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PAPER

New angular momentum conservation laws for electromagnetic waves interacting with dirac fields

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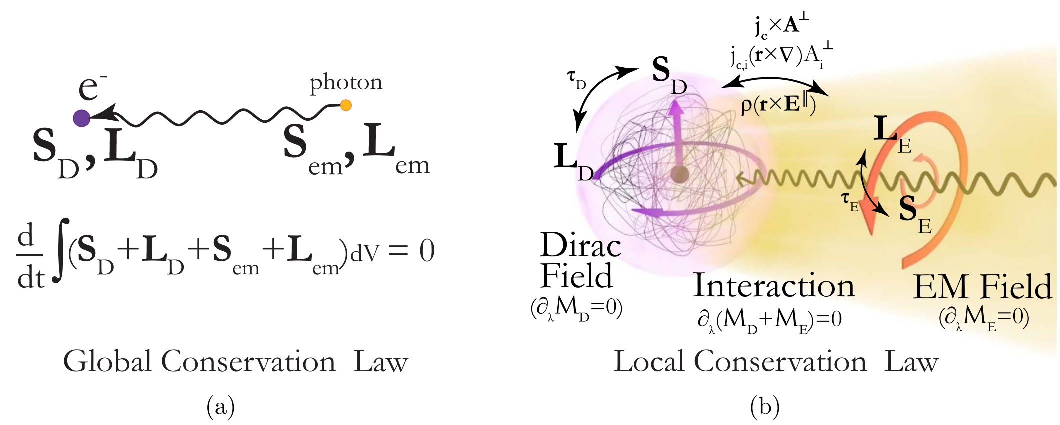
Abstract

Global conservation laws of angular momentum (AM) are well-known in the theory of light–matter interaction. However, local conservation laws, i.e. the conservation law of AM at every point in space, remain unexplored especially in the context of relativistic Dirac–Maxwell fields. Here, we use the QED Lagrangian and Noether's theorem to derive a new local conservation law of AM for Dirac–Maxwell fields in the form of the continuity relation for linear momentum. We separate this local conservation law into four coupled motion equations for spin and orbital AM (OAM) densities. We introduce a helicity current tensor, OAM current tensor, and spin–orbit torque in the motion equations to shed light on the local dynamics of spin-OAM interaction and AM exchange between Maxwell and Dirac fields. We elucidate how our results translate to classical electrodynamics using the example of plane wave interference as well as a dual-mode optical fiber. Our results shine light on AM phenomena related to the relativistic interaction of electromagnetic waves and Dirac fields.

1. Introduction

Global conservation laws for angular momentum (AM) are well known in the theory of light matter interaction [1]. However, the spatial distribution of AM, i.e. AM density, is relatively unexplored especially in the relativistic interaction between electromagnetic waves and Dirac fields. This interaction is the building block for quantum electrodynamics (QED). Figure 1 highlights the difference between global and local conservation of total AM. It also highlights unique physical quantities related to AM for both Dirac fields and Maxwell fields. We emphasize that in the presence of sources, a comprehensive, gauge-independent, locally applicable conservation equation becomes essential in understanding the local dynamics of the AM (figure 1(b)).

On the other hand, the developments in classical electrodynamics have focused on both global and local AM. The local approach towards field quantities such as helicity, chirality, and AM has proven useful in the study of conservation laws as well as geometrical properties of EM fields [2–10]. However, the procedure employed in these derivations is based on the duality symmetry of the source-free EM Lagrangian. Since the duality symmetry is only maintained in source-free regions, this form of Lagrangian neglects any form of interaction with fermionic fields and cannot be employed to present a local conservation law for the AM of gauge fields in the QED Lagrangian describing classical Dirac-Maxwell fields. These properties are captured in the manifestly covariant construction of the Dirac equation [11]. Connections between the fermionic field of the Dirac equation and the bosonic fields of Maxwell's equations is the focus of Dirac-Maxwell correspondence where the relativistic parallels between electromagnetic and Dirac fields are studied [6, 12, 13]. These studies show that, although different in nature, electronic and electromagnetic fields can exhibit



Behavior	Scalar	Vector			Tensor	
Physical quantity	Chirality / Helicity	Spin Angular Momentum	Orbital Angular Momentum	Spin-orbit Torque	Helicity Current	Orbital Angular Momentum Current
Dirac Field	$\frac{\hbar}{2}\psi^\dagger\gamma^5\psi$	$\frac{\hbar}{2}\psi^\dagger\boldsymbol{\Sigma}\psi$	$\mathcal{R}\{\psi^\dagger(\boldsymbol{r}\times\boldsymbol{p}^\parallel)\psi\}$	$-c\mathcal{R}\left\{\bar{\psi}\left(\boldsymbol{\gamma}\times\boldsymbol{p}^\parallel\right)\psi\right\}$	0	$\mathcal{R}\left\{\bar{\psi}\gamma^i(\boldsymbol{r}\times\boldsymbol{p}^\parallel)_j\psi\right\}$
Maxwell Field	$\frac{1}{\mu_0 c}\left(\boldsymbol{A}^\perp\cdot\boldsymbol{B}\right)$	$\epsilon_0(\boldsymbol{E}^\perp\times\boldsymbol{A}^\perp)$	$-\epsilon_0\left[E^{\perp,i}(\boldsymbol{r}\times\nabla)A_i^\perp\right]$	$\frac{1}{\mu_0}(\boldsymbol{B}\cdot\nabla)\boldsymbol{A}^\perp$ $-\epsilon_0\left(\frac{\partial\boldsymbol{E}^\parallel}{\partial t}\times\boldsymbol{A}^\perp\right)$	$-\frac{1}{\mu_0}(A^{\perp,i}B_j)$	$-\frac{1}{\mu_0}\left[\varepsilon^{ik}{}_l B_k(\boldsymbol{r}\times\boldsymbol{\nabla})_j A^{\perp,l}\right]$ $-\epsilon_0 A^{\perp,i}\left(\boldsymbol{r}\times\frac{\partial\boldsymbol{E}^\parallel}{\partial t}\right)_j$

(c)

Figure 1. Conservation laws of angular momentum in light-matter interacting systems (a) conventional conservation of global total electromagnetic and electronic angular momentum. This conservation law applies to closed systems and does not include angular momentum exchange due to near-field interactions. (b) Local conservation of angular momentum applicable to all regions of interaction. The conventional conservation of sum of angular momenta is replaced by the conservation equation (8). (c) Table of quantities defined in this paper, representing the spatial and temporal densities of the scalar, vector, or tensor observables pertinent to angular momentum. Highlighted quantities are the new terms defined in this paper. The local conservation of angular momentum equation (equation (8)) connects the well-known terms such as spin density, OAM density, helicity and chirality to these newly defined quantities.

many analogous properties. This correspondence is evident in phenomena such as spin-momentum locking which emerges in the Dirac equation [14] as well as the Maxwell's equations [15, 16].

In this paper, we study AM of light using the Lorentz symmetry of the QED Lagrangian and local $U(1)$ gauge invariance [17]. With the application of Noether's theorem [18], we find unique local conservation laws pertinent to the AM of the gauge field interacting with fermionic fields. We show that, in the near-field, the electronic AM can be transferred not only to the optical AM, but also to other field quantities that represent AM current [19]. These extra terms include electromagnetic helicity density [20], fermionic chirality density [11], electromagnetic *helicity current tensor*, as well as the electromagnetic and electronic *orbital AM (OAM) current tensors*. We further study these conservation laws and angular-momentum-carrying terms for the classical electromagnetic solutions of a dual-mode optical fiber. This connection between electromagnetic waves and Dirac fields will be of interest for future experiments related to OAM and SAM of light.

2. Noether's theorem, QED Lagrangian, and Lorentz transformation

An important problem in QED relates to the AM of the gauge field. Particularly, the goal is to decompose the total AM of gauge bosons into spin and orbital contributions [21]. In early work, in both Belinfante's [22] and Ji's [23] decompositions, only the total AM of photons has been explored in detail. Jaffe and Manohar split the photonic AM into spin and orbital parts [24]. However, their decomposition does not satisfy the gauge-invariant requirement. By splitting the vector potential into transverse and longitudinal parts, Chen *et al* [25] and Wakamatsu [26] propose gauge-invariant decompositions of electromagnetic AM. However, these existing decompositions have not resolved the conflict between the gauge-invariant requirement and canonical AM commutation relations. Recently, this problem has been solved by quantizing the U(1) gauge field in the Lorenz gauge and merging the spin and OAM of virtual photons with the OAM of Dirac fermions [27, 28]. Our present results are consistent with the results presented in these works, after the quantization of the electromagnetic field.

The application of Noether's theorem to the Dirac Lagrangian has been used in the Quantum Chromodynamics (QCD) and QED communities to derive the expressions for the AM of the nuclei and the gauge fields [23, 25]. In QCD and QED interactions, conservation of the total integrated AM suffices to describe the AM transfer in scattering processes. On the other hand, for a fast moving electron, its spin and momentum degrees of freedom are highly correlated [29]. Explorations of a generalized definition for the spin operator of an electron as well as for gauge bosons is still a problem of widespread interest [30, 31] in QED and QCD [21, 23, 25]. While the focus has mainly been on global conservation laws for AM, our goal in this paper is to focus on local spatially dependent AM density effects. In nanophotonic and condensed matter systems, atoms can interact locally with an external EM field. Therefore, for such systems, a local conservation equation for the AM density is necessary in the realization of nanoscale applications.

In this paper, we start with the real (also called symmetrized) Lagrangian density of a Dirac field coupled to EM field [32],

$$\mathfrak{L} = \bar{\psi}\left[c\gamma^{\mu}\left(\frac{1}{2}i\hbar\overleftrightarrow{\partial}_{\mu} - eA_{\mu}\right) - mc^2\right]\psi - \frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu}, \tag{1}$$

where $\bar{\psi} = \psi^{\dagger}\gamma^0$ and γ^{μ} are the gamma matrices, $F_{\mu\nu}$ the electromagnetic tensor, $A_{\mu} = (\phi/c, -\boldsymbol{A})$ the electromagnetic four-potential, ψ Dirac fields, $\hbar$ Planck's constant, and μ_0 the vacuum permeability (see appendix A). Note that summation over repeated indices is assumed throughout this paper.

In order to derive the angular momenta conservation law, we consider the the change in the Maxwell and Dirac fields under Lorentz transformations:

$$\delta A^{\mu}(x) = -\frac{i}{2}\omega_{\kappa\sigma}\left(\hat{M}^{\kappa\sigma}_{\text{em}}\right)^{\mu}{}_{\nu}A^{\nu}(x), \quad \delta\psi(x) = -\frac{i}{2}\omega_{\mu\nu}\hat{M}^{\mu\nu}_{\text{D}}\psi(x). \tag{2}$$

Lorentz transformations are defined as the coordinate transformations such that the coordinates transform as $x^{\mu} \rightarrow \Lambda^{\mu}{}_{\nu}x^{\mu}$, with

$$\Lambda^{\mu}{}_{\nu} = e^{-\frac{i}{2}\omega_{\kappa\sigma}\left(\hat{S}^{\kappa\sigma}\right)^{\mu}{}_{\nu}}, \tag{3}$$

where $(\hat{S}^{\kappa\sigma})^{\mu}{}_{\nu} = i(\eta^{\kappa\mu}\eta^{\sigma}{}_{\nu} - \eta^{\sigma\mu}\eta^{\kappa}{}_{\nu})$ are the generators of rotation and boost in the four-dimensional real space [33] and $\omega_{\kappa\sigma}$ are the rotation and boost parameters. These expressions can be used to find the AM operators of the Maxwell, $\hat{M}_{\text{em}}$, and Dirac, $\hat{M}_{\text{D}}$, fields (see appendix A). By applying Noether's theorem [18], we obtain the conserved current for the AM for the Lagrangian in equation (1) by finding,

$$\mathcal{M}^{\lambda} = \frac{\partial\mathfrak{L}}{\partial(\partial_{\lambda}\psi)}\delta\psi + \delta\bar{\psi}\frac{\partial\mathfrak{L}}{\partial(\partial_{\lambda}\bar{\psi})} + \frac{\partial\mathfrak{L}}{\partial(\partial_{\lambda}A^{\mu})}\delta A^{\mu} + \mathfrak{L}\delta x^{\lambda}, \tag{4}$$

under Lorentz transformations.

Specializing to the rotations ($\mu,\nu = i,j = 1,2,3$), we obtain the general continuity equation associated with rotational symmetry of the QED Lagrangian [33](see appendix A),

$$\partial_{\lambda}\mathcal{M}^{ij,\lambda} = \partial_{\lambda}\left(\mathcal{S}^{ij,\lambda}_{\text{D}} + \mathcal{L}^{ij,\lambda}_{\text{D}} + \mathcal{S}^{ij,\lambda}_{\text{em}} + \mathcal{L}^{ij,\lambda}_{\text{em}}\right) = 0, \tag{5}$$

where $\mathcal{M}^{ij,\lambda}$ is the total angular momentum current tensor. Note that the roman indices take the values $i,j = 1,2,3$ while the Greek indices are $\lambda = 0,1,2,3$. This tensor can be split into spin $\mathcal{S}^{ij,\lambda}_{\text{D(em)}}$ (related to the rotation of the internal degrees of freedom) and OAM $\mathcal{L}^{ij,\lambda}_{\text{D(em)}}$ (related to the coordinate dependence of the fields) parts of the Dirac and EM fields, respectively, with individual components defined as (figure 1),

$$\mathcal{S}^{\mu\nu,\lambda}_{\text{D}} = \frac{\hbar c}{4}\bar{\psi}\left(\gamma^{\lambda}\sigma^{\mu\nu} + \sigma^{\mu\nu}\gamma^{\lambda}\right)\psi, \tag{6a}$$

$$\mathcal{L}^{\mu\nu,\lambda}_{\text{D}} = \hbar c\mathcal{R}\left\{\bar{\psi}\gamma^{\lambda}\left(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}\right)\psi\right\} + \left(\eta^{\lambda\mu}x^{\nu} - \eta^{\lambda\nu}x^{\nu}\right)\mathcal{L}_{\text{D}}, \tag{6b}$$

$$\mathcal{S}^{\mu\nu,\lambda}_{\text{em}} = -\frac{1}{\mu_0}\left(F^{\lambda\mu}A^{\nu} - F^{\lambda\nu}A^{\mu}\right), \tag{6c}$$

$$\mathcal{L}^{\mu\nu,\lambda}_{\text{em}} = -\frac{1}{\mu_0}\left[F^{\lambda}{}_{\kappa}\left(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}\right)A^{\kappa}\right] + \left(\eta^{\lambda\mu}x^{\nu} - \eta^{\lambda\nu}x^{\mu}\right)\mathcal{L}_{\text{em}}, \tag{6d}$$

where $\mathcal{L}_{\text{D}}$ and $\mathcal{L}_{\text{em}}$ are the Dirac and electromagnetic contributions to the total Lagrangian in equation (1), respectively. Here $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^{\mu},\gamma^{\nu}]$, $\partial^{\mu} = (\frac{\partial_t}{c}, -\boldsymbol{\nabla})$, $F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$ is the anyti-symmetric electromagnetic tensor, and $\mathcal{R}\{\cdots\}$ takes the real part of its argument. It is important to note that splitting the tensor $\mathcal{M}^{ij,\lambda}$ into the four components for the electromagnetic and Dirac spin and OAM, as demonstrated in equations (5) and (6), is arbitrary since Noether's theorem does not prescribe how spin and OAM should be defined individually. Instead it defines the conservation law for the total AM of the combined fields. We use this fact to derive gauge-independent expressions for the spin and OAM of the Dirac and Maxwell fields in the next section by rearranging the longitudinal (labelled by the symbol $\|$) and the transverse (labelled by the symbol $\perp$) components of the electric field and the vector potential between these four equations (see appendix A). For non-interacting fields, however, we obtain separate continuity equations for the Dirac and EM fields.

The time-components ($\lambda = 0$) of these four AM tensors in equation (6) give the common spin and OAM densites for the Dirac and EM fields, which are generally gauge dependent. The electronic part of the spin and OAM respectively are $\mathcal{S}_{\mathrm{D}}^{ij,0} = \varepsilon^{ijk}\hbar\psi^\dagger\Sigma_k\psi/2$ and $\mathcal{L}_{\mathrm{D}}^{ij,0} = -i\varepsilon^{ijk}\hbar\mathcal{R}\left\{\psi^\dagger(\boldsymbol{r}\times\nabla)_k\psi\right\}$, where $\boldsymbol{\Sigma}$ is the spin operator in Dirac equation. The EM part of the spin and OAM, on the other hand, are $\mathcal{S}_{\mathrm{em}}^{ij,0} = \varepsilon^{ijk}(\epsilon_0\boldsymbol{E}\times\boldsymbol{A})_k$ and $\mathcal{L}_{\mathrm{em}}^{ij,0} = -\varepsilon^{ijk}\left[\epsilon_0 E^l(\boldsymbol{r}\times\nabla)_k A_l\right]$, respectively, where ε_0 is vacuum permittivity. An important observation is that OAM densities of the Dirac fields as well as the spin and OAM of the EM field are gauge dependent and thus do not represent observable physical quantities in a local frame. In fact, one can see that most of the terms in the AM tensors $\mathcal{S}_{\mathrm{D(E)}}^{ij,\lambda}$ and $\mathcal{L}_{\mathrm{D(E)}}^{ij,\lambda}$ are gauge dependent. Due to this problem, we rearrange the expressions in these four tensors to write the conservation law in equation (5) in terms of gauge-independent and physically observable quantities.

We note that throughout this paper, ε^{ijk} represents the Levi–Civita symbol with the convention that $\varepsilon^{ijk} = \varepsilon_{ijk}$, and is normalized such that $\varepsilon_{123} = 1$ together with all of its even permutations; this means that any of its indices can be lowered or raised without any sign change. Also, our convention for outer product is $(a\times b)_k = \varepsilon_{ijk}a^i b^j$ for arbitrary vectors a and b.

3. Local conservation law of AM

The essence of Noether's theorem in field theory lies in the local conservation law with a general form $\partial_\mu j^\mu = 0$. The global charge $Q \equiv \int \mathrm{d}^3 x j^0(x)$ conserves only if all fields vanish at spatial infinity, i.e. the vanishing boundary condition. As we mentioned above, the local conservation law of AM has not received the attention it deserves both in experiments and in theory. We note that the density of the conserved charge in the original continuity equation (5) of AM is a rank-2 tensor and the corresponding current is a rank-3 tensor. This local conservation law is difficult to apply to practical systems and challenging to verify in experiments. In this section, we re-express this relation as a continuity equation of a vector density similar to the local conservation law of linear momentum. The corresponding current reduces to a rank-2 tensor akin to the Maxwell stress tensor. We then go one step further by splitting our local conservation law into four coupled motion equations, which describe the spin-OAM interaction and AM exchanging between Maxwell–Dirac fields.

By splitting the electromagnetic vector potential into transverse and longitudinal parts, $\boldsymbol{A} = \boldsymbol{A}^\perp + \boldsymbol{A}^\|$, defined as $\nabla\cdot\boldsymbol{A}^\perp = 0$ and $\nabla\times\boldsymbol{A}^\| = 0$ [1], Chen *et al* defined the gauge-independent AM densities for both the Dirac field and EM field [25]. However, a gauge-independent form for the continuity equation has not been addressed. Here, following the same approach used in [25], and using equation (6) and Dirac and Maxwell's equations, we obtain the following gauge-independent spin and OAM continuity equations (see appendix A):

$$\partial_\lambda \mathcal{S}_{\mathrm{D}}^{ij,\lambda} = \varepsilon^{ijk}\left[\hbar\frac{\partial}{\partial t}\left(\psi^\dagger\boldsymbol{\Sigma}\psi\right) + \frac{\hbar c}{2}\nabla\left(\psi^\dagger\gamma^5\psi\right)\right]_k, \tag{7a}$$

$$\partial_\lambda \mathcal{L}_{\mathrm{D}}^{ij,\lambda} = \varepsilon^{ijk}\left[\frac{\partial}{\partial t}\mathcal{R}\left\{\psi^\dagger\left(\boldsymbol{r}\times\boldsymbol{p}^\|\right)\psi\right\} + c\nabla\cdot\mathcal{R}\left\{\bar{\psi}\gamma\left(\boldsymbol{r}\times\boldsymbol{p}^\|\right)\psi\right\}\right]_k, \tag{7b}$$

$$\partial_\lambda \mathcal{S}_{\mathrm{em}}^{ij,\lambda} = \varepsilon^{ijk}\left[\epsilon\frac{\partial}{\partial t}\left(\boldsymbol{E}^\perp\times\boldsymbol{A}^\perp\right) - \frac{1}{\mu_0}\nabla\cdot\left(\boldsymbol{A}^\perp\boldsymbol{B}\right) + \frac{1}{\mu_0}\nabla\left(\boldsymbol{A}^\perp\cdot\boldsymbol{B}\right)\right]_k, \tag{7c}$$

$$\begin{aligned}\partial_\lambda \mathcal{L}_{\mathrm{em}}^{ij,\lambda} = \varepsilon^{ijk}\Bigg\{&\epsilon\frac{\partial}{\partial t}\left[\boldsymbol{E}^\perp\cdot(\boldsymbol{r}\times\nabla)\boldsymbol{A}^\perp\right] - \nabla\cdot\left[\frac{1}{\mu_0}\boldsymbol{B}\times(\boldsymbol{r}\times\nabla)\boldsymbol{A}^\perp + \epsilon_0\boldsymbol{A}^\perp\left(\boldsymbol{r}\times\frac{\partial\boldsymbol{E}^\|}{\partial t}\right)\right]\\ &+(\boldsymbol{r}\times\nabla)\left[-\frac{1}{2\mu_0}\boldsymbol{B}\cdot\boldsymbol{B} + \frac{\epsilon_0}{2}\boldsymbol{E}^\perp\cdot\boldsymbol{E}^\perp + \epsilon_0\frac{\partial\boldsymbol{E}^\|}{\partial t}\cdot\boldsymbol{A}^\perp\right]\Bigg\}_k.\end{aligned} \tag{7d}$$

The longitudinal and transverse components of the electric field can be defined similarly to those of the vector potential. In the Lorenz gauge, they take the form $\boldsymbol{E}^\| = -\boldsymbol{\nabla}\phi - \frac{\partial\boldsymbol{A}^\|}{\partial t}$ and $\boldsymbol{E}^\perp = -\frac{\partial\boldsymbol{A}^\perp}{\partial t}$. Adding these equations together, we obtain a standard (vector) continuity equation for the AM density of the combined system,

$$\frac{\partial M_j}{\partial t} + \nabla_i T^i{}_j = 0, \tag{8}$$

where M_j are the components of the AM density vector, written as

$$\boldsymbol{M} = \frac{\hbar}{2}\left(\psi^\dagger \boldsymbol{\Sigma}\psi\right) + \mathcal{R}\left\{\psi^\dagger\left(\boldsymbol{r}\times\boldsymbol{p}^{\parallel}\right)\psi\right\} + \epsilon_0\left(\boldsymbol{E}^{\perp}\times\boldsymbol{A}^{\perp}\right) - \epsilon_0\left[E^{i,\perp}\left(\boldsymbol{r}\times\nabla\right)A_i^{\perp}\right], \tag{9}$$

with gauge-independent transverse vector potential $\boldsymbol{A}^{\perp}$ and gauge-independent electronic momentum operator $\boldsymbol{p}^{\parallel} = \left(-i\hbar\boldsymbol{\nabla} - e\boldsymbol{A}^{\parallel}\right)$. The first two terms in equation (9) are the spin and OAM densities of the Dirac field, while the last two terms are the gauge-independent spin and OAM densities of the EM field [21, 25]. This shows that only the transverse part of the vector potential contributes to the physically observable spin and OAM of EM field. Note that here $A_i^{\perp} = -A^{\perp,i} = (-\boldsymbol{A})$ for $i = 1,2,3$.

The AM current tensor, $T^i{}_j$, is a second rank tensor, similar to the Maxwell stress tensor (EM momentum current) [14, 34]. The tensor $T^i{}_j$ is composed of three parts :$\nabla_i T^i{}_j = \nabla_i\left(\chi^i{}_j + J^i{}_j + N^i{}_j\right)$ with

$$\chi^i{}_j = \left[\overbrace{\frac{\hbar}{2}\left(\psi^\dagger\gamma^5\psi\right)}^{\text{chirality}} + \overbrace{\frac{1}{\mu_0}\left(\boldsymbol{A}^{\perp}\cdot\boldsymbol{B}\right)}^{\text{helicity}}\right]\delta^i{}_j \equiv \chi\ \delta^i{}_j, \tag{10a}$$

$$\begin{aligned} J^i{}_j = &\overbrace{\mathcal{R}\left\{\bar{\psi}\gamma^i\left(\boldsymbol{r}\times\boldsymbol{p}^{\parallel}\right)_j\psi\right\}}^{\text{OAM current tensor (Dirac)}} - \overbrace{\frac{1}{\mu_0}\left(A^{\perp,i}B_j\right)}^{\text{helicity current tensor (Maxwell)}} \\ &+ \underbrace{\frac{1}{\mu_0}\left[\varepsilon^{ikl}B_k\left(\boldsymbol{r}\times\boldsymbol{\nabla}\right)_j A_l^{\perp}\right] - \epsilon_0 A^{\perp,i}\left(\boldsymbol{r}\times\frac{\partial\boldsymbol{E}^{\parallel}}{\partial t}\right)_j}_{\text{OAM current tensor (Maxwell)}}, \end{aligned} \tag{10b}$$

$$\nabla_i N^i{}_j = \left(\boldsymbol{r}\times\nabla\right)_j\mathfrak{N}_{\text{em}}, \quad \mathfrak{N}_{\text{em}} = \underbrace{\frac{\epsilon_0}{2}\boldsymbol{E}^{\perp}\cdot\boldsymbol{E}^{\perp} - \frac{1}{2\mu_0}\boldsymbol{B}\cdot\boldsymbol{B}}_{\text{Maxwell Lagrangian in free space}} + \epsilon_0\frac{\partial\boldsymbol{E}^{\parallel}}{\partial t}\cdot\boldsymbol{A}^{\perp}. \tag{10c}$$

where γ^5 is the chirality operator in Dirac equation (see appendix A).

The term χ in equation (10a) describes the chirality/helicity of the fields. The first term, $\hbar(\psi^\dagger\gamma^5\psi)/2$, is the density of expectation value (in the classical sense of quantum mechanics) of the chirality operator of the Dirac field. The operator γ^5 is widely used to find the projections of fermionic fields into left-handed and right-handed chiral states [17]. Therefore, the expecation value $\psi^\dagger\gamma^5\psi$ represents the chirality of the Dirac field; a negative (postivie) value means a left-handed (right-handed) chiral state. The second term in equation (10a), on the other hand, is widely known as the helicity of EM field [35–37]. Interestingly, equation (10a) shows that the chirality of the fermionic field is identified with the helicity of the electromagnetic field and the gradient of these quantities enter the continuity equation.

Note that the chirality of EM field defined in [38] does not appear in the continuity equation of AM. This shows that the helicity is more fundamental for interacting systems. It is also important to note that helicity can be defined for the Dirac field. Being defined as the projection of spin on the momentum of the field, it is fundamentally different from chirality as it is momentum dependent, and unlike chirality, it is not Lorentz covariant. Interestingly, however, for the massless Dirac field, the definition of chirality and helicity become identical [39]. This points to a similar property in the massless EM fields of the Maxwell's equations.

In the next three sections we define the new terms encountered in the conservation equation.

3.1. Helicity current density tensor

The second terms in equation (10b) is reminiscent of the EM helicity in equation (10a) with the difference that this quantity is a tensor, while helicity is a scalar. In fact, helicity density in equation (10a) is equal to the trace of $\frac{1}{\mu_0}A^{\perp,i}B_j$. We thus name this tensor quantity *helicity current density tensor* since it transforms as a helicity. Furthermore, the divergence of the helicity current tensor enters the conservation of AM equation, meaning that it serves as a current between different forms of Dirac or Maxwell AM.

One important fact regarding this term is that it has no analog in the Dirac field representation, as one might expect. The equivalent expression for helicity current tensor, in the Dirac field, is normally considered to be spin current represented by a tensor involving γ matrices and spin operators. However, our derivation shows that this term vanishes due to the spinor nature of the Dirac fields (see appendix A). This, of course, does not put the numerous works on spin currents in condensed matter physics into question [40–43], since spin currents are generated by the collective motion of electrons through spatially separating electrons with opposite spins in a quantum cavity [44] or using ferromagnets with strong electron–electron interactions [45]. The relativistic treatment of the Dirac equation in this paper, however, only incorporates the wave function of a single electron in vacuum and thus cannot account for the observed electronic spin currents. This shows that a single electron in vacuum does not exhibit spin current and that many-body Dirac equation should be considered for a relativistic account of electronic spin currents [46, 47].

3.2. OAM current density tensor

Equation (10*b*) specifies the AM current tensors for the fields. This current tensor is specified by two directions: direction of AM and direction of the current. For instance, in the first term of equation (10*b*), the direction of propagation of the current is specified by the gamma matrices γ^i, while the direction of the AM is specified by the operator $(\boldsymbol{r}\times\boldsymbol{p}^{\|})$. Thus we call this term *OAM current density tensor* of Dirac field. Similarly, the third term in equation (10*b*) is the OAM current tensor density of the EM field. The last term is the AM current tensor due to longitudinal component of electric field and we incorporate that into the OAM current tensor of the EM field. Equation (10*c*) shows that the electromagnetic Lagrangian also contributes to the continuity equation (8).

Barnett introduced the AM carrying terms in the source-free electromagnetism [19] as quantities that behave as the currents in the conservation equations written for AM of light. The terms helicity and OAM current tensors introduced here for the electromagnetic field have a similar tensorial nature and extend to the QED interactions. These quantities are gauge-invariant and connect to similar terms coming from the Dirac fields.

Note that the expression for the AM due to the electromagnetic field in equation (9) originally has contribution from the longitudinal electric field, $\boldsymbol{E}^{\|}$. This is the contribution due to the free charges $(\nabla\cdot\boldsymbol{E}^{\|}=\rho/\epsilon_0)$ which decays as a function of $1/r^2$ outside of the region of charges [34]. We have shown that the contribution from this component, namely the term $\boldsymbol{E}^{\|}\times\boldsymbol{A}^{\perp}-E^{\|,i}(\boldsymbol{r}\times\nabla)A_i^{\perp}$, can be written as the divergence of some quantities involving $\boldsymbol{E}^{\|}$ (see appendix A). Therefore, when integrated over the entire space, these terms vanish and thus the total global AM of the interacting field does not involve longitudinal electric field components. Therefore, we have shown that the AM density of the EM field only depends on the transverse component of the electric field $\boldsymbol{E}^{\perp}$ since it is only these terms that give nonzero contribution to the total AM.

It should be emphasized that, when working with global quantities (integrated over entire space), the contribution from the longitudinal electric field as well as the contribution from $T^i{}_j$ in equation (10) become surface terms and vanish, assuming that the field quantities are zero on the surface of the integration volume. In such cases, the global AM can be written in equivalent forms which are different only by the divergence of a function of the fields. In the local case, however, all of the terms in equation (10) are observable quantities and each have a different interpretation.

3.3. Spin–orbit torque

We now study the spin-OAM exchange and the AM transfer between the Dirac and EM fields. The continuity equation in equations (5) and (8) only give the dynamics of the total AM density. To obtain insight into the detailed interaction between spin and OAM of the Dirac and EM fields, we separately write the four-divergence of the gauge-independent spin and OAM tensors of the Dirac ($\mathrm{S}_\mathrm{D}^{ij,\lambda}$ and $\mathrm{L}_\mathrm{D}^{ij,\lambda}$) as well as the EM fields ($\mathrm{S}_\mathrm{em}^{ij,\lambda}$ and $\mathrm{L}_\mathrm{em}^{ij,\lambda}$). By incorporating the Dirac and Maxwell's equations into the gauge- invariant AM densities in equations (9) and (10), we obtain their detailed conservation equations (see appendix B),

$$\partial_\lambda\mathcal{S}_\mathrm{D}^{ij,\lambda}=\varepsilon^{ijk}\left[\boldsymbol{\tau}_\mathrm{D}+\boldsymbol{j}_c\times\boldsymbol{A}^{\perp}\right]_k,\tag{11a}$$

$$\partial_\lambda\mathcal{L}_\mathrm{D}^{ij,\lambda}=\varepsilon^{ijk}\left[-\boldsymbol{\tau}_\mathrm{D}+\rho\left(\boldsymbol{r}\times\boldsymbol{E}^{\|}\right)-j_c^l\left(\boldsymbol{r}\times\boldsymbol{\nabla}\right)A_l^{\perp}\right]_k,\tag{11b}$$

$$\partial_\lambda\mathcal{S}_\mathrm{em}^{ij,\lambda}=\varepsilon^{ijk}\left[\boldsymbol{\tau}_\mathrm{em}-\boldsymbol{j}_c\times\boldsymbol{A}^{\perp}\right]_k,\tag{11c}$$

$$\partial_\lambda\mathcal{L}_\mathrm{em}^{ij,\lambda}=\varepsilon^{ijk}\left[-\boldsymbol{\tau}_\mathrm{em}-\rho\left(\boldsymbol{r}\times\boldsymbol{E}^{\|}\right)+j_c^l\left(\boldsymbol{r}\times\boldsymbol{\nabla}\right)A_l^{\perp}\right]_k,\tag{11d}$$

where we have defined

$$\boldsymbol{\tau}_\mathrm{D}=-c\mathcal{R}\left\{\bar{\psi}\left(\boldsymbol{\gamma}\times\boldsymbol{p}^{\|}\right)\psi\right\}\tag{12}$$

and

$$\boldsymbol{\tau}_\mathrm{em}=\frac{1}{\mu_0}\left(\boldsymbol{B}\cdot\nabla\right)\boldsymbol{A}^{\perp}-\epsilon_0\left(\frac{\partial\boldsymbol{E}^{\|}}{\partial t}\times\boldsymbol{A}^{\perp}\right)\tag{13}$$

as the *Dirac* and *Maxwell spin–orbit torque*, respectively, since it gives the amount of torque exerted on the spin from the OAM of the fields and vice versa. This nomenclature is further motivated by the resemblance of the Dirac spin–orbit torque (equation (12)) of the Rashba spin–orbit coupling Hamiltonian [49]. The direct connection between these terms, however, is out of scope of this article and is the focus of a future work. Note that $\boldsymbol{\gamma} = \gamma^1\hat{x} + \gamma^2\hat{y} + \gamma^3\hat{z}$ and $\boldsymbol{j}_c = ec\bar{\psi}\boldsymbol{\gamma}\psi$ is the electric charge current density in equations (11) and (12).

Equation (11) clearly shows that the spin–orbit torques contribute to the spin-OAM exchange in both Dirac and EM fields. Moreover, it is evident from equations (11*a*) – (11*c*) and (11*d*) that the charge-field coupling terms, $\boldsymbol{j}_c \times \boldsymbol{A}^{\perp}$, $j_c^l(\boldsymbol{r}\times\nabla)A_l^{\perp}$, and $\rho(\boldsymbol{r}\times\boldsymbol{E}^{||})$, are responsible for the AM transfer between the Dirac and EM fields [50]. In fact, the first terms gives rise to an optical torque exerted on dipoles due to a circularly polarized optical field [51]. Our results are significantly different from scalar continuity equations of EM helicity in [4] and [5], which derive a scalar conservation law for the dual-symmetric expressions of spin and helicity of EM field. The helicity continuity equation obtained from the free-space Maxwell equations cannot characterize the spin-OAM exchange and specifically the AM transfer between Dirac and EM fields.

4. Source-free problems

We now show the importance of these continuity equations by demonstrating the spin-OAM exchange via the spin–orbit torque for propagating EM fields. We evaluate the terms in $\partial_\lambda S_{\text{em}}^{ij,\lambda}$ and the EM spin–orbit torque $\boldsymbol{\tau}_{\text{em}}$ (equation (13)) for two simple EM problems in source-free regions. Similar spin–orbit signature can also be observed in a cylindrical geometry for the Dirac fields [14]. However, a thorough study of each individual term in equation (10) for Dirac and EM fields is outside the scope of this paper.

The general form of $\partial_\lambda S_{\text{em}}^{ij,\lambda}$, in a source-free regions, can be found from equations (7*c*) – (11*c*) and by taking $\boldsymbol{j}_c = 0$:

$$\epsilon\frac{\partial}{\partial t}\left(\boldsymbol{E}^{\perp}\times\boldsymbol{A}^{\perp}\right) - \frac{1}{\mu_0}\nabla_i\left(A^{\perp,i}\boldsymbol{B}\right) + \frac{1}{\mu_0}\nabla\left(\boldsymbol{A}^{\perp}\cdot\boldsymbol{B}\right) = \boldsymbol{\tau}_{\text{em}}. \tag{14}$$

Here, the extra term on the right hand side results from the coupling to OAM. The spin–orbit torque in the source-free case reduces to $\boldsymbol{\tau}_{\text{em}} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{A}^{\perp}/\mu_0$. We emphasize that different from free-space case [4, 5], the near-field spin-OAM exchange can still exist in the presence of sources.

4.1. Dual-mode optical fiber

We first consider a dual-mode optical fiber, which is placed along the z axis with radius a. The two modes have the same propagation constant, β, and different frequencies of ω_1 and ω_2. Standard solutions of source-free Maxwell equations in cylindrical coordinates have been used to evaluate the local angular momenta distributions [48]. Figure 2 shows the four terms in equation (14). The three rows show the components of these quantities along the radius $\hat{\rho}$, azimuthal $\hat{\phi}$, and z axis of the fiber. Note that for all three rows, the sum of the first three terms is equal to $\boldsymbol{\tau}_{\text{em}}$; thus satisfying equation (14). These figures show that, for a dual-mode optical fiber, the helicity current current tensor, helicity density, spin density, as well as the spin–orbit torque are all non-zero and can play a role in a local interaction between optical modes and atomic sources. Note that all three components become zero as the fiber radius $a \to \infty$, which confirms that the spin–orbit torque $\boldsymbol{\tau}_{\text{em}}$ becomes zero for plane wave solutions.

Figure 3 shows the dynamics of the time-derivative of spin, divergence of helicity current density, and gradient of helicity (the terms in equation (14)) as a function of z (figure 3(a)) and time (figure 3(b)). The curves show the value of these terms at a point 0.05 a from the fiber as shown in the insets. The black curve with the circle markers show the EM spin–orbit torque, $\boldsymbol{\tau}_{\text{em}}$, which is essentially the torque that is transferred to the OAM of the fields. Note that the sum of other curves equals the black curve $\boldsymbol{\tau}_{\text{em}}$; satisfying the continuity equation (14).

4.2. Double plane wave interference

The second example is the interference of two circularly polarized plane waves at two different frequencies. For the plane wave solutions, the current term $\nabla_i(A_i^{\perp}\boldsymbol{B})$ and the spin–orbit torque $\boldsymbol{\tau}_{\text{em}}$ vanish (see appendix C). Therefore, we arrive at a simpler conservation law for the second problem,

$$\frac{\partial}{\partial t}\left(\epsilon_0\boldsymbol{E}^{\perp}\times\boldsymbol{A}^{\perp}\right) + c\boldsymbol{\nabla}\left(\frac{1}{\mu_0 c}\boldsymbol{A}^{\perp}\cdot\boldsymbol{B}\right) = 0, \tag{15}$$

which shows the spin density propagating in space with light speed c. For circularly polarized plane waves propagating along z direction, the electric field is written as $\boldsymbol{E} = \mathcal{R}\{\boldsymbol{\mathcal{E}}_1 e^{-i(\omega_1 t - k_1 z)} + \boldsymbol{\mathcal{E}}_2 e^{-i(\omega_2 t - k_2 z)}\}$, where

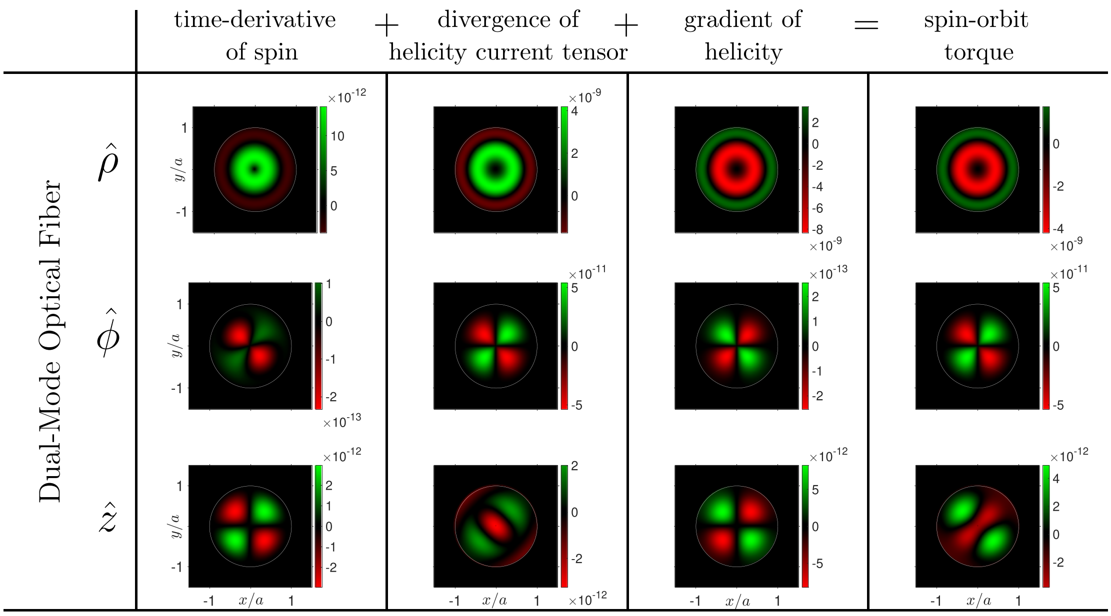


Figure 2. Local conservation law for the spin in a dual-mode optical fiber. The individual terms in equation (14) are plotted on the first three columns from left: the first column is the time-derivative of spin ($\frac{\partial}{\partial t}(\boldsymbol{E}^{\perp} \times A^{\perp})$), the second column is the divergence of helicity current tensor $(-\nabla_i(A_i^{\perp}\boldsymbol{B})/\mu_0)$, and the third column is the gradient of the helicity density $(\nabla(\boldsymbol{A}^{\perp} \cdot \boldsymbol{B})/\mu_0)$. The fourth column is the EM spin–orbit torque given by equation (13) (in this case $\tau_{\mathrm{em}} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{A}^{\perp}/\mu_0$). The three rows show the local value of each vector along the three axes of optical fiber problem: $\hat{\rho}$ radial direction, $\hat{\phi}$ azimuthal direction, and $\hat{z}$ axis of the fiber. Note that adding the first three column on each row together gives the last column $\boldsymbol{\tau}_{\mathrm{em}}$; thus confirming equation (14). The results are for an optical fiber of radius 50 μm with the two modes at the wavelengths 4.3 μm and 4.29 μm. Solutions of source-free Maxwell equations in cylindrical coordinates have been used to evaluate these terms [48].

$\boldsymbol{\mathcal{E}}_i = \mathcal{E}_i(\hat{x} + i\hat{y})/\sqrt{2}$ are the complex electric field amplitudes of the two modes with frequencies $\omega_i/c = k_i$. For these fields, we find (see appendix C),

$$\frac{1}{\mu_0}\nabla\left(\boldsymbol{A}^{\perp} \cdot \boldsymbol{B}\right) = -\epsilon_0 \frac{\partial}{\partial t}\left(\boldsymbol{E} \times \boldsymbol{A}^{\perp}\right) = \epsilon_0 \frac{\omega_1^2 - \omega_2^2}{2\omega_1\omega_2} \mathcal{I}\left\{\mathcal{E}_1 \mathcal{E}_2^* e^{-i[(\omega_1-\omega_2)t-(k_1-k_2)z]}\right\}, \tag{16}$$

hence satisfying the conservation equation (15). This clearly shows that the change in time of the spin is compensated by the gradient of the helicity of the EM field.

5. Experimental perspective

The interaction between the electronic spin and optical AM has been widely used in devices such as optical circulators, insulators, electro-optic modulators, and Faraday rotators. In these devices, conservation of AM combined with the field properties of the electromagnetic (EM) radiation gives rise to non-reciprocal effects that can be manipulated for a wide variety of applications. These devices incorporate the interaction between EM radiation and the magnetic materials in the far-field regions where a conservation law based on the total electronic and EM angular momenta suffices to explain the underlying physical phenomena [52].

Due to the possibility of optical control of the quantum spin states, the dynamics of AM between the electronic and optical fields has recently gained attention for the systems interacting in the near-field region. Coupling to the local spin of EM field has been observed for cold atoms in the vicinity of an optical fiber [53], magnons interacting with spherical whispering gallery modes [54], quantum dots in photonic crystals [55], as well as quantum sources coupling to other waveguide systems [9, 56, 57]. Non-classical spin texture and non-local photonic spin noise has been shown in quantum structured light [58, 59], which can be measured in experiment via nano-scale quantum sensors for photonic spin density [60]. In such systems, since the interaction between the source and the EM field occurs in the near-field rather than the far-field regions, a local approach to the governing dynamical equations of AM becomes important. These studies reveal the new insight that the local interaction with classical EM field has to offer.

With recent advances in the cold atom and quantum dot communities, local interactions between atomic sources and optical fields are gaining more attention due to the emergent new phenomena. These experiments emphasize the need for a local conservation law describing the dynamics of AM in light-matter interactions.

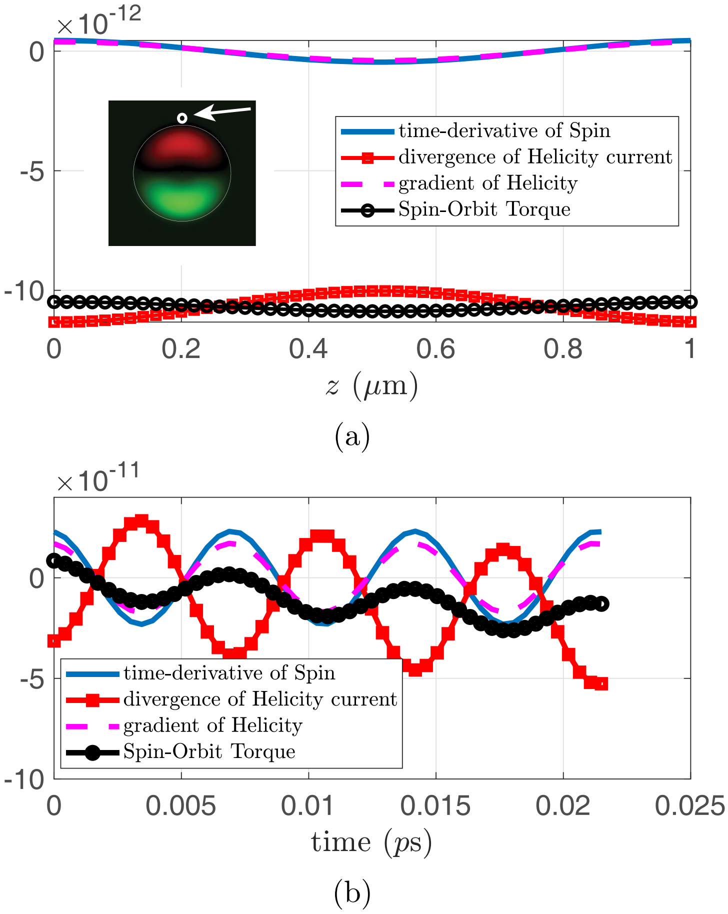


Figure 3. Dynamics of the terms in equation (14) versus (a) z and (b) time. The plots only show the $\hat{z}$ component of each term. These plots show that the spin–orbit torque ($\boldsymbol{\tau}_{\mathrm{em}}$) is equal to the sum of the other terms. The inset in panel (a) shows the location where the terms are evaluated for both of the figures.

We applied the Noether's theorem to the Dirac Lagrangian interacting with the EM field to find the local conservation laws of AM for the most general electrodynamics problem. The results developed here can be applied to near-field as well as far-field to study the transformation of AM between different fields in these regions. Our results show that, in consideration of local conservation laws, other quantities including helicity and OAM current tensors, EM helicity, and electronic chirality should also be considered in addition to the spin and OAM of the EM and Dirac fields.

Equation (8) holds everywhere in space and time. This shows that the total AM density $\boldsymbol{M}$ is not locally conserved. In other words, $\frac{\partial \boldsymbol{M}}{\partial t} \neq 0$ and the conservation law can only be written after including all the other terms in equations (8) and (10). Integrating equation (8) over some volume V, on the surface of which both Dirac and EM fields become zero, $\psi \to 0$, $\boldsymbol{E} \to 0$, gives the usual global conservation law, $\frac{\partial}{\partial t}\int_V \boldsymbol{M}\mathrm{d}^3x = 0$. This states that the integrated values of the spin and OAM densities of electronic and EM field over the entire space is a conserved quantity (figure 1(a)).

We can also get a simpler continuity equation than equation (8) if we limit ourselves to regions outside the Dirac fields. To do so, we take the integral of equation (8) in the volume V' on which only the Dirac fields become zero $\psi = 0$. Doing so we get the *semi-local* conservation law (see appendix D),

$$\frac{\partial \tilde{\boldsymbol{M}}_{\mathrm{D}}}{\partial t} = -\left[\frac{\partial \tilde{\boldsymbol{M}}_{\mathrm{em}}}{\partial t} + \tilde{\boldsymbol{J}} + \tilde{\boldsymbol{h}} + \int_{S'} \hat{\boldsymbol{n}} \times (\boldsymbol{r}\mathfrak{N}_{\mathrm{em}})\,\mathrm{d}a\right] \tag{17}$$

which expresses the time-evolution of the AM of the Dirac field $\tilde{\boldsymbol{M}}_{\mathrm{D}}$ in terms of EM dependent quantities. $\tilde{\boldsymbol{M}}_{\mathrm{em}}$ is the EM AM in the region of the source, while $\tilde{\boldsymbol{J}}$ and $\tilde{\boldsymbol{h}}$ are the EM AM current tensor (projected onto the normal of the surface) and helicity (multiplied by the normal of the surface), integrated on the surface of the volume surrounding the current charges. Also, $\mathfrak{N}_{\mathrm{em}}$ is given by equation (10*c*). Equation (17) can be used to find the conservation of AM in the near-field of current sources and how is it transferred to the EM fields.

It is also important to note that the conservation of AM derived here is based on the spatial components of Lorentz transformations (i.e. rotations). Time-space components of the Lorentz transformation can also give conservation equations related to the boost operators in the relativistic equations of motion. These equations provide a new conservation equation for the Dirac–Maxwell fields which can be of interest to applications investigating relativistic behaviour of the particles and their pertinent conserved quantities when interacting. A brief discussion about these conservation equations is given in appendix E.

The method presented in this paper can be further extended to find the dynamics of magnetization in different materials. The simplest system can be regarded as the interaction between an externally applied EM field and the electrons in a non-magnetic metal. A weak probe signal can then be used to study spin dynamics of the metal. Such a system can be closely modeled as a non-interacting electron gas whose dynamics can be described by the Dirac equation. Although the solutions of the Dirac equation interacting with an externally applied plane wave can be rigorously found [61, 62], these solutions are extremely complicated and only apply to the particular case of free Dirac field described by plane wave solutions. Our method, circumvents the problem of solving the Dirac Hamiltonian to find the spin dynamics of electronic fields, by using Noether's theorem and finding the conservation equation governing the AM dynamics. By knowing the properties of the externally applied EM field, a simpler equation such as equation (17) can be used to find the dynamics of AM without the need of electronic wavefunctions. Since in our derivations we make no simplifying assumption on the EM field, these equations can be easily used for interactions that take place in the near-field region and where EM fields cannot be regarded as plane waves.

The Dirac equation can be further extended to model ferromagnetic materials. Dirac–Kohn–Sham (DKS) equation is an extension to the Dirac equation which accounts for the Kohn–Sham potential as well as the spin-polarized part of the exchange correlation potential inside a magnetic material [63, 64]. Corrections from DKS equation can be added to the usual Pauli Hamiltonian to account for the terms in the Landau–Lifshitz–Gilbert equation and to add corrections accounting for higher order terms dependent on external and internal parameters [65]. The method presented in this paper can also be applied to the DKS Hamiltonian to derive the conservation equation for the AM of electrons in magnetic materials.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Acknowledgments

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Appendix A. Conserved angular momentum tensor

Symmetrized Dirac Lagrangian, with the minimal coupling term [32], is given by equation (1), where $\overleftrightarrow{\partial} = \overleftarrow{\partial} + \overrightarrow{\partial}$ with $\overleftarrow{\partial}$ and $\overrightarrow{\partial}$ acting only on $\bar{\psi}$ and ψ, respectively,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{A.1}$$

is the EM tensor, and γ^μ are the Dirac gamma matrices with the property $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, where

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{A.2}$$

is the Minkowski metric tensor with the signature $(+---)$. We can get the conserved currents related to the rotational symmetry of the Lagrangian, using the Noether's theorem, as [18, 32]:

$$\mathcal{M}^{\mu\nu,\lambda} = \left(-\frac{i}{2}\frac{\partial\mathfrak{L}}{\partial(\partial_\lambda\psi)}\hat{M}_{\mathrm{D}}^{\mu\nu}\psi\right) + \left(-\frac{i}{2}\bar{\psi}\hat{M}_{\mathrm{D}}^{\mu\nu}\frac{\partial\mathfrak{L}}{\partial(\partial_\lambda\bar{\psi})}\right) + \left(-\frac{i}{2}\frac{\partial\mathfrak{L}}{\partial(\partial_\lambda A^\kappa)}\right)\left(\hat{M}_{\mathrm{em}}^{\mu\nu}\right)^{\kappa}{}_{\sigma}A^\sigma + \frac{1}{2}\left(\eta^{\lambda\mu}\mathfrak{L}x^\nu - \eta^{\lambda\nu}\mathfrak{L}x^\mu\right) \tag{A.3}$$

where

$$\hat{M}_{\mathrm{D}}^{\mu\nu} = \hat{L}^{\mu\nu} + \hat{\Sigma}^{\mu\nu} \tag{A.4}$$

is the AM operator for the Dirac fields with

$$\hat{\Sigma}^{\mu\nu} = \frac{1}{2}\sigma^{\mu\nu}, \quad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \tag{A.5}$$

$$\hat{L}^{\mu\nu} = x^\mu\partial^\nu - x^\nu\partial^\mu, \tag{A.6}$$

and

$$\left(\hat{M}_{\mathrm{em}}^{\mu\nu}\right)^{\kappa}{}_{\sigma} = \hat{L}^{\mu\nu}\delta^\kappa_\sigma + \left(\hat{S}^{\mu\nu}\right)^{\kappa}{}_{\sigma} \tag{A.7}$$

is the AM operator for the EM fields with $\hat{L}^{\mu\nu}$ given by equation (A.6), δ^κ_σ being the Kronecker delta function, and

$$\left(\hat{S}^{\mu\nu}\right)^{\kappa}{}_{\sigma} = i\left(\eta^{\mu\kappa}\eta^{\nu}{}_{\sigma} - \eta^{\mu}{}_{\sigma}\eta^{\nu\kappa}\right). \tag{A.8}$$

Plugging these equations into equation (A.3), we get for the AM currents

$$\mathcal{M}^{\mu\nu,\lambda} = \mathcal{M}_{\mathrm{D}}^{\mu\nu,\lambda} + \mathcal{M}_{\mathrm{em}}^{\mu\nu,\lambda} \tag{A.9}$$

where

$$\mathcal{M}_{\mathrm{D}}^{\mu\nu,\lambda} = \mathcal{S}_{\mathrm{D}}^{\mu\nu,\lambda} + \mathcal{L}_{\mathrm{D}}^{\mu\nu,\lambda} \tag{A.10}$$

and

$$\mathcal{M}_{\mathrm{em}}^{\mu\nu,\lambda} = \mathcal{S}_{\mathrm{em}}^{\mu\nu,\lambda} + \mathcal{L}_{\mathrm{em}}^{\mu\nu,\lambda} \tag{A.11}$$

are the contributions to the spin and OAM from the Dirac and Maxwell's fields, respectively, given in equation (6). In these equations, $\mathcal{L}_{\mathrm{D}}$ and $\mathcal{L}_{\mathrm{em}}$ are contributions due to the Dirac field and the EM field to the total Lagrangian, given by,

$$\mathcal{L}_{\mathrm{D}} = \bar{\psi}\left[i\hbar c\frac{1}{2}\gamma^\mu\overleftrightarrow{\partial}_\mu - mc^2\right]\psi - ce\bar{\psi}\gamma^\mu\psi A_\mu \tag{A.12}$$

and

$$\mathcal{L}_{\mathrm{em}} = -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2\mu_0}\left(\frac{\boldsymbol{E}\cdot\boldsymbol{E}}{c^2} - \boldsymbol{B}\cdot\boldsymbol{B}\right). \tag{A.13}$$

As a consequence of Noether theorem, the AM tensor $\mathcal{M}^{\mu\nu,\lambda}$ is conserved. In other words,

$$\partial_\lambda\mathcal{M}^{\mu\nu,\lambda} = 0. \tag{A.14}$$

For the tensors given in equation (6), and using Maxwell and Dirac equations, one can show that equation (A.14) holds for the total AM tensor.

For the case that $\mu\nu = ij$, where $i, j = 1, 2, 3$, we find the AM currents due to rotations. We find for the spin and OAM currents of the Dirac field

$$\partial_\lambda \mathcal{S}_{\mathrm{D}}^{ij,\lambda} = \varepsilon^{ijk}\left[\hbar\frac{\partial}{\partial t}\left(\psi^\dagger \boldsymbol{\Sigma}\psi\right) + \frac{\hbar c}{2}\nabla\left(\psi^\dagger\gamma^5\psi\right)\right]_k, \tag{A.15a}$$

$$\partial_\lambda \mathcal{L}_{\mathrm{D}}^{ij,\lambda} = \varepsilon^{ijk}\left[-\hbar\frac{\partial}{\partial t}\mathcal{R}\left\{i\psi^\dagger\left(\boldsymbol{r}\times\nabla\right)\psi\right\} - \hbar c\nabla\cdot\mathcal{R}\left\{i\bar{\psi}\boldsymbol{\gamma}\left(\boldsymbol{r}\times\nabla\right)\psi\right\}\right]_k \tag{A.15b}$$

where

$$\Sigma_i = \frac{1}{2}\varepsilon_{ijk}\sigma^{jk} = \frac{i}{4}\varepsilon_{ijk}\left[\gamma^j,\gamma^k\right] = \frac{1}{2}\begin{pmatrix}\sigma_i & 0\\ 0 & \sigma_i\end{pmatrix}, \tag{A.16}$$

with σ_i being the Pauli matrices, $\mathcal{R}\{\cdots\}$ takes the real part of its argument, $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ is the chirality operator in the Dirac equation [32], and

$$\boldsymbol{\gamma} = \gamma^1\hat{x} + \gamma^2\hat{y} + \gamma^3\hat{z}. \tag{A.17}$$

Note that we have used the fact that, using the Dirac equation,

$$\left(i\hbar\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu - mc\right)\psi = 0, \tag{A.18}$$

we get $\mathcal{L}_{\mathrm{D}} = 0$ for the fields that follow Dirac equation of motion. This is straightforward to show by multiplying equation (A.18) from left by $c\bar{\psi}$. For the spin and OAM currents of the EM field we find

$$\partial_\lambda \mathcal{S}_{\mathrm{em}}^{ij,\lambda} = \varepsilon^{ijk}\left[\epsilon\frac{\partial}{\partial t}\left(\boldsymbol{E}\times\boldsymbol{A}\right) - \frac{1}{\mu_0}\nabla\cdot\left(\boldsymbol{AB}\right) + \frac{1}{\mu_0}\nabla\left(\boldsymbol{A}\cdot\boldsymbol{B}\right)\right]_k \tag{A.19a}$$

$$\partial_\lambda \mathcal{L}_{\mathrm{em}}^{ij,\lambda} = \varepsilon^{ijk}\left\{\epsilon\frac{\partial}{\partial t}\left[\boldsymbol{E}\cdot\left(\boldsymbol{r}\times\nabla\right)\boldsymbol{A}\right] - \nabla\cdot\left[\frac{1}{\mu_0}\boldsymbol{B}\times\left(\boldsymbol{r}\times\nabla\right)\boldsymbol{A} - \epsilon\boldsymbol{E}\left(\boldsymbol{r}\times\nabla\phi\right)\right] + \left(\boldsymbol{r}\times\nabla\right)\mathcal{L}_{\mathrm{em}}\right\}_k. \tag{A.19b}$$

The problem with equations (A.15b) and (A.19) is that they are gauge dependent, which means that under the transformations $\psi \to \psi e^{\frac{i}{\epsilon}\zeta}$ and $A_\mu \to A_\mu + \partial_\mu\zeta$ they do not remain invariant. Therefore, the individual terms do not represent any physically meaningful quantity. The fundamental equation to hold is equation (A.14). Therefore, as long as this relation is satisfied, we can cast equations (A.15) and (A.19) into gauge-independent forms. This means that by rearranging some terms between equations (A.15) and (A.19) we can produce gauge-independent terms while still satisfying the continuity equation in equation (A.14). To do so, we break $\boldsymbol{A}$ into two longitudinal and transverse parts as $\boldsymbol{A} = \boldsymbol{A}^{\parallel} + \boldsymbol{A}^{\perp}$, where $\nabla\cdot\boldsymbol{A}^{\perp} = 0$ and $\nabla\times\boldsymbol{A}^{\parallel} = 0$ by definition. From equations (A.15) and (A.19), the time component of AM tensor $\mathcal{M}^{ij,0}$ is given by:

$$\boldsymbol{M} = \frac{\hbar}{2}\left(\psi^\dagger\boldsymbol{\Sigma}\psi\right) - \hbar\mathcal{R}\left\{\psi^\dagger\left(\boldsymbol{r}\times\nabla\right)\psi\right\} + \epsilon\left(\boldsymbol{E}\times\boldsymbol{A}\right) + \epsilon\left[\boldsymbol{E}\cdot\left(\boldsymbol{r}\times\nabla\right)\boldsymbol{A}\right]. \tag{A.20}$$

Splitting A into transverse and longitudinal parts, we can write the terms containing $\boldsymbol{A}^{\parallel}$ as:

$$\boldsymbol{E}\times\boldsymbol{A}^{\parallel} + \boldsymbol{E}\cdot\left(\boldsymbol{r}\times\nabla\right)\boldsymbol{A}^{\parallel} = \nabla\cdot\left[\boldsymbol{E}\left(\boldsymbol{r}\times\boldsymbol{A}^{\parallel}\right)\right] - \left(\nabla\cdot E\right)\left(\boldsymbol{r}\times\boldsymbol{A}^{\parallel}\right) \tag{A.21}$$

where we have used the following identity for arbitrary vectors $\boldsymbol{X}$ and $\boldsymbol{X}$:

$$\nabla\cdot\left[\boldsymbol{X}\left(\boldsymbol{r}\times\boldsymbol{X}\right)\right] = \left(\nabla\cdot\boldsymbol{X}\right)\left(\boldsymbol{r}\times\boldsymbol{X}\right) + \left(\boldsymbol{X}\times\boldsymbol{X}\right) + \boldsymbol{X}\cdot\left(\boldsymbol{r}\times\nabla\right)\boldsymbol{X} + \left(\boldsymbol{X}\cdot\boldsymbol{r}\right)\left(\nabla\times\boldsymbol{X}\right) - \boldsymbol{X}\left[\boldsymbol{r}\cdot\left(\nabla\times\boldsymbol{X}\right)\right], \tag{A.22}$$

and the fact that $\nabla\times\boldsymbol{A}^{\parallel} = 0$. Using the fact that $\nabla\cdot\boldsymbol{E} = \rho/\epsilon = \frac{e}{\epsilon}\psi^\dagger\psi$, we can write the last term in equation (A.21) as:

$$\left(\nabla\cdot\boldsymbol{E}\right)\left(\boldsymbol{r}\times\boldsymbol{A}^{\parallel}\right) = \frac{e}{\epsilon}\psi^\dagger\left(\boldsymbol{r}\times\boldsymbol{A}^{\parallel}\right)\psi \tag{A.23}$$

Plugging these into equation (A.20) we get

$$\begin{aligned}\boldsymbol{M} &= \frac{\hbar}{2}\left(\psi^\dagger\boldsymbol{\Sigma}\psi\right) - \hbar\mathcal{R}\left\{\psi^\dagger\left(\boldsymbol{r}\times\nabla\right)\psi\right\} + \epsilon\nabla\cdot\left[\boldsymbol{E}\left(\boldsymbol{r}\times\boldsymbol{A}^{\parallel}\right)\right] - e\psi^\dagger\left(\boldsymbol{r}\times\boldsymbol{A}^{\parallel}\right)\psi.\\ &= \frac{\hbar}{2}\left(\psi^\dagger\boldsymbol{\Sigma}\psi\right) + \mathcal{R}\left\{\psi^\dagger\left[\boldsymbol{r}\times\boldsymbol{p}^{\parallel}\right]\psi\right\} + \epsilon\nabla\cdot\left[\boldsymbol{E}\left(\boldsymbol{r}\times\boldsymbol{A}^{\parallel}\right)\right],\end{aligned} \tag{A.24}$$

where $\boldsymbol{p}^{\parallel} = -i\hbar\boldsymbol{\nabla} - e\boldsymbol{A}^{\parallel}$ is the gauge-independent covariant momentum operator of the electronic field. Note that the first two terms are now gauge invariant. The last term is written as the divergence of a vector and can thus be absorbed into the rest of the continuity equations written as the divergence of the space components of the AM tensor $\mathcal{M}^{ij,\lambda}$.

Repeating a similar procedure as above for the other components of the continuity equation, namely $\nabla_k\mathcal{M}^{ij,k}$, we find for the new spin and OAM tensors of Dirac and EM field after some algebra:

$$\partial_\lambda \mathcal{S}_{\mathrm{D}}^{ij,\lambda} = \varepsilon^{ijk}\left[\hbar\frac{\partial}{\partial t}\left(\psi^\dagger\boldsymbol{\Sigma}\psi\right) + \frac{\hbar c}{2}\nabla\left(\psi^\dagger\gamma^5\psi\right)\right]_k, \tag{A.25a}$$

$$\partial_\lambda \mathcal{L}_{\mathrm{D}}^{ij,\lambda} = \varepsilon^{ijk}\left[\frac{\partial}{\partial t}\mathcal{R}\left\{\psi^\dagger\left(\boldsymbol{r}\times\boldsymbol{p}^{\parallel}\right)\psi\right\} + c\nabla\cdot\mathcal{R}\left\{\bar{\psi}\gamma\left(\boldsymbol{r}\times\boldsymbol{p}^{\parallel}\right)\psi\right\}\right]_k, \tag{A.25b}$$

$$\partial_\lambda \mathcal{S}_{\mathrm{em}}^{ij,\lambda} = \varepsilon^{ijk}\left[\epsilon\frac{\partial}{\partial t}\left(\boldsymbol{E}\times\boldsymbol{A}^\perp\right) - \frac{1}{\mu_0}\nabla\cdot\left(\boldsymbol{A}^\perp\boldsymbol{B}\right) + \frac{1}{\mu_0}\nabla\left(\boldsymbol{A}^\perp\cdot\boldsymbol{B}\right)\right]_k, \tag{A.25c}$$

$$\partial_\lambda \mathcal{L}_{\mathrm{em}}^{ij,\lambda} = \varepsilon^{ijk}\left\{\epsilon\frac{\partial}{\partial t}\left[\boldsymbol{E}\cdot\left(\boldsymbol{r}\times\nabla\right)\boldsymbol{A}^\perp\right] - \nabla\cdot\left[\frac{1}{\mu_0}\boldsymbol{B}\times\left(\boldsymbol{r}\times\nabla\right)\boldsymbol{A}^\perp + \epsilon\boldsymbol{E}\left(\boldsymbol{r}\times\boldsymbol{E}^{\parallel}\right)\right] + \left(\boldsymbol{r}\times\nabla\right)\mathcal{L}_{\mathrm{em}}\right\}_k. \tag{A.25d}$$

We can further separate the contribution of electric field to the AM into its transverse and longitudinal components as $\boldsymbol{E} = \boldsymbol{E}^\perp + \boldsymbol{E}^\parallel$. Making use of the vector identity in equation (A.22) again, this contribution can be written as:

$$\boldsymbol{E}^\parallel\times\boldsymbol{A}^\perp - E^{\parallel,i}\left(\boldsymbol{r}\times\nabla\right)A_i^\perp = -\nabla\times\left[\boldsymbol{r}\left(\boldsymbol{E}^\parallel\cdot\boldsymbol{A}^\perp\right)\right] - \nabla\cdot\left[\boldsymbol{A}^\perp\left(\boldsymbol{r}\times\boldsymbol{E}^\parallel\right)\right]. \tag{A.26}$$

When integrated over the entire space, both of these terms on r.h.s of this expression become zero due to the Stokes theorem and thus the longitudinal electric field does not contribute to the global AM of the EM field. For this reason, we move this term into the AM current terms so that the global AM represents the integrated AM density. Making this change, we get for the new components of the EM AM currents:

$$\partial_\lambda \mathcal{S}_{\mathrm{em}}^{ij,\lambda} = \varepsilon^{ijk}\left[\epsilon\frac{\partial}{\partial t}\left(\boldsymbol{E}^\perp\times\boldsymbol{A}^\perp\right) - \frac{1}{\mu_0}\nabla\cdot\left(\boldsymbol{A}^\perp\boldsymbol{B}\right) + \frac{1}{\mu_0}\nabla\left(\boldsymbol{A}^\perp\cdot\boldsymbol{B}\right)\right]_k, \tag{A.27a}$$

$$\begin{aligned}\partial_\lambda \mathcal{L}_{\mathrm{em}}^{ij,\lambda} = \varepsilon^{ijk}\Bigg\{&\epsilon\frac{\partial}{\partial t}\left[\boldsymbol{E}^\perp\cdot\left(\boldsymbol{r}\times\nabla\right)\boldsymbol{A}^\perp\right] - \nabla\cdot\left[\frac{1}{\mu_0}\boldsymbol{B}\times\left(\boldsymbol{r}\times\nabla\right)\boldsymbol{A}^\perp + \epsilon_0\boldsymbol{A}^\perp\left(\boldsymbol{r}\times\frac{\partial\boldsymbol{E}^\parallel}{\partial t}\right)\right]\\ &+\left(\boldsymbol{r}\times\nabla\right)\left[-\frac{1}{2\mu_0}\boldsymbol{B}\cdot\boldsymbol{B} + \frac{\epsilon_0}{2}\boldsymbol{E}^\perp\cdot\boldsymbol{E}^\perp + \epsilon_0\frac{\partial\boldsymbol{E}^\parallel}{\partial t}\cdot\boldsymbol{A}^\perp\right]\Bigg\}_k.\end{aligned} \tag{A.27b}$$

Adding these four equations together we get equation (5),

$$\frac{\partial M_j}{\partial t} + \nabla_j J^i{}_j + \nabla_i\chi + \nabla_i N^i{}_j = 0, \tag{A.28}$$

where the terms are given in equations (9) and (10). Equation (A.28) can be written as

$$\frac{\partial\boldsymbol{M}}{\partial t} + \nabla\cdot\overleftrightarrow{\boldsymbol{T}} = 0 \tag{A.29}$$

where

$$\overleftrightarrow{\boldsymbol{T}} = \chi\overleftrightarrow{I} + \overleftrightarrow{\boldsymbol{J}} + \overleftrightarrow{\boldsymbol{N}} \tag{A.30}$$

with $\overleftrightarrow{I} = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z} = \delta^i{}_j\hat{x}_i\hat{x}^j$, and

$$\overleftrightarrow{N} = \varepsilon^i{}_{jk}\hat{x}_i\hat{x}^j x^k\mathfrak{N}_{\mathrm{em}} = \begin{pmatrix}0 & z & -y\\ -z & 0 & x\\ y & -x & 0\end{pmatrix}\mathfrak{N}_{\mathrm{em}},\quad \mathfrak{N}_{em} = \frac{\epsilon_0}{2}\boldsymbol{E}^\perp\cdot\boldsymbol{E}^\perp - \frac{1}{2\mu_0}\boldsymbol{B}\cdot\boldsymbol{B} + \epsilon_0\frac{\partial\boldsymbol{E}^\parallel}{\partial t}\cdot\boldsymbol{A}^\perp. \tag{A.31}$$

Note that the first two terms of $\mathfrak{N}_{em}$ describe the Lagrangian due to the transverse electric field, which can be interpreted as the Lagrangian of the free EM field [66], while the last term shows the interaction between the transverse vector potential and the currents due to the longitudinal electric fields. Note that the term $\frac{\epsilon_0}{2}\boldsymbol{E}^\parallel\cdot\boldsymbol{E}^\parallel$ in the Lagrangian of the EM field cancels out with the same term coming from $\nabla_i J_{ij}$. The expressions for $\overleftrightarrow{\boldsymbol{J}}$ and χ are given in the main manuscript.

Also note that the EM Lagrangian does not include the interaction term $j_c^\mu A_\mu$ because it appears in the Dirac part of the Lagrangian $\mathcal{L}_\mathrm{D}$. Dirac Lagrangian, as mentioned earlier, vanishes from the expression for the OAM of the Dirac field since it satisfies the Dirac equation. For this reason, the contribution from the Dirac Lagrangian $\mathcal{L}_\mathrm{D}$ disappears from the conservation equations.

Equation (A.29) describes the local conservation law for the AM currents $\overleftrightarrow{\boldsymbol{T}}$ and the AM density (charge) $\boldsymbol{M}$. When integrated over the entire space, and assuming that the fields vanish on the boundary of the this surface, the second term becomes an integral over this surface and thus vanishes. In this case, we arrive at the usual global conservation of AM equation which states that the total AM of the Dirac and Maxwell fields is a constant. However, in situations where the problem under consideration is an open dissipative system, this simplification cannot be made and surface terms of the AM current can carry AM out of the system.

Appendix B. Spin–orbit torque

In this section we show that equation (7) leads to equation (11). Starting with the equation for the spin of the Dirac fields, we get for the z component for instance,

$$\partial_\lambda \mathcal{S}_\mathrm{D}^{12,\lambda} = \frac{\hbar}{2}\partial_t\left(\psi^\dagger\Sigma_z\psi\right) + \frac{\hbar c}{2}\partial_z\left(\psi^\dagger\gamma^5\psi\right) \tag{B.1}$$

where $\partial_t \equiv \frac{\partial}{\partial t}$ and $\partial_i \equiv \frac{\partial}{\partial x_i}$. Using Dirac equation we get

$$\hbar\partial_t\psi = -\hbar c\gamma^0\left(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}\psi\right) - iceA_\mu\gamma^0\gamma^\mu\psi - imc^2\gamma^0\psi \tag{B.2a}$$

$$\hbar\partial_t\psi^\dagger = \hbar c\left(\boldsymbol{\nabla}\psi^\dagger\cdot\boldsymbol{\gamma}\right)\gamma^0 + iceA_\mu\psi^\dagger\gamma^0\gamma^\mu + imc^2\psi^\dagger\gamma^0 \tag{B.2b}$$

Using these equations we find

$$\begin{aligned}
\partial_\lambda \mathcal{S}_\mathrm{D}^{12,\lambda} &= \frac{\hbar}{2}\left\{\left(\partial_t\psi^\dagger\right)\Sigma_z\psi + \psi^\dagger\Sigma_z\left(\partial_t\psi\right) + c\partial_z\left(\psi^\dagger\gamma^5\psi\right)\right\}\\
&\quad + \frac{1}{2}\left\{\hbar c\left(\boldsymbol{\nabla}\psi^\dagger\cdot\boldsymbol{\gamma}\right)i\gamma^0\gamma^1\gamma^2\psi - ceA_\mu\psi^\dagger\gamma^0\gamma^\mu\gamma^1\gamma^2\psi - mc^2\psi^\dagger\gamma^0\gamma^1\gamma^2\psi\right.\\
&\quad \left.-i\hbar c\psi^\dagger\gamma^1\gamma^2\gamma^0\left(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}\psi\right) + ceA_\mu\psi^\dagger\gamma^1\gamma^2\gamma^0\gamma^\mu\psi + mc^2\psi^\dagger\gamma^1\gamma^2\gamma^0\psi + \hbar c\partial_z\left(\psi^\dagger\gamma^5\psi\right)\right\}\\
&= \frac{1}{2}\left\{-\hbar c\partial_z\left(\psi^\dagger\gamma^5\psi\right) + i\hbar c\left(\partial_x\psi^\dagger\right)\gamma^0\gamma^2\psi - i\hbar c\psi^\dagger\gamma^0\gamma^2\left(\partial_x\psi\right)\right.\\
&\quad -i\hbar c\left(\partial_y\psi^\dagger\right)\gamma^0\gamma^1\psi + i\hbar c\psi^\dagger\gamma^0\gamma^1\left(\partial_y\psi\right)\\
&\quad \left.+2ceA_1\psi^\dagger\gamma^0\gamma^2\psi - 2ceA_2\psi^\dagger\gamma^0\gamma^1\psi + \hbar c\partial_z\left(\psi^\dagger\gamma^5\psi\right)\right\}\\
&= \hbar c\mathcal{R}\left\{i\bar\psi\left(\gamma^1\partial_y - \gamma^2\partial_x\right)\psi\right\} - ce\left(A_x^{\|}\bar\psi\gamma^2\psi - A_y^{\|}\bar\psi\gamma^1\psi\right) - \left(A_x^\perp j_{c,y} - A_y^\perp j_{c,x}\right)\\
&= -c\mathcal{R}\left\{\bar\psi\left[\gamma^1\left(-i\hbar\partial_y - eA_y^{\|}\right) - \gamma^2\left(-i\hbar\partial_x - eA_x^{\|}\right)\right]\psi\right\} - \left(A_x^\perp j_{c,y} - A_y^\perp j_{c,x}\right)\\
&= -c\mathcal{R}\left\{\bar\psi\left(\boldsymbol{\gamma}\times\boldsymbol{p}^{\|}\right)_z\psi\right\} + \left(\boldsymbol{j}_c\times\boldsymbol{A}^\perp\right)_z.
\end{aligned} \tag{B.3}$$

In this derivation we have used the facts that $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}$, $(\gamma^i)^2 = -1$, $A_1 = -A_x$, $A_2 = -A_y$, and $\boldsymbol{j}_c = ce\bar\psi\boldsymbol{\gamma}\psi$. We can do a similar derivation for other components of $\partial_\lambda \mathcal{S}_\mathrm{D}^{ij,\lambda}$. Doing so we find equation (11a). Note that the first term on the r.h.s. of equation (B.3) is nothing but the spin–orbit torque of the Dirac field $\boldsymbol{\tau}_\mathrm{D}$ given in equation (12).

We now turn into the equation for the OAM of the Dirac field equation (7b). We get, for instance, for the z component,

$$\begin{aligned}
\partial_\lambda \mathcal{L}_\mathrm{D}^{12,\lambda} &= \partial_t\mathcal{R}\left\{\psi^\dagger(xp_y^{\|} - yp_x^{\|})\psi\right\} + c\boldsymbol{\nabla}\cdot\mathcal{R}\left\{\bar\psi\boldsymbol{\gamma}(xp_y^{\|} - yp_x^{\|})\psi\right\}\\
&= \mathcal{R}\Big\{(\partial_t\psi^\dagger)(xp_y^{\|} - yp_x^{\|})\psi + \psi^\dagger(xp_y^{\|} - yp_x^{\|})(\partial_t\psi) + c\boldsymbol{\nabla}\cdot\bar\psi\boldsymbol{\gamma}(xp_y^{\|} - yp_x^{\|})\psi\\
&\quad - e\psi^\dagger\psi\left(x\partial_tA_y^{\|} - y\partial_tA_x^{\|}\right)\Big\}\\
&= \mathcal{R}\Big\{c(\boldsymbol{\nabla}\psi^\dagger\cdot\boldsymbol{\gamma})\gamma^0(xp_y^{\|} - yp_x^{\|})\psi + \frac{ice}{\hbar}A_\mu\psi^\dagger\gamma^0\gamma^\mu(xp_y^{\|} - yp_x^{\|})\psi + \frac{imc^2}{\hbar}\psi^\dagger\gamma^0(xp_y^{\|} - yp_x^{\|})\psi\\
&\quad - c\psi^\dagger(xp_y^{\|} - yp_x^{\|})\gamma^0(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}\psi) - \frac{ice}{\hbar}\psi^\dagger(xp_y^{\|} - yp_x^{\|})\gamma^0\gamma^\mu(A_\mu\psi) - \frac{imc^2}{\hbar}\psi^\dagger(xp_y^{\|} - yp_x^{\|})\gamma^0\psi\\
&\quad + c\boldsymbol{\nabla}\cdot\bar\psi\boldsymbol{\gamma}(xp_y^{\|} - yp_x^{\|})\psi - e\psi^\dagger\psi\left(x\partial_tA_y^{\|} - y\partial_tA_x^{\|}\right)\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{R}\Big\{-c\boldsymbol{\nabla}\cdot\bar{\psi}\boldsymbol{\gamma}(xp_y^{\|}-yp_x^{\|})\psi + c\bar{\psi}\left[\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}(xp_y^{\|}-yp_x^{\|})\right]\psi - ce\bar{\psi}\gamma^{\mu}\psi\,(x\partial_y - y\partial_x)A_{\mu}\\
&\quad + c\boldsymbol{\nabla}\cdot\bar{\psi}\boldsymbol{\gamma}(xp_y^{\|}-yp_x^{\|})\psi - e\psi^{\dagger}\psi\,(x\partial_t A_y^{\|} - y\partial_t A_x^{\|})\Big\}\\
&= \mathcal{R}\left\{c\bar{\psi}(\gamma^1 p_y^{\|} - \gamma^2 p_x^{\|})\psi\right\} - ce\bar{\psi}\boldsymbol{\gamma}\psi\,(x\boldsymbol{\nabla}A_y^{\|} - y\boldsymbol{\nabla}A_x^{\|}) - ce\bar{\psi}\gamma^{\mu}\psi\,(x\partial_y - y\partial_x)A_{\mu}\\
&\quad - e\psi^{\dagger}\psi\,(x\partial_t A_y^{\|} - y\partial_t A_x^{\|})\\
&= \mathcal{R}\left\{c\bar{\psi}(\gamma^1 p_y^{\|} - \gamma^2 p_x^{\|})\psi\right\} - ce\bar{\psi}\Big[\gamma^1(x\partial_x A_y^{\|} - y\partial_x A_x^{\|}) + \gamma^2(x\partial_y A_y^{\|} - y\partial_y A_x^{\|})\\
&\quad + \gamma^3(x\partial_z A_y^{\|} - y\partial_z A_x^{\|}) - \gamma^1(x\partial_y A_x - y\partial_x A_x) - \gamma^2(x\partial_y A_y - y\partial_x A_y) - \gamma^3(x\partial_y A_z - y\partial_x A_z)\\
&\quad + \frac{1}{c}\gamma^0(x\partial_y\phi - y\partial_x\phi) + \frac{1}{c}\psi^{\dagger}(x\partial_t A_y^{\|} - y\partial_t A_x^{\|})\Big]\psi\\
&= \mathcal{R}\left\{c\bar{\psi}(\gamma^1 p_y^{\|} - \gamma^2 p_x^{\|})\psi\right\} - ce\bar{\psi}\gamma^1\psi x(\partial_x A_y^{\|} - \partial_y A_x^{\|}) + ce\bar{\psi}\gamma^1\psi\,(x\partial_y - y\partial_x)A_x^{\perp}\\
&\quad - ce\bar{\psi}\gamma^2\psi y(\partial_x A_y^{\|} - \partial_y A_x^{\|}) + ce\bar{\psi}\gamma^2\psi\,(x\partial_y - y\partial_x)A_y^{\perp} - ce\bar{\psi}\gamma^3\psi x(\partial_z A_y^{\|} - \partial_y A_z^{\|})\\
&\quad + ce\bar{\psi}\gamma^3\psi y(\partial_z A_x^{\|} - \partial_x A_z^{\|}) + ce\bar{\psi}\gamma^3\psi\,(x\partial_y - y\partial_x)A_z^{\perp} + e\psi^{\dagger}\psi\left[x(-\partial_y\phi - \partial_t A_y^{\|}) - y(-\partial_x\phi - \partial_t A_x^{\|})\right]\\
&= c\mathcal{R}\left\{\bar{\psi}(\boldsymbol{\gamma}\times\boldsymbol{p}^{\|})_z\psi\right\} - j_c^i(\boldsymbol{r}\times\boldsymbol{\nabla})_z A_i^{\perp} + \rho(\boldsymbol{r}\times\boldsymbol{E}^{\|})_z
\end{aligned}
\tag{B.4}
$$

where we have used the facts that $\boldsymbol{\nabla}\times\boldsymbol{A}^{\|} = 0$ and $\boldsymbol{E}^{\|} = -\boldsymbol{\nabla}\phi - \partial_t\boldsymbol{A}^{\|}$. Writing similar equations for the other components we get equation (11*b*).

We can repeat this derivation for the EM spin and OAM currents as well. Using equation (7*c*) we find, again for the z component for instance,

$$
\begin{aligned}
\partial_{\lambda}\mathcal{S}_{\mathrm{em}}^{12,\lambda} &= \epsilon_0\partial_t\left(\boldsymbol{E}^{\perp}\times\boldsymbol{A}^{\perp}\right)_z - \frac{1}{\mu_0}\boldsymbol{\nabla}\cdot\left(\boldsymbol{A}^{\perp}B_z\right) + \frac{1}{\mu_0}\partial_z\left(\boldsymbol{A}^{\perp}\cdot\boldsymbol{B}\right)\\
&= \epsilon_0\partial_t\left(\boldsymbol{E}^{\perp}\times\boldsymbol{A}^{\perp}\right)_z - \frac{1}{\mu_0}B_z\left(\boldsymbol{\nabla}\cdot\boldsymbol{A}^{\perp}\right) - \frac{1}{\mu_0}\left(\boldsymbol{A}^{\perp}\cdot\boldsymbol{\nabla}\right)B_z\\
&\quad + \frac{1}{\mu_0}\left\{\left(\boldsymbol{A}^{\perp}\cdot\boldsymbol{\nabla}\right)B_z + \left(\boldsymbol{B}\cdot\boldsymbol{\nabla}\right)A_z^{\perp} + \left[\boldsymbol{A}^{\perp}\times\left(\boldsymbol{\nabla}\times\boldsymbol{B}\right)\right]_z + \left[\boldsymbol{B}\times\left(\boldsymbol{\nabla}\times\boldsymbol{A}^{\perp}\right)\right]_z\right\}\\
&= \epsilon_0\partial_t\left(\boldsymbol{E}^{\perp}\times\boldsymbol{A}^{\perp}\right) + \frac{1}{\mu_0}\left(\boldsymbol{B}\cdot\boldsymbol{\nabla}\right)A_z^{\perp} - \epsilon_0\left[\left(\partial_t\boldsymbol{E}\right)\times\boldsymbol{A}^{\perp}\right]_z - \left(\boldsymbol{j}_c\times\boldsymbol{A}^{\perp}\right)_z\\
&= \epsilon_0\left[\boldsymbol{E}^{\perp}\times\left(\partial_t\boldsymbol{A}^{\perp}\right)\right]_z + \frac{1}{\mu_0}\left(\boldsymbol{B}\cdot\boldsymbol{\nabla}\right)A_z^{\perp} - \epsilon_0\left[\frac{\partial\boldsymbol{E}^{\|}}{\partial t}\times\boldsymbol{A}^{\perp}\right] - \left(\boldsymbol{j}_c\times\boldsymbol{A}^{\perp}\right)_z\\
&= \frac{1}{\mu_0}\left(\boldsymbol{B}\cdot\boldsymbol{\nabla}\right)A_z^{\perp} - \epsilon_0\left(\frac{\partial\boldsymbol{E}^{\|}}{\partial t}\times\boldsymbol{A}^{\perp}\right)_z - \left(\boldsymbol{j}_c\times\boldsymbol{A}^{\perp}\right)_z,
\end{aligned}
\tag{B.5}
$$

where we have used the facts that $\boldsymbol{\nabla}\cdot\boldsymbol{A}^{\perp} = 0$, $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}^{\perp}$, $\boldsymbol{E}^{\perp} = -\partial_t\boldsymbol{A}^{\perp}$, and $\boldsymbol{\nabla}\times\boldsymbol{B} = \frac{1}{c^2}\partial_t\boldsymbol{E} + \mu_0\boldsymbol{j}_c$. By repeating this derivation for the other components we get equation (11*c*). Note that the first two terms are the spin–orbit torque of the EM field, $\boldsymbol{\tau}_{\mathrm{em}}$, given in equation (13).

We can repeat this derivation to find the equation for $\partial_{\lambda}\mathrm{L}_{\mathrm{em}}^{ij,\lambda}$. However, using the continuity condition,

$$
\partial_{\lambda}\left(\mathcal{S}_{\mathrm{D}}^{ij,\lambda} + \mathcal{L}_{\mathrm{D}}^{ij,\lambda} + \mathcal{S}_{\mathrm{em}}^{ij,\lambda} + \mathcal{L}_{\mathrm{em}}^{ij,\lambda}\right) = 0,
\tag{B.6}
$$

it is straightforward to show that equation (11*d*) holds.

Appendix C. Two plane wave interference

We now evaluate the terms in equation (11*c*) for the interference of two plane waves at different frequencies. For a plane wave propagating along $\boldsymbol{k}/|\boldsymbol{k}|$, with the wavevector k we have $\boldsymbol{k}\cdot\boldsymbol{E} = \boldsymbol{k}\cdot\boldsymbol{A}^{\perp} = \boldsymbol{k}\cdot\boldsymbol{B} = 0$. Therefore we get

$$
\nabla\cdot\left(\boldsymbol{A}^{\perp}\boldsymbol{B}\right) = \left(\boldsymbol{k}\cdot\boldsymbol{A}^{\perp}\right)\boldsymbol{B} = 0.
\tag{C.1}
$$

Note also that the EM spin–orbit torque $\boldsymbol{\tau}_{\text{em}}$ also vanishes for planes waves because

$$(\boldsymbol{B}\cdot\nabla)\boldsymbol{A}^{\perp}=(\boldsymbol{B}\cdot\boldsymbol{k})\boldsymbol{A}^{\perp}=0. \tag{C.2}$$

Thus the only relevant terms in finding the conservation law for the spin current of the two plane wave interference are the time-derivative of spin and gradient of helicity.

The electric field for two plane waves propagating along z direction can be written as

$$\boldsymbol{E}=\mathcal{R}\left\{\boldsymbol{E}_1 e^{-i(\omega_1 t-k_1 z)}+\boldsymbol{E}_2 e^{-i(\omega_2 t-k_2 z)}\right\} \tag{C.3}$$

where $\boldsymbol{\mathcal{E}}_i=\frac{\mathcal{E}_i}{\sqrt{2}}(\hat{x}+i\hat{y})$ are the complex electric field amplitudes of the two modes with frequencies $\omega_i/c=k_i$. Using Maxwell equation $\nabla\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t}$ and $\boldsymbol{E}=-\frac{\partial\boldsymbol{A}}{\partial t}$, we get

$$\boldsymbol{B}=\mathcal{R}\left\{-\frac{k_1}{i\omega_1}\boldsymbol{E}_1 e^{-i(\omega_1 t-k_1 z)}-\frac{k_2}{i\omega_2}\boldsymbol{E}_2 e^{-i(\omega_2 t-k_2 z)}\right\} \tag{C.4a}$$

$$\boldsymbol{A}^{\perp}=\mathcal{R}\left\{\frac{1}{i\omega_1}\boldsymbol{E}_1 e^{-i(\omega_1 t-k_1 z)}+\frac{1}{i\omega_2}\boldsymbol{E}_2 e^{-i(\omega_2 t-k_2 z)}\right\}. \tag{C.4b}$$

Using these equations we get for the spin of the two plane waves

$$\epsilon\left(\boldsymbol{E}^{\perp}\times\boldsymbol{A}^{\perp}\right)=\frac{\epsilon}{2}\left[\frac{1}{\omega_1}\mathcal{I}\{\boldsymbol{\mathcal{E}}_1^*\times\boldsymbol{\mathcal{E}}_1\}+\frac{1}{\omega_2}\mathcal{I}\{\boldsymbol{\mathcal{E}}_2^*\times\boldsymbol{\mathcal{E}}_2\}+\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right)\mathcal{I}\left\{\boldsymbol{\mathcal{E}}_1^*\times\boldsymbol{\mathcal{E}}_2 e^{+i[(\omega_1-\omega_2)t-(k_1-k_2)z]}\right\}\right] \tag{C.5}$$

and thus we get

$$\begin{aligned}\epsilon\frac{\partial}{\partial t}\left(\boldsymbol{E}^{\perp}\times\boldsymbol{A}^{\perp}\right)&=-\epsilon\frac{\omega_1^2-\omega_2^2}{2\omega_1\omega_2}\mathcal{R}\left\{\boldsymbol{\mathcal{E}}_1^*\times\boldsymbol{\mathcal{E}}_2 e^{+i[(\omega_1-\omega_2)t-(k_1-k_2)z]}\right\}\\&=-\epsilon\frac{\omega_1^2-\omega_2^2}{2\omega_1\omega_2}\mathcal{I}\left\{\mathcal{E}_1\mathcal{E}_2^* e^{-i[(\omega_1-\omega_2)t-(k_1-k_2)z]}\right\}\hat{z}.\end{aligned} \tag{C.6}$$

For the helicity density we find

$$\boldsymbol{A}^{\perp}\cdot\boldsymbol{B}=-\frac{k_1}{\omega_1^2}|\mathcal{E}_1|^2-\frac{k_2}{\omega_2^2}|\mathcal{E}_2|^2-\frac{k_1+k_2}{2\omega_1\omega_2}\mathcal{R}\left\{\mathcal{E}_1\mathcal{E}_2^* e^{-i[(\omega_1-\omega_2)t-(k_1-k_2)z]}\right\} \tag{C.7}$$

and thus

$$\begin{aligned}\frac{1}{\mu_0}\nabla\left(\boldsymbol{A}^{\perp}\cdot\boldsymbol{B}\right)&=\frac{1}{\mu_0}\frac{k_1^2-k_2^2}{2\omega_1\omega_2}\mathcal{I}\left\{\mathcal{E}_1\mathcal{E}_2^* e^{-i[(\omega_1-\omega_2)t-(k_1-k_2)z]}\right\}\hat{z}\\&=\epsilon\frac{\omega_1^2-\omega_2^2}{2\omega_1\omega_2}\mathcal{I}\left\{\mathcal{E}_1\mathcal{E}_2^* e^{-i[(\omega_1-\omega_2)t-(k_1-k_2)z]}\right\}\end{aligned} \tag{C.8}$$

which confirms the conservation equation (16).

Appendix D. Semi-local conservation law

We can get the semi-local conservation laws by integrating the terms in equation (7) over the volume V', on which surface the Dirac eigenfunctions ψ become zero. Using Gauss's theorem, the integral of the terms $\nabla(\psi^\dagger\gamma^5\psi)$ and $\nabla\cdot\mathcal{R}\{\bar{\psi}\boldsymbol{\gamma}(\boldsymbol{r}\times\boldsymbol{p})\psi\}$ vanish because they become surface integrals of functions of ψ. We therefore arrive at the semi-local conservation law

$$\frac{\partial\tilde{\boldsymbol{M}}_{\text{D}}}{\partial t}+\frac{\partial\tilde{\boldsymbol{M}}_{\text{em}}}{\partial t}+\tilde{\boldsymbol{J}}_{\text{A}}+\tilde{\boldsymbol{h}}+\int_{V'}\nabla\times(\boldsymbol{r}\mathfrak{N}_{\text{em}})\,\text{d}V'=0 \tag{D.1}$$

where $\tilde{\boldsymbol{M}}_{\text{D}}$ is the total AM of the electronic field given by

$$\tilde{\boldsymbol{M}}_{\text{D}}=\hbar\int_{V'}\left[\frac{1}{2}\left(\psi^\dagger\boldsymbol{\Sigma}\psi\right)+\mathcal{R}\left\{\psi^\dagger(\boldsymbol{r}\times\boldsymbol{p})\psi\right\}\right]\text{d}V', \tag{D.2}$$

$\tilde{\boldsymbol{M}}_{\text{em}}$ is the EM total AM in the volume V',

$$\tilde{\boldsymbol{M}}_{\text{em}}=\int_{V'}\epsilon\left(\boldsymbol{E}^{\perp}\times\boldsymbol{A}^{\perp}\right)\text{d}^3x+\int_{V'}\epsilon\left[\boldsymbol{E}^{\perp}\cdot(\boldsymbol{r}\times\nabla)\boldsymbol{A}^{\perp}\right]\text{d}^3x, \tag{D.3}$$

and

$$\tilde{\boldsymbol{J}}_{\mathrm{A}} = -\int_{V'} \nabla \cdot \left[\frac{1}{\mu_0}\boldsymbol{A}^{\perp}\boldsymbol{B} + \frac{1}{\mu_0}\boldsymbol{B}\times(\boldsymbol{r}\times\nabla)\boldsymbol{A}^{\perp} + \epsilon\boldsymbol{A}^{\perp}\left(\boldsymbol{r}\times\frac{\partial E^{\|}}{\partial t}\right)\right]\mathrm{d}V', \tag{D.4a}$$

$$\tilde{\boldsymbol{h}} = \int_{V'} \nabla\left(\boldsymbol{A}^{\perp}\cdot\boldsymbol{B}\right)\mathrm{d}V'. \tag{D.4b}$$

Using Gauss's theorem the volume integrals in equations (D.1), (D.4a) and (D.4b) can be converted into surface integrals. We find:

$$\tilde{\boldsymbol{J}}_{\mathrm{A}} = -\int_{S'} \hat{n}_i\left[A^{\perp,i}\boldsymbol{B} + \epsilon^{ij}{}_{k}B_j(\boldsymbol{r}\times\nabla)A^{\perp,k} + \epsilon A^{\perp,i}\left(\boldsymbol{r}\times\frac{\partial E^{\|}}{\partial t}\right)\right]\mathrm{d}a, \tag{D.5a}$$

$$\tilde{\boldsymbol{h}} = \int_{S'} \hat{\boldsymbol{n}}\left(\boldsymbol{A}^{\perp}\cdot\boldsymbol{B}\right)\mathrm{d}a, \tag{D.5b}$$

where $\hat{\boldsymbol{n}}$ is the unit vector normal to the surface of the volume V', and da its surface element. Equation (D.1) presents an equation for the time evolution of AM of the electronic field in terms of the EM fields.

Appendix E. Boost conservation relations

We started our derivation by applying the Noether's theorem to the Dirac-Maxwell fields under the Lorentz transformation. To derive the AM conservation equations, however, we only focused on the spatial rotations of the coordinates i.e. the space-space components of the AM tensor current $\mathcal{M}^{\mu\nu,\lambda}$. In this section, we look at the conservation equations for the time-space components of the Lorentz transformations, namely the boosts, of the Dirac–Maxwell fields.

Using a similar approach to the one used in the first section, the conservation equation resulting from the boost components of the Lorentz transformation is given by,

$$\begin{aligned}\partial_\lambda\mathcal{M}^{i0,\lambda} &= \partial_\lambda\left(\mathcal{M}_{\mathrm{D}}\right)^{i0,\lambda} + \partial_\lambda\left(\mathcal{M}_{\mathrm{E}}\right)^{i0,\lambda}\\ &= \frac{\partial}{\partial t}\mathcal{R}\left\{i\psi^{\dagger}\left[\frac{\boldsymbol{r}}{c}p_0 - ct\boldsymbol{p}^{\|}\right]\psi\right\} - c\nabla\times\left(\frac{\hbar}{2}\psi^{\dagger}\boldsymbol{\Sigma}\psi\right) + c\nabla.\mathcal{R}\left\{i\bar{\psi}\gamma\left[\frac{\boldsymbol{r}}{c}p_0 - ct\boldsymbol{p}^{\|}\right]\psi\right\}\\ &\quad - \epsilon\frac{\partial}{\partial t}\left[\boldsymbol{E}.\left(\frac{\boldsymbol{r}}{c}\partial_t + ct\nabla\right)\boldsymbol{A}^{\perp}\right] + \frac{1}{\mu_0}\nabla.\left[\boldsymbol{B}\times\left(\frac{\boldsymbol{r}}{c}\partial_t + ct\nabla\right)\boldsymbol{A}^{\perp}\right] + \frac{1}{\mu_0 c}\boldsymbol{B}\times\boldsymbol{E}^{\perp} + \frac{1}{\mu_0 c}\boldsymbol{E}\cdot(\nabla)\boldsymbol{A}^{\perp}\\ &\quad + c\rho\boldsymbol{A}^{\perp} - \left(\frac{\boldsymbol{r}}{c}\right)\boldsymbol{J}.\boldsymbol{E}^{\|} + ct\rho\boldsymbol{E}^{\|} + \epsilon\boldsymbol{E}.\left(\frac{\boldsymbol{r}}{c}\partial_t + ct\nabla\right)\boldsymbol{E}^{\|} + c\left(\frac{\boldsymbol{r}}{c^2}\partial_t + t\nabla\right)\mathcal{L}_{\mathrm{E}} = 0,\end{aligned} \tag{E.1}$$

where $p_0 = i\hbar\partial_t - e\phi$ and $\boldsymbol{p}^{\|} = -i\hbar\boldsymbol{\nabla} - e\boldsymbol{A}^{\|}$ are the gauge-independent time-derivative and momentum operators of the Dirac field, respectively. It is a matter of straightforward algebra to show that $\boldsymbol{r}\times\partial_\lambda\boldsymbol{M}^{0,\lambda} = \varepsilon_{ijk}x^j\partial_\lambda\mathcal{M}^{k0,\lambda}$ gives the conservation equation of the AM in equation (8).

Appendix F. Symmetrized angular momentum tensor

In our derivation, we have used the canonical form of the AM tensor which is derived directly from the application of Noether theorem to the QED Lagrangian in equation (1). The AM tensor can be written in terms of the canonical energy-momentum tensor, $T^{\mu\nu}$, as

$$\mathcal{M}^{\mu\nu,\lambda} = x^{\mu}T^{\lambda\nu} - x^{\nu}T^{\lambda\mu} + S^{\mu\nu,\lambda}. \tag{F.1}$$

Conservation of AM implies that, when $\partial_\lambda S^{\mu\nu,\lambda} \neq 0$,

$$T^{\mu\nu} \neq T^{\nu\mu}. \tag{F.2}$$

This in unpleasant because in general relativity, the energy-momentum tensor is directly proportional to the metric tensor which is symmetric in μ and ν. To overcome this problem, the energy momentum tensor can be modified to the so-called Bellifante–Resenfeld energy-momentum tensor as [22, 67]

$$T'^{\mu\nu} = T^{\mu\nu} + \frac{1}{2}\partial_\lambda\left(S^{\nu\lambda,\mu} + S^{\mu\lambda,\nu} - S^{\nu\mu,\lambda}\right). \tag{F.3}$$

It can be shown that this new energy-momentum tensor is symmetric and does not change the conservation law of the energy-momentum tensor, $\partial_\mu T^{'\mu\nu} = 0$. We can therefore write a new symmetrized AM tensor, $M^{'\mu\nu,\lambda}$ as

$$\mathcal{M}^{'\mu\nu,\lambda} = x^\mu T^{'\lambda\nu} - x^\nu T^{'\lambda\mu} \tag{F.4}$$

where we do not need to include the additional spin tensor because it is already present in the symmetric energy-momentum tensor. This symmetrized AM tensor is of course different from the canonical one we derived in equation (6). However, it is still gauge invariant and the conservation law of AM still holds. In other words,

$$\partial_\lambda \mathcal{M}^{'\mu\nu,\lambda} = \partial_\lambda \mathcal{M}^{\mu\nu,\lambda} = 0. \tag{F.5}$$

We can follow a similar procedure as we did in the previous section to derive the expression for the gauge-independent forms of the symmetrized AM tensor for the EM and Dirac fields. We find, setting $\mu, \nu = i, j$,

$$\frac{\partial \boldsymbol{M}'}{\partial t} + \nabla \cdot \overleftrightarrow{\boldsymbol{J}}' + \nabla \chi' - \nabla \times (\boldsymbol{r}\mathcal{U}_{\text{em}}) = 0 \tag{F.6}$$

where

$$\boldsymbol{M}' = \frac{\hbar}{2}\left(\psi^\dagger \boldsymbol{\Sigma} \psi\right) + \mathcal{R}\left\{\psi^\dagger \left(\boldsymbol{r} \times \boldsymbol{p}^{\|}\right)\psi\right\} + \epsilon_0\, \boldsymbol{r} \times (\boldsymbol{E} \times \boldsymbol{B}) - \rho\left(\boldsymbol{r} \times \boldsymbol{A}^{\perp}\right) \tag{F.7a}$$

$$\nabla \cdot \overleftrightarrow{\boldsymbol{J}}' = c\nabla_i \mathcal{R}\left\{\bar{\psi}\gamma^i \left(\boldsymbol{r} \times \boldsymbol{p}^{\|}\right)\psi\right\} + \epsilon_0\, \nabla_i E_i \left(\boldsymbol{r} \times \boldsymbol{E}\right) + \frac{1}{\mu_0}\nabla_i \left[B_i \left(\boldsymbol{r} \times \boldsymbol{B}\right)\right] - \nabla_i \left[J_i \left(\boldsymbol{r} \times \boldsymbol{A}^{\perp}\right)\right] \tag{F.7b}$$

$$\tilde{\chi} = \frac{\hbar c}{2}\left(\psi^\dagger \gamma^5 \psi\right) \tag{F.7c}$$

$$\mathcal{U}_{\text{em}} = \frac{1}{2}\left(\epsilon_0 \boldsymbol{E} \cdot \boldsymbol{E} + \frac{1}{\mu_0}\boldsymbol{B} \cdot \boldsymbol{B}\right). \tag{F.7d}$$

We emphasize that equation (F.6) is identical to the conservation equation (8). In fact, the terms related to the Dirac field are exactly the same as the one with the four-divergence of the canonical AM tensor. The main difference is that we lose separate physically observable expressions for the spin and OAM densities and currents of the EM field and instead we get expressions for the total AM,

$$\epsilon_0 \boldsymbol{r} \times (\boldsymbol{E} \times \boldsymbol{B}) - \rho\left(\boldsymbol{r} \times \boldsymbol{A}^{\perp}\right), \tag{F.8}$$

and total AM currents,

$$\epsilon_0\, \boldsymbol{E}(\boldsymbol{r} \times \boldsymbol{E}) + \frac{1}{\mu_0}\boldsymbol{B}(\boldsymbol{r} \times \boldsymbol{B}) - \boldsymbol{J}\left(\boldsymbol{r} \times \boldsymbol{A}^{\perp}\right), \tag{F.9}$$

of the EM field. Assuming that the symmetric energy-momentum tensor is more fundamental than the canonical one, however, does not prevent us from writing conservation laws that involve canonical AM tensor and taking its gauge-independent terms as physically meaningful quantities.

Appendix G. Meaning of the notations

Throughout this paper, the usual expressions for dot and cross product are assumed. Terms like $\boldsymbol{A}^{\perp}\boldsymbol{B}$ are tensorial expressions which can be expanded as

$$\begin{aligned}\boldsymbol{A}^{\perp}\boldsymbol{B} = {} & A_x^{\perp}B_x\hat{x}\hat{x} + A_x^{\perp}B_y\hat{x}\hat{y} + A_x^{\perp}B_z\hat{x}\hat{z} \\ & + A_y^{\perp}B_x\hat{y}\hat{x} + A_y^{\perp}B_y\hat{y}\hat{y} + A_y^{\perp}B_z\hat{y}\hat{z} \\ & + A_z^{\perp}B_x\hat{z}\hat{x} + A_z^{\perp}B_y\hat{z}\hat{y} + A_z^{\perp}B_z\hat{z}\hat{z}.\end{aligned} \tag{G.1}$$

Similar expressions can be written the terms like $\boldsymbol{E}(\boldsymbol{r} \times \boldsymbol{E})$, $\boldsymbol{E}(\boldsymbol{r} \times \boldsymbol{E})$, $\boldsymbol{J}(\boldsymbol{r} \times \boldsymbol{A}^{\perp})$, and so on. Therefore, the expression $\nabla \cdot (\boldsymbol{A}^{\perp}\boldsymbol{B})$ means

$$\begin{aligned}\nabla \cdot (\boldsymbol{A}^{\perp}\boldsymbol{B}) = {} & \left[\partial_x \left(A_x^{\perp}B_x\right) + \partial_y \left(A_y^{\perp}B_x\right) + \partial_z \left(A_z^{\perp}B_x\right)\right]\hat{x} \\ & + \left[\partial_x \left(A_x^{\perp}B_y\right) + \partial_y \left(A_y^{\perp}B_y\right) + \partial_z \left(A_z^{\perp}B_y\right)\right]\hat{y} \\ & + \left[\partial_x \left(A_x^{\perp}B_z\right) + \partial_y \left(A_y^{\perp}B_z\right) + \partial_z \left(A_z^{\perp}B_z\right)\right]\hat{x}\end{aligned} \tag{G.2}$$

which is a vector. We can similar expand

$$\begin{aligned}\boldsymbol{B}\times(\boldsymbol{r}\times\nabla)\boldsymbol{A}^{\perp}=&\left[B_y\left(y\partial_z-z\partial_y\right)A_z^{\perp}-B_z\left(y\partial_z-z\partial_y\right)A_y^{\perp}\right]\hat{x}\hat{x}\\&+\left[B_z\left(y\partial_z-z\partial_y\right)A_x^{\perp}-B_x\left(y\partial_z-z\partial_y\right)A_z^{\perp}\right]\hat{y}\hat{x}\\&+\left[B_x\left(y\partial_z-z\partial_y\right)A_y^{\perp}-B_y\left(y\partial_z-z\partial_y\right)A_x^{\perp}\right]\hat{z}\hat{x}\\&+\left[B_y\left(z\partial_x-x\partial_z\right)A_z^{\perp}-B_z\left(z\partial_x-x\partial_z\right)A_y^{\perp}\right]\hat{x}\hat{y}\\&+\left[B_z\left(z\partial_x-x\partial_z\right)A_x^{\perp}-B_x\left(z\partial_x-x\partial_z\right)A_z^{\perp}\right]\hat{y}\hat{y}\\&+\left[B_x\left(z\partial_x-x\partial_z\right)A_y^{\perp}-B_y\left(z\partial_x-x\partial_z\right)A_x^{\perp}\right]\hat{z}\hat{y}\\&+\left[B_y\left(x\partial_y-y\partial_x\right)A_z^{\perp}-B_z\left(x\partial_y-y\partial_x\right)A_y^{\perp}\right]\hat{x}\hat{z}\\&+\left[B_z\left(x\partial_y-y\partial_x\right)A_x^{\perp}-B_x\left(x\partial_y-y\partial_x\right)A_z^{\perp}\right]\hat{y}\hat{z}\\&+\left[B_x\left(x\partial_y-y\partial_x\right)A_y^{\perp}-B_y\left(x\partial_y-y\partial_x\right)A_x^{\perp}\right]\hat{z}\hat{z}\end{aligned}\tag{G.3}$$

and we can take its divergence similar to equation (G.2).

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