THE LATENT MEANING OF FORCING IN QUANTUM MECHANICS*

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We analyze random forcing in QM from the dual perspective of the measure and category correspondence. The dual Cohen forcing allows interpreting the real numbers in a model M and its Cohen extension $M[G]$ as absolute subtrees of the binary tree (Cantor space). The trees are spanning non-trivial Casson handles of smooth exotic 4-manifolds, like $\mathbb{R}^4$. We formulate the consequences for the cosmological model with random forcing where dual smooth non-standard and non-flat Riemannian geometries have to appear.

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1. Introduction

Searching for physics beyond the Standard Model (SM) of particle physics is a far-reaching project. Even though supersymmetry signals have not been found at the LHC data yet, the expectation is to unveil partially the nature of dark matter (DM). The tremendous effort of experimentalists at micro-scales will allow better understanding of the deep mysteries of the expanding Universe at the cosmological scales. However, this fascinating scenario has to be augmented by substantial activity of theorists. The direct reason is that we still do not have any satisfactory theory of quantum gravity and

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cosmology has to deal with some fundamental issues like cosmological constant (CC) problem or the nature of dark energy (DE). Both these problems are touched in this paper, however the method we follow differs from usually applied. Namely, we explore the formalism of quantum mechanics (QM) and its relation to mathematics of general relativity (GR) from the point of view of some deep results in set theory and differential topology of dimension four. It appears that the mathematical structure of the line of real numbers $\mathbb{R}$, that essentially is in use in GR (but also in QM), is highly non-trivial, which indicates the new and fundamental relation between both theories [1]. This mathematical fact, without deforming QM and GR into entirely new theories, helps to understand some subtleties in the relation between QM and GR, and also sheds some new light on CC and DE problems.

As shown in [2], the semiclassical state in the local hidden variables (LHV) program (if it exists) has to be given by a generic filter in an atomless Boolean algebra $\mathcal{B}$ containing pairwise commuting self-adjoint operators made of QM projections [3]. It was analyzed and explained in detail in our previous work [1], where we showed (*without* assumptions of LHV) that there indeed exists a generic filter in the measure algebra on $\mathbb{R}$ when passing from (quantum) micro-scales to the (classical) large-scale regimes. A measure algebra is *not* absolute in countable transitive models (CTMs) of Zermelo–Fraenkel set theory (ZF) which gives dependence on the choice of such model (see *e.g.* [4]).

In the paper, we dualize this measure algebra into the Cohen algebra and analyze its meaning in the cosmological model where the change of ZFC (ZF with the axiom of choice) models is allowed [1]. We find unexpectedly that Cohen forcing, adding Cohen reals to a CTM of ZFC, is represented by deformed trees which correspond to Casson handles known from 4-dimensional topology [5, 6]. The appearance of Casson handles shows that exotic smooth Riemannian geometry on $\mathbb{R}^4$ is an important complementation of the model of the Universe with forcing. Such smooth exotic $\mathbb{R}^4$s cannot be flat (Riemannian tensor cannot vanish totally, though they are topologically like $\mathbb{R}^4$) and it can have some gravitational impact. Indeed, it was shown recently that they generate realistic cosmological parameters, like the value of CC [7] (see also [8]). The connection between forcing, Casson handles, and exotic 4-smoothness was already studied some time ago (see *e.g.* [9–11]).

We explain some symbols from set theory since they are not in common use within the physics community. Symbol 2^ω denotes the set of all subsets of the natural numbers $\mathbb{N}$ and ω is the order type of $\mathbb{N}$. 2^ω can be also seen as a set of real numbers $\mathbb{R}$. For a CTM M of ZFC, $M[G]$ is its generic extension by a forcing and $\mathbb{R}_M$ denotes the real line in M. Note that if M is countable, then $\mathbb{R}_M$ is countable too (when seen from the outside of M).

2. Dual Cohen forcing vs. random forcing

The duality of measure and category (or null sets and 1st Baire category sets) is the classic and well-studied phenomenon in mathematics (*e.g.* [12]). Subsets of $\mathbb{R}$ of Lebesgue measure 0 are null sets and they constitute the ideal $\mathcal{N}$, whereas meager (1st Baire category) sets are countable unions of nowhere-dense sets and constitute the ideal $\mathcal{M}$. The Boolean algebra (BA) $\mathrm{Bor}(2^\omega)/\mathcal{N}$ is the measure algebra (MA) and the $\mathrm{Bor}(2^\omega)/\mathcal{M}$ is the Cohen algebra (CA). Here, $\mathrm{Bor}(2^\omega)$ is the σ-algebra of Borel subsets of $\mathbb{R} \simeq 2^\omega$. Both MA and CA are atomless BAs, but CA is the unique (up to isomorphism) atomless BA that has a countable dense subset [13]. As atomless BAs they serve as forcing algebras for the Boolean-valued models of ZFC. In [1], we worked out the physical meaning of the forcing based on MA, namely the forcing is ever-present when passing from QM regime (though in Boolean contexts) to the classical macro-scales of GR. Thus, one refers to different real lines in Boolean contexts of QM and in GR. Moreover, the inner measure of $\mathbb{R}_M$ in the generic extension $M[G]$ is 0, while the outer measure is 1 (full measure) [4]. Let us dualize the random forcing into the Cohen forcing according to the following Erdős–Sierpiński theorem [12, p. 75]. Assuming the continuum hypothesis, it holds that:

Theorem 1 (Sierpiński 1934, Erdős 1943)**.** *There exists a one-to-one mapping f of $\mathbb{R}$ onto itself such that $f = f^{-1}$ and such that $f(E)$ is a null set if and only if E is meager. And conversely, $f(E)$ is meager if and only if E is a null set.*

$\mathcal{M}$ is switched to $\mathcal{N}$, but also $f\colon \mathrm{Bor}(\mathbb{R}) \to \mathrm{Bor}(\mathbb{R})$ and hence $f\colon \mathrm{MA} \to \mathrm{CA}$. From the point of view of physics, f is not a local transformation (a differentiable function of spacetime coordinates). It is rather the choice of a complementary mathematical formalism. Thus, one has Cohen or random forcing pictures. Choosing the Cohen picture, we have the immediate reformulation of the result in [1]:

Lemma 1. *$\mu_{M[G]}(\mathbb{R}_M) = 0$ in $M[G]$, where M and $M[G]$ are a CTM model and its Cohen generic extension, respectively, and $\mu_{M[G]}(\mathbb{R}_M)$ is the Lebesgue measure of $\mathbb{R}_M$ in $M[G]$.*

Since $\mathbb{R}_M \subset \mathbb{R}_{M[G]} \subset \mathbb{R}$ and the set $\mathbb{R}_M$ is dense in $\mathbb{R}_{M[G]}$ (and hence in $\mathbb{R}$), we consider the quantity $\mu_{M[G]}(\mathbb{R}_M)$ as the measure of density of $\mathbb{R}_M$ in $\mathbb{R}_{M[G]}$. Note that since models M and $M[G]$ are countable, then the densities of $\mathbb{R}_M$ and $\mathbb{R}_{M[G]}$ in $\mathbb{R}$ also vanish (*i.e.* $\mu_{\mathbb{R}}(\mathbb{R}_M) = \mu_{\mathbb{R}}(\mathbb{R}_{M[G]}) = 0$). Following [1], we assume that the density of $\mathbb{R}_M$ in $\mathbb{R}_{M[G]}$ is the measure of density of zero-modes of quantum fields in the cosmological model where the change $M \to M[G]$ is assumed. The change of models switches the presentation of the real line without affecting the physical fields ($\sim$ no new interactions).

Theorem 2. *In the dual Cohen picture, the density of zero-modes of the quantum fields of particles of mass m vanishes in the extended model $M[G]$.*

Proof. The statement follows from the direct calculation of the integrals

$$\frac{E}{V} = \int\limits_{\mathbb{R}^3_M} \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{\sqrt{\boldsymbol{k}^2 + m^2}}{2}\,, \qquad m \in \mathbb{R}_{M[G]}\,, \qquad \boldsymbol{k} \in \mathbb{R}^3_M\,, \tag{1}$$

which all vanish, since we integrate over the null set $\mathbb{R}^3_M \subset \mathbb{R}^3_{M[G]}$. □

3. Casson handles, Cantor sets and Cohen forcing

The set of all subsets of $\mathbb{N}$, *i.e.* $2^\omega = \{0,1\}^\omega$ is the real line [4]. It is the Cantor space since there exists a homeomorphism of 2^ω onto the standard Cantor set (CS). One can represent 2^ω by an infinite binary tree (BT). Taking this binary tree as CS and noting that every real number is represented in 2^ω by some infinite branch, we ask the question: which real numbers from 2^ω are still present in $2^\omega|_M$ (real numbers in the model M)? Another question is which real numbers are in $2^\omega|_{M[G]}$, $2^\omega|_M$ and 2^ω? We can compare all these sets, since

$$\mathbb{R}_M \subset \mathbb{R}_{M[G]} \subset \mathbb{R}\,, \qquad \mathbb{R}_M = 2^\omega|_M\,, \qquad \mathbb{R}_{M[G]} = 2^\omega|_{M[G]}\,, \qquad \mathbb{R} = 2^\omega\,.$$

Moreover, for CTMs M and $M[G]$, the subsets $\mathbb{R}_M$ and $\mathbb{R}_{M[G]}$ of 2^ω are countable. In M and $M[G]$, these sets of real numbers are 'internal' trees. Are they still infinite subtrees of 2^ω as seen outside of the model? The answer is *yes* and comes from the following absoluteness result:

Lemma 2 ([14])**.** *Let M be a transitive model of ZFC. If T is a tree in M, then T has an infinite branch in M if and only if it has an infinite branch.*

Thus, the change of reals corresponding to the change of CTMs is coded in the structure of infinite trees, seen as model-independent absolute trees. For countable models, the trees are infinite (length of branches) and with countably many branches. But we know more: the set of rational numbers $\mathbb{Q}$ is a dense subtree of each $2^\omega|_M$, $2^\omega|_{M[G]}$, and 2^ω. How does the tree generated by the rational numbers look like? Its branches are infinite *eventually zero* sequences of 0s and 1s, meaning that if $\langle a_i | i \in \mathbb{N}\rangle$ is such a sequence, then there exists $k \in \mathbb{N}$ such that $\forall_{i>k}\ a_i = 0$. Since all rational numbers are already present in every M, the reals added by a forcing are irrational. In fact, added numbers are *transcendental* (*i.e.* not algebraic). The trees corresponding to $\mathbb{R}_M$ and $\mathbb{R}_{M[G]}$ differ as subtrees of 2^ω by branches which represent such transcendental numbers. The corresponding deformations of the trees, representing reals in the models M and $M[G]$, are, in general, hard to describe.

Let us now turn to Casson handles. They are represented by an infinite $(+,-)$-signed trees. Such trees are created due to the infinite geometric constructions [9, 15]. The important fact is the following: *given any infinite tree, one can always build a Casson handle (CH) corresponding to it* [16]. So the deformed trees representing real numbers in M and $M[G]$ generate Casson handles. For *small exotic* $\mathbb{R}^4$*s*, there always exist infinite Casson handles in their handle-decompositions [5]. The question is whether the deformed trees as above give rise (or not) to non-diffeomorphic smoothness structures on $\mathbb{R}^4$ via Casson handles representing them [17]. The trees T_1, T_2 corresponding to CHs are *smoothly deformed* whenever none of T_1, T_2 is homeomorphic to any embedded subtree of T_1, T_2 [17]. In such a case, the smoothly deformed trees represent non-diffeomorphic CHs. In general, the following scenarios are possible: *(I)* at least for some CTMs M of ZFC and their Cohen extensions $M[G]$, the trees $2^\omega|_M$, $2^\omega|_{M[G]}$, 2^ω are pairwise smoothly deformed. *(II)* the deformations represent the trees of non-diffeomorphic CHs which, in turn, correspond to the pairwise non-diffeomorphic smooth exotic small $\mathbb{R}^4$s. The opposite possibility is the following: even though *(I)* holds (or instead *(I')* is true: the trees $2^\omega|_M$, $2^\omega|_{M[G]}$, 2^ω are pairwise smoothly undeformed), the *(II')* holds: there is no change of exotic smoothness on $\mathbb{R}^4$. The smooth structure on $\mathbb{R}^4$ still can be the standard one, even though CH in the handle-body is exotic. We do not decide these claims here, instead, we discuss some of their consequences for the cosmological model with forcing. Applying *(I)* and *(II)* one has:

Corollary 1. *(i) Suppose that the evolution of the Universe begins with the standard smoothness of* $\mathbb{R}^4$*. The presence of forcing at some stage of the evolution changes the smoothness into a small exotic one. (ii) Suppose that the evolution begins with some small exotic* $\mathbb{R}^4$*. Then, the smooth evolution requires large exotic* $\mathbb{R}^4$*.*

(ii) follows from the fact that there exists universal large exotic $\mathbb{R}^4$ containing all other exotics [5]. The conditions *(I')* and *(II')* lead to:

Corollary 2. *(a) The evolution of the Universe begins with exotic smoothness of* $\mathbb{R}^4$*. Then, the forcing does not change the smoothness on* $\mathbb{R}^4$ *which remains exotic. (b) The evolution of the Universe begins with the standard smoothness of* $\mathbb{R}^4$*. Then, the non-trivial forcing is represented by exotic CH even though* $\mathbb{R}^4$ *remains standard.*

(a) is obvious but, regarding (b), a CH with infinite tree labeled by (+)-signs preserves the attaching region and embeds into the simplest one [17]. Such CHs lead also to exotic smooth $\mathbb{R}^4$ [5, 6]. Here, one has infinite (exotic) CHs but with different Akbulut cork, so that $\mathbb{R}^4$ would remain standard.

These formal possibilities show that non-vanishing components of 4D Riemann tensor are generated in the cosmological models with forcing changing the structure of the real line. Such Riemann tensor represents the curvature of small exotic $\mathbb{R}^4$, *i.e.* the density of gravitational energy. One can derive some realistic values of cosmological parameters from these densities (such as CC) [7].

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