{11} &= \frac{s_{13}{+}s_{23}}{s_{13}{+}s_{23}{+}s_{34}}F^{(4,1)}(s_{12}, s_{24})F^{(4,1)}(s_{35}, s_{13}{+}s_{23}{+}s_{34}) \\ &\quad - \frac{\sin(\pi(s_{13}{+}s_{23}))}{\sin(\pi(s_{13}{+}s_{23}{+}s_{34}))}F^{(5,2)}_{11} - \frac{\sin(\pi s_{13})}{\sin(\pi(s_{13}{+}s_{23}{+}s_{34}))}F^{(5,2)}_{21} \\ H^{(6,2)}_{12} &= \frac{s_{13}}{s_{13}{+}s_{23}{+}s_{34}}F^{(4,1)}(s_{12}, s_{24})F^{(4,1)}(s_{35}, s_{13}{+}s_{23}{+}s_{34}) \\ &\quad - \frac{\sin(\pi(s_{13}{+}s_{23}))}{\sin(\pi(s_{13}{+}s_{23}{+}s_{34}))}F^{(5,2)}_{12} - \frac{\sin(\pi s_{13})}{\sin(\pi(s_{13}{+}s_{23}{+}s_{34}))}F^{(5,2)}_{22}\,, \end{aligned}$$

while the entries $H^{(6,2)}_{2j}$ can be obtained from $H^{(6,2)}_{1j}$ by relabeling $2 \leftrightarrow 3$ at the level of the Mandelstam variables throughout and are thus given by

$$\begin{aligned} H^{(6,2)}_{21} &= \frac{s_{12}}{s_{12}{+}s_{23}{+}s_{24}}F^{(4,1)}(s_{13}, s_{34})F^{(4,1)}(s_{25}, s_{12}{+}s_{23}{+}s_{24}) \\ &\quad - \frac{\sin(\pi(s_{12}{+}s_{23}))}{\sin(\pi(s_{12}{+}s_{23}{+}s_{24}))}F^{(5,2)}_{21} - \frac{\sin(\pi s_{12})}{\sin(\pi(s_{12}{+}s_{23}{+}s_{24}))}F^{(5,2)}_{11} \\ H^{(6,2)}_{22} &= \frac{s_{12}{+}s_{23}}{s_{12}{+}s_{23}{+}s_{24}}F^{(4,1)}(s_{13}, s_{34})F^{(4,1)}(s_{25}, s_{12}{+}s_{23}{+}s_{24}) \end{aligned}$$

$$
- \frac{\sin(\pi(s_{12}+s_{23}))}{\sin(\pi(s_{12}+s_{23}+s_{24}))} F_{22}^{(5,2)} - \frac{\sin(\pi s_{12})}{\sin(\pi(s_{12}+s_{23}+s_{24}))} F_{12}^{(5,2)}
$$

The entries $J_{1j}^{(6,2)}$ are given by

$$
\begin{aligned}
J_{11}^{(6,2)} &= u_{11} \hat{F}_{11}^{(5,2)} + u_{12} \hat{F}_{12}^{(5,2)} \\
&\quad - \frac{\sin(\pi s_{12})}{\sin(\pi(s_{12}+s_{24}))} \frac{s_{13}+s_{23}}{s_{13}+s_{23}+s_{34}} F^{(4,1)}(s_{12}, s_{24}) F^{(4,1)}(s_{35}, s_{13}+s_{23}+s_{34}) \\
&\quad + \frac{\left[-\sin(\pi s_{12}) \sin(\pi s_{34}) F_{11}^{(5,2)} + \sin(\pi s_{13}) \sin(\pi(s_{123}+s_{34})) F_{21}^{(5,2)} \right]}{\sin(\pi(s_{13}+s_{23}+s_{34})) \sin(\pi(s_{123}+s_{24}+s_{34}))} \\
J_{12}^{(6,2)} &= u_{21} \hat{F}_{11}^{(5,2)} + u_{22} \hat{F}_{12}^{(5,2)} \\
&\quad - \frac{\sin(\pi s_{12})}{\sin(\pi(s_{12}+s_{24}))} \frac{s_{13}}{s_{13}+s_{23}+s_{34}} F^{(4,1)}(s_{12}, s_{24}) F^{(4,1)}(s_{35}, s_{13}+s_{23}+s_{34}) \\
&\quad + \frac{\left[-\sin(\pi s_{12}) \sin(\pi s_{34}) F_{12}^{(5,2)} + \sin(\pi s_{13}) \sin(\pi(s_{123}+s_{34})) F_{22}^{(5,2)} \right]}{\sin(\pi(s_{13}+s_{23}+s_{34})) \sin(\pi(s_{123}+s_{24}+s_{34}))} ,
\end{aligned}
$$

where the u_{ij} are defined below

$$
u_{ij} = \begin{pmatrix} \frac{s_{12}(s_{123}+s_{24})}{(s_{12}+s_{24})(s_{123}+s_{24}+s_{34})} & \frac{-s_{12}s_{34}}{(s_{13}+s_{34})(s_{123}+s_{24}+s_{34})} \\ \frac{-s_{13}s_{24}}{(s_{12}+s_{24})(s_{123}+s_{24}+s_{34})} & \frac{s_{13}(s_{123}+s_{34})}{(s_{13}+s_{34})(s_{123}+s_{24}+s_{34})} \end{pmatrix}_{ij} .
$$

Also, the hat denotes the following replacement of the arguments of $F^{(5,2)}$,

$$
\hat{F}_{ab}^{(5,2)}(s_{12}, s_{13}, s_{23}, s_{24}, s_{34}) = F_{ab}^{(5,2)}(s_{12}+s_{24}, s_{13}+s_{34}, s_{23}, s_{25}, s_{35}) .
$$

The entries $J_{2j}^{(6,2)}$ can again be obtained from $J_{1j}^{(6,2)}$ by relabeling $2 \leftrightarrow 3$:

$$
\begin{aligned}
J_{21}^{(6,2)} &= J_{12}^{(6,2)} \big|_{(2\leftrightarrow 3)} = u_{12} \hat{F}_{22}^{(5,2)} + u_{11} \hat{F}_{21}^{(5,2)} \\
&\quad - \frac{\sin(\pi s_{13})}{\sin(\pi(s_{13}+s_{34}))} \frac{s_{12}}{s_{12}+s_{23}+s_{24}} F^{(4,1)}(s_{13}, s_{34}) F^{(4,1)}(s_{25}, s_{12}+s_{23}+s_{24}) \\
&\quad + \frac{\left[-\sin(\pi s_{13}) \sin(\pi s_{24}) F_{21}^{(5,2)} + \sin(\pi s_{12}) \sin(\pi(s_{123}+s_{24})) F_{11}^{(5,2)} \right]}{\sin(\pi(s_{12}+s_{23}+s_{24})) \sin(\pi(s_{123}+s_{24}+s_{34}))} \\
J_{22}^{(6,2)} &= J_{11}^{(6,2)} \big|_{(2\leftrightarrow 3)} = u_{22} \hat{F}_{22}^{(5,2)} + u_{21} \hat{F}_{21}^{(5,2)} \\
&\quad - \frac{\sin(\pi s_{13})}{\sin(\pi(s_{13}+s_{34}))} \frac{s_{12}+s_{23}}{s_{12}+s_{23}+s_{24}} F^{(4,1)}(s_{13}, s_{34}) F^{(4,1)}(s_{25}, s_{12}+s_{23}+s_{24})
\end{aligned}
$$

$$+\frac{\left[-\sin(\pi s_{13})\sin(\pi s_{24})F^{(5,2)}_{22}+\sin(\pi s_{12})\sin(\pi(s_{123}+s_{24}))F^{(5,2)}_{12}\right]}{\sin(\pi(s_{12}+s_{23}+s_{24}))\sin(\pi(s_{123}+s_{24}+s_{34}))}\,.$$

Lastly, the entries $K^{(6,2)}_{ii}$ related by $2\leftrightarrow 3$ are given by

$$\begin{aligned}
K^{(6,2)}_{11} &= \frac{s_{12}\hat{F}^{(5,2)}_{11}}{s_{12}+s_{24}}-\frac{\sin(\pi s_{12})}{\sin(\pi(s_{12}+s_{24}))}F^{(4,1)}(s_{12},s_{24})F^{(4,1)}(s_{35},s_{13}+s_{23}+s_{34})\\
K^{(6,2)}_{22} &= \frac{s_{13}\hat{F}^{(5,2)}_{22}}{s_{13}+s_{34}}-\frac{\sin(\pi s_{13})}{\sin(\pi(s_{13}+s_{34}))}F^{(4,1)}(s_{13},s_{34})F^{(4,1)}(s_{25},s_{12}+s_{23}+s_{24})\,.
\end{aligned}$$

Naturally we also need to look at the $z_4\to 1$ limit. So we define

$$C^{(6,2)}_1 := \lim_{z_4\to 1}F^{(6,2)}(z_4)\,(1-z_4)^{-E_1} \tag{6.2.15}$$

and recall that it satisfies the relation

$$C_1 = C_0\,\Phi(E_0,E_1)\,. \tag{6.2.16}$$

As in the previous chapter we can compute C_1 by looking at a change of variables

$$y_j = 1-z_j\,,\; j\in\{1,\ldots,6\}\,. \tag{6.2.17}$$

which gives

$$\gamma_a(y_j)=\gamma_{7-a}(z_j) \tag{6.2.18}$$

and

$$\omega_a(y_j)=\omega_{7-a,1\leftrightarrow 5}(z_j) \tag{6.2.19}$$

for $a\in\{1,\ldots,6\}$.

We define the $\check{A}$ notation for 6×6 matrices so that

$$\check{A}_{ij}=A_{(7-i)(7-j),1\leftrightarrow 5}\,,\; 1\le i,j\le 6\,, \tag{6.2.20}$$

which gives $E_1=\check{E}_0$ and $C_1=\check{C}_0$.

Finally, we set $s_{24}=s_{34}=0$ and write $\tilde{C}_0=C_0(s_{24},s_{34}\to 0)$, $\tilde{C}_1=C_1(s_{24},s_{34}\to 0)$, $\tilde{E}_0=E_0(s_{24},s_{34}\to 0)$ and $\tilde{E}_1=E_1(s_{24},s_{34}\to 0)$.

Then, the first two rows of $\tilde{C}_0$ is of the form $(A|0)$ where A is the 2×2 matrix

$$A=\begin{pmatrix} F^{(4,1)}(s_{12},s_{23}) & 0\\ 0 & F^{(4,1)}(s_{13},s_{23})\end{pmatrix} \tag{6.2.21}$$

and the entries to the right of A are all zero.

On the other hand the first two rows of $\tilde{C_1}$ is of the form where $(B|*)$ where B is the 2×2 matrix

$$B = \begin{pmatrix} F_{11}^{(5,2)}(s_{12}, s_{13}, s_{23}, s_{25}, s_{35}) & F_{12}^{(5,2)}(s_{12}, s_{13}, s_{23}, s_{25}, s_{35}) \\ F_{21}^{(5,2)}(s_{12}, s_{13}, s_{23}, s_{25}, s_{35}) & F_{22}^{(5,2)}(s_{12}, s_{13}, s_{23}, s_{25}, s_{35}) \end{pmatrix} \tag{6.2.22}$$

that is we get the period matrix $F^{(5,2)}$ except for relabelling the index 4 by 5. We won't need to worry about the entries to the right of B so we ignore them for now. **However, do note that the entries in the first two columns below the sub-matrix B are all zero.**

So the relation $\check{C_1} = \check{C_0}\,\Phi(\check{E_0}, \check{E_1})$ gives us access to the coaction of the $(5,2)$ period matrix by focusing on the top-left (2×2) sub-matrix on both sides.

Theorem 6.2.1 (D. Kamlesh)**.** *The coaction of the initial value matrix $C_0^{(5,2)}$ is given by*

$$\Delta(C_0^{(5,2),\mathfrak{m}}) = C_0^{(5,2),\mathfrak{m}}\, C_0^{(5,2),\mathfrak{dr}}\,. \tag{6.2.23}$$

Proof. First of all, replacing the sine factors coming from monodromy relations by their arguments one can check that we have

$$[E_1, C_1^{\mathfrak{dr}}] = [E_0, C_0^{\mathfrak{dr}}] = 0 \tag{6.2.24}$$

Therefore, the coaction on the Drinfeld associator is given by

$$\Delta(\Phi^{\mathfrak{m}}(\tilde{E_0}, \tilde{E_1})) = \tilde{C_0}^{-\mathfrak{dr}}\, \Phi^{\mathfrak{m}}(\tilde{E_0}, \tilde{E_1})\, \tilde{C_1}^{\mathfrak{dr}}\,. \tag{6.2.25}$$

using identical arguments as in the previous section 6.1.47.

Similarly we get the coaction on $\check{C_1}$ by following the same argument as 6.1.48.

$$\begin{aligned} \Delta(\tilde{C_1}^{\mathfrak{m}}) &= \Delta(\tilde{C_0}^{\mathfrak{m}})\, \Delta(\Phi^{\mathfrak{m}}(\tilde{E_0}, \tilde{E_1})) \\ &= \Delta(\tilde{C_0}^{\mathfrak{m}})\, \tilde{C_0}^{-\mathfrak{dr}} \Phi^{\mathfrak{m}}\, (\tilde{E_0}, \tilde{E_1})\, \tilde{C_1}^{\mathfrak{dr}}\,. \end{aligned} \tag{6.2.26}$$

We have $\Delta(A^{\mathfrak{m}}) = A^{\mathfrak{m}}\, A^{\mathfrak{dr}}$ since A is a diagonal matrix with $(4,1)$ periods, that is Beta function as its entries, see 6.2.21. Therefore, $\Delta(\tilde{C_0}^{\mathfrak{m}}) = \tilde{C_0}^{\mathfrak{m}}\tilde{C_0}^{\mathfrak{dr}}$ is verified up to the first two rows. Also, the first two rows of $(\tilde{C_0}^{\mathfrak{dr}})^{-1}$ is of the form $((A^{\mathfrak{dr}})^{-1}|0)$. So $\Delta(\tilde{C_0}^{\mathfrak{m}})(\tilde{C_0}^{\mathfrak{dr}})^{-1} = \tilde{C_0}^{\mathfrak{m}}$ is verified up to the first two rows. Putting this relation in 6.2.26 we get

$$\Delta(\tilde{C_1}^{\mathfrak{m}}) = \tilde{C_0}^{\mathfrak{m}} \Phi^{\mathfrak{m}}(\tilde{E}_0, \tilde{E}_1) \tilde{C_1}^{\mathfrak{dr}} \tag{6.2.27}$$

$$= \tilde{C_1}^{\mathfrak{m}} \tilde{C_1}^{\mathfrak{dr}} \tag{6.2.28}$$

also verified up to the first two rows. However, the form of the $\tilde{C_1}$ (see 6.2.22) then implies that $\Delta(B^{\mathfrak{m}}) = B^{\mathfrak{m}} B^{\mathfrak{dr}}$ which gives that $\Delta(F^{(5,2),\mathfrak{m}}) = F^{(5,2),\mathfrak{m}} F^{(5,2),\mathfrak{dr}}$ and proves the coaction conjecture for $(n, p) = (5, 2)$.

Now that we know the coaction of both $(4, 1)$ and $(5, 2)$ periods we can put this back in the original initial value matrix C_0 to get $\Delta(C_0^{\mathfrak{m}}) = C_0^{\mathfrak{m}} C_0^{\mathfrak{dr}}$ and $\Delta(C_1^{\mathfrak{m}}) = C_1^{\mathfrak{m}} C_1^{\mathfrak{dr}}$. □

Finally, the same argument as in the proof 6.1.2 gives the coaction conjecture for $(n, p) = (6, 2)$.

Theorem 6.2.2 (D. Kamlesh)**.** *The motivic coaction of the period matrix $F^{(6,2)}$ is given by*

$$\Delta(F^{(6,2),\mathfrak{m}}) = F^{(6,2),\mathfrak{m}} F^{(6,2),\mathfrak{dr}}. \tag{6.2.29}$$

Corollary 6.2.3. *The single-valued image of the period matrix $F^{(5,2)}$ and $F^{(6,2)}$ is given by*

$$\mathrm{sv}F^{\mathfrak{m}} = \overline{(F^{\mathfrak{dr}})^t}\, F^{\mathfrak{dr}} \tag{6.2.30}$$

Proof. This follows from theorem 5.6.8 and theorem 6.2.2. □

Remark 6.2.4. Application to open superstring amplitudes - In the work of Schlotterer-Stieberger in [51], open string amplitudes in genus 0 were observed to have series expansion in multiple zeta values with elegant properties. To be concrete, note the following expression for the series expansion of $C_0^{(5,2)}$ that was observed in [51].

$$\begin{aligned} C_0^{(5,2)} &= 1 + \zeta_2 P_2 + \zeta_3 M_3 + \zeta_4 P_4 + \zeta_5 M_5 + \zeta_2\zeta_3 P_2 M_3 + \frac{1}{2}\zeta_3^2 M_3 M_3 + \zeta_6 P_6 \\ &+ \zeta_7 M_7 + \zeta_2\zeta_5 P_2 M_5 + \zeta_4\zeta_3 P_4 M_3 + \frac{1}{5}\zeta_{3,5}[M_5, M_3] + \zeta_5\zeta_3 M_5 M_3 \\ &+ \frac{1}{2}\zeta_2\zeta_3^2 P_2 M_3 M_3 + \zeta_8 P_8 + \zeta_9 M_9 + \frac{1}{6}\zeta_3^3 M_3 M_3 M_3 + \zeta_2\zeta_7 P_2 M_7 \\ &+ \zeta_4\zeta_5 P_4 M_5 + \zeta_6\zeta_3 P_6 M_3 + \left(\frac{3}{14}\zeta_5^2 + \frac{1}{14}\zeta_{3,7}\right)[M_7, M_3] + \zeta_7\zeta_3 M_7 M_3 \\ &+ \frac{1}{2}\zeta_5^2 M_5 M_5 + \frac{1}{5}\zeta_2\zeta_{3,5} P_2 [M_5, M_3] + \zeta_2\zeta_5\zeta_3 P_2 M_5 M_3 \\ &+ \frac{1}{2}\zeta_4\zeta_3^2 P_4 M_3 M_3 + \zeta_{10} P_{10} + \ldots \end{aligned} \tag{6.2.31}$$

where the letters P_{2k} and M_{2k+1} are matrix coefficients of $\zeta(2k)$ and $\zeta(2k+1)$ respectively. The matrices themselves have entries which are homogeneous polynomials in s_{ij} with degree equal to the weight of the corresponding zeta value which is denoted by the index. One can see that the higher depth MZVs have coefficients determined by those of depth one zeta values. This can be reformulated in terms of the f-alphabet as follows.

$$C_0^{(5,2)} = \Big(\sum_{j=0}^{\infty}(f_2^j)P_{2j}\Big) \sum_{i_1,i_2,\ldots,i_r\in 2\mathbb{N}+1} (f_{i_1}f_{i_2}\ldots f_{i_r})^{\mathfrak{m}} M_{i_1}\ \ldots\ M_{i_r}\,. \tag{6.2.32}$$

While the above was verified to be true in low orders, the fact that it continues to hold at all orders was left as a conjecture. This conjecture was reformulated by Drummond-Ragoucy [34] to be equivalent to the following coaction property.

$$\Delta(F^{(5,2),\mathfrak{m}}) = F^{(5,2),\mathfrak{m}}\, F^{(5,2),\mathfrak{dr}}\,. \tag{6.2.33}$$

Since we have already shown this property to be true earlier in this section, this resolves the above conjecture on open superstring amplitudes.

Chapter 7

Appendix : Infinitesimal braid relations and commutator identities

In this section, we collect together some consequences of the infinitesimal braid relations 5.1 and commutator relations 5.2.9, 5.2.10 that are used in the proof of theorem 5.5.3.

Lemma 7.0.1.

$$[X_{2,F}^{(n)}, X_{1,F}^{(n)}] = [\sum_{j=0,j\neq 2}^{n+1} \frac{e_{2,j}}{z_2 - z_j}, \sum_{k=0,k\neq 1}^{n+1} \frac{e_{1,k}}{z_1 - z_k}] = 0\,. \tag{7.0.1}$$

Proof. First, we collect together terms with z_1 in the denominator so we split the left factor of the commutator into two parts, corresponding to $j \neq 1$ and $j = 1$.

$$[X_{2,F}^{(n)}, X_{1,F}^{(n)}] = [\sum_{j=0,j\neq 2}^{n+1} \frac{e_{2,j}}{z_2 - z_j}, \sum_{k=0,k\neq 1}^{n+1} \frac{e_{1,k}}{z_1 - z_k}] \tag{7.0.2}$$

$$= [\sum_{j=0,j\neq 1,2}^{n+1} \frac{e_{2,j}}{z_2 - z_j} + \frac{e_{2,1}}{z_2 - z_1}, \sum_{k=0,k\neq 1}^{n+1} \frac{e_{1,k}}{z_1 - z_k}] \tag{7.0.3}$$

$$= [\sum_{j=0,j\neq 1,2}^{n+1} \frac{e_{2,j}}{z_2 - z_j}, \sum_{k=0,k\neq 1}^{n+1} \frac{e_{1,k}}{z_1 - z_k}] + [\frac{e_{2,1}}{z_2 - z_1}, \sum_{k=0,k\neq 1}^{n+1} \frac{e_{1,k}}{z_1 - z_k}] \tag{7.0.4}$$

$$= \sum_{j=0,j\neq 1,2}^{n+1} \sum_{k=0,k\neq 1}^{n+1} \frac{[e_{2,j}, e_{1,k}]}{(z_2 - z_j)(z_1 - z_k)} + \sum_{k=0,k\neq 1}^{n+1} \frac{[e_{2,1}, e_{1,k}]}{(z_2 - z_1)(z_1 - z_k)} . \tag{7.0.5}$$

We denote the terms in the last equation by I and II respectively.

Observe that

$$\frac{1}{(z_2 - z_1)(z_1 - z_k)} = \frac{1}{(z_2 - z_k)}\left(\frac{1}{z_1 - z_k} - \frac{1}{z_1 - z2}\right) \tag{7.0.6}$$

so we can rewrite II as

$$\sum_{k=0,k\neq 1}^{n+1} \frac{[e_{2,1}, e_{1,k}]}{(z_2 - z_1)(z_1 - z_k)} = \sum_{k=0,k\neq 1}^{n+1} \frac{[e_{2,1}, e_{1,k}]}{(z_2 - z_k)}\left(\frac{1}{(z_1 - z_k)} - \frac{1}{(z_1 - z_2)}\right) \tag{7.0.7}$$

$$= \sum_{k=0,k\neq 1}^{n+1} \frac{[e_{2,1}, e_{1,k}]}{(z_2 - z_k)(z_1 - z_k)} + \sum_{k=0,k\neq 1}^{n+1} \frac{[e_{1,k}, e_{1,2}]}{(z_2 - z_k)(z_1 - z_2)} . \tag{7.0.8}$$

We denote the two terms above by $II(a)$ and $II(b)$ respectively.

Next, we want to collect together terms with common denominators so we split I into three parts according to the following condition on the summation indices.

a. $j = k$, $k \neq 2$.

b. $j \neq k$, $k \neq 2$.

c. $j \neq k$, $k = 2$.

To be precise, we get $I = I(a) + I(b) + I(c)$ with

$$I(a) = \sum_{k=0,\, k\neq 1,2}^{n+1} \frac{[e_{2,k}, e_{1,k}]}{(z_2 - z_k)(z_1 - z_k)}, \tag{7.0.9}$$

$$I(b) = \sum_{j=0,\, j\neq 1,2,k}^{n+1} \sum_{k=0,\, k\neq 1,2}^{n+1} \frac{[e_{2,j}, e_{1,k}]}{(z_2 - z_j)(z_1 - z_k)}, \tag{7.0.10}$$

$$I(c) = \sum_{j=0,\, j\neq 1,2}^{n+1} \frac{[e_{2,j}, e_{1,2}]}{(z_2 - z_j)(z_1 - z_2)} . \tag{7.0.11}$$

Now, consider the sum below.

$$I(a) + II(a) = \sum_{k=0,\, k\neq 1,2}^{n+1} \frac{[e_{2,k}, e_{1,k}]}{(z_2 - z_k)(z_1 - z_k)} + \sum_{k=0,k\neq 1}^{n+1} \frac{[e_{2,1}, e_{1,k}]}{(z_2 - z_k)(z_1 - z_k)} \tag{7.0.12}$$

$$= \sum_{k=0,\,k\neq 1,2}^{n+1} \frac{[e_{2,k}, e_{1,k}] + [e_{2,1}, e_{1,k}]}{(z_2 - z_k)(z_1 - z_k)} + \frac{[e_{2,1}, e_{1,2}]}{(z_2 - z_k)(z_1 - z_k)} \tag{7.0.13}$$

$$= \sum_{k=0,\,k\neq 1,2}^{n+1} \frac{[e_{2,k} + e_{2,1}, e_{1,k}]}{(z_2 - z_k)(z_1 - z_k)} + \frac{[e_{2,1}, e_{1,2}]}{(z_2 - z_k)(z_1 - z_k)} = 0 \tag{7.0.14}$$

because of the infinitesimal braid relations $[e_{2,k} + e_{2,1}, e_{1,k}] = 0$, $[e_{2,1}, e_{1,2}] = 0$.

Next, we have

$$I(b) = \sum_{j=0,\,j\neq 1,2,k}^{n+1} \sum_{k=0,\,k\neq 1,2}^{n+1} \frac{[e_{2,j}, e_{1,k}]}{(z_2 - z_j)(z_1 - z_k)} = 0\,. \tag{7.0.15}$$

This is because $[e_{2,j}, e_{1,k}] = 0$ for $1 \neq 2 \neq j \neq k$.

Finally, we are left with the sum $I(c) + II(b)$ which we denote by III.

$$III = \sum_{j=0,\,j\neq 1,2}^{n+1} \frac{[e_{2,j}, e_{1,2}]}{(z_2 - z_j)(z_1 - z_2)} + \sum_{k=0,\,k\neq 1}^{n+1} \frac{[e_{1,k}, e_{1,2}]}{(z_2 - z_k)(z_1 - z_2)} \tag{7.0.16}$$

We split III into two parts according to whether: a. $j = k$, b. $j \neq k$. Then, we get

$$III(a) = \sum_{j=0,\,j\neq 1,2}^{n+1} \frac{[e_{2,j} + e_{1,j}, e_{1,2}]}{(z_2 - z_j)(z_1 - z_2)} = 0 \tag{7.0.17}$$

since $[e_{2,j} + e_{1,j}, e_{1,2}] = 0$ for $j \neq 2$.

For each pair $(j,\, k)$ with $j \neq k$ we also have the opposite pair $(k,\, j)$ and adding the corresponding terms we get

$$\left(\frac{[e_{2,j}, e_{1,2}]}{(z_2 - z_j)(z_1 - z_2)} + \frac{[e_{1,k}, e_{1,2}]}{(z_2 - z_k)(z_1 - z_2)}\right) + \left(\frac{[e_{2,k}, e_{1,2}]}{(z_2 - z_k)(z_1 - z_2)} + \frac{[e_{1,j}, e_{1,2}]}{(z_2 - z_j)(z_1 - z_2)}\right) \tag{7.0.18}$$

$$= \left(\frac{[e_{2,j}, e_{1,2}]}{(z_2 - z_j)(z_1 - z_2)} + \frac{[e_{1,j}, e_{1,2}]}{(z_2 - z_j)(z_1 - z_2)}\right) + \left(\frac{[e_{2,k}, e_{1,2}]}{(z_2 - z_k)(z_1 - z_2)} + \frac{[e_{1,k}, e_{1,2}]}{(z_2 - z_k)(z_1 - z_2)}\right) \tag{7.0.19}$$

$$= \frac{[e_{2,j} + e_{1,j}, e_{1,2}]}{(z_2 - z_j)(z_1 - z_2)} + \frac{[e_{2,k} + e_{1,k}, e_{1,2}]}{(z_2 - z_k)(z_1 - z_2)} = 0 \tag{7.0.20}$$

since $[e_{2,j} + e_{1,j}, e_{1,2}] = 0$, $[e_{2,k} + e_{1,k}, e_{1,2}] = 0$. This proves the lemma. □

Lemma 7.0.2. *Let l, m, n be non-negative integers such that $0 \leq l \neq m \leq n$, then we have*

$$[\sum_{j=0}^{n} \sum_{k=j+1}^{n} e_{j,k},\, e_{l,m}] = 0\,. \tag{7.0.21}$$

Proof. Without loss of generality suppose that $l < m$. If $j \neq k \neq l \neq m$, then $[e_{j,k}, e_{l,m}] = 0$ by 5.1.6. If $(j,k) = (l,m)$, then clearly $[e_{j,k}, e_{l,m}] = 0$. So we are only left with the terms $e_{j,k}$ where exactly one of j or k is equal to l or m and we get

$$[\sum_{j=0}^{n} \sum_{k=j+1}^{n} e_{j,k},\, e_{l,m}] = [\sum_{a=1}^{n}(e_{l,a} + e_{a,m}),\, e_{l,m}] = 0 \tag{7.0.22}$$

by relation 5.1.7. □

Lemma 7.0.3. *For $m, n \in \mathbb{N}$ with $n \geq 3$ and $m = 1$ or $m \geq n$, we have*

$$[\sum_{j=2}^{n-1}(e_{j,0} + \sum_{k=j+1}^{n-1} e_{j,k}), e_{m,0} + \sum_{l=2}^{n-1} e_{m,l}] = 0\,. \tag{7.0.23}$$

Proof. For $j, l = 2, \ldots, n-1$ and $l \neq j$, we have

$$[e_{j,0}, e_{m,0} + e_{m,j}] = 0, \quad [e_{j,0}, e_{m,l}] = 0\,. \tag{7.0.24}$$

Therefore,

$$[\sum_{j=2}^{n-1} e_{j,0}, e_{m,0} + \sum_{l=2}^{n-1} e_{m,l}] = 0\,. \tag{7.0.25}$$

Further, for $j, k, l = 2 \ldots, n-1$ with $l \neq j, k$ we have

$$[e_{j,k}, e_{m,0}] = 0, \; [e_{j,k}, e_{m,l}] = 0, \; [e_{j,k}, e_{m,j} + e_{m,k}] = 0. \tag{7.0.26}$$

Therefore,

$$[\sum_{j=2}^{n-1} \sum_{k=j+1}^{n-1} e_{j,k}, e_{m,0} + \sum_{l=2}^{n-1} e_{m,l}] = 0 \tag{7.0.27}$$

and we are done. □

Lemma 7.0.4. *For $k \geq 3$ odd and symbols x, y, z with $x + y + z = 0$, we have*

$$[y, w_k(x,y)] + [z, w_k(x,z)] = 0 \tag{7.0.28}$$

where w_k are the commutators that appear in the circle operator expansion of the deRham Drinfeld associator in Theorem 4.2.1.

Before we prove this result we need to recall some background information on associator relations [32, 33].

Let K be a field. Let $K\langle\langle e_0, e_1\rangle\rangle$ be a non-commutative formal power series ring in variables e_0 and e_1.

Let $\lambda \in K^*$ and $\Phi(e_0, e_1) \in K\langle\langle e_0, e_1\rangle\rangle$ a power series in e_0 and e_1. A pair (λ, Φ) is said to satisfy the associator equations if the following relations are satisfied.

1. The constant term in $\Phi(e_0, e_1)$ is 1.
2. The power series Φ is group-like, that is,
$$\Delta(\Phi) = \Phi \otimes \Phi \tag{7.0.29}$$
where Δ is the coproduct on the Hopf algebra $K\langle\langle e_0, e_1\rangle\rangle$ defined by $\Delta(e_j) = e_j \otimes 1 + 1 \otimes e_j$ for $j = 0, 1$.
3. Pentagon equation :
$$\Phi(t_{12}, t_{23} + t_{24})\, \Phi(t_{13} + t_{23}, t_{34}) = \Phi(t_{23}, t_{34})\, \Phi(t_{12} + t_{13}, t_{23} + t_{34})\, \Phi(t_{12}, t_{23}) \tag{7.0.30}$$
where the t_{ij}'s satisfy the infinitesimal braid relations 5.1.
4. Hexagon equation :
$$\exp(\lambda x/2)\, \Phi(z, x)\, \exp(\lambda z/2)\, \Phi(y, z)\, \exp(\lambda y/2)\, \Phi(x, y) = 1 \tag{7.0.31}$$
provided $x + y + z = 0$.

From Furusho's work on associator equations (lemma 5 of [38]), we know that the associator equations imply the 5-cycle relation

$$\Phi(X_{12}, X_{23})\, \Phi(X_{34}, X_{45})\, \Phi(X_{51}, X_{12})\, \Phi(X_{23}, X_{34})\, \Phi(X_{45}, X_{51}) = 1 \tag{7.0.32}$$

where the X_{ij}'s satisfy the relations of the pure sphere braid Lie algebra

$$X_{ij} = X_{ji},\ [X_{ij}, X_{kl}] = 0 \text{ if } \{i, j\} \cap \{k, l\} = \emptyset \text{ and } \sum_{j=1}^{5} X_{ij} = 0 \text{ for } i = 1, \dots, 5. \tag{7.0.33}$$

We have enough information now to prove lemma 7.0.4. Let w_k denote the commutators that appear in the circle operator expansion of the deRham Drinfeld associator in Theorem 4.2.1.

Proof. The deRham Drinfeld associator $\Phi^{\mathfrak{dr}}$ satisfies the associator equations [33] with $\lambda = (2\pi i)^{\mathfrak{dr}} = 0$ and Furusho's 5-cycle relation. . Therefore, the $w_k's$ satisfy the following relations

$$w_k(X_{12}, X_{23}) + w_k(X_{34}, X_{45}) + w_k(X_{51}, X_{12}) + w_k(X_{23}, X_{34}) + w_k(X_{45}, X_{51}) = 0\,, \tag{7.0.34}$$

$$w_k(x, y) + w_k(y, z) + w_k(z, x) = 0 \tag{7.0.35}$$

whenever $x + y + z = 0$.

Recall also that

$$\Phi^{\mathfrak{dr}}(e_0, e_1)\,\Phi^{\mathfrak{dr}}(e_1, e_0) = 1 \tag{7.0.36}$$

and thus we have

$$w_k(e_0, e_1) + w_k(e_1, e_0) = 0\,. \tag{7.0.37}$$

Taken together, equation 7.0.34, 7.0.36 and 7.0.37 constitute the relations of the normalized form of the stable derivation algebra (see section 2.1 of [37], also see remark 2.2.1). Further, by Ihara's work on the stable derivation algebra [45], it follows that the commutator w_k satisfy

$$[y, w_k(x, y)] + [z, w_k(x, z)] = 0 \tag{7.0.38}$$

whenever $x + y + z = 0$. □

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