Derivation of the Symmetric Stress-Energy-Momentum Tensor in Exterior Algebra

Ivano Colombaro, Josep Font-Segura and Alfonso Martinez

Department of Information and Communications Technologies, Universitat Pompeu Fabra, Barcelona, Spain

E-mail: `ivano.colombaro@upf.edu`

Abstract. We present a derivation of a manifestly symmetric form of the stress-energy-momentum using the mathematical tools of exterior algebra and exterior calculus, bypassing the standard symmetrizations of the canonical tensor. In a generalized flat space-time with arbitrary time and space dimensions, the tensor is found by evaluating the invariance of the action to infinitesimal space-time translations, using Lagrangian densities that are linear combinations of dot products of multivector fields. An interesting coordinate-free expression is provided for the divergence of the tensor, in terms of the interior and exterior derivatives of the multivector fields that form the Lagrangian density. A generalized Leibniz rule, applied to the variation of action, allows to obtain a conservation law for the derived stress-energy-momentum tensor. We finally show an application to the generalized theory of electromagnetism.

1. Introduction

The stress-energy-momentum (SEM) tensor, a symmetric rank-2 tensor whose divergence is related to the conservation laws in isolated systems, is an important mathematical object in field theories that describes the flux of energy and momentum across regions of space-time [1]. Despite tensor symmetry is a relevant property in many physical contexts [2] —for instance it is needed in the context of general relativity to describe the coupling with gravity—, the usual derivations of the energy-momentum tensor such as the popular canonical procedure to get the tensor, may lead to a not necessarily symmetric and not necessarily gauge-independent tensor.

In the context of field theories, it is rather common to find the so called canonical stress-energy tensor, which is obtained by means of the Noether's theorem as a conserved current from the invariance of the action with respect to infinitesimal space-time translations [1], [3, Sect. 3.2], [4, Sect. 2.5]. However, the canonical tensor is not symmetric by definition, and requires symmetrization using, e. g. the Belinfante-Rosenfeld procedure [1, 5], [4, Sect. 2.5]. An alternative method that directly leads to a symmetric stress-energy-momentum tensor builds on the invariance of the action to variations in the Einstein-Hilbert space-time metric [1, 5]. In parallel, the study of generalized SEM tensors using the formalism of fiber bundles is a fertile area of research in mathematical physics [6, 7].

Using the tools provided by exterior algebra, briefly described in Sec. 2 together with the problem setup, we derive the symmetric stress-energy-momentum tensor with a rather direct method that is valid for a generic grade r of the multivector field, and arbitrary n space and k time dimensions in Sec. 3. The conservation laws for the SEM tensor and the appearance in electromagnetism are discussed in Sec. 4, and finally Sec. 5 concludes the paper.

2. Mathematical setup

We consider the tools of exterior calculus, allowing to combine the simplicity and intuitiveness of standard vector calculus with the power of tools as tensors and differential forms, as motivated in [8]. In a (k, n)-space-time, where k is the number of time coordinates and n is the number of space coordinates, we consider two multivector fields, or r-vector fields, $\mathbf{a}$ and $\mathbf{b}$, respectively

$$\mathbf{a} = \sum_{I \in \mathcal{I}_r} a_I \mathbf{e}_I, \tag{1}$$

$$\mathbf{b} = \sum_{I \in \mathcal{I}_r} b_I \mathbf{e}_I, \tag{2}$$

whose basis is defined by means of the exterior product, also known as wedge product $\wedge$, as

$$\mathbf{e}_I = \mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_r}, \tag{3}$$

and where $\mathcal{I}_r$ is the family of permuted ordered lists of r indexes. For two given ordered lists I and J, we have two multivector basis elements of grades, respectively, $|I|$ and $|J|$, satisfying the relation $\mathbf{e}_I \wedge \mathbf{e}_J = \sigma(I, J)\mathbf{e}_{\varepsilon(I,J)}$, where $\sigma(I, J)$ is the signature if the permutation sorting the elements of I and J and $\mathbf{e}_{\varepsilon(I,J)}$ denotes the resulting sorting list.

We consider a Lagrangian density $\mathcal{L}$ given by the linear combination of dot products between r-graded multivector fields $\mathbf{a}$ and $\mathbf{b}$ with coefficients $\gamma_{\mathbf{a},\mathbf{b}}$, as

$$\mathcal{L} = \sum_{\mathbf{a},\mathbf{b}} \gamma_{\mathbf{a},\mathbf{b}} \, (\mathbf{a} \cdot \mathbf{b}), \tag{4}$$

where the dot product between two multivectors $\mathbf{e}_I$ and $\mathbf{e}_J$ is defined for case $|I| = |J| = r$, as

$$\mathbf{e}_I \cdot \mathbf{e}_J = \Delta_{IJ} = \Delta_{i_1 j_1} \Delta_{i_2 j_2} \ldots \Delta_{i_r j_r}, \tag{5}$$

where Δ_{ij} is the metric diagonal tensor giving $+1$ when both i-th and j-th coordinates coincide and they are spatial, giving -1 when both are the same time coordinates, and zero otherwise.

The dot product $\cdot$ is defined for two multivectors of the same grade. In exterior algebra, we also have the interior product between multivectors of different grade. The left-interior product is defined as

$$\mathbf{e}_I \lrcorner\, \mathbf{e}_J = \Delta_{II} \sigma(J \setminus I, I) \mathbf{e}_{J \setminus I}, \tag{6}$$

where the new basis $\mathbf{e}_{J\setminus I}$ is a vector of grade $|J| - |I|$ and it includes the indexes of J excluding those in common with I, for $I \subseteq J$. As we defined the left interior product, we can also write the right interior product, denoted by $\llcorner$, which can be found by means of the relation $\mathbf{e}_J \llcorner \mathbf{e}_I = \mathbf{e}_I \llcorner \mathbf{e}_J (-1)^{|I|(|J|-|I|)}$ [8, Sec. 2.2].

The exterior and interior products define the exterior and interior derivatives of a multivector field $\mathbf{a}$ as $\boldsymbol{\partial} \wedge \mathbf{a}$ and $\boldsymbol{\partial} \lrcorner\, \mathbf{a}$, where $\boldsymbol{\partial}$ is the derivative operator given by

$$\boldsymbol{\partial} = \sum_{i \in \mathcal{I}} \Delta_{ii} \partial_i \mathbf{e}_i, \tag{7}$$

and $\mathcal{I}$ is the set of integers from 0 to $k + n - 1$.

The Lagrangian density in (4) and the derivative operator (7) play an important part in the rest of the paper. We start by defining the generalized action in terms of the Lagrangian density.

3. The symmetric stress-energy-momentum tensor

Let us consider the generalized action of the system $\mathcal{S}_{\text{sys}}$ as the integral over the whole space-time of the Lagrangian density in (4), that is

$$\mathcal{S}_{\text{sys}} = \int_{\mathcal{R}} \mathrm{d}^{k+n}\mathbf{x} \left(\sum_{\mathbf{a},\mathbf{b}\in\mathcal{I}_r} \gamma_{\mathbf{a},\mathbf{b}} \left(\mathbf{a}\cdot\mathbf{b}\right) \right), \tag{8}$$

where $\mathcal{R}$ is a suitable integration domain where fields decay fast enough and that its boundary $\partial\mathcal{R}$ is arbitrary small. We next apply the principle of stationary action. We start by shifting the origin of coordinates by an infinitesimal perturbation vector $\boldsymbol{\varepsilon}$ and we denote by $\{\mathbf{e}\}$ and $\{\mathbf{e}'\}$ respectively the original and shifted basis elements. As proved in [9, Sec. 3.3], we are able to find that, in general, the r-vector field $\mathbf{a}$ transforms under infinitesimal translation as

$$\mathbf{a}' = \mathbf{a} - \mathbf{G}^r_{\boldsymbol{\varepsilon}} \times \mathbf{a}, \tag{9}$$

where $\mathbf{G}^r_{\boldsymbol{\varepsilon}}$ is a matrix having rows and columns indexed by couples of r-tuples and components $G^r_{I,J}$. We write such matrix in terms of the the tensor product $\mathbf{w}_{ij} = \mathbf{e}_i \otimes \mathbf{e}_j$ [9, Sec. 2.4], namely

$$\mathbf{G}^r_{\boldsymbol{\varepsilon}} = \sum_{I,J\in\mathcal{I}_r} G^r_{I,J}\mathbf{w}_{I,J}, \tag{10}$$

After some calculations as in [9, Sec. 3.3], we obtain the explicit expression

$$\mathbf{G}^r_{\boldsymbol{\varepsilon}} = \sum_{I\in\mathcal{I}_r} \Delta_{II} \left[\left(\sum_{i\in I} \partial_i\varepsilon_i\right) \mathbf{w}_{I,I} + \sum_{K\in\mathcal{I}_{r-1}:\varepsilon(i,K)=I, j\notin K} \sigma\left(I_{i\leftrightarrow j}\right) \partial_i\varepsilon_j\, \mathbf{w}_{I,\varepsilon(j,K)} \right], \tag{11}$$

where $I_{i\leftrightarrow j}$ corresponds to the list of indexes where the index i in I is replaced by j. For $r = 0$, we have $\mathbf{G}^0_{\boldsymbol{\varepsilon}} = 0$, as there is only one option for $I = J$, the empty set $\varnothing$. In other words, $1' = 1$, as in classical field theory the scalar field is not affected by space-time translations. Similarly, for the vector case $r = 1$, the matrix is given by $\mathbf{G}^1_{\boldsymbol{\varepsilon}} = \boldsymbol{\partial} \otimes \boldsymbol{\varepsilon}$, e. g., $\boldsymbol{\partial}' = (\mathbf{1} - \boldsymbol{\partial}\otimes\boldsymbol{\varepsilon}) \times \boldsymbol{\partial}$, and therefore $G^1_{I,J} = G_{ij} = \Delta_{ii}\partial_i\varepsilon_j$. It is then easy to identify the cases $\mathbf{G}^0_{\boldsymbol{\varepsilon}}$ and $\mathbf{G}^1_{\boldsymbol{\varepsilon}}$ with the classical scalar and vector transformations, respectively. To obtain the transformation (9), we consider that a vector basis element transforms as

$$\mathbf{e}'_i = \mathbf{e}_i \times (\mathbf{1} + \boldsymbol{\partial}\otimes\boldsymbol{\varepsilon}), \tag{12}$$

where $\mathbf{1}$ is the square identity matrix, defined for a general grade r, $\mathbf{1}_r = \sum_{I\in\mathcal{I}_r} \Delta_{II}\mathbf{w}_{I,I}$, the operation $\times$ is defined by means fo the basis elements as $\mathbf{e}_i \times \mathbf{w}_{j,\ell} = \Delta_{ij}\mathbf{e}_\ell$, and the matrix $\boldsymbol{\partial}\otimes\boldsymbol{\varepsilon}$ represents the Jacobian-partial derivative matrix. In general, the transformation in (12) may also be written as $\mathbf{e}'_i = \sum_j \tau_{i,j}\mathbf{e}_j$ for some $\tau_{i,j} = \delta_{ij} + \partial_i\varepsilon_j$. The transformation (12) is extended to multivector basis elements as the exterior product of transformations, namely

$$\mathbf{e}'_I = \mathbf{e}'_{i_1} \wedge \mathbf{e}'_{i_2} \cdots \wedge \mathbf{e}'_{i_r} \tag{13}$$

It can be shown that the transformation (13) can be written in terms of the determinant of a matrix $\tau_{I\otimes J}$, that is

$$\mathbf{e}'_I = \sum_{J\in\mathcal{I}_r} \det\tau_{I\otimes J}\,\mathbf{e}_J, \tag{14}$$

where

$$\tau_{I\otimes J} = \begin{pmatrix} \tau_{i_1,j_1} & \cdots & \tau_{i_1,j_r} \\ \vdots & & \vdots \\ \tau_{i_r,j_1} & \cdots & \tau_{i_r,j_r} \end{pmatrix}.$$

To evaluate (14), we keep terms not exceeding the first derivative of $\boldsymbol{\varepsilon}$, and we note that, for each list I, we need only to consider J such that $I = J$ or $|I \cap J| = s - 1$, in order to remove unneeded terms. Thus, for $I = J$, we have that $\det \tau_{I\otimes I} = 1 + \sum_{i\in I} \partial_i \varepsilon_i$, while for $|I \cap J| = s-1$ we obtain $\det \tau_{I\otimes J} = \sigma\big(I_{i\leftrightarrow j}\big)\, \partial_i \varepsilon_j$ as the weights in the transformation (13).

Now, we consider the action of the single term

$$\mathcal{S}_{\mathbf{a}\cdot\mathbf{b}} = \int_{\mathcal{R}} \mathrm{d}^{k+n}\mathbf{x}\,(\mathbf{a}\cdot\mathbf{b}), \tag{15}$$

and we write its transformed action as

$$\mathcal{S}_{\mathbf{a}'\cdot\mathbf{b}'} = \int_{\mathcal{R}'} \mathrm{d}^{k+n}\mathbf{x}'\,(\mathbf{a}'\cdot\mathbf{b}'), \tag{16}$$

where the differential follows the transformation law $\mathrm{d}^{k+n}\mathbf{x}' = \mathrm{d}^{k+n}\mathbf{x}\big(1 + \boldsymbol{\partial}\cdot\boldsymbol{\varepsilon}\big)$ and the multivector fields $\mathbf{a}$ and $\mathbf{b}$ transform according to (9). We evaluate the variation

$$\delta\mathcal{S}_{\mathbf{a}\cdot\mathbf{b}} = \mathcal{S}_{\mathbf{a}'\cdot\mathbf{b}'} - \mathcal{S}_{\mathbf{a}\cdot\mathbf{b}}, \tag{17}$$

which, substituting the relations for the transformation, can be written

$$\delta\mathcal{S}_{\mathbf{a}\cdot\mathbf{b}} = \int_{\mathcal{R}} \mathrm{d}^{k+n}\mathbf{x}(\boldsymbol{\partial}\cdot\boldsymbol{\varepsilon})\,(\mathbf{a}\cdot\mathbf{b}) - \int_{\mathcal{R}} \mathrm{d}^{k+n}\mathbf{x}\,\big((\mathbf{G}^r_{\boldsymbol{\varepsilon}}\times\mathbf{a})\cdot\mathbf{b}\big) - \int_{\mathcal{R}} \mathrm{d}^{k+n}\mathbf{x}\,\big(\mathbf{a}\cdot(\mathbf{G}^r_{\boldsymbol{\varepsilon}}\times\mathbf{b})\big), \tag{18}$$

having replaced $\mathcal{R}'$ by $\mathcal{R}$, since we assumed that the difference between the integration regions $\mathcal{R}'$ and $\mathcal{R}$ lies far from the origin, coupled with the rapid decay of the fields.

The properties of exterior algebra and calculus imply that (17) can be expressed as

$$\delta\mathcal{S}_{\mathbf{a}\cdot\mathbf{b}} = \int_{\mathcal{R}} \mathrm{d}^{k+n}\mathbf{x}\,(\boldsymbol{\partial}\otimes\boldsymbol{\varepsilon})\cdot\mathbf{T}_{\mathbf{a}\cdot\mathbf{b}}, \tag{19}$$

where $\mathbf{T}_{\mathbf{a}\cdot\mathbf{b}}$ is a tensor given by

$$\mathbf{T}_{\mathbf{a}\cdot\mathbf{b}} = (-1)^r\big(\mathbf{a}\odot\mathbf{b} + \mathbf{a}\oslash\mathbf{b}\big), \tag{20}$$

in terms of two special tensorial operations, $\odot$ and $\oslash$, respectively defined as

$$\mathbf{a}\odot\mathbf{b} = \sum_{i\leq j}(\Delta_{ii}\mathbf{e}_i \,\lrcorner\, \mathbf{a})\cdot(\mathbf{b}\,\llcorner\,\mathbf{e}_j\Delta_{jj})\mathbf{w}_{ij}, \tag{21}$$

$$\mathbf{a}\oslash\mathbf{b} = \sum_{i\leq j}(\Delta_{ii}\mathbf{e}_i \wedge \mathbf{a})\cdot(\mathbf{b}\wedge\mathbf{e}_j\Delta_{jj})\mathbf{w}_{ij}. \tag{22}$$

For completeness, we can also write the components of (20), which are

$$T^{\mathbf{a}\cdot\mathbf{b}}_{ii} = \Delta_{ii}\left(\sum_{K\in\mathcal{I}_s:i\notin K}\Delta_{KK}a_K b_K - \sum_{K\in\mathcal{I}_s:i\in K}\Delta_{KK}a_K b_K\right), \tag{23}$$

$$T^{\mathbf{a}\cdot\mathbf{b}}_{ij} = -\sum_{K\in\mathcal{I}_{s-1}:i,j\notin K}\Delta_{KK}\sigma\big(\varepsilon(i,K)_{i\leftrightarrow j}\big)\big(a_{\varepsilon(i,K)}b_{\varepsilon(j,K)} + b_{\varepsilon(i,K)}a_{\varepsilon(j,K)}\big). \tag{24}$$

Finally, from linearity in (8), we find that the SEM tensor is given by

$$\mathbf{T}_{\text{sys}} = \sum_{\mathbf{a},\mathbf{b}\in\mathcal{I}_r}\gamma_{\mathbf{a},\mathbf{b}}\mathbf{T}_{\mathbf{a}\cdot\mathbf{b}}. \tag{25}$$

The identity between (18) and (19) is established thanks to the relation

$$(\boldsymbol{\partial} \otimes \boldsymbol{\varepsilon}) \cdot \mathbf{T}_{\mathbf{a}\cdot\mathbf{b}} = (\boldsymbol{\partial} \cdot \boldsymbol{\varepsilon})\,(\mathbf{a} \cdot \mathbf{b}) - (\mathbf{G}^r_{\boldsymbol{\varepsilon}} \times \mathbf{a}) \cdot \mathbf{b} - \mathbf{a} \cdot (\mathbf{G}^r_{\boldsymbol{\varepsilon}} \times \mathbf{b}), \tag{26}$$

which can be proved simply by expanding the left hand side and the right hand side.

It is worth noting that the tensor product $\mathbf{w}_{ij}$ is not necessarily symmetric, in fact (21) and (22) are not separately, but they are when summed, since (20) is symmetric, as can be manifestly deduced by (23) and (24). We next discuss a conservation law for the SEM tensor and an application to electromagnetism.

4. Conservation law for energy-momentum and generalized electromagnetism

We finally study the change of action integral as the flux of the tensor across a slice of space-time [10, Sec. 4.3]. We apply the generalized Leibniz rule [9, Eq. (120)]

$$\boldsymbol{\partial} \cdot (\boldsymbol{\varepsilon} \,\lrcorner\, \mathbf{T}) = \boldsymbol{\varepsilon} \cdot (\boldsymbol{\partial} \,\lrcorner\, \mathbf{T}) + (\boldsymbol{\partial} \otimes \boldsymbol{\varepsilon}) \cdot \mathbf{T}, \tag{27}$$

in order to write the action variation (19) into

$$\delta\mathcal{S}_{\mathbf{a}\cdot\mathbf{b}} = -\int_{\mathcal{R}} \mathrm{d}^{k+n}\mathbf{x}\,(\boldsymbol{\partial} \,\lrcorner\, \mathbf{T}_{\mathbf{a}\cdot\mathbf{b}}) \cdot \boldsymbol{\varepsilon}, \tag{28}$$

removing the total derivative term because of boundary conditions. We also provide an interesting coordinate-free expression of the divergence of the tensor $\mathbf{T}_{\mathbf{a}\cdot\mathbf{b}}$ in (20), that corresponds to the interior derivative in exterior calculus, in terms of the interior $\boldsymbol{\partial}\lrcorner$ and exterior $\boldsymbol{\partial}\wedge$ derivatives, given by

$$\boldsymbol{\partial} \,\lrcorner\, \mathbf{T}_{\mathbf{a}\cdot\mathbf{b}} = \mathbf{a} \,\lrcorner\, (\boldsymbol{\partial} \wedge \mathbf{b}) + \mathbf{b} \,\lrcorner\, (\boldsymbol{\partial} \wedge \mathbf{a}) - \mathbf{a} \,\llcorner\, (\boldsymbol{\partial} \,\lrcorner\, \mathbf{b}) - \mathbf{b} \,\llcorner\, (\boldsymbol{\partial} \,\lrcorner\, \mathbf{a}).$$

Under the assumption that space-time translations are a symmetry of the system and having selected fields that decay sufficiently fast at the boundary of the region $\mathcal{R}$, then we have that $\delta\mathcal{S}_{\mathbf{a}\cdot\mathbf{b}} = 0$ and, as a consequence, the divergence of the SEM tensor of the system vanishes because of linearity, namely

$$\boldsymbol{\partial} \,\lrcorner\, \mathbf{T}_{\text{sys}} = \sum_{\mathbf{a},\mathbf{b}\in\mathcal{I}_r} \gamma_{\mathbf{a},\mathbf{b}} \boldsymbol{\partial} \,\lrcorner\, \mathbf{T}_{\mathbf{a}\cdot\mathbf{b}} = 0. \tag{29}$$

We have hence obtained a conservation law for the derived stress-energy-momentum tensor.

As an example, we consider the free Lagrangian density for generalized electromagnetism [11], which for any k, n, given the generalized Maxwell field $\mathbf{F}$, with grade r, written

$$\mathcal{L}_{\text{free-gem}} = -\frac{1}{2}\mathbf{F} \cdot \mathbf{F}. \tag{30}$$

This single term Lagrangian density of type (4), where $\mathbf{a} = \mathbf{b} = \mathbf{F}$, leads to the SEM tensor

$$\mathbf{T}_{\text{free-gem}} = -\frac{1}{2}\big(\mathbf{F} \odot \mathbf{F} + \mathbf{F} \oslash \mathbf{F}\big), \tag{31}$$

and, as we have seen in (23) and (24), we can write on- and off-diagonal components as

$$T^{\text{free-gem}}_{ii} = \frac{(-1)^r}{2}\Delta_{ii}\left(\sum_{I\in\mathcal{I}_r : i\in I} F_I^2 \Delta_{II} - \sum_{I\in\mathcal{I}_r : i\notin I} F_I^2 \Delta_{II}\right) \tag{32}$$

$$T^{\text{free-gem}}_{ij} = -\sum_{L\in\mathcal{I}_{r-1} : i,j\neq L} \sigma(L,i)\sigma(j,L) F_{i+L} F_{j+L} \Delta_{LL}, \tag{33}$$

according to [10] and to the stress-energy tensor for standard electromagnetism with bivectors, or the Faraday tensor, which can be simply recovered by setting $r = 2$, $k = 1$, $n = 3$, [11, Sect. 12.10], [12, Sect. 33]. In generalized electromagnetism, the symmetric structure of the tensor (31) is found using the Maxwell equations

$$\boldsymbol{\partial} \lrcorner \mathbf{F} = \mathbf{J}, \tag{34}$$

$$\boldsymbol{\partial} \wedge \mathbf{F} = 0, \tag{35}$$

where $\mathbf{J}$ is a $(r-1)$-multivectorial source. We identify the generalized Lorentz force density $\mathbf{f} = \mathbf{J} \lrcorner \mathbf{F}$, which using (34) and (35), we write as

$$\mathbf{f} = (\boldsymbol{\partial} \lrcorner \mathbf{F}) \lrcorner \mathbf{F} + (\boldsymbol{\partial} \wedge \mathbf{F}) \llcorner \mathbf{F}. \tag{36}$$

The right-hand side of (36) corresponds to the divergence of a symmetric tensor $\mathbf{T}_{\text{free-gem}}$ in (31), that is a conservation law of electromagnetism [10, Sec. A.2]

$$\boldsymbol{\partial} \lrcorner \mathbf{T}_{\text{free-gem}} + \mathbf{f} = 0. \tag{37}$$

5. Conclusions

For a Lagrangian density expressed as dot product between multivector fields, we provided an exterior-algebraic derivation of the symmetric stress-energy-momentum tensor, considering a closed physical system and imposing the invariance of the action to infinitesimal space-time translations. Our method to obtain the stress-energy-momentum tensor and its divergence is suited not only for free field theories, but also for interaction phenomena mathematically accommodated by the dot product $\mathbf{a} \cdot \mathbf{b}$, which is something not so common to find in classical field theories. The physical interpretation of the terms due to interaction, including their gauge invariance and possible connection to the tensor of the matter fields, will be object of further investigations, as the derivation of the stress-energy-momentum tensor for other interesting models of mathematical physics is left as a future work.

Acknowledgments

This work has been funded in part by the Spanish Ministry of Science, Innovation and Universities under grants TEC2016-78434-C3-1-R and BES-2017-081360.

References

[1] Forger M and Römer H 2004 *Annals of Physics* **309** 306–389 ISSN 0003-4916
[2] Montesinos M and Flores E 2006 *Revista Mexicana de Fisica* **52**(1) 29–36
[3] Maggiore M 2005 *A Modern Introduction to Quantum Field Theory* Oxford Master Series in Statistical, Computational, and Theoretical Physics (Oxford: Oxford Univ. Press)
[4] Di Francesco P, Mathieu P and Sénéchal D 1997 *Conformal Field Theory* Graduate texts in contemporary physics (New York, NY: Springer)
[5] Blaschke D N, Gieres F, Reboud M and Schweda M 2016 *Nuclear Physics B* **912** 192–223 ISSN 0550-3213
[6] Gotay M J and Marsden J E 1992 *Contemp. Math.* **132** 367–392
[7] Voicu N 2015 *International Journal of Geometric Methods in Modern Physics* **13** 1640001
[8] Colombaro I, Font-Segura J and Martinez A 2019 *Mathematics* **7** 564–583
[9] Martinez A, Font-Segura J and Colombaro I 2021 *European Physical Journal Plus* **212**
[10] Colombaro I, Font-Segura J and Martinez A 2020 *European Physical Journal Plus* **135**
[11] Jackson J D 1999 *Classical Electrodynamics* 3rd ed (John Wiley & Sons)
[12] Landau L D and Lifshitz E M 1987 *The Classical Theory of Fields* 4th ed (*Course of Theoretical Physics* no 2) (Butterworth-Heinemann)