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Thermodynamics of Brans–Dicke–BTZ black holes coupled to conformal-invariant electrodynamics

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The field equations of Brans–Dicke conformal-invariant theory in (2+1)-dimensions are highly nonlinear and difficult to solve directly. They are related to those of Einstein–dilaton theory, where the solutions can be obtained easily, by use of a mathematical tool known as the conformal transformation. The exact solutions of three-dimensional Brans–Dicke theory, which are obtained from their Einstein-dilaton counterparts, give two novel classes of conformal-invariant black holes. When the scalar potential is absent (or is considered constant) in the action, it has been shown that the exact solution of this theory is just the conformal-invariant BTZ black hole with a trivial constant scalar field. This issue corresponds to the four-dimensional Brans–Dicke–Maxwell theory discussed in Ref. [R.-G. Cai, Y. S. Myung, Phys. Rev. D **56**, 3466 (1997)]. The Brans–Dicke conformal-invariant black holes' thermodynamic quantities have been calculated by use of the appropriator methods, and it has been shown that they satisfy the first law of black hole thermodynamics in its standard form. The thermal stability of Brans–Dicke black holes has been studied by use of the canonical ensemble method and noting the signature of the black holes' heat capacity.

Subject Index E01, E03

1. Introduction

Einstein's theory of gravity is the most successful theory of twentieth century. It can explain the dynamics of our Solar System with sufficient accuracy. The existence of gravitational waves, as an outstanding prediction of this theory, was confirmed in 2016 by LIGO and Virgo collaborations [1]. Now, advantages of the LIGO collaboration could be used for testing the validity of Einstein's general relativity and any other modified theories of gravity [2–5]. It is well known that our universe experiences an accelerated expansion phase at the present, which cannot be explained by Einstein's theory of relativity [6–8]. Moreover, it is not consistent with either Mach's principle or Dirac's large number hypothesis [9,10]. The aforementioned failures, and some other ones, signal that this theory may be incomplete, and a generalized theory of gravity is needed for resolving the related problems. Thus, various alternatives of gravitational theories such as the Lovelock gravity [11–14], the braneworld scenario [15–17], the $f(R)$ and $f(T)$ gravity theories [18–25], the quadratic curvature theories [26–29], and the scalar–tensor gravity theory [30–33], have been explored. Einstein's theory was initially extended to the well-known scalar–tensor theory by Brans and Dicke, in which the gravity is coupled to a scalar field (fields). The Brans–Dicke (BD) modified gravity theory includes both Mach's principle and Dirac's large number hypothesis [34]. Also, it can be viewed as the simplest extension which in one hand recovers Einstein's theory as the BD parameter ω grows to infinity, and coincides with the low energy limit of superstring theory on the other hand [35]. Various aspects of BHs and gravitational collapse in BD theory have been investigated by many authors [36–38]. Also, exact black hole (BH) solutions, related thermodynamic behavior and stability properties of higher-dimensional BD BHs have been studied in the presence of linear and non-linear theories of electrodynamics [39–43]. But three-dimensional BD BH solutions and related thermodynamic properties have not been studied up to now.

The present work is aimed to obtain the exact charged BH solutions of BD modified gravity theory in a three-dimensional spacetime with a conformal-invariant (CI) electrodynamics, and to investigate physical and thermodynamical properties the BD-CI BHs. This is motivated by the fact that, in comparison with the four- and higher-dimensional spacetimes, the lower dimensional models can lead to a deep understanding of the salient problems of quantum BHs [44,45]. In addition, according to the anti-de Sitter/conformal-invariant field theory (AdS/CFT) correspondence, there is a relationship between CI quantum field theory on one side and gravity theory in an AdS spacetime with lower dimension on the other. Thus, for a better understanding of quantum aspects of gravity in the four-dimensional universe, the gravity in lower-dimensional spacetimes can play important roles. In other words, from the quantum gravitational point of view, three-dimensional BHs are even more important than the spacetimes with dimensions equal to or more than four. Therefore, noting the fundamental importance of conformal invariance in constructing AdS/CFT correspondence and also in establishing conformal quantum gravity [46,47], we study impacts of this symmetry, via consideration of CI electrodynamics, on thermodynamic properties of BD BHs. Since BHs are systems with high energy levels, it is expected that such quantum corrected classical perspectives lead to more realistic advantages about thermal stability and thermodynamic behavior of the BHs [48,49].

This paper is outlined as follows: In Sect. 2, the field equations of three-dimensional BD-CI theory has been obtained by use of the variational principle on the related action. Since the field equations are strongly coupled and cannot be solved directly, they have been translated from the Jordan to the Einstein frame by use of a mathematical tool named as the conformal transformation (CT). In Sec. 3, the Einstein frame equations have be solved to obtain the exact BH solutions. As a special case, it has been shown that this theory, just like the 4D BD–Maxwell theory, reduces to the CI-BTZ BH with a trivial constant scalar field. In Sect. 4, the general exact solutions of three-dimensional BD theory have been obtained from their Einstein-dilaton correspondences by applying the inverse CT. By direct calculations, it has been shown that the BH charge and temperature of the BD and Einstein-dilaton are identical. Also, despite the fact that the entropy of BD BHs violates the entropy-area law, the Euclidean action method shows that BHs' mass, entropy and electric potential remain invariant under CTs. It has been shown that the first law of BH thermodynamics is valid for the BD BHs too. Then, stability of the BD-BTZ BHs has been investigated by use of the canonical ensemble method and noting the signature of the BHs' heat capacity. Some concluding remarks are summarized and discussed in Sect. 5.

2. The general formalism

The BD theory can be formulated in either Einstein or Jordan frames which are related via CTs. In the Jordan frame, the action is non-minimally coupled to a scalar field and the field equations are highly non-linear. By use of the CTs, the Jordan frame translates to the Einstein frame where the field equations are decoupled and can be solved more easily. Here, we start with the Jordan frame action of BD-BTZ BHs. That is [50,51],

$$I^{(BD)} = -\frac{1}{16\pi}\int \sqrt{-\tilde{g}}\, d^3x \left[\psi\tilde{\mathcal{R}} - U(\psi) - \frac{\omega}{\psi}\,\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\psi\tilde{\nabla}_\nu\psi + L(\tilde{\mathcal{F}})\right], \tag{2.1}$$

in which ψ, $\tilde{g}_{\mu\nu}$ and $\tilde{\mathcal{R}} = \tilde{g}^{\mu\nu}\tilde{\mathcal{R}}_{\mu\nu}$ are the scalar field, metric tensor and Ricci scalar of the BD spacetime, respectively. The BD parameter is denoted by ω, and $\tilde{\nabla}$ reveals the covariant derivative associated with the BD metric. The electromagnetic Lagrangian is considered as $L(\tilde{\mathcal{F}}) = \left(s\tilde{\mathcal{F}}\right)^p$, named as the power-law electrodynamics, in which $\tilde{\mathcal{F}} = \tilde{F}^{\alpha\beta}\tilde{F}_{\alpha\beta}$ is the Maxwell invariant with $\tilde{F}_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$, and $\tilde{F}^{\rho\lambda} = \tilde{g}^{\rho\alpha}\tilde{g}^{\lambda\beta}\tilde{F}_{\alpha\beta}$ [52,53]. Note that the case corresponding to $s = -1$ and $p = 1$ recovers the Maxwell's electromagnetic Lagrangian.

By applying variational principle on the action (2.1), the field equations of electromagnetic, gravitation and scalar are achieved as follows [54] :

$$\tilde{\nabla}_\alpha\left(L'(\tilde{\mathcal{F}})\tilde{F}^{\alpha\beta}\right) = 0, \tag{2.2}$$

$$\psi\left(\tilde{\mathcal{R}}_{\alpha\beta} - \frac{1}{2}\tilde{\mathcal{R}}\tilde{g}_{\alpha\beta}\right) - \left(\tilde{\nabla}_\alpha\tilde{\nabla}_\beta - \tilde{g}_{\alpha\beta}\tilde{\Box}\right)\psi = T^{(s)}_{\alpha\beta} + T^{(em)}_{\alpha\beta}, \tag{2.3}$$

$$2(\omega+2)\tilde{\Box}\psi = \psi U'(\psi) - 3U(\psi) + (3-4p)L(\tilde{\mathcal{F}}), \tag{2.4}$$

where $\tilde{\Box} = \tilde{\nabla}_\mu\tilde{\nabla}^\mu$, the prime symbol means the derivative relative to the argument, $T^{(s)}_{\alpha\beta}$ and $T^{(em)}_{\alpha\beta}$ are the energy-momentum tensor of scalar and electromagnetic fields [55,56], respectively, and

$$T^{(s)}_{\alpha\beta} = \frac{\omega}{\psi}\,\tilde{\nabla}_\alpha\psi\tilde{\nabla}_\beta\psi - \frac{1}{2}\left[U(\psi) + \frac{\omega}{\psi}\,(\tilde{\nabla}\psi)^2\right]\tilde{g}_{\alpha\beta}, \tag{2.5}$$

$$T^{(em)}_{\alpha\beta} = \frac{1}{2}L(\tilde{\mathcal{F}})\tilde{g}_{\alpha\beta} - 2L'(\tilde{\mathcal{F}})\tilde{F}_{\alpha\nu}\tilde{F}_\beta{}^\nu. \tag{2.6}$$

Since the field equations of BD-BTZ are highly nonlinear, the solutions can not be obtained easily. Thus, by use of a mathematical tool named as CT, we translate the Jordan frame action (2.1) to that of the well-known Einstein-dilaton theory where the field equations are decoupled and can be solved more easily [57,58]. The suitable form of the CT, in terms of a smooth function $\Omega(\psi)$, which is named as the conformal factor, can be written as [59–61]

$$\tilde{g}_{\mu\nu} \to \tilde{g}_{\mu\nu} = (\Omega(\psi))^2 g_{\mu\nu}, \tag{2.7}$$

where $g_{\mu\nu}$ is the Einstein frame metric tensor. By use of the CTs (2.7), one can show that

$$\tilde{\mathcal{R}} \to \tilde{\mathcal{R}} = \Omega^{-2}\mathcal{R} + 2\Omega^{-4}(\nabla\Omega)^2 - 4\Omega^{-3}\Box\Omega. \tag{2.8}$$

Now, making use of the CTs (2.7) in the BD action (2.1), we define

$$\Omega(\psi) = \psi^{-1}, \quad \text{and} \quad V(\phi) = U(\psi)\psi^{-3}. \tag{2.9}$$

By introducing the scalar field ϕ in the spacetime identified by $g_{\mu\nu}$ and assuming $\psi = \psi(\phi)$, after doing by-part integration on the term including $\Box\Omega$, we obtain [62]

$$\left(\frac{d\ln\Omega(\psi)}{d\psi}\right)^2 + \frac{\omega}{2\psi^2}\left(\frac{d\psi}{d\phi}\right)^2 = 1, \tag{2.10}$$

from which one can conclude that

$$\psi = e^{2\beta\phi}, \quad \text{with} \quad 2\beta = \pm\sqrt{\frac{2}{2+\omega}}. \tag{2.11}$$

Note that ω is restricted to $\omega > -2$, while in (3+1)-dimensional spacetimes it must be chosen as $\omega > -3/2$. By taking the limit $\omega \to \infty$ we have $\psi = 1$, and by consideration of $V(\psi = 1)$

$= constant = 2\Lambda$ the action (2.1) reduces to that of Einstein-Λ theory. Thus, the BD theory recovers Einstein-Λ theory as ω goes to infinity.

Also, for the electromagnetic Lagrangian by assuming $\tilde{F}_{\mu\nu} \to F_{\mu\nu}$, one can show that in general it transforms to $L(\mathcal{F}, \phi)$ such that

$$L(\mathcal{F}, \phi) = \psi^{-3} L\left(\psi^4 \mathcal{F}\right). \tag{2.12}$$

In our case, with power-law electrodynamics, we have $L(\mathcal{F}, \phi) = \psi^{-3}(s\psi^4\mathcal{F})^p = \psi^{4p-3}L(\mathcal{F})$. Thus by letting $p = 3/4$ the electromagnetic Lagrangian remains invariant (i.e. $L(\tilde{\mathcal{F}}) = L(\mathcal{F})$), and the electromagnetic energy–momentum tensor becomes traceless (i.e. $\tilde{g}^{\alpha\beta} T^{(em)}_{\alpha\beta} = 0$). Similar situations can be achieved in the higher dimensional spacetimes by letting $D = 4p$ [63]. Therefore, the action (2.1) takes the form [64,65]

$$I = -\frac{1}{16\pi} \int \sqrt{-g} d^3x \left[\mathcal{R} - V(\phi) - 2g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + L(\mathcal{F})\right], \tag{2.13}$$

which is nothing but the action of three-dimensional Einstein-dilaton theory in the presence of CI electrodynamics. Here, $\mathcal{R}$ and ∇ denote the Ricci scalar and covariant derivative in the Einstein frame with the metric $g_{\mu\nu}$. ϕ is the scalar field with a scalar potential $V(\phi)$. The Maxwell invariant $\mathcal{F} = F^{\mu\nu}F_{\mu\nu}$ is defined in terms of the Faraday's tensor $F_{\mu\nu}$ and gauge potential A_μ as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The Lagrangian of CI electrodynamics $L(\mathcal{F})$is written as [66]

$$L(\mathcal{F}) = (s\mathcal{F})^{\frac{3}{4}}. \tag{2.14}$$

Clearly it does not depend on the scalar field ϕ.

3. The BHs in the Einstein frame

By applying variational principle to the action (2.13), the equations of gravitational, electromagnetic and scalar fields can be achieved as follows:

$$\mathcal{R}_{\mu\nu} = V(\phi)g_{\mu\nu} + 2\partial_\mu\phi(r)\partial_\nu\phi(r) - g_{\mu\nu}(s\mathcal{F})^{\frac{3}{4}} + \frac{3}{2}s\left(\mathcal{F}g_{\mu\nu} - F_{\mu\sigma}F_\nu{}^\sigma\right)(s\mathcal{F})^{-\frac{1}{4}}, \tag{3.1}$$

$$\nabla_\mu\left[L'(\mathcal{F})F^{\mu\nu}\right] = 0, \tag{3.2}$$

$$4\Box\phi(r) = \frac{dV(\phi)}{d\phi}. \tag{3.3}$$

Here, we are interested in obtaining the exact solutions in a static and circularly symmetric spacetime. Thus we start with the following three-dimensional ansutz [67]

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2a^2(r)d\theta^2. \tag{3.4}$$

In this spacetime, the only non-vanishing component of the Faraday's tensor is $F_{tr} = E(r) = -A_t'(r)$, and we have $\mathcal{F} = -2F_{tr}^2$. Therefore, noting the fractional power of electromagnetic Lagrangian, to avoid imaginary solutions we fix the parameter s as $s = -1$. Also, in this spacetime various components of Eq. (3.1) are governed by the following differential equations:

$$e_{tt} \equiv f''(r) + \left[\frac{1}{r} + \frac{a'(r)}{a(r)}\right]f'(r) + 2V(\phi) - \frac{1}{2}(-\mathcal{F})^{\frac{3}{4}} = 0, \tag{3.5}$$

$$e_{rr} \equiv e_{tt} + 2f(r)\left[\frac{a''(r)}{a(r)} + \frac{2a'(r)}{ra(r)} + 2\phi'^2(r)\right] = 0, \tag{3.6}$$

$$e_{\theta\theta} \equiv \left[\frac{1}{r} + \frac{a'(r)}{a(r)}\right]f'(r) + \left[\frac{a''(r)}{a(r)} + \frac{2a'(r)}{ra(r)}\right]f(r) + V(\phi) + \frac{1}{2}(-\mathcal{F})^{\frac{3}{4}} = 0. \tag{3.7}$$

After some algebraic calculations, one can show that [66]

$$\frac{de_{\theta\theta}}{dr} = \left(\frac{1}{r} + \frac{a'(r)}{a(r)}\right)(e_{tt} - 2e_{\theta\theta}), \tag{3.8}$$

which signals that Eqs. (3.5) and (3.7) are not independent, and the solution of $C_{\theta\theta} = 0$ satisfies $C_{tt} = 0$. Thus, while we have five unknowns—$a(r)$, F_{tr}, $\phi(r)$, $V(\phi)$ and $f(r)$—there are only four unique equations, and we are confronted with the problem of indeterminacy [66]. For resolving this problem, as is customary, we proceed by use of an exponential ansatz function in the form of $a(r) = e^{\pm 2\alpha\phi}$. Then the solution of Eq. (3.6) is obtained as

$$\phi = \gamma \ln\left(\frac{b}{r}\right). \tag{3.9}$$

with positive b and $\gamma = \pm\alpha/(1+2\alpha^2)$. After replacing in Eq. (2.11) for ψ, as the physical scalar field, we have

$$\psi = \left(\frac{b}{r}\right)^{2\beta\gamma}. \tag{3.10}$$

Evidently, ψ must vanish at infinite distance from the source. Thus, by choosing positive (negative) sign for β in Eq. (2.11), γ must be positive (negative) too. It must be noted that by changing the sign of α, all of the quantities that have been calculated in the Einstein-dilaton theory remain unchanged, because they have appeared proportional to α^2. Hereafter, we consider both parameters γ and β with negative signs.

By solving Eq. (3.2), we obtain [66]

$$F_{tr} = q\, r^{-\frac{2}{1+2\alpha^2}}, \tag{3.11}$$

and thus we obtain the non-zero component of electromagnetic four-potential, that is,

$$A_t = \begin{cases} -q \ln\left(\frac{r}{\ell}\right), & \alpha^2 = \frac{1}{2}, \\ q\left(\frac{1+2\alpha^2}{1-2\alpha^2}\right) r^{\frac{2\alpha^2-1}{2\alpha^2+1}}, & \alpha^2 < \frac{1}{2}. \end{cases} \tag{3.12}$$

At this stage, we consider the special case corresponding to $V(\phi) = 0$ or $V(\phi) = 2\Lambda = constant$. It reduces the action (2.1) to the original one considered by Cai and Myung in Ref. [68] for the four-dimensional BD theory with CI/ Maxwell electrodynamics. They proved that its exact solution is just the RN BHs with a trivial constant scalar field $\psi = 1$.

First, it must be noted that the relation (3.8) is still valid for this special case. Then the scalar field Eq. (3.3) becomes source-free and, by use of the ansatz $a(r) = e^{-2\alpha\phi}$ in Eqs. (3.3) and (3.7), we have

$$\frac{\gamma}{r}\left(f'(r) + \frac{2\alpha\gamma}{r} f(r)\right) = 0, \tag{3.13}$$

$$\frac{1+2\alpha\gamma}{r}\left(f'(r) + \frac{2\alpha\gamma}{r} f(r)\right) + 2\Lambda + \frac{1}{2}(-\mathcal{F})^{\frac{3}{4}} = 0. \tag{3.14}$$

These equations are compatible only for $\gamma = 0 = \alpha$, and noting (3.9), $\phi = 0$. Therefore, Eq. (2.11) yields $\psi = 1$ and the Jordan frame action (2.1) reduces to that of Einstein-Λ-CI theory with the metric function [66]

$$f(r) = -m - \Lambda r^2 + \frac{(2q^2)^{\frac{3}{4}}}{2r}. \tag{3.15}$$

This means that the BD-BTZ theory, in the absence of scalar potential $U(\psi)$, is just the Einstein-Λ-CI theory with a trivial scalar field $\psi = 1$. A similar result has been reported for the four-dimensional BD–Maxwell theory in Ref. [68].

Now, we return to the general solution of the field equations in the Einstein-dilaton theory. Hereafter we continue with $a(r) = e^{-2\alpha\phi}$ and $\gamma = -\alpha/(1+2\alpha^2)$. The scalar potential $V(\phi)$ and the metric function $f(r)$ are obtained as [66]

$$V(\phi) = 2\Lambda e^{-4\alpha\phi} + 2\Lambda_0 e^{-3\alpha_0\phi}, \qquad \alpha^2 \leq \frac{1}{2} \tag{3.16}$$

where

$$\alpha_0 = \alpha^{-1}, \qquad \Lambda_0 = \frac{\alpha^2 \left(2q^2\right)^{\frac{3}{4}}}{(3-4\alpha^2)\, b^{\frac{3}{1+2\alpha^2}}}, \tag{3.17}$$

$$f(r) = \begin{cases} -mr^{\frac{1}{2}} - 8\Lambda br - 3\left(2q^2\right)^{\frac{3}{4}} r^{\frac{1}{2}} \ln\left(\frac{r}{\ell}\right), & \alpha^2 = \frac{1}{2}, \\ -m\, r^{-2\alpha\gamma} - \left(1+2\alpha^2\right)^2 \left[\frac{\Lambda b^2}{1-\alpha^2}\left(\frac{r}{b}\right)^{4\alpha\gamma+2} + \frac{3\Lambda_0 b^2}{2\alpha^2\left(2\alpha^2-1\right)}\left(\frac{r}{b}\right)^{3\alpha_0\gamma+2}\right], & \alpha^2 < \frac{1}{2}. \end{cases} \tag{3.18}$$

4. Thermodynamics of the Jordan frame BHs

With the exact BH solutions of Einstein-dilaton gravity theory, we are in the position to introduce our novel BD-BTZ BHs by applying inverse CTs. Then, we explore their thermodynamic and stability properties. To this end, we start with the following line element as the solution to the field equations of three-dimensional BD theory [69]:

$$d\tilde{s}^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu = -A(r)dt^2 + \frac{1}{B(r)}dr^2 + r^2C^2(r)d\theta^2. \tag{4.1}$$

The functions $A(r)$, $B(r)$, and $C(r)$ can be determined by use of the corresponding quantities in the Einstein frame. Regarding Eqs. (2.7), (2.9) and (3.10) we have $\tilde{g}_{\mu\nu} = g_{\mu\nu}\,(b/r)^{-4\beta\gamma}$, and

$$A(r) = \left(\frac{b}{r}\right)^{-4\beta\gamma} f(r), \tag{4.2}$$

$$B(r) = \left(\frac{b}{r}\right)^{4\beta\gamma} f(r), \tag{4.3}$$

$$C(r) = \left(\frac{b}{r}\right)^{-2\beta\gamma} a(r). \tag{4.4}$$

Here, $a(r) = e^{-2\alpha\phi}$, and $f(r)$ was presented in Eq. (3.18). The plots of $B(r)$ versus r are shown in Fig. 1 to illustrate the existence of event horizons. The plots show that our BD-BTZ BH solutions can produce extreme, two-horizon and horizon-less ones with appropriate choices of the parameters.

Now, we study thermodynamic properties of the BD-BTZ BHs. Thus, we have to calculate the mass, charge, temperature, entropy and electric potential of the BD-BTZ BHs.

The BH temperature which is defined via the BH's surface gravity can be calculated as [70,71]

$$\tilde{T} = \frac{1}{4\pi}\left(\sqrt{\frac{B(r)}{A(r)}}\frac{dA(r)}{dr}\right)_{r=r_+} = \frac{1}{4\pi}\left(\frac{b}{r_+}\right)^{4\beta\gamma}\frac{d}{dr}\left[\left(\frac{b}{r}\right)^{-4\beta\gamma} f(r)\right]_{r=r_+}$$
$$= \frac{1}{4\pi}\frac{df(r)}{dr}\Big|_{r=r_+} = T. \tag{4.5}$$

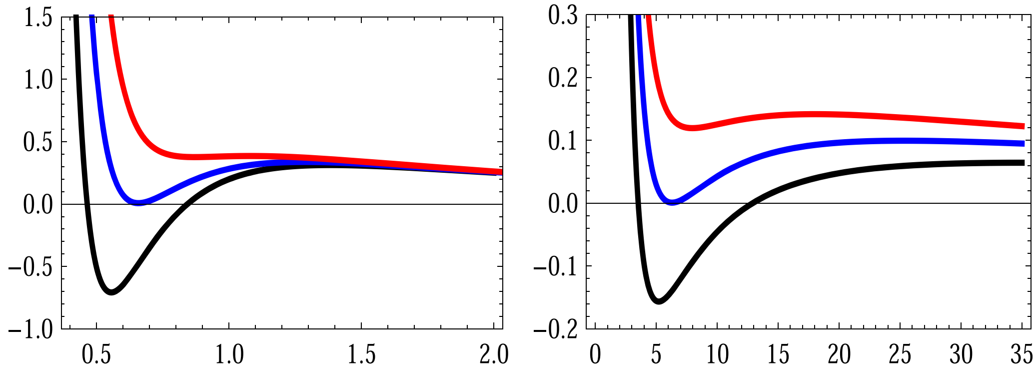

Fig. 1. $B(r)$ versus r: Left: $B(r)$, $\Lambda = -1, m = 2.1, b = 1, \alpha = 0.25, \beta = -4, q = 0.8$ (black), 0.846 (blue), 0.9 (red), Right: $B(r)$, $\Lambda = -1, m = 2.3, \ell = 1, q = 2, \beta = -1.5, \alpha^2 = 0.5, b = 1.35$ (black), 1.423 (blue), 1.48 (red).

Note that we have used the relation $f(r_+) = 0$ in the last step. Thus, the BH temperature remains invariant under CTs, and takes the following explicit form [66]:

$$T = -\frac{1 + 2\alpha^2}{4\pi r_+}\left[2\Lambda b^2\left(\frac{r_+}{b}\right)^{4\alpha\gamma+2} + \frac{3\Lambda_0 b^2}{2\alpha^2}\left(\frac{r_+}{b}\right)^{3\alpha_0\gamma+2}\right], \qquad \alpha^2 \leq \frac{1}{2}. \tag{4.6}$$

The electric charge of BD-BTZ BHs can be calculated by use of the Gauss's electric law and noting Eq. (2.2). That is [72],

$$\begin{aligned}\tilde{Q} &= \frac{1}{4\pi}\int \sqrt{-\tilde{g}}\, \partial_{\tilde{\mathcal{F}}}(-\tilde{\mathcal{F}})^{\frac{3}{4}}\frac{B(r)}{A(r)}F_{tr}d\theta,\\ &= \frac{3}{16\pi}\int rC(r)\sqrt{\frac{B(r)}{A(r)}}\left(2\frac{B(r)}{A(r)}F_{tr}^2\right)^{-\frac{1}{4}}F_{tr}d\theta,\end{aligned} \tag{4.7}$$

and noting Eqs. (3.11) and (4.2)–(4.4), we have

$$\tilde{Q} = \frac{3b^{-2\alpha\gamma}}{8\sqrt[4]{2}}q^{\frac{1}{2}} = Q, \tag{4.8}$$

which is just the same as obtained for the Einstein frame BHs. The results of Eqs. (4.5) and (4.8) show that BH charge and Hawking temperature remain unchanged under CTs. The BH entropy, mass and electric potential are the other quantities which need to be calculated for the BD-CI BHs. It is well known that the entropy of BD BHs cannot be calculated by use of the entropy-area law [68,73]. However, following the works of Refs. [42,43], the BH entropy, mass and electric potential can be calculated by use of the Euclidean action method. As a matter of calculation, one can show the aforementioned quantities remain invariant under CTs [42,43]. That is,

$$\tilde{S}(r_+) = S(r_+), \qquad \tilde{\Phi}(r_+) = \Phi(r_+), \qquad \tilde{M}(r_+, q) = M(r_+, q). \tag{4.9}$$

Note that the explicit form of these quantities have been presented in Ref. [66]. They are

$$\tilde{S} = \frac{\pi r_+ a(r_+)}{2} = \frac{\pi r_+}{2}e^{-2\alpha\phi}, \tag{4.10}$$

$$\tilde{\Phi}(r_+) = \begin{cases} -cq\ln\left(\frac{r_+}{\ell}\right), & \text{for} \quad \alpha^2 = \frac{1}{2},\\[2ex] cq\left(\frac{1+2\alpha^2}{1-2\alpha^2}\right)r_+^{\frac{2\alpha^2-1}{2\alpha^2+1}}, & \text{for} \quad \alpha^2 < \frac{1}{2},\end{cases} \tag{4.11}$$

with $c = 3/(3-4\alpha^2)$, and

$$\tilde{M} = \frac{mb^{-2\alpha\gamma}}{8\left(1+2\alpha^2\right)}. \tag{4.12}$$

By imposing the condition $B(r_+) = 0$, after obtaining $m(r_+)$ and replacing in Eq. (4.12), the Smarr mass formula is obtained as

$$\tilde{M}(r_+, q) = \begin{cases} -\frac{\sqrt{b}}{16}\left[8\Lambda b r_+^{\frac{1}{2}} + 3\left(2q^2\right)^{\frac{3}{4}}\ln\left(\frac{r_+}{\ell}\right)\right], & \alpha^2 = \frac{1}{2}, \\ -\frac{1+2\alpha^2}{8}\left(\frac{r_+}{b}\right)^{2\alpha\gamma}\left[\frac{\Lambda b^2}{1-\alpha^2}\left(\frac{r_+}{b}\right)^{4\alpha\gamma+2} + \frac{3\Lambda_0 b^2}{2\alpha^2\left(2\alpha^2-1\right)}\left(\frac{r_+}{b}\right)^{3\alpha_0\gamma+2}\right], & \alpha^2 < \frac{1}{2}. \end{cases} \tag{4.13}$$

Since q and r_+ are functions of $\tilde{Q}$ and $\tilde{S}$, it can be regarded as a function of these quantities. An immediate result is that

$$\tilde{\Phi} = \frac{\partial \tilde{M}(\tilde{S}, \tilde{Q})}{\partial \tilde{Q}}, \qquad \text{and} \qquad \tilde{T} = \frac{\partial \tilde{M}(\tilde{S}, \tilde{Q})}{\partial \tilde{S}}. \tag{4.14}$$

This proves that the first law of black thermodynamics is valid for our new BD-BTZ BHs in the following form:

$$d\tilde{M}(\tilde{S}, \tilde{Q}) = \tilde{T}d\tilde{S} + \tilde{\Phi}d\tilde{Q}. \tag{4.15}$$

Thermal stability of the BD-BTZ BHs can be studied by use of the canonical ensemble method and noting the signature of heat capacity. The BH heat capacity can be calculated by use of the following definition [74]:

$$\tilde{C}_{\tilde{Q}} = \tilde{T}\left(\frac{\partial \tilde{S}}{\partial \tilde{T}}\right)_{\tilde{Q}} = \frac{\tilde{T}}{\tilde{M}_{\tilde{S}\tilde{S}}}, \qquad \text{with} \quad \tilde{M}_{\tilde{S}\tilde{S}} = \left(\frac{\partial^2 \tilde{M}}{\partial \tilde{S}^2}\right)_{\tilde{Q}}. \tag{4.16}$$

The physical BHs with positive heat capacity are locally stable. Those with zero heat capacity undergo first-order phase transition, and vanishing points of $\tilde{M}_{\tilde{S}\tilde{S}}$, or divergent points of $\tilde{C}_{\tilde{Q}}$, are the horizon radius of those BHs which experience second-order phase transition [75,76]. Therefore, we have to calculate the BH heat capacity by use of the relation (4.16). It can be shown that

$$\tilde{C}_{\tilde{Q}} = -\frac{\pi r_+(\Lambda + \Gamma)}{2\left[\left(2\alpha^2-1\right)\Lambda + 2\left(1-\alpha^2\right)\Gamma\right]}\left(\frac{r_+}{b}\right)^{2\alpha\gamma}, \qquad \alpha^2 \leq \frac{1}{2}. \tag{4.17}$$

Here, we have used the definition $\Gamma = [(3\Lambda_0)/(4\alpha^2)]\left(r_+/b\right)^{(3-4\alpha^2)\alpha_0\gamma}$ for simplicity. Noting the allowed values of α, in an AdS background with $\Lambda < 0$, the denominator of Eq. (4.17) has a positive value. Therefore, there is no divergence point for $\tilde{C}_{\tilde{Q}}$ and no second-order phase transition occurs. while the nominator of $\tilde{C}_{\tilde{Q}}$ vanishes at $r_+ = b\left[(4\alpha^2\Lambda)/(-3\Lambda_0)\right]^{1/[(3-4\alpha^2)\alpha_0\gamma]}$. It is a point of first-order phase transition. Diagrams of $\tilde{C}_{\tilde{Q}}$ and $\tilde{T}$ are depicted in Fig. 2 for a complete stability analysis of the BD-BTZ BHs. The plots show that there is only one point of first-order phase transition located at $r_+ = r_{ext}$ (r_{ext} is the horizon radius of extreme BHs), and non-extreme BHs with horizon radii greater than r_{ext} are locally stable. The BHs with horizon radii smaller than r_{ext} have negative temperatures, and are not physically reasonable. They maybe named as unphysical BHs.

5. Conclusion

We studied the exact BH solutions of the three-dimensional BD-CI gravity theory and their thermodynamic properties. To this end, we obtained the related field equations by applying the

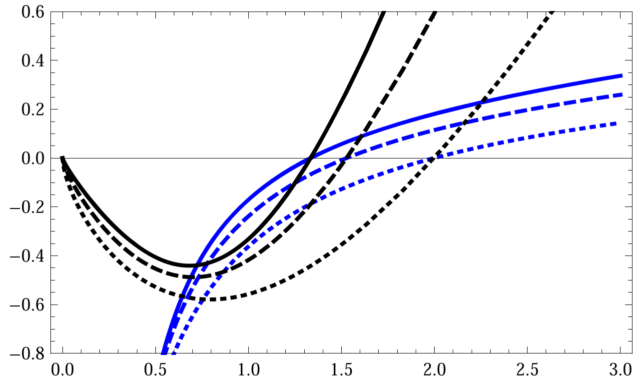

Fig. 2. $\tilde{C}_{\tilde{Q}}$ (black) and $\tilde{T}$ (blue) versus r_+: $\Lambda = -1, b = 1, Q = 0.5, \alpha = 0.3$ (solid line), 0.4 (dashed), 0.5 (dotted).

variational principle on the appropriate action of this theory. Because of the nonlinearity and strong coupling of the Jordan frame scalar and gravitational field equations, the exact solutions cannot be obtained easily. To overcome this problem, we have obtained the corresponding equations in the Einstein-dilaton theory by use of a set of CTs. The spacetime identified with the metric tensor $\tilde{g}_{\mu\nu}$ is known as the Jordan frame, and its conformal related one with the metric $g_{\mu\nu}$ is conventionally named as the Einstein frame. The CT decouples the field equations and they can be solved easily in the Einstein frame, where the theory is named as the Einstein-dilaton gravity theory. Through this procedure, we have introduced a three-dimensional CI electromagnetic Lagrangian inspired by power-law nonlinear electrodynamics. It has been found that the special case corresponding to the absence or constancy of scalar potential in the action leads to the CI-BTZ BHs with a constant trivial scalar field. This result is the three-dimensional correspondence of that obtained in the four-dimensional BD–Maxwell theory considered in Ref. [68]. The BH solutions of Einstein-dilaton theory have been obtained in its general form (i.e. in the presence of an arbitrary self-interacting scalar potential) and it has been shown that the exact BH solutions are asymptotically neither flat nor A(dS). This means that inclusion of scalar field affects the asymptotic behavior of the solutions.

The exact solutions of Jordan frame equations have been extracted by use of the inverse CTs, and noting their Einstein frame correspondence. It has been shown that the BD-BTZ BHs, introduced here, can reveal two-horizon, extreme and horizon-less ones if the parameters are fixed properly. Although the BH entropy violates entropy–area law, it has been found that the thermodynamic and conserved quantities of the BD-CI BHs are just those obtained for the Einstein-dilaton ones. It means that the BHs' conserved and thermodynamic quantities remain invariant under CTs. Through a Smarr-type mass formula, which gives the BH mass as a function of charge and entropy, it has been proved that the first law of BH thermodynamics is valid for the BD BHs too. Thermal stability of the BD-CI BHs has been analyzed by use of the canonical ensemble method and signature of BHs' heat cthe apacity. The results show that there is only one point of first-order phase transition which occu,rs at $r_+ = r_{ext}$. The BH heat capacity does not diverge, thus no second-order phase transition can take place for the BD BHs. The larger BHs with horizon radii greater than r_{ext} are locally stable, while the smaller ones are unstable. The aforementioned stability properties are just like those already obtained for three-dimensional Einstein-dilaton-CI BHs, which reveals conformal invariance of these properties.

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