E11 Invariant Field Theories

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E_{11} Invariant Field Theories

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Abstract

It has been proposed that the low energy effective action of the theory of strings and branes possesses a large symmetry described by the Kac-Moody algebra E_{11}. The non-linear realisation of this algebra and its vector representation determines the fields and coordinates of the theory, as well as the equations that describe their dynamics. In order to construct the generators of E_{11} algebra it is split into representations of its $GL(d) \times E_{11-d}$ subalgebra. Here d is an integer that determines the dimension of the corresponding E_{11} theory. The low levels of the non-linear realisation contain the set of equations of the supergravity theory in corresponding space-time dimension, while the higher levels introduce an infinite number of fields that are connected to the supergravity ones via a chain of duality relations, as well as standalone fields that have no counterparts in standard supergravity theory.

In this thesis we derive the set of commutators of E_{11} algebra and its vector representation up to a certain level in five and ten-dimensional cases. We use the non-linear realisation approach to construct the generalised vielbein and the Cartan forms of the E_{11} theory in four, five, ten and eleven dimensions.

We then build a set of E_{11} invariant equations in five and eleven-dimensional theories from the non-linear realisation of E_{11}. The low level equations, when appropriately truncated, are shown to perfectly reproduce the dynamics of the standard supergravity theories in corresponding dimensions. The dynamics of certain higher level fields are considered, including the dual graviton field and an eleven-dimensional field that, when reduced to ten dimensions, gives rise to the Romans mass term in type IIA theory.

Lastly, we describe the non-linear realisation of very extended A_1 algebra, called A_1^{+++}, together with its commutators, Cartan forms and generalised vielbein.

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1 Overview

1.1 Maximal supergravities and their exceptional symmetries

It is believed that all superstring theories are different manifestations of a single theory, called the M theory. M theory, however, has no dynamical description of its own and relies on the existence of supergravity theories that provide the low energy limit description for it. In particular, the unique eleven-dimensional supergravity [1] and maximally supersymmetric ten-dimensional supergravity theories, which are the IIA [2, 3, 4] and IIB [5, 6, 7] supergravities, describe the low-energy limit of the M theory, type IIA and type IIB string theories respectively. An important feature of supergravity theories is the occurence of coset space symmetries that determine the way the scalars enter these theories [8, 9]. In particular, four-dimensional maximal supergravity possesses an E_7 symmetry [10, 11]. Similarly, three and two-dimensional maximal supergravities possess E_8 [12] and E_9 [13, 14] symmetries respectively. Scalars of type IIB supergravity belong to the coset space of $\frac{SL(2,\mathbb{R})}{SO(2)}$ [5]. These symmetries, usually referred to as exceptional symmetries, and the corresponding coset spaces are given in the following table

Dimension	Exceptional symmetry group	Coset space
10D IIA	$O\,(1,\,1)$	—
10D IIB	$SL\,(2)$	$\frac{SL(2)}{SO(2)}$
9D	$GL\,(2)$	$\frac{GL(2)}{SO(2)}$
8D	$SL\,(2) \times SL\,(3)$	$\frac{SL(2)\times SL(3)}{SO(2)\times SO(3)}$
7D	$SL\,(5)$	$\frac{SL(5)}{SO(5)}$
6D	$SO\,(5,\,5)$	$\frac{SO(5,5)}{SO(5)\times SO(5)}$
5D	E_6	$\frac{E_6}{USp(8)}$
4D	E_7	$\frac{E_7}{SU(8)}$
3D	E_8	$\frac{E_8}{SO(16)}$

The coset construction was later extended [15] to include the gauge fields of the su-

pergravity theories into a coset of an algebra, whose generators carried no space-time indexes and involved both commutators and anticommutators. Using a different approach it was then shown [16] that the entire bosonic sector of eleven and ten-dimensional IIA supergravity could be formulated as a non-linear realisation of an infinite-dimensional algebra. This led to a proposition [17] that that the low energy effective action of the theory of strings and branes possesses a large symmetry described by the Kac-Moody algebra E_{11}. This theory can be formulated as a non-linear realisation, in which all the fields form a coset space of a certain subalgebra in the E_{11} algebra. It was subsequently proposed [18] that in order to incorporate the generalised space-time of this theory one should also introduce generators transforming in the l_1 fundamental representation of E_{11}. Both the fields and the coordinates of the theory emerge from the non-linear realisation of the semidirect product of E_{11} with its l_1 representation, denoted as $E_{11} \ltimes l_1$.

Theories in different dimensions emerge from the different possible decompositions of E_{11} into the subalgebras that correspond to deleting different nodes from the E_{11} Dynkin diagram [19, 20, 21, 22, 23, 24]. The fields at low levels of these decompositions are those of the maximal supergravity theories in corresponding dimension and the lowest level coordinates are just the coordinates of the usual space-time. The higher levels of the theory involve an infinite number of fields, some of which are connected by infinite chains of duality relations, while others are standalone fields that could potentially describe new physical phenomena. The dynamics of these fields are entirely determined by the symmetries of the non-linear realisation.

This thesis is based on the series of works in which we investigate the properties of the non-linear realisation of E_{11} by building the generalised vielbein of the theory [25], investigating the connection to exceptional field theories [26] and, finally, constructing the dynamics of the E_{11} theory in eleven and five dimensions [27, 28, 29] and showing that at the low levels they are identical to ones of the corresponding supergravity theories.

We will start by giving a brief introduction to Kac-Moody algebras, their representations and the non-linear realisations. We will then discuss a particular non-linear

realisation that leads to the d-dimensional pure gravity theory under certain conditions. This specific example will illustrate how E_{11} manages to incorporate the description of gravity into its non-linear realisation. In the second chapter we will construct the commutation relations of the E_{11} algebra and its l_1 representation in five and ten dimensional case. In Chapter 3 we will build the non-linear realisation of $E_{11} \ltimes l_1$ in eleven, ten and five and dimensional cases. This leads to the main part of this work, Chapter 4, in which we investigate the dynamics of the eleven and five-dimensional non-linear realisation of $E_{11} \ltimes l_1$ and show that it includes the description of the corresponding maximal supergravity fields at low levels. In the last chapter of the thesis we will construct the non-linear realisation of the very extended A_1 algebra, denoted as A_1^{+++}. At low levels this model contains a description of pure four-dimensional gravity, supplemented with the dual graviton field [30].

1.2 Kac-Moody algebras

A Kac-Moody algebra is a Lie algebra (usually infinite dimensional) that is characterised by a generalised Cartan matrix, that is a $r \times r$ matrix $A = \{a_{ij}\}$ with integer entries that has the following properties

1. For diagonal elements $a_{ii} = 2$.
2. For non-diagonal elements $a_{ij} \leq 0$.
3. $a_{ij} = 0 \iff a_{ji} = 0$.
4. A can be written as $D\,S$, where D is diagonal and S is symmetric.

A Kac-Moody algebra g is then defined by the following set of relations

$$\begin{aligned} &[H_a,\, H_b] = 0, && [E_a,\, F_b] = \delta_{ab}\, H_a, \\ &[H_a,\, E_b] = A_{ab}\, E_b, && [H_a,\, F_b] = -\, A_{ab}\, F_b, \end{aligned} \tag{1.2.1}$$

as well as the Serre relation

$$ad\,(E_a)^{1-A_{ab}}\; E_b = ad\,(F_a)^{1-A_{ab}}\; F_b = 0. \tag{1.2.2}$$

The Cartan matrix can be used to construct the Dynkin diagram, which is uniquely determined by the algebra. It also affects the dimension of the Kac-Moody algebra in the following way

1. If symmetrised Cartan matrix S is positive definite $\forall\, v \neq 0 :\ v_a\, S_{ab}\, v_b > 0$, the algebra is a finite-dimensional semi-simple Lie algebra. In this case $\det A > 0$.

2. If symmetrised Cartan matrix S is semi-definite $\forall\, v \neq 0 :\ v_a\, S_{ab}\, v_b \geq 0$ with exactly one zero eigenvalue, the algebra is an affine Lie algebra. In this case $\det A = 0$.

3. In all other cases we are dealing with a general Kac-Moody algebra.

Very little is known about the general Kac-Moody algebras, even a list of the generators is impossible to construct for algebras of this class. Therefore, in order to be able to develop a workable approach to studying them we will only consider a certain subclass of Kac-Moody algebras, called Lorentzian algebras. These are the Kac-Moody algebras whose Dynkin diagram contains at least one node whose deletion yields a Dynkin diagram with connected components of finite type except for at most one of affine type. In other words, their Cartan matrix possesses at most one negative eigenvalue. The Dynkin diagram with one node removed is referred to as reduced Dynkin diagram.

Lorentzian algebras can be studied by decomposing their generators into representations of their subalgebras that correspond to the reduced Dynkin diagram (see, for instance, Chapter 16.3 of [32]). All of the generators then become parametrised with an integer parameter m, called **level**. This parameter indicates how many times the Chevalley generators that correspond to the deleted node enter the generator in question. Level $m = 0$ generators form the adjoint representation of the reduced subalgebra. Positive level generators correspond to the rising generator E_r, while negative level ones — to F_r, where index r labels the deleted node. The exact representation content of the algebra on any given level is uniquely determined by the choice of the deleted node. It can be found by either theoretical analysis, or using a specialised program like SimpLie [31].

1.2.1 The l_1 representation

The representations of the Kac-Moody algebras can be studied using the same technique as the one that we used in the previous section to study the algebra itself (see, for instance, Chapter 16.6 of [32]). In order to construct the fundamental representation associated with node e of a Lorentzian algebra g we first construct an enlarged Dynkin diagram $D^\star$ by adding a new node, called $\star$, connected by a single line to the node e in the original Dynkin diagram D. We can decompose the adjoint representation of the enlarged algebra $g^\star$ in terms of the original algebra g by deleting node $\star$ and using the techniques above. That is, we can introduce a new level parameter $m^\star$ associated with the added node. At level $m^\star = 0$ we find the adjoint representation of g. It can be shown, that at level $m^\star = 1$ of the decomposition we find the fundamental representation of g, associated with node e. In this thesis we will be mainly interested in the fundamental representation, associated with the leftmost node of the Dynkin diagram, called the l_1 representation. Unlike the adjoint representation, discussed in the previous section, the l_1 representation is a highest weight representation. This implies that the level parameter for it takes only non-negative values $m \geq 0$. The importance of this representation comes from the fact that it contains the translation generators associated with the algebra g. As we will see later in Chapter 2, at level $m = 0$ of this representation we find the regular space-time translation generators P_a, while the higher levels $m > 0$ are associated with the translations of the generalised coordinates, which will be introduced below.

We will illustrate the procedure outlined above with the following diagram

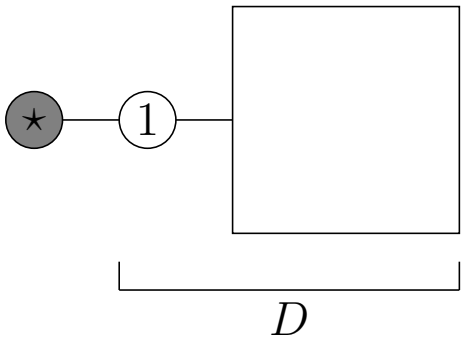


Figure 1: Enlarged Dynkin Diagram $D^{\star}$ for the l_1 representation

1.3 Non-linear realisations

The theory of non-linear realisations has been historically used to describe effective theories with spontaneously broken symmetry [33, 34]. In this section we will briefly discuss the structure and the symmetries of the non-linear realisation. We will then illustrate how d-dimensional pure gravity theory is related to the non-linear realisation of $GL\,(d)$ with the local subgroup $SO\,(d)$. This is an important example that lays groundwork for implementing the non-linear realisation approach for the E_{11} theory.

1.3.1 Non-linear realisation of $G \ltimes l_1$ over H

An arbitrary group element of $G \ltimes l_1$ can be parametrised in the following way

$$g = g_L\, g_E, \quad g_L = e^{x^A\, L_A}, \quad g_E = \prod_{R^{\underline{\alpha}} \in G} e^{A_{\underline{\alpha}}(x)\, R^{\underline{\alpha}}}. \tag{1.3.1}$$

Here $R^{\underline{\alpha}}$ are the generators of G, while L_A is the set of generators of the vector (l_1) representation of G. They obey the following general commutation relations

$$\left[R^{\underline{\alpha}},\, R^{\underline{\beta}}\right] = f^{\underline{\alpha}\underline{\beta}}{}_{\underline{\gamma}}\, R^{\underline{\gamma}}, \quad \left[R^{\underline{\alpha}},\, L_A\right] = -\left(D^{\underline{\alpha}}\right)_A{}^B\, L_B. \tag{1.3.2}$$

Matrices $\left(D^{\underline{\alpha}}\right)_A{}^B = D\left(R^{\underline{\alpha}}\right)_A{}^B$ form the l_1 representation of G. The coefficients $A_{\underline{\alpha}}$ play the role of the fields in the resulting theory, while x^A are the generalised coordinates

that these fields depend on. The group element from equation (1.3.1) forms a non-linear realisation when subject to the following transformations

- Rigid (global) $G \ltimes l_1$ transformations

$$g \longrightarrow g_0\, g, \quad g_0 \in G \ltimes l_1, \tag{1.3.3}$$

- Local H transformations

$$g \longrightarrow g\, h, \quad h \in H. \tag{1.3.4}$$

The equations of motion that describe the dynamics of the fields from equation (1.3.1) are invariant under the rigid and local transformations defined above. The role of the local transformations subgroup H (tangent group) is usually played by the Cartan involution subalgebra of G, called $I_c\,(G)$. It is defined as the subalgebra of G that is invariant under the Cartan involution, which acts on the Chevalley generators (H_α, E_α, F_α) of G in the following way

$$I_c\,(H_\alpha) = -\,H_\alpha, \quad I_c\,(E_\alpha) = -\,F_\alpha, \quad I_c\,(F_\alpha) = -\,E_\alpha. \tag{1.3.5}$$

A noteworthy example of this structure is the Cartan involution subalgebra of $GL\,(d)$, which happens to be the Lorentz algebra in the same dimension: $I_c\,(GL\,(d)) = SO\,(d)$. We will later use this fact to illustrate how the d-dimensional pure gravity is related to the non-linear realisation of $IGL\,(d)$.

In order to construct the dynamics of the non-linear realisation one has to consider the Cartan form, which satisfies the Maurer–Cartan equation.

$$\mathcal{V} = g^{-1}\, dg, \quad d\mathcal{V} + \mathcal{V} \wedge \mathcal{V} = 0. \tag{1.3.6}$$

where g is the group element from equation (1.3.1). The Cartan form is invariant under the rigid transformations and transforms as follows under the local ones

$$\mathcal{V} \longrightarrow h^{-1}\, \mathcal{V}\, h + h^{-1}\, dh. \tag{1.3.7}$$

The Cartan form can be split into the adjoint and the l_1 parts.

$$\mathcal{V} = \mathcal{V}_L + \mathcal{V}_E = dx^\Pi \left(E_\Pi{}^A\, L_A + G_{\Pi,\,\underline{\alpha}}\, R^{\underline{\alpha}}\right) = dx^\Pi\, E_\Pi{}^A\, \left(L_A + G_{A,\,\underline{\alpha}}\, R^{\underline{\alpha}}\right), \tag{1.3.8}$$

where comma in $G_{\Pi,\,\underline{\alpha}}$ separates the l_1 index Π from the adjoint index $\underline{\alpha}$. $G_{\Pi,\,\underline{\alpha}}$ can also be written as a differential form: $G_{\underline{\alpha}} = dx^{\Pi}\, G_{\Pi,\,\underline{\alpha}}$. $E_{\Pi}{}^{A}$ is the generalised vielbein of the theory, which is determined by the following relation

$$E_{\Pi}{}^{A}\, L_A = g_E^{-1}\, L_{\Pi}\, g_E. \tag{1.3.9}$$

The generalised vielbein can be also expressed through the $(D^{\underline{\alpha}})_{A}{}^{B}$ matrices from equation (1.3.2).

$$E_{\Pi}{}^{A} = \left(\prod_{\underline{\alpha}} e^{A_{\underline{\alpha}}\, D^{\underline{\alpha}}}\right)_{\Pi}^{A}. \tag{1.3.10}$$

This object will be calculated for the E_{11} theory in various dimensions in Chapter 3. $G_{A,\,\alpha}$ are the Cartan forms that generalise the field strengths of the fields of the theory. The latter fact can be justified by observing that at the linearised level $G_{A,\,\underline{\alpha}} = \partial_A\, A_{\underline{\alpha}}$. Knowing the generalised vielbein one can construct the most general expression for the Cartan forms

$$\begin{aligned} -\,[\mathcal{V}_E,\, L_A] &= dx^{\Pi}\, G_{\Pi,\,\underline{\alpha}}\, (D^{\underline{\alpha}})_{A}{}^{B}\, L_B = -\,\left[g_E^{-1} dg_E,\, L_A\right] \\ &= -\, g_E^{-1}\, d\left(g_E\, L_A\, g_E^{-1}\right)\, g_E = E_{A}{}^{\Sigma}\, dE_{\Sigma}{}^{B}\, L_B, \end{aligned} \tag{1.3.11}$$

and so

$$G_{\Pi,\,A}{}^{B} = G_{\Pi,\,\underline{\alpha}}\, (D^{\underline{\alpha}})_{A}{}^{B} = \left(E^{-1}\,\partial_{\Pi}\, E\right)_{A}{}^{B}. \tag{1.3.12}$$

The adjoint part of the Cartan form can be further split into the coset and the subalgebra parts

$$\mathcal{V}_E = \mathcal{P} + \mathcal{Q}, \quad \mathcal{P} \in G/H, \quad \mathcal{Q} \in H. \tag{1.3.13}$$

The benefit of this decomposition is that under the local transformations the coset part transforms homogeneously, while the subalgebra part — as a connection.

$$\begin{aligned} \mathcal{P} &\longrightarrow h^{-1}\, \mathcal{P}\, h, \\ \mathcal{Q} &\longrightarrow h^{-1}\, \mathcal{Q}\, h + h^{-1}\, dh. \end{aligned} \tag{1.3.14}$$

The transformation law of the general vielbein under the transformations from equations (1.3.3, 1.3.4) is determined by the following equation

$$E_{\Pi}{}^{A} \longrightarrow D\,(g_0)_{\Pi}{}^{\Lambda}\, E_{\Lambda}{}^{B}\, D\,(h)_{B}{}^{A}, \tag{1.3.15}$$

where D matrices were defined in (1.3.2). From this equation one can observe that the generalised vielbein transforms on its upper, "flat", index under the local transformations, and on its lower, "world", index under the rigid transformations as well as the general coordinate transformations. This makes this object a natural generalisation of the regular vierbein from general relativity. Using equations (1.3.12, 1.3.15) one can derive the most general transformation law of the Cartan forms under the local transformations.

$$\begin{aligned} G_{A,\,B}{}^{C} \longrightarrow\; & D\left(h^{-1}\right)_{A}{}^{D}\, D\left(h^{-1}\right)_{B}{}^{E}\, G_{D,\,E}{}^{F}\, D\left(h\right)_{F}{}^{C} \\ & + D\left(h^{-1}\right)_{A}{}^{D}\, D\left(h^{-1}\right)_{B}{}^{E}\, E_{D}{}^{\Pi}\, \partial_{\Pi}\, D\left(h\right)_{E}{}^{C}. \end{aligned} \tag{1.3.16}$$

Lastly, we will describe how the general coordinate and gauge transformations can be generalised for the non-linear realisation of G. The generalised vielbein transforms as follows [35]

$$E_{A}{}^{\Sigma}\, \delta_{g} E_{\Pi}{}^{A} = \left(D^{\underline{\alpha}}\right)_{\Pi}{}^{\Xi}\, \left(D_{\underline{\alpha}}\right)_{\Xi}{}^{\Theta}\, \partial_{\Theta}\, \Lambda^{\Sigma} + \Lambda^{\Theta}\, E_{A}{}^{\Sigma}\, \partial_{\Theta}\, E_{\Pi}{}^{A}. \tag{1.3.17}$$

Here index $\underline{\alpha}$ is lowered with the Killing metric, defined as a scalar product on the algebra $g^{\underline{\alpha}\underline{\beta}} = \left(R^{\underline{\alpha}},\, R^{\underline{\beta}}\right)$. It is straightforward to calculate using the invariance property, $\left(R^{\underline{\alpha}},\, \left[R^{\underline{\beta}},\, R^{\underline{\gamma}}\right]\right) = \left(\left[R^{\underline{\alpha}},\, R^{\underline{\beta}}\right],\, R^{\underline{\gamma}}\right)$.

These transformations take a rather simple form when reduced to the flat indexes and linearised. One finds

$$\delta_{g} A_{\underline{\alpha}} = \left(D_{\underline{\alpha}}\right)_{A}{}^{B}\, \partial_{B}\, \Lambda^{A}. \tag{1.3.18}$$

Parameter Λ^{Π} belongs to the l_1 representation of G. In the E_{11} case this transformation law simultaneously encodes both the general coordinate transformations of the regular space-time vielbein $e_{\mu}{}^{a}$, as well as the gauge transformations of the vector fields. Hence we will later refer to it as "gauge transformations". Note that the construction of the non-linear realisation does not imply that the dynamics of the system have to be invariant under these transformations, however, we will later discover that the E_{11} symmetry automatically imposes the gauge invariance.

1.3.2 Gravity as a non-linear realisation of $IGL\,(d)$

We are now going to show how the pure gravity in d dimensions is related to the non-linear realisation of $IGL\,(d)$. This connection was first established in [36] for the four-dimensional case and later generalised for d dimensions in [16]. In this section we are going to follow the narrative of Chapter 13.2 from [32]. $IGL\,(d)$ algebra consists of translation generators P_a and $GL\,(d)$ generators $K^a{}_b$. The commutators of this algebra are

$$[K^a{}_b,\, K^c{}_d] = \delta^c_b\, K^a{}_d - \delta^a_d\, K^c{}_b, \quad [K^a{}_b,\, P_c] = -\delta^a_c\, P_b. \tag{1.3.19}$$

For the local subalgebra we are going to choose the Cartan involution subalgebra of $GL\,(d)$, which is generated by the following set of generators $J_{ab} = K^a{}_b - K^b{}_a$. One can easily recognise these as the generators of $SO\,(d)$. Alternatively, one could choose $J_{ab} = \eta_{ac}\, K^c{}_b - \eta_{bc}\, K^c{}_a$, which would result in the tangent group $SO\,(1,\, d-1)$. The first choice leads to the Euclidean gravity, while the second one — to the Minkowski gravity. For simplicity, we are going to stick with the Euclidean case. The coset generator orthogonal to J_{ab} is $T_{ab} = K^a{}_b + K^b{}_a$. The group element can be written as

$$g = e^{x^a\, P_a}\, e^{h_a{}^b(x)\, K^a{}_b}. \tag{1.3.20}$$

The Cartan form is given by

$$\mathcal{V} = dx^\mu\, e_\mu{}^a\, \left(P_a + \Omega_{a,\,b}{}^c\, K^b{}_c\right) = dx^\mu\, e_\mu{}^a\, \left(P_a + \frac{1}{2}\, S_{a,\,bc}\, T_{bc} + \frac{1}{2}\, Q_{a,\,bc}\, J_{bc}\right), \tag{1.3.21}$$

where $\Omega_{a,\,b}{}^c = e_a{}^\mu\, \left(e^{-1}\partial_\mu\, e\right)_b{}^c$ is split, according to equation (1.3.13), into its coset and subalgebra parts: $S_{a,\,bc} = \Omega_{a,\,(bc)}$, $Q_{a,\,bc} = \Omega_{a,\,[bc]}$. $e_\mu{}^a$ is the vielbein of the theory. The most general form of the action that is second order in derivatives and is invariant under the tangent group transformations of equation (1.3.4) is

$$\begin{aligned} S = \int d^d x\, \det e \Big(& d_1\, D_a\, S_{a,\,bb} + d_2\, D_a\, S_{b,\,ab} + d_3\, S_{a,\,ac}\, S_{b,\,bc} \\ & + d_4\, S_{a,\,bc}\, S_{b,\,ac} + d_5\, S_{a,\,bc}\, S_{a,\,bc} + d_6\, S_{a,\,ab}\, S_{b,\,cc} + d_7\, S_{a,\,bb}\, S_{a,\,cc} \Big), \end{aligned} \tag{1.3.22}$$

where the covariant derivative is defined as follows

$$D_a\, S_{b,\,cd} = \partial_a\, S_{b,\,cd} + Q_{a,\,be}\, S_{e,\,cd} + Q_{a,\,ce}\, S_{a,\,ed} + Q_{a,\,de}\, S_{a,\,ce}. \tag{1.3.23}$$

As was mentioned before, the subalgebra part of the Cartan form $Q_{a,b}{}^{c}$ plays the role of the connection, due to the fact that it transforms inhomogeneously under the local subalgebra transformations. This action is generally not diffeomorphism invariant. Requiring that it is invariant under the general coordinate transformations results in specific values of coefficients d_i: $(d_1,\, d_2,\, d_3,\, d_4,\, d_5,\, d_6,\, d_7) = \left(-\frac{1}{2},\, \frac{1}{2},\, -\frac{1}{2},\, \frac{1}{2},\, -\frac{1}{4},\, \frac{1}{2},\, -\frac{1}{4}\right)$. With these values of the coefficients the action then takes the familiar form

$$S = \int d^d x \, \det e \, R, \tag{1.3.24}$$

where R is the Ricci scalar built from the metric tensor $g_{\mu\nu} = e_\mu{}^a \, e_\nu{}^a$. To summarise, we'd like to point out that the non-linear realisation of $IGL\,(d)$ does include the description of gravity, but the local symmetry of the non-linear realisation is not sufficient to fix the lagrangian uniquely. One has to impose additional symmetries in order to derive the well-known $R\sqrt{g}$ action. Alternatively, one can consider a simultaneous non-linear realisation of $IGL\,(d)$ and the conformal group [36], which also fixes the lagrangian uniquely.

As we will see later, the situation is drastically different in the E_{11} case. The E_{11} symmetry is infinitely richer than the finite $GL\,(d)$ group. Consequently, the E_{11} theory is capable of predicting the correct set of supergravity equations without requiring any additional symmetries to be imposed. This means that the gauge (and diffemomorpism) invariance of the E_{11} theory is actually an emerging property that follows from the local $I_c\,(E_{11})$ symmetry of the non-linear realisation.

1.4 E_{11} algebra and its non-linear realisation

The E_{11} is a Lorentzian Kac-Moody algebra of rank 11 that belongs to E series. The Dynkin diagram of this algebra is

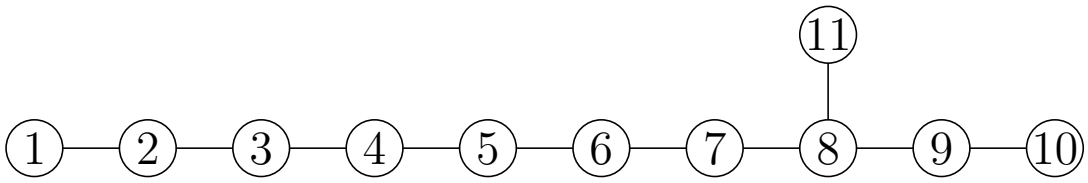


Figure 2: Dynkin diagram of E_{11}

As is the case with all Kac-Moody algebras the full listing of the generators and their commutators is unknown. The generators can be classified with respect to the level parameter that correspond to deletion of one of the nodes from the diagram. The choice of the deleted node plays an important role in the E_{11} model, as it determines the dimension of the resulting theory. In the general case, deleting node d from the diagram results in the $GL\,(d) \times E_{11-d}$ subalgebra.

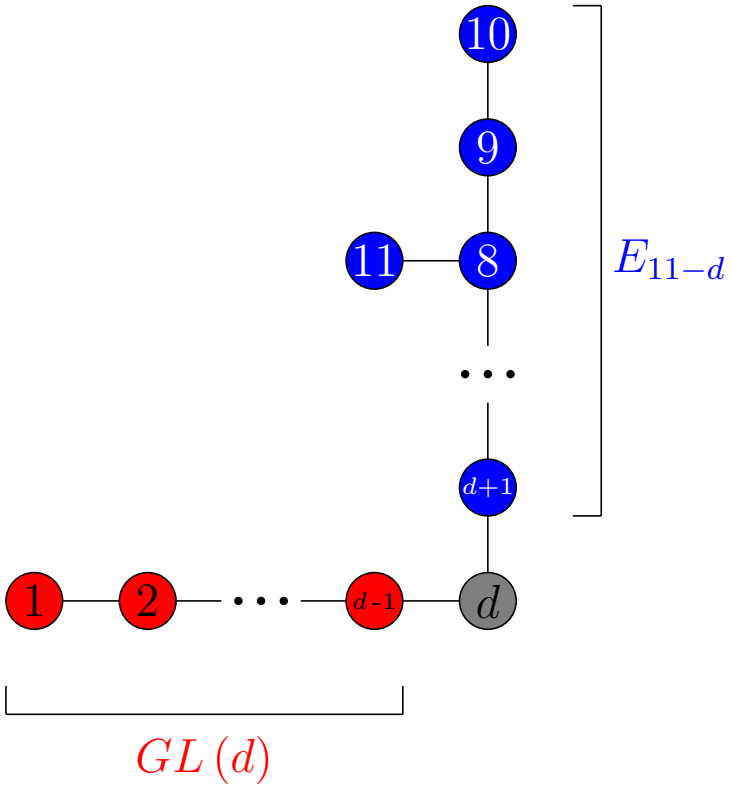


Figure 3: E_{11} algebra in d dimensions

More specifically, all the decompositions of the E_{11} algebra that result in the generalisations of the maximal supergravity theories in different dimensions are given in the following table.

Dimension	Deleted node	Level 0 subalgebra	I_c (Level 0 subalgebra)
11D	11	$GL\,(11)$	$SO\,(11)$
10D IIA	10	$O\,(10,\,10)$	$SO\,(10) \times SO\,(10)$
10D IIB	9	$GL\,(10) \times SL\,(2)$	$SO\,(10) \times SO\,(2)$
9D	9, 11	$GL\,(9) \times GL\,(2)$	$SO\,(9) \times SO\,(2)$
8D	8	$GL\,(8) \times SL\,(2) \times SL\,(3)$	$SO\,(8) \times SO\,(2) \times SO\,(3)$
7D	7	$GL\,(7) \times SL\,(5)$	$SO\,(7) \times SO\,(5)$
6D	6	$GL\,(6) \times SO\,(5,\,5)$	$SO\,(6) \times SO\,(5) \times SO\,(5)$
5D	5	$GL\,(5) \times E_6$	$SO\,(5) \times USp\,(8)$
4D	4	$GL\,(4) \times E_7$	$SO\,(4) \times SU\,(8)$
3D	3	$GL\,(3) \times E_8$	$SO\,(3) \times SO\,(16)$

Table 1: Decompositions of the E_{11} algebra

The subalgebras colored in red describe the symmetry of the space-time coordinates. As was shown in Section 1.3.2, the non-linear realisation of these $GL\,(d)$ algebras produces the equations for the gravitational sector of the supergravity theories. Consequently, they are usually referred to as the "gravity line". The blue colored algebras, on the other hand, represent the internal symmetries of the theory. As one can see they match perfectly the exceptional symmetry groups, given earlier in the introduction. In Chapter 2 we will present the commutation relations of E_{11} algebra for different space-time dimensions. They were constructed by first finding the representations that the generators belong to on each level, then assuming the most general commutation relations between them and, finally, implementing the Jacobi identity in order to fix all the free coefficients.

The non-linear realisation of E_{11} is conjectured to be the low energy limit of the

theory of strings and branes. As such, it has to contain all maximal supergravity theories and their exceptional symmetry groups. In Chapter 4 we will illustrate how the low levels of the five and eleven-dimensional non-linear realisations of E_{11} produce the field content and the dynamics of maximal supergravity theories in the corresponding dimensions. The higher levels of the decomposition contain an infinite number of fields, some of which are related to the supergravity ones by infinite chains of duality relations, while others are standalone fields, some of which could describe new physical phenomena.

$I_c\left(E_{11}\right)$ is an infinite-dimensional algebra, whose generators have the following form: $E_{\underline{\alpha}} - F_{\underline{\alpha}}$. Therefore, each generator of this algebra is a combination of a level m generator of E_{11} with the corresponding level $-m$ generator. We will refer to the resulting combination as level m generator of $I_c\left(E_{11}\right)$. This implies that the E_{11} theory possesses an infinite number of local invariances defined in equation (1.3.7) and classified by level parameter $m = 0,\ 1,\ \ldots$. On level 0 these symmetry simply ensures the Lorentz covariance of the equations of the theory. Level 1 transformations, on the other hand, play the key role in the E_{11} model, as they transform equations of the theory into each other and, therefore, determine the structure of the E_{11} multiplet that describes the dynamics of the fields. Higher level transformations can be obtained as compositions of level 1 transformations. Unlike the $IGL\left(d\right)$ model from Section 1.3.2, E_{11} theory does not admit a description in terms of the lagrangian. In order to find the dynamics of the fields one has to construct the equations of motion directly from the non-linear realisation. These equations transform into each other under the local transformations and, therefore, form a multiplet of $I_c\left(E_{11}\right)$. The power of E_{11} symmetry ensures that selecting one of the equations as a starting point allows one to reconstruct the whole multiplet by repeatedly applying the local transformations and demanding their closure. We will also see that the resulting theory will exhibit the gauge invariance, despite the fact gauge transformations are not encoded into the structure of the non-linear realisation. This method will be implemented for five and eleven-dimensional cases in Chapter 4.

2 E_{11} algebra in 5D and 10D Type IIB theories

2.1 5D

The E_{11} algebra in five dimensions can be obtained by deleting node 5 from the Dynkin diagram [21] as shown in Figure 4.

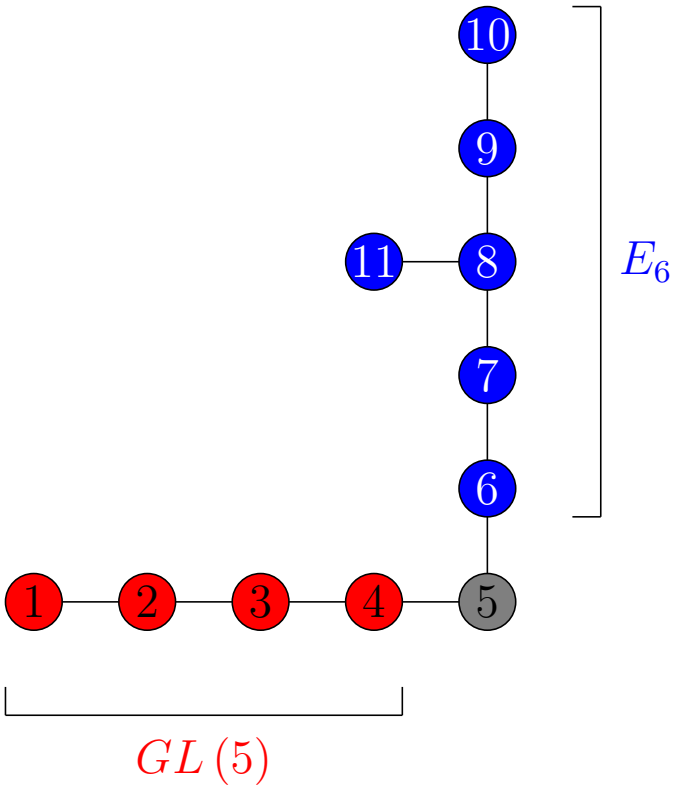


Figure 4: E_{11} algebra in 5 dimensions

The E_{11} algebra is then decomposed into representations of its $GL\,(5) \times E_6$ subalgebra. To describe the E_6 part of the algebra we are going to further decompose it in terms of its Cartan involution subalgebra $USp\,(8)$. These notations were proposed in [37] and are extremely useful for the implementation of Cartan involution transformations. **78** generators of E_6 are split into **36** adjoint representation of $USp\,(8)$ and its **42**

representation. They correspond to the following tensors

$$R^{\alpha_1\alpha_2}, \quad R^{\alpha_1\ldots\alpha_4}. \tag{2.1.1}$$

Here Greek indexes (α, β, γ, ...) range from 1 to 8. Generator $R^{\alpha_1\alpha_2}$ is symmetric, while $R^{\alpha_1\ldots\alpha_4}$ is antisymmetric and $USp\,(8)$-traceless: $\Omega_{\beta_1\beta_2}\,R^{\beta_1\beta_2\alpha_1\alpha_2} = 0$, where $\Omega_{\alpha_1\alpha_2} = \Omega_{[\alpha_1\alpha_2]}$ is the invariant $USp\,(8)$ metric. It will later be used to raise and lower $USp\,(8)$ indexes: $T_\alpha = \Omega_{\alpha\beta}\,T^\beta$, $T^\alpha = \Omega^{\alpha\beta}\,T_\beta$, $\Omega^{\alpha\gamma}\,\Omega_{\gamma\beta} = \delta^\alpha_\beta$. The generators of E_{11} up to level 3 are

Level	Generators
0	$K^a{}_b, \quad R^{\alpha_1\alpha_2}, \quad R^{\alpha_1\ldots\alpha_4}$
1	$R^{a\,\alpha_1\alpha_2}$
-1	$R_{a\,\alpha_1\alpha_2}$
2	$R^{a_1a_2\,\alpha_1\alpha_2}$
-2	$R_{a_1a_2\,\alpha_1\alpha_2}$
3	$R^{a_1a_2,\,b}, \quad R^{a_1a_2a_3\,\alpha_1\alpha_2}, \quad R^{a_1a_2a_3\,\alpha_1\ldots\alpha_4}$
-3	$R_{a_1a_2,\,b}, \quad R_{a_1a_2a_3\,\alpha_1\alpha_2}, \quad R_{a_1a_2a_3\,\alpha_1\ldots\alpha_4}$

where Latin indexes (a, b, c, ... $= 1, \ldots, 5$) label the $GL\,(5)$ representations. Generators $R^{\alpha_1\alpha_2}$, $R^{a_1a_2a_3\,\alpha_1\alpha_2}$ and $R_{a_1a_2a_3\,\alpha_1\alpha_2}$ are symmetric in $\alpha_1\alpha_2$, while others are antisymmetric in Greek indexes. All the generators with antisymmetric Greek indexes are $USp\,(8)$-traceless. Level 3 generator $R^{a_1a_2,\,b}$ obeys $R^{[a_1a_2,\,b]} = R_{[a_1a_2,\,b]} = 0$. The generators of the l_1 representation are

Level	Generators
0	P_a
1	$Z^{\alpha_1\alpha_2}$
2	$Z^{a\,\alpha_1\alpha_2}$
3	$Z^{ab}, \quad Z^{a_1a_2\,\alpha_1\alpha_2}, \quad Z^{a_1a_2\,\alpha_1\ldots\alpha_4}$

Here $Z^{a_1a_2\,\alpha_1\alpha_2}$ is symmetric in $\alpha_1\alpha_2$. We now give the E_{11} algebra when written in terms of the above generators. The commutators of the E_{11} generators with the GL(4) generators $K^a{}_b$ are

$$
\begin{aligned}
&[K^a{}_b,\, K^c{}_d] = \delta^c_b\, K^a{}_d - \delta^a_d\, K^c{}_b, && [K^a{}_b,\, R^{\alpha_1\alpha_2}] = 0, \quad [K^a{}_b,\, R^{\alpha_1\ldots\alpha_4}] = 0,\\
&[K^a{}_b,\, R^{c\,\alpha_1\alpha_2}] = \delta^c_b\, R^{a\,\alpha_1\alpha_2}, && [K^a{}_b,\, R_{c\,\alpha_1\alpha_2}] = -\,\delta^a_c\, R_{b\,\alpha_1\alpha_2},\\
&[K^a{}_b,\, R^{a_1a_2\,\alpha_1\alpha_2}] = 2\,\delta^{[a_1}_b\, R^{|a|a_2]\,\alpha_1\alpha_2}, && [K^a{}_b,\, R_{a_1a_2\,\alpha_1\alpha_2}] = -\,2\,\delta^a_{[a_1}\, R_{|b|a_2]\,\alpha_1\alpha_2},\\
&[K^a{}_b,\, R^{a_1a_2a_3\,\alpha_1\alpha_2}] = 3\,\delta^{[a_1}_b\, R^{|a|a_2a_3]\,(\alpha_1\alpha_2)}, && [K^a{}_b,\, R_{a_1a_2a_3\,\alpha_1\alpha_2}] = -\,3\,\delta^a_{[a_1}\, R_{|b|a_2a_3]\,(\alpha_1\alpha_2)},\\
&[K^a{}_b,\, R^{a_1a_2a_3\,\alpha_1\ldots\alpha_4}] = 3\,\delta^{[a_1}_b\, R^{|a|a_2a_3]\,\alpha_1\ldots\alpha_4}, && [K^a{}_b,\, R_{a_1a_2a_3\,\alpha_1\ldots\alpha_4}] = -\,3\,\delta^a_{[a_1}\, R_{|b|a_2a_3]\,\alpha_1\ldots\alpha_4},\\
&[K^a{}_b,\, R^{a_1a_2,\,c}] = 2\,\delta^{[a_1}_b\, R^{|a|a_2],\,c} + \delta^c_b\, R^{a_1a_2,\,a}, && [K^a{}_b,\, R_{a_1a_2,\,c}] = -\,2\,\delta^a_{[a_1}\, R_{|b|a_2],\,c} - \delta^a_c\, R_{a_1a_2,\,b}.
\end{aligned}
\tag{2.1.2}
$$

The commutators of the E_{11} generators with E_6 generators are determined by the representation of $USp\,(8)$ that this generator belongs to. With $R^{\alpha_1\alpha_2}$ we have

$$
\begin{aligned}
&\left[R^{\alpha_1\alpha_2},\, R^{\beta_1\beta_2}\right] = 2\,\Omega^{(\alpha_1(\beta_1}\, R^{\alpha_2)\beta_2)}, && \left[R^{\alpha_1\alpha_2},\, R^{\beta_1\ldots\beta_4}\right] = 4\,\Omega^{(\alpha_1[\beta_1}\, R^{\alpha_2)\beta_2\beta_3\beta_4]},\\
&\left[R^{\alpha_1\alpha_2},\, R^{a\,\beta_1\beta_2}\right] = 2\,\Omega^{(\alpha_1[\beta_1}\, R^{a\,\alpha_2)\beta_2]}, && [R^{\alpha_1\alpha_2},\, R_{a\,\beta_1\beta_2}] = 2\,\Omega^{(\alpha_1\gamma}\,\delta^{\alpha_2)}_{[\beta_1}\, R_{a\,\beta_2]\gamma},\\
&\left[R^{\alpha_1\alpha_2},\, R^{a_1a_2\,\beta_1\beta_2}\right] = 2\,\Omega^{(\alpha_1[\beta_1}\, R^{a_1a_2\,\alpha_2)\beta_2]}, && [R^{\alpha_1\alpha_2},\, R_{a_1a_2\,\beta_1\beta_2}] = 2\,\Omega^{(\alpha_1\gamma}\,\delta^{\alpha_2)}_{[\beta_1}\, R_{a_1a_2\,\beta_2]\gamma},\\
&\left[R^{\alpha_1\alpha_2},\, R^{a_1a_2,\,b}\right] = 0, && [R^{\alpha_1\alpha_2},\, R_{a_1a_2,\,b}] = 0,\\
&\left[R^{\alpha_1\alpha_2},\, R^{a_1a_2a_3\,\beta_1\beta_2}\right] = 2\,\Omega^{(\alpha_1(\beta_1}\, R^{a_1a_2a_3\,\alpha_2)\beta_2)}, &&\\
&[R^{\alpha_1\alpha_2},\, R_{a_1a_2a_3\,\beta_1\beta_2}] = 2\,\Omega^{(\alpha_1\gamma}\,\delta^{\alpha_2)}_{(\beta_1}\, R_{a_1a_2a_3\,\beta_2)\gamma}, &&\\
&\left[R^{\alpha_1\alpha_2},\, R^{a_1a_2a_3\,\beta_1\ldots\beta_4}\right] = 4\,\Omega^{(\alpha_1[\beta_1}\, R^{a_1a_2a_3\,\alpha_2)\beta_2\beta_3\beta_4]}, &&\\
&[R^{\alpha_1\alpha_2},\, R_{a_1a_2a_3\,\beta_1\ldots\beta_4}] = 4\,\Omega^{(\alpha_1\gamma}\,\delta^{\alpha_2)}_{[\beta_1}\, R_{a_1a_2a_3\,\beta_2\beta_3\beta_4]\gamma}. &&
\end{aligned}
\tag{2.1.3}
$$

For $R^{\alpha_1\ldots\alpha_4}$ we find

$$
\begin{aligned}
\left[R^{\alpha_1\ldots\alpha_4},\, R^{\beta_1\ldots\beta_4}\right] &= \frac{1}{2}\,\Omega^{[\alpha_1\alpha_2}\,\Omega^{[\beta_1\beta_2}\,\Omega^{\alpha_3\beta_3}\,R^{\alpha_4]\beta_4]} - \frac{2}{3}\,\Omega^{[\alpha_1[\beta_1}\,\Omega^{\alpha_2\beta_2}\,\Omega^{\alpha_3\beta_3}\,R^{\alpha_4]\beta_4]},\\
\left[R^{\alpha_1\ldots\alpha_4},\, R^{a\,\beta_1\beta_2}\right] &= \Omega^{[\alpha_1\alpha_2}\,\Omega^{\alpha_3[\beta_1}\,R^{a\,\alpha_4]\beta_2]} + \Omega^{[\alpha_1[\beta_1}\,\Omega^{\alpha_2\beta_2]}\,R^{a\,\alpha_3\alpha_4]}\\
&\quad - \frac{1}{4}\,\Omega^{\beta_1\beta_2}\,\Omega^{[\alpha_1\alpha_2}\,R^{a\,\alpha_3\alpha_4]} - \frac{1}{12}\,\Omega^{[\alpha_1\alpha_2}\,\Omega^{\alpha_3\alpha_4]}\,R^{a\,\beta_1\beta_2},\\
[R^{\alpha_1\ldots\alpha_4},\, R_{a\,\beta_1\beta_2}] &= -\,\Big(\Omega^{[\alpha_1\alpha_2}\,\Omega^{\alpha_3[\gamma_1}\,\delta^{\alpha_4]\gamma_2]}_{\beta_1\ \beta_2} + \Omega^{[\alpha_1[\gamma_1}\,\Omega^{\alpha_2\gamma_2]}\,\delta^{\alpha_3\alpha_4]}_{\beta_1\beta_2}
\end{aligned}
$$

$$
\begin{aligned}
&\qquad - \frac{1}{12}\,\Omega^{[\alpha_1\alpha_2}\,\Omega^{\alpha_3\alpha_4]}\,\delta^{\gamma_1\gamma_2}_{\beta_1\beta_2} + \frac{1}{4}\,\Omega_{\beta_1\beta_2}\,\Omega^{[\alpha_1\alpha_2}\,\Omega^{\alpha_3[\gamma_1}\,\Omega^{\alpha_4]\gamma_2]}\Big)\,R_{a\,\gamma_1\gamma_2},\\
\big[R^{\alpha_1\ldots\alpha_4},\,R^{a_1a_2\,\beta_1\beta_2}\big] &= -\Big(\Omega^{[\alpha_1\alpha_2}\,\Omega^{\alpha_3[\beta_1}\,R^{a_1a_2\,\alpha_4]\beta_2]} + \Omega^{[\alpha_1[\beta_1}\,\Omega^{\alpha_2\beta_2]}\,R^{a_1a_2\,\alpha_3\alpha_4]}\\
&\qquad - \frac{1}{4}\,\Omega^{\beta_1\beta_2}\,\Omega^{[\alpha_1\alpha_2}\,R^{a_1a_2\,\alpha_3\alpha_4]} - \frac{1}{12}\,\Omega^{[\alpha_1\alpha_2}\,\Omega^{\alpha_3\alpha_4]}\,R^{a_1a_2\,\beta_1\beta_2}\Big),
\end{aligned}
$$

$$
\begin{aligned}
[R^{\alpha_1\ldots\alpha_4},\,R_{a_1a_2\,\beta_1\beta_2}] &= \Big(\Omega^{[\alpha_1\alpha_2}\,\Omega^{\alpha_3[\gamma_1}\,\delta^{\alpha_4]\gamma_2]}_{\beta_1\ \ \beta_2} + \Omega^{[\alpha_1[\gamma_1}\,\Omega^{\alpha_2\gamma_2]}\,\delta^{\alpha_3\alpha_4]}_{\beta_1\beta_2}\\
&\qquad - \frac{1}{12}\,\Omega^{[\alpha_1\alpha_2}\,\Omega^{\alpha_3\alpha_4]}\,\delta^{\gamma_1\gamma_2}_{\beta_1\beta_2} + \frac{1}{4}\,\Omega_{\beta_1\beta_2}\,\Omega^{[\alpha_1\alpha_2}\,\Omega^{\alpha_3[\gamma_1}\,\Omega^{\alpha_4]\gamma_2]}\Big)\,R_{a_1a_2\,\gamma_1\gamma_2},\\
\big[R^{\alpha_1\ldots\alpha_4},\,R^{a_1a_2a_3\,\beta_1\ldots\beta_4}\big] &= \frac{1}{2}\,\Omega^{[\alpha_1\alpha_2}\,\Omega^{[\beta_1\beta_2}\,\Omega^{\alpha_3\beta_3}\,R^{a_1a_2a_3\,\alpha_4]\beta_4]}\\
&\qquad - \frac{2}{3}\,\Omega^{[\alpha_1[\beta_1}\,\Omega^{\alpha_2\beta_2}\,\Omega^{\alpha_3\beta_3}\,R^{a_1a_2a_3\,\alpha_4]\beta_4]},\\
[R^{\alpha_1\ldots\alpha_4},\,R_{a_1a_2a_3\,\beta_1\ldots\beta_4}] &= \frac{1}{2}\,\delta^{[\alpha_1}_{[\beta_1}\,\Omega^{\alpha_2\alpha_3}\,\Omega^{\alpha_4]\gamma}\,\Omega_{\beta_2\beta_3}\,R_{a_1a_2a_3\,\beta_4]\gamma} + \frac{2}{3}\,\delta^{[\alpha_1\alpha_2\alpha_3}_{[\beta_1\beta_2\beta_3}\,\Omega^{\alpha_4]\gamma}\,R_{a_1a_2a_3\,\beta_4]\gamma},\\
\big[R^{\alpha_1\ldots\alpha_4},\,R^{a_1a_2,\,b}\big] &= 0,\\
[R^{\alpha_1\ldots\alpha_4},\,R_{a_1a_2,\,b}] &= 0.
\end{aligned}
\tag{2.1.4}
$$

Note that level 2 and 3 generators $R^{a\,\alpha_1\alpha_2}$ and $R^{a_1a_2\,\alpha_1\alpha_2}$ belong to the same $USp\,(8)$ representation **27**, but different E_6 representations **27** and $\overline{\mathbf{27}}$. Consequently, they transform identically under $R^{\alpha_1\alpha_2}$, but differently under $R^{\alpha_1\ldots\alpha_4}$. l_1 generators $Z^{\alpha_1\alpha_2}$ and $Z^{a\,\alpha_1\alpha_2}$ have the same property. The commutation relations of the positive level E_{11} generators are given by

$$
\begin{aligned}
\big[R^{a\,\alpha_1\alpha_2},\,R^{b\,\beta_1\beta_2}\big] &= 4\,\Omega^{[\alpha_1[\beta_1}\,R^{ab\,\alpha_2]\beta_2]} - \frac{1}{2}\,\Omega^{\beta_1\beta_2}\,R^{ab\,\alpha_1\alpha_2} - \frac{1}{2}\,\Omega^{\alpha_1\alpha_2}\,R^{ab\,\beta_1\beta_2},\\
\big[R^{a\,\alpha_1\alpha_2},\,R^{b_1b_2\,\beta_1\beta_2}\big] &= 4\,\Omega^{[\alpha_1[\beta_1}\,R^{ab_1b_2\,\alpha_2]\beta_2]} + R^{ab_1b_2\,\alpha_1\alpha_2\beta_1\beta_2}\\
&\qquad + 2\,\left(\Omega^{\alpha_1[\beta_1}\,\Omega^{\alpha_2\beta_2]} - \frac{1}{8}\,\Omega^{\alpha_1\alpha_2}\,\Omega^{\beta_1\beta_2}\right)\,R^{b_1b_2,\,a},
\end{aligned}
\tag{2.1.5}
$$

The commutators of negative-level E_{11} generators are

$$
\begin{aligned}
[R_{a\,\alpha_1\alpha_2},\,R_{b\,\beta_1\beta_2}] &= 4\,\Omega_{[\alpha_1[\beta_1}\,R_{ab\,\alpha_2]\beta_2]} - \frac{1}{2}\,\Omega_{\beta_1\beta_2}\,R_{ab\,\alpha_1\alpha_2} - \frac{1}{2}\,\Omega_{\alpha_1\alpha_2}\,R_{ab\,\beta_1\beta_2},\\
[R_{a\,\alpha_1\alpha_2},\,R_{b_1b_2\,\beta_1\beta_2}] &= 4\,\Omega_{[\alpha_1[\beta_1}\,R_{ab_1b_2\,\alpha_2]\beta_2]} + R_{ab_1b_2\,\alpha_1\alpha_2\beta_1\beta_2}\\
&\qquad + 2\,\left(\Omega_{\alpha_1[\beta_1}\,\Omega_{\alpha_2\beta_2]} - \frac{1}{8}\,\Omega_{\alpha_1\alpha_2}\,\Omega_{\beta_1\beta_2}\right)\,R_{b_1b_2,\,a}.
\end{aligned}
\tag{2.1.6}
$$

The commutators between the positive and negative level generators of E_{11} up to level 4 with level ± 1 generator are given by

$$
\begin{aligned}
[R^{a\,\alpha_1\alpha_2},\, R_{b\,\beta_1\beta_2}] &= 4\,\delta^a_b\,\delta^{[\alpha_1}_{[\beta_1}\,\Omega_{\beta_2]\gamma}\,R^{\alpha_2]\gamma} + 12\,\delta^a_b\,\Omega_{\beta_1\gamma_1}\,\Omega_{\beta_2\gamma_2}\,R^{\alpha_1\alpha_2\gamma_1\gamma_2} \\
&\quad + 2\,\left(\delta^{\alpha_1\alpha_2}_{\beta_1\beta_2} + \frac{1}{8}\,\Omega^{\alpha_1\alpha_2}\,\Omega_{\beta_1\beta_2}\right)\left(K^a{}_b - \frac{1}{3}\delta^a_b\,K^c{}_c\right), \\
\left[R_{a\,\alpha_1\alpha_2},\, R^{b_1b_2\,\beta_1\beta_2}\right] &= -\,8\,\delta^{[\beta_1}_{[\alpha_1}\,\Omega_{\alpha_2]\gamma}\,\delta^{[b_1}_a\,R^{b_2]\,\beta_2]\gamma} - \Omega^{\beta_1\beta_2}\,\Omega_{\alpha_1\gamma_1}\,\Omega_{\alpha_2\gamma_2}\,\delta^{[b_1}_a\,R^{b_2]\,\gamma_1\gamma_2} \\
&\quad + \Omega_{\alpha_1\alpha_2}\,\delta^{[b_1}_a\,R^{b_2]\,\beta_1\beta_2}, \\
[R^{a\,\alpha_1\alpha_2},\, R_{b_1b_2\,\beta_1\beta_2}] &= -\,8\,\delta^{[\alpha_1}_{[\beta_1}\,\Omega^{\alpha_2]\gamma}\,\delta^a_{[b_1}\,R_{b_2]\,\beta_2]\gamma} - \Omega_{\beta_1\beta_2}\,\Omega^{\alpha_1\gamma_1}\,\Omega^{\alpha_2\gamma_2}\,\delta^a_{[b_1}\,R_{b_2]\,\gamma_1\gamma_2} \\
&\quad + \Omega^{\alpha_1\alpha_2}\,\delta^a_{[b_1}\,R_{b_2]\,\beta_1\beta_2}, \\
\left[R_{a\,\alpha_1\alpha_2},\, R^{b_1b_2b_3\,\beta_1\beta_2}\right] &= 6\,\Omega_{[\alpha_1\gamma}\,\delta^{(\beta_1}_{\alpha_2]}\,\delta^{[b_1}_a\,R^{b_2b_3]\,\beta_2)\gamma}, \\
[R^{a\,\alpha_1\alpha_2},\, R_{b_1b_2b_3\,\beta_1\beta_2}] &= 6\,\Omega^{[\alpha_1\gamma}\,\delta^{\alpha_2]}_{(\beta_1}\,\delta^a_{[b_1}\,R_{b_2b_3]\,\beta_2)\gamma}, \\
\left[R_{a\,\alpha_1\alpha_2},\, R^{b_1b_2b_3\,\beta_1\ldots\beta_4}\right] &= 36\,\Big(\delta^{[\beta_1\beta_2}_{\alpha_1\alpha_2}\,\delta^{[b_1}_a\,R^{b_2b_3]\,\beta_3\beta_4]} + \Omega^{[\beta_1\beta_2}\,\Omega_{[\alpha_1\gamma}\,\delta^{\beta_3}_{\alpha_2]}\,\delta^{[b_1}_a\,R^{b_2b_3]\,\beta_4]\gamma} \\
&\quad + \frac{1}{4}\,\Omega_{\alpha_1\alpha_2}\,\Omega^{[\beta_1\beta_2}\,\delta^{[b_1}_a\,R^{b_2b_3]\,\beta_3\beta_4]} \\
&\quad - \frac{1}{12}\,\Omega^{[\beta_1\beta_2}\,\Omega^{\beta_3\beta_4]}\,\Omega_{\alpha_1\gamma_1}\,\Omega_{\alpha_2\gamma_2}\,\delta^{[b_1}_a\,R^{b_2b_3]\,\gamma_1\gamma_2}\Big), \\
[R^{a\,\alpha_1\alpha_2},\, R_{b_1b_2b_3\,\beta_1\ldots\beta_4}] &= 36\,\Big(\delta^{\alpha_1\alpha_2}_{[\beta_1\beta_2}\,\delta^a_{[b_1}\,R_{b_2b_3]\,\beta_3\beta_4]} + \Omega_{[\beta_1\beta_2}\,\Omega^{[\alpha_1\gamma}\,\delta^{\alpha_2]}_{\beta_3]}\,\delta^a_{[b_1}\,R_{b_2b_3]\,\beta_4]\gamma} \\
&\quad + \frac{1}{4}\,\Omega^{\alpha_1\alpha_2}\,\Omega_{[\beta_1\beta_2}\,\delta^a_{[b_1}\,R_{b_2b_3]\,\beta_3\beta_4]} \\
&\quad - \frac{1}{12}\,\Omega_{[\beta_1\beta_2}\,\Omega_{\beta_3\beta_4]}\,\Omega^{\alpha_1\gamma_1}\,\Omega^{\alpha_2\gamma_2}\,\delta^a_{[b_1}\,R_{b_2b_3]\,\gamma_1\gamma_2}\Big), \\
\left[R_{a\,\alpha_1\alpha_2},\, R^{b_1b_2,\,b}\right] &= \Omega_{\alpha_1\beta_1}\,\Omega_{\alpha_2\beta_2}\,\left(\delta^b_a\,R^{b_1b_2\,\beta_1\beta_2} - \delta^{[b}_a\,R^{b_1b_2]\,\beta_1\beta_2}\right), \\
[R^{a\,\alpha_1\alpha_2},\, R_{b_1b_2,\,b}] &= \Omega^{\alpha_1\beta_1}\,\Omega^{\alpha_2\beta_2}\,\left(\delta^a_b\,R_{b_1b_2\,\beta_1\beta_2} - \delta^a_{[b}\,R_{b_1b_2]\,\beta_1\beta_2}\right),
\end{aligned}
\tag{2.1.7}
$$

The Cartan involution acts on the generators of E_{11} as follows

Generator	$I_c\big(\text{Generator}\big)$	Generator	$I_c\big(\text{Generator}\big)$
$K^a{}_b$	$-K^b{}_a$	$R^{a_1a_2\,\alpha_1\alpha_2}$	$-\Omega^{\alpha_1\beta_1}\,\Omega^{\alpha_2\beta_2}\,R_{a_1a_2\,\beta_1\beta_2}$
$R^{\alpha_1\alpha_2}$	$R^{\alpha_1\alpha_2}$	$R^{a_1a_2,\,b}$	$R_{a_1a_2,\,b}$
$R^{\alpha_1\ldots\alpha_4}$	$-R^{\alpha_1\ldots\alpha_4}$	$R^{a_1a_2a_3\,\alpha_1\alpha_2}$	$-\Omega^{\alpha_1\beta_1}\,\Omega^{\alpha_2\beta_2}\,R_{a_1a_2a_3\,\beta_1\beta_2}$
$R^{a\,\alpha_1\alpha_2}$	$-\Omega^{\alpha_1\beta_1}\,\Omega^{\alpha_2\beta_2}\,R_{a\,\beta_1\beta_2}$	$R^{a_1a_2a_3\,\alpha_1\ldots\alpha_4}$	$\Omega^{\alpha_1\beta_1}\,\Omega^{\alpha_2\beta_2}\,\Omega^{\alpha_3\beta_3}\,\Omega^{\alpha_4\beta_4}\,R_{a_1a_2a_3\,\beta_1\ldots\beta_4}$

We now give the commutators between the generators of E_{11} and those of the l_1 representation up to level 2. The commutation relations between the later and the generators of $GL\,(5)$ are given by

$$\begin{aligned}
&[K^a{}_b,\,P_c] = -\,\delta^a_c\,P_b + \tfrac{1}{2}\,\delta^a_b\,P_c, \quad [K^a{}_b,\,Z^{\alpha_1\alpha_2}] = \tfrac{1}{2}\,\delta^a_b\,Z^{\alpha_1\alpha_2},\\
&[K^a{}_b,\,Z^{c\,\alpha_1\alpha_2}] = \delta^c_b\,Z^{a\,\alpha_1\alpha_2} + \tfrac{1}{2}\,\delta^a_b\,Z^{c\,\alpha_1\alpha_2},
\end{aligned} \tag{2.1.8}$$

while with the generators of E_6 we have

$$\begin{aligned}
[R^{\alpha_1\alpha_2},\,P_a] &= 0, \qquad \big[R^{\alpha_1\alpha_2},\,Z^{\beta_1\beta_2}\big] = 2\,\Omega^{(\alpha_1[\beta_1}\,Z^{\alpha_2)\beta_2]},\\
\big[R^{\alpha_1\alpha_2},\,Z^{a\,\beta_1\beta_2}\big] &= 2\,\Omega^{(\alpha_1[\beta_1}\,Z^{a\,\alpha_2)\beta_2]}, \qquad [R^{\alpha_1\ldots\alpha_4},\,P_a] = 0,\\
\big[R^{\alpha_1\ldots\alpha_4},\,Z^{\beta_1\beta_2}\big] &= \Omega^{[\alpha_1\alpha_2}\,\Omega^{\alpha_3[\beta_1}\,Z^{\alpha_4]\beta_2]} + \Omega^{[\alpha_1[\beta_1}\,\Omega^{\alpha_2\beta_2]}\,Z^{\alpha_3\alpha_4]}\\
&\quad - \frac{1}{4}\,\Omega^{\beta_1\beta_2}\,\Omega^{[\alpha_1\alpha_2}\,Z^{\alpha_3\alpha_4]} - \frac{1}{12}\,\Omega^{[\alpha_1\alpha_2}\,\Omega^{\alpha_3\alpha_4]}\,Z^{\beta_1\beta_2},\\
\big[R^{\alpha_1\ldots\alpha_4},\,Z^{a\,\beta_1\beta_2}\big] &= -\,\Big(\Omega^{[\alpha_1\alpha_2}\,\Omega^{\alpha_3[\beta_1}\,Z^{a\,\alpha_4]\beta_2]} + \Omega^{[\alpha_1[\beta_1}\,\Omega^{\alpha_2\beta_2]}\,Z^{a\,\alpha_3\alpha_4]}\\
&\quad - \frac{1}{4}\,\Omega^{\beta_1\beta_2}\,\Omega^{[\alpha_1\alpha_2}\,Z^{a\,\alpha_3\alpha_4]} - \frac{1}{12}\,\Omega^{[\alpha_1\alpha_2}\,\Omega^{\alpha_3\alpha_4]}\,Z^{a\,\beta_1\beta_2}\Big).
\end{aligned} \tag{2.1.9}$$

Commutators with the level rising generators are

$$\begin{aligned}
&[R^{a\,\alpha_1\alpha_2},\,P_b] = \delta^a_b\,Z^{\alpha_1\alpha_2}, \qquad [R^{a_1a_2\,\alpha_1\alpha_2},\,P_b] = -\,2\,\delta^{[a_1}_b\,Z^{a_2]\,\alpha_1\alpha_2},\\
&\big[R^{a\,\alpha_1\alpha_2},\,Z^{\beta_1\beta_2}\big] = 4\,\Omega^{[\alpha_1[\beta_1}\,Z^{a\,\alpha_2]\beta_2]} - \tfrac{1}{2}\,\Omega^{\beta_1\beta_2}\,Z^{a\,\alpha_1\alpha_2} - \tfrac{1}{2}\,\Omega^{\alpha_1\alpha_2}\,Z^{a\,\beta_1\beta_2}.
\end{aligned} \tag{2.1.10}$$

Commutators with the negative level generators are

$$\begin{aligned}
&[R_{a\,\alpha_1\alpha_2},\,P_b] = 0, \qquad \big[R_{a\,\alpha_1\alpha_2},\,Z^{\beta_1\beta_2}\big] = 2\,\left(\delta^{\beta_1\beta_2}_{\alpha_1\alpha_2} + \tfrac{1}{8}\,\Omega_{\alpha_1\alpha_2}\,\Omega^{\beta_1\beta_2}\right)\,P_a,\\
&\big[R_{a\,\alpha_1\alpha_2},\,Z^{b\,\beta_1\beta_2}\big] = 4\,\delta^b_a\,\left(\Omega_{[\alpha_1\gamma}\,\delta^{[\beta_1}_{\alpha_2]}\,Z^{\beta_2]\gamma} + \tfrac{1}{8}\,\Omega_{\alpha_1\alpha_2}\,Z^{\beta_1\beta_2} - \tfrac{1}{8}\,\Omega^{\beta_1\beta_2}\,\Omega_{\alpha_1\gamma_1}\,\Omega_{\alpha_2\gamma_2}\,Z^{\gamma_1\gamma_2}\right),\\
&\big[R_{a_1a_2\,\alpha_1\alpha_2},\,Z^{b\,\beta_1\beta_2}\big] = 4\,\left(\delta^{\beta_1\beta_2}_{\alpha_1\alpha_2} + \tfrac{1}{8}\,\Omega_{\alpha_1\alpha_2}\,\Omega^{\beta_1\beta_2}\right)\,\delta^b_{[a_1}\,P_{a_2]}.
\end{aligned} \tag{2.1.11}$$

2.2 10D

In order to construct the ten-dimensional theory from the E_{11} algebra one has to delete a node from the Dynkin diagram in such a way that after the deletion the diagram still contains a sequence of nine connected nodes, which will lead to the ten-dimensional gravity. There are two different ways to do this [19, 38]: deleting node 10 and deleting node 9. Deleting node 10 results in Type IIA theory.

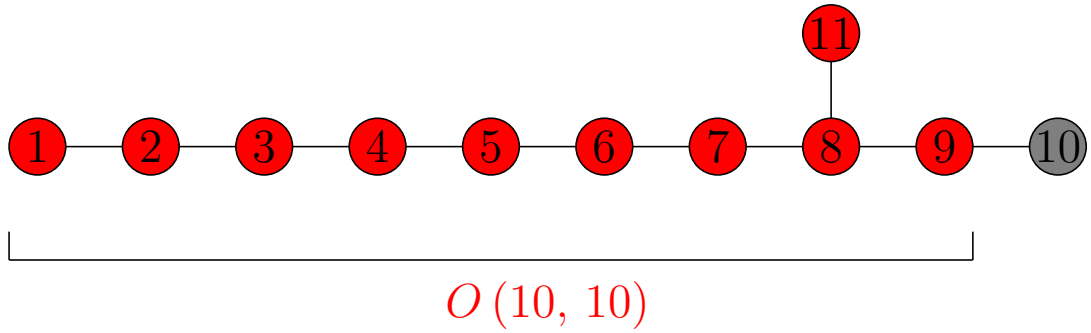


Figure 5: Type IIA E_{11} algebra in 10 dimensions

Deleting node 9, on the other hand, yields Type IIB theory with an internal symmetry group $SL(2, \mathbb{R})$.

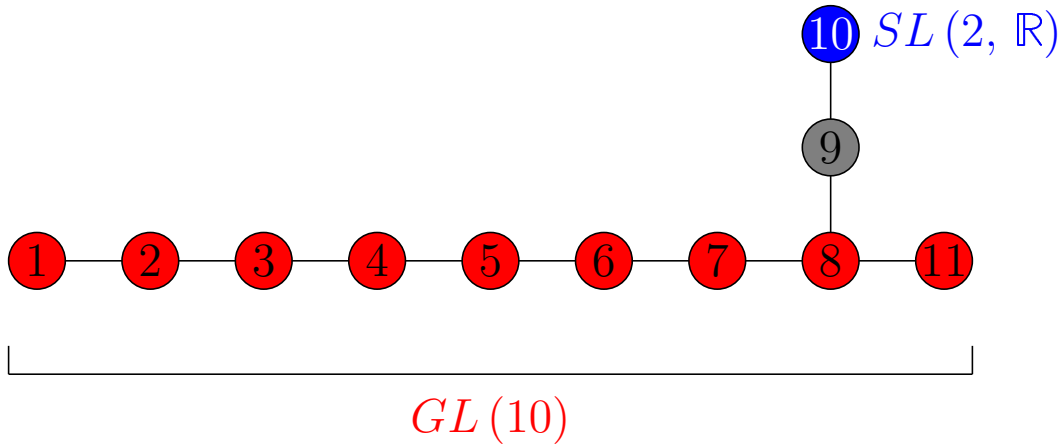


Figure 6: Type IIB E_{11} algebra in 10 dimensions

Type IIB algebra was first partially constructed in [19] and later expanded in [38]. Here we give the complete set of commutators for this algebra up to level 4. The commutators of $E_{11} \ltimes l_1$, decomposed into representations of $GL(10) \times SL(2, \mathbb{R})$ subalgebra are

Level	Generators
0	$K^a{}_b, \quad R_{\alpha\beta}$
1	$R^{a_1 a_2}_{\alpha}$
-1	$R^{\alpha}_{a_1 a_2}$
2	$R^{a_1 \ldots a_4}$
-2	$R_{a_1 \ldots a_4}$
3	$R^{a_1 \ldots a_6}_{\alpha}$
-3	$R^{\alpha}_{a_1 \ldots a_6}$
4	$R^{a_1 \ldots a_8}_{\alpha\beta}, \quad R^{a_1 \ldots a_7, c}$
-4	$R^{\alpha\beta}_{a_1 \ldots a_8}, \quad R_{a_1 \ldots a_7, c}$

Here Latin indexes (a, b, c, ... $= 1, \ldots, 10$) label the $GL(10)$ representations, while the Greek indexes (α, β, γ, ... $= 1, 2$) correspond to the spinor representation of $SL(2)$. All the generators with two Greek indexes are symmetric in them, e.g. $R_{\alpha\beta} = R_{\beta\alpha}$. Level 4 generator obeys $R^{[a_1 \ldots a_7, c]} = R_{[a_1 \ldots a_7, c]} = 0$. The generators of l_1 representation are

Level	Generators
0	P_a
1	Z^a_{α}
2	$Z^{a_1 a_2 a_3}$
3	$Z^{a_1 \ldots a_5}_{\alpha}$
4	$Z^{a_1 \ldots a_7}, \quad Z^{a_1 \ldots a_6, b}, \quad Z^{a_1 \ldots a_7}_{\alpha\beta}$

where $Z^{[a_1 \ldots a_6, c]} = Z_{[a_1 \ldots a_6, c]} = 0$ and $Z^{a_1 \ldots a_7}_{\alpha\beta}$ is symmetric in α and β. The commutators

with the SL(10) generators $K^a{}_b$ are

$$
\begin{aligned}
&[K^a{}_b,\, K^c{}_d] = \delta^c_b\, K^a{}_d - \delta^a_d\, K^c{}_b, \qquad && [K^a{}_b,\, R_{\alpha\beta}] = 0,\\
&[K^a{}_b,\, R^{a_1a_2}_{\alpha}] = 2\,\delta^{[a_1}_b\, R^{|a|a_2]}_{\alpha}, && \left[K^a{}_b,\, R^{\alpha}_{a_1a_2}\right] = -\,2\,\delta^{a}_{[a_1}\, R^{\alpha}_{|b|a_2]},\\
&[K^a{}_b,\, R^{a_1\ldots a_4}] = 4\,\delta^{[a_1}_b\, R^{|a|a_2a_3a_4]}, && [K^a{}_b,\, R_{a_1\ldots a_4}] = -\,4\,\delta^{a}_{[a_1}\, R_{|b|a_2a_3a_4]},\\
&[K^a{}_b,\, R^{a_1\ldots a_6}_{\alpha}] = 6\,\delta^{[a_1}_b\, R^{|a|a_2\ldots a_6]}_{\alpha}, && \left[K^a{}_b,\, R^{\alpha}_{a_1\ldots a_6}\right] = -\,6\,\delta^{a}_{[a_1}\, R^{\alpha}_{|b|a_2\ldots a_6]},\\
&\left[K^a{}_b,\, R^{a_1\ldots a_8}_{\alpha\beta}\right] = 8\,\delta^{[a_1}_b\, R^{|a|a_2\ldots a_8]}_{\alpha\beta}, && \left[K^a{}_b,\, R^{\alpha\beta}_{a_1\ldots a_8}\right] = -\,8\,\delta^{a}_{[a_1}\, R^{\alpha\beta}_{|b|a_2\ldots a_8]},\\
&[K^a{}_b,\, R^{a_1\ldots a_7,\,c}] = 7\,\delta^{[a_1}_b\, R^{|a|a_2\ldots a_7],\,c} + \delta^c_b\, R^{a_1\ldots a_7,\,a}, &&\\
&[K^a{}_b,\, R_{a_1\ldots a_7,\,c}] = -\,7\,\delta^{a}_{[a_1}\, R_{|b|a_2\ldots a_7],\,c} - \delta^a_c\, R_{a_1\ldots a_7,\,b}. &&
\end{aligned}
\tag{2.2.1}
$$

The commutators of the E_{11} generators with the $SL\,(2)$ generators $R_{\alpha\beta}$ are

$$
\begin{aligned}
&[R_{\alpha\beta},\, R_{\gamma\delta}] = \delta^{\sigma}_{(\alpha}\,\varepsilon_{\beta)\gamma}\, R_{\sigma\delta} + \delta^{\sigma}_{(\alpha}\,\varepsilon_{\beta)\delta}\, R_{\gamma\sigma},\\
&\left[R_{\alpha\beta},\, R^{a_1a_2}_{\gamma}\right] = \delta^{\delta}_{(\alpha}\,\varepsilon_{\beta)\gamma}\, R^{a_1a_2}_{\delta}, && \left[R_{\alpha\beta},\, R^{\gamma}_{a_1a_2}\right] = -\,\delta^{\gamma}_{(\alpha}\,\varepsilon_{\beta)\delta}\, R^{\delta}_{a_1a_2},\\
&[R_{\alpha\beta},\, R^{a_1\ldots a_4}] = 0, && [R_{\alpha\beta},\, R_{a_1\ldots a_4}] = 0,\\
&\left[R_{\alpha\beta},\, R^{a_1\ldots a_6}_{\gamma}\right] = \delta^{\delta}_{(\alpha}\,\varepsilon_{\beta)\gamma}\, R^{a_1\ldots a_6}_{\delta}, && \left[R_{\alpha\beta},\, R^{\gamma}_{a_1\ldots a_6}\right] = -\,\delta^{\gamma}_{(\alpha}\,\varepsilon_{\beta)\delta}\, R^{\delta}_{a_1\ldots a_6},\\
&\left[R_{\alpha\beta},\, R^{a_1\ldots a_7,\,b}\right] = 0, && [R_{\alpha\beta},\, R_{a_1\ldots a_7,\,b}] = 0,\\
&\left[R_{\alpha\beta},\, R^{a_1\ldots a_8}_{\gamma\delta}\right] = \delta^{\sigma}_{(\alpha}\,\varepsilon_{\beta)\gamma}\, R^{a_1\ldots a_8}_{\sigma\delta} + \delta^{\sigma}_{(\alpha}\,\varepsilon_{\beta)\delta}\, R^{a_1\ldots a_8}_{\gamma\sigma},\\
&\left[R_{\alpha\beta},\, R^{\gamma\delta}_{a_1\ldots a_8}\right] = -\,\delta^{\gamma}_{(\alpha}\,\varepsilon_{\beta)\sigma}\, R^{\sigma\delta}_{a_1\ldots a_8} - \delta^{\delta}_{(\alpha}\,\varepsilon_{\beta)\sigma}\, R^{\gamma\sigma}_{a_1\ldots a_8}.
\end{aligned}
\tag{2.2.2}
$$

The commutators of the positive level E_{11} generators with each other are given by

$$
\begin{aligned}
&\left[R^{a_1a_2}_{\alpha},\, R^{a_3a_4}_{\beta}\right] = -\,\varepsilon_{\alpha\beta}\, R^{a_1\ldots a_4}, && [R^{a_1a_2}_{\alpha},\, R^{a_3\ldots a_6}] = 4\, R^{a_1\ldots a_6}_{\alpha},\\
&\left[R^{a_1a_2}_{\alpha},\, R^{a_3\ldots a_8}_{\beta}\right] = -\,R^{a_1\ldots a_8}_{\alpha\beta} - \varepsilon_{\alpha\beta}\, R^{a_1a_2[a_3\ldots a_7,\,a_8]}, && [R^{a_1\ldots a_4},\, R^{a_5\ldots a_8}] = \tfrac{8}{3}\, R^{a_1\ldots a_4[a_5a_6a_7,\,a_8]}.
\end{aligned}
\tag{2.2.3}
$$

For negative level generators we have

$$
\begin{aligned}
&\left[R^{\alpha}_{a_1a_2},\, R^{\beta}_{a_3a_4}\right] = -\,\varepsilon^{\alpha\beta}\, R_{a_1\ldots a_4}, && \left[R^{\alpha}_{a_1a_2},\, R_{a_3\ldots a_6}\right] = 4\, R^{\alpha}_{a_1\ldots a_6},\\
&\left[R^{\alpha}_{a_1a_2},\, R^{\beta}_{a_3\ldots a_8}\right] = -\,R^{\alpha\beta}_{a_1\ldots a_8} - \varepsilon^{\alpha\beta}\, R_{a_1a_2[a_3\ldots a_7,\,a_8]}, && [R_{a_1\ldots a_4},\, R_{a_5\ldots a_8}] = \tfrac{8}{3}\, R_{a_1\ldots a_4[a_5a_6a_7,\,a_8]}.
\end{aligned}
\tag{2.2.4}
$$

To find the commutators between positive and negative level generators we need to

utilize the Jacobi identities. These commutators up to level 3 are given by

$$
\begin{aligned}
&\Big[R_{\alpha}^{a_1a_2},\, R_{b_1b_2}^{\beta}\Big] = 4\,\delta_{\alpha}^{\beta}\,\delta_{[b_1}^{[a_1}\,K^{a_2]}{}_{b_2]} - \tfrac{1}{2}\,\delta_{\alpha}^{\beta}\,\delta_{b_1b_2}^{a_1a_2}\,K^{d}{}_{d} - 2\,\delta_{b_1b_2}^{a_1a_2}\,\varepsilon^{\beta\gamma}\,R_{\alpha\gamma},\\
&[R^{a_1\ldots a_4},\, R_{b_1\ldots b_4}] = 12\,\delta_{b_1\ldots b_4}^{a_1\ldots a_4}\,K^{d}{}_{d} - 96\,\delta_{[b_1b_2b_3}^{[a_1a_2a_3}\,K^{a_4]}{}_{b_4]},\\
&\Big[R_{\alpha}^{a_1\ldots a_6},\, R_{b_1\ldots b_6}^{\beta}\Big] = 270\,\delta_{\alpha}^{\beta}\,\delta_{[b_1\ldots b_5}^{[a_1\ldots a_5}\,K^{a_6]}{}_{b_6]} - \tfrac{135}{4}\,\delta_{\alpha}^{\beta}\,\delta_{b_1\ldots b_6}^{a_1\ldots a_6}\,K^{d}{}_{d} - 45\,\delta_{b_1\ldots b_6}^{a_1\ldots a_6}\,\varepsilon^{\beta\gamma}\,R_{\alpha\gamma},\\
&[R_{\alpha}^{a_1a_2},\, R_{b_1\ldots b_4}] = -12\,\varepsilon_{\alpha\beta}\,\delta_{[b_1b_2}^{a_1a_2}\,R_{b_3b_4]}^{\beta}, \quad \Big[R_{a_1a_2}^{\alpha},\, R^{b_1\ldots b_4}\Big] = -12\,\varepsilon^{\alpha\beta}\,\delta_{[a_1a_2}^{[b_1b_2}\,R_{\beta}^{b_3b_4]},\\
&\Big[R_{\alpha}^{a_1a_2},\, R_{b_1\ldots b_6}^{\beta}\Big] = \tfrac{15}{2}\,\delta_{\alpha}^{\beta}\,\delta_{[b_1b_2}^{a_1a_2}\,R_{b_3\ldots b_6]}, \quad \Big[R_{a_1a_2}^{\alpha},\, R_{\beta}^{b_1\ldots b_6}\Big] = \tfrac{15}{2}\,\delta_{\beta}^{\alpha}\,\delta_{[a_1a_2}^{[b_1b_2}\,R^{b_3\ldots b_6]},\\
&\Big[R^{a_1\ldots a_4},\, R_{b_1\ldots b_6}^{\alpha}\Big] = 90\,\delta_{[b_1\ldots b_4}^{a_1\ldots a_4}\,R_{b_5b_6]}^{\alpha}, \qquad \Big[R_{a_1\ldots a_4},\, R_{\alpha}^{b_1\ldots b_6}\Big] = 90\,\delta_{[a_1\ldots a_4}^{[b_1\ldots b_4}\,R_{\alpha}^{b_5b_6]}.
\end{aligned}
\tag{2.2.5}
$$

The commutators of level ∓ 4 generators with level ± 1 ones are

$$
\begin{aligned}
&\Big[R_{\alpha}^{a_1a_2},\, R_{b_1\ldots b_8}^{\beta\gamma}\Big] = -56\,\delta_{\alpha}^{(\beta}\,\delta_{[b_1b_2}^{a_1a_2}\,R_{b_3\ldots b_8]}^{\gamma)}, \quad \Big[R_{a_1a_2}^{\alpha},\, R_{\beta\gamma}^{b_1\ldots b_8}\Big] = -56\,\delta_{(\beta}^{\alpha}\,\delta_{[a_1a_2}^{[b_1b_2}\,R_{\gamma)}^{b_3\ldots b_8]},\\
&[R_{\alpha}^{a_1a_2},\, R_{b_1\ldots b_7,\,b}] = -252\,\varepsilon_{\alpha\beta}\,\delta_{[b_1b_2}^{a_1a_2}\,R_{b_3\ldots b_7]b}^{\beta} + 252\,\varepsilon_{\alpha\beta}\,\delta_{[b_1b_2}^{a_1a_2}\,R_{b_3\ldots b_7b]}^{\beta},\\
&\Big[R_{a_1a_2}^{\alpha},\, R^{b_1\ldots b_7,\,b}\Big] = -252\,\varepsilon^{\alpha\beta}\,\delta_{[a_1a_2}^{[b_1b_2}\,R_{\beta}^{b_3\ldots b_7]b} + 252\,\varepsilon^{\alpha\beta}\,\delta_{[a_1a_2}^{[b_1b_2}\,R_{\beta}^{b_3\ldots b_7b]},
\end{aligned}
\tag{2.2.6}
$$

with levels ± 2:

$$
\begin{aligned}
&\Big[R^{a_1\ldots a_4},\, R_{b_1\ldots b_8}^{\alpha\beta}\Big] = 0, \qquad\qquad \Big[R_{a_1\ldots a_4},\, R_{\alpha\beta}^{b_1\ldots b_8}\Big] = 0,\\
&[R^{a_1\ldots a_4},\, R_{b_1\ldots b_7,\,b}] = -1260\,\delta_{[b_1\ldots b_4}^{a_1\ldots a_4}\,R_{b_5b_6b_7]b} + 1260\,\delta_{[b_1\ldots b_4}^{a_1\ldots a_4}\,R_{b_5b_6b_7b]},\\
&\Big[R_{a_1\ldots a_4},\, R^{b_1\ldots b_7,\,b}\Big] = -1260\,\delta_{[a_1\ldots a_4}^{[b_1\ldots b_4}\,R^{b_5b_6b_7]b} + 1260\,\delta_{[a_1\ldots a_4}^{[b_1\ldots b_4}\,R^{b_5b_6b_7b]},
\end{aligned}
\tag{2.2.7}
$$

with levels ± 3:

$$
\begin{aligned}
&\Big[R_{\alpha}^{a_1\ldots a_6},\, R_{b_1\ldots b_8}^{\beta\gamma}\Big] = 1260\,\delta_{\alpha}^{(\beta}\,\delta_{[b_1\ldots b_6}^{a_1\ldots a_6}\,R_{b_7b_8]}^{\gamma)}, \quad \Big[R_{a_1\ldots a_6}^{\alpha},\, R_{\beta\gamma}^{b_1\ldots b_8}\Big] = 1260\,\delta_{(\beta}^{\alpha}\,\delta_{[a_1\ldots a_6}^{[b_1\ldots b_6}\,R_{\gamma)}^{b_7b_8]},\\
&[R_{\alpha}^{a_1\ldots a_6},\, R_{b_1\ldots b_7,\,b}] = 1890\,\varepsilon_{\alpha\beta}\,\delta_{[b_1\ldots b_6}^{a_1\ldots a_6}\,R_{b_7]b}^{\beta} - 1890\,\varepsilon_{\alpha\beta}\,\delta_{[b_1\ldots b_6}^{a_1\ldots a_6}\,R_{b_7b]}^{\beta},\\
&\Big[R_{a_1\ldots a_6}^{\alpha},\, R^{b_1\ldots b_7,\,b}\Big] = 1890\,\varepsilon^{\alpha\beta}\,\delta_{[a_1\ldots a_6}^{[b_1\ldots b_6}\,R_{\beta}^{b_7]b} - 1890\,\varepsilon^{\alpha\beta}\,\delta_{[a_1\ldots a_6}^{[b_1\ldots b_6}\,R_{\beta}^{b_7b]},
\end{aligned}
\tag{2.2.8}
$$

and, finally, the commutators of level ± 4 generators between themselves are

$$
\begin{aligned}
\Big[R_{\alpha_1\alpha_2}^{a_1\ldots a_8},\, R_{b_1\ldots b_8}^{\beta_1\beta_2}\Big] = &-20160\,\delta_{\alpha_1\alpha_2}^{(\beta_1\beta_2)}\,\delta_{[b_1\ldots b_7}^{[a_1\ldots a_7}\,K^{a_8]}{}_{b_8]} + 2520\,\delta_{\alpha_1\alpha_2}^{(\beta_1\beta_2)}\,\delta_{b_1\ldots b_8}^{a_1\ldots a_8}\,K^{d}{}_{d}\\
&+ 5040\,\delta_{b_1\ldots b_8}^{a_1\ldots a_8}\,\delta_{(\alpha_1}^{(\beta_1}\,\varepsilon^{\beta_2)\gamma}\,R_{\alpha_2)\gamma},
\end{aligned}
$$

$$
\begin{aligned}
[R^{a_1\ldots a_7,\,a},\, R_{b_1\ldots b_7,\,b}] = & \; -\,11340\,\delta^{a_1\ldots a_7}_{b_1\ldots b_7}\,K^{a}{}_{b} + 11340\,\delta^{a_1\ldots a_7}_{[b_1\ldots b_7}\,K^{a}{}_{b]} + 11340\,\delta^{[a_1\ldots a_7}_{b_1\ldots b_7}\,K^{a]}{}_{b} \\
& + 11340\,\delta^{a_1\ldots a_7}_{b_1\ldots b_7}\,\delta^{a}_{b}\,K^{d}{}_{d} - 11340\,\delta^{a_1\ldots a_7 a}_{b_1\ldots b_7 b}\,K^{d}{}_{d} \\
& - 79380\,\delta^{a}_{b}\,\delta^{[a_1\ldots a_6}_{[b_1\ldots b_6}\,K^{a_7]}{}_{b_7]} + 79380\,\delta^{a[a_1\ldots a_6}_{[bb_1\ldots b_6}\,K^{a_7]}{}_{b_7]} \\
& + 79380\,\delta^{[aa_1\ldots a_6}_{b[b_1\ldots b_6}\,K^{a_7]}{}_{b_7]} - 90720\,\delta^{[a_1\ldots a_7}_{[b_1\ldots b_7}\,K^{a]}{}_{b]}, \\
\big[R^{a_1\ldots a_8}_{\alpha\beta},\, R_{b_1\ldots b_7,\,b}\big] = 0, & \qquad\qquad \big[R^{\alpha\beta}_{a_1\ldots a_8},\, R^{b_1\ldots b_7,\,b}\big] = 0.
\end{aligned}
\tag{2.2.9}
$$

The action of the Cartan involution on the adjoint generators is given by

Generator	$I_c\big(\text{Generator}\big)$
$K^{a}{}_{b}$	$-\,K^{b}{}_{a}$
$R_{\alpha\beta}$	$\varepsilon_{\alpha\gamma}\,\varepsilon_{\beta\delta}\,R_{\gamma\delta}$
$R^{a_1a_2}_{\alpha}$	$-\,R^{\alpha}_{a_1a_2}$
$R^{a_1\ldots a_4}$	$R_{a_1\ldots a_4}$
$R^{a_1\ldots a_6}_{\alpha}$	$-\,R^{\alpha}_{a_1\ldots a_6}$
$R^{a_1\ldots a_8}_{\alpha_1\alpha_2}$	$R^{\alpha_1\alpha_2}_{a_1\ldots a_8}$
$R^{a_1\ldots a_7,\,b}$	$R_{a_1\ldots a_7,\,b}$

We now consider the commutators of the E_{11} generators with those of the l_1 representation. The commutators of the l_1 representation generators with the level 0 $SL(11)$ generators $K^{a}{}_{b}$ are given by

$$
\begin{aligned}
& [K^{a}{}_{b},\, P_c] = -\,\delta^{a}_{c}\,P_b + \tfrac{1}{2}\,\delta^{a}_{b}\,P_c, \qquad [K^{a}{}_{b},\, Z^{c}_{\alpha}] = \delta^{c}_{b}\,Z^{a}_{\alpha} + \tfrac{1}{2}\,\delta^{a}_{b}\,Z^{c}_{\alpha}, \\
& [K^{a}{}_{b},\, Z^{a_1a_2a_3}] = 3\,\delta^{[a_1}_{b}\,Z^{|a|a_2a_3]} + \tfrac{1}{2}\,\delta^{a}_{b}\,Z^{a_1a_2a_3}, \\
& [K^{a}{}_{b},\, Z^{a_1\ldots a_5}_{\alpha}] = 5\,\delta^{[a_1}_{b}\,Z^{|a|a_2\ldots a_5]}_{\alpha} + \tfrac{1}{2}\,\delta^{a}_{b}\,Z^{a_1\ldots a_5}_{\alpha}, \\
& \big[K^{a}{}_{b},\, Z^{a_1\ldots a_7}_{\alpha\beta}\big] = 7\,\delta^{[a_1}_{b}\,Z^{|a|a_2\ldots a_7]}_{\alpha\beta} + \tfrac{1}{2}\,\delta^{a}_{b}\,Z^{a_1\ldots a_7}_{\alpha\beta}, \\
& [K^{a}{}_{b},\, Z^{a_1\ldots a_7}] = 7\,\delta^{[a_1}_{b}\,Z^{|a|a_2\ldots a_7]} + \tfrac{1}{2}\,\delta^{a}_{b}\,Z^{a_1\ldots a_7}, \\
& [K^{a}{}_{b},\, Z^{a_1\ldots a_6,\,c}] = 6\,\delta^{[a_1}_{b}\,Z^{|a|a_2\ldots a_6],\,c} + \delta^{c}_{b}\,Z^{a_1\ldots a_6,\,a} + \tfrac{1}{2}\,\delta^{a}_{b}\,Z^{a_1\ldots a_6,\,c}.
\end{aligned}
\tag{2.2.10}
$$

The commutators with the $SL\,(2)$ generators $R_{\alpha\beta}$ are

$$\begin{aligned}
&[R_{\alpha\beta},\, P_a] = 0, && \left[R_{\alpha\beta},\, Z^a_\gamma\right] = \delta^\delta_{(\alpha}\,\varepsilon_{\beta)\gamma}\,Z^a_\delta,\\
&[R_{\alpha\beta},\, Z^{a_1a_2a_3}] = 0, && \left[R_{\alpha\beta},\, Z^{a_1\ldots a_5}_\gamma\right] = \delta^\delta_{(\alpha}\,\varepsilon_{\beta)\gamma}\,Z^{a_1\ldots a_5}_\delta,\\
&[R_{\alpha\beta},\, Z^{a_1\ldots a_7}] = 0, && \left[R_{\alpha\beta},\, Z^{a_1\ldots a_6,\,b}\right] = 0,\\
&\left[R_{\alpha\beta},\, Z^{a_1\ldots a_7}_{\gamma\delta}\right] = \delta^\sigma_{(\alpha}\,\varepsilon_{\beta)\gamma}\,Z^{a_1\ldots a_7}_{\sigma\delta} + \delta^\sigma_{(\alpha}\,\varepsilon_{\beta)\delta}\,Z^{a_1\ldots a_7}_{\gamma\sigma}.
\end{aligned} \tag{2.2.11}$$

The commutators with level one E_{11} generators can be taken as

$$\begin{aligned}
&[R^{a_1a_2}_\alpha,\, P_a] = \delta^{[a_1}_a\,Z^{a_2]}_\alpha, \quad \left[R^{a_1a_2}_\alpha,\, Z^{a_3}_\beta\right] = -\,\varepsilon_{\alpha\beta}\,Z^{a_1a_2a_3}, \quad [R^{a_1a_2}_\alpha,\, Z^{a_3a_4a_5}] = Z^{a_1\ldots a_5}_\alpha,\\
&\left[R^{a_1a_2}_\alpha,\, Z^{a_3\ldots a_7}_\beta\right] = Z^{a_1\ldots a_7}_{\alpha\beta} - \varepsilon_{\alpha\beta}\,Z^{a_1\ldots a_7} - \varepsilon_{\alpha\beta}\,Z^{a_1a_2[a_3\ldots a_6,\,a_7]}.
\end{aligned} \tag{2.2.12}$$

The commutators with other positive-level generators can be found using the Jacobi identities to be given by

$$\begin{aligned}
&[R^{a_1\ldots a_4},\, P_a] = 2\,\delta^{[a_1}_a\,Z^{a_2a_3a_4]}, \quad [R^{a_1\ldots a_4},\, Z^{a_5}_\alpha] = -\,Z^{a_1\ldots a_5}_\alpha,\\
&[R^{a_1\ldots a_6}_\alpha,\, P_a] = \tfrac{3}{4}\,\delta^{[a_1}_a\,Z^{a_2\ldots a_6]}_\alpha, \quad \left[R^{a_1\ldots a_8}_{\alpha\beta}, P_a\right] = -\,\delta^{[a_1}_a\,Z^{a_2\ldots a_8]}_{\alpha\beta},\\
&\left[R^{a_1\ldots a_7,\,b}, P_a\right] = -\,3\,\delta^b_a Z^{a_1\ldots a_7} + 3\,\delta^{[b}_a\,Z^{a_1\ldots a_7]} + \tfrac{21}{20}\,\delta^{[a_1}_a\,Z^{a_2\ldots a_7],\,b},\\
&[R^{a_1\ldots a_4},\, Z^{a_5a_6a_7}] = 2\,Z^{a_1\ldots a_7} + \tfrac{3}{5}\,Z^{a_1\ldots a_4[a_5a_6,\,a_7]},\\
&\left[R^{a_1\ldots a_6}_\alpha, Z^{a_7}_\beta\right] = -\tfrac{1}{4}Z^{a_1\ldots a_7}_{\alpha\beta} + \tfrac{3}{4}\varepsilon_{\alpha\beta}Z^{a_1\ldots a_7} + \tfrac{1}{20}\varepsilon_{\alpha\beta}Z^{a_1\ldots a_6,\,a_7}.
\end{aligned} \tag{2.2.13}$$

The commutators with level -1 E_{11} generators are given by

$$\begin{aligned}
&\left[R^\alpha_{a_1a_2},\, P_a\right] = 0, && \left[R^\alpha_{a_1a_2},\, Z^b_\beta\right] = -\,4\,\delta^\alpha_\beta\,\delta^b_{[a_1}\,P_{a_2]},\\
&\left[R^\alpha_{a_1a_2},\, Z^{b_1b_2b_3}\right] = -\,6\,\varepsilon^{\alpha\beta}\,\delta^{[b_1b_2}_{a_1a_2}\,Z^{b_3]}_\beta, && \left[R^\alpha_{a_1a_2},\, Z^{b_1\ldots b_5}_\beta\right] = 20\,\delta^\alpha_\beta\,\delta^{[b_1b_2}_{a_1a_2}\,Z^{b_3b_4b_5]},\\
&\left[R^\alpha_{a_1a_2},\, Z^{b_1\ldots b_7}_{\alpha_1\alpha_2}\right] = 42\,\delta^\alpha_{(\alpha_1}\,\delta^{[b_1b_2}_{a_1a_2}\,Z^{b_3\ldots b_7]}_{\alpha_2)}, && \left[R^\alpha_{a_1a_2},\, Z^{b_1\ldots b_7}\right] = -\,3\,\varepsilon^{\alpha\beta}\,\delta^{[b_1b_2}_{a_1a_2}\,Z^{b_3\ldots b_7]}_\beta,\\
&\left[R^\alpha_{a_1a_2},\, Z^{b_1\ldots b_6,\,b}\right] = -\,150\,\varepsilon^{\alpha\beta}\,\delta^{[b_1b_2}_{a_1a_2}\,Z^{b_3\ldots b_6]b}_\beta + 150\,\varepsilon^{\alpha\beta}\,\delta^{[b_1b_2}_{a_1a_2}\,Z^{b_3\ldots b_6b]}_\beta,
\end{aligned} \tag{2.2.14}$$

while the commutators with level -2 generators are

$$\begin{aligned}
&[R_{a_1\ldots a_4},\, P_a] = 0, && \left[R_{a_1\ldots a_4},\, Z^b_\beta\right] = 0,\\
&\left[R_{a_1\ldots a_4},\, Z^{b_1b_2b_3}\right] = 48\,\delta^{[b_1b_2b_3}_{a_1a_2a_3}\,P_{b_4]}, && \left[R_{a_1\ldots a_4},\, Z^{b_1\ldots b_5}_\alpha\right] = 120\,\delta^{[b_1\ldots b_4}_{a_1\ldots a_4}\,Z^{b_5]}_\alpha,\\
&\left[R_{a_1\ldots a_4},\, Z^{b_1\ldots b_7}_{\alpha_1\alpha_2}\right] = 0, && \left[R_{a_1\ldots a_4},\, Z^{b_1\ldots b_7}\right] = -\,120\,\delta^{[b_1\ldots b_4}_{a_1\ldots a_4}\,Z^{b_5b_6b_7]},\\
&\left[R_{a_1\ldots a_4},\, Z^{b_1\ldots b_6,\,b}\right] = -\,1800\,\delta^{[b_1\ldots b_4}_{a_1\ldots a_4}\,Z^{b_5b_6]b} + 1800\,\delta^{[b_1\ldots b_4}_{a_1\ldots a_4}\,Z^{b_5b_6b]},
\end{aligned} \tag{2.2.15}$$

with level -3 generators

$$\begin{aligned}
&\left[R^{\alpha}_{a_1\ldots a_6},\, P_a\right] = 0, && \left[R^{\alpha}_{a_1\ldots a_6},\, Z^{b}_{\beta}\right] = 0,\\
&\left[R^{\alpha}_{a_1\ldots a_6},\, Z^{b_1b_2b_3}\right] = 0, && \left[R^{\alpha}_{a_1\ldots a_6},\, Z^{b_1\ldots b_5}_{\beta}\right] = -\,360\,\delta^{\alpha}_{\beta}\,\delta^{b_1\ldots b_5}_{[a_1\ldots a_5}\,P_{a_6]},\\
&\left[R^{\alpha}_{a_1\ldots a_6},\, Z^{b_1\ldots b_7}_{\alpha_1\alpha_2}\right] = -\,1260\,\delta^{\alpha}_{(\alpha_1}\,\delta^{[b_1\ldots b_6}_{a_1\ldots a_6}\,Z^{b_7]}_{\alpha_2)}, && \left[R^{\alpha}_{a_1\ldots a_6},\, Z^{b_1\ldots b_7}\right] = 270\,\varepsilon^{\alpha\beta}\,\delta^{[b_1\ldots b_6}_{a_1\ldots a_6}\,Z^{b_7]}_{\beta},\\
&\left[R^{\alpha}_{a_1\ldots a_6},\, Z^{b_1\ldots b_6,\,b}\right] = 900\,\varepsilon^{\alpha\beta}\,\delta^{b_1\ldots b_6}_{a_1\ldots a_6}\,Z^{b}_{\beta} - 900\,\varepsilon^{\alpha\beta}\,\delta^{[b_1\ldots b_6}_{a_1\ldots a_6}\,Z^{b]}_{\beta},
\end{aligned} \tag{2.2.16}$$

and, finally, with level -4 generators

$$\begin{aligned}
&\left[R^{\alpha_1\alpha_2}_{a_1\ldots a_8},\, P_a\right] = 0, && \left[R^{\alpha_1\alpha_2}_{a_1\ldots a_8},\, Z^{b}_{\beta}\right] = 0, && \left[R^{\alpha_1\alpha_2}_{a_1\ldots a_8},\, Z^{b_1b_2b_3}\right] = 0,\\
&\left[R^{\alpha_1\alpha_2}_{a_1\ldots a_8},\, Z^{b_1\ldots b_5}_{\beta}\right] = 0, && \left[R^{\alpha_1\alpha_2}_{a_1\ldots a_8},\, Z^{b_1\ldots b_7}\right] = 0, && \left[R^{\alpha_1\alpha_2}_{a_1\ldots a_8},\, Z^{b_1\ldots b_6,\,b}\right] = 0,\\
&\left[R_{a_1\ldots a_7,\,a},\, P_a\right] = 0, && \left[R_{a_1\ldots a_7,\,a},\, Z^{b}_{\beta}\right] = 0, && \left[R_{a_1\ldots a_7,\,a},\, Z^{b_1b_2b_3}\right] = 0,\\
&\left[R_{a_1\ldots a_7,\,a},\, Z^{b_1\ldots b_5}_{\beta}\right] = 0, && \left[R_{a_1\ldots a_7,\,a},\, Z^{b_1\ldots b_7}_{\alpha_1\alpha_2}\right] = 0, &&\\
&\left[R^{\alpha_1\alpha_2}_{a_1\ldots a_8},\, Z^{b_1\ldots b_7}_{\beta_1\beta_2}\right] = -\,20160\,\delta^{(\alpha_1\alpha_2)}_{\beta_1\beta_2}\,\delta^{b_1\ldots b_7}_{[a_1\ldots a_7}\,P_{a_8]}, &&&&\\
&\left[R_{a_1\ldots a_7,\,a},\, Z^{b_1\ldots b_7}\right] = 4320\,\delta^{b_1\ldots b_7}_{a_1\ldots a_7}\,P_a - 4320\,\delta^{b_1\ldots b_7}_{[a_1\ldots a_7}\,P_{a]}, &&&&\\
&\left[R_{a_1\ldots a_7,\,a},\, Z^{b_1\ldots b_6,\,b}\right] = -\,75600\,\delta^{b}_{a}\,\delta^{b_1\ldots b_6}_{[a_1\ldots a_6}\,P_{a_7]} + 75600\,\delta^{bb_1\ldots b_6}_{a[a_1\ldots a_6}\,P_{a_7]}. &&&&
\end{aligned} \tag{2.2.17}$$

3 Non-linear realisation, generalized vielbein and $I_c(E_{11})$ transformations in different dimensions

3.1 11D

In this chapter we will construct the non-linear realisation of E_{11} algebra in eleven, ten type IIB, five and four dimensions. This includes the exact expressions for the Cartan forms (1.3.8) and the generalised vielbein (1.3.9), as well as their transformations under the local $I_c(E_{11})$ transformations (1.3.7). Later in Chapter 4 this will allow us to combine them into the set of equations that is closed under said transformations and, therefore, describes the dynamics of the non-linear realisation.

3.1.1 Cartan forms

The eleven-dimensional theory is obtained by deleting node 11 from the Dynkin diagram [17, 18].

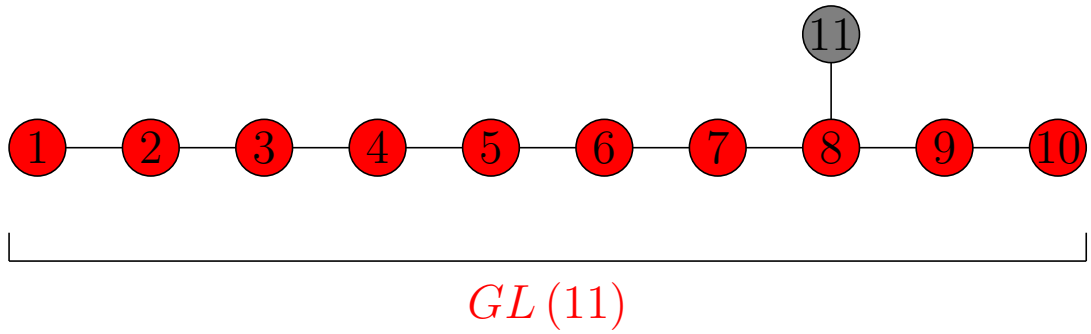


Figure 7: E_{11} algebra in 11 dimensions

The E_{11} algebra is then decomposed into its $GL(11)$ subalgebra. The generators of this algebra up to level 4 in this decomposition are given by

Level	Generators
0	$K^{a}{}_{b}$
1	$R^{a_1a_2a_3}$
-1	$R_{a_1a_2a_3}$
2	$R^{a_1\ldots a_6}$
-2	$R_{a_1\ldots a_6}$
3	$R^{a_1\ldots a_8,\,b}$
-3	$R_{a_1\ldots a_8,\,b}$
4	$R^{a_1\ldots a_9,\,b_1b_2b_3}$, $R^{a_1\ldots a_{10},\,b_1b_2}$, $R^{a_1\ldots a_{11},\,b}$
-4	$R_{a_1\ldots a_9,\,b_1b_2b_3}$, $R_{a_1\ldots a_{10},\,b_1b_2}$, $R_{a_1\ldots a_{11},\,b}$

All the blocks of indexes separated by commas are fully antisymmetric, with an exception of the level 4 generator $R^{a_1\ldots a_{10},\,b_1b_2}$, which is symmetric in b_1b_2: $R^{a_1\ldots a_{10},\,[b_1b_2]} = R_{a_1\ldots a_{10},\,[b_1b_2]} = 0$. All the generators with several blocks of indexes obey the following $GL\,(11)$ irreducibility constraints

$$R^{[a_1\ldots a_8,\,b]} = R^{[a_1\ldots a_9,\,b_1]b_2b_3} = R^{[a_1\ldots a_{10},\,b_1]b_2} = 0,$$
$$R_{[a_1\ldots a_8,\,b]} = R_{[a_1\ldots a_9,\,b_1]b_2b_3} = R_{[a_1\ldots a_{10},\,b_1]b_2} = 0. \qquad (3.1.1)$$

The generators of the l_1 representation are given by the following table

Level	Generators
0	P_a
1	$Z^{a_1a_2}$
2	$Z^{a_1\ldots a_5}$
3	$Z^{a_1\ldots a_8}$, $Z^{a_1\ldots a_7,\,b}$
4	$Z^{a_1\ldots a_8,\,b_1b_2b_3}$, $Z^{a_1\ldots a_9,\,b_1b_2}$, $\hat{Z}^{a_1\ldots a_9,\,b_1b_2}$, $Z^{a_1\ldots a_{10},\,b}_{(1)}$, $Z^{a_1\ldots a_{10},\,b}_{(2)}$, $Z^{a_1\ldots a_{11}}$

Generator $Z^{a_1\ldots a_{10},\,b_1b_2}$ is antisymmetric in b_1b_2, while $\hat{Z}^{a_1\ldots a_{10},\,b_1b_2}$ is symmetric in them:

$Z^{a_1\dots a_{10},(b_1b_2)}=0$, $\hat{Z}^{a_1\dots a_{10},[b_1b_2]}=0$. The lower index on $Z^{a_1\dots a_{10},\,b}_{(i)}$ implies that it comes with the multiplicity 2. All the generators obey the corresponding $GL\,(11)$ irreducibility constraints, similar to the ones given in equation (3.1.1).

The algebra of these generators was found up to level 3 in [17] and extended up to level 4 in [39]. The reader can find the full set of commutators up to level 4 in Appendix A. The construction of the non-linear realisation starts from the group element $g\in E_{11}\ltimes l_1$ which is subject to the transformations $g\to g_0gh$ where $g_0\in E_{11}\ltimes l_1$ is a rigid transformation and $h\in I_c\,(E_{11})$ is a local transformation. The group element $g=g_L\,g_E$ from equation (1.3.1) truncated by level 4 can be parametrised as follows

$$\begin{aligned}
g_E &= e^{R^{a_1\dots a_{11},\,b}\,A_{a_1\dots a_{11},\,b}}\,e^{R^{a_1\dots a_{10},\,b_1b_2}\,A_{a_1\dots a_{10},\,b_1b_2}}\,e^{R^{a_1\dots a_9,\,b_1b_2b_3}\,A_{a_1\dots a_9,\,b_1b_2b_3}}\\
&\quad\cdot e^{h_{a_1\dots a_8,\,b}\,R^{a_1\dots a_8,\,b}}\,e^{A_{a_1\dots a_6}\,R^{a_1\dots a_6}}\,e^{A_{a_1a_2a_3}\,R^{a_1a_2a_3}}\,e^{h_a{}^b\,K^a{}_b}=e^{A_{\underline{\alpha}}\,R^{\underline{\alpha}}},\\
g_L &= e^{x^a\,P_a}\,e^{x_{a_1a_2}\,Z^{a_1a_2}}\,e^{x_{a_1\dots a_5}\,Z^{a_1\dots a_5}}\,e^{x_{a_1\dots a_8}\,Z^{a_1\dots a_8}}\,e^{x_{a_1\dots a_7,\,b}\,Z^{a_1\dots a_7,\,b}}\\
&\quad\cdot e^{x_{a_1\dots a_8,\,b_1b_2b_3}\,Z^{a_1\dots a_8,\,b_1b_2b_3}}\,e^{x_{a_1\dots a_9,\,b_1b_2}\,Z^{a_1\dots a_9,\,b_1b_2}}\,e^{\hat{x}_{a_1\dots a_9,\,b_1b_2}\,\hat{Z}^{a_1\dots a_9,\,b_1b_2}}\\
&\quad\cdot e^{x^{(1)}_{a_1\dots a_{10},\,b}\,Z^{a_1\dots a_{10},\,b}_{(1)}}\,e^{x^{(2)}_{a_1\dots a_{10},\,b}\,Z^{a_1\dots a_{10},\,b}_{(2)}}\,e^{x_{a_1\dots a_{11}}\,Z^{a_1\dots a_{11}}}=e^{x^A\,L_A}.
\end{aligned}\tag{3.1.2}$$

We used the local $I_c\,(E_{11})$ invariance to eliminate the negative level generators from the group element g_E. This is equivalent to fixing the gauge for the higher level $I_c\,(E_{11})$ transformations. x^A are the generalised coordinates of the theory. On level 0 we find the regular space-time coordinates x^a, while the higher levels contain the coordinates that parametrise the extended space-time. Historically, the realisation that strings could wrap around circles lead to the introduction of additional momenta associated with the wrapping. This corresponds to an additional set of space-time coordinates, which make the $SO(D,\,D)$ T-duality symmetry of string theory manifest [40, 41, 42, 43, 44]. This approach gave rise to theories like double field theory [45, 46, 47].

The adjoint part of the Cartan form $\mathcal{V}_E$ defined in equation (1.3.8) is given by

$$\begin{aligned}
\mathcal{V}_E &= G_a{}^b\,K^a{}_b+G_{a_1a_2a_3}\,R^{a_1a_2a_3}+G_{a_1\dots a_6}\,R^{a_1\dots a_6}+G_{a_1\dots a_8,\,b}\,R^{a_1\dots a_8,\,b}\\
&\quad+G_{a_1\dots a_9,\,b_1b_2b_3}\,R^{a_1\dots a_9,\,b_1b_2b_3}+G_{a_1\dots a_{10},\,b_1b_2}\,R^{a_1\dots a_{10},\,b_1b_2}+G_{a_1\dots a_{11},\,b}\,R^{a_1\dots a_{11},\,b},\\
\mathcal{V}_L &= dx^\Pi\,E_\Pi{}^A\,L_A,
\end{aligned}\tag{3.1.3}$$

where $E_{\Pi}{}^{A}$ is the generalised vielbein defined in equation (1.3.9). It will be calculated up to level 3 in the next section. The Cartan forms expressed through fields are given by

$$\begin{aligned}
G_a{}^b &= \left(e^{-1}\, de\right)_a{}^b, \\
G_{a_1a_2a_3} &= e_{a_1a_2a_3}^{\mu_1\mu_2\mu_3}\, dA_{\mu_1\mu_2\mu_3}, \\
G_{a_1\ldots a_6} &= e_{a_1\ldots a_6}^{\mu_1\ldots\mu_6}\, \left(dA_{\mu_1\ldots\mu_6} - A_{[\mu_1\mu_2\mu_3}\, dA_{\mu_4\mu_5\mu_6]}\right), \\
G_{a_1\ldots a_8,\, b} &= e_{a_1\ldots a_8,\, b}^{\mu_1\ldots\mu_8,\,\nu}\, \big(dh_{\mu_1\ldots\mu_8,\,\nu} - A_{[\mu_1\mu_2\mu_3}\, dA_{\mu_4\mu_5\mu_6}\, A_{\mu_7\mu_8]\nu} + 3\, A_{[\mu_1\ldots\mu_6}\, dA_{\mu_7\mu_8]\nu} \\
&\qquad + A_{[\mu_1\mu_2\mu_3}\, dA_{\mu_4\mu_5\mu_6}\, A_{\mu_7\mu_8\nu]} - 3\, A_{[\mu_1\ldots\mu_6}\, dA_{\mu_7\mu_8\nu]}\big),
\end{aligned} \tag{3.1.4}$$

where $e_\mu{}^a = \left(e^h\right)_\mu{}^a$, $e_a{}^\mu = \left(e^{-h}\right)_a{}^\mu$ and

$$e_{a_1\ldots a_n}^{\mu_1\ldots\mu_n} = e_{[a_1}^{\mu_1}\ldots e_{a_n]}^{\mu_n}, \quad e_{a_1\ldots a_7,\, b}^{\mu_1\ldots\mu_7,\,\nu} = e_{[a_1}^{\mu_1}\ldots e_{a_7]}^{\mu_7}\, e_b{}^\nu - e_{[a_1}^{\mu_1}\ldots e_{a_7}^{\mu_7}\, e_{b]}{}^\nu. \tag{3.1.5}$$

These definitions will carry over to the other sections. We also give the **linearised** level 4 Cartan forms. They are given by

$$G_{a_1\ldots a_9,\, b_1b_2b_3} = dA_{a_1\ldots a_9,\, b_1b_2b_3}, \quad G_{a_1\ldots a_{10},\, b_1b_2} = dA_{a_1\ldots a_{10},\, b_1b_2}, \quad G_{a_1\ldots a_{11},\, b} = dA_{a_1\ldots a_{11},\, b}. \tag{3.1.6}$$

The parameter of the level 1 local $I_c\left(E_{11}\right)$ transformations of equation (1.3.4) is given by

$$h = 1 - \Lambda_{a_1a_2a_3}\, S^{a_1a_2a_3}, \quad \text{where} \quad S^{a_1a_2a_3} = R^{a_1a_2a_3} - \eta^{a_1b_1}\, \eta^{a_2b_2}\, \eta^{a_3b_3}\, R_{b_1b_2b_3}. \tag{3.1.7}$$

Under these transformations the Cartan form of equation (3.1.3) transforms as follows

$$\delta\, \mathcal{V}_E = [S^{a_1a_2a_3}\, \Lambda_{a_1a_2a_3}, \mathcal{V}_E] - S^{a_1a_2a_3}\, d\Lambda_{a_1a_2a_3}. \tag{3.1.8}$$

Written in terms of the Cartan forms of equations (3.1.4, 3.1.6) these transformations take the following form

$$\begin{aligned}
\delta G_a{}^b &= 18\, \Lambda^{c_1c_2b}\, G_{c_1c_2a} - 2\, \delta_a^b\, \Lambda^{c_1c_2c_3}\, G_{c_1c_2c_3}, \\
\delta G_{a_1a_2a_3} &= 60\, G_{a_1a_2a_3b_1b_2b_3}\, \Lambda^{b_1b_2b_3} - 3\, G^c{}_{[a_1}\, \Lambda_{|c|a_2a_3]} - d\Lambda_{a_1a_2a_3} \\
&= 60\, G_{a_1a_2a_3b_1b_2b_3}\, \Lambda^{b_1b_2b_3} - 6\, G_{(c[a_1)}\, \Lambda^c{}_{a_2a_3]},
\end{aligned}$$

$$
\begin{aligned}
\delta G_{a_1\dots a_6} &= 2\,\Lambda_{[a_1a_2a_3}\,G_{a_4a_5a_6]} - 336\,G_{b_1b_2b_3[a_1\dots a_5,\,a_6]}\,\Lambda^{b_1b_2b_3},\\
\delta G_{a_1\dots a_8,\,b} &= -3\,G_{[a_1\dots a_6}\,\Lambda_{a_7a_8]b} + 3\,G_{[a_1\dots a_6}\,\Lambda_{a_7a_8b]}\\
&\quad - 440\,\left(G_{a_1\dots a_8c_1,\,c_2c_3b} + G_{[a_1\dots a_7|bc_1,\,c_2c_3|a_8]}\right)\,\Lambda^{c_1c_2c_3}\\
&\quad - 120\,\left(G_{a_1\dots a_8c_1c_2,\,c_3b} + G_{[a_1\dots a_7|bc_1c_2,\,c_3|a_8]}\right)\,\Lambda^{c_1c_2c_3}\\
&\quad - 110\,\left(G_{a_1\dots a_8c_1c_2c_3,\,b} + G_{[a_1\dots a_7|bc_1c_2c_3,\,|a_8]}\right)\,\Lambda^{c_1c_2c_3},
\end{aligned}
\tag{3.1.9}
$$

where we have taken into account the gauge fixing condition that we introduced in equation (3.1.2). Since the group element and the Cartan form do not contain any negative level generators, we have to ensure that they are not produced by the transformation from equation (3.1.8). We can do that by making parameter $\Lambda^{a_1a_2a_3}$ obey the following constraint

$$
d\Lambda^{a_1a_2a_3} - 3\,G_b{}^{[a_1|}\,\Lambda^{b|a_2a_3]} = 0. \tag{3.1.10}
$$

This allows us to get rid of the $d\Lambda^{a_1a_2a_3}$ term in the variation of $G_{a_1a_2a_3}$ from equation (3.1.9). One can also notice that this equation has another important implication. It ensures that parameter $\Lambda^{\mu_1\mu_2\mu_3}$ with the curved indexes is a world constant.

$$
\Lambda^{\mu_1\mu_2\mu_3} = e_{a_1a_2a_3}^{\mu_1\mu_2\mu_3}\,\Lambda^{a_1a_2a_3}, \quad d\Lambda^{\mu_1\mu_2\mu_3} = 0. \tag{3.1.11}
$$

We will also give the transformation law of the level 4 Cartan forms

$$
\begin{aligned}
\delta G_{a_1\dots a_9,\,b_1b_2b_3} &= \frac{7}{8}\,\Lambda_{[b_1b_2[a_1}\,G_{a_2\dots a_9],\,b_3]} + 2\,\Lambda_{[b_1[a_1a_2}\,G_{a_3\dots a_9]b_2,\,b_3]}\\
&\quad + \frac{7}{6}\,\Lambda_{[a_1a_2a_3}\,G_{a_4\dots a_9][b_1b_2,\,b_3]} + \dots,\\
\delta G_{a_1\dots a_{10},\,b_1b_2} &= \frac{24}{11}\,G_{[a_1\dots a_7(b_1,\,b_2)}\,\Lambda_{a_8a_9a_{10}]} - \frac{24}{11}\,G_{[a_1\dots a_8,\,(b_1}\,\Lambda_{b_2)a_9a_{10}]} + \dots,\\
\delta G_{a_1\dots a_{11},\,b} &= \Lambda_{[a_1a_2a_3}\,G_{a_4\dots a_{11}],\,b} + \dots,
\end{aligned}
\tag{3.1.12}
$$

where $+\dots$ indicates the presence of level 5 terms, which we haven't considered.

The Cartan forms, discussed above, were forms written as $G_{\underline{\alpha}} = dx^{\Pi}\,G_{\Pi,\,\underline{\alpha}}$. Although the Cartan forms when written in form notation are invariant under the rigid transformations of equation $g \to g_0\,g$, once written as $G_{\Pi,\,\underline{\alpha}}$ they are no longer invariant. We can remedy this by taking the first index to be a tangent index, that is,

$G_{A,\underline{\alpha}} = E_A{}^{\Pi}\, G_{\Pi,\underline{\alpha}}$ which is inert under the rigid E_{11} transformations, but transforms under the local $I_c(E_{11})$ transformations. Using equation (1.3.15) one finds that the Cartan forms, when referred to the tangent space, transform on their l_1 index up to level 1 in coordinates as follows

$$\delta G_{a,\bullet} = -\,3\, G^{b_1 b_2,}{}_{\bullet}\, \Lambda_{b_1 b_2 a}, \quad \delta G^{a_1 a_2,}{}_{\bullet} = 6\, \Lambda^{a_1 a_2 b}\, G_{b,\bullet} + \ldots\,, \tag{3.1.13}$$

where $+\ldots$ refers to the level 2 terms that we have neglected.

3.1.2 Generalised vielbein

In this section we will construct the eleven-dimensional general vielbein up to level 3 in coordinates. According to equation (1.3.10) we have

$$E_{\Pi}{}^{A} = \left(e^{\mathcal{A}_3}\, e^{\mathcal{A}_2}\, e^{\mathcal{A}_1}\, e^{\mathcal{A}_0}\right)_{\Pi}{}^{A}, \tag{3.1.14}$$

where

$$\begin{aligned} \mathcal{A}_0 &= h_a{}^b\, D^a{}_b, & \mathcal{A}_1 &= A_{a_1 a_2 a_3}\, D^{a_1 a_2 a_3}, \\ \mathcal{A}_2 &= A_{a_1 \ldots a_6}\, D^{a_1 \ldots a_6}, & \mathcal{A}_3 &= A_{a_1 \ldots a_8,\, b}\, D^{a_1 \ldots a_8,\, b}. \end{aligned} \tag{3.1.15}$$

The level zero matrix is given by the expression

$$\begin{aligned} dx \cdot \mathcal{A}_0 \cdot L = -\big[& h_a{}^b K^a{}_b,\, dx^c\, P_c + dx_{c_1 c_2}\, Z^{c_1 c_2} + dx_{c_1 \ldots c_5}\, Z^{c_1 \ldots c_5} \\ & + \, dx_{c_1 \ldots c_8}\, Z^{c_1 \ldots c_8} + dx_{c_1 \ldots c_7,\, c}\, Z^{c_1 \ldots c_7,\, c}\big]. \end{aligned} \tag{3.1.16}$$

Using the commutators from appendix A.1, we find

$$\mathcal{A}_0 = \begin{pmatrix} h_a{}^b & 0 & 0 & 0 & 0 \\ 0 & -2\delta^{[b_1}_{[a_1}\, h_{a_2]}{}^{b_2]} & 0 & 0 & 0 \\ 0 & 0 & -\,5\,\delta^{[b_1 \ldots b_4}_{[a_1 \ldots a_4}\, h_{a_5]}{}^{b_5]} & 0 & 0 \\ 0 & 0 & 0 & -\,8\,\delta^{[b_1 \ldots b_7}_{[a_1 \ldots a_7}\, h_{a_8]}{}^{b_8]} & 0 \\ 0 & 0 & 0 & 0 & -\,k^{b_1 \ldots b_7,\, d}_{a_1 \ldots a_7,\, c} \end{pmatrix} - \frac{1}{2}\, h_c{}^c\, \mathbb{1}, \tag{3.1.17}$$

where $k^{b_1\ldots b_7,\,d}_{a_1\ldots a_7,\,c} = 7\,\delta^d_c\,\delta^{[b_1\ldots b_6}_{[a_1\ldots a_6}\,h_{a_7]}{}^{b_7]} + \delta^{a_1\ldots a_7}_{b_1\ldots b_7}\,h_c{}^d - 8\,\delta^{[a_1\ldots a_7}_{[b_1\ldots b_7}\,h_{c]}{}^{d]}$. Exponentiating this matrix results in

$$e^{\mathcal{A}_0} = (\det e)^{-\frac{1}{2}} \begin{pmatrix} e_\mu{}^a & 0 & 0 & 0 & \\ 0 & e^{\mu_1\mu_2}_{a_1a_2} & 0 & 0 & 0 \\ 0 & 0 & e^{\mu_1\ldots\mu_5}_{a_1\ldots a_5} & 0 & 0 \\ 0 & 0 & 0 & e^{\mu_1\ldots\mu_8}_{a_1\ldots a_8} & 0 \\ 0 & 0 & 0 & 0 & e^{\mu_1\ldots\mu_7,\,\nu}_{a_1\ldots a_7,\,b} \end{pmatrix}. \tag{3.1.18}$$

The combinations of vielbeins used in this formula were defined in (3.1.5). We now compute $\mathcal{A}_1$ in a similar way by considering

$$\begin{aligned} dx\cdot\mathcal{A}_1\cdot L = -\big[A_{a_1a_2a_3}\,R^{a_1a_2a_3},\, dx^c\,P_c + dx_{c_1c_2}\,Z^{c_1c_2} + dx_{c_1\ldots c_5}\,Z^{c_1\ldots c_5} \\ +\; dx_{c_1\ldots c_8}\,Z^{c_1\ldots c_8} + dx_{c_1\ldots c_7,\,c}\,Z^{c_1\ldots c_7,\,c}\big], \end{aligned} \tag{3.1.19}$$

from which we conclude, using the commutators of appendix A.1, that

$$\mathcal{A}_1 = \begin{pmatrix} 0 & -3\,A_{ab_1b_2} & 0 & 0 & 0 \\ 0 & 0 & -\delta^{a_1a_2}_{[b_1b_2}\,A_{b_3b_4b_5]} & 0 & 0 \\ 0 & 0 & 0 & -\delta^{a_1\ldots a_5}_{[b_1\ldots b_5}\,A_{b_6b_7b_8]} & -\delta^{a_1\ldots a_5}_{[b_1\ldots b_5}\,A_{b_6b_7]b} + \delta^{a_1\ldots a_5}_{[b_1\ldots b_5}\,A_{b_6b_7b]} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.1.20}$$

Proceeding in a similar way we find that

$$\mathcal{A}_2 = \begin{pmatrix} 0 & 0 & 3\,A_{ab_1\ldots b_5} & 0 & 0 \\ 0 & 0 & 0 & \delta^{a_1a_2}_{[b_1b_2}\,A_{b_3\ldots b_8]} & \delta^{a_1a_2}_{[b_1b_2}\,A_{b_3\ldots b_7]b} - \delta^{a_1a_2}_{[b_1b_2}\,A_{b_3\ldots b_7b]} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.1.21}$$

and

$$\mathcal{A}_3 = \begin{pmatrix} 0 & 0 & 0 & \frac{3}{2}\, A_{b_1\ldots b_8,\, a} & -\frac{4}{3}\, A_{a[b_1\ldots b_7],\, b} + \frac{4}{3}\, A_{a[b_1\ldots b_7,\, b]} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.1.22}$$

To compute the generalised vielbein we just need to evaluate the matrix expression of equation (3.1.14). We find that

$$E_{\Pi}{}^{A} = (\det e)^{-\frac{1}{2}} \begin{pmatrix} e_\mu{}^a & e_\mu{}^c\, \alpha_{c|a_1a_2} & e_\mu{}^c\, \alpha_{c|a_1\ldots a_5} & e_\mu{}^c\, \alpha_{c|a_1\ldots a_8} & e_\mu{}^c\, \alpha_{c|a_1\ldots a_7,\, b} \\ 0 & e_{a_1a_2}^{\mu_1\mu_2} & e_{c_1c_2}^{\mu_1\mu_2}\, \beta^{c_1c_2}{}_{a_1\ldots a_5} & e_{c_1c_2}^{\mu_1\mu_2}\, \beta^{c_1c_2}{}_{a_1\ldots a_8} & e_{c_1c_2}^{\mu_1\mu_2}\, \beta^{c_1c_2}{}_{a_1\ldots a_7,\, b} \\ 0 & 0 & e_{a_1\ldots a_5}^{\mu_1\ldots\mu_5} & e_{c_1\ldots c_5}^{\mu_1\ldots\mu_5}\, \gamma^{c_1\ldots c_5}{}_{a_1\ldots a_8} & e_{c_1\ldots c_5}^{\mu_1\ldots\mu_5}\, \gamma^{c_1\ldots c_5}{}_{a_1\ldots a_7,\, b} \\ 0 & 0 & 0 & e_{a_1\ldots a_8}^{\mu_1\ldots\mu_8} & 0 \\ 0 & 0 & 0 & 0 & e_{a_1\ldots a_7,\, b}^{\mu_1\ldots\mu_7,\nu} \end{pmatrix}, \tag{3.1.23}$$

where the symbols in the first line of this matrix are given by

$$\begin{aligned} \alpha_{a|a_1a_2} &= -\, 3\, A_{aa_1a_2}, \quad \alpha_{a|a_1\ldots a_5} = 3\, A_{aa_1\ldots a_5} + \frac{3}{2}\, A_{a[a_1a_2}\, A_{a_3a_4a_5]}, \\ \alpha_{a|a_1\ldots a_8} &= \frac{3}{2}\, A_{a_1\ldots a_8,\, a} - 3\, A_{a[a_1\ldots a_5}\, A_{a_6a_7a_8]}, \\ \alpha_{a|a_1\ldots a_7,\, b} &= \frac{4}{3}\, A_{a[a_1\ldots a_7,\, b]} + 3\, A_{a[a_1\ldots a_5}\, A_{a_6a_7b]} - \frac{4}{3}\, A_{a[a_1\ldots a_7],\, b} \\ &\quad - 3\, A_{a[a_1\ldots a_5}\, A_{a_6a_7]b} - \frac{1}{2}\, A_{a[a_1a_2}\, A_{a_3a_4a_5}\, A_{a_6a_7]b}, \end{aligned} \tag{3.1.24}$$

the symbols in the second line are given by

$$\begin{aligned} \beta^{b_1b_2}{}_{a_1\ldots a_5} &= -\, \delta^{b_1b_2}_{[a_1a_2}\, A_{a_3a_4a_5]}, \qquad \beta^{b_1b_2}{}_{a_1\ldots a_8} = \delta^{b_1b_2}_{[a_1a_2}\, A_{a_3\ldots a_8]}, \\ \beta^{b_1b_2}{}_{a_1\ldots a_7,\, b} &= \delta^{b_1b_2}_{[a_1a_2}\, A_{a_3\ldots a_7]b} + \frac{1}{2}\, \delta^{b_1b_2}_{[a_1a_2}\, A_{a_3a_4a_5}\, A_{a_6a_7]b} - \delta^{b_1b_2}_{[a_1a_2}\, A_{a_3\ldots a_7b]}, \end{aligned} \tag{3.1.25}$$

and, finally, the symbols in the third line are given by

$$\gamma^{b_1\ldots b_5}{}_{a_1\ldots a_8} = -\, \delta^{b_1\ldots b_5}_{[a_1\ldots a_5}\, A_{a_6a_7a_8]}, \quad \gamma^{b_1\ldots b_5}{}_{a_1\ldots a_7,\, b} = \delta^{b_1\ldots b_5}_{[a_1\ldots a_5}\, A_{a_6a_7b]} - \delta^{b_1\ldots b_5}_{[a_1\ldots a_5}\, A_{a_6a_7]b}. \tag{3.1.26}$$

3.2 10D Type IIB

3.2.1 Cartan forms

The level 4 group element in ten-dimensional type IIB case can be parametrised in the following way

$$
\begin{aligned}
g_L &= \exp\Big(x^a\, P_a + x^\alpha_a\, Z^a_\alpha + x_{a_1a_2a_3}\, Z^{a_1a_2a_3} + x^\alpha_{a_1\dots a_5}\, Z^{a_1\dots a_5}_\alpha \\
&\qquad + x^{\alpha\beta}_{a_1\dots a_7}\, Z^{a_1\dots a_7}_{\alpha\beta} + x_{a_1\dots a_7}\, Z^{a_1\dots a_7} + x_{a_1\dots a_6,\,b}\, Z^{a_1\dots a_6,\,b}\Big) = e^{x^A\, L_A}, \\
g_E &= \exp\Big(h_a{}^b\, K^a{}_b\Big)\, \exp\Big(\varphi^{\alpha\beta}\, R_{\alpha\beta}\Big)\, \exp\Big(A_{a_1\dots a_7,\,b}\, K^{a_1\dots a_7,\,b}\Big)\, \exp\Big(A^{\alpha\beta}_{a_1\dots a_8}\, R^{a_1\dots a_8}_{\alpha\beta}\Big) \\
&\quad \times \exp\Big(A^\alpha_{a_1\dots a_6}\, R^{a_1\dots a_6}_\alpha\Big)\, \exp\Big(A_{a_1\dots a_4}\, R^{a_1\dots a_4}\Big)\, \exp\Big(A^\alpha_{a_1a_2}\, R^{a_1a_2}_\alpha\Big) = e^{A_{\underline{\alpha}}\, R^{\underline{\alpha}}}, \qquad (3.2.1)
\end{aligned}
$$

Once again, we used the local $I_c\,(E_{11})$ invariance to eliminate the negative level generators from the group element. The notations used here were defined in Section 2.2.

The Cartan form is given by

$$
\begin{aligned}
\mathcal{V}_E &= G_a{}^b\, K^a{}_b + G^{\alpha\beta}\, R_{\alpha\beta} + G^\alpha_{a_1a_2}\, R^{a_1a_2}_\alpha + G_{a_1\dots a_4}\, R^{a_1\dots a_4} \\
&\qquad + G^\alpha_{a_1\dots a_6}\, R^{a_1\dots a_6}_\alpha + G_{a_1\dots a_7,\,b}\, R^{a_1\dots a_7,\,b} + G^{\alpha\beta}_{a_1\dots a_8}\, R^{a_1\dots a_8}_{\alpha\beta}, \\
\mathcal{V}_L &= dx^\Pi\, E_\Pi{}^A\, L_A, \qquad (3.2.2)
\end{aligned}
$$

The parameter of the level 1 local $I_c\,(E_{11})$ transformations of equation (1.3.4) has the following form

$$
h = 1 - \Lambda^\alpha_{a_1a_2a}\, S^{a_1a_2}_\alpha, \quad \text{where} \quad S^{a_1a_2}_\alpha = R^{a_1a_2}_\alpha - \eta^{a_1b_1}\, \eta^{a_2b_2}\, R^\alpha_{b_1b_2b_3}. \qquad (3.2.3)
$$

Under these transformations the Cartan form of equation (3.1.3) transforms as follows

$$
\delta\mathcal{V}_E = \big[S^{a_1a_2}_\alpha\, \Lambda^\alpha_{a_1a_2}, \mathcal{V}_E\big] - S^{a_1a_2}_\alpha\, d\Lambda^\alpha_{a_1a_2}. \qquad (3.2.4)
$$

The condition that these transformations should not create negative level terms in the Cartan form is equivalent to

$$
d\Lambda^{a_1a_2}_\alpha - 2\,\Lambda^{[a_1|b|}_\alpha\, G_b{}^{a_2]} + \varepsilon_{\alpha\beta}\, \Lambda^{a_1a_2}_\gamma\, G^{\beta\gamma} = 0 \quad \text{or} \quad d\Lambda^{\mu_1\mu_2}_{\dot\alpha} = 0, \qquad (3.2.5)
$$

Where dot on α indicates that it is a world index, rather than a flat one. Up to level 4 in fields these transformations give

$$\begin{aligned}
\delta G_a{}^b &= 4\,\Lambda^{bc}_{\alpha}\,G^{\alpha}_{ac} - \frac{1}{2}\,\delta^b_a\,\Lambda^{c_1c_2}_{\alpha}\,G^{\alpha}_{c_1c_2}, \qquad\qquad \delta G^{\alpha\beta} = -\,2\,\varepsilon^{(\alpha|\gamma|}\,\Lambda^{c_1c_2}_{\gamma}\,G^{\beta)}_{c_1c_2},\\
\delta G^{\alpha}_{a_1a_2} &= -\,2\,\Lambda^{\alpha}_{[a_1|b|}\,G_{a_2]}{}^b + \varepsilon_{\beta\gamma}\,\Lambda^{\beta}_{a_1a_2}\,G^{\gamma\alpha} - 12\,\varepsilon^{\alpha\beta}\,\Lambda^{b_1b_2}_{\beta}\,G_{b_1b_2a_1a_2} - d\Lambda^{\alpha}_{a_1a_2}\\
&= -\,4\,\Lambda^{\alpha}_{[a_1}{}^{b}\,G_{(|b|a_2])} - 12\,\varepsilon^{\alpha\beta}\,\Lambda^{b_1b_2}_{\beta}\,G_{b_1b_2a_1a_2} + \varepsilon_{\beta\gamma}\,\Lambda^{\beta}_{a_1a_2}\,G^{\gamma\alpha} + \varepsilon^{\alpha\beta}\,\Lambda^{\gamma}_{a_1a_2}\,G_{\beta\gamma},\\
\delta G_{a_1\ldots a_4} &= -\,\varepsilon_{\alpha\beta}\,\Lambda^{\alpha}_{[a_1a_2}\,G^{\beta}_{a_2a_3]} - \frac{15}{2}\,\Lambda^{b_1b_2}_{\alpha}\,G^{\alpha}_{b_1b_2a_1\ldots a_4}, \qquad (3.2.6)\\
\delta G^{\alpha}_{a_1\ldots a_6} &= 4\,\Lambda^{\alpha}_{[a_1a_2}\,G_{a_3\ldots a_6]} - 252\,\varepsilon^{\alpha\beta}\,\Lambda^{b_1b_2}_{\beta}\,G_{b_1b_2[a_1\ldots a_5,\,a_6]} + 56\,\Lambda^{b_1b_2}_{\beta}\,G^{\alpha\beta}_{b_1b_2a_1\ldots a_6},\\
\delta G^{\alpha\beta}_{a_1\ldots a_8} &= -\,\Lambda^{(\alpha}_{a_1a_2}\,G^{\beta)}_{a_3\ldots a_8]}, \quad \delta G^{\alpha}_{a_1\ldots a_7,\,b} = -\,\varepsilon_{\alpha\beta}\,\Lambda_{a_1a_2}\,G_{a_3\ldots a_7]b} + \varepsilon_{\alpha\beta}\,\Lambda_{a_1a_2}\,G_{a_3\ldots a_7b]}.
\end{aligned}$$

The Cartan forms transform as follows on their l_1 index

$$\delta G_{a,\bullet} = -\,\Lambda^{\alpha}_{ab}\,G^{b}_{\alpha,\bullet}, \quad \delta G^{a}_{\alpha,\bullet} = -\,4\,\Lambda^{ab}_{\alpha}\,G_{b,\bullet} + \ldots\,, \qquad (3.2.7)$$

where $+\ldots$ refers to the level 2 terms that we have neglected.

3.2.2 Generalised vielbein

In this section we are going to calculate the generalised vielbein using its definition from equation (1.3.9) rather than the matrix method we used in the previous section. In this approach the generalised vielbein is computed by conjugating the l_1 generators with the E_{11} group element. Using the algebra from Section 2.3 we can perform this conjugation for the $D = 10$ case. Conjugation with level 0 group element gives

$$\begin{aligned}
&e^{-\,\varphi^{\alpha\beta}\,R_{\alpha\beta}}e^{-\,h_a{}^b\,K^a{}_b}\Big\{P_\mu,\;Z^{\mu}_{\dot\alpha},\;Z^{\mu_1\mu_2\mu_3},\;Z^{\mu_1\ldots\mu_5}_{\dot\alpha},\;Z^{\mu_1\ldots\mu_7}_{\dot\alpha_1\dot\alpha_2},\;Z^{\mu_1\ldots\mu_7},\;Z^{\mu_1\ldots\mu_6,\,\nu}\Big\}e^{h_a{}^b\,K^a{}_b}e^{\varphi^{\alpha\beta}\,R_{\alpha\beta}}\\
&\quad= (\det e)^{-\,\frac{1}{2}}\,\Big\{e_\mu{}^a\,P_a,\;e_a{}^\mu\,g_{\dot\alpha}{}^{\beta}\,Z^{a}_{\beta},\;e^{\mu_1\mu_2\mu_3}_{a_1a_2a_3}\,Z^{a_1a_2a_3},\;e^{\mu_1\ldots\mu_5}_{a_1\ldots a_5}\,g_{\dot\alpha}{}^{\beta}\,Z^{a_1\ldots a_5}_{\beta},\\
&\qquad\qquad e^{\mu_1\ldots\mu_7}_{a_1\ldots a_7}\,g^{\beta_1\beta_2}_{\dot\alpha_1\dot\alpha_2}\,Z^{a_1\ldots a_7}_{\beta_1\beta_2},\;e^{\mu_1\ldots\mu_7}_{a_1\ldots a_7}\,Z^{a_1\ldots a_7},\;e^{\mu_1\ldots\mu_6,\,\nu}_{a_1\ldots a_6,\,b}\,Z^{a_1\ldots a_6,\,b}\Big\}, \qquad (3.2.8)
\end{aligned}$$

where $g_{\dot\alpha}{}^{\beta} = \left(e^{\varepsilon_\bullet\gamma\varphi^{\gamma\bullet}}\right)_{\dot\alpha}{}^{\beta}$ and $g^{\beta_1\ldots\beta_n}_{\dot\alpha_1\ldots\dot\alpha_n} = g^{\beta_1}_{[\dot\alpha_1}\ldots g^{\beta_n}_{\dot\alpha_n]}$. The coordinate vielbein $e_\mu{}^a$, its inverse and their combinations were defined in equation (3.1.5) of the previous section.

In the above equation and what follows we denote world, rather than tangent, $SL\,(2)$ indices with a dot, that is $\dot{\alpha}$, $\dot{\beta}$, ... Conjugating with positive level generators can be obtained by Taylor-expanding the exponents and truncating the series by level 4. For level one E_{11} generator we have

$$
\begin{aligned}
& e^{-\,A^{\alpha}_{b_1b_2}\,R^{b_1b_2}_{\alpha}}\,\Big\{P_a,\,Z^{a_1}_{\alpha}\,,\,Z^{a_1a_2a_3},\,Z^{a_1\ldots a_5}_{\alpha}\Big\}\,e^{A^{\alpha}_{b_1b_2}\,R^{b_1b_2}_{\alpha}}\,= \\
&= \Big\{P_a - A^{\alpha}_{ab}\,Z^{b}_{\alpha} + \frac{1}{2}\,\varepsilon_{\alpha\beta}\,A^{\alpha}_{aa_1}\,A^{\beta}_{a_2a_3}\,Z^{a_1a_2a_3} - \frac{1}{6}\,\varepsilon_{\alpha\beta}\,A^{\alpha}_{aa_1}\,A^{\beta}_{a_2a_3}\,A^{\gamma}_{a_4a_5}\,Z^{a_1\ldots a_5}_{\gamma} \\
&\quad + \frac{1}{24}\,\varepsilon_{\alpha\beta}\,A^{\alpha}_{aa_1}\,A^{\beta}_{a_2a_3}\,A^{\alpha_1}_{a_4a_5}\,A^{\alpha_2}_{a_6a_7}\,Z^{a_1\ldots a_7}_{\alpha_1\alpha_2} - \frac{1}{60}\,\varepsilon_{\alpha\beta}\,\varepsilon_{\sigma\lambda}\,A^{\alpha}_{aa_1}\,A^{\beta}_{a_2a_3}\,A^{\sigma}_{a_4a_5}\,A^{\lambda}_{a_6b}\,Z^{a_1\ldots a_6,\,b}, \\
&\quad Z^{a_1}_{\alpha} - \varepsilon_{\alpha\beta}\,A^{\beta}_{a_2a_3}\,Z^{a_1a_2a_3} + \frac{1}{2}\,\varepsilon_{\alpha\beta}\,A^{\beta}_{a_2a_3}\,A^{\gamma}_{a_4a_5}\,Z^{a_1\ldots a_5}_{\gamma} \\
&\quad - \frac{1}{6}\,\varepsilon_{\alpha\beta}\,A^{\beta}_{a_2a_3}\,A^{\alpha_1}_{a_4a_5}\,A^{\alpha_2}_{a_6a_7}\,Z^{a_1\ldots a_7}_{\alpha_1\alpha_2} + \frac{1}{15}\,\varepsilon_{\alpha\beta}\,\varepsilon_{\sigma\lambda}\,A^{\beta}_{a_2a_3}\,A^{\sigma}_{a_4a_5}\,A^{\lambda}_{a_6b}\,Z^{a_1\ldots a_6,\,b}, \\
&\quad Z^{a_1a_2a_3} - A^{\alpha}_{a_4a_5}\,Z^{a_1\ldots a_5}_{\alpha} + \frac{1}{2}\,A^{\alpha_1}_{a_4a_5}\,A^{\alpha_2}_{a_6a_7}\,Z^{a_1\ldots a_7}_{\alpha_1\alpha_2} - \frac{1}{5}\,\varepsilon_{\alpha\beta}A^{\alpha}_{a_4a_5}\,A^{\beta}_{a_6b}\,Z^{a_1\ldots a_6,\,b}, \\
&\quad Z^{a_1\ldots a_5}_{\alpha} - A^{\beta}_{a_6a_7}Z^{a_1\ldots a_7}_{\alpha\beta} - \varepsilon_{\alpha\beta}A^{\beta}_{a_6a_7}Z^{a_1\ldots a_7} + \frac{2}{5}\varepsilon_{\alpha\beta}A^{\beta}_{a_6b}Z^{a_1\ldots a_6,\,b}\Big\},
\end{aligned}
\tag{3.2.9}
$$

for level 2 generator:

$$
\begin{aligned}
& e^{-\,A_{b_1\ldots b_4}\,R^{b_1\ldots b_4}}\,\Big\{P_a,\,Z^{a_1}_{\alpha}\,,\,Z^{a_1a_2a_3}\Big\}\,e^{A_{b_1\ldots b_4}\,R^{b_1\ldots b_4}}\,= \\
&= \Big\{P_a - 2\,A_{aa_1a_2a_3}\,Z^{a_1a_2a_3} + 2\,A_{aa_1a_2a_3}\,A_{a_4\ldots a_7}\,Z^{a_1\ldots a_7} - \frac{4}{5}\,A_{aa_1a_2a_3}\,A_{a_4a_5a_6b}\,Z^{a_1\ldots a_6,\,b}, \\
&\quad Z^{a_1}_{\alpha} + A_{a_2\ldots a_5}\,Z^{a_1\ldots a_5}_{\alpha},\,Z^{a_1a_2a_3} - 2\,A_{a_4\ldots a_7}\,Z^{a_1\ldots a_7} + \frac{4}{5}\,A_{a_4a_5a_6b}\,Z^{a_1\ldots a_6,\,b}\Big\},
\end{aligned}
\tag{3.2.10}
$$

for level 3 generator:

$$
\begin{aligned}
& e^{-\,A^{\beta}_{b_1\ldots b_6}\,R^{b_1\ldots b_6}_{\beta}}\,\Big\{P_a,\,Z^{a_1}_{\alpha}\Big\}\,e^{A^{\beta}_{b_1\ldots b_6}\,R^{b_1\ldots b_6}_{\beta}}\,= \\
&= \Big\{P_a - \frac{3}{4}\,A^{\alpha}_{aa_1\ldots a_5}\,Z^{a_1\ldots a_5}_{\alpha},\,Z^{a_1}_{\alpha} + \frac{1}{4}\,A^{\beta}_{a_2\ldots a_7}\,Z^{a_1\ldots a_7}_{\alpha\beta} + \\
&\quad + \frac{3}{4}\,\varepsilon_{\alpha\beta}\,A^{\beta}_{a_2\ldots a_7}\,Z^{a_1\ldots a_7} + \frac{1}{20}\,\varepsilon_{\alpha\beta}\,A^{\beta}_{a_2\ldots a_7}\,Z^{a_2\ldots a_7,\,a_1}\Big\},
\end{aligned}
\tag{3.2.11}
$$

and, finally, for level 4 generators:

$$e^{-A^{\beta_1\beta_2}_{b_1\ldots b_8}R^{b_1\ldots b_8}_{\beta_1\beta_2}}\,P_a\,e^{A^{\beta_1\beta_2}_{b_1\ldots b_8}R^{b_1\ldots b_8}_{\beta_1\beta_2}} = P_a + A^{\alpha_1\alpha_2}_{aa_1\ldots a_7}\,Z^{a_1\ldots a_7}_{\alpha_1\alpha_2}, \tag{3.2.12}$$

$$e^{-A_{b_1\ldots b_7,\,b}R^{b_1\ldots b_7,\,b}}\,P_a\,e^{A_{b_1\ldots b_7,\,b}R^{b_1\ldots b_7,\,b}} = P_a + 3\,A_{a_1\ldots a_7,\,a}\,Z^{a_1\ldots a_7} - \frac{21}{20}\,A_{aa_1\ldots a_6,b}\,Z^{a_1\ldots a_6,\,b}.$$

Using all these results we find, from equation (3.2.5), that the generalised vielbein is given by

$$E_{\Pi}{}^{A} = (\det e)^{-\frac{1}{2}}\begin{pmatrix} e_\mu{}^a & e_\mu{}^c\,\alpha_{c|a}{}^{\beta} & e_\mu{}^c\,\alpha_{c|a_1a_2a_3} & e_\mu{}^c\,\alpha_{c|a_1\ldots a_5}{}^{\beta} & e_\mu{}^c\,\alpha_{c|a_1\ldots a_7}{}^{\beta_1\beta_2} & e_\mu{}^c\,\alpha_{c|a_1\ldots a_7} & e_\mu{}^c\,\alpha_{c|a_1\ldots a_6,\,b} \\ 0 & e_a{}^\mu\,g_{\dot\alpha}{}^\beta & e_c{}^\mu\,g_{\dot\alpha}{}^\gamma\,\beta^{c}_{\gamma|a_1a_2a_3} & e_c{}^\mu\,g_{\dot\alpha}{}^\gamma\,\beta^{c|\beta}_{\gamma|a_1\ldots a_5} & e_c{}^\mu\,g_{\dot\alpha}{}^\gamma\,\beta^{c|\beta_1\beta_2}_{\gamma|a_1\ldots a_7} & e_c{}^\mu\,g_{\dot\alpha}{}^\gamma\,\beta^{c}_{\gamma|a_1\ldots a_7} & e_c{}^\mu\,g_{\dot\alpha}{}^\gamma\,\beta^{c}_{\gamma|a_1\ldots a_6,\,b} \\ 0 & 0 & e^{\mu_1\mu_2\mu_3}_{a_1a_2a_3} & e^{\mu_1\mu_2\mu_3}_{c_1c_2c_3}\,\gamma^{c_1c_2c_3|\beta}{}_{a_1\ldots a_5} & e^{\mu_1\mu_2\mu_3}_{c_1c_2c_3}\,\gamma^{c_1c_2c_3|\beta_1\beta_2}{}_{a_1\ldots a_7} & e^{\mu_1\mu_2\mu_3}_{c_1c_2c_3}\,\gamma^{c_1c_2c_3}{}_{a_1\ldots a_7} & e^{\mu_1\mu_2\mu_3}_{c_1c_2c_3}\,\gamma^{c_1c_2c_3}{}_{a_1\ldots a_6,\,b} \\ 0 & 0 & 0 & e^{\mu_1\ldots\mu_5}_{a_1\ldots a_5}\,g_{\dot\alpha}{}^\beta & e^{\mu_1\ldots\mu_5}_{c_1\ldots c_5}\,g_{\dot\alpha}{}^\gamma\,\chi^{c_1\ldots c_5|\beta_1\beta_2}_{\gamma\qquad|a_1\ldots a_7} & e^{\mu_1\ldots\mu_5}_{c_1\ldots c_5}\,g_{\dot\alpha}{}^\gamma\,\chi^{c_1\ldots c_5}_{\gamma\qquad|a_1\ldots a_7} & e^{\mu_1\ldots\mu_5}_{c_1\ldots c_5}\,g_{\dot\alpha}{}^\gamma\,\chi^{c_1\ldots c_5}_{\gamma\qquad|a_1\ldots a_6,\,b} \\ 0 & 0 & 0 & 0 & e^{\mu_1\ldots\mu_7}_{a_1\ldots a_7}\,g_{\dot\alpha_1\dot\alpha_2}{}^{\beta_1\beta_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{\mu_1\ldots\mu_7}_{a_1\ldots a_7} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{\mu_1\ldots\mu_6,\,\nu}_{a_1\ldots a_6,\,b} \end{pmatrix}, \tag{3.2.13}$$

The symbols in the first line of the above matrix are given by

$$\begin{aligned}
\alpha_{a|b}{}^{\alpha} &= -\,A^{\alpha}_{ab}, \qquad \alpha_{a|a_1a_2a_3} = -\,2\,A_{aa_1a_2a_3} + \frac{1}{2}\,\varepsilon_{\alpha\beta}\,A^{\alpha}_{a[a_1}\,A^{\beta}_{a_2a_3]},\\
\alpha_{a|a_1\ldots a_5}{}^{\alpha} &= -\,\frac{3}{4}\,A^{\alpha}_{aa_1\ldots a_5} + 2\,A_{a[a_1a_2a_3}\,A^{\alpha}_{a_4a_5]} - \frac{1}{6}\varepsilon_{\beta\gamma}A^{\beta}_{a[a_1}\,A^{\gamma}_{a_2a_3}A^{\alpha}_{a_4a_5]},\\
\alpha_{a|a_1\ldots a_7}{}^{\alpha_1\alpha_2} &= A^{\alpha_1\alpha_2}_{aa_1\ldots a_7} + \frac{3}{4}A^{(\alpha_1}_{a[a_1\ldots a_5}A^{\alpha_2)}_{a_6a_7]} - A_{a[a_1a_2a_3}A^{\alpha_1}_{a_4a_5}A^{\alpha_2}_{a_6a_7]}\\
&\quad + \frac{1}{24}\varepsilon_{\beta\gamma}A^{\beta}_{a[a_1}A^{\gamma}_{a_2a_3}A^{\alpha_1}_{a_4a_5}A^{\alpha_2}_{a_6a_7]},\\
\alpha_{a|a_1\ldots a_7} &= 3\,A_{a_1\ldots a_7,\,a} + \frac{3}{4}\,\varepsilon_{\alpha\beta}\,A^{\alpha}_{a[a_1\ldots a_5}\,A^{\beta}_{a_6a_7]} + 2\,A_{a[a_1a_2a_3}\,A_{a_4\ldots a_7]},\\
\alpha_{a|a_1\ldots a_6,\,b} &= -\,\frac{21}{20}\,A_{aa_1\ldots a_6,\,b} - \frac{3}{10}\,\varepsilon_{\alpha\beta}\,A^{\alpha}_{a[a_1\ldots a_5}\,A^{\beta}_{a_6]b} + \frac{3}{10}\,\varepsilon_{\alpha\beta}\,A^{\alpha}_{a[a_1\ldots a_5}\,A^{\beta}_{a_6b]}\\
&\quad + \frac{2}{5}\,\varepsilon_{\alpha\beta}\,A_{a[a_1a_2a_3}\,A^{\alpha}_{a[a_4a_5}\,A^{\beta}_{a_6]b} - \frac{4}{5}\,A_{a[a_1a_2a_3}\,A_{a_4a_5a_6]b}\\
&\quad + \frac{4}{5}\,A_{a[a_1a_2a_3}\,A_{a_4a_5a_6b]} - \frac{1}{60}\varepsilon_{\alpha\beta}\,\varepsilon_{\sigma\lambda}\,A^{\alpha}_{a[a_1}\,A^{\beta}_{a_2a_3}\,A^{\sigma}_{a_4a_5}\,A^{\lambda}_{a_6]b},
\end{aligned} \tag{3.2.14}$$

in the second line are

$$\begin{aligned}
\beta^{a}_{\alpha|a_1a_2a_3} &= -\,\varepsilon_{\alpha\beta}\,\delta^{a}_{[a_1}\,A^{\beta}_{a_2a_3]}, \qquad \beta^{a|\beta}_{\alpha|a_1\ldots a_5} = \delta^{\beta}_{\alpha}\,\delta^{a}_{[a_1}\,A_{a_2\ldots a_5]} + \frac{1}{2}\,\varepsilon_{\alpha\gamma}\,\delta^{a}_{[a_1}\,A^{\gamma}_{a_2a_3}\,A^{\beta}_{a_4a_5]},\\
\beta^{a|\beta_1\beta_2}_{\alpha|a_1\ldots a_7} &= \frac{1}{4}\,\delta^{a}_{[a_1}\,\delta^{(\beta_1}_{\alpha}\,A^{\beta_2)}_{a_2\ldots a_7]} - \delta^{a}_{[a_1}\,\delta^{(\beta_1}_{\alpha}\,A_{a_2\ldots a_5}\,A^{\beta_2)}_{a_6a_7]} - \frac{1}{6}\,\varepsilon_{\alpha\gamma}\,\delta^{a}_{[a_1}\,A^{\gamma}_{a_2a_3}\,A^{\beta_1}_{a_4a_5}\,A^{\beta_2}_{a_6a_7]},
\end{aligned}$$

$$
\begin{aligned}
\beta^{a}_{\alpha|a_1\ldots a_7} &= \frac{3}{4}\,\varepsilon_{\alpha\beta}\,\delta^{a}_{[a_1}\,A^{\beta}_{a_2\ldots a_7]} - \varepsilon_{\alpha\beta}\,\delta^{a}_{[a_1}\,A_{a_2\ldots a_5}\,A^{\beta}_{a_6a_7]},\\
\beta^{a}_{\alpha|a_1\ldots a_6,\,b} &= \frac{1}{20}\,\varepsilon_{\alpha\beta}\,\delta^{a}_{b}\,A^{\beta}_{a_1\ldots a_6} - \frac{1}{20}\,\varepsilon_{\alpha\beta}\,\delta^{a}_{[b}\,A^{\beta}_{a_1\ldots a_6]} + \frac{2}{5}\,\varepsilon_{\alpha\beta}\,\delta^{a}_{[a_1}\,A_{a_2\ldots a_5}\,A^{\beta}_{a_6]b}\\
&\quad - \frac{2}{5}\,\varepsilon_{\alpha\beta}\,\delta^{a}_{[a_1}\,A_{a_2\ldots a_5}\,A^{\beta}_{a_6b]} + \frac{1}{15}\,\varepsilon_{\alpha\beta}\,\varepsilon_{\sigma\lambda}\,\delta^{a}_{[a_1}\,A^{\beta}_{a_2a_3}\,A^{\sigma}_{a_4a_5}\,A^{\lambda}_{a_6]b},
\end{aligned}
\tag{3.2.15}
$$

in the third line are

$$
\begin{aligned}
\gamma^{b_1b_2b_3|\beta}{}_{a_1\ldots a_5} &= -\,\delta^{b_1b_2b_3}_{[a_1a_2a_3}\,A^{\beta}_{a_4a_5]}, \qquad & \gamma^{b_1b_2b_3|\beta_1\beta_2}{}_{a_1\ldots a_7} &= \frac{1}{2}\,\delta^{b_1b_2b_3}_{[a_1a_2a_3}\,A^{\beta_1}_{a_4a_5}\,A^{\beta_2}_{a_6a_7]},\\
\gamma^{b_1b_2b_3}{}_{a_1\ldots a_7} &= -2\,\delta^{b_1b_2b_3}_{[a_1a_2a_3}\,A_{a_4\ldots a_7]}, & &\\
\gamma^{b_1b_2b_3}{}_{a_1\ldots a_6,\,b} &= \frac{4}{5}\,\delta^{b_1b_2b_3}_{[a_1a_2a_3}\,A_{a_4a_5a_6]b} - \frac{4}{5}\,\delta^{b_1b_2b_3}_{[a_1a_2a_3}\,A_{a_4a_5a_6b]} - \frac{1}{5}\,\varepsilon_{\alpha\beta}\,\delta^{b_1b_2b_3}_{[a_1a_2a_3}\,A^{\alpha}_{a_4a_5}\,A^{\beta}_{a_6]b}, & &
\end{aligned}
\tag{3.2.16}
$$

and, finally, in the fourth line are

$$
\begin{aligned}
\chi^{b_1\ldots b_5|\alpha_1\alpha_2}_{\alpha}{}_{|a_1\ldots a_7} &= -\,\delta^{(\alpha_1}_{\alpha}\,\delta^{b_1\ldots b_5}_{[a_1\ldots a_5}\,A^{\alpha_2)}_{a_6a_7]},\\
\chi^{b_1\ldots b_5}_{\alpha}{}_{|a_1\ldots a_7} &= -\varepsilon_{\alpha\beta}\,\delta^{b_1\ldots b_5}_{[a_1\ldots a_5}\,A^{\beta}_{a_6a_7]},\\
\chi^{b_1\ldots b_5}_{\alpha}{}_{|a_1\ldots a_6,\,b} &= -\,\varepsilon_{\alpha\beta}\,\delta^{b_1\ldots b_5}_{[a_1\ldots a_5}\,A^{\beta}_{a_6]b} + \varepsilon_{\alpha\beta}\,\delta^{b_1\ldots b_5}_{[a_1\ldots a_5}\,A^{\beta}_{a_6b]}.
\end{aligned}
\tag{3.2.17}
$$

3.3 5D

3.3.1 Cartan forms

In this section the non-linear realisation of E_{11} in five dimensions will be constructed, including the Cartan form, generalised vielbein and level 1 local $I_c\,(E_{11})$ transformations. Later in Section 4.2 we will use these results to find the dynamics of the five-dimensional E_{11} theory.

The general $E_{11} \ltimes l_1$ group element can be written as $g = g_L\,g_E$, where

$$
\begin{aligned}
g_E &= e^{A_{a_1a_2a_3\,\alpha_1\alpha_2}\,R^{a_1a_2a_3\,\alpha_1\alpha_2} + A_{a_1a_2a_3\,\alpha_1\ldots\alpha_4}\,R^{a_1a_2a_3\,\alpha_1\ldots\alpha_4} + A_{a_1a_2,\,b}R^{a_1a_2,\,b}}\\
&\quad\cdot e^{A_{a_1a_2\,\alpha_1\alpha_2}\,R^{a_1a_2\,\alpha_1\alpha_2}}\,e^{A_{a\,\alpha_1\alpha_2}\,R^{a\,\alpha_1\alpha_2}}\,e^{\varphi_{\alpha_1\alpha_2}\,R^{\alpha_1\alpha_2} + \varphi_{\alpha_1\ldots\alpha_4}\,R^{\alpha_1\ldots\alpha_4}}\,e^{h_a{}^b\,K^a{}_b},\\
g_L &= \exp\Big\{x^a\,P_a + x_{\alpha_1\alpha_2}\,Z^{\alpha_1\alpha_2} + x_{a\,\alpha_1\alpha_2}\,Z^{a\,\alpha_1\alpha_2}\\
&\qquad + x_{a_1a_2\,\alpha_1\alpha_2}\,Z^{a_1a_2\,\alpha_1\alpha_2} + x_{a_1a_2\,\alpha_1\ldots\alpha_4}\,Z^{a_1a_2\,\alpha_1\ldots\alpha_4} + x_{ab}\,Z^{ab}\Big\},
\end{aligned}
\tag{3.3.1}
$$

The Cartan form is given by

$$\begin{aligned}\mathcal{V}_E &= G_a{}^b\, K^a{}_b + G_{a\,\alpha_1\alpha_2}\, R^{a\,\alpha_1\alpha_2} + G_{a_1a_2\,\alpha_1\alpha_2}\, R^{a_1a_2\,\alpha_1\alpha_2} \\ &\quad + G_{a_1a_2a_3\,\alpha_1\alpha_2}\, R^{a_1a_2a_3\,\alpha_1\alpha_2} + G_{a_1a_2a_3\,\alpha_1\ldots\alpha_4}\, R^{a_1a_2a_3\,\alpha_1\ldots\alpha_4} + G_{a_1a_2,\,b}\, R^{a_1a_2,\,b}, \\ \mathcal{V}_L &= dx^{\Pi}\, E_{\Pi}{}^A\, L_A, \end{aligned} \tag{3.3.2}$$

The higher level Cartan forms can be determined by using the following relation

$$\begin{aligned}&G_{a\,\alpha_1\alpha_2}\, R^{a\,\alpha_1\alpha_2} + G_{a_1a_2\,\alpha_1\alpha_2}\, R^{a_1a_2\,\alpha_1\alpha_2} + G_{a_1a_2a_3\,\alpha_1\alpha_2}\, R^{a_1a_2a_3\,\alpha_1\alpha_2} \\ &\quad + G_{a_1a_2a_3\alpha_1\ldots\alpha_4}\, R^{a_1a_2a_3\,\alpha_1\ldots\alpha_4} + G_{a_1a_2,\,b}\, R^{a_1a_2,\,b} \\ &= g_0^{-1}\Big(\bar{G}_{a\,\alpha_1\alpha_2}\, R^{a\,\alpha_1\alpha_2} + \bar{G}_{a_1a_2\,\alpha_1\alpha_2}\, R^{a_1a_2\,\alpha_1\alpha_2} + \bar{G}_{a_1a_2a_3\,\alpha_1\alpha_2}\, R^{a_1a_2a_3\,\alpha_1\alpha_2} \\ &\quad + \bar{G}_{a_1a_2a_3\,\alpha_1\ldots\alpha_4}\, R^{a_1a_2a_3\,\alpha_1\ldots\alpha_4} + \bar{G}_{a_1a_2,\,b}\, R^{a_1a_2,\,b}\Big)\, g_0, \end{aligned} \tag{3.3.3}$$

where

$$g_0 = \exp\left(\varphi_{\alpha_1\alpha_2}\, R^{\alpha_1\alpha_2} + \varphi_{\alpha_1\ldots\alpha_4}\, R^{\alpha_1\ldots\alpha_4}\right) \exp\left(h_a{}^b\, K^a{}_b\right), \tag{3.3.4}$$

and

$$\begin{aligned}\bar{G}_{\mu\,\dot\alpha_1\dot\alpha_2} &= dA_{\mu\,\dot\alpha_1\dot\alpha_2}, \\ \bar{G}_{\mu_1\mu_2\,\dot\alpha_1\dot\alpha_2} &= dA_{\mu_1\mu_2\,\dot\alpha_1\dot\alpha_2} - 2\,A_{[\mu_1\,[\dot\alpha_1|\dot\gamma|}\, dA_{\mu_2]\,\dot\alpha_2]}{}^{\dot\gamma} - \frac{1}{4}\,\Omega_{\dot\alpha_1\dot\alpha_2}\, A_{[\mu_1\,\dot\gamma_1\dot\gamma_2}\, dA_{\mu_2]}{}^{\dot\gamma_1\dot\gamma_2}, \\ \bar{G}_{\mu_1\mu_2\mu_3\,\dot\alpha_1\dot\alpha_2} &= dA_{\mu_1\mu_2\mu_3\,\dot\alpha_1\dot\alpha_2} - 4\,A_{[\mu_1\,(\dot\alpha_1|\dot\gamma|}\, dA_{\mu_2\mu_3]\,\dot\alpha_2)}{}^{\dot\gamma} \\ &\quad + \frac{4}{3}\,A_{[\mu_1\,\dot\alpha_1\dot\gamma_1}\, A_{\mu_2\,\dot\alpha_2\dot\gamma_2}\, dA_{\mu_3]}{}^{\dot\gamma_1\dot\gamma_2} - \frac{4}{3}\,A_{[\mu_1\,(\dot\alpha_1|\dot\gamma_1}\, A_{\mu_2}{}^{\dot\gamma_1}{}_{\dot\gamma_2}\, dA_{\mu_3]\,|\dot\alpha_2)}{}^{\dot\gamma_2}, \\ \bar{G}_{\mu_1\mu_2\mu_3\,\dot\alpha_1\ldots\dot\alpha_4} &= \Big(dA_{\mu_1\mu_2\mu_3\,\dot\alpha_1\ldots\dot\alpha_4} - A_{[\mu_1\,\dot\alpha_1\dot\alpha_2}\, dA_{\mu_2\mu_3]\,\dot\alpha_3\dot\alpha_4} \\ &\quad + \frac{2}{3}\,A_{[\mu_1\,\dot\alpha_1\dot\alpha_2}\, A_{\mu_2\,\dot\alpha_3|\dot\gamma|}\, dA_{\mu_3]\,\dot\alpha_4}{}^{\dot\gamma}\Big)_{\text{proj }\mathbf{42}}, \\ \bar{G}_{\mu_1\mu_2,\,\nu} &= dA_{\mu_1\mu_2,\,\nu} - 2\,A_{\nu\,\dot\alpha_1\dot\alpha_2}\, dA_{\mu_1\mu_2}{}^{\dot\alpha_1\dot\alpha_2} + 2\,A_{[\nu\,\dot\alpha_1\dot\alpha_2}\, dA_{\mu_1\mu_2]}{}^{\dot\alpha_1\dot\alpha_2} \\ &\quad + \frac{4}{3}\,A_{\nu\,\dot\alpha_1\dot\alpha_2}\, A_{[\mu_1\,\dot\gamma}{}^{\dot\alpha_1}\, dA_{\mu_2]}{}^{\dot\gamma\dot\alpha_2} - \frac{4}{3}\,A_{[\nu\,\dot\alpha_1\dot\alpha_2}\, A_{\mu_1\,\dot\gamma}{}^{\dot\alpha_1}\, dA_{\mu_2]}{}^{\dot\gamma\dot\alpha_2}. \end{aligned} \tag{3.3.5}$$

As in previous section, indexes with the dot on them refer to curved extended space-time coordinates. "proj **42**" implies that the corresponding Cartan form has to be made irreducible with respect to $\dot\alpha_1\ldots\dot\alpha_4$ indexes.

Level one $I_c\,(E_{11})$ transformation of the Cartan form is given by

$$\begin{aligned} S^{a\,\alpha_1\alpha_2} &= R^{a\,\alpha_1\alpha_2} - \eta^{ab}\,\Omega^{\alpha_1\beta_1}\,\Omega^{\alpha_2\beta_2}\,R_{b\,\beta_1\beta_2}, \\ \delta\mathcal{V} &= [S^{a\,\alpha_1\alpha_2}\,\Lambda_{a\,\alpha_1\alpha_2},\,\mathcal{V}] - S^{a\,\alpha_1\alpha_2}\,d\Lambda_{a\,\alpha_1\alpha_2}. \end{aligned} \tag{3.3.6}$$

The level one $I_c\,(E_{11})$ is restricted by the gauge choice in the following way

$$d\Lambda_{a\,\alpha_1\alpha_2} - G_{ba}\,\Lambda^{b}{}_{\alpha_1\alpha_2} - G_{\alpha_1\ldots\alpha_4}\,\Lambda_{a}{}^{\alpha_3\alpha_4} - 2\,G_{[\alpha_1|\gamma|}\,\Lambda_{a\,\alpha_2]}{}^{\gamma} = 0. \tag{3.3.7}$$

This constraint ensures that the Cartan form doesn't acquire negative level terms under the $I_c\,(E_{11})$ transformation. The generalised vielbein transforms in the following way under the $I_c\,(E_{11})$ transformation:

$$\begin{aligned} \delta E^{a} &= -\,2\,E_{\alpha_1\alpha_2}\,\Lambda^{a\,\alpha_1\alpha_2}, \\ \delta E_{\alpha_1\alpha_2} &= E^{a}\,\Lambda_{a\,\alpha_1\alpha_2} - 4\,E_{a\,[\alpha_1|\gamma|}\,\Lambda^{a}{}_{\alpha_2]}{}^{\gamma} - \frac{1}{2}\,\Omega_{\alpha_1\alpha_2}\,E_{a\,\gamma_1\gamma_2}\,\Lambda^{a\,\gamma_1\gamma_2}, \\ \delta E_{a\,\alpha_1\alpha_2} &= 4\,E_{[\alpha_1|\gamma|}\,\Lambda_{a\,\alpha_2]}{}^{\gamma} + \frac{1}{2}\,\Omega_{\alpha_1\alpha_2}\,E_{\gamma_1\gamma_2}\,\Lambda_{a}{}^{\gamma_1\gamma_2}. \end{aligned} \tag{3.3.8}$$

Here we are using the form notation for the generalised vielbein: $E^{A} = dx^{\Pi}\,E_{\Pi}{}^{A}$. This corresponds to the following transformations of the Cartan forms with respect to their l_1 index.

$$\begin{aligned} \delta G_{a,\bullet} &= -\,G_{\alpha_1\alpha_2,\bullet}\,\Lambda_{a}{}^{\alpha_1\alpha_2}, \\ \delta G_{\alpha_1\alpha_2,\bullet} &= 2\,G_{a,\bullet}\,\Lambda^{a}{}_{\alpha_1\alpha_2} - 4\,G_{a\,[\alpha_1|\gamma|,\bullet}\,\Lambda^{a}{}_{\alpha_2]}{}^{\gamma} - \frac{1}{2}\,\Omega_{\alpha_1\alpha_2}\,G_{a\,\gamma_1\gamma_2,\bullet}\,\Lambda^{a\,\gamma_1\gamma_2}, \\ \delta G_{a\,\alpha_1\alpha_2,\bullet} &= 4\,G_{[\alpha_1|\gamma|,\bullet}\,\Lambda_{a\,\alpha_2]}{}^{\gamma} + \frac{1}{2}\,\Omega_{\alpha_1\alpha_2}\,G_{\gamma_1\gamma_2,\bullet}\,\Lambda_{a}{}^{\gamma_1\gamma_2}. \end{aligned} \tag{3.3.9}$$

We now give the variations of the Cartan forms with respect to their adjoint index. For level 0 Cartan forms we have

$$\begin{aligned} \delta G_{a}{}^{b} &= 2\,G_{a\,\alpha_1\alpha_2}\,\Lambda^{b\,\alpha_1\alpha_2} - \frac{2}{3}\,\delta^{b}_{a}\,G_{c\,\alpha_1\alpha_2}\,\Lambda^{c\,\alpha_1\alpha_2}, \\ \delta G_{\alpha_1\alpha_2} &= -\,4\,G_{a\,(\alpha_1|\gamma|}\,\Lambda^{a}{}_{\alpha_2)}{}^{\gamma}, \\ \delta G_{\alpha_1\ldots\alpha_4} &= 12\,G_{a\,[\alpha_1\alpha_2}\,\Lambda^{a}{}_{\alpha_3\alpha_4]} - 12\,\Omega_{[\alpha_1\alpha_2}\,G_{a\,\alpha_3|\gamma|}\,\Lambda^{a}{}_{\alpha_4]}{}^{\gamma} - \Omega_{[\alpha_1\alpha_2}\,\Omega_{\alpha_3\alpha_4]}\,G_{a\,\gamma_1\gamma_2}\,\Lambda^{a\,\gamma_1\gamma_2}. \end{aligned} \tag{3.3.10}$$

Variation of the level 1 Cartan form is

$$\begin{aligned}\delta G_{a\,\alpha_1\alpha_2} = & -G_a{}^b\,\Lambda_{b\,\alpha_1\alpha_2} + 2\,G_{[\alpha_1|\gamma|}\,\Lambda_{a\,\alpha_2]}{}^{\gamma} - G_{\alpha_1\alpha_2\gamma_1\gamma_2}\,\Lambda_a{}^{\gamma_1\gamma_2} \\ & + 8\,G_{ab\,[\alpha_1|\gamma|}\,\Lambda^b{}_{\alpha_2]}{}^{\gamma} + \Omega_{\alpha_1\alpha_2}\,G_{ab\,\gamma_1\gamma_2}\,\Lambda^{b\,\gamma_1\gamma_2} - d\Lambda_{a\,\alpha_1\alpha_2} \\ = & -2\,G_{(ab)}\,\Lambda^b{}_{\alpha_1\alpha_2} - 2\,G_{\alpha_1\alpha_2\gamma_1\gamma_2}\,\Lambda_a{}^{\gamma_1\gamma_2} \\ & + 8\,G_{ab\,[\alpha_1|\gamma|}\,\Lambda^b{}_{\alpha_2]}{}^{\gamma} + \Omega_{\alpha_1\alpha_2}\,G_{ab\,\gamma_1\gamma_2}\,\Lambda^{b\,\gamma_1\gamma_2}, \end{aligned} \tag{3.3.11}$$

where we have used the gauge fixing condition (3.3.7). Lastly, variation of the level 2 Cartan form is

$$\begin{aligned}\delta G_{a_1a_2\,\alpha_1\alpha_2} = & -4\,G_{[a_1\,[\alpha_1|\gamma|}\,\Lambda_{a_2]\,\alpha_2]}{}^{\gamma} - \frac{1}{2}\,\Omega_{\alpha_1\alpha_2}\,G_{[a_1\,\gamma_1\gamma_2}\,\Lambda_{a_2]}{}^{\gamma_1\gamma_2} - G_{a_1a_2,\,b}\,\Lambda^b{}_{\alpha_1\alpha_2} \\ & + 6\,G_{a_1a_2b\,[\alpha_1|\gamma|}\,\Lambda^b{}_{\alpha_2]}{}^{\gamma} - 36\,G_{a_1a_2b\,\alpha_1\alpha_2\gamma_1\gamma_2}\,\Lambda^{b\,\gamma_1\gamma_2}. \end{aligned} \tag{3.3.12}$$

3.3.2 Generalized vielbein

We will now build the generalised vielbein of the five-dimensional theory up to level 2 in coordinates. Conjugating the l_1 generators with the level 0 group element results in

$$\begin{aligned} & e^{-\,\varphi_{\alpha_1\alpha_2}\,R^{\alpha_1\alpha_2} - \varphi_{\alpha_1\dots\alpha_4}\,R^{\alpha_1\dots\alpha_4}}\,e^{-\,h_a{}^b\,K^a{}_b}\left\{P_\mu,\ Z^{\dot\alpha_1\dot\alpha_2},\ Z^{\mu\,\dot\alpha_1\dot\alpha_2}\right\}e^{h_a{}^b\,K^a{}_b}\,e^{\varphi_{\alpha_1\alpha_2}\,R^{\alpha_1\alpha_2} + \varphi_{\alpha_1\dots\alpha_4}\,R^{\alpha_1\dots\alpha_4}} \\ & \qquad = (\det e)^{-\frac{1}{2}}\left\{e_\mu{}^a\,P_a,\ f^{\dot\alpha_1\dot\alpha_2}{}_{\beta_1\beta_2}\,Z^{\beta_1\beta_2},\ e_a{}^\mu\,\hat f^{\dot\alpha_1\dot\alpha_2}{}_{\beta_1\beta_2}\,Z^{a\,\beta_1\beta_2}\right\}, \end{aligned} \tag{3.3.13}$$

where $f^{\dot\alpha_1\dot\alpha_2}{}_{\beta_1\beta_2}$ and $\hat f^{\dot\alpha_1\dot\alpha_2}{}_{\beta_1\beta_2}$ are the solutions of the following equations

$$\begin{aligned} \left(f^{-1}\right)^{\alpha_1\alpha_2}{}_{\dot\gamma_1\dot\gamma_2}\,df^{\dot\gamma_1\dot\gamma_2}{}_{\beta_1\beta_2} &= 2\,\delta^{[\alpha_1}_{[\beta_1}\,G^{\alpha_2]}{}_{\beta_2]} - G^{\alpha_1\alpha_2}{}_{\beta_1\beta_2}, \\ \left(\hat f^{-1}\right)^{\alpha_1\alpha_2}{}_{\dot\gamma_1\dot\gamma_2}\,d\hat f^{\dot\gamma_1\dot\gamma_2}{}_{\beta_1\beta_2} &= 2\,\delta^{[\alpha_1}_{[\beta_1}\,G^{\alpha_2]}{}_{\beta_2]} + G^{\alpha_1\alpha_2}{}_{\beta_1\beta_2}. \end{aligned} \tag{3.3.14}$$

Conjugating with the level 1 group element gives

$$\begin{aligned} & e^{-\,A_b\,\beta_1\beta_2\,R^{b\,\beta_1\beta_2}}\left\{P_a,\ Z^{\alpha_1\alpha_2}\right\}e^{A_b\,\beta_1\beta_2\,R^{b\,\beta_1\beta_2}} \\ & \qquad = \Bigg\{P_a - A_{a\,\beta_1\beta_2}\,Z^{\beta_1\beta_2} + 2\,A_{a\,\beta_1\gamma}\,A_{b\,\beta_2}{}^{\gamma}\,Z^{b\,\beta_1\beta_2}, \\ & \qquad\qquad Z^{\alpha_1\alpha_2} - \left(4\,\delta^{[\alpha_1}_{[\beta_1}\,A_{b\,\beta_2]}{}^{\alpha_2]} - \frac{1}{2}\,\Omega^{\alpha_1\alpha_2}\,A_{b\,\beta_1\beta_2}\right)\,Z^{b\,\beta_1\beta_2}\Bigg\}. \end{aligned} \tag{3.3.15}$$

For the level 2 group element we have

$$e^{-A_{b\,\beta_1\beta_2}\,R^{b\,\beta_1\beta_2}}\,P_a\,e^{A_{b\,\beta_1\beta_2}\,R^{b\,\beta_1\beta_2}}=P_a+2\,A_{ab,\,\beta_1\beta_2}\,Z^{b\,\beta_1\beta_2}. \tag{3.3.16}$$

Putting all these results together one finds

$$E_{\Pi}{}^{A}=(\det e)^{-\frac{1}{2}}\begin{pmatrix} e_\mu{}^a & -e_\mu{}^b\,A_{b\,\alpha_1\alpha_2} & e_\mu{}^b\left(2\,A_{ba,\,\alpha_1\alpha_2}+2\,A_{b\,[\alpha_1|\gamma|}\,A_{a\,\alpha_2]}{}^{\gamma}\right. \\ & & \left.+\frac{1}{4}\,\Omega_{\alpha_1\alpha_2}\,A_{b\,\gamma_1\gamma_2}\,A_a{}^{\gamma_1\gamma_2}\right) \\ 0 & f^{\dot\alpha_1\dot\alpha_2}{}_{\alpha_1\alpha_2} & -f^{\dot\alpha_1\dot\alpha_2}{}_{\beta_1\beta_2}\left(4\,\delta^{[\beta_1}_{[\alpha_1}\,A_{a\,\alpha_2]}{}^{\beta_2]}\right. \\ & & \left.+\frac{1}{2}\,\Omega_{\alpha_1\alpha_2}\,A_a{}^{\beta_1\beta_2}-\frac{1}{2}\,\Omega^{\beta_1\beta_2}\,A_{a\,\alpha_1\alpha_2}\right) \\ 0 & 0 & e_a{}^\mu\,\hat f^{\dot\alpha_1\dot\alpha_2}{}_{\alpha_1\alpha_2} \end{pmatrix}. \tag{3.3.17}$$

3.4 4D

3.4.1 Cartan forms

The four dimensional theory is obtained by deleting node 4 from the Dynkin diagram and decomposing the E_{11} algebra into representations of $GL\,(4)\times E_7$ [48]. However, it is easier to work with $SL\,(8)$ subalgebra of E_7, instead of E_7 itself. In this case all the generators belong to representations of $GL\,(4)\times SL\,(8)$.

The generators of E_{11} in four dimensions are

Level	Generators
0	$K^a{}_b,\quad R^I{}_J,\quad R^{I_1\ldots I_4}$
1	$R^{aI_1I_2},\quad R^a{}_{I_1I_2}$
-1	$R_{aI_1I_2},\quad R_a{}^{I_1I_2}$
2	$R^{a_1a_2I}{}_J,\quad R^{a_1a_2I_1\ldots I_4},\quad \hat K^{(ab)}$
-2	$R_{a_1a_2}{}^I{}_J,\quad R_{a_1a_2I_1\ldots I_4},\quad \hat K_{(ab)}$

Here capital Latin indexes (I, J, ... $=1,\ldots,8$) label the representations of $SL\,(8)$. All the generators with blocks of Latin indexes are antisymmetric in them. For the l_1 representation we have

Level	Generators
0	P_a
1	$Z^{I_1 I_2}, \quad Z_{I_1 I_2}$
2	$Z^a, \quad Z^{aI}{}_J, \quad Z^{aI_1 \ldots I_4}$

The algebra of these generators is given in Appendix B. The corresponding Dynkin diagram is

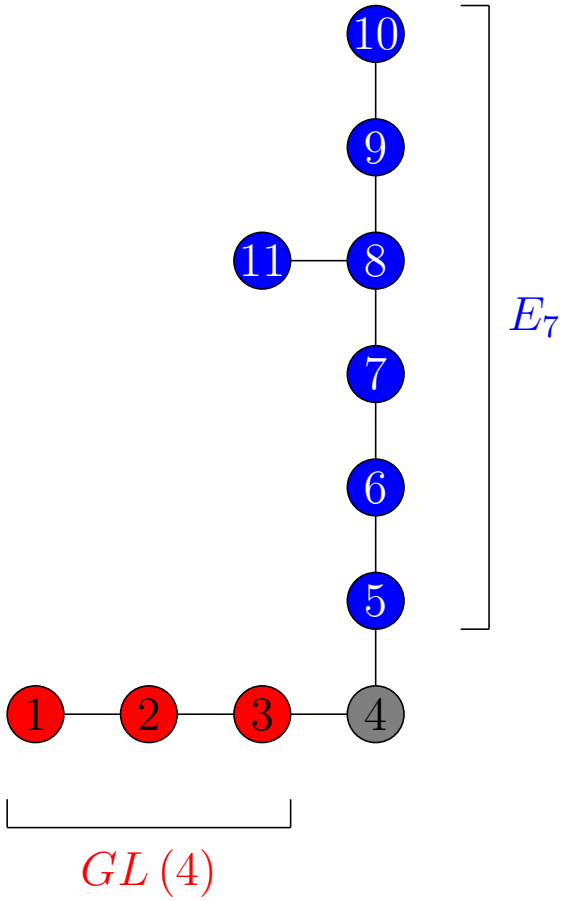


Figure 8: E_{11} algebra in 4 dimensions

The parametrisation of an arbitrary level 2 group element is of the form

$$
\begin{aligned}
g_E = &\ \exp\left(\hat{h}_{(ab)}\,\hat{K}^{(ab)}\right)\,\exp\left(A_{a_1a_2}{}^{J}{}_{I}\,R^{a_1a_2I}{}_{J}\right)\,\exp\left(A_{a_1a_2I_1\ldots I_4}\,R^{a_1a_2I_1\ldots I_4}\right) \qquad (3.4.1)\\
&\times\exp\left(A_{aI_1I_2}\,R^{aI_1I_2}+A_{a}{}^{I_1I_2}\,R^{a}{}_{I_1I_2}\right)\,\exp\left(h_a{}^b\,K^a{}_b\right)\,\exp\left(\varphi^I{}_J\,R^I{}_J\right)\,\exp\left(\varphi_{I_1\ldots I_4}\,R^{I_1\ldots I_4}\right),\\
g_L = &\ \exp\left(x^a\,P_a+x_{I_1I_2}\,Z^{I_1I_2}+x^{I_1I_2}\,Z_{I_1I_2}+\hat{x}_a\,Z^a+x_a{}^{J}{}_{I}\,Z^{aI}{}_{J}+x_{aI_1\ldots I_4}\,Z^{aI_1\ldots I_4}\right),
\end{aligned}
$$

The corresponding Cartan form is

$$
\begin{aligned}
\mathcal{V}_E = &\ G_a{}^b\,K^a{}_b+\Omega^I{}_J\,R^J{}_I+\Omega_{I_1\ldots I_4}\,R^{I_1\ldots I_4}+G_{aI_1I_2}\,R^{aI_1I_2}+G_a{}^{I_1I_2}\,R^a{}_{I_1I_2}\\
&+G_{a_1a_2}{}^{I}{}_{J}\,R^{a_1a_2J}{}_{I}+G_{a_1a_2I_1\ldots I_4}\,R^{a_1a_2I_1\ldots I_4}+\hat{G}_{ab}\,\hat{K}^{ab},\\
\mathcal{V}_L = &\ dx^{\Pi}\,E_{\Pi}{}^{A}\,L_A. \qquad (3.4.2)
\end{aligned}
$$

The $I_c\,(E_{11})$ transformations of the Cartan forms have been found and discussed in [48].

3.4.2 Generalised vielbein

We will now build the generalised vielbein of the five-dimensional theory up to level 2 in coordinates. Conjugating the l_1 generators with the level 0 generators $K^a{}_b$ and $R^I{}_J$ gives the following

$$
\begin{aligned}
&e^{-\,\varphi^I{}_J\,R^I{}_J}\,e^{-\,h_a{}^b\,K^a{}_b}\left\{P_\mu,\,Z^{\dot{I}_1\dot{I}_2},\,Z_{\dot{I}_1\dot{I}_2},\,Z^{\mu},\,Z^{\mu\dot{I}}{}_{j},\,Z^{\mu\dot{I}_1\ldots\dot{I}_4}\right\}e^{h_a{}^b\,K^a{}_b}\,e^{\varphi^I{}_J\,R^I{}_J}\\
&\quad=(\det e)^{-\frac{1}{2}}\,\Big\{e_\mu{}^a\,P_a,\,f^{\dot{I}_1\dot{I}_2}_{\ \ J_1J_2}\,Z^{J_1J_2},\,f^{J_1J_2}_{\ \ \dot{I}_1\dot{I}_2},\,Z_{J_1J_2},\,e_a{}^\mu\,Z^a,\\
&\qquad\qquad e_a{}^\mu\,f^{\dot{I}}{}_K\,f^L{}_J\,Z^{aK}{}_L,\,e_a{}^\mu\,f^{\dot{I}_1\ldots\dot{I}_4}_{\ \ J_1\ldots J_4}\,Z^{aJ_1\ldots J_4}\Big\}, \qquad (3.4.3)
\end{aligned}
$$

where $e_\mu{}^b=\left(e^h\right)_\mu{}^b$, $f^I{}_{\dot{j}}=\left(e^{\varphi^\bullet{}_\bullet}\right)^I{}_{\dot{j}}$, $f^{\dot{I}}{}_J=\left(e^{-\,\varphi^\bullet{}_\bullet}\right)^{\dot{I}}{}_J$ and

$$
e_{a_1\ldots a_n}^{\ \ \mu_1\ldots\mu_n}=e_{[a_1}{}^{\mu_1}\ldots e_{a_n]}{}^{\mu_n},\quad f^{\dot{I}_1\ldots\dot{I}_n}_{\ \ J_1\ldots J_n}=f^{\dot{I}_1}{}_{[J_1}\ldots f^{\dot{I}_n}{}_{J_n]}. \qquad (3.4.4)
$$

We place a dot on a $SL(8)$ index to denote that it is a world index, rather than a tangent one. Conjugation with $R^{I_1\ldots I_4}$ generator gives

$$
e^{-\,\varphi_{I_1\ldots I_4}\,R^{I_1\ldots I_4}}\left\{P_a,\,Z^{I_1I_2},\,Z_{I_1I_2},\,Z^a,\,Z^{aI}{}_J,\,Z^{aI_1\ldots I_4}\right\}e^{\varphi_{I_1\ldots I_4}\,R^{I_1\ldots I_4}}
$$

$$
\begin{aligned}
= \Big\{ & P_a,\ \beta^{I_1 I_2}_{\ \ J_1 J_2}\, Z^{J_1 J_2} + \beta^{I_1 I_2 | J_1 J_2}\, Z_{J_1 J_2}, \\
& \beta^{J_1 J_2}_{\ \ I_1 I_2}\, Z_{J_1 J_2} + \beta_{I_1 I_2 | J_1 J_2}\, Z^{J_1 J_2},\ Z^a, \\
& \gamma^I{}_{J|K}{}^L\, Z^{aK}{}_L + \gamma^I{}_{J|J_1 \ldots J_4}\, Z^{aJ_1 \ldots J_4}, \\
& \gamma^{I_1 \ldots I_4}_{\ \ \ \ J_1 \ldots J_4}\, Z^{aJ_1 \ldots J_4} + \gamma^{I_1 \ldots I_4}{}_K{}^L\, Z^{aK}{}_L \Big\},
\end{aligned}
\tag{3.4.5}
$$

where the β-matrices that mix level 1 elements are defined as

$$
\begin{aligned}
\beta^{I_1 I_2}_{\ \ J_1 J_2} &= \left(1 + \frac{1}{2}\, P + \frac{1}{4!}\, P^2 + \frac{1}{6!}\, P^3 + \ldots\right)^{I_1 I_2}_{\ \ J_1 J_2}, \\
\beta^{I_1 I_2 | J_1 J_2} &= -\frac{1}{24}\, \varepsilon^{J_1 \ldots J_8} \left(1 + \frac{1}{3!}\, P + \frac{1}{5!}\, P^2 + \frac{1}{7!}\, P^3 + \ldots\right)^{I_1 I_2}_{\ \ J_3 J_4} \varphi_{J_5 \ldots J_8}, \\
\beta_{I_1 I_2 | J_1 J_2} &= -\,\varphi_{J_1 \ldots J_4} \left(1 + \frac{1}{3!}\, P + \frac{1}{5!}\, P^2 + \frac{1}{7!}\, P^3 + \ldots\right)^{J_3 J_4}_{\ \ I_1 I_2},
\end{aligned}
\tag{3.4.6}
$$

where

$$
P^{I_1 I_2}_{\ \ J_1 J_2} = \frac{1}{24}\, \varepsilon^{I_1 \ldots I_8}\, \varphi_{I_3 \ldots I_6}\, \varphi_{I_7 I_8 J_1 J_2},
\tag{3.4.7}
$$

while the γ-matrices, responsible for mixing of level 2 elements, are given by

$$
\begin{aligned}
\gamma^I{}_{J|K}{}^L &= \left(1 + \frac{1}{2}\, Q + \frac{1}{4!}\, Q^2 + \frac{1}{6!}\, Q^3 + \ldots\right)^{I\ \ \ L}_{\ \ J|K}, \\
\gamma^{I_1 \ldots I_4}_{\ \ \ \ J_1 \ldots J_4} &= \left(1 + \frac{1}{2}\, R + \frac{1}{4!}\, R^2 + \ldots\right)^{I_1 \ldots I_4}_{\ \ \ \ J_1 \ldots J_4}, \\
\gamma^I{}_{J|J_1 \ldots J_4} &= \left(1 + \frac{1}{3!}\, Q + \frac{1}{5!}\, Q^2 + \frac{1}{7!}\, Q^3 + \ldots\right)^{I\ \ \ L}_{\ \ J|K} \left(\frac{4}{3}\, \delta^K_{[J_1}\, \varphi_{|L|J_2 J_3 J_4]} - \frac{1}{6}\, \delta^K_L\, \varphi_{J_1 \ldots J_4}\right), \\
\gamma^{I_1 \ldots I_4}{}_I{}^J &= -\,\frac{1}{12} \left(1 + \frac{1}{3!}\, R + \frac{1}{5!}\, R^2 + \frac{1}{7!}\, R^3 + \ldots\right)^{I_1 \ldots I_4}_{\ \ \ \ J_1 \ldots J_4} \varepsilon^{J_1 \ldots J_4 K_1 K_2 K_3 J}\, \varphi_{K_1 K_2 K_3 I}.
\end{aligned}
\tag{3.4.8}
$$

where

$$
\begin{aligned}
Q^I{}_{J|K}{}^L &= \left(\frac{1}{72}\, \delta^I_J\, \varphi_{I_1 \ldots I_4} - \frac{1}{9}\, \delta^I_{I_1}\, \varphi_{J I_2 I_3 I_4}\right) \varepsilon^{I_1 \ldots I_4 J_1 J_2 J_3 L}\, \varphi_{J_1 J_2 J_3 K}, \\
R^{I_1 \ldots I_4}_{\ \ \ \ J_1 \ldots J_4} &= \varepsilon^{I_1 \ldots I_4 K_1 K_2 K_3 J}\, \varphi_{K_1 K_2 K_3 I} \left(\frac{1}{72}\, \delta^I_J\, \varphi_{J_1 \ldots J_4} - \frac{1}{9}\, \delta^I_{[J_1}\, \varphi_{|J|J_2 J_3 J_4]}\right),
\end{aligned}
\tag{3.4.9}
$$

Conjugation with level 1 and level 2 elements is performed by Taylor-expanding the exponents. The generalised vielbein is

$$E_{\Pi}{}^{A} = (\det e)^{-\frac{1}{2}} \begin{pmatrix} e_{\mu}{}^{a} & e_{\mu}{}^{b}\,\alpha_{b|J_1J_2} & e_{\mu}{}^{b}\,\alpha_{b}{}^{J_1J_2} & e_{\mu}{}^{b}\,\alpha_{b|a} & e_{\mu}{}^{b}\,\alpha_{b|aK}{}^{L} & e_{\mu}{}^{b}\,\alpha_{b|aJ_1\ldots J_4} \\ 0 & f^{\dot I_1\dot I_2}{}_{K_1K_2}\,\beta^{K_1K_2}{}_{J_1J_2} & f^{\dot I_1\dot I_2}{}_{K_1K_2}\,\beta^{K_1K_2|J_1J_2} & f^{\dot I_1\dot I_2}{}_{K_1K_2}\,\beta^{K_1K_2}{}_{a} & f^{\dot I_1\dot I_2}{}_{K_1K_2}\,\beta^{K_1K_2}{}_{aK}{}^{L} & f^{\dot I_1\dot I_2}{}_{K_1K_2}\,\beta^{K_1K_2}{}_{aJ_1\ldots J_4} \\ 0 & f^{K_1K_2}{}_{\dot I_1\dot I_2}\,\beta_{K_1K_2|J_1J_2} & f^{K_1K_2}{}_{\dot I_1\dot I_2}\,\beta^{J_1J_2}{}_{K_1K_2} & f^{K_1K_2}{}_{\dot I_1\dot I_2}\,\beta_{K_1K_2|a} & f^{K_1K_2}{}_{\dot I_1\dot I_2}\,\beta_{K_1K_2|aK}{}^{L} & f^{K_1K_2}{}_{\dot I_1\dot I_2}\,\beta_{K_1K_2|aJ_1\ldots J_4} \\ 0 & 0 & 0 & e_{a}{}^{\mu} & 0 & 0 \\ 0 & 0 & 0 & 0 & e_{a}{}^{\mu}\,f^{\dot I}{}_{M}\,f^{N}{}_{j}\,\gamma^{M}{}_{N|K}{}^{L} & e_{\mu}{}^{a}\,f^{\dot I}{}_{M}\,f^{N}{}_{j}\,\gamma^{M}{}_{N|J_1\ldots J_4} \\ 0 & 0 & 0 & 0 & e_{a}{}^{\mu}\,f^{\dot I_1\ldots\dot I_4}{}_{K_1\ldots K_4}\,\gamma^{K_1\ldots K_4}{}_{K}{}^{L} & e_{a}{}^{\mu}\,f^{\dot I_1\ldots\dot I_4}{}_{K_1\ldots K_4}\,\gamma^{K_1\ldots K_4}{}_{J_1\ldots J_4} \end{pmatrix}, \tag{3.4.10}$$

where the symbols in the first line of the matrix are given by

$$\begin{aligned} \alpha_{a|I_1I_2} &= -\,A_{aI_1I_2}, \quad \alpha_{a}{}^{I_1I_2} = -\,A_{a}{}^{I_1I_2}, \\ \alpha_{a|b} &= -\,\hat h_{(ab)} - \frac{1}{2}\,A_{[aI_1I_2}\,A_{b]}{}^{I_1I_2}, \\ \alpha_{a|bI}{}^{J} &= \frac{1}{2}\,A_{abI}{}^{J} + \frac{1}{2}\,A_{(aKI}\,A_{b)}{}^{KJ}, \\ \alpha_{a|bI_1\ldots I_4} &= \frac{1}{6}\,A_{abI_1\ldots I_4} - \frac{1}{2}\,A_{a[I_1I_2}\,A_{bI_3I_4]} + \frac{1}{48}\,\varepsilon_{I_1\ldots I_8}\,A_{a}{}^{I_5I_6}\,A_{b}{}^{I_7I_8}, \end{aligned} \tag{3.4.11}$$

and the second line by

$$\begin{aligned} \beta^{I_1I_2}{}_{a} &= \beta^{I_1I_2}_{J_1J_2}\,A_{a}^{J_1J_2} - \beta^{I_1I_2|J_1J_2}\,A_{aJ_1J_2}, \\ \beta^{I_1I_2}{}_{aI}{}^{J} &= -\,\beta^{I_1I_2}_{KI}\,A_{a}^{KJ} - \beta^{I_1I_2|KJ}\,A_{aKI}, \\ \beta^{I_1I_2}_{aJ_1\ldots J_4} &= \beta^{I_1I_2}_{[J_1J_2}\,A_{aJ_3J_4]} - \frac{1}{24}\,\varepsilon_{J_1\ldots J_8}\,\beta^{I_1I_2|J_5J_6}\,A_{a}^{J_7J_8}, \\ \beta_{I_1I_2|a} &= -\,\beta^{J_1J_2}_{I_1I_2}\,A_{aJ_1J_2} + \beta_{I_1I_2|J_1J_2}\,A_{a}^{J_1J_2}, \\ \beta_{I_1I_2|aI}{}^{J} &= -\,\beta^{KJ}_{I_1I_2}\,A_{aKI} - \beta_{I_1I_2|KI}\,A_{a}^{KJ}, \\ \beta_{I_1I_2|aJ_1\ldots J_4} &= -\,\frac{1}{24}\,\varepsilon_{J_1\ldots J_8}\,\beta^{J_5J_6}_{I_1I_2}\,A_{a}^{J_7J_8} + \beta_{I_1I_2|[J_1J_2}\,A_{aJ_3J_4]}. \end{aligned} \tag{3.4.12}$$

4 Non-linear realisation and equations of motion

4.1 Non-linear realisation of E_{11} in 11D

The dynamics of the non-linear realisation of E_{11} are described by a set of equations that are invariant under both the rigid transformations and the local $I_c\,(E_{11})$ symmetry. Since the Cartan forms of equation (1.3.8) are invariant under the rigid transformations, we only have to ensure that the equations we construct from them possess the local $I_c\,(E_{11})$ invariance. On level 0 the $I_c\,(E_{11})$ algebra is isomorphic to $SO\,(1,\,10)$. This symmetry ensures that the equations of motion are invariant under the local Lorentz transformations. The higher level transformations, on the other hand, transform the equations into each other. We are particularly interested in level 1 transformations that were derived in Section 3.1.1. This symmetry ensures that the equations in the theory form a multiplet under the local $I_c\,(E_{11})$ transformations. We can express this in the following general form

$$\delta E_i = \Lambda_{ij}\, E_j, \tag{4.1.1}$$

where E_i, $i = 1,\, 2,\, \ldots$, are the equations of motion in the multiplet. Equation (4.1.1) ensures that all E_i can be set to zero without breaking the $I_c\,(E_{11})$ invariance. Thus, the dynamics of the theory are described by the following set of equations: $E_i = 0$, $i = 1,\, 2,\, \ldots$. To illustrate this point we will present a segment of the multiplet that consists of the following equations: three form - six form duality relation $D_{a_1 \ldots a_4}$, second order three form equation $E_{a_1 a_2 a_3}$ and the Einstein equation E_{ab}. The linearised versions of these equations are

$$D_{a_1 \ldots a_4} = G_{[a_1,\, a_2 a_3 a_4]} - \frac{1}{2 \cdot 4!}\, \varepsilon_{a_1 \ldots a_4}{}^{b_1 \ldots b_7}\, G_{b_1,\, b_2 \ldots b_7}, \quad E_{a_1 a_2 a_3} = \partial^b\, G_{[b,\, a_1 a_2 a_3]}, \quad E_{ab} = R_{ab}, \tag{4.1.2}$$

where R_{ab} is the Ricci tensor. The Cartan forms used in this equation were defined in (3.1.4). The $I_c\,(E_{11})$ transformations of these equations are illustrated by the following diagram

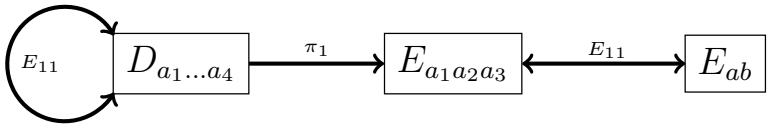


Figure 9: $3 - 6$ duality, vector and gravity equations

If one neglects other equations of the multiplet, under $I_c\,(E_{11})$ transformations equations $E_{a_1a_2a_3}$ and E_{ab} transform into each other, while equation $D_{a_1\ldots a_4}$ is self-dual. $E_{a_1a_2a_3}$ can be obtained from it by applying a projector, denoted as π_1, that eliminates the six form. In the next two sections of this chapter we will derive all the equations presented above and show that their variation closes at full **non-linear** level.

However, not all equations behave in this simple manner. As we will see later, the scheme illustrated by equation (4.1.1) will require certain generalisations. More specifically, some of the equations that involve higher level fields do not completely close on the other equations of the multiplet. Instead, they produce an additional "modulo" term [48, 49, 50] in the following way

$$\delta E_i = \Lambda_{ij}\, E_j + \partial\tilde{\Lambda}_i. \tag{4.1.3}$$

In order to ensure the complete closure of these equations one has to apply additional derivatives to them in order to eliminate the modulo term $\partial\tilde{\Lambda}_i$ from the variation. The higher level the field is, the more derivatives it requires to eliminate the corresponding modulo term. This mechanism explains why higher level fields in the E_{11} theory have equations of motion with multiple derivatives.

We will later argue that the modulo term is determined by the generalised gauge transformations, defined in equation (1.3.17). The gauge invariant equations do not produce any modulo terms, while the ones that are not invariant produce the modulo term that can be compensated by a gauge transformation with an appropriate parameter. This point will be illustrated in Section 4.1.3.

As E_{11} theory contains an infinite set of fields and coordinates, we are only going to work out the equations up to a certain level. Thus, we will truncate all the fields

above level 4 and all the coordinates above level 1. The resulting set of equations and connections between them is given in Figure 10.

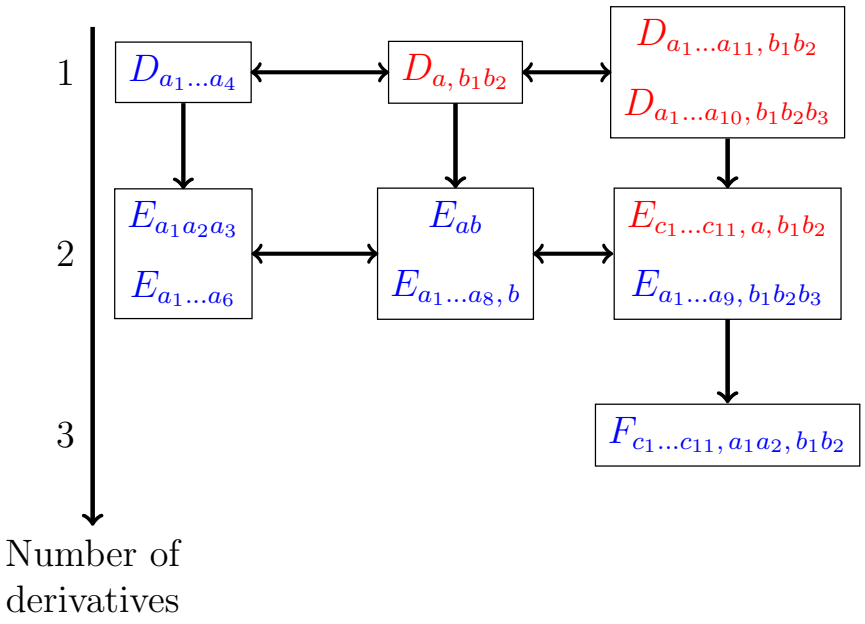


Figure 10: 11D E_{11} multiplet

The equations in blue hold exactly, while equations in red possess a modulo term. These equations are classified with respect to the number of derivatives they have. Equations with one derivative are denoted as D, with two derivatives — as E, and the one equation with three derivatives we are considering — as F. The vertical arrows indicate the projectors that allow us to construct higher order equations by applying derivatives to lower order ones. The horizontal lines indicate the level 1 $I_c\,(E_{11})$ transformations. In the next three sections we are going to go over the different sectors of this diagram in detail, while deriving the equations listed in Figure 10 and calculating their variations.

Due to the power of the E_{11} symmetry we will be able to reconstruct the whole multiplet from a single equation by repeatedly applying the level 1 $I_c\,(E_{11})$ transformations and projectors to it. Any of the equations can serve as this starting point, so we will try picking the simplest one. To do so we are going to look for an equation that involves the three form field $A_{a_1 a_2 a_3}$ and is first order in derivatives. According to equation

(3.1.9), under the $I_c\left(E_{11}\right)$ transformations $A_{a_1a_2a_3}$ transforms into the six form $A_{a_1\ldots a_6}$, and vice versa. This implies that they have to be connected by a duality relation that transforms into itself under the $I_c\left(E_{11}\right)$ transformations. The most general form of this relation that is invariant under the local Lorentz transformations is the $3-6$ duality relation, mentioned above

$$D_{a_1\ldots a_4} = G_{[a_1,\, a_2a_3a_4]} + c\,\varepsilon_{a_1\ldots a_4}{}^{b_1\ldots b_7}\, G_{b_1,\, b_2\ldots b_7}. \tag{4.1.4}$$

Coefficient c cannot be fixed by considering exclusively the local Lorentz invariance. In the next section we will determine its value by imposing the level 1 $I_c\left(E_{11}\right)$ symmetry and demanding that this equation transforms into itself. In the eleven-dimensional supergravity theory the six form field is the magnetic dual of the three form field. One can see that equation (4.1.4) mirrors the corresponding supergravity duality relation.

We only work up to and including level 1 in the coordinates. As we can see from equation (3.1.13), the variation of the Cartan forms that involve the level 1 derivative contains terms with level 0 derivatives. This implies that in order to ensure the level 0 closure of the variation we have to keep the level 1 l_1 terms in the **equation that we are varying**. On the other hand, all the level 1 l_1 terms **in the variation** are truncated out as we do not have enough accuracy to check their closure above level 0. This means that we have to consider two different versions of each equation in the multiplet. We start with an equation of motion that has no higher level derivatives, generically denoted as D, E or F. In order to close the variation of these equations we then build the l_1-extended versions of them, denoted as $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ respectively, that differ from their level 0 counterparts by addition of the most general level 1 l_1 term. We then carry out the variation and demand their closure on the other equations in the multiplet. This fixes the level 1 l_1 terms that we're adding.

In order to avoid overly complicated calculations we are often going to adopt the correct form of the level 1 derivative terms in the l_1-extended equations from the beginning. The reader has to keep in mind that these terms are actually **uniquely determined** by the calculation itself.

4.1.1 $3-6$ duality relation and the second order equations for 3 and 6 forms

The starting point of this calculation is the $3-6$ duality relation $D_{a_1\dots a_4}$ from equation (4.1.4). We will first consider its $I_c\,(E_{11})$ variation and construct the first order gravity - dual gravity relation $D_{a,\,b_1b_2}$ that arises from it, then we will build two projectors π_1, π_2 that eliminate $A_{a_1\dots a_6}$ or $A_{a_1a_2a_3}$ from the equation and, therefore, produce two equations $E_{a_1a_2a_3}$ and $E_{a_1\dots a_6}$, which describe the dynamics of the remaining field. Lastly, we will consider the variation of these equations to find the second order equations for the graviton E_{ab} (Einstein equation) and the dual graviton $E_{a_1\dots a_8,\,b}$.

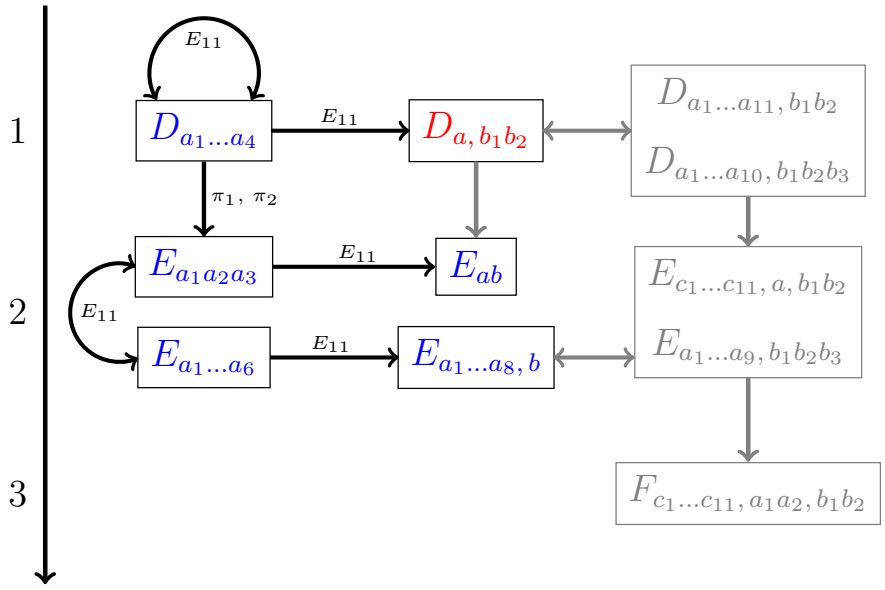


Figure 11: Vector sector of the E_{11} multiplet

According to the procedure described earlier, we have to add certain terms with higher l_1 derivatives to equation (4.1.4) in order to ensure that its variation closes. The equation with these terms added in, denoted now as $\mathcal{D}$, takes the following form.

$$\mathcal{D}_{a_1\dots a_4} = \mathcal{G}_{a_1\dots a_4} + c\,\varepsilon_{a_1\dots a_4}{}^{b_1\dots b_7}\,\mathcal{G}_{b_1\dots b_7} + \frac{1}{2}\,G_{[a_1a_2,\,a_3a_4]}, \tag{4.1.5}$$

where

$$\mathcal{G}_{a_1\ldots a_4} = G_{[a_1,\,a_2a_3a_4]} + \frac{15}{2}\,G^{b_1b_2,}{}_{b_1b_2a_1\ldots a_4},$$
$$\mathcal{G}_{a_1\ldots a_7} = G_{[a_1,\,a_2\ldots a_7]} + 28\,G^{b_1b_2,}{}_{b_1b_2[a_1\ldots a_6,\,a_7]}. \tag{4.1.6}$$

The coefficients in these combinations are fixed uniquely during the calculation, but for the sake of simplicity we will fix their values from the start. The E_{11} variations of these objects are

$$\delta\mathcal{G}_{a_1\ldots a_4} = -\,6\,G_{[a_1,\,(a_2|c|)}\,\Lambda^{c}{}_{a_3a_4]} + 105\,G_{[a_1,\,a_2a_3a_4b_1b_2b_3]}\,\Lambda^{b_1b_2b_3}, \tag{4.1.7}$$
$$\delta\mathcal{G}_{a_1\ldots a_7} = -\,2\,G_{[a_1,\,a_2a_3a_4}\,\Lambda_{a_5a_6a_7]} + 168\,\left(G_{[a_1,\,a_2\ldots a_7]b_1b_2,\,b_3} + G_{b_1,\,b_2b_3[a_1\ldots a_6,\,a_7]}\right)\,\Lambda^{b_1b_2b_3}.$$

Variation of equation (4.1.5) is given by

$$\begin{aligned}\delta\mathcal{D}_{a_1\ldots a_4} = {}& 105\,G_{[a_1,\,a_2a_3a_4b_1b_2b_3]}\,\Lambda^{b_1b_2b_3} - 2\,c\,\varepsilon_{a_1..a_4}{}^{b_1\ldots b_7}\,G_{b_1,\,b_2b_3b_4}\,\Lambda_{b_5b_6b_7}\\ &+ 168\,c\,\varepsilon_{a_1..a_4}{}^{b_1\ldots b_7}\,\left(G_{b_1,\,b_2\ldots b_7c_1c_2,\,c_3} + G_{c_1,\,c_2c_3b_1\ldots b_6,\,b_7}\right)\,\Lambda^{c_1c_2c_3}\\ &+ 3\,(\det e)^{\frac{1}{2}}\,\omega_{c,\,[a_1a_2}\,\Lambda^{c}{}_{a_3a_4]}.\end{aligned} \tag{4.1.8}$$

Here $\omega_{a,\,b_1b_2}$ is the standard general relativity spin connection defined as

$$\omega_{a,\,b_1b_2} = (\det e)^{-\frac{1}{2}}\left(-\,G_{b_1,\,(b_2a)} + G_{b_2,\,(b_1a)} + G_{a,\,[b_1b_2]}\right). \tag{4.1.9}$$

When linearised it takes the following form:

$$\omega_{a,\,b_1b_2} = -\,\partial_{b_1}\,h_{(b_2a)} + \partial_{b_2}\,h_{(b_1a)} + \partial_a\,h_{[b_1b_2]}. \tag{4.1.10}$$

The terms in the first line of equation (4.1.8) can be recombined in the following way

$$\begin{aligned}&105\,G_{[a_1,\,a_2a_3a_4b_1b_2b_3]}\,\Lambda^{b_1b_2b_3} - 2\,c\,\varepsilon_{a_1..a_4}{}^{b_1\ldots b_7}\,G_{b_1,\,b_2b_3b_4}\,\Lambda_{b_5b_6b_7}\\ = {}& -\,2\,c\,\varepsilon_{a_1\ldots a_4}{}^{b_1\ldots b_7}\left(G_{b_1,\,b_2b_3b_4} + \frac{1}{(2\cdot 4!)^2\,c}\,\varepsilon_{b_1\ldots b_4}{}^{c_1\ldots c_7}\,G_{c_1,\,c_2\ldots c_7}\right)\,\Lambda_{b_5b_6b_7}.\end{aligned} \tag{4.1.11}$$

The term in the brackets reproduces the original equation (4.1.4), given that the following condition is satisfied

$$c = \pm\frac{1}{2\cdot 4!}. \tag{4.1.12}$$

We are going to pick the solution with the minus and investigate the properties of the multiplet that arises from it. The solution with the plus can be processed in a similar fashion.

If one neglects the level 0 and level 3 terms in the variation, which as we will later see combine into gravity - dual gravity relation, one finds that the variation of $D_{a_1\ldots a_4}$ closes on itself

$$\delta\mathcal{D}_{a_1\ldots a_4} = \frac{1}{4!}\,\varepsilon_{a_1\ldots a_4}{}^{b_1\ldots b_7}\,D_{b_1\ldots b_4}\,\Lambda_{b_5b_6b_7} + \ldots\,, \tag{4.1.13}$$

where ... refers to the neglected level 0 and level 3 terms. This justifies the statement made in the introduction to this chapter that the $3-6$ duality relation is self-dual under $I_c\,(E_{11})$ transformations in the absence of gravity.

We will now process the remaining terms in equation (4.1.8). By using the $8,\,1$ irreducibility of $G_{a_1\ldots a_8,\,b}$ and performing several manipulations with ε one can rewrite the terms in the second line of this equation in the following way

$$\begin{aligned} & -\frac{7}{2}\,\varepsilon_{a_1..a_4}{}^{b_1\ldots b_7}\,\left(G_{b_1,\,b_2\ldots b_7c_1c_2,\,c_3} + G_{c_1,\,c_2c_3b_1\ldots b_6,\,b_7}\right)\,\Lambda^{c_1c_2c_3} \\ = & -\frac{3}{4}\,\varepsilon_{[a_1a_2|}{}^{b_1\ldots b_9}\,G_{b_1,\,b_2\ldots b_9,\,c}\,\Lambda^{c}{}_{|a_3a_4]}. \end{aligned} \tag{4.1.14}$$

Here we picked $c = -\frac{1}{2\cdot 4!}$.

Combining all the results together we find the following expressions for the $3-6$ duality equation $D_{a_1\ldots a_4}$ and its l_1-extended version $\mathcal{D}_{a_1\ldots a_4}$.

$$\begin{aligned} D_{a_1\ldots a_4} &= G_{[a_1,\,a_2a_3a_4]} - \frac{1}{2\cdot 4!}\,\varepsilon_{a_1\ldots a_4}{}^{b_1\ldots b_7}\,G_{b_1,\,b_2\ldots b_7}, \\ \mathcal{D}_{a_1\ldots a_4} &= \mathcal{G}_{a_1\ldots a_4} - \frac{1}{2\cdot 4!}\,\varepsilon_{a_1\ldots a_4}{}^{b_1\ldots b_7}\,\mathcal{G}_{b_1\ldots b_7} + \frac{1}{2}\,G_{[a_1a_2,\,a_3a_4]}. \end{aligned} \tag{4.1.15}$$

Its variation is given by

$$\delta\mathcal{D}_{a_1\ldots a_4} = \frac{1}{4!}\,\varepsilon_{a_1\ldots a_4}{}^{b_1\ldots b_7}\,D_{b_1\ldots b_4}\,\Lambda_{b_5b_6b_7} + 3\,D_{c,\,[a_1a_2}\,\Lambda^{c}{}_{a_3a_4]}, \tag{4.1.16}$$

where we have introduced a new duality relation that connects the graviton field $h_a{}^b$ with the dual graviton $A_{a_1\ldots a_8,\,b}$:

$$D_{a,\,b_1b_2} = (\det e)^{\frac{1}{2}}\;\omega_{a,\,b_1b_2} - \frac{1}{4}\,\varepsilon_{b_1b_2}{}^{c_1\ldots c_9}\,G_{c_1,\,c_2\ldots c_9,\,a}. \tag{4.1.17}$$

$D_{a,\,b_1b_2}$ is the first equation in the E_{11} multiplet that cannot be set to zero exactly. Instead it is satisfied modulo the local Lorentz transformations. This subtle point will be later explained in Section 4.1.2. Under the local $I_c\,(E_{11})$ transformations equation (4.1.16) varies into itself and produces a new duality relation. This variation only closes for the particular value of the coefficient given in equation (4.1.12).

$$\begin{aligned} E^{a_1a_2a_3} &= (\pi_1\,D)^{a_1a_2a_3} = e_{\mu_1\mu_2\mu_3}^{\;\;a_1a_2a_3}\,\partial_\nu\left((\det e)^{\frac{1}{2}}\;D^{\nu\mu_1\mu_2\mu_3}\right), \\ E^{a_1\ldots a_6} &= (\pi_2\,D)^{a_1\ldots a_6} = \frac{2}{7!}\,e_{\mu_1\ldots\mu_6}^{\;\;\;\;a_1\ldots a_6}\,\varepsilon^{\mu_1\ldots\mu_6\nu\sigma_1\ldots\sigma_4}\,\partial_\nu\left((\det e)^{-\frac{1}{2}}\;D_{\sigma_1\ldots\sigma_4}\right). \end{aligned} \tag{4.1.18}$$

These projectors take an exceptionally simple form in the linearised case. We have

$$\begin{aligned} E_{a_1a_2a_3} &= (\pi_1\,D)_{a_1a_2a_3} = \partial^b\,D_{ba_1a_2a_3}, \\ E_{a_1\ldots a_6} &= (\pi_2\,D)_{a_1\ldots a_6} = \frac{2}{7!}\,\varepsilon_{a_1\ldots a_6}{}^{bc_1\ldots c_4}\,\partial_b\,D_{c_1\ldots c_4}. \end{aligned} \tag{4.1.19}$$

Note that from now on we will always lower the indexes when working with linearised equations, as their positioning doesn't make any difference in this case. Applying the non-linear projectors we find the following:

$$\begin{aligned} E^{a_1a_2a_3} =\;& (\det e)^{\frac{1}{2}}\;e_b{}^\mu\,\partial_\mu\,G^{[b,\,a_1a_2a_3]} + \frac{1}{2\cdot 4!}\,\varepsilon^{a_1a_2a_3b_1\ldots b_8}\,G_{[b_1,\,b_2b_3b_4]}\,G_{[b_5,\,b_6b_7b_8]} \\ &+ \frac{1}{2}\,G_{b,\,c}{}^{c}\,G^{[b,\,a_1a_2a_3]} - 3\,G_{b,\,c}{}^{[a_1|}\,G^{[b,\,c|a_2a_3]]} - G_{c,\,b}{}^{c}\,G^{[b,\,a_1a_2a_3]}, \\ E^{a_1\ldots a_6} =\;& (\det e)^{\frac{1}{2}}\;e_b{}^\mu\,\partial_\mu\,G^{[b,\,a_1\ldots a_6]} \\ &+ \frac{1}{2}\,G_{b,\,c}{}^{c}\,G^{[b,\,a_1\ldots a_6]} - 6\,G_{b,\,c}{}^{[a_1|}\,G^{[b,\,c|a_2\ldots a_6]]} - G_{c,\,b}{}^{c}\,G^{[b,\,a_1\ldots a_6]}, \end{aligned} \tag{4.1.20}$$

while their linearised counterparts give

$$E_{a_1a_2a_3} = \partial_b\,G_{[b,\,a_1a_2a_3]}, \quad E_{a_1\ldots a_6} = \partial_b\,G_{[b,\,a_1\ldots a_6]}. \tag{4.1.21}$$

We can now see a general pattern that emerges in the E_{11} theory. We have the duality relation (4.1.15) that relates the six form $A_{a_1\ldots a_6}$ to the three form $A_{a_1a_2a_3}$. This equation is then projected in two different ways, each of them giving us a second order equation that describes a single field. Note that even after the second order equations (4.1.20, 4.1.21) are derived, one cannot simply exclude the original duality equation

(4.1.15) from the multiplet. There are two reasons for this. First, duality relation (4.1.15) ensures that equations (4.1.20) describe the same physical degrees of freedom. Removing it would effectively double the number of fields. Second, it is impossible to close the E_{11} multiplet without the duality relations, since, as we will see later, some of the equations vary into them. This includes both the other first order dualities as well as the second and even third order equations that will be found later on.

In the final part of this section we are going to find the variations of the second order equations (4.1.20). We will start with the non-linear $E^{a_1a_2a_3}$ equation. We are going to perform the procedure of l_1-extension of this equation step by step. We will start by varying the unextended equation $E^{a_1a_2a_3}$ from (4.1.20). It contains two different Cartan forms: $G_a{}^b$ and $G_{a_1a_2a_3}$. We'll start by explicitly performing the variation with respect to $G_a{}^b$. Using equation (3.1.9) we get

$$\begin{aligned}\delta E^{a_1a_2a_3} = {} & (\det e)^{\frac{1}{2}}\ e_b{}^\mu\,\partial_\mu\,\delta G^{[b,\,a_1a_2a_3]} + \frac{1}{4!}\,\varepsilon^{a_1a_2a_3b_1\ldots b_8}\,G_{b_1,\,b_2b_3b_4}\,\delta G_{b_5,\,b_6b_7b_8} \\ & + \frac{1}{2}\,G_{b,\,c}{}^{c}\,\delta G^{[b,\,a_1a_2a_3]} - 3\,G_{b,\,c}{}^{[a_1|}\,\delta G^{[b,\,c|a_2a_3]]} - G_{c,\,b}{}^{c}\,\delta G^{[b,\,a_1a_2a_3]} \\ & + 24\,\Lambda^{b_1b_2b_3}\,G_{[c,\,b_1b_2b_3]}\,G^{[c,\,a_1a_2a_3]} - 108\,\Lambda^{b_1b_2[a_1|}\,G_{[b_1,\,b_2c_1c_2]}\,G^{[c_1,\,c_2|a_2a_3]]} \\ & + 54\,\Lambda^{b_1b_2[a_1|}\,G_{b_1,\,b_2c_1c_2}\,G^{[c_1,\,c_2|a_2a_3]]}. \end{aligned} \tag{4.1.22}$$

Here we did not vary the pre-factor $(\det e)^{\frac{1}{2}}\ e_b{}^\mu$ in the derivative term in equation $E^{a_1a_2a_3}$ from (4.1.20). The reason for this is explained in the following short calculation

$$\begin{aligned}\delta\left((\det e)^{\frac{1}{2}}\ e_b{}^\mu\,\partial_\mu\,G^{[b,\,a_1a_2a_3]}\right) &= \delta\left(E_b{}^\mu\,\partial_\mu\,G^{[b,\,a_1a_2a_3]}\right) \\ &= \delta E_b{}^\mu\,\partial_\mu\,G^{[b,\,a_1a_2a_3]} + E_b{}^\mu\,\partial_\mu\,\delta G^{[b,\,a_1a_2a_3]} \\ &= -\,3\,\Lambda_{bc_1c_2}\,\partial^{c_1c_2}\,G^{[b,\,a_1a_2a_3]} + (\det e)^{\frac{1}{2}}\ e_b{}^\mu\,\partial_\mu\,\delta G^{[b,\,a_1a_2a_3]},\end{aligned} \tag{4.1.23}$$

whre $E_A{}^\Pi$ is the inverse generalised vielbein, whose transformations were given in equation (3.1.13). The first term in the last line contains a derivative with respect to the level 1 coordinate, and, therefore, can be dropped out in our truncation.

One can see that the last term in equation (4.1.22) has a derivative (index b_1 in $G_{b_1,\,b_2c_1c_2}$) contracted with Λ. This implies that it can be cancelled by adding an appropriate l_1 term to the left hand side. Another useful observation is that all the terms

that involve the variation of $G_{a_1a_2a_3}$ can be rewritten in a compact way. This can be achieved by replacing the flat indexes with the world ones in $\delta G^{[b,\, a_1a_2a_3]}$ from the first term and pulling the resulting vielbeins through the derivative. We get

$$\begin{aligned}\delta\Big(E^{a_1a_2a_3} - 9\, G^{b[a_1|,}{}_{bc_1c_2}\, G^{[c_1,\, c_2|a_2a_3]]}\Big) &= e^{\,a_1a_2a_3}_{\mu_1\mu_2\mu_3}\, \partial_\nu \left((\det e)^{\frac{1}{2}}\ \delta G^{[\nu,\, \mu_1\mu_2\mu_3]}\right) \\ &+ \frac{1}{4!}\, \varepsilon^{a_1a_2a_3b_1\ldots b_8}\, G_{b_1,\, b_2b_3b_4}\, \delta G_{b_5,\, b_6b_7b_8} \\ &+ 24\, \Lambda^{b_1b_2b_3}\, G_{[c,\, b_1b_2b_3]}\, G^{[c,\, a_1a_2a_3]} - 108\, \Lambda^{b_1b_2[a_1|}\, G_{[b_1,\, b_2c_1c_2]}\, G^{[c_1,\, c_2|a_2a_3]]}.\end{aligned} \tag{4.1.24}$$

We will now process the term from the second line of equation (4.1.24). We have

$$\begin{aligned}\frac{1}{4!}\, \varepsilon^{a_1a_2a_3b_1\ldots b_8}\, G_{b_1,\, b_2b_3b_4}\, \delta G_{b_5,\, b_6b_7b_8} = &-\frac{1}{4}\, \varepsilon^{a_1a_2a_3b_1\ldots b_8}\, G_{b_1,\, b_2b_3b_4}\, G_{b_5,\, (b_6|c|)}\, \Lambda^{c}{}_{b_7b_8} \\ &+ \frac{35}{8}\, \varepsilon^{a_1a_2a_3b_1\ldots b_8}\, G_{b_1,\, b_2b_3b_4}\, G_{[b_5,\, b_6b_7b_8c_1c_2c_3]}\, \Lambda^{c_1c_2c_3} \\ &- \frac{15}{8}\, \varepsilon^{a_1a_2a_3b_1\ldots b_8}\, G_{b_1,\, b_2b_3b_4}\, G_{c_1,\, c_2c_3b_5\ldots b_8}\, \Lambda^{c_1c_2c_3}.\end{aligned} \tag{4.1.25}$$

The term in the third line is also an l_1 term and can be cancelled in the variation. In the second line we are going to replace the six form $G_{[b_5,\, b_6b_7b_8c_1c_2c_3]}$ with the three form using the duality relation (4.1.15). Rewriting it in terms of the six form we get

$$G_{[a_1,\, a_2\ldots a_7]} = \frac{2}{7!}\, \varepsilon_{a_1\ldots a_7}{}^{b_1\ldots b_4}\ (D_{b_1\ldots b_4} - G_{b_1,\, b_2b_3b_4})\,. \tag{4.1.26}$$

Equation (4.1.25) becomes

$$\begin{aligned}&\frac{1}{4!}\, \varepsilon^{a_1a_2a_3b_1\ldots b_8}\, G_{b_1,\, b_2b_3b_4}\, \delta G_{b_5,\, b_6b_7b_8} + \delta\Big(\frac{5}{16}\, \varepsilon^{a_1a_2a_3b_1\ldots b_8}\, G_{b_1,\, b_2b_3b_4}\, G^{c_1c_2,}{}_{c_1c_2b_5\ldots b_8}\Big) \\ &= -\frac{1}{4}\, \varepsilon^{a_1a_2a_3b_1\ldots b_8}\, G_{b_1,\, b_2b_3b_4}\, G_{b_5,\, (b_6|c|)}\, \Lambda^{c}{}_{b_7b_8} \\ &\quad - 210\, G_{b_1,\, b_2b_3b_4}\, D^{[b_1\ldots b_4}\, \Lambda^{a_1a_2a_3]} + 210\, G_{b_1,\, b_2b_3b_4}\, G^{[b_1,\, b_2b_3b_4}\, \Lambda^{a_1a_2a_3]}.\end{aligned} \tag{4.1.27}$$

The second term in the last line of equation (4.1.27) can be further processed by ex-

panding the antisymmetrisation. This gives the following

$$\begin{aligned}
&\frac{1}{4!}\,\varepsilon^{a_1a_2a_3b_1\ldots b_8}\,G_{b_1,\,b_2b_3b_4}\,\delta G_{b_5,\,b_6b_7b_8}+\delta\Big(\frac{5}{16}\,\varepsilon^{a_1a_2a_3b_1\ldots b_8}\,G_{b_1,\,b_2b_3b_4}\,G^{c_1c_2,}{}_{c_1c_2b_5\ldots b_8}\Big)\\
&=\;-\frac{1}{4}\,\varepsilon^{a_1a_2a_3b_1\ldots b_8}\,G_{b_1,\,b_2b_3b_4}\,G_{b_5,\,(b_6|c|)}\,\Lambda^{c}{}_{b_7b_8}-210\,G_{b_1,\,b_2b_3b_4}\,D^{[b_1\ldots b_4}\,\Lambda^{a_1a_2a_3]}\\
&\quad-24\,\Lambda^{b_1b_2b_3}\,G_{[c,\,b_1b_2b_3]}\,G^{[c,\,a_1a_2a_3]}+108\,\Lambda^{b_1b_2[a_1|}\,G_{[b_1,\,b_2c_1c_2]}\,G^{[c_1,\,c_2|a_2a_3]]}\\
&\quad-72\,\Lambda^{b[a_1a_2}\,G_{[b,\,c_1c_2c_3]}\,G^{[a_3],\,c_1c_2c_3]}+6\,\Lambda^{a_1a_2a_3}\,G_{[c_1,\,c_2c_3c_4]}\,G^{[c_1,\,c_2c_3c_4]}. \qquad (4.1.28)
\end{aligned}$$

Two of the new terms cancel precisely with the corresponding ones from equation (4.1.24). Combining equations (4.1.24) and (4.1.28) we get

$$\begin{aligned}
&\delta\Big(E^{a_1a_2a_3}-9\,G^{b[a_1|,}{}_{bc_1c_2}\,G^{[c_1,\,c_2|a_2a_3]]}+\frac{5}{16}\,\varepsilon^{a_1a_2a_3b_1\ldots b_8}\,G_{b_1,\,b_2b_3b_4}\,G^{c_1c_2,}{}_{c_1c_2b_5\ldots b_8}\Big)\\
&\qquad=e^{\,a_1a_2a_3}_{\mu_1\mu_2\mu_3}\,\partial_\nu\Big((\det e)^{\frac{1}{2}}\;\delta G^{[\nu,\,\mu_1\mu_2\mu_3]}\Big)-210\,G_{b_1,\,b_2b_3b_4}\,D^{[b_1\ldots b_4}\,\Lambda^{a_1a_2a_3]}\\
&\qquad\quad+\frac{3}{2}\,\Big(-48\,G_{[b,\,c_1c_2c_3]}\,G^{[[a_1|,\,c_1c_2c_3]}+4\,\delta^{[a_1|}_{b}\,G_{[c_1,\,c_2c_3c_4]}\,G^{[c_1,\,c_2c_3c_4]}\Big)\;\Lambda^{|b|a_2a_3]}\\
&\qquad\quad-\frac{1}{4}\,\varepsilon^{a_1a_2a_3b_1\ldots b_8}\,G_{b_1,\,b_2b_3b_4}\,G_{b_5,\,(b_6|c|)}\,\Lambda^{c}{}_{b_7b_8}. \qquad (4.1.29)
\end{aligned}$$

Lastly, we will now process the remaining term. Taking the variation we find

$$\begin{aligned}
&e^{\,a_1a_2a_3}_{\mu_1\mu_2\mu_3}\,\partial_\nu\Big((\det e)^{\frac{1}{2}}\;\delta G^{[\nu,\,\mu_1\mu_2\mu_3]}\Big)+\delta\Big(\frac{15}{2}\,e^{\,a_1a_2a_3}_{\mu_1\mu_2\mu_3}\,\partial_\nu\Big((\det e)^{\frac{1}{2}}\;G^{\sigma_1\sigma_2,}{}_{\sigma_1\sigma_2}{}^{\nu\mu_1\mu_2\mu_3}\Big)\Big)\\
&\qquad=3\,e^{\,a_1a_2a_3}_{\mu_1\mu_2\mu_3}\,\partial_\nu\Big((\det e)\;\omega_{\sigma,}{}^{[\nu\mu_1|}-(\det e)^{\frac{1}{2}}\;G_{\sigma,}{}^{[\nu\mu_1|}\Big)\;\Lambda^{\sigma|\mu_2\mu_3]}\\
&\qquad\quad+105\,e^{\,a_1a_2a_3}_{\mu_1\mu_2\mu_3}\,\partial_\nu\Big((\det e)^{\frac{1}{2}}\;G^{[\nu,\,\mu_1\mu_2\mu_3\sigma_1\sigma_2\sigma_3]}\,\Lambda_{\sigma_1\sigma_2\sigma_3}\Big). \qquad (4.1.30)
\end{aligned}$$

As per usual, we have extracted an l_1 term. $\omega_{a,\,b_1b_2}$ has been defined in (4.1.9). In order to process the term in the last line we're going to use equation (4.1.26) again. After some calculations we find

$$\begin{aligned}
&105\,e^{\,a_1a_2a_3}_{\mu_1\mu_2\mu_3}\,\partial_\nu\Big((\det e)^{\frac{1}{2}}\;G^{[\nu,\,\mu_1\mu_2\mu_3\sigma_1\sigma_2\sigma_3]}\,\Lambda_{\sigma_1\sigma_2\sigma_3}\Big)\\
&\qquad=\frac{1}{24}\,e^{\,a_1a_2a_3}_{\mu_1\mu_2\mu_3}\,\varepsilon^{\mu_1\mu_2\mu_3\sigma_1\sigma_2\sigma_3\nu\lambda_1\ldots\lambda_4}\,\partial_\nu\Big((\det e)^{-\frac{1}{2}}\;D_{\lambda_1\ldots\lambda_4}\,g_{\sigma_1\rho_1}\,g_{\sigma_2\rho_2}\,g_{\sigma_3\rho_3}\Big)\;\Lambda^{\rho_1\rho_2\rho_3}\\
&\qquad\quad+\frac{1}{4}\,\varepsilon^{a_1a_2a_3b_1\ldots b_8}\,G_{b_1,\,b_2b_3b_4}\,G_{b_5,\,(b_6|c|)}\,\Lambda^{c}{}_{b_7b_8}. \qquad (4.1.31)
\end{aligned}$$

Note that $\Lambda_{\sigma_1\sigma_2\sigma_3}$ cannot be taken out of the derivative, as only $\Lambda^{\sigma_1\sigma_2\sigma_3}$ with its indexes up is a constant. The final step is to process the remaining term from equation equation

(4.1.30). By extracting two more l_1 terms one finds

$$\begin{aligned}
3\, e_{\mu_1\mu_2\mu_3}^{a_1a_2a_3}\, \partial_\nu \Big((\det e)\; \omega_{\sigma,}{}^{[\nu\mu_1} - (\det e)^{\frac{1}{2}}\; G_{\sigma,}{}^{[\nu\mu_1} \Big)\; \Lambda^{|\sigma|\mu_2\mu_3]} \\
+ \delta \left(\frac{1}{2}\, e_{\mu_1\mu_2\mu_3}^{a_1a_2a_3}\, \partial_\nu \Big((\det e)^{\frac{1}{2}}\; G^{[\nu\mu_1,\, \mu_2\mu_3]} \Big) + \frac{1}{4}\, e_{\mu_1\mu_2\mu_3}^{a_1a_2a_3}\, \partial^{\sigma\mu_1} \left((\det e)\; \omega_{\sigma,}{}^{\mu_2\mu_3} \right) \right) \\
= \frac{3}{2}\, e_{\mu_1\mu_2\mu_3}^{a_1a_2a_3}\, \partial_\nu \left((\det e)\; \omega_{\sigma,}{}^{\nu\mu_1} \right)\; \Lambda^{\sigma\mu_2\mu_3} \\
= \frac{3}{2}\; (\det e)\; \Big(R_b{}^{[a_1|} - e_c{}^\nu\, e_b{}^\sigma\, \partial_\sigma\, \omega_{\nu,}{}^{[a_1|c} \Big)\; \Lambda^{b|a_2a_3]}, \qquad (4.1.32)
\end{aligned}$$

where $R_a{}^b$ is the Ricci tensor, defined as a contraction of the Riemann curvature.

$$R_{\mu\nu,}{}^{ab} = \partial_\mu\, \omega_{\nu,}{}^{ab} - \partial_\nu\, \omega_{\mu,}{}^{ab} + \omega_{\mu,}{}^a{}_c\, \omega_{\nu,}{}^{cb} - \omega_{\nu,}{}^a{}_c\, \omega_{\mu,}{}^{cb}, \quad R_\mu{}^a = e_b{}^\nu\, R_{\mu\nu,}{}^{ab}. \qquad (4.1.33)$$

One of the l_1 terms in equation (4.1.32) contains a derivative with respect to the level 1 coordinate $\partial^{a_1a_2}$, that is separate from G. In order to process the variation of this kind of terms on has to first rewrite them as a sum of terms that have the following form $f_1\left(G_{\text{tangent}}\right)\, E^{a_1a_2,\,\Pi}\, \partial_\Pi\, f_2\left(G_{\text{tangent}}\right)$. After that one takes the variation of these terms

$$\begin{aligned}
\delta \left(f_1\left(G_{\text{tangent}}\right)\, E^{a_1a_2,\,\Pi}\, \partial_\Pi\, f_2\left(G_{\text{tangent}}\right) \right) = f_1\left(G_{\text{tangent}}\right)\, 6\, \Lambda^{a_1a_2b}\, E_b{}^\Pi\, \partial_\Pi\, f_2\left(G_{\text{tangent}}\right) \\
+ \text{higher } l_1 \text{ terms}, \qquad (4.1.34)
\end{aligned}$$

and, finally, groups them back together into one term. The result of this procedure gives the following simple recipe for taking the variation of these terms

$$\delta\left(\ldots\, \partial^{\mu_1\mu_2}\, \ldots \right) \longrightarrow \ldots\, 6\, \Lambda^{\mu_1\mu_2\nu}\, \partial_\nu\, \ldots\,. \qquad (4.1.35)$$

The second term in the brackets in the last line of equation (4.1.32) also happens to be an l_1 term of this kind. Putting all the variations together we find

$$\begin{aligned}
\delta\mathcal{E}^{a_1a_2a_3} = \frac{3}{2}\, E_b{}^{[a_1}\, \Lambda^{|b|a_2a_3]} - 210\, G_{[b_1,\, b_2b_3b_4]}\, D^{[b_1\ldots b_4}\, \Lambda^{a_1a_2a_3]} \qquad (4.1.36) \\
+ \frac{1}{24}\, e_{\mu_1\mu_2\mu_3}^{a_1a_2a_3}\, \varepsilon^{\mu_1\mu_2\mu_3\sigma_1\sigma_2\sigma_3\nu\lambda_1\ldots\lambda_4}\, \partial_\nu \Big((\det e)^{-\frac{1}{2}}\; D_{\lambda_1\ldots\lambda_4}\, g_{\sigma_1\rho_1}\, g_{\sigma_2\rho_2}\, g_{\sigma_3\rho_3} \Big)\, \Lambda^{\rho_1\rho_2\rho_3},
\end{aligned}$$

where $E_a{}^b$ is the Einstein equation that has the following form

$$E_a{}^b = (\det e)\, R_a{}^b - 48\, G_{[a,\, c_1c_2c_3]}\, G^{[b,\, c_1c_2c_3]} + 4\, \delta_a^b\, G_{[c_1,\, c_2c_3c_4]}\, G^{[c_1,\, c_2c_3c_4]}. \qquad (4.1.37)$$

The last two terms in this equation reproduce the correct expression for the stress-energy tensor of the three form in eleven dimensions. We can now group all the l_1 terms to construct the l_1-extended three form equation. The result is

$$\begin{aligned}\mathcal{E}^{a_1a_2a_3} &= E^{a_1a_2a_3} - 9\, G^{b[a_1|,}{}_{bc_1c_2}\, G^{[c_1,\, c_2|a_2a_3]]} + \frac{5}{16}\, \varepsilon^{a_1a_2a_3b_1\ldots b_8}\, G_{b_1,\, b_2b_3b_4}\, G^{c_1c_2,}{}_{c_1c_2b_5\ldots b_8} \\ &+ \frac{1}{2}\, e^{\ a_1a_2a_3}_{\mu_1\mu_2\mu_3}\, \partial_\nu \left((\det e)^{\frac{1}{2}}\ G^{[\nu\mu_1,\, \mu_2\mu_3]} \right) + \frac{15}{2}\, e^{\ a_1a_2a_3}_{\mu_1\mu_2\mu_3}\, \partial_\nu \left((\det e)^{\frac{1}{2}}\ G^{\sigma_1\sigma_2,}{}_{\sigma_1\sigma_2}{}^{\nu\mu_1\mu_2\mu_3} \right) \\ &+ \frac{1}{4}\, e^{\ a_1a_2a_3}_{\mu_1\mu_2\mu_3}\, \partial^{\sigma\mu_1} \left((\det e)\ \omega_{\sigma,}{}^{\mu_2\mu_3} \right) + \frac{1}{4}\ (\det e)\ e_b{}^\nu\, \partial^{[a_1a_2}\, \omega_{\nu,}{}^{a_3]b}. \end{aligned} \tag{4.1.38}$$

Equations (4.1.20) were obtained from equation (4.1.15) by applying the π projectors from equation (4.1.18). This implies that the l_1 terms in equation (4.1.38) should also be obtainable by the same procedure. However, instead of projecting the l_1 terms we have independently applied the procedure of l_1-extension to it. We did that because the projectors from equation (4.1.18) are only defined on level 0. In order to match the l_1 terms in equations (4.1.15) and (4.1.38) one has to introduce the l_1 extended versions of π projectors.

This calculation simplifies a lot in the linear case. The linearised vector equation is given by

$$\begin{aligned}\mathcal{E}_{a_1a_2a_3} &= E_{a_1a_2a_3} + \frac{15}{2}\, \partial^b\, G^{c_1c_2,}{}_{c_1c_2ba_1a_2a_3} \\ &+ \frac{1}{4}\, \partial^b\, G_{[a_1a_2,\, (a_3]b)} + \frac{1}{2}\, \partial_{[a_1}\, G^b{}_{a_2,\, (a_3]b)} - \frac{1}{4}\, \partial_{[a_1}\, G_{a_2a_3],\, b}{}^b. \end{aligned} \tag{4.1.39}$$

It is important to point out that the l_1 terms of the following form $\partial_\mu\, \partial^{\mu\nu}\, ...$ can be neglected in the limit that we are working in, as their level 0 variations vanish: $\delta\, (\partial_\mu\, \partial^{\mu\nu}\, ...) = 6\, \Lambda^{\mu\nu\lambda}\, \partial_\mu\, \partial_\lambda\, ... = 0$. The l_1 terms in equation (4.1.39) can be obtained by by linearising the l_1 terms from equation (4.1.38) and throwing out all the terms of this kind. The variation of equation (4.1.39) is given by

$$\delta \mathcal{E}_{a_1a_2a_3} = 105\, E_{a_1a_2a_3b_1b_2b_3}\, \Lambda^{b_1b_2b_3} + \frac{3}{2}\, E_{b[a_1}\, \Lambda^b{}_{a_2a_3]}. \tag{4.1.40}$$

One can see that in the linear case the variation of equation (4.1.38) closes on the other second order equations and doesn't require having the first order equations in the multiplet. The non-linear variation of equation (4.1.38), however, cannot be closed without

the first order equations. This is the first example of a second order equation that has the first order dualities in its variation. Later we will see that E_{11} multiplet contains equations, whose variation contains the first order equations even at the linearised level.

The linearised variation of the six form equation of motion from (4.1.20) under the $I_c\left(E_{11}\right)$ transformations is given by

$$\delta\mathcal{E}_{a_1\ldots a_6} = \frac{8}{7}\,\Lambda_{[a_1a_2a_3}\,E_{a_4a_5a_6]} - 1728\,E_{a_1\ldots a_6b_1b_2,\,b_3}\,\Lambda^{b_1b_2b_3}, \tag{4.1.41}$$

where we used the following expression for the l_1 extension of this equation

$$\mathcal{E}_{a_1\ldots a_6} = E_{a_1\ldots a_6} + \frac{1}{7}\,\partial_{[a_1}\,G_{a_2a_3,\,a_4a_5a_6]} - 24\,\partial^{c_1c_2}\,G_{[a_1,\,a_2\ldots a_6]c_1c_2d,}{}^{d} + 24\,\partial^{c_1c_2}\,G^{d,}{}_{dc_1c_2[a_1\ldots a_5,\,a_6]}, \tag{4.1.42}$$

and we introduced a new second order equation, that describes the dynamics of the dual graviton

$$E_{a_1\ldots a_8,\,b} = -\frac{1}{4}\,\partial_{[c}\,G_{[c,\,a_1\ldots a_8],\,b]} = -\frac{1}{4}\,\partial_{[c}\,\partial_{[c}\,A_{a_1\ldots a_8],\,b]}. \tag{4.1.43}$$

One can show that this equation obeys the same $GL\left(11\right)$ irreducibility condition as the $A_{a_1\ldots a_8,\,b}$ field itself. In order to do that we manually expand the antisymmetrisations in equation (4.1.43) to get

$$\begin{aligned} E_{a_1\ldots a_8,\,b} = & -\frac{1}{72}\,\partial^2\,A_{a_1\ldots a_8,\,b} + \frac{1}{9}\,\partial_b\,\partial_{[a_1}\,A_{a_2\ldots a_8]c,}{}^{c} \\ & + \frac{1}{72}\,\partial^c\,\partial_b\,A_{a_1\ldots a_8,\,c} - \frac{1}{9}\,\partial^c\,\partial_{[a_1}\,A_{a_2\ldots a_8]c,\,b}. \end{aligned} \tag{4.1.44}$$

We can now see that

$$\begin{aligned} E_{[a_1\ldots a_8,\,b]} = & -\frac{1}{72}\,\partial^2\,A_{[a_1\ldots a_8,\,b]} + \frac{1}{9}\,\partial_{[b}\,\partial_{a_1}\,A_{a_2\ldots a_8]c,}{}^{c} \\ & - \frac{5}{36}\,\partial^c\,\partial_{[c}\,A_{a_1\ldots a_8,\,b]} - \frac{1}{72}\,\partial^2\,A_{[a_1\ldots a_8,\,b]} = 0, \end{aligned} \tag{4.1.45}$$

where we used the following identity

$$\partial_{[b}\,A_{a_1\ldots a_8],\,c} - 8\,\partial_{[a_1}\,A_{a_2\ldots a_8|c|,\,b]} = 10\,\partial_{[b}\,A_{a_1\ldots a_8,\,c]} + \partial_c\,A_{[a_1\ldots a_8,\,b]}. \tag{4.1.46}$$

This implies that $E_{[a_1\ldots a_8,\,b]} = 0$ and, therefore, equation (4.1.43) belongs to the irreducible **1760** representation of $GL\left(11\right)$.

4.1.2 Gravity - dual duality relation and the second order equations for the graviton and the dual graviton fields

We are going to start this section by examining how the gravity - dual gravity relation $D_{a,\,b_1b_2}$ is affected by the modulo transformations mentioned before. We will then proceed to find its variation and show that it closes on the previously derived vector duality relation $D_{a_1\ldots a_4}$ from equation (4.1.15), as well as two new level 4 equations $D_{a_1\ldots a_{11},\,b_1b_2}$ and $D_{a_1\ldots a_{10},\,b_1b_2b_3}$. We will later build two new projectors θ_1 and θ_2, one of which will give us the Einstein equation (4.1.37), previously obtained from the variation of the second order three form equation $E_{a_1a_2a_3}$, while the other — the second order dual graviton equation (4.1.43), found before by varying the second order six form equation $E_{a_1\ldots a_6}$. We will then take the variations of these two equations to find that they are closing on the previously found equations from the vector sector and produce two new second order level 4 equations $E_{a_1\ldots a_9,\,b_1b_2b_3}$ and $E_{c_1\ldots c_{11},\,a,\,b_1b_2}$. Lastly, we will discuss two of the commutative diagrams that arise in the E_{11} multiplet. All the calculations in this section will be done at the **linearised** level, except for the variation of the Einstein equation which will be done both linearly and non-linearly.

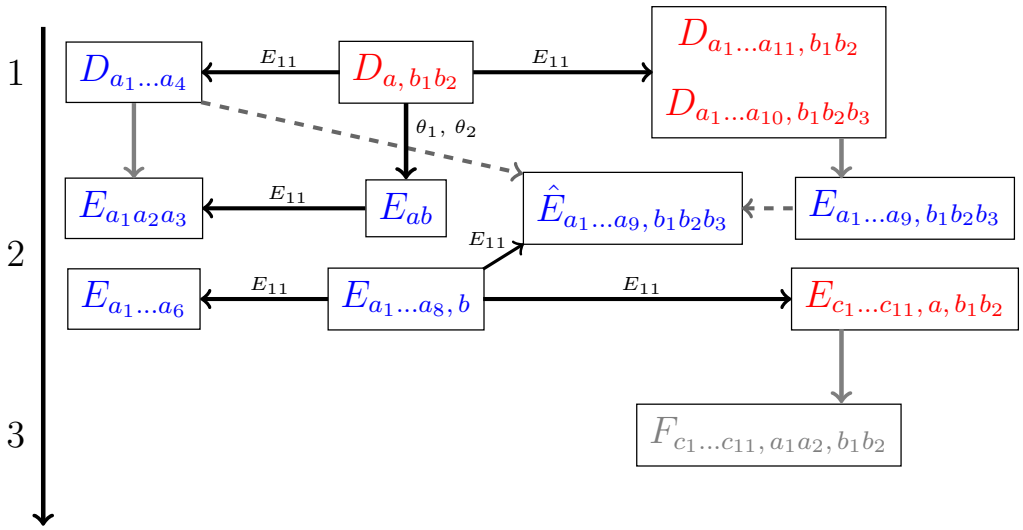


Figure 12: Gravity sector of the E_{11} multiplet

We recall that the local Lorentz transformations were not used to fix our choice of group element of equation (3.1.2) and as such they are still an explicit symmetry. These transform the spin connection of equation (4.1.17) by an inhomogeneous term. At the linearised level we have $\delta\omega_{a,\,b_1b_2} = \partial_a\,\hat{\Lambda}_{b_1b_2} + \ldots$, where $+\ldots$ indicates the homogeneous part of the variation. As a result equation (4.1.17) is not invariant under the local Lorentz transformations and we should consider it as being valid only modulo said transformations. In other words, instead of setting it to zero directly, we find that

$$D_{a,\,b_1b_2} \doteq 0 \quad \Longleftrightarrow \quad D_{a,\,b_1b_2} - \partial_a\,\hat{\Lambda}_{b_1b_2} = 0, \tag{4.1.47}$$

where we have introduced a new notation $\doteq$ which means that the equation is equal to zero up to a certain transformation. This strategy was first advocated in [49]. This new approach poses a difficulty when it comes to the interpretation of equation (4.1.16). It contains two different equations ($D_{a_1\ldots a_4}$ and $D_{a,\,b_1b_2}$). The first one is satisfied exactly ($D_{a_1\ldots a_4} = 0$), while the second one — modulo local Lorentz transformations ($D_{a,\,b_1b_2} \doteq 0$). In order to preserve the consistency of this equation we can make the following adjustment

$$\delta\mathcal{D}_{a_1\ldots a_4} = \frac{1}{4!}\,\varepsilon_{a_1\ldots a_4}{}^{b_1\ldots b_7}\,D_{b_1\ldots b_4}\,\Lambda_{b_5b_6b_7} + 3\,\left(D_{c,\,[a_1a_2} - \partial_c\,\hat{\Lambda}_{[a_1a_2}\right)\,\Lambda^{c}{}_{a_3a_4]}. \tag{4.1.48}$$

By moving $\hat{\Lambda}$ into the variation we get

$$\delta\left(\mathcal{D}_{a_1\ldots a_4} + \frac{1}{2}\,\partial_{[a_1a_2}\,\hat{\Lambda}_{a_3a_4]}\right) = \frac{1}{4!}\,\varepsilon_{a_1\ldots a_4}{}^{b_1\ldots b_7}\,D_{b_1\ldots b_4}\,\Lambda_{b_5b_6b_7} + 3\,D_{c,\,[a_1a_2}\,\Lambda^{c}{}_{a_3a_4]}. \tag{4.1.49}$$

As one can see, we were able to compensate for the modulo invariance of equation $D_{a,\,b_1b_2}$ by adding an extra l_1 term to equation $D_{a_1\ldots a_4}$. As we are truncating all the derivatives with respect to higher level coordinates, this change doesn't affect the dynamics of the fields, while ensuring that the variation of $D_{a_1\ldots a_4}$ is consistent with the modulo transformations of $D_{a,\,b_1b_2}$.

Modulo equations like (4.1.17) also acquire the modulo terms in their own variations. We will now illustrate this by taking the variation of the linearised equation (4.1.17).

First we extend this equation by the necessary l_1 terms. We get

$$\begin{aligned}\mathcal{D}_{a,\,b_1b_2} = \;& (\det e)^{\frac{1}{2}}\;\Omega_{a,\,b_1b_2} - \tfrac{1}{4}\,\varepsilon_{b_1b_2}{}^{c_1\ldots c_9}\,G_{c_1,\,c_2\ldots c_9,\,a} \\ & - \varepsilon_{b_1b_2}{}^{c_1\ldots c_9}\;\;\Big[\tfrac{55}{3}\,\big(\tfrac{1}{9}\,G^{d_1d_2,}{}_{c_1\ldots c_9,\,d_1d_2a} + \tfrac{1}{8}\,G^{d_1d_2,}{}_{c_1\ldots c_8a,\,d_1d_2c_9}\big) \\ & \qquad + 10\,\big(\tfrac{1}{9}\,G^{d_1d_2,}{}_{c_1\ldots c_9d_1,\,d_2a} + \tfrac{1}{8}\,G^{d_1d_2,}{}_{c_1\ldots c_8ad_1,\,d_2c_9}\big) \\ & \qquad + \tfrac{55}{4}\,\big(\tfrac{1}{9}\,G^{d_1d_2,}{}_{d_1d_2c_1\ldots c_9,\,a} + \tfrac{1}{8}\,G^{d_1d_2,}{}_{d_1d_2ac_1\ldots c_8,\,c_9}\big)\Big],\end{aligned} \tag{4.1.50}$$

where we have introduced the l_1-extended spin-connection $\Omega_{a,\,b_1b_2}$.

$$(\det e)^{\frac{1}{2}}\;\Omega_{a,\,b_1b_2} = (\det e)^{\frac{1}{2}}\;\omega_{a,\,b_1b_2} - 3\,G^{c}{}_{a,\,b_1b_2c} + 6\,G^{c}{}_{[b_1,\,b_2]ac} + 2\,\eta_{a[b_1}\,G^{c_1c_2,}{}_{b_2]c_1c_2}. \tag{4.1.51}$$

Its variation is given by

$$\begin{aligned}\delta\Big((\det e)^{\frac{1}{2}}\;\Omega_{a,\,b_1b_2}\Big) = \;& -36\,\Lambda^{c_1c_2}{}_{a}\,G_{[b_1,\,b_2c_1c_2]} + 72\,\Lambda^{c_1c_2}{}_{[b_1}\,G_{[b_2],\,ac_1c_2]} \\ & - 16\,\eta_{a[b_1}\,\Lambda^{c_1c_2c_3}\,G_{[b_2],\,c_1c_2c_3]}.\end{aligned} \tag{4.1.52}$$

The variation of equation (4.1.50) is

$$\begin{aligned}\delta\mathcal{D}_{a,\,b_1b_2} = \;& -\;36\,\Lambda_a{}^{c_1c_2}\,D_{b_1b_2c_1c_2} - 8\,\eta_{a[b_1}\,D_{b_2]c_1c_2c_3}\,\Lambda^{c_1c_2c_3} \\ & - \frac{55}{2}\,\varepsilon^{d_1\ldots d_{10}}{}_{[b_1}\,\Lambda_{b_2]}{}^{c_1c_2}\,D_{d_1\ldots d_{10},\,ac_1c_2} - \frac{55}{18}\,\eta_{a[b_1}\,\varepsilon_{b_2]}{}^{d_1\ldots d_{10}}\,\Lambda^{c_1c_2c_3}\,D_{d_1\ldots d_{10},\,c_1c_2c_3} \\ & + \frac{3}{4}\,\varepsilon^{d_1\ldots d_{11}}\,\Lambda_{b_1b_2}{}^{c}\,D_{d_1\ldots d_{11},\,ac} + \partial_a\,\tilde{\Lambda}_{b_1b_2},\end{aligned} \tag{4.1.53}$$

where we have introduced a new $3-9$, 3 duality equation, as well as an equation that describes the dynamics of the $A_{a_1\ldots a_{10},\,b_1b_2}$ field

$$\begin{aligned}D_{a_1\ldots a_{10},\,b_1b_2b_3} &= G_{[a_1,\,a_2\ldots a_{10}],\,b_1b_2b_3} - \frac{144}{5\cdot 11!}\,\varepsilon_{a_1\ldots a_{10}}{}^{c}\,G_{[c,\,b_1b_2b_3]}, \\ D_{a_1\ldots a_{11},\,b_1b_2} &= G_{[a_1,\,a_2\ldots a_{11}],\,b_1b_2}.\end{aligned} \tag{4.1.54}$$

Both these equations are also subject to the modulo transformations. They will be investigated in the next section. The 11, 1 field was eliminated by the l_1 corrections and didn't appear in the final variation. As we will see later this is a general pattern that indicates that this field is non-dynamical.

The modulo term of equation (4.1.53) $\partial_a \tilde{\Lambda}_{b_1 b_2}$ is given by

$$\partial_a \tilde{\Lambda}_{b_1 b_2} = -\varepsilon_{b_1 b_2}{}^{c_1 \ldots c_9} \left[\frac{1}{12} G_{a,\, c_1 \ldots c_6} \Lambda_{c_7 c_8 c_9} + \frac{55}{36} G_{a,\, c_1 \ldots c_9,\, d_1 d_2 d_3} \Lambda^{d_1 d_2 d_3} + \frac{55}{8} G_{a,\, d_1 d_2 d_3 c_1 \ldots c_8,\, c_9} \Lambda^{d_1 d_2 d_3} \right]. \tag{4.1.55}$$

We were able to account for the modulo transformations of equation (4.1.17) both in its variation and in the variation of the $3-6$ duality equation (4.1.15). The general form of the modulo transformations will be discussed later in Section 4.1.3.

Next we are going to define two projectors θ_1, θ_2, similar to ones defined in (4.1.18), which will allow us to obtain the second order equations for the graviton and dual graviton fields. At the **linearised** level they have the following form

$$E_a{}^b = (\theta_1\, D)_a{}^b = \partial_a\, D_{c,}{}^{bc} - \partial_c\, D_{a,}{}^{bc} = \partial_a\, \omega_{c,}{}^{bc} - \partial_c\, \omega_{a,}{}^{bc} = R_a{}^b, \tag{4.1.56}$$

$$E_{a_1 \ldots a_8,\, b} = (\theta_2\, D)_{a_1 \ldots a_8,\, b} = \frac{1}{2 \cdot 9!}\, \varepsilon_{a_1 \ldots a_8}{}^{c d_1 d_2}\, \partial_{[b}\, D_{c],\, d_1 d_2} = -\frac{1}{4}\, \partial_{[c}\, G_{[c,\, a_1 \ldots a_8],\, b]}.$$

The non-linear form of these projectors hasn't been investigated.

Now we will consider the variations of the non-linear second order gravity equation (4.1.37). The l_1 extension of this equation has the following form.

$$\begin{aligned} \mathcal{E}_a{}^b = {} & (\det e)\ \mathcal{R}_a{}^b - 48\, G_{[a,\, c_1 c_2 c_3]}\, G^{[b,\, c_1 c_2 c_3]} + 4\, \delta_a^b\, G_{[c_1,\, c_2 c_3 c_4]}\, G^{[c_1,\, c_2 c_3 c_4]} \\ & - 360\, G^{d_1 d_2,}{}_{d_1 d_2 a c_1 c_2 c_3}\, G^{[b,\, c_1 c_2 c_3]} - 360\, G^{d_1 d_2,}{}_{d_1 d_2}{}^{b c_1 c_2 c_3}\, G_{[a,\, c_1 c_2 c_3]} \\ & + 60\, \delta_a^b\, G^{d_1 d_2}{}_{,\, d_1 d_2 c_1 \ldots c_4}\, G^{[c_1,\, c_2 c_3 c_4]} - 12\, G_{c_1 c_2,\, a c_3}\, G^{[b,\, c_1 c_2 c_3]} + 3\, G_{c_1 c_2,\, d}{}^{d}\, G_{[a,}{}^{b c_1 c_2]} \\ & - 6\ (\det e)\ e_a{}^{\lambda}\, e^{b\mu}\, \partial_{[\mu|} \left[(\det e)^{-\frac{1}{2}}\ G^{\sigma_1 \sigma_2,}{}_{|\sigma_1 \sigma_2 \lambda]} \right] \\ & - (\det e)^{\frac{1}{2}}\, \omega_{c,\, b}{}^{c}\, G^{d_1 d_2,}{}_{d_1 d_2 a} - 3\ (\det e)^{\frac{1}{2}}\, \omega_{a,\, b}{}^{c}\, G^{d_1 d_2,}{}_{d_1 d_2 c}, \end{aligned} \tag{4.1.57}$$

where $\mathcal{R}_a{}^b$ is the l_1-extended Ricci tensor, obtained by taking the standard definition (4.1.33) and replacing $\omega_{a,\, b_1 b_2}$ with its l_1-extended counterpart $\Omega_{a,\, b_1 b_2}$ from equation (4.1.51).

The variation of equation (4.1.57) is

$$\begin{aligned}\delta\mathcal{E}_a{}^b = & -36\,E_{ac_1c_2}\,\Lambda^{bc_1c_2} - 36\,E^{bc_1c_2}\,\Lambda_{ac_1c_2} + 8\,\delta^b_a\,\Lambda^{c_1c_2c_3}\,E_{c_1c_2c_3} \\ & - 2\,\varepsilon_{ac_1\ldots c_7d_1d_2d_3}\,G^{[b,\,c_1c_2c_3]}\,D^{c_4\ldots c_7}\,\Lambda^{d_1d_2d_3} \\ & - 2\,\varepsilon^{bc_1\ldots c_7d_1d_2d_3}\,G_{[a,\,c_1c_2c_3]}\,D_{c_4\ldots c_7}\,\Lambda_{d_1d_2d_3} \\ & + \frac{1}{3}\,\delta^b_a\,\varepsilon^{c_1\ldots c_8d_1d_2d_3}\,G_{c_1,\,c_2c_3c_4}\,D_{c_5\ldots c_8}\,\Lambda_{d_1d_2d_3}. \end{aligned} \tag{4.1.58}$$

When linearised, equation (4.1.57) and its variation take the following form

$$\begin{aligned}\mathcal{E}_{ab} &= \mathcal{R}_{ab} + 6\,\partial^{c_1c_2}\,G_{[c_1,\,c_2ab]}, \\ \delta\mathcal{E}_{ab} &= -36\,E_{ac_1c_2}\,\Lambda_b{}^{c_1c_2} - 36\,E_{bc_1c_2}\,\Lambda_a{}^{c_1c_2} + 8\,\eta_{ab}\,\Lambda^{c_1c_2c_3}\,E_{c_1c_2c_3}. \end{aligned} \tag{4.1.59}$$

This equation has non-trivial symmetry properties. Unextended Einstein equation (4.1.37) with its second index lowered $E_{ab} = \eta_{bc}\,E_a{}^c$ is clearly symmetric in ab. This follows from the corresponding property of the Ricci tensor. On the other hand, the extended version of the Ricci tensor $\mathcal{R}_{ab}$ no longer possesses the ab symmetry. However, it turns out that this symmetry is explicitly restored by the additional l_1 terms, introduced in equation (4.1.57). Although the full non-linear proof is quite complicated, this can be easily shown at the **linearised** level. We have

$$\begin{aligned}\mathcal{E}_{ab} &= \mathcal{R}_{ab} + 6\,\partial^{c_1c_2}\,G_{[c_1,\,c_2ab]} \\ &= R_{ab} - 3\,\partial_{(a}\,G^{c_1c_2,}{}_{b)c_1c_2} + 6\,\partial^{c_1}\,G^{c_2}{}_{(a,\,b)c_1c_2} \\ &\quad - \eta_{ab}\,\partial^d\,G^{c_1c_2,}{}_{c_1c_2d} + 6\,\partial_{c_1}\,G^{c_1c_2,}{}_{c_2ab}. \end{aligned} \tag{4.1.60}$$

The last term in this equation is antisymmetric, but it also is of the form $\partial_\mu\,\partial^{\mu\nu}\,\ldots$, and, therefore, according to the argument given in the previous section, can be dismissed. The rest of the terms are symmetric. This shows that the l_1 terms added to equation (4.1.57) possess the same ab symmetry as the level 0 part of the equation.

We will now find the variation of the **linearised** equation (4.1.43) for the dual

graviton field $A_{a_1\ldots a_8,\,b}$. Carrying out the l_1-extension procedure we find

$$\begin{aligned}\mathcal{E}_{a_1\ldots a_8,\,b} = E_{a_1\ldots a_8,\,b} &- \frac{7}{72}\,\eta_{b[a_1}\,\partial^{c}{}_{a_2}\,G_{[|c|,\,a_3\ldots a_8]]}\\ &+\frac{55}{48}\,\partial^{c_1c_2}\Big(20\,G_{[d,\,a_1\ldots a_8c_1],\,c_2b}{}^{d}+\frac{10}{3}\,G_{[d,\,a_1\ldots a_8b],\,c_1c_2}{}^{d}\\ &\qquad\qquad+G_{b,\,a_1\ldots a_8d,\,c_1c_2}{}^{d}-G^{d,}{}_{da_1\ldots a_8,\,c_1c_2b}\Big)\\ &+\frac{5}{8}\,\partial^{c_1c_2}\,\Big(G^{d,}{}_{dc_1a_1\ldots a_8,\,c_2b}+\frac{1}{9}\,G^{d,}{}_{c_1a_1\ldots a_8b,\,c_2d}\Big)\\ &-\frac{5}{9}\,\partial^{c_1c_2}\Big(G_{[a_1,\,a_2\ldots a_8]bc_1d,\,c_2}{}^{d}-G_{b,\,a_1\ldots a_8c_1d,\,c_2}{}^{d}\Big)\\ &+\frac{55}{72}\,\partial^{c_1c_2}\Big(G^{d,}{}_{c_1c_2da_1\ldots a_8,\,b}-G^{d,}{}_{c_1c_2db[a_1\ldots a_7,\,a_8]}\\ &\qquad\qquad+\frac{1}{8}\,G^{d,}{}_{c_1c_2ba_1\ldots a_8,\,d}-\frac{9}{8}\,G_{b,\,c_1c_2a_1\ldots a_8d,}{}^{d}\Big).\end{aligned}\tag{4.1.61}$$

The variation of this equation is given by

$$\begin{aligned}\delta\mathcal{E}_{a_1\ldots a_8,\,b} = &-\frac{7}{4}\,\eta_{b[a_1}\,E_{a_2\ldots a_6|c|}\,\Lambda^{c}{}_{a_7a_8]}\\ &+275\,\Big(\hat{E}^{d}{}_{a_1\ldots a_8c_1,\,dc_2c_3b}-\frac{1}{9}\,\hat{E}^{d}{}_{a_1\ldots a_8b,\,dc_1c_2c_3}\Big)\,\Lambda^{c_1c_2c_3}\\ &+\frac{165}{8}\,\Big(\partial^{d}\,G_{[d,\,a_1\ldots a_8c_1c_2],\,c_3b}-\partial_b\,G_{[d,\,a_1\ldots a_8c_1c_2],\,c_3}{}^{d}\\ &\qquad-\frac{2}{9}\,\partial_{c_1}\,G_{[c_2,\,a_1\ldots a_8bd],\,c_3}{}^{d}\Big)\,\Lambda^{c_1c_2c_3},\end{aligned}\tag{4.1.62}$$

where we have introduced the second order $3-9,\,3$ duality equation

$$E_{a_1\ldots a_{10},\,b_1\ldots b_4}=\partial_{[b_1}\,G_{[a_1,\,a_2\ldots a_{10}],\,b_2b_3b_4]}-\frac{36}{5\cdot 11!}\,\varepsilon_{a_1\ldots a_{10}}{}^{c}\,\partial_c\,G_{[b_1,\,b_2b_3b_4]}.\tag{4.1.63}$$

The variation of $8,\,1$ equation, given in equation (4.1.62), doesn't contain equation (4.1.63) directly, but rather its combination with $3-6$ duality equation (4.1.15) that forms the alternative $6-9,\,3$ duality, defined as follows

$$\begin{aligned}\hat{E}_{a_1\ldots a_{10},}{}^{b_1\ldots b_4} &= E_{a_1\ldots a_{10},}{}^{b_1\ldots b_4}+\frac{36}{5\cdot 11!}\,\varepsilon_{a_1\ldots a_{10}}{}^{c}\,\partial_c\,D^{b_1\ldots b_4}\\ &=\partial^{[b_1}\,G_{[a_1,\,a_2\ldots a_{10}]}{}^{b_2b_3b_4]}+\frac{3}{55}\,\partial^{c}\,G_{[a_1,\,a_2\ldots a_7}\,\delta^{b_1\ \ldots\ b_4}_{a_8a_9a_{10}]c}\\ &\quad+\frac{21}{220}\,\partial^{c}\,G_{[c,\,[a_1\ldots a_6]}\,\delta^{b_1\ldots\,b_4}_{a_7\ldots a_{10}]},\end{aligned}\tag{4.1.64}$$

As one can see from the variation (4.1.62), equation (4.1.63) enters it in a contracted form.

$$\begin{aligned} E_{a_1\dots a_9,}{}^{b_1b_2b_3} &= E_{ca_1\dots a_9,}{}^{cb_1b_2b_3} \\ &= \partial^{[c}\, G_{[c,\, a_1\dots a_9],}{}^{b_1b_2b_3]} - \frac{36}{5\cdot 11!}\,\varepsilon_{a_1\dots a_9c_1c_2}\,\partial^{c_1}\, G^{[c_2,\, b_1b_2b_3]}. \end{aligned} \tag{4.1.65}$$

This is the real dynamical equation that belongs to the E_{11} multiplet and describes the dynamics of the $A_{a_1\dots a_9,\, b_1b_2b_3}$ field. The three form can be fully eliminated from equation (4.1.63) by tracing it on four indexes. The resulting equation contains exclusively the Cartan form of the $A_{a_1\dots a_9,\, b_1b_2b_3}$ field.

$$E_{b_1\dots b_4a_1\dots a_6,}{}^{b_1\dots b_4} = \partial^{b_1}\, G_{[b_1,\, b_2b_3b_4a_1\dots a_6],}{}^{b_2b_3b_4}. \tag{4.1.66}$$

Further analysing equation (4.1.61) and its variation (4.1.62) one can notice that the contribution of the 11, 1 field $A_{a_1\dots a_{11},\, b}$ has been completely eliminated by the l_1 corrections. This implies that there is no equation that describes the dynamics of this field in the E_{11} multiplet, i.e. it is non-dynamical.

The last field that we have to process is the 10, 2 field $A_{a_1\dots a_{10},\, b_1b_2}$. From equation (4.1.62) one can conclude that the dynamics of this field are described by the following equation

$$\partial^d\, G_{[d,\, a_1\dots a_8[c_1c_2],\, c_3]b} - \partial_b\, G_{[d,\, a_1\dots a_8[c_1c_2],\, c_3]}{}^d - \frac{2}{9}\,\partial_{[c_1}\, G_{[c_2|,\, a_1\dots a_8bd],\, |c_3]}{}^d \doteq 0. \tag{4.1.67}$$

One, however, can show that this equation is equivalent to a much simpler one

$$E_{c_1\dots c_{11},\, a,\, b_1b_2} \doteq 0, \quad \text{where} \quad E_{c_1\dots c_{11},\, a,\, b_1b_2} = -\,2\,\partial_{[b_1}\, G_{[c_1,\, c_2\dots c_{11}],\, b_2]a}. \tag{4.1.68}$$

This is the second order equation that describes the dynamics of the $A_{a_1\dots a_{10},\, b_1b_2}$ field. Unlike all the other second order equations we considered before this equation is subject to modulo transformations. The exact form of these transformations will be discussed in the next section. Using this equation we can rewrite the variation from equation

(4.1.62) in the following way

$$\begin{aligned}\delta\mathcal{E}_{a_1\dots a_8,\,b} = & -\frac{7}{4}\,\eta_{b[a_1}\,E_{a_2\dots a_6|c|}\,\Lambda^{c}{}_{a_7a_8]} \\ & + 275\left(\hat{E}^{d}{}_{a_1\dots a_8c_1,\,dc_2c_3b} - \frac{1}{9}\,\hat{E}^{d}{}_{a_1\dots a_8b,\,dc_1c_2c_3}\right)\Lambda^{c_1c_2c_3} \\ & + \frac{165}{8}\left(E^{d}{}_{a_1\dots a_8c_1c_2,\,c_3,\,db} - \frac{1}{9}\,E^{d}{}_{c_1a_1\dots a_8b,\,d,\,c_2c_3}\right)\Lambda^{c_1c_2c_3}. \qquad (4.1.69)\end{aligned}$$

All the second order equations we obtained so far were derived in two independent ways: first, by projecting the corresponding first order equations and second, by taking the variations of the other second order equations. This fact gives rise to an internal consistency check of the E_{11} multiplet that can be illustrated by the following commutative diagrams

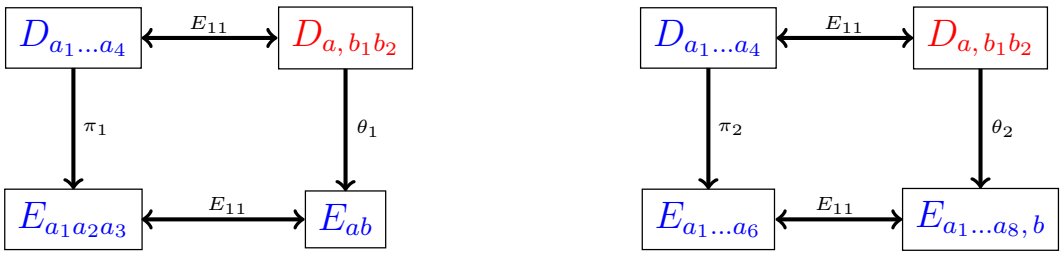


Figure 13: Two commutative diagrams of the E_{11} multiplet

We have shown that the first diagram commutes non-linearly, except for the projector θ_1, which is only known at the linear level. The second diagram has been shown to commute linearly.

4.1.3 Level 4 equations of the E_{11} theory. Modulo transformations. Eleven-dimensional origin of the mass term in Romans supergravity.

We will start this section by discussing the variations of the first order level 4 equations $D_{a_1\dots a_{10},\,b_1b_2b_3}$ and $D_{a_1\dots a_{11},\,b_1b_2}$, found previously in equation (4.1.54). We will then construct a set of projectors σ_1, σ_2 that transform these equations into their second order counterparts $E_{a_1\dots a_{10},\,b_1\dots b_4}$ from equation (4.1.63) and $E_{a_1\dots a_{11},\,b,\,c_1c_2}$ from equation

(4.1.68). The variation of $E_{a_1\ldots a_{11},\,b,\,c_1c_2}$ is then considered. It is shown to contain a modulo term, which forces us to define another projector ρ in order to obtain a third order equation $F_{a_1\ldots a_{11},\,b_1b_2,\,c_1c_2}$ that doesn't produce any modulo terms in its variation. We will finish this section by discussing the general properties of the modulo transformations and the connection of $A_{a_1\ldots a_{10},\,b_1b_2}$ field to Romans supergravity.

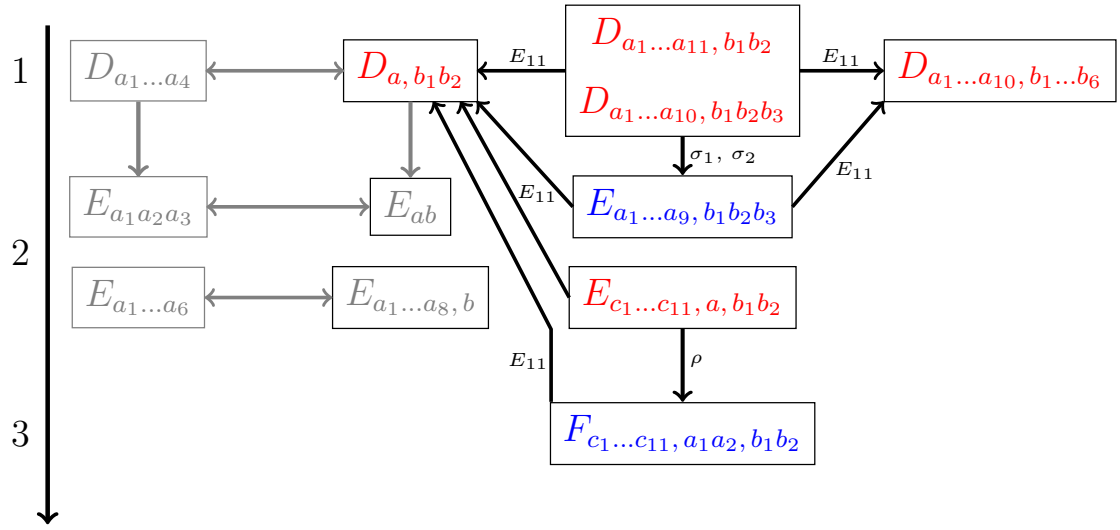


Figure 14: Level 4 sector of the E_{11} multiplet

The biggest difficulty we are going to encounter when varying equation $D_{a_1\ldots a_{10},\,b_1b_2b_3}$ and its second order counterpart $E_{a_1\ldots a_{10},\,b_1\ldots b_4}$ is that there is a new duality relation in their variations. The E_{11} theory predicts an infinite number of fields, some of which are connected by infinite chains of dualities. Two of these chains are known and were proposed in [51]. The vector duality chain, pictured in Figure 15, starts with the $3-6$ duality equation (4.1.15). It links certain fields on levels $3n+1$ and $3n+2$ for $n = 0,\ 1,\ 2,\ \ldots$. The gravity chain, which starts with gravity - dual gravity relation (4.1.17), links certain fields on levels $3n$ and $3n+3$ for $n = 0,\ 1,\ 2,\ \ldots$.

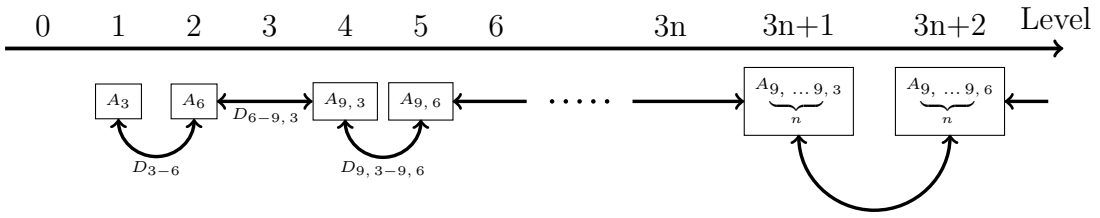


Figure 15: The vector duality chain

Some of the dualities presented in Figure 15 have been discussed earlier: D_{3-6} is the original vector - dual vector relation (4.1.15), while $D_{6-9,\,3}$ can be obtained as a composition of the $3-6$ duality from equation (4.1.15) with the $3-9,\,3$ duality from equation (4.1.54). As the variation of equation (4.1.63) contains both the terms on level 2 and level 5 we are expecting it to close on the $6-9,\,6$ duality relation $D_{6-9,\,6}$, which can be obtained from Figure 15 as a composition of $D_{6-9,\,3}$ and $D_{9,\,3-9,\,6}$. This equation has the following form

$$D_{a_1\ldots a_{10},\,b_1\ldots b_6} = G_{[a_1,\,a_2\ldots a_{10}],\,b_1\ldots b_6} + p\,\varepsilon_{a_1\ldots a_{10}}{}^{c}\,G_{[c,\,b_1\ldots b_6]}. \tag{4.1.70}$$

As we are truncating all the fields on level 5 and higher, we are unable to determine the value of the coefficient p or to show that the variation actually closes. Equation $D_{a_1\ldots a_{11},\,b_1b_2}$ from (4.1.54) also has level 5 equations in its variations. These equations, however, do not belong to any duality chains and, therefore, do not contain any terms on level 4 and lower. This implies that according to our truncation procedure they can simply be removed from the variation. Equation $D_{a_1\ldots a_{11},\,b_1b_2}$ can then be varied precisely. l_1-extension of this equation results in the following

$$\begin{aligned}
\mathcal{D}_{c_1\ldots c_{11},\,a_1a_2} &= G_{[c_1,\,c_2\ldots c_{11}],\,a_1a_2} + \frac{20}{11\cdot 11!}\,\varepsilon_{c_1\ldots c_{11}}\,G^{d}{}_{(a_1,\,(a_2)d)},\\
\delta\mathcal{D}_{c_1\ldots c_{11},\,a_1a_2} &= -\frac{60}{11\cdot 11!}\,\varepsilon_{c_1\ldots c_{11}}\,D_{(a_1|,\,d_1d_2}\,\Lambda_{|a_2)}{}^{d_1d_2} - \varepsilon_{c_1\ldots c_{11}}\,\partial_{(a_1}\,\tilde{\Lambda}_{a_2)},
\end{aligned} \tag{4.1.71}$$

where $D_{a,\,b_1b_2}$ is the gravity - dual gravity relation (4.1.17) and $\partial_{(a_1}\,\tilde{\Lambda}_{a_2)}$ is the modulo

term given by

$$\partial_{(a_1}\tilde{\Lambda}_{a_2)} = -\frac{60}{11\cdot 11!}\left(G_{(a_1,|d_1d_2|}\,\Lambda_{a_2)}{}^{d_1d_2} + \frac{1}{20}\,\varepsilon^{d_1\ldots d_{11}}\,G_{(a_1,|d_1\ldots d_8|,a_2)}\,\Lambda_{d_9d_{10}d_{11}}\right). \tag{4.1.72}$$

As we stated before, appearance of the modulo term in the variation implies that the exact equation that describes the dynamics of the fields is higher order in derivatives. Therefore, we will now define two projectors that act on the first order level 4 equations in order to obtain the second order ones, previously obtained from the variation of the second order dual graviton equation (4.1.61). Projectors σ_1 and σ_2 acts on the first order level 4 equations (4.1.54) in the following way

$$\begin{aligned} E_{a_1\ldots a_{10},\,b_1\ldots b_4} &= (\sigma_1\,D)_{a_1\ldots a_{10},\,b_1\ldots b_4} = \partial_{[b_1|}\,D_{a_1\ldots a_{10},\,|b_2b_3b_4]} \\ &= \partial_{[b_1}\,G_{[a_1,\,a_2\ldots a_{10}],\,b_2b_3b_4]} - \frac{36}{5\cdot 11!}\,\varepsilon_{a_1\ldots a_{10}}{}^{c}\,\partial_c\,G_{[b_1,\,b_2b_3b_4]}, \\ E_{a_1\ldots a_{11},\,b,\,c_1c_2} &= (\sigma_2\,D)_{a_1\ldots a_{11},\,b,\,c_1c_2} = -\,2\,\partial_{[c_1|}\,D_{a_1\ldots a_{11},\,|c_2]b} = -\,2\,\partial_{[c_1}\,G_{[a_1,\,a_2\ldots a_{11}],\,c_2]b}. \end{aligned} \tag{4.1.73}$$

The first projector gives us equation (4.1.63), while the second one reconstructs the second order 10, 2 equation (4.1.68). We will now vary equation (4.1.68) to see whether projector σ_2 eliminates the modulo term or not. l_1-extension procedure results in

$$\mathcal{E}_{c_1\ldots c_{11},\,a,\,b_1b_2} = -\,2\,\partial_{[b_1}\,G_{[c_1,\,c_2\ldots c_{11}],\,b_2]a} - \frac{20}{11\cdot 11!}\,\varepsilon_{c_1\ldots c_{11}}\,\left(\partial^d{}_a\,G_{[b_1,\,(b_2]d)} - \partial^d{}_{[b_1}\,G_{b_2],\,(ad)}\right). \tag{4.1.74}$$

The variation of this equation is then given by

$$\begin{aligned} \delta\mathcal{E}_{c_1\ldots c_{11},\,a,\,b_1b_2} = {}& \frac{60}{11\cdot 11!}\,\varepsilon_{c_1\ldots c_{11}}\,\partial_{[b_1|}\,\left(D_{|b_2],\,d_1d_2}\,\Lambda_a{}^{d_1d_2} + D_{a,\,d_1d_2}\,\Lambda_{|b_2]}{}^{d_1d_2}\right) \\ & + \varepsilon_{c_1\ldots c_{11}}\,\partial_a\,\partial_{[b_1}\,\tilde{\Lambda}_{b_2]}, \end{aligned} \tag{4.1.75}$$

where $\partial_a\,\partial_{[b_1}\,\tilde{\Lambda}_{b_2]}$ is the modulo term given by

$$\partial_a\,\partial_{[b_1}\tilde{\Lambda}_{b_2]} = -\,\frac{60}{11\cdot 11!}\,\partial_a\,\left(G_{[b_1,\,|d_1d_2|}\,\Lambda_{b_2]}{}^{d_1d_2} + \frac{1}{20}\,\varepsilon^{d_1\ldots d_{11}}\,G_{[b_1,\,|d_1\ldots d_8|,\,b_2]}\,\Lambda_{d_9d_{10}d_{11}}\right). \tag{4.1.76}$$

The fact that the second order equation still retains the modulo term implies that one has to apply another derivative to it. Hence we define projector ρ that acts on the

second order equation (4.1.68) in the following way

$$F_{a_1\ldots a_{11},\,b_1b_2,\,c_1c_2} = (\rho\, E)_{a_1\ldots a_{11},\,b_1b_2,\,c_1c_2} = -\,\partial_{[b_1}\, E_{|a_1\ldots a_{11}|,\,b_2],\,c_1c_2}$$
$$= 2\,\partial_{[b_1}\,\partial_{[c_1}\, D_{|a_1\ldots a_{11}|,\,c_2]b_2]} = 2\,\partial_{[b_1}\,\partial_{[c_1}\, G_{[a_1,\,a_2\ldots a_{11}],\,c_2]b_2]}. \tag{4.1.77}$$

The l_1-extension procedure gives the following result

$$\delta\mathcal{F}_{c_1\ldots c_{11},\,a_1a_2,\,b_1b_2} = -\,\frac{60}{11\cdot 11!}\,\varepsilon_{c_1\ldots c_{11}}\,\partial_{[a_1|}\,\partial_{[b_1|}\,\left(D_{|a_2],\,d_1d_2}\,\Lambda_{|b_2]}{}^{d_1d_2} + D_{|b_2],\,d_1d_2}\,\Lambda_{|a_2]}{}^{d_1d_2}\right), \tag{4.1.78}$$

where

$$\mathcal{F}_{c_1\ldots c_{11},\,a_1a_2,\,b_1b_2} = 2\,\partial_{[a_1}\,\partial_{[b_1}\,G_{[c_1,\,c_2\ldots c_{11}],\,b_2]a_2]} \tag{4.1.79}$$
$$-\,\frac{20}{11\cdot 11!}\,\varepsilon_{c_1\ldots c_{11}}\,\left(\partial^{d}{}_{[b_1}\,\partial_{b_2]}\,G_{[a_1,\,(a_2]d)} + \partial^{d}{}_{[a_1}\,\partial_{a_2]}\,G_{[b_1,\,(b_2]d)}\right).$$

As one can see, ρ has eliminated the modulo term. This means that the dynamics of the 10, 2 field are described by an exact equation that is third order in derivatives. One can also construct the contructed version of this equation, namely

$$F_{c_1\ldots c_{11},\,ab} = F_{c_1\ldots c_{11},\,ad,\,b}{}^{d}. \tag{4.1.80}$$

The dynamical implications of these equations will be discussed at the end of this section.

We will now investigate the general properties of the modulo transformations and their dynamical implications. The clearest sign that an equation is subject to modulo transformations is it acquiring additional $\partial\tilde{\Lambda}$ term in its variation. A prime example of this would be equation (4.1.53). In [51] it was proposed that this happens due to the fact that these equations are not invariant under the gauge transformations, defined in equation (1.3.17). Under this assumption, the modulo transformation works as a compensation mechanism for the gauge transformation. Applying projectors to these equations eliminates the gauge degrees of freedom and makes the equations exact. As we do not have a way to prove this general statement, we will illustrate it at the **linearised** level using the particular set of equations that we've just derived. The set

of gauge parameters that belong to the l_1 multiplet is

$$\underbrace{\Lambda^a}_{\text{Level 0}}, \quad \underbrace{\Lambda_{a_1 a_2}}_{\text{Level 1}}, \quad \underbrace{\Lambda_{a_1 \dots a_5}}_{\text{Level 2}}, \quad \underbrace{\Lambda_{a_1 \dots a_8}, \quad \Lambda_{a_1 \dots a_7, b}}_{\text{Level 3}}, \tag{4.1.81}$$

$$\underbrace{\Lambda_{a_1 \dots a_8, b_1 b_2 b_3}, \quad \Lambda_{a_1 \dots a_9, b_1 b_2}, \quad \hat{\Lambda}_{a_1 \dots a_9, b_1 b_2}, \quad \Lambda^{(1)}_{a_1 \dots a_{10}, b}, \quad \Lambda^{(2)}_{a_1 \dots a_{10}, b}, \quad \Lambda_{a_1 \dots a_{11}}}_{\text{Level 4}}.$$

These parameters mirror the set of the l_1 generators given in Section 3.1. We will start by finding the gauge transformations laws of all the fields in the multiplet using equation (1.3.18). Up to some renormalisations of Λ's we have

$$\begin{aligned}
\delta_g h_a{}^b &= \partial_a \Lambda^b, \quad \delta_g A_{a_1 a_2 a_3} = \partial_{[a_1} \Lambda_{a_2 a_3]}, \quad \delta_g A_{a_1 \dots a_6} = \partial_{[a_1} \Lambda_{a_2 \dots a_6]}, \\
\delta_g A_{a_1 \dots a_8, b} &= \partial_{[a_1} \Lambda_{a_2 \dots a_8], b} + \partial_b \Lambda_{a_1 \dots a_8} - \partial_{[a_1} \Lambda_{a_2 \dots a_8] b}, \\
\delta_g A_{a_1 \dots a_9, b_1 b_2 b_3} &= \partial_{[a_1} \Lambda_{a_2 \dots a_9], b_1 b_2 b_3} + \partial_{[b_1} \Lambda_{|a_1 \dots a_9|, b_2 b_3]} + \frac{9}{7} \partial_{[a_1} \Lambda_{a_2 \dots a_9][b_1, b_2 b_3]}, \\
\delta_g A_{a_1 \dots a_{10}, b_1 b_2} &= \partial_{[a_1} \hat{\Lambda}_{a_2 \dots a_{10}], b_1 b_2} + \partial_{(b_1} \Lambda_{|a_1 \dots a_{10}|, b_2)} - \frac{10}{11} \partial_{[a_1} \Lambda_{a_2 \dots a_{10}](b_1, b_2)}, \\
\delta_g A_{a_1 \dots a_{11}, b} &= \partial_{[a_1} \Lambda_{a_2 \dots a_{11}], b} + \partial_b \Lambda_{a_1 \dots a_{11}}.
\end{aligned} \tag{4.1.82}$$

Here we truncated all the terms with the higher level derivatives. We used the equations of Appendix A in order to calculate $(D_{\underline{\alpha}})_A{}^B$. We also combined $\Lambda^{(1)}_{a_1 \dots a_{10}, b}$ and $\Lambda^{(2)}_{a_1 \dots a_{10}, b}$ into $\Lambda_{a_1 \dots a_{10}, b}$, as their contributions are proportional to each other. Note that the normalisations of $\Lambda_{a_1 \dots a_{10}, b}$ are different in the last two lines. This is not going to affect the result of the calculation, as none of the equations contain $A_{a_1 \dots a_{11}, b}$.

A very important pattern emerges from equation (4.1.82): all the fields with one block of indexes have a single gauge parameter in their transformations, while the fields with two blocks — two. The number of gauge parameters is always equal to the number of blocks of indexes that the field has. This also implies that we will need to consider an increasing number of derivatives to construct the gauge invariant objects out of these fields. We will now illustrate this point by applying the gauge transformations of equation (4.1.82) to all the equations in the multiplet. For the first order equations we

have

$$\begin{aligned}
&\delta_g D_{a_1\ldots a_4} = 0, \quad \delta_g D_{a,\,b_1b_2} = -\,\tfrac{1}{4}\,\partial_a\,\varepsilon_{b_1b_2}{}^{c_1\ldots c_9}\,\partial_{c_1}\,\Lambda_{c_2\ldots c_9} = \partial_a\,\tilde{\Lambda}_{b_1b_2},\\
&\delta_g D_{a_1\ldots a_{10},\,b_1b_2b_3} = \partial_{[b_1}\,\partial_{[a_1}\,\Lambda_{a_2\ldots a_{10}],\,b_2b_3]},\\
&\delta_g D_{a_1\ldots a_{11},\,b_1b_2} = \partial_{(b_1}\,\partial_{[a_1}\,\Lambda_{a_2\ldots a_{11}],\,b_2)} = -\,\varepsilon_{a_1\ldots a_{11}}\,\partial_{(b_1}\,\tilde{\Lambda}_{b_2)},
\end{aligned} \tag{4.1.83}$$

where we have simplified two of the equations by rewriting them using the following parameters: $\tilde{\Lambda}_{a_1a_2} = -\,\frac{1}{4}\,\varepsilon_{a_1a_2}{}^{c_1\ldots c_9}\,\partial_{c_1}\,\Lambda_{c_2\ldots c_9}$ and $\tilde{\Lambda}_a = \frac{1}{11!}\,\varepsilon^{c_1\ldots c_{11}}\,\partial_{c_1}\,\Lambda_{c_2\ldots c_{11},\,a}$. Equation (4.1.83) confirms earlier statement that $3-6$ duality equation $D_{a_1\ldots a_4}$ is gauge invariant and, therefore, doesn't produce a modulo term in its variation. The remaining four equations are subject to modulo transformations. Transformation of the gravity - dual gravity duality relation $D_{a,\,b_1b_2}$ is consistent with the modulo term found previously in equation (4.1.55), while transformation of $D_{a_1\ldots a_{11},\,b_1b_2}$ is consistent with equation (4.1.72). Applying projectors to the first order equations gives us the second order equations and eliminates the majority of the modulo terms.

$$\begin{aligned}
&\delta_g E_{a_1a_2a_3} = (\pi_1\,\delta_g D)_{a_1a_2a_3} = 0, \quad \delta_g E_{a_1\ldots a_6} = (\pi_2\,\delta_g D)_{a_1\ldots a_6} = 0,\\
&\delta_g E_{ab} = (\theta_1\,\delta_g D)_{ab} = 0, \qquad\qquad \delta_g E_{a_1\ldots a_8,\,b} = (\theta_2\,\delta_g D)_{a_1\ldots a_8,\,b} = 0,\\
&\delta_g E_{a_1\ldots a_{10},\,b_1\ldots b_4} = (\sigma_1\,\delta_g D)_{a_1\ldots a_{10},\,b_1\ldots b_4} = 0,\\
&\delta_g E_{a_1\ldots a_{11},\,b,\,c_1c_2} = (\sigma_2\,\delta_g D)_{a_1\ldots a_{11},\,b,\,c_1c_2} = \varepsilon_{a_1\ldots a_{11}}\,\partial_b\,\partial_{[c_1}\,\tilde{\Lambda}_{c_2]}.
\end{aligned} \tag{4.1.84}$$

The modulo terms are eliminated by the projectors in all cases, except for the 10, 2 equation $E_{a_1\ldots a_{11},\,b,\,c_1c_2}$. It's gauge transformation matches the result found in equation (4.1.76). Applying projector ρ to $E_{a_1\ldots a_{11},\,b,\,c_1c_2}$ results in

$$\delta_g F_{a_1\ldots a_{11},\,b_1b_2,\,c_1c_2} = (\rho\,\delta_g D)_{a_1\ldots a_{11},\,b_1b_2,\,c_1c_2} = 0. \tag{4.1.85}$$

Consequently, the contracted version of this equation (4.1.80) $F_{a_1\ldots a_{11},\,b_1b_2}$ is also gauge invariant. Transformations (4.1.83, 4.1.84, 4.1.85) imply that the 10, 2 equation $\varepsilon^{c_1\ldots c_{11}}\,D_{c_1\ldots c_{11},\,a_1a_2}$ transforms like a symmetrised gravity field $h_{(a_1a_2)}$, equation $\varepsilon^{c_1\ldots c_{11}}\,E_{c_1\ldots c_{11},\,a,\,b_1b_2}$ — like the spin connection $\omega_{a,\,b_1b_2}$, equation $\varepsilon^{c_1\ldots c_{11}}\,F_{c_1\ldots c_{11},\,a_1a_2,\,b_1b_2}$ — like the Riemann tensor $R_{a_1a_2,\,b_1b_2}$ and, finally, equation $\varepsilon^{c_1\ldots c_{11}}\,F_{c_1\ldots c_{11},\,a_1a_2}$ — like the Ricci tensor $R_{a_1a_2}$.

We now consider the consequences of the field equation (4.1.77) for the field $A_{a_1\ldots a_{10},\,b_1b_2}$, which essentially states that the analogue of the Riemann tensor vanishes. We recall that if the Riemann tensor vanishes then one can find a coordinate system in which the space-time is flat. Applying this to our setting we can conclude that there is a gauge in which

$$G_{[a_1,\,a_2\ldots a_{11}],\,b_1b_2} = m\,\varepsilon_{a_1\ldots a_{11}}\,\eta_{b_1b_2}, \tag{4.1.86}$$

where m is a parameter. This makes it clear that equation (4.1.77) implies that $A_{a_1\ldots a_{10},\,b_1b_2}$ field carries no degrees of freedom. Even though we have a field equation with three derivatives E_{11} ensures that there are no additional degrees of freedom coming from this level 4 field.

We now consider the dimensional reduction to ten dimensions, that is, to the IIA theory. We find the eleven-dimensional field $A_{a_1\ldots a_{10},\,b_1b_2}$ gives rise to the following ten-dimensional fields $A_{\hat{a}_1\ldots\hat{a}_{10},\,\hat{b}_1\hat{b}_2}$, $A_{\hat{a}_1\ldots\hat{a}_{10},\,\hat{b}}$, $A_{\hat{a}_1\ldots\hat{a}_{10}}$ and $A_{\hat{a}_1\ldots\hat{a}_9}$, where $\hat{a}$, $\hat{b} = 1, \ldots, 10$. In listing these fields we have taken into account the irreducibility condition of the $A_{a_1\ldots a_{10},\,b_1b_2}$ field. From equation (4.1.86) we see from that the field $A_{\hat{a}_1\ldots\hat{a}_9}$ obeys the equation

$$F_{\hat{a}_1\ldots\hat{a}_{10}} = m'\,\varepsilon_{\hat{a}_1\ldots\hat{a}_{10}}, \tag{4.1.87}$$

where $F_{\hat{a}_1\ldots\hat{a}_{10}} = \partial_{[\hat{a}_1}\,A_{\hat{a}_2\ldots\hat{a}_{10}]}$ and $m' = \frac{11}{10}\,m$.

Type IIA supergravity with cosmological term is known as Romans theory [52]. Equation (4.1.87) implies that dimensional reduction of the $A_{a_1\ldots a_{10},\,b_1b_2}$ field to ten dimensions produces a cosmological term and, therefore, has to contain Romans supergravity. The proposition that the $A_{a_1\ldots a_{10},\,b_1b_2}$ field is related to Romans supergravity was first made in [53].

4.2 Non-linear realisation of E_{11} in 5D

To illustrate the point that the non-linear realisation of E_{11} provides a description for theories in all dimensions up to eleven we will construct it in five-dimensional case. We will build the first order vector duality equation, analogues to the $3-6$ duality

equation (4.1.15) in elven-dimensional theory. We will then vary it to derive other duality relations in the multiplet and apply a projector to it to obtain the second order equation for the vector field. Finally, we will vary this second order equation to show that, like in eleven dimensions, its variation conteins the Einstein equation. In this section we are using the commutators from Section 2.1 and the Cartan forms from Section 3.3.

Vector duality equation is uniquely fixed by Lorentz and $Usp\,(8)$ invariances. It establishes a connection between level 1 $A_{a\,\alpha_1\alpha_2}$ field and level 2 $A_{a_1a_2\,\alpha_1\alpha_2}$ field. It has the following form

$$D^V_{a_1a_2\,\alpha_1\alpha_2} = G_{[a_1,\,a_2]\,\alpha_1\alpha_2} \pm \frac{1}{2}\,\varepsilon_{a_1a_2}{}^{b_1b_2b_3}\,G_{b_1,\,b_2b_3\,\alpha_1\alpha_2}. \tag{4.2.1}$$

Like in elven-dimensional case, the coefficient between the terms is fixed by the condition of closure under the $I_c\,(E_{11})$ transformations. Here, however, we did not make a choice of sign in the duality relation. In order to find the variation of this equation one has to apply the $I_c\,(E_{11})$ transformations found in equations (3.3.9, 3.3.10, 3.3.11, 3.3.12) and perform the l_1-extension. We get

$$\mathcal{D}^V_{a_1a_2\,\alpha_1\alpha_2} = \mathcal{G}_{a_1a_2\,\alpha_1\alpha_2} \pm \frac{1}{2}\,\varepsilon_{a_1a_2}{}^{b_1b_2b_3}\,\mathcal{G}_{b_1b_2b_3\,\alpha_1\alpha_2} + \frac{1}{2}\,G_{\alpha_1\alpha_2,\,[a_1a_2]}, \tag{4.2.2}$$

where

$$\begin{aligned}
\mathcal{G}_{a_1a_2\,\alpha_1\alpha_2} &= G_{[a_1,\,a_2]\,\alpha_1\alpha_2} + 2\,G_{[\alpha_1\gamma,\,a_1a_2\,\alpha_2]}{}^{\gamma} + \frac{1}{4}\,\Omega_{\alpha_1\alpha_2}\,G^{\gamma_1\gamma_2}{}_{,a_1a_2\,\gamma_1\gamma_2}, \\
\mathcal{G}_{a_1a_2a_3\,\alpha_1\alpha_2} &= G_{[a_1,\,a_2a_3]\,\alpha_1\alpha_2} - G_{[\alpha_1|\gamma,\,a_1a_2a_3\,|\alpha_2]}{}^{\gamma} + 6\,G^{\gamma_1\gamma_2,}{}_{a_1a_2a_3\,\alpha_1\alpha_2\gamma_1\gamma_2}.
\end{aligned} \tag{4.2.3}$$

Under the local $I_c(E_{11})$ transformations this equation transforms as

$$\begin{aligned}
\delta\mathcal{D}^V_{a_1a_2\,\alpha_1\alpha_2} = {} & \mp 2\,\varepsilon_{a_1a_2}{}^{a_3a_4a_5}\,D^V_{a_3a_4\,[\alpha_1\gamma}\,\Lambda_{a_5\,\alpha_2]}{}^{\gamma} \mp \frac{1}{4}\,\Omega_{\alpha_1\alpha_2}\,\varepsilon_{a_1a_2}{}^{a_3a_4a_5}\,D^V_{a_3a_4\,\gamma_1\gamma_2}\,\Lambda_{a_5}{}^{\gamma_1\gamma_2} \\
& - 2\,D^S_{[a_1\,\alpha_1\alpha_2\gamma_1\gamma_2}\,\Lambda_{a_2]}{}^{\gamma_1\gamma_2} \mp 2\,\varepsilon_{[a_1|}{}^{b_1\ldots b_4}\,\hat{D}^S_{b_1\ldots b_4\,[\alpha_1|\gamma}\,\Lambda_{|a_2]\,|\alpha_2]}{}^{\gamma} \\
& + D^G_{b,\,a_1a_2}\,\Lambda^{b}{}_{\alpha_1\alpha_2}.
\end{aligned} \tag{4.2.4}$$

where $D^S_{a\,\alpha_1\ldots\alpha_4}$ and $\hat{D}^S_{a_1\ldots a_4\,\alpha_1\alpha_2}$ are the scalar equations which are given by

$$\begin{aligned} D^S_{a\,\alpha_1\ldots\alpha_4} &= G_{a,\,\alpha_1\ldots\alpha_4} \mp 6\,\varepsilon_a{}^{b_1\ldots b_4}\,G_{b_1,\,b_2b_3b_4\,\alpha_1\ldots\alpha_4}, \\ \hat{D}^S_{a_1\ldots a_4\,\alpha_1\alpha_2} &= G_{[a_1,\,a_2a_3a_4]\,\alpha_1\alpha_2}, \end{aligned} \tag{4.2.5}$$

and $D^G_{a,\,b_1b_2}$ is the gravity - dual gravity relation similar to the eleven-dimensional one introduced in equation (4.1.17). It is given by

$$D^G_{a,\,b_1b_2} = (\det e)^{\frac{1}{2}}\;\omega_{a,\,b_1b_2} \mp \frac{1}{2}\,\varepsilon_{b_1b_2}{}^{c_1c_2c_3}\,G_{c_1,\,c_2c_3,\,a}. \tag{4.2.6}$$

$\omega_{a,\,b_1b_2}$ was defined in (4.1.9). The variation of the vector duality relation (4.2.4) contains two equations that describe the dynamics of the scalar fields in the theory. These equations did not appear in previous sections, as eleven-dimensional theory has no scalar fields.

As in the elven-dimensional case the gravity - dual gravity relation is subject to modulo transformations. One can obtain an exact second order equation that describes gravity by applying a projector to it. In order to build this equation we will first construct the second order vector equation and then apply the $I_c\,(E_{11})$ variation to it. The second order vector and scalar equations are given by

$$\begin{aligned} E^a{}_{\alpha_1\alpha_2} = e_{\mu_2}{}^a\,\partial_{\mu_1}\,\Big((\det e)^{\frac{1}{2}}\;G^{[\mu_1,\,\mu_2]}{}_{\alpha_1\alpha_2}\Big) + G^{[b,\,a]\,\gamma_1\gamma_2}\,G_{b,\,\gamma_1\gamma_2\alpha_1\alpha_2} - 2\,G^{[b,\,a]}{}_{[\beta_1|\gamma|}\,G_{b,\,\alpha_2]}{}^{\gamma} \\ \pm\,(\det e)^{-1}\;\varepsilon^{ac_1\ldots c_4}\,\Big(G_{c_1,\,c_2\,[\alpha_1|\gamma|}\,G_{c_3,\,c_4\,\alpha_2]}{}^{\gamma} + \frac{1}{8}\,\Omega_{\alpha_1\alpha_2}\,G_{c_1,\,c_2\gamma_1\gamma_2}\,G_{c_3,\,c_4}{}^{\gamma_1\gamma_2}\Big), \end{aligned} \tag{4.2.7}$$

and

$$\begin{aligned} E_{\alpha_1\ldots\alpha_4} = \partial_\mu\,\Big((\det e)^{\frac{1}{2}}\;G^{\mu}{}_{\alpha_1\ldots\alpha_4}\Big) + 4\,\Big(G_{a,\,\gamma[\alpha_1}\,G^{a,\,\gamma}{}_{\alpha_2\alpha_3\alpha_4]} \\ + 3\,G_{[c_1,\,c_2]\dot{\gamma}_1\dot{\gamma}_2}\,G^{[c_1,\,c_2]}{}_{\dot{\delta}_1\dot{\delta}_2}f^{\dot{\gamma}_1\dot{\gamma}_2}{}_{[\alpha_1\alpha_2}\,f^{\dot{\delta}_1\dot{\delta}_2}{}_{\alpha_3\alpha_4]}\Big)_{\text{proj }\mathbf{42}}, \end{aligned} \tag{4.2.8}$$

where "proj **42**" implies that the expression in the brackets has to be made irreducible with respect to $\alpha_1\ldots\alpha_4$ indexes. We observe that these are precisely the vector and scalar equations of five-dimensional maximal supergravity.

We now vary the vector equation of equation (4.2.7) under the local $I_c(E_{11})$ transformations of equation to find that we recover the scalar equation of motion (4.2.8) as well as the gravity equation which occurs as the coefficient of $\Lambda^b_{\alpha_1\alpha_2}$. The results of this long and subtle calculation that involves several $Usp(8)$ identities is the equation

$$\begin{aligned} E_a{}^b = (\det e)\; R_a{}^b - 4\, G_{[a,\,c]\,\alpha_1\alpha_2}\, G^{[b,\,c]\,\alpha_1\alpha_2} + \frac{2}{3}\,\delta^b_a\, G_{[c_1,\,c_2]\,\alpha_1\alpha_2}\, G^{[c_1,\,c_2]\,\alpha_1\alpha_2} \\ - G_{a,\,\alpha_1\ldots\alpha_4}\, G^{b,\,\alpha_1\ldots\alpha_4}, \end{aligned} \tag{4.2.9}$$

where R_{ab} is the Ricci tensor. At the linearised level this calculation gives the following result

$$\begin{aligned} \mathcal{E}_{a\,\alpha_1\alpha_2} &= \partial^c\, G_{[c,\,a]\,\alpha_1\alpha_2} - \frac{1}{2}\,\partial^{\gamma_1\gamma_2}\, G_{a,\,\alpha_1\alpha_2\gamma_1\gamma_2} + \frac{1}{2}\,\partial_{\alpha_1\alpha_2}\, G^{c,}{}_{(ca)} - \frac{1}{2}\,\partial_{\alpha_1\alpha_2}\, G_{a,\,c}{}^{c}, \\ \delta\mathcal{E}_{a\,\alpha_1\alpha_2} &= R_{ba}\,\Lambda^b_{\alpha_1\alpha_2} - E_{\alpha_1\alpha_2\gamma_1\gamma_2}\,\Lambda_a{}^{\gamma_1\gamma_2}, \end{aligned} \tag{4.2.10}$$

where $R_{ab} = \partial_a\,\omega_{c,\,b}{}^c - \partial_c\,\omega_{a,\,b}{}^c$. The linearised variation of the Einstein equation closes back on the vector equation. In order to see this we first perform the l_1 extension procedure. It gives the following

$$\mathcal{E}_{ab} = \mathcal{R}_{ab} + \partial^{\alpha_1\alpha_2}\, G_{[a,\,b]\,\alpha_1\alpha_2} = R_{ab} - \partial^{\alpha_1\alpha_2}\, G_{(a,\,b)\,\alpha_1\alpha_2}, \tag{4.2.11}$$

where

$$\mathcal{R}_{ab} = \partial_a\,\Omega_{c,\,b}{}^c - \partial_c\,\Omega_{a,\,b}{}^c, \quad \Omega_{a,\,b_1b_2} = \omega_{a,\,b_1b_2} + \frac{2}{3}\,\eta_{a[b_1}\, G^{\alpha_1\alpha_2,}{}_{b_2]\,\alpha_1\alpha_2}. \tag{4.2.12}$$

The variation of this equation is

$$\delta\mathcal{E}_{ab} = -\,2\,E_{a\,\alpha_1\alpha_2}\,\Lambda_b{}^{\alpha_1\alpha_2} -\,2\,E_{b\,\alpha_1\alpha_2}\,\Lambda_a{}^{\alpha_1\alpha_2} + \frac{4}{3}\,\eta_{ab}\,E_{c\,\alpha_1\alpha_2}\,\Lambda^{c\,\alpha_1\alpha_2}. \tag{4.2.13}$$

The linearised variation of the scalar equation (4.2.8) likewise closes on the vector equation. We have

$$\begin{aligned} \mathcal{E}_{\alpha_1\ldots\alpha_4} &= \partial^c\, G_{c,\,\alpha_1\ldots\alpha_4} - 6\,\partial_{[\alpha_1\alpha_2}\, G^{c,}{}_{c\,\alpha_3\alpha_4]} + 6\,\Omega_{[\alpha_1\alpha_2}\,\partial_{\alpha_3|\gamma|}\, G^{c,}{}_{c\,\alpha_4]}{}^{\gamma} \\ &\quad + \frac{1}{2}\,\Omega_{[\alpha_1\alpha_2}\,\Omega_{\alpha_3\alpha_4]}\,\partial^{\gamma_1\gamma_2}\, G^{c,}{}_{c\,\gamma_1\gamma_2}, \\ \delta\mathcal{E}_{\alpha_1\ldots\alpha_4} &= 24\,E_{a\,[\alpha_1\alpha_2}\,\Lambda^a{}_{\alpha_3\alpha_4]} - 24\,\Omega_{[\alpha_1\alpha_2}\,E_{a\,\alpha_3|\gamma|}\,\Lambda^a{}_{\alpha_4]}{}^{\gamma} \\ &\quad - 2\,\Omega_{[\alpha_1\alpha_2}\,\Omega_{\alpha_3\alpha_4]}\,E_{a\,\gamma_1\gamma_2}\,\Lambda^{a\,\gamma_1\gamma_2}. \end{aligned} \tag{4.2.14}$$

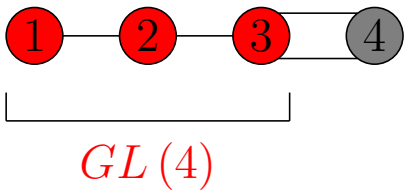
One can see that the five dimensional E_{11} theory also leads to a set of equations closed under the local $I_c\left(E_{11}\right)$ transformations. At low levels these equations mirror the dynamics of the corresponding supergravity theory.

5 Non-linear realisation of the A_1^{+++} algebra

The basic properties of the E_{11} theory can be illustrated using a simple example of the very extended A_1 algebra, denoted as A_1^{+++}. It is proposed [30] this model contains a description of pure four-dimensional gravity, supplemented with the dual graviton field . It might prove useful for studying this duality relation independently of the other duality relations that appear in more general E_{11} case. We will construct the commutation relations of this algebra and its l_1 representation up to level 2 in fields and coordinates and use them to build the non-linear realisation of this algebra.

5.1 A_1^{+++} algebra

A_1^{+++} algebra has the following Dynkin diagram



which corresponds to the Cartan matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -2 & 2 \end{pmatrix}. \tag{5.1.1}$$

Deleting node four from the diagram given in Figure 5.1 results in a four dimensional theory that includes the pure four-dimensional gravity, supplemented with the dual graviton field that can be found on level 1 of the decomposition. The list of generators up to level 2 is given by

Level	Generators
0	$K^{a}{}_{b}$
1	$R^{a_1a_2}$
-1	$R_{a_1a_2}$
2	$R^{a_1a_2,\,b_1b_2}$
-2	$R_{a_1a_2,\,b_1b_2}$

where the generators obey the conditions $R^{a_1a_2} = R^{(a_1a_2)}$, $R_{a_1a_2} = R_{(a_1a_2)}$ and $R^{a_1a_2,\,b_1b_2} = R^{[a_1a_2],\,b_1b_2} = R^{a_1a_2,\,(b_1b_2)}$, $R_{a_1a_2,\,b_1b_2} = R_{[a_1a_2],\,b_1b_2} = R_{a_1a_2,\,(b_1b_2)}$. On top of that, level 2 generator obeys the irreducibility constraint

$$R^{[a_1a_2,\,b_1]b_2} = R_{[a_1a_2,\,b_1]b_2} = 0. \tag{5.1.2}$$

We will now give the algebra formed by these generators. Taking the commutators with $K^{a}{}_{b}$ one finds

$$\begin{aligned}
[K^{a}{}_{b},\, K^{c}{}_{d}] &= \delta^{c}_{b}\, K^{a}{}_{d} - \delta^{a}_{d}\, K^{c}{}_{b},\\
[K^{a}{}_{b},\, R^{a_1a_2}] &= 2\,\delta^{(a_1}_{b}\, R^{|a|a_2)}, \quad [K^{a}{}_{b},\, R_{a_1a_2}] = -\,2\,\delta^{a}_{(a_1}\, R_{|b|a_2)},\\
\left[K^{a}{}_{b},\, R^{a_1a_2,\,b_1b_2}\right] &= 2\,\delta^{[a_1|}_{b}\, R^{a|a_2],\,b_1b_2} + 2\,\delta^{(b_1|}_{b}\, R^{a_1a_2,\,a|b_2)},\\
[K^{a}{}_{b},\, R_{a_1a_2,\,b_1b_2}] &= -\,2\,\delta^{a}_{[a_1|}\, R_{b|a_2],\,b_1b_2} - 2\,\delta^{a}_{(b_1|}\, R_{a_1a_2,\,b|b_2)}.
\end{aligned} \tag{5.1.3}$$

The level 2 (-2) two commutators must appear from the following commutators of level 1 (-1) generators

$$\left[R^{a_1a_2},\, R^{b_1b_2}\right] = R^{a_1b_1,\,a_2b_2} + R^{a_2b_2,\,a_1b_1}, \quad [R_{a_1a_2},\, R_{b_1b_2}] = R_{a_1b_1,\,a_2b_2} + R_{a_2b_2,\,a_1b_1}. \tag{5.1.4}$$

The normalisation of the level 2 (-2) generators is fixed by these relations. The reader may verify that the right-hand side of these commutators do indeed have the symmetries of the generators which occur in the left-hand side using the constraints on the generators given in equation (5.1.2). The commutators between the positive and

negative level generators are given by

$$
\begin{aligned}
[R^{a_1a_2},\, R_{b_1b_2}] &= 2\,\delta^{(a_1}_{(b_1}\,K^{a_2)}{}_{b_2)} - \delta^{(a_1a_2)}_{b_1b_2}\,K^{c}{}_{c},\\
\big[R^{a_1a_2,\,b_1b_2},\, R_{c_1c_2}\big] &= \delta^{(a_2b_2)}_{c_1c_2}\,R^{a_1b_1} + \delta^{(a_2b_1)}_{c_1c_2}\,R^{a_1b_2} - \delta^{(a_1b_1)}_{c_1c_2}\,R^{a_2b_2} - \delta^{(a_1b_2)}_{c_1c_2}\,R^{a_2b_1},\\
[R_{a_1a_2,\,b_1b_2},\, R^{c_1c_2}] &= \delta^{\,c_1c_2}_{(a_2b_2)}\,R_{a_1b_1} + \delta^{\,c_1c_2}_{(a_2b_1)}\,R_{a_1b_2} - \delta^{\,c_1c_2}_{(a_1b_1)}\,R_{a_2b_2} - \delta^{\,c_1c_2}_{(a_1b_2)}\,R_{a_2b_1}. \qquad (5.1.5)
\end{aligned}
$$

The relation of the above generators to the Chevalley generators of A_1^{+++} is given by

$$
\begin{aligned}
&H_1 = K^1{}_1 - K^2{}_2, \quad H_2 = K^2{}_2 - K^3{}_3, \quad H_3 = K^3{}_3 - K^4{}_4,\\
&H_4 = -\,K^1{}_1 - K^2{}_2 - K^3{}_3 + K^4{}_4,\\
&E_1 = K^1{}_2, \quad E_2 = K^2{}_3, \quad E_3 = K^3{}_4, \quad E_4 = R^{44},\\
&F_1 = K^2{}_1, \quad F_2 = K^3{}_2, \quad F_3 = K^4{}_3, \quad F_4 = R_{44}. \qquad (5.1.6)
\end{aligned}
$$

One can verify that the satisfy the defining relations

$$
[H_a,\, E_b] = A_{ab}\,E_b, \quad [E_a,\, F_b] = \delta_{ab}\,H_a, \quad [H_a,\, F_b] = -\,A_{ab}\,F_b. \qquad (5.1.7)
$$

were A_{ab} is the Cartan matrix of A_1^{+++} given in equation (5.1.1). The Cartan involution acts on the generators of A_1^{+++} as follows

Generator	$I_c\big(\text{Generator}\big)$
$K^a{}_b$	$-\,K^b{}_a$
$R_{a_1a_2}$	$-\,R^{a_1a_2}$
$R^{a_1a_2,\,b_1b_2}$	$R_{a_1a_2,\,b_1b_2}$

The reader may verify that it leaves invariant the above commutators. The l_1 representation generators up to level two are given by

Level	Generators
0	P_a
1	Z^a
2	$Z^{a_1a_2a_3}, \quad Z^{a_1a_2,\,b}$

where $Z^{a_1a_2a_3} = Z^{(a_1a_2a_3)}$, $Z^{a_1a_2,\,b} = Z^{[a_1a_2],\,b}$ and $Z^{[a_1a_2,\,b]} = 0$. Their commutators with the level 0 generator $K^a{}_b$ are given by

$$\begin{aligned}
[K^a{}_b,\, P_c] &= -\,\delta^a_c\, P_b + \frac{1}{2}\,\delta^a_b\, P_c,\\
[K^a{}_b,\, Z^c] &= \delta^c_b\, Z^a + \frac{1}{2}\,\delta^a_b\, Z^c,\\
[K^a{}_b,\, Z^{a_1a_2a_3}] &= 3\,\delta^{(a_1}_b\, Z^{|a|a_2a_3)} + \frac{1}{2}\,\delta^a_b\, Z^{a_1a_2a_3},\\
[K^a{}_b,\, Z^{a_1a_2,\,c}] &= 2\,\delta^{[a_1}_b\, Z^{|a|a_2],\,c} + \delta^c_b\, Z^{a_1a_2,\,a} + \frac{1}{2}\,\delta^a_b\, Z^{a_1a_2,\,c}.
\end{aligned} \tag{5.1.8}$$

The commutators of the level one A_1^{+++} generators with the l_1 generators can be chosen to be of the form

$$[R^{a_1a_2},\, P_c] = \delta^{(a_1}_c\, Z^{a_2)}, \quad \left[R^{a_1a_2},\, Z^b\right] = Z^{a_1a_2b} + Z^{b(a_1,\,a_2)}. \tag{5.1.9}$$

Using the Jacobi identities, the commutator of P_a with the level 2 generator of A_1^{+++} is found to be

$$\left[R^{a_1a_2,\,b_1b_2},\, P_c\right] = -\,\delta^{[a_1}_c\, Z^{a_2]b_1b_2} + \frac{1}{2}\,\delta^{[a_1}_c\, Z^{a_2](b_1,\,b_2)} - \frac{3}{4}\,\delta^{(b_1|}_c\, Z^{a_1a_2,\,|b_2)}. \tag{5.1.10}$$

The commutators with level-lowering generators are given by

$$\begin{aligned}
&[R_{a_1a_2},\, P_b] = 0, && \left[R_{a_1a_2},\, Z^b\right] = 2\,\delta^b_{(a_1}\, P_{a_2)},\\
&\left[R_{a_1a_2},\, Z^{b_1b_2b_3}\right] = 2\,\delta^{(b_1b_2}_{a_1a_2}\, Z^{b_3)}, && \left[R_{a_1a_2},\, Z^{b_1b_2,\,c}\right] = -\,\frac{8}{3}\,\delta^{\,c\,[b_1}_{(a_1a_2)}\, Z^{b_2]}.
\end{aligned} \tag{5.1.11}$$

The very first relation reflects the fact that the l_1 representation is a lowest weight representation.

5.2 Cartan forms

Having constructed the $A_1^{+++} \ltimes l_1$ algebra up to level two we can build its non-linear realisation. The group element $g = g_L\, g_E$ can, up to level two, be written in the form

$$\begin{aligned}
g_L &= \exp\left(x^a\, P_a + y_a\, Z^a + x_{a_1a_2a_3}\, Z^{a_1a_2a_3} + x_{a_1a_2,\,b}\, Z^{a_1a_2,\,b}\right),\\
g_E &= \exp\left(A_{a_1a_2,\,b_1b_2}\, R^{a_1a_2,\,b_1b_2}\right)\, \exp\left(A_{ab}\, R^{ab}\right)\, \exp\left(h_a{}^b\, K^a{}_b\right),
\end{aligned} \tag{5.2.1}$$

The Cartan form is

$$\mathcal{V}_E = G_a{}^b\, K^a{}_b + G_{a_1a_2}\, R^{a_1a_2} + G_{a_1a_2,\,b_1b_2}\, R^{a_1a_2,\,b_1b_2},$$
$$\mathcal{V}_L = dx^\Pi\, E_\Pi{}^A\, L_A, \tag{5.2.2}$$

where

$$G_a{}^b = \left(e^{-1}\, de\right)_a{}^b, \quad G_{a_1a_2} = e^{(\mu_1\mu_2)}_{(a_1a_2)}\, dA_{\mu_1\mu_2},$$
$$G_{a_1a_2,\,b_1b_2} = e^{\mu_1\mu_2}_{a_1a_2}\, e^{(\nu_1\nu_2)}_{(b_1b_2)} \Big(dA_{\mu_1\mu_2,\,\nu_1\nu_2} - A_{\mu_1\nu_1}\, dA_{\mu_2\nu_2}\Big). \tag{5.2.3}$$

Local level 1 $I_c\left(A_1^{+++}\right)$ transformation is parametrised by

$$h = 1 - \Lambda_{a_1a_2}\, S^{a_1a_2}, \quad \text{where} \quad S^{a_1a_2} = R^{a_1a_2} - \eta^{a_1b_1}\,\eta^{a_2b_2}\, R_{b_1b_2}. \tag{5.2.4}$$

Under these transformations the Cartan form transforms as follows

$$\delta\mathcal{V}_E = [S^{a_1a_2}\,\Lambda_{a_1a_2}, \mathcal{V}_E] - S^{a_1a_2}\, d\Lambda_{a_1a_2}. \tag{5.2.5}$$

As per usual, we have chosen a gauge in which the group element (5.2.1) is free from the negative level generators. Parameter $\Lambda_{a_1a_2}$ is then restricted in order not to break this gauge choice. We have

$$d\Lambda_{a_1a_2} - 2\,\Lambda_{(a_1}{}^b\, G_{|b|a_2)} = 0. \tag{5.2.6}$$

For the transformations of the Cartan forms we have

$$\begin{aligned}
\delta G_a{}^b &= 2\,\Lambda^{cb}\,\bar{G}_{ca} - \delta^b_a\,\Lambda^{c_1c_2}\,\bar{G}_{c_1c_2},\\
\delta\bar{G}_{a_1a_2} &= -2\,\Lambda_{(a_1}{}^b\, G_{a_2)b} - 4\, G_{(a_1|b_1|,\,a_2)b_2}\,\Lambda^{b_1b_2} - d\Lambda_{a_1a_2}\\
&= -4\Lambda_{(a_1}{}^b\, G_{(a_2)b)} - 4\, G_{(a_1|b_1|,\,a_2)b_2}\,\Lambda^{b_1b_2},\\
\delta G_{a_1a_2,\,b_1b_2} &= 2\,\Lambda_{[a_1(b_1}\, G_{a_2]b_2)} + \dots,
\end{aligned} \tag{5.2.7}$$

where $+\dots$ corresponds to the level 3 field term that we're not considering. Bar is used to distinguish between level 1 and level 0 Cartan forms.

With respect to their l_1 index the Cartan forms transform in the following way

$$\delta G_{a,\bullet} = -\,\Lambda_{ab}\,\hat{G}^{b,}{}_\bullet, \quad \delta\hat{G}^{a,}{}_\bullet = 2\,\Lambda^{ab}\, G_{b,\bullet} + \dots. \tag{5.2.8}$$

Here $+\dots$ corresponds to the level 2 l_1 term. Hat indicates that the l_1 index corresponds to the level 1 generalised coordinate y_a.

5.3 Generalised vielbein

We will use the definition of equation (1.3.9) which involves conjugating the l_1 generators with g_E using the above algebra. Conjugation with level 0

$$\begin{aligned}&\exp\left(-h_a{}^b\,K^a{}_b\right)\left\{P_\mu,\,Z^\mu,\,Z^{\mu_1\mu_2\mu_3},\,Z^{\mu_1\mu_2,\,\nu}\right\}\exp\left(h_a{}^b\,K^a{}_b\right)\\&\quad=(\det e)^{-\frac{1}{2}}\left\{e_\mu{}^a\,P_a,\,e_a{}^\mu\,Z^a,\,e^{(\mu_1\mu_2\mu_3)}_{(a_1a_2a_3)}\,Z^{a_1a_2a_3},\,e^{\mu_1\mu_2,\,\nu}_{a_1a_2,\,b}\,Z^{a_1a_2,\,b}\right\},\end{aligned}\tag{5.3.1}$$

where vielbein $e_a{}^b$ and its combinations were defined in equation (3.1.5). Conjugating with positive level generators can be obtained by Taylor-expanding the exponents and truncating the series by level 2. For the E_{11} level one generators we have

$$\begin{aligned}&\exp\left(-A_{b_1b_2}\,R^{b_1b_2}\right)\left\{P_a,\,Z^a\right\}\exp\left(A_{b_1b_2}\,R^{b_1b_2}\right)\\&\qquad=\Big\{P_a-A_{ab}\,Z^b+\frac{1}{2}\,A_{ab_1}\,A_{b_2b_3}\,Z^{b_1b_2b_3}+\frac{1}{2}\,A_{ab_1}\,A_{b_2c}\,Z^{b_1b_2,\,c},\\&\qquad\quad Z^a-A_{b_1b_2}\,Z^{ab_1b_2}-A_{b_1b_2}\,Z^{ab_1,\,b_2}\Big\}.\end{aligned}\tag{5.3.2}$$

while for the level 2 generator:

$$\begin{aligned}&\exp\left(A_{a_1a_2,\,b_1b_2}\,R^{a_1a_2,\,b_1b_2}\right)P_a\,\exp\left(A_{a_1a_2,\,b_1b_2}\,R^{a_1a_2,\,b_1b_2}\right)\\&=P_a+A_{ab_1,\,b_2b_3}\,Z^{b_1b_2b_3}+A_{b_1b_2,\,ac}\,Z^{b_1b_2,\,c}.\end{aligned}\tag{5.3.3}$$

Combining these results together we find that the generalised vielbein up to level two is given by

$$E_\Pi{}^A=(\det e)^{-\frac{1}{2}}\begin{pmatrix}e_\mu{}^a & e_\mu{}^c\,\alpha_{c|a} & e_\mu{}^c\,\alpha_{c|a_1a_2a_3} & e_\mu{}^c\,\alpha_{c|a_1a_2,\,b}\\0 & e_a{}^\mu & e_c{}^\mu\,\beta^c{}_{a_1a_2a_3} & e_c{}^\mu\,\beta^c{}_{a_1a_2,\,b}\\0 & 0 & e^{(\mu_1\mu_2\mu_3)}_{(a_1a_2a_3)} & 0\\0 & 0 & 0 & e^{\mu_1\mu_2,\,\nu}_{a_1a_2,\,b}\end{pmatrix},\tag{5.3.4}$$

where the symbols in the first line are given by

$$\begin{aligned}\alpha_{a|b}&=-A_{ab},\\\alpha_{a|a_1a_2a_3}&=A_{a(a_1,\,a_2a_3)}+\frac{1}{2}\,A_{a(a_1}\,A_{a_2a_3)},\\\alpha_{a|a_1a_2,\,b}&=A_{a_1a_2,\,ab}+\frac{1}{2}\,A_{a[a_1}\,A_{a_2]b},\end{aligned}\tag{5.3.5}$$

while the symbols in the second line are given by

$$\beta^{a}{}_{a_1a_2a_3} = -\,\delta^{a}_{(a_1}\,A_{a_2a_3)}, \quad \beta^{a}{}_{a_1a_2,\,b} = -\,\delta^{a}_{[a_1}\,A_{a_2]b}. \tag{5.3.6}$$

6 Conclusions

We have studied the non-linear realisation of the E_{11} algebra and its vector representation. The set of generators of this algebra and their commutators has been found in ten-dimensional case up to level 4, which corresponds to its decomposition into representations of $GL(10) \times SL(2)$, and in five-dimensional case up to level 3, decomposed into representations of $GL(5) \times E_6$. We have then constructed the non-linear realisation in eleven, then and five dimensions, as well as for the A_1^{+++} algebra. This included the generalised vielben, the Cartan forms and their transformations under the local symmetry group of the non-linear realisation. The generalised vielbein was also found in four dimensions.

In eleven dimensions the non-linear realisation of $E_{11} \ltimes l_1$ has been studied in detail up to level 4 in fields. One finds a set of E_{11} invariant equations that are first order in derivatives and transform into each other under the local transformations of the non-linear realisation. The majority of these equations are duality relations that involve two different fields, except for one standalone equation that involves a single level 4 field $A_{a_1 \dots a_{10}, b_1 b_2}$. One then finds that these equations only hold modulo certain transformations, with an exception of the $3 - 6$ duality relation $D_{a_1 \dots a_4}$ that holds exactly. The modulo transformations have been shown to be related to the gauge transformations of the E_{11} fields. One then introduces a set of projectors that acts on the first order equations in such a way that modulo terms are eliminated from the variation. This leads us to a system of the second order equations that is likewise closed under the local transformations. The resulting system is shown to contain the set of equations of eleven-dimensional supergravity theory. An exception to this is the second order equations that describes the dynamics of the level 4 field $A_{a_1 \dots a_{10}, b_1 b_2}$, which is still subject to modulo transformations, even after being projected. This forces us to apply another projector to it, which leads to a third order equation for this field. The solution of this equation indicates that the $A_{a_1 \dots a_{10}, b_1 b_2}$ field is non-dynamical, and, therefore, doesn't create any new degrees of freedom in the theory. Its dimensional

reduction to ten dimensions is shown to generate the cosmological term that matches the one found in Romans supergravity. Likewise, all the other level 4 fields are shown to have no dynamical degrees of freedom, as they either drop out from the variations or are dual to the lower level fields. This fact supports the proposition that the only dynamical degrees of freedom in the E_{11} are the ones of the supergravity theory. A similar calculation has been performed in the five-dimensional case. The multiplet has been shown to have the similar structure to the eleven-dimensional one.

In carrying out this calculation we began from the E_{11} algebra and its vector representation l_1 and constructed the dynamics of the non-linear realisation at low levels with only one other assumption was that the local subgroup in the non-linear realisation is the Cartan involution invariant subgroup $I_c\left(E_{11}\right)$. The bosonic sectors of the maximal supergravity theories follow from this construction, at low levels, in a unique way and, therefore, one can even say that they are encoded in the E_{11} Dynkin diagram. This result strongly supports the proposition that the low-energy limit of the theory of strings and branes possesses the E_{11} symmetry.

There is a number of directions in which the results obtained in case of the eleven-dimensional non-linear realisation can be taken. The most interesting of them are

- Constructing the non-linear version dual of the dual graviton equation $E_{a_1\ldots a_8,\,b}$ (4.1.43). In order to do so one has to close the non-linear variation of $E^{a_1\ldots a_6}$ equation from (4.1.20). Another way to obtain it is to construct the non-linear versions of θ_1, θ_2 projectors from equation (4.1.56). Matching the results of these calculations will ensure that both commutative diagrams from Figure 13 are satisfied at the non-linear level.

- Understanding the modulo transformations at the non-linear level.

- Investigating the higher level duality relations like equation (4.1.70) and demonstrating the closure of the variations of $D_{a_1\ldots a_{10},\,b_1b_2b_3}$ and $E_{a_1\ldots a_9,\,b_1b_2b_3}$ from equations (4.1.54, 4.1.65).

- Considering the multiplet up to level 5 in fields and testing the hypothesis that the higher level E_{11} fields do not contain any extra degrees of freedom.

- Matching the l_1 terms in the first and second order equations by ensuring that the projectors that we have defined act correctly on the l_1 part of the equations.

One can also consider a number of directions in different dimensions.

- Testing the equations of the five-dimensional theory in the non-linear case. Apart from the non-linear variation of the second order vector equation found in Section 4.2 one can consider variations of the second order scalar $E_{\alpha_1 \dots \alpha_4}$ and gravity equations $E_a{}^b$ (4.2.8, 4.2.9), as well as the first order gravity - dual gravity relation $D^G_{a,\, b_1 b_2}$ and scalar equation $D_{a\, \alpha_1 \dots \alpha_4}$(4.2.6, 4.2.5). One could also build the projectors that connect them and investigate the modulo terms in their variations.

- Building the first order graviton - dual graviton equation in the A_1^{+++} case. Unlike the E_{11} case, in which this equation varies into the vector duality relation, in the A_1^{+++} case it must be self-dual under the $I_c\left(A_1^{+++}\right)$ transformations. Thus, it might provide some new insights into the dynamics of the dual graviton field.

- Finally, one could build the dynamics of the non-linear realisation of E_{11} in different important cases, like $4D$, $10D$ type IIA or $10D$ type IIB.

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Appendixes

A E_{11} algebra in 11D

In this appendix we give the $E_{11} \ltimes_s l_1$ algebra decomposed into representations of $GL\,(11)$. It has previously been constructed up to level 3 in [17] and then finalised in Chapter 16.7 of [32]. Here we give all the commutators of the generators up to level 4. This is a result of a joint work with Nikolay Gromov and Peter West that is yet to appear [39]. The commutators of the E_{11} generators with the generators of $K^a{}_b$ are

$$
\begin{aligned}
[K^a{}_b,\, K^c{}_d] &= \delta^c_b\, K^a{}_d - \delta^a_d\, K^c{}_b,\\
[K^a{}_b,\, R^{a_1a_2a_3}] &= 3\,\delta^{[a_1}_b\, R^{|a|a_2a_3]}, \qquad [K^a{}_b,\, R_{a_1a_2a_3}] = -\,3\,\delta^a_{[a_1}\, R_{|b|a_2a_3]},\\
[K^a{}_b,\, R^{a_1\ldots a_5}] &= 5\,\delta^{[a_1}_b\, R^{|a|a_2\ldots a_5]}, \quad [K^a{}_b,\, R_{a_1\ldots a_5}] = -\,5\,\delta^a_{[a_1}\, R_{|b|a_2\ldots a_5]},\\
[K^a{}_b,\, R^{a_1\ldots a_8}] &= 8\,\delta^{[a_1}_b\, R^{|a|a_2\ldots a_8]}, \quad [K^a{}_b,\, R_{a_1\ldots a_8}] = -\,8\,\delta^a_{[a_1}\, R_{|b|a_2\ldots a_8]},\\
[K^a{}_b,\, R^{a_1\ldots a_7,\,c}] &= 7\,\delta^{[a_1}_b\, R^{|a|a_2\ldots a_7],\,c} + \delta^c_b\, R^{a_1\ldots a_7,\,a},\\
[K^a{}_b,\, R_{a_1\ldots a_7,\,c}] &= -\,7\,\delta^a_{[a_1}\, R_{|b|a_2\ldots a_7],\,c} - \delta^a_c\, R_{a_1\ldots a_7,\,b},\\
[K^a{}_b,\, R^{a_1\ldots a_9,\,c_1c_2c_3}] &= 9\,\delta^{[a_1}_b\, R^{|a|a_2\ldots a_9],\,c_1c_2c_3} + 3\,\delta^{[c_1|}_b\, R^{a_1\ldots a_9,\,a|c_2c_3]},\\
[K^a{}_b,\, R_{a_1\ldots a_9,\,c_1c_2c_3}] &= -\,9\,\delta^a_{[a_1}\, R_{|b|a_2\ldots a_9],\,c_1c_2c_3} - 3\,\delta^a_{[c_1|}\, R_{a_1\ldots a_9,\,b|c_2c_3]},\\
[K^a{}_b,\, R^{a_1\ldots a_{10},\,c_1c_2}] &= 10\,\delta^{[a_1}_b\, R^{|a|a_2\ldots a_{10}],\,c_1c_2} + 2\,\delta^{(c_1|}_b\, R^{a_1\ldots a_{10},\,a|c_2)},\\
[K^a{}_b,\, R_{a_1\ldots a_{10},\,c_1c_2}] &= -\,10\,\delta^a_{[a_1}\, R_{|b|a_2\ldots a_{10}],\,c_1c_2} - 2\,\delta^a_{(c_1|}\, R_{a_1\ldots a_{10},\,b|c_2)},\\
[K^a{}_b,\, R^{a_1\ldots a_{11},\,c}] &= 11\,\delta^{[a_1}_b\, R^{|a|a_2\ldots a_{11}],\,c} + \delta^c_b\, R^{a_1\ldots a_{11},\,a},\\
[K^a{}_b,\, R_{a_1\ldots a_{11},\,c}] &= -\,11\,\delta^a_{[a_1}\, R_{|b|a_2\ldots a_{11}],\,c} - \delta^a_c\, R_{a_1\ldots a_{11},\,b}.
\end{aligned}
\tag{A.1}
$$

The positive level commutators are given by

$$
\begin{aligned}
[R^{a_1a_2a_3},\, R^{a_4a_5a_6}] &= 2\,R^{a_1\ldots a_6}, \qquad \left[R^{a_1a_2a_3},\, R^{b_1\ldots b_6}\right] = 6\,R^{a_1a_2a_3[b_1\ldots b_5,\,b_6]},\\
\left[R^{a_1a_2a_3},\, R^{b_1\ldots b_8,\,c}\right] &= \frac{3}{2}\,R^{b_1\ldots b_8[a_1,\,a_2a_3]c} - \frac{1}{6}\,R^{b_1\ldots b_8c,\,a_1a_2a_3}\\
&\quad + R^{b_1\ldots b_8[a_1a_2,\,a_3]c} + R^{a_1a_2a_3b_1\ldots b_8,\,c} - \frac{1}{3}\,R^{b_1\ldots b_8c[a_1a_2,\,a_3]},\\
\left[R^{a_1\ldots a_6},\, R^{b_1\ldots b_6}\right] &= 5\,R^{a_1\ldots a_6[b_1\ldots b_5,\,b_6]} + 4\,R^{a_1\ldots a_6[b_1b_2b_3,\,b_4b_5b_6]},
\end{aligned}
\tag{A.2}
$$

while the negative level ones are

$$
\begin{aligned}
&[R_{a_1a_2a_3},\, R_{a_4a_5a_6}] = 2\, R_{a_1\ldots a_6}, \qquad [R_{a_1a_2a_3},\, R_{b_1\ldots b_6}] = 6\, R_{a_1a_2a_3[b_1\ldots b_5,\, b_6]},\\
&[R_{a_1a_2a_3},\, R_{b_1\ldots b_8,\, c}] = \frac{3}{2}\, R_{b_1\ldots b_8[a_1,\, a_2a_3]c} - \frac{1}{6}\, R_{b_1\ldots b_8c,\, a_1a_2a_3}\\
&\qquad\qquad + R_{b_1\ldots b_8[a_1a_2,\, a_3]c} + R_{a_1a_2a_3b_1\ldots b_8,\, c} - \frac{1}{3}\, R_{b_1\ldots b_8c[a_1a_2,\, a_3]},\\
&[R_{a_1\ldots a_6},\, R_{b_1\ldots b_6}] = 5\, R_{a_1\ldots a_6[b_1\ldots b_5,\, b_6]} + 4\, R_{a_1\ldots a_6[b_1b_2b_3,\, b_4b_5b_6]},
\end{aligned}
\tag{A.3}
$$

The commutators between the positive and negative level generators up to level 3 are given by

$$
\begin{aligned}
[R^{a_1a_2a_3},\, R_{b_1b_2b_3}] &= 18\, \delta^{[a_1a_2}_{[b_1b_2}\, K^{a_3]}{}_{b_3]} - 2\, \delta^{a_1a_2a_3}_{b_1b_2b_3}\, K^{a}{}_{a},\\
[R^{a_1a_2a_3}, R_{b_1\ldots b_6}] &= 60\, \delta^{a_1a_2a_3}_{[b_1b_2b_3}\, R_{b_4b_5b_6]}, \quad \left[R_{a_1a_2a_3},\, R^{b_1\ldots b_6}\right] = 60\, \delta^{[b_1b_2b_3}_{a_1a_2a_3}\, R^{b_4b_5b_6]},\\
[R^{a_1a_2a_3},\, R_{b_1\ldots b_8,\, b}] &= 112\, \delta^{a_1a_2a_3}_{[b_1b_2b_3}\, R_{b_4\ldots b_8]b} - 112\, \delta^{a_1a_2a_3}_{[b_1b_2|b|}\, R_{b_3\ldots b_8]},\\
\left[R_{a_1a_2a_3},\, R^{b_1\ldots b_8,\, b}\right] &= 112\, \delta^{[b_1b_2b_3}_{a_1a_2a_3}\, R^{b_4\ldots b_8]b} - 112\, \delta^{[b_1b_2|b|}_{a_1a_2a_3}\, R^{b_3\ldots b_8]},\\
[R^{a_1\ldots a_6},\, R_{b_1\ldots b_6}] &= -1080\, \delta^{[a_1\ldots a_5}_{[b_1\ldots b_5}\, K^{a_6]}{}_{b_6]} + 120\, \delta^{a_1\ldots a_6}_{b_1\ldots b_6}\, K^{a}{}_{a},\\
\left[R_{a_1\ldots a_6},\, R^{b_1\ldots b_8,\, b}\right] &= -3360\, \delta^{[b_1\ldots b_6}_{a_1\ldots a_6}\, R^{b_7b_8]b} - 3360\, \delta^{[b_1\ldots b_5|b|}_{a_1\ldots a_6}\, R^{b_6b_7b_8]},\\
[R^{a_1\ldots a_6},\, R_{b_1\ldots b_8,\, b}] &= -3360\, \delta^{a_1\ldots a_6}_{[b_1\ldots b_6}\, R_{b_7b_8]b} - 3360\, \delta^{a_1\ldots a_6}_{[b_1\ldots b_5|b|}\, R_{b_6b_7b_8]},\\
[R^{a_1\ldots a_8,\, c},\, R_{b_1\ldots b_8,\, d}] &= \frac{16\cdot 7!}{9}\Big(\, \delta^{a_1\ldots a_8}_{b_1\ldots b_8}\, K^{c}{}_{d} + 8\, \delta^{c}_{d}\, \delta^{[a_1\ldots a_7}_{[b_1\ldots b_7}\, K^{a_8]}{}_{b_8]}\\
&\quad + 7\, \delta^{[a_1}_{d}\, \delta^{a_2\ldots a_7|c|}_{[b_1\,\ldots\, b_7}\, K^{a_8]}{}_{b_8]} + \delta^{a_1\,\ldots\, a_8}_{[b_1\ldots b_7|d|}\, K^{c}{}_{b_8]} + \delta^{[a_1\ldots a_7|c|}_{b_1\,\ldots\, b_8}\, K^{a_8]}{}_{d}\\
&\quad - \delta^{c}_{d}\, \delta^{a_1\ldots a_8}_{b_1\ldots b_8}\, K^{p}{}_{p} + \delta^{[a_1}_{d}\, \delta^{a_2\ldots a_8]c}_{b_1\,\ldots\, b_8}\, K^{p}{}_{p}\Big),
\end{aligned}
\tag{A.4}
$$

The level 4 commutators with lower level ones are

$$
\begin{aligned}
\left[R_{a_1a_2a_3},\, R^{b_1\ldots b_9,\, c_1c_2c_3}\right] &= 189\, \delta^{[c_1c_2[b_1}_{a_1a_2\,a_3}\, R^{b_2\ldots b_9],\, c_3]} + 432\, \delta^{[c_1[b_1b_2}_{a_1\,a_2a_3}\, R^{b_3\ldots b_9]c_2,\, c_3]}\\
&\quad + 252\, \delta^{[b_1b_2b_3}_{a_1a_2a_3}\, R^{b_4\ldots b_9][c_1c_2,\, c_3]},\\
[R^{a_1a_2a_3},\, R_{b_1\ldots b_9,\, c_1c_2c_3}] &= 189\, \delta^{a_1a_2\,a_3}_{[c_1c_2[b_1}\, R_{b_2\ldots b_9],\, c_3]} + 432\, \delta^{a_1\,a_2a_3}_{[c_1[b_1b_2}\, R_{b_3\ldots b_9]c_2,\, c_3]}\\
&\quad + 252\, \delta^{a_1a_2a_3}_{[b_1b_2b_3}\, R_{b_4\ldots b_9][c_1c_2,\, c_3]},
\end{aligned}
$$

$$
\begin{aligned}
\big[R_{a_1\ldots a_6},\, R^{b_1\ldots b_9,\,c_1c_2c_3}\big] &= 3\cdot 7!\,\delta^{[c_1[b_1\ldots b_5}_{a_1\ \ldots\ a_6}\,R^{b_6\ldots b_9]c_2c_3]} - 15\cdot 7!\,\delta^{[c_1c_2[b_1\ldots b_4}_{a_1\ \ldots\ a_6}\,R^{b_5\ldots b_9]c_3]}\\
&\quad - 4\cdot 7!\,\delta^{[b_1\ldots b_6}_{a_1\ldots a_6}\,R^{b_7\ldots b_9]c_1c_2c_3},\\
[R^{a_1\ldots a_6},\, R_{b_1\ldots b_9,\,c_1c_2c_3}] &= 3\cdot 7!\,\delta^{a_1\ \ldots\ a_6}_{[c_1[b_1\ldots b_5}\,R_{b_6\ldots b_9]c_2c_3]} - 15\cdot 7!\,\delta^{a_1\ \ldots\ a_6}_{[c_1c_2[b_1\ldots b_4}\,R_{b_5\ldots b_9]c_3]}\\
&\quad - 4\cdot 7!\,\delta^{a_1\ldots a_6}_{[b_1\ldots b_6}\,R_{b_7\ldots b_9]c_1c_2c_3},\\
\big[R_{a_1\ldots a_8,\,d},\, R^{b_1\ldots b_9,\,c_1c_2c_3}\big] &= 56\cdot 7!\,\Big(\delta^{[a_1}_{d}\,\delta^{a_2\ldots a_6|c_1c_2c_3|}_{b_1\ \ldots\ b_8}\,R^{a_7a_8a_9]} - \delta^{[c_1}_{d}\,\delta^{c_2c_3][a_1\ldots a_6}_{b_1\ \ldots\ b_8}\,R^{a_7a_8a_9]}\\
&\quad + 2\,\delta^{[a_1}_{d}\,\delta^{a_2\ldots a_7[c_1c_2}_{b_1\ \ldots\ b_8}\,R^{c_3]a_8a_9]} + \delta^{[a_1}_{d}\,\delta^{a_2\ldots a_8[c_1}_{b_1\ \ldots\ b_8}\,R^{c_2c_3]a_9]}\\
&\quad - \delta^{[c_1}_{d}\,\delta^{[a_1\ldots a_8}_{b_1\ldots b_8}\,R^{a_9]c_2c_3]} - 2\,\delta^{[c_1}_{d}\,\delta^{c_2[a_1\ldots a_7}_{b_1\ \ldots\ b_8}\,R^{a_8a_9]c_3]}\Big),\\
\big[R^{a_1\ldots a_8,\,d},\, R_{b_1\ldots b_9,\,c_1c_2c_3}\big] &= 56\cdot 7!\,\Big(\delta^{d}_{[a_1}\,\delta^{b_1\ \ldots\ b_8}_{a_2\ldots a_6|c_1c_2c_3|}\,R_{a_7a_8a_9]} - \delta^{d}_{[c_1}\,\delta^{b_1\ \ldots\ b_8}_{c_2c_3][a_1\ldots a_6}\,R_{a_7a_8a_9]}\\
&\quad + 2\,\delta^{d}_{[a_1}\,\delta^{b_1\ \ldots\ b_8}_{a_2\ldots a_7[c_1c_2}\,R_{c_3]a_8a_9]} + \delta^{d}_{[a_1}\,\delta^{b_1\ \ldots\ b_8}_{a_2\ldots a_8[c_1}\,R_{c_2c_3]a_9]}\\
&\quad - \delta^{d}_{[c_1}\,\delta^{b_1\ldots b_8}_{[a_1\ldots a_8}\,R_{a_9]c_2c_3]} - 2\,\delta^{d}_{[c_1}\,\delta^{b_1\ \ldots\ b_8}_{c_2[a_1\ldots a_7}\,R_{a_8a_9]c_3]}\Big),\\
\big[R_{a_1a_2a_3},\, R^{b_1\ldots b_{10},\,c_1c_2}\big] &= \frac{405}{2}\,\delta^{[b_1b_2(c_1}_{a_1a_2a_3}\,R^{b_3\ldots b_{10}],\,c_2)} - 180\,\delta^{[b_1b_2b_3}_{a_1a_2a_3}\,R^{b_4\ldots b_{10}](c_1,\,c_2)},\\
[R^{a_1a_2a_3},\, R_{b_1\ldots b_{10},\,c_1c_2}] &= \frac{405}{2}\,\delta^{a_1a_2a_3}_{[b_1b_2(c_1}\,R_{b_3\ldots b_{10}],\,c_2)} - 180\,\delta^{a_1a_2a_3}_{[b_1b_2b_3}\,R_{b_4\ldots b_{10}](c_1,\,c_2)},\\
\big[R_{a_1\ldots a_6},\, R^{b_1\ldots b_{10},\,c_1c_2}\big] &= 0, \qquad\qquad \big[R^{a_1\ldots a_6},\, R_{b_1\ldots b_{10},\,(c_1c_2)}\big] = 0,\\
\big[R_{a_1\ldots a_8,\,d},\, R^{b_1\ldots b_{10},\,c_1c_2}\big] &= 4\cdot 7!\,\Big(\delta^{(c_1}_{d}\,\delta^{c_2)[a_1\ldots a_7}_{b_1\ \ldots\ b_8}\,R^{a_8a_9a_{10}]} + \delta^{[a_1}_{d}\,\delta^{a_2\ldots a_8](c_1}_{b_1\ \ldots\ b_8}\,R^{c_2)a_9a_{10}]}\\
&\quad - \delta^{(c_1}_{d}\,\delta^{a_2\ldots a_8](c_1}_{b_1\ \ldots\ b_8}\,R^{a_9a_{10}]c_2)}\Big),\\
\big[R^{a_1\ldots a_8,\,d},\, R_{b_1\ldots b_{10},\,c_1c_2}\big] &= 4\cdot 7!\,\Big(\delta^{d}_{(c_1}\,\delta^{b_1\ \ldots\ b_8}_{c_2)[a_1\ldots a_7}\,R_{a_8a_9a_{10}]} + \delta^{d}_{[a_1}\,\delta^{b_1\ \ldots\ b_8}_{a_2\ldots a_8](c_1}\,R_{c_2)a_9a_{10}]}\\
&\quad - \delta^{d}_{(c_1}\,\delta^{b_1\ \ldots\ b_8}_{a_2\ldots a_8](c_1}\,R_{a_9a_{10}]c_2)}\Big),\\
\big[R_{a_1a_2a_3},\, R^{b_1\ldots b_{11},\,c}\big] &= \frac{495}{4}\,\delta^{[b_1b_2b_3}_{a_1a_2a_3}\,R^{b_4\ldots b_{11}],\,c},\\
[R^{a_1a_2a_3},\, R_{b_1\ldots b_{11},\,c}] &= \frac{495}{4}\,\delta^{a_1a_2a_3}_{[b_1b_2b_3}\,R_{b_4\ldots b_{11}],\,c},\\
\big[R_{a_1\ldots a_6},\, R^{b_1\ldots b_{11},\,c}\big] &= -\frac{11\cdot 7!}{2}\,\delta^{[b_1\ldots b_6}_{a_1\ldots a_6}\,R^{b_7\ldots b_{11}]c}\\
[R^{a_1\ldots a_6},\, R_{b_1\ldots b_{11},\,c}] &= -\frac{11\cdot 7!}{2}\,\delta^{a_1\ldots a_6}_{[b_1\ldots b_6}\,R_{b_7\ldots b_{11}]c},\\
\big[R_{a_1\ldots a_8,\,d},\, R^{b_1\ldots b_{11},\,c}\big] &= \frac{11!}{216}\,\Big(\delta^{c}_{[b_1}\,\delta^{[a_1\ \ldots\ a_8}_{b_2\ldots b_8]d}\,R^{a_9a_{10}a_{11}]} - \delta^{c}_{d}\,\delta^{[a_1\ldots a_8}_{b_1\ldots b_8}\,R^{a_9a_{10}a_{11}]}\Big),\\
\big[R^{a_1\ldots a_8,\,d},\, R_{b_1\ldots b_{11},\,c}\big] &= \frac{11!}{216}\,\Big(\delta^{[b_1}_{c}\,\delta^{b_2\ldots b_8]d}_{[a_1\ \ldots\ a_8}\,R_{a_9a_{10}a_{11}]} - \delta^{d}_{c}\,\delta^{b_1\ldots b_8}_{[a_1\ldots a_8}\,R_{a_9a_{10}a_{11}]}\Big). \qquad \text{(A.5)}
\end{aligned}
$$

The level 4 commutators with level -4 ones are

$$
\begin{aligned}
[R^{a_1\ldots a_9,\,c_1c_2c_3},\,R_{b_1\ldots b_9,\,d_1d_2d_3}] = &-786\cdot 7!\,\delta^{a_1\ldots a_9}_{b_1\ldots b_9}\,\delta^{[c_1c_2}_{[d_1d_2}\,K^{c_3]}_{\ \ d_3]} - \frac{87\cdot 7!^2}{140}\,\delta^{[c_1[a_1\ldots a_9}_{b_1\ \ldots\ b_9}\,\delta^{a_9]c_2}_{[d_1\,d_2}\,K^{c_3]}_{\ \ d_3]}\\
&-\frac{129\cdot 7!^2}{280}\,\delta^{c_1c_2c_3}_{d_1d_2d_3}\,\delta^{[a_1\ldots a_8}_{[b_1\ldots b_8}\,K^{a_9]}_{\ \ b_9]} + \frac{3\cdot 7!^2}{5}\,\delta^{[a_1a_2[c_1}_{d_1d_2d_3}\,\delta^{c_2c_3]a_3\ldots a_8}_{[b_1\ \ldots\ b_8}\,K^{a_9]}_{\ \ b_9]}\\
&-\frac{93\cdot 7!^2}{35}\,\delta^{[a_1[c_1c_2}_{d_1d_2d_3}\,\delta^{c_3]a_2\ldots a_8}_{[b_1\ \ldots\ b_8}\,K^{a_9]}_{\ \ b_9]} - \frac{3\cdot 7!^2}{5}\,\delta^{[a_1a_2a_3}_{d_1d_2d_3}\,\delta^{a_4\ldots a_9][c_1c_2}_{[b_1\ \ldots\ b_8}\,K^{c_3]}_{\ \ b_9]}\\
&+\frac{24\cdot 7!^2}{35}\,\delta^{[c_1[a_1a_2}_{d_1\,d_2d_3}\,\delta^{a_3\ldots a_9]c_2}_{[b_1\ \ldots\ b_8}\,K^{c_3]}_{\ \ b_9]} - \frac{111\cdot 7!^2}{280}\,\delta^{[a_1\ldots a_8}_{[b_1\ldots b_8}\,\delta^{a_9][c_1c_2}_{d_1\,d_2d_3}\,K^{c_3]}_{\ \ b_9]}\\
&-1674\cdot 7!\,\delta^{[a_1\ldots a_8[c_1}_{b_1\ \ldots\ b_9}\,\delta^{c_2c_3]}_{[d_1d_2}\,K^{a_9]}_{\ \ d_3]} + 864\cdot 7!\,\delta^{[a_1\ldots a_7[c_1c_2}_{b_1\ \ldots\ b_9}\,\delta^{c_3]a_8}_{[d_1\,d_2}\,K^{a_9]}_{\ \ d_3]}\\
&+360\cdot 7!\,\delta^{c_1c_2c_3}_{d_1d_2d_3}\,\delta^{a_1\ldots a_9}_{b_1\ldots b_9}\,K^{p}_{\ p} + \frac{27\cdot 7!^2}{70}\,\delta^{[c_1c_2[a_1}_{d_1d_2\,d_3}\,\delta^{a_2\ldots a_9]c_3]}_{b_1\ \ldots\ b_9}\,K^{p}_{\ p},
\end{aligned}
$$

$$
\begin{aligned}
[R^{a_1\ldots a_{11},\,c},\,R_{b_1\ldots b_{11},\,d}] &= \frac{55\cdot 11!}{16}\left(\delta^{[a_1}_{d}\,\delta^{|c|a_2\ldots a_{10}}_{[b_1\ \ldots\ b_{10}}\,K^{a_{11}]}_{\ \ b_{11}]} - \frac{1}{15}\,\delta^{c}_{d}\,\delta^{[a_1\ldots a_{10}}_{[b_1\ldots b_{10}}\,K^{a_{11}]}_{\ \ b_{11}]}\right),\\
[R^{a_1\ldots a_{10},\,c_1c_2},\,R_{b_1\ldots b_{10},\,d_1d_2}] &= -\frac{5\cdot 7!^2}{14}\,\delta^{(c_1c_2)}_{d_1d_2}\,\delta^{[a_1\ldots a_9}_{[b_1\ldots b_9}\,K^{a_{10}]}_{\ \ b_{10}]} - \frac{11!}{24}\,\delta^{a_1\ldots a_{10}}_{b_1\ldots b_{10}}\,\delta^{(c_1}_{(d_1}\,K^{c_2)}_{\ \ d_2)},\\
&\quad -\frac{5\cdot 7!^2}{84}\,\delta^{[a_1}_{(d_1}\,\delta^{|(c_1|a_2\ldots a_{10}]}_{b_1\ \ldots\ b_{10}}\,K^{c_2)}_{\ \ d_2)} + 230\cdot 7!\,\delta^{(c_1c_2)}_{d_1d_2}\,\delta^{a_1\ldots a_{10}}_{b_1\ldots b_{10}}\,K^{p}_{\ p}\\
&\quad +100\cdot 7!\,\delta^{[a_1(c_1}_{(d_1d_2)}\,\delta^{c_2)a_2\ldots a_{10}]}_{b_1\ \ldots\ b_{10}}\,K^{p}_{\ p},\\
[R^{a_1\ldots a_{10},\,c_1c_2},\,R_{b_1\ldots b_9,\,d_1d_2d_3}] &= \left[R_{a_1\ldots a_{10},\,c_1c_2},\,R^{b_1\ldots b_9,\,d_1d_2d_3}\right] = 0,\\
[R^{a_1\ldots a_{10},\,c_1c_2},\,R_{b_1\ldots b_{11},\,d}] &= \left[R_{a_1\ldots a_{10},\,c_1c_2},\,R^{b_1\ldots b_{11},\,d}\right] = 0,\\
[R^{a_1\ldots a_9,\,c_1c_2c_3},\,R_{b_1\ldots b_{11},\,d}] &= \left[R_{a_1\ldots a_9,\,c_1c_2c_3},\,R^{b_1\ldots b_{11},\,d}\right] = 0.
\end{aligned}
\tag{A.6}
$$

We will now list the commutators of the l_1 representation. The commutators of the l_1 representation generators with those of $GL\,(11)$ are given by

$$
\begin{aligned}
[K^{a}{}_{b},\,P_c] &= -\,\delta^{a}_{c}\,P_b + \frac{1}{2}\,\delta^{a}_{b}\,P_c,\\
[K^{a}{}_{b},\,Z^{a_1a_2}] &= 2\,\delta^{[a_1}_{b}\,Z^{|a|a_2]} + \frac{1}{2}\,\delta^{a}_{b}\,Z^{a_1a_2},\\
[K^{a}{}_{b},\,Z^{a_1\ldots a_5}] &= 5\,\delta^{[a_1}_{b}\,Z^{|a|a_2\ldots a_5]} + \frac{1}{2}\,\delta^{a}_{b}\,Z^{a_1\ldots a_5},\\
[K^{a}{}_{b},\,Z^{a_1\ldots a_8}] &= 8\,\delta^{[a_1}_{b}\,Z^{|a|a_2\ldots a_8]} + \frac{1}{2}\,\delta^{a}_{b}\,Z^{a_1\ldots a_8},\\
[K^{a}{}_{b},\,Z^{a_1\ldots a_7,\,c}] &= 7\,\delta^{[a_1}_{b}\,Z^{|a|a_2\ldots a_7],\,c} + \delta^{c}_{b}\,Z^{a_1\ldots a_7,\,a} + \frac{1}{2}\,\delta^{a}_{b}\,Z^{a_1\ldots a_7,\,c},
\end{aligned}
$$

$$
\begin{aligned}
\left[K^a{}_b,\, Z^{a_1\ldots a_8,\, b_1b_2b_3}\right] &= 8\,\delta_b^{[a_1}\, Z^{|a|a_2\ldots a_8],\, b_1b_2b_3} + 3\,\delta_b^{[b_1}\, Z^{|a_1\ldots a_8,\, a|b_2b_3]} + \frac{1}{2}\,\delta_b^a\, Z^{a_1\ldots a_8,\, b_1b_2b_3},\\
\left[K^a{}_b,\, Z^{a_1\ldots a_9,\, b_1b_2}\right] &= 9\,\delta_b^{[a_1}\, Z^{|a|a_2\ldots a_9],\, b_1b_2} + 2\,\delta_b^{[b_1}\, Z^{|a_1\ldots a_9,\, a|b_2]} + \frac{1}{2}\,\delta_b^a\, Z^{a_1\ldots a_9,\, b_1b_2},\\
\left[K^a{}_b,\, \hat{Z}^{a_1\ldots a_9,\, b_1b_2}\right] &= 9\,\delta_b^{[a_1}\, \hat{Z}^{|a|a_2\ldots a_9],\, b_1b_2} + 2\,\delta_b^{(b_1}\, \hat{Z}^{|a_1\ldots a_9,\, a|b_2)} + \frac{1}{2}\,\delta_b^a\, \hat{Z}^{a_1\ldots a_9,\, b_1b_2},\\
\left[K^a{}_b,\, Z_{(1)}^{a_1\ldots a_{10},\, c}\right] &= 10\,\delta_b^{[a_1}\, Z_{(1)}^{|a|a_2\ldots a_{10}],\, c} + \delta_b^c\, Z_{(1)}^{a_1\ldots a_{10},\, a} + \frac{1}{2}\,\delta_b^a\, Z_{(1)}^{a_1\ldots a_{10},\, c},\\
\left[K^a{}_b,\, Z_{(2)}^{a_1\ldots a_{10},\, c}\right] &= 10\,\delta_b^{[a_1}\, Z_{(2)}^{|a|a_2\ldots a_{10}],\, c} + \delta_b^c\, Z_{(2)}^{a_1\ldots a_{10},\, a} + \frac{1}{2}\,\delta_b^a\, Z_{(2)}^{a_1\ldots a_{10},\, c},\\
\left[K^a{}_b,\, Z^{a_1\ldots a_{11}}\right] &= \frac{3}{2}\,\delta_b^a\, Z^{a_1\ldots a_{11}}.
\end{aligned}
\tag{A.7}
$$

The commutators of the level 1 generator of E_{11} with l_1 generators are given by

$$
\begin{aligned}
\left[R^{a_1a_2a_3},\, P_a\right] &= 3\,\delta_a^{[a_1}\, Z^{a_2a_3]}, \quad \left[R^{a_1a_2a_3},\, Z^{a_4a_5}\right] = Z^{a_1\ldots a_5},\\
\left[R^{a_1a_2a_3},\, Z^{b_1\ldots b_5}\right] &= Z^{b_1\ldots b_5a_1a_2a_3} + Z^{b_1\ldots b_5[a_1a_2,\, a_3]},\\
\left[R^{a_1a_2a_3},\, Z^{b_1\ldots b_8}\right] &= Z^{a_1a_2a_3b_1\ldots b_8} + \frac{4}{135}\, Z^{a_1a_2a_3[b_1\ldots b_5,\, b_6b_7b_8]}\\
&\quad - \frac{20}{63}\, Z^{a_1a_2a_3[b_1\ldots b_6,\, b_7b_8]} - Z_{(2)}^{a_1a_2a_3[b_1\ldots b_7,\, b_8]},\\
\left[R^{a_1a_2a_3},\, Z^{b_1\ldots b_7,\, c}\right] &= Z^{a_1a_2a_3c[b_1\ldots b_4,\, b_5b_6b_7]} + \frac{1}{2}\, Z^{c[b_1\ldots b_5[a_1a_2,\, a_3]b_6b_7]}\\
&\quad + Z^{a_1a_2a_3c[b_1\ldots b_5,\, b_6b_7]} - \frac{3}{7}\, Z^{c[b_1\ldots b_6[a_1a_2,\, a_3]b_7]}\\
&\quad - \hat{Z}^{c[b_1\ldots b_6[a_1a_2,\, a_3]b_7]}\\
&\quad - Z_{(1)}^{a_1a_2a_3c[b_1\ldots b_6,\, b_7]} - \frac{3}{8}\, Z_{(1)}^{cb_1\ldots b_7[a_1a_2,\, a_3]}.
\end{aligned}
\tag{A.8}
$$

Level 2 generator gives

$$
\begin{aligned}
\left[R^{a_1\ldots a_6},\, P_a\right] &= -\,3\,\delta_a^{[a_1}\, Z^{a_2\ldots a_6]},\\
\left[R^{a_1\ldots a_6},\, Z^{b_1b_2}\right] &= -\, Z^{a_1\ldots a_6b_1b_2} + \frac{1}{3}\, Z^{a_1\ldots a_6[b_1,\, b_2]},\\
\left[R^{a_1\ldots a_6},\, Z^{b_1\ldots b_5}\right] &= -\, Z^{a_1\ldots a_6b_1\ldots b_5} + \frac{4}{189}\, Z^{a_1\ldots a_6[b_1b_2b_3,\, b_4b_5]},\\
&\quad - \frac{40}{441}\, Z^{a_1\ldots a_6[b_1b_2b_3,\, b_4b_5]}\\
&\quad - \frac{55}{336}\, Z_{(1)}^{a_1\ldots a_6[b_1\ldots b_4,\, b_5]} + \frac{5}{16}\, Z_{(2)}^{a_1\ldots a_6[b_1\ldots b_4,\, b_5]}.
\end{aligned}
\tag{A.9}
$$

For level 3 generator we have

$$
\begin{aligned}
[R^{a_1\ldots a_8,\,c},\,P_a] =&\ -\frac{4}{3}\,\delta^c_a\,Z^{a_1\ldots a_8}+\frac{4}{3}\,\delta^{[a_1}_a\,Z^{a_2\ldots a_8]c}+\frac{4}{3}\,\delta^{[a_1}_b\,Z^{a_2\ldots a_8],\,c},\\
\left[R^{a_1\ldots a_8,\,c},\,Z^{b_1b_2}\right] =&\ -\frac{16}{135}\,Z^{b_1b_2c[a_1\ldots a_5,\,a_6a_7a_8]}+\frac{4}{63}\,\hat{Z}^{b_1b_2[a_1\ldots a_7,\,a_8]c}\\
&+\frac{16}{189}\,Z^{b_1b_2c[a_1\ldots a_6,\,a_7a_8]}-\frac{16}{189}\,Z^{b_1b_2[a_1\ldots a_7,\,a_8]c}\\
&+\frac{1}{42}\,Z^{b_1b_2c[a_1\ldots a_7,\,a_8]}_{(1)}-\frac{1}{42}\,Z^{b_1b_2a_1\ldots a_8,\,c}_{(1)}\\
&-\frac{1}{6}\,Z^{b_1b_2c[a_1\ldots a_7,\,a_8]}_{(2)}+\frac{1}{6}\,Z^{b_1b_2a_1\ldots a_8,\,c}_{(2)}.
\end{aligned}
\tag{A.10}
$$

Finally, level 4 generators give

$$
\begin{aligned}
[R^{a_1\ldots a_{11},\,c},\,P_a] =&\ -\frac{3}{2}\,\delta^c_a\,Z^{a_1\ldots a_{11}}+\frac{121}{448}\,\delta^{[a_1}_a\,Z^{a_2\ldots a_{11}],\,c}_{(1)}-\frac{33}{64}\,\delta^{[a_1}_a\,Z^{a_2\ldots a_{11}],\,c}_{(2)},\\
[R^{a_1\ldots a_{10},\,c_1c_2},\,P_a] =&\ \frac{5}{7}\,\delta^{[a_1}_a\,Z^{a_2\ldots a_9(c_1,\,c_2)a_{10}]}\\
&+\frac{11}{2016}\,\delta^{(c_1}_a\,Z^{|a_1\ldots a_{10}|,\,c_2)}_{(1)}-\frac{5}{1008}\,\delta^{[a_1}_a\,Z^{a_2\ldots a_{10}](c_1,\,c_2)}_{(1)}\\
&-\frac{11}{32}\,\delta^{(c_1}_a\,Z^{|a_1\ldots a_{10}|,\,c_2)}_{(2)}+\frac{5}{16}\,\delta^{[a_1}_a\,Z^{a_2\ldots a_{10}](c_1,\,c_2)}_{(2)},\\
[R^{a_1\ldots a_9,\,c_1c_2c_3},\,P_a] =&\ -\frac{4}{105}\,\delta^{[a_1}_a\,Z^{a_2\ldots a_9],\,c_1c_2c_3}\\
&+\frac{2}{21}\,\delta^{[c_1}_a\,Z^{|a_1\ldots a_9|,\,c_2c_3]}+\frac{3}{49}\,\delta^{[a_1}_a\,Z^{a_2\ldots a_9][c_1,\,c_2c_3]}.
\end{aligned}
\tag{A.11}
$$

The commutators with the level -1 generator are

$$
\begin{aligned}
[R_{a_1a_2a_3},\,P_a] =&\ 0, \qquad \left[R_{a_1a_2a_3},\,Z^{b_1b_2}\right] = 6\,\delta^{b_1b_2}_{[a_1a_2}\,P_{a_3]},\\
\left[R_{a_1a_2a_3},\,Z^{b_1\ldots b_5}\right] =&\ 60\,\delta^{b_1b_2b_3}_{a_1a_2a_3}\,Z^{b_4b_5]}, \quad \left[R_{a_1a_2a_3},\,Z^{b_1\ldots b_8}\right] = -42\,\delta^{b_1b_2b_3}_{a_1a_2a_3}\,Z^{b_4\ldots b_8]},\\
\left[R_{a_1a_2a_3},\,Z^{b_1\ldots b_7,\,c}\right] =&\ \frac{945}{4}\,\delta^{[b_1b_2|c|}_{a_1a_2a_3}\,Z^{b_3\ldots b_7]}+\frac{945}{4}\,\delta^{[b_1b_2b_3}_{a_1a_2a_3}\,Z^{b_3\ldots b_7]c},\\
\left[R_{a_1a_2a_3},\,Z^{b_1\ldots b_8,\,c_1c_2c_3}\right] =&\ -945\,\delta^{c_1c_2c_3}_{a_1a_2a_3}\,Z^{b_1\ldots b_8}-2835\,\delta^{[b_1[c_1c_2}_{a_1a_2a_3}\,Z^{c_3]b_2\ldots b_8]},\\
&-2835\,\delta^{[b_1b_2[c_1}_{a_1a_2a_3}\,Z^{c_2c_3]b_3\ldots b_8]}+\delta^{[b_1b_2b_3}_{a_1a_2a_3}\,Z^{b_4\ldots b_8]c_1c_2c_3}\\
&+6075\,\delta^{[c_1c_2[b_1}_{a_1a_2a_3}\,Z^{b_2\ldots b_8],\,c_3]}+42525\,\delta^{[b_1b_2[c_1}_{a_1a_2a_3}\,Z^{c_2c_3]b_3\ldots b_7,\,b_8]}\\
&+14175\,\delta^{[b_1b_2b_3}_{a_1a_2a_3}\,Z^{|c_1c_2c_3|b_4\ldots b_7,\,b_8]},\\
\left[R_{a_1a_2a_3},\,\hat{Z}^{b_1\ldots b_9,\,c_1c_2}\right] =&\ \frac{36288}{11}\,\delta^{(c_1[b_1b_2}_{a_1a_2a_3}\,Z^{b_3\ldots b_9],\,c_2)}+\frac{31752}{11}\,\delta^{[b_1b_2b_3}_{a_1a_2a_3}\,Z^{b_4\ldots b_9](c_1,\,c_2)},
\end{aligned}
$$

$$
\begin{aligned}
\left[R_{a_1a_2a_3},\, Z^{b_1\ldots b_9,\,c_1c_2}\right] =\ & -2646\,\delta^{c_1c_2[b_1}_{a_1a_2a_3}\,Z^{b_2\ldots b_9]} - 2646\,\delta^{[b_1b_2b_3}_{a_1a_2a_3}\,Z^{b_4\ldots b_9]c_1c_2} \\
& - 5292\,\delta^{[c_1[b_1b_2}_{a_1a_2a_3}\,Z^{b_2\ldots b_9]c_2]} - 2646\,\delta^{[b_1b_2b_3}_{a_1a_2a_3}\,Z^{|c_1c_2|b_4\ldots b_8,\,b_9]} \\
& + 756\,\delta^{[c_1[b_1b_2}_{a_1a_2a_3}\,Z^{b_4\ldots b_9],\,c_2]}, \\
\left[R_{a_1a_2a_3},\, Z^{b_1\ldots b_{10},\,c}_{(1)}\right] =\ & -\frac{8820}{11}\,\delta^{[b_1b_2b_3}_{a_1a_2a_3}\,Z^{b_4\ldots b_{11}]} \\
& + \frac{18900}{11}\,\delta^{[b_1b_2|c|}_{a_1a_2a_3}\,Z^{b_3\ldots b_{10}]} - \frac{18900}{11}\,\delta^{[b_1b_2b_3}_{a_1a_2a_3}\,Z^{b_4\ldots b_{10}]c}, \\
\left[R_{a_1a_2a_3},\, Z^{b_1\ldots b_{10},\,c}_{(2)}\right] =\ & -100\,\delta^{[b_1b_2b_3}_{a_1a_2a_3}\,Z^{b_4\ldots b_{11}]} \\
& + \frac{8940}{11}\,\delta^{[b_1b_2|c|}_{a_1a_2a_3}\,Z^{b_3\ldots b_{10}]} - \frac{8940}{11}\,\delta^{[b_1b_2b_3}_{a_1a_2a_3}\,Z^{b_4\ldots b_{10}]c}, \\
\left[R_{a_1a_2a_3},\, Z^{b_1\ldots b_{11}}\right] =\ & 120\,\delta^{[b_1b_2b_3}_{a_1a_2a_3}\,Z^{b_4\ldots b_{11}]}.
\end{aligned}
\tag{A.12}
$$

For the level -2 generator we have

$$
\begin{aligned}
[R_{a_1\ldots a_6},\, P_a] = 0, \qquad & \left[R_{a_1\ldots a_6},\, Z^{b_1b_2}\right] = 0, \\
\left[R_{a_1\ldots a_6},\, Z^{b_1\ldots b_5}\right] =\ & -360\,\delta^{b_1\ldots b_5}_{[a_1\ldots a_5}\,P_{a_6]}, \quad \left[R_{a_1a_2a_3},\, Z^{b_1\ldots b_8}\right] = 2520\,\delta^{[b_1\ldots b_6}_{a_1\ldots a_6}\,Z^{b_7b_8]}, \\
\left[R_{a_1\ldots a_6},\, Z^{b_1\ldots b_7,\,c}\right] =\ & 5670\,\delta^{[b_1\ldots b_5|c|}_{a_1\ \ldots\ a_6}\,Z^{b_6b_7]} - 5670\,\delta^{[b_1\ldots b_6}_{a_1\ldots a_6}\,Z^{b_7]c}, \\
\left[R_{a_1\ldots a_6},\, Z^{b_1\ldots b_8,\,c_1c_2c_3}\right] =\ & \frac{5\cdot 10!}{32}\,\delta^{c_1c_2c_3[b_1b_2b_3}_{a_1\ \ldots\ a_6}\,Z^{b_6\ldots b_8]} - \frac{15\cdot 10!}{32}\,\delta^{[c_1c_2[b_1\ldots b_4}_{a_1\ \ldots\ a_6}\,Z^{b_5\ldots b_8]c_3]} \\
& + \frac{15\cdot 10!}{32}\,\delta^{[c_1[b_1\ldots b_5}_{a_1\ \ldots\ a_6}\,Z^{b_6b_7b_8]c_2c_3]} - \frac{5\cdot 10!}{32}\,\delta^{[b_1\ldots b_6}_{a_1\ldots a_6}\,Z^{b_7b_8]c_1c_2c_3}, \\
\left[R_{a_1\ldots a_6},\, Z^{b_1\ldots b_9,\,c_1c_2}\right] =\ & -\frac{63\cdot 7!}{2}\,\delta^{c_1c_2[b_1\ldots b_4}_{a_1\ \ldots\ a_6}\,Z^{b_5\ldots b_9]} - \frac{63\cdot 7!}{2}\,\delta^{[b_1\ldots b_6}_{a_1\ldots a_6}\,Z^{b_7b_8b_9]c_1c_2} \\
& + 63\cdot 7!\,\delta^{[c_1[b_1\ldots b_5}_{a_1\ \ldots\ a_6}\,Z^{b_6\ldots b_9]c_2]}, \\
\left[R_{a_1\ldots a_6},\, Z^{b_1\ldots b_{10},\,c}_{(1)}\right] =\ & \frac{315\cdot 7!}{11}\,\delta^{[b_1\ldots b_6}_{a_1\ldots a_6}\,Z^{b_7\ldots b_{10}]c} + \frac{315\cdot 7!}{11}\,\delta^{[b_1\ldots b_5|c|}_{a_1\ \ldots\ a_6}\,Z^{b_6\ldots b_{10}]} \\
\left[R_{a_1\ldots a_6},\, Z^{b_1\ldots b_{10},\,c}_{(2)}\right] =\ & \frac{5\cdot 7!}{11}\,\delta^{[b_1\ldots b_6}_{a_1\ldots a_6}\,Z^{b_7\ldots b_{10}]c} + \frac{5\cdot 7!}{11}\,\delta^{[b_1\ldots b_5|c|}_{a_1\ \ldots\ a_6}\,Z^{b_6\ldots b_{10}]} \\
\left[R_{a_1\ldots a_6},\, \hat{Z}^{b_1\ldots b_9,\,c_1c_2}\right] = 0, \qquad & \left[R_{a_1\ldots a_6},\, Z^{b_1\ldots b_{11}}\right] = 5040\,\delta^{[b_1\ldots b_6}_{a_1\ldots a_6}\,Z^{b_7\ldots b_{11}]}.
\end{aligned}
\tag{A.13}
$$

Next, we have the commutators with level -3 generator.

$$
\begin{aligned}
[R_{a_1\ldots a_8,\,d},\, P_a] = 0, \quad & \left[R_{a_1\ldots a_8,\,d},\, Z^{b_1b_2}\right] = 0, \quad \left[R_{a_1\ldots a_8,\,d},\, Z^{b_1\ldots b_5}\right] = 0, \\
\left[R_{a_1\ldots a_8,\,d},\, Z^{b_1\ldots b_8}\right] =\ & -6720\,\delta^{b_1\ldots b_8}_{a_1\ldots a_8}\,P_d - 6720\,\delta^{b_1\ \ldots\ b_8}_{[a_1\ldots a_7|d|}\,P_{a_8]}, \\
\left[R_{a_1\ldots a_8,\,d},\, Z^{b_1\ldots b_7,\,c}\right] =\ & -\frac{21\cdot 7!}{2}\,\delta^c_d\,\delta^{b_1\ldots b_7}_{[a_1\ldots a_7}\,P_{a_8]} - \frac{21\cdot 7!}{2}\,\delta^{b_1\ \ldots\ b_8}_{[a_1\ldots a_7|d|}\,P_{a_8]},
\end{aligned}
$$

$$
\begin{aligned}
\Big[R_{a_1\ldots a_8,\,d},\, Z^{b_1\ldots b_8,\,c_1c_2c_3}\Big] =\; & -\frac{21\cdot 10!}{4}\,\delta_d^{[b_1}\,\delta_{a_1\ \ldots\ a_8}^{b_2\ldots b_7[c_1c_2}\,Z^{c_3]b_8]} - \frac{21\cdot 10!}{4}\,\delta_d^{[c_1}\,\delta_{a_1\ \ldots\ a_8}^{c_2[b_1\ldots b_7}\,Z^{b_8]c_3]} \\
& + \frac{21\cdot 10!}{8}\,\delta_d^{[c_1}\,\delta_{a_1\ldots a_8}^{|b_1\ldots b_8|}\,Z^{c_2c_3]} - \frac{21\cdot 10!}{8}\,\delta_d^{[b_1}\,\delta_{a_1\ \ldots\ a_8}^{b_2\ldots b_6|c_1c_2c_3|}\,Z^{b_7b_8]} \\
& - \frac{21\cdot 10!}{8}\,\delta_d^{[b_1}\,\delta_{a_1\ \ldots\ a_8}^{b_2\ldots b_8][c_1}\,Z^{c_2c_3]} + \frac{21\cdot 10!}{8}\,\delta_d^{[c_1}\,\delta_{a_1\ \ldots\ a_8}^{c_2c_3][b_1\ldots b_6}\,Z^{b_7b_8]}, \\
\Big[R_{a_1\ldots a_8,\,d},\, \hat{Z}^{b_1\ldots b_9,\,c_1c_2}\Big] =\; & -\frac{126\cdot 10!}{55}\,\delta_d^{(c_1}\,\delta_{a_1\ \ldots\ a_8}^{c_2)[b_1\ldots b_7}\,Z^{b_8b_9]} - \frac{126\cdot 10!}{55}\,\delta_d^{(c_1}\,\delta_{a_1\ldots a_8}^{[b_1\ldots b_8}\,Z^{b_9]c_2)} \\
& - \frac{126\cdot 10!}{55}\,\delta_d^{[b_1}\,\delta_{a_1\ \ldots\ a_8}^{b_2\ldots b_8(c_1}\,Z^{c_2)b_9]}, \\
\Big[R_{a_1\ldots a_8,\,d},\, Z^{b_1\ldots b_9,\,c_1c_2}\Big] =\; & \frac{49\cdot 9!}{3}\,\delta_d^{[b_1}\,\delta_{a_1\ \ldots\ a_8}^{b_2\ldots b_7|c_1c_2|}\,Z^{b_8b_9]} - \frac{49\cdot 9!}{3}\,\delta_d^{[c_1}\,\delta_{a_1\ \ldots\ a_8}^{c_2][b_1\ldots b_7}\,Z^{b_8b_9]} \\
& - \frac{49\cdot 9!}{3}\,\delta_d^{[c_1}\,\delta_{a_1\ldots a_8}^{[b_1\ldots b_8}\,Z^{b_9]c_2]} - \frac{49\cdot 9!}{3}\,\delta_d^{[b_1}\,\delta_{a_1\ \ldots\ a_8}^{b_2\ldots b_8[c_1}\,Z^{c_2]b_9]}, \\
\Big[R_{a_1\ldots a_8,\,d},\, Z_{(1)}^{b_1\ldots b_{10},\,c}\Big] =\; & -\frac{7\cdot 10!}{66}\,\delta_d^c\,\delta_{a_1\ldots a_8}^{[b_1\ldots b_8}\,Z^{b_9b_{10}]} + \frac{7\cdot 10!}{66}\,\delta_{[a_1}^c\,\delta_{a_2\ldots a_8]d}^{[b_1\ \ldots\ b_8}\,Z^{b_9b_{10}]}, \\
\Big[R_{a_1\ldots a_8,\,d},\, Z_{(2)}^{b_1\ldots b_{10},\,c}\Big] =\; & -\frac{13\cdot 10!}{54}\,\delta_d^c\,\delta_{a_1\ldots a_8}^{[b_1\ldots b_8}\,Z^{b_9b_{10}]} + \frac{13\cdot 10!}{54}\,\delta_{[a_1}^c\,\delta_{a_2\ldots a_8]d}^{[b_1\ \ldots\ b_8}\,Z^{b_9b_{10}]}, \\
\Big[R_{a_1\ldots a_8,\,d},\, Z^{b_1\ldots b_{11}}\Big] =\; & 0.
\end{aligned}
\tag{A.14}
$$

The commutators with level -4 generator $R_{a_1\ldots a_9,\,c_1c_2c_3}$ are given by

$$
\begin{aligned}
\big[R_{a_1\ldots a_9,\,c_1c_2c_3},\, P_a\big] &= 0, \quad \Big[R_{a_1\ldots a_9,\,c_1c_2c_3},\, Z^{b_1b_2}\Big] = 0, \\
\Big[R_{a_1\ldots a_9,\,c_1c_2c_3},\, Z^{b_1\ldots b_5}\Big] &= 0, \quad \Big[R_{a_1\ldots a_9,\,c_1c_2c_3},\, Z^{b_1\ldots b_8}\Big] = 0, \\
\Big[R_{a_1\ldots a_9,\,c_1c_2c_3},\, Z^{b_1\ldots b_7,\,c}\Big] &= 0, \quad \Big[R_{a_1\ldots a_9,\,c_1c_2c_3},\, \hat{Z}^{b_1\ldots b_9,\,d_1d_2}\Big] = 0, \\
\Big[R_{a_1\ldots a_9,\,c_1c_2c_3},\, Z_{(1)}^{b_1\ldots b_{10},\,d}\Big] &= 0, \quad \Big[R_{a_1\ldots a_9,\,c_1c_2c_3},\, Z_{(2)}^{b_1\ldots b_{10},\,d}\Big] = 0, \\
\Big[R_{a_1\ldots a_9,\,c_1c_2c_3},\, Z^{b_1\ldots b_9,\,d_1d_2}\Big] =\; & -\frac{1071\cdot 9!}{8}\,\delta_{[c_1c_2}^{d_1d_2}\,\delta_{|a_1\ldots a_9|}^{b_1\ldots b_9}\,P_{c_3]} - \frac{189\cdot 10!}{16}\,\delta_{[c_1c_2}^{d_1d_2}\,\delta_{c_3][a_1\ldots a_8}^{b_1\ \ldots\ b_9}\,P_{a_9]} \\
& - 378\cdot 9!\,\delta_{[a_1[c_1}^{d_1d_2}\,\delta_{c_2c_3]a_2\ldots a_8}^{b_1\ \ldots\ b_9}\,P_{a_9]} \\
& - 189\cdot 9!\,\delta_{[a_1a_2}^{d_1d_2}\,\delta_{a_3\ldots a_9][c_1c_2}^{b_1\ \ldots\ b_9}\,P_{c_3]} \\
& - \frac{189\cdot 10!}{8}\,\delta_{[c_1[a_1}^{d_1d_2}\,\delta_{a_2\ldots a_9]c_2}^{b_1\ \ldots\ b_9}\,P_{c_3]}, \\
\Big[R_{a_1\ldots a_9,\,c_1c_2c_3},\, Z^{b_1\ldots b_8,\,d_1d_2d_3}\Big] =\; & \frac{189\cdot 10!}{2}\Big(\delta_{c_1c_2c_3}^{d_1d_2d_3}\,\delta_{[a_1\ldots a_8}^{b_1\ldots b_8}\,P_{a_9]} - \delta_{c_1c_2c_3}^{[b_1b_2b_3}\,\delta_{[a_1\ \ldots\ a_8}^{b_4\ldots b_8]d_1d_2d_3}\,P_{a_9]} \\
& + 3\,\delta_{c_1c_2c_3}^{[d_1[b_1b_2}\,\delta_{[a_1\ \ldots\ a_8}^{b_3\ldots b_8]d_2d_3]}\,P_{a_9]} - 3\,\delta_{c_1c_2c_3}^{[d_1d_2[b_1}\,\delta_{[a_1\ \ldots\ a_8}^{b_2\ldots b_8]d_3]}\,P_{a_9]}\Big), \\
\Big[R_{a_1\ldots a_9,\,c_1c_2c_3},\, Z^{b_1\ldots b_{11}}\Big] &= 0.
\end{aligned}
\tag{A.15}
$$

Taking commutators with level -4 generator $R_{a_1\ldots a_{10},\,c_1c_2}$ one finds

$$
\begin{aligned}
[R_{a_1\ldots a_{10},\,c_1c_2},\,P_a] &= 0, \quad \left[R_{a_1\ldots a_{10},\,c_1c_2},\,Z^{b_1b_2}\right] = 0,\\
\left[R_{a_1\ldots a_{10},\,c_1c_2},\,Z^{b_1\ldots b_5}\right] &= 0, \quad \left[R_{a_1\ldots a_{10},\,c_1c_2},\,Z^{b_1\ldots b_8}\right] = 0,\\
\left[R_{a_1\ldots a_{10},\,c_1c_2},\,Z^{b_1\ldots b_7,\,c}\right] &= 0, \quad \left[R_{a_1\ldots a_{10},\,c_1c_2},\,Z^{b_1\ldots b_9,\,d_1d_2}\right] = 0,\\
\left[R_{a_1\ldots a_{10},\,c_1c_2},\,\hat{Z}^{b_1\ldots b_9,\,d_1d_2}\right] &= \frac{189\cdot 12!}{88}\left(\delta^{\,d_1d_2}_{(c_1c_2)}\,\delta^{\,b_1\ldots b_9}_{[a_1\ldots a_9}\,P_{a_{10}]} + 2\,\delta^{\,[b_1(d_1}_{(c_1\,c_2)}\,\delta^{\,d_2)b_2\ldots b_9]}_{[a_1\,\ldots\,a_9}\,P_{a_{10}]}\right),\\
\left[R_{a_1\ldots a_{10},\,c_1c_2},\,Z^{b_1\ldots b_{10},\,d}_{(1)}\right] &= \frac{105\cdot 10!}{44}\Big(\delta^{\,b_1\ldots b_{10}}_{a_1\ldots a_{10}}\,\delta^{\,d}_{(c_1}\,P_{c_2)} + \delta^{\,[b_1\ldots b_9|d|}_{a_1\,\ldots\,a_{10}}\,\delta^{\,b_{10}]}_{(c_1}\,P_{c_2)}\\
&\qquad + \delta^{\,b_1\,\ldots\,b_{10}}_{[a_1\ldots a_9(c_1}\,\delta^{\,d}_{c_2)}\,P_{a_{10}]}\Big),\\
\left[R_{a_1\ldots a_{10},\,c_1c_2},\,Z^{b_1\ldots b_{10},\,d}_{(2)}\right] &= \frac{5\cdot 10!}{4}\Big(\delta^{\,b_1\ldots b_{10}}_{a_1\ldots a_{10}}\,\delta^{\,d}_{(c_1}\,P_{c_2)} + \delta^{\,[b_1\ldots b_9|d|}_{a_1\,\ldots\,a_{10}}\,\delta^{\,b_{10}]}_{(c_1}\,P_{c_2)}\\
&\qquad + \delta^{\,b_1\,\ldots\,b_{10}}_{[a_1\ldots a_9(c_1}\,\delta^{\,d}_{c_2)}\,P_{a_{10}]}\Big),\\
\left[R_{a_1\ldots a_{10},\,c_1c_2},\,Z^{b_1\ldots b_8,\,d_1d_2d_3}\right] &= 0, \quad \left[R_{a_1\ldots a_{10},\,c_1c_2},\,Z^{b_1\ldots b_{11}}\right] = 0.
\end{aligned}
\tag{A.16}
$$

Finally, commutators with level -4 generator $R_{a_1\ldots a_{11},\,b}$ are

$$
\begin{aligned}
[R_{a_1\ldots a_{11},\,c},\,P_a] &= 0, \quad \left[R_{a_1\ldots a_{11},\,c},\,Z^{b_1b_2}\right] = 0,\\
\left[R_{a_1\ldots a_{11},\,c},\,Z^{b_1\ldots b_5}\right] &= 0, \quad \left[R_{a_1\ldots a_{11},\,c},\,Z^{b_1\ldots b_8}\right] = 0,\\
\left[R_{a_1\ldots a_{11},\,c},\,Z^{b_1\ldots b_7,\,c}\right] &= 0, \quad \left[R_{a_1\ldots a_{11},\,c},\,Z^{b_1\ldots b_9,\,d_1d_2}\right] = 0,\\
\left[R_{a_1\ldots a_{11},\,c},\,\hat{Z}^{b_1\ldots b_9,\,d_1d_2}\right] &= 0, \quad \left[R_{a_1\ldots a_{11},\,c},\,Z^{b_1\ldots b_8,\,d_1d_2d_3}\right] = 0,\\
\left[R_{a_1\ldots a_{11},\,c},\,Z^{b_1\ldots b_{10},\,d}_{(1)}\right] &= -\frac{105\cdot 10!}{8}\left(\delta^{\,d}_{c}\,\delta^{\,b_1\ldots b_{10}}_{[a_1\ldots a_{10}}\,P_{a_{11}]} - \delta^{\,[b_1}_{c}\,\delta^{\,b_2\ldots b_{10}]d}_{[a_1\,\ldots\,a_{10}}\,P_{a_{11}]}\right),\\
\left[R_{a_1\ldots a_{11},\,c},\,Z^{b_1\ldots b_{10},\,d}_{(2)}\right] &= -\frac{5\cdot 10!}{24}\left(\delta^{\,d}_{c}\,\delta^{\,b_1\ldots b_{10}}_{[a_1\ldots a_{10}}\,P_{a_{11}]} - \delta^{\,[b_1}_{c}\,\delta^{\,b_2\ldots b_{10}]d}_{[a_1\,\ldots\,a_{10}}\,P_{a_{11}]}\right),\\
\left[R_{a_1\ldots a_{11},\,c},\,Z^{b_1\ldots b_{11}}\right] &= 180\cdot 7!\,\delta^{\,b_1\ldots b_{11}}_{a_1\ldots a_{11}}\,P_c.
\end{aligned}
\tag{A.17}
$$

B E_{11} algebra in 4D

In this appendix we give the $E_{11} \ltimes_s l_1$ algebra decomposed into representations of $GL\,(4) \times SL\,(8)$. The set of generators of this algebra was given in Section 3.4 up to level 2. This algebra was previously constructed in [48, 54]. We will first give

the commutation relations of level 0 generators with the rest of E_{11} algebra. The commutation relations of any generator with $K^a{}_b$ are

$$
\begin{aligned}
&[K^a{}_b,\, K^c{}_d] = \delta^c_b\, K^a{}_d - \delta^a_d\, K^c{}_b, && \left[K^a{}_b,\, R^I{}_J\right] = 0, \quad \left[K^a{}_b,\, R^{I_1\ldots I_4}\right] = 0,\\
&\left[K^a{}_b,\, R^{cI_1I_2}\right] = \delta^c_b\, R^{aI_1I_2}, && [K^a{}_b,\, R^c{}_{I_1I_2}] = \delta^c_b\, R^a{}_{I_1I_2},\\
&[K^a{}_b,\, R_{cI_1I_2}] = -\,\delta^a_c\, R_{bI_1I_2}, && \left[K^a{}_b,\, R_c{}^{I_1I_2}\right] = -\,\delta^a_c\, R_b{}^{I_1I_2},\\
&\left[K^a{}_b,\, \hat{K}^{cd}\right] = 2\,\delta^{(c}_b\, \hat{K}^{|a|d)}, && \left[K^a{}_b,\, \hat{K}_{cd}\right] = -\,2\,\delta^a_{(c}\, \hat{K}_{|b|d)},\\
&\left[K^a{}_b,\, R^{a_1a_2I}{}_J\right] = 2\,\delta^{[a_1}_b\, R^{|a|a_2]I}{}_J, && \left[K^a{}_b,\, R_{a_1a_2}{}^I{}_J\right] = -\,2\,\delta^a_{[a_1}\, R_{|b|a_2]}{}^I{}_J,\\
&\left[K^a{}_b,\, R^{a_1a_2I_1\ldots I_4}\right] = 2\,\delta^{[a_1}_b\, R^{|a|a_2]I_1\ldots I_4}, && [K^a{}_b,\, R_{a_1a_2I_1\ldots I_4}] = -\,2\,\delta^a_{[a_1}\, R_{|b|a_2]I_1\ldots I_4}.
\end{aligned}
\tag{B.1}
$$

The commutators with $SL(8)$ generator $R^I{}_J$ are given by

$$
\begin{aligned}
\left[R^I{}_J,\, R^K{}_L\right] &= \delta^K_J\, R^I{}_L - \delta^I_L\, R^K{}_J,\\
\left[R^I{}_J,\, R^{I_1\ldots I_4}\right] &= 4\,\delta^{[I_1}_J\, R^{|I|I_2I_3I_4]} - \frac{1}{2}\,\delta^I_J\, R^{I_1\ldots I_4},\\
\left[R^I{}_J,\, R^{aI_1I_2}\right] &= 2\,\delta^{[I_1}_J\, R^{a|I|I_2]} - \frac{1}{4}\,\delta^I_J\, R^{aI_1I_2},\\
\left[R^I{}_J,\, R^a{}_{I_1I_2}\right] &= -\,2\,\delta^I_{[I_1}\, R^a{}_{|J|I_2]} + \frac{1}{4}\,\delta^I_J\, R^a{}_{I_1I_2},\\
\left[R^I{}_J,\, R_{aI_1I_2}\right] &= -\,2\,\delta^I_{[I_1}\, R_{a|J|I_2]} + \frac{1}{4}\,\delta^I_J\, R_{aI_1I_2},\\
\left[R^I{}_J, R_a{}^{I_1I_2}\right] &= 2\,\delta^{[I_1}_J\, R_a{}^{|I|I_2]} - \frac{1}{4}\,\delta^I_J\, R_a{}^{I_1I_2},\\
\left[R^I{}_J,\, \hat{K}^{(ab)}\right] &= 0, \quad \left[R^I{}_J,\, \hat{K}_{(ab)}\right] = 0,\\
\left[R^I{}_J,\, R^{a_1a_2K}{}_L\right] &= \delta^K_J\, R^{a_1a_2I}{}_L - \delta^I_L\, R^{a_1a_2K}{}_J,\\
\left[R^I{}_J,\, R_{a_1a_2}{}^K{}_L\right] &= \delta^K_J\, R_{a_1a_2K}{}^I{}_L - \delta^I_L\, R_{a_1a_2}{}^K{}_J,\\
\left[R^I{}_J,\, R^{a_1a_2I_1\ldots I_4}\right] &= 4\,\delta^{[I_1}_J\, R^{a_1a_2|I|I_2I_3I_4]} - \frac{1}{2}\,\delta^I_J\, R^{a_1a_2I_1\ldots I_4},\\
\left[R^I{}_J,\, R_{a_1a_2I_1\ldots I_4}\right] &= -\,4\,\delta^I_{[I_1}\, R_{a_1a_2|J|I_2I_3I_4]} + \frac{1}{2}\,\delta^I_J\, R_{a_1a_2I_1\ldots I_4},
\end{aligned}
\tag{B.2}
$$

The commutators with the other E_7 generators $R^{I_1\ldots I_4}$ generators are given by

$$\begin{aligned}
\left[R^{I_1\ldots I_4},\, R^{J_1\ldots J_4}\right] &= -\frac{1}{36}\,\varepsilon^{I_1\ldots I_4[J_1J_2J_3|L|}\,R^{J_4]}{}_L,\\
\left[R^{I_1\ldots I_4},\, R^{aJ_1J_2}\right] &= \frac{1}{24}\,\varepsilon^{I_1\ldots I_4J_1\ldots J_4}\,R^a{}_{J_3J_4},\\
\left[R^{I_1\ldots I_4},\, R^a{}_{J_1J_2}\right] &= \delta^{[I_1I_2}_{J_1J_2}\,R^{aI_3I_4]},\\
\left[R^{I_1\ldots I_4},\, R_{aJ_1J_2}\right] &= \delta^{[I_1I_2}_{J_1J_2}\,R_a{}^{I_3I_4]},\\
\left[R^{I_1\ldots I_4},\, R_a{}^{J_1J_2}\right] &= \frac{1}{24}\,\varepsilon^{I_1\ldots I_4J_1\ldots J_4}\,R_{aJ_3J_4},\\
\left[R^{I_1\ldots I_4},\, R^{a_1a_2I}{}_J\right] &= -4\,\delta^{[I_1}_J\,R^{a_1a_2|I|I_2I_3I_4]} + \frac{1}{2}\,\delta^I_J\,R^{a_1a_2I_1\ldots I_4},\\
\left[R^{I_1\ldots I_4},\, R_{a_1a_2}{}^I{}_J\right] &= -\frac{1}{6}\,\varepsilon^{I_1\ldots I_4J_1J_2J_3I}\,R_{a_1a_2J_1J_2J_3J} + \frac{1}{48}\,\delta^I_J\,\varepsilon^{I_1\ldots I_4J_1\ldots J_4}\,R_{a_1a_2J_1\ldots J_4},\\
\left[R^{I_1\ldots I_4}, R^{a_1a_2J_1\ldots J_4}\right] &= \frac{1}{36}\varepsilon^{I_1\ldots I_4[J_1J_2J_3|L|}R^{a_1a_2J_4]}{}_L,\\
\left[R^{I_1\ldots I_4},\, R_{a_1a_2J_1\ldots J_4}\right] &= -\frac{2}{3}\,\delta^{[I_1I_2I_3}_{[J_1J_2J_3}\,R_{a_1a_2}{}^{I_4]}{}_{J_4]},\\
\left[R^{I_1\ldots I_4},\, \hat{K}^{(ab)}\right] &= 0, \qquad \left[R^{I_1\ldots I_4},\, \hat{K}_{(ab)}\right] = 0.
\end{aligned} \tag{B.3}$$

The commutators of the positive level one E_{11} generators with each other are given by

$$\begin{aligned}
\left[R^{aI_1I_2},\, R^{bI_3I_4}\right] &= -12\,R^{abI_1\ldots I_4},\\
[R^a{}_{I_1I_2}, R^b{}_{I_3I_4}] &= \frac{1}{2}\,\varepsilon_{I_1\ldots I_4J_1\ldots J_4}\,R^{abJ_1\ldots J_4},\\
\left[R^{aI_1I_2},\, R^b{}_{J_1J_2}\right] &= 4\,\delta^{[I_1}_{[J_1}R^{abI_2]}{}_{J_2]} + 2\,\delta^{I_1I_2}_{J_1J_2}\,\hat{K}^{(ab)}.
\end{aligned} \tag{B.4}$$

The equivalent commutators for the negative level E_{11} generators are

$$\begin{aligned}
[R_{aI_1I_2},\ R_{bI_3I_4}] &= -12\,R_{abI_1\ldots I_4},\\
[R_a{}^{I_1I_2},\, R_b{}^{I_3I_4}] &= \frac{1}{2}\,\varepsilon^{I_1\ldots I_4J_1\ldots J_4}\,R_{a_1a_2J_1\ldots J_4},\\
\left[R_{aI_1I_2},\, R_b{}^{J_1J_2}\right] &= 4\,\delta^{[J_1}_{[I_1}\,R_{ab}{}^{J_2]}{}_{I_2]} + 2\,\delta^{J_1J_2}_{I_1I_2}\,\hat{K}_{(ab)}.
\end{aligned} \tag{B.5}$$

The commutators between the level 1 and -1 E_{11} generators are given by

$$\begin{aligned}
&\left[R^{aI_1I_2},\, R_b{}^{I_3I_4}\right] = -12\,\delta^a_b\,R^{I_1\ldots I_4}, \quad \left[R^a_{I_1I_2},\, R_{bI_3I_4}\right] = \frac{1}{2}\,\delta^a_b\,\varepsilon_{I_1\ldots I_4J_1\ldots J_4}\,R^{J_1\ldots J_4},\\
&\left[R^{aI_1I_2},\, R_{bJ_1J_2}\right] = 2\,\delta^{I_1I_2}_{J_1J_2}\,K^a{}_b + 4\,\delta^a_b\,\delta^{[I_1}_{[J_1}\,K^{I_2]}{}_{J_2]} - \delta^a_b\,\delta^{I_1I_2}_{J_1J_2}\,K^c{}_c,\\
&\left[R^a{}_{I_1I_2},\, R_b{}^{J_1J_2}\right] = -2\,\delta^{J_1J_2}_{I_1I_2}\,K^a{}_b + 4\,\delta^a_b\,\delta^{[J_1}_{[I_1}\,K^{J_2]}{}_{I_2]} + \delta^a_b\,\delta^{J_1J_2}_{I_1I_2}\,K^c{}_c,
\end{aligned} \tag{B.6}$$

The commutators with the level 2 and level -1 E_{11} generators are given by

$$\begin{aligned}
\big[R^{abI}{}_J,\, R_{cI_1I_2}\big] &= -\,4\,\delta^{[a}_c\,\delta^I_{[I_1}R^{b]}{}_{|J|I_2]} + \frac{1}{2}\,\delta^{[a}_c\,\delta^I_J\,R^{b]}{}_{I_1I_2},\\
\big[R^{abI}{}_J,\, R_c{}^{I_1I_2}\big] &= 4\,\delta^{[a}_c\delta^{[I_1}_J\,R^{b]|I|I_2]} - \frac{1}{2}\,\delta^{[a}_c\,\delta^I_J\,R^{b]I_1I_2},\\
\big[R^{abI_1\ldots I_4},\, R_{cJ_1J_2}\big] &= 2\,\delta^{[a}_c\,\delta^{[I_1I_2}_{J_1J_2}\,R^{b]I_3I_4]},\\
\big[R^{abI_1\ldots I_4},\, R_c{}^{I_5I_6}\big] &= \frac{1}{12}\,\varepsilon^{I_1\ldots I_8}\,\delta^{[a}_c\,R^{b]}{}_{I_7I_8},\\
\Big[\hat{K}^{ab},\, R_{cI_1I_2}\Big] &= -\,\delta^{(a}_c\,R^{b)}{}_{I_1I_2},\\
\Big[\hat{K}^{ab},\, R_c{}^{I_1I_2}\Big] &= -\,\delta^{(a}_c\,R^{b)J_1J_2}.
\end{aligned} \tag{B.7}$$

Finally, the commutators of level -2 with the level 1 E_{11} generators are

$$\begin{aligned}
\big[R_{ab}{}^I{}_J,\, R^{cI_1I_2}\big] &= -\,4\,\delta^c_{[a}\delta^{[I_1}_J\,R_{b]}{}^{|I|I_2]} + \frac{1}{2}\,\delta^c_{[a}\,\delta^I_J\,R_{b]}{}^{I_1I_2},\\
\big[R_{ab}{}^I{}_J,\, R^c{}_{I_1I_2}\big] &= 4\,\delta^c_{[a}\,\delta^I_{[I_1}\,R_{b]|J|I_2]} - \frac{1}{2}\,\delta^c_{[a}\,\delta^I_J\,R_{b]I_1I_2},\\
\big[R_{ab}{}^{I_1\ldots I_4},\, R^c{}_{J_1J_2}\big] &= 2\,\delta^c_{[a}\,\delta^{[I_1I_2}_{J_1J_2}\,R_{b]}{}^{I_3I_4]},\\
\big[R_{ab}{}^{I_1\ldots I_4},\, R^{cI_5I_6}\big] &= \frac{1}{12}\,\varepsilon^{I_1\ldots I_8}\,\delta^c_{[a}\,R_{b]I_7I_8},\\
\Big[\hat{K}_{ab},\, R^c{}_{I_1I_2}\Big] &= -\,\delta^c_{(a}\,R_{b)I_1I_2},\\
\Big[\hat{K}_{ab},\, R^{cI_1I_2}\Big] &= -\,\delta^c_{(a}\,R_{b)}{}^{I_1I_2}.
\end{aligned} \tag{B.8}$$

The Cartan involution preserves the above commutators and is given by

Generator	$I_c\big(\text{Generator}\big)$
$K^a{}_b$	$-\,K^b{}_a$
$R^I{}_J$	$-\,R^J{}_I$
$R^{I_1\ldots I_4}$	$-\,\star\,R^{I_1\ldots I_4}$
$R^{aI_1I_2}$	$-\,R_{aI_1I_2}$
$R^a{}_{I_1I_2}$	$R_a{}^{I_1I_2}$
$R^{a_1a_2I}{}_J$	$-\,R_{a_1a_2}{}^J{}_I$
$R^{a_1a_2I_1\ldots I_4}$	$R_{a_1a_2I_1\ldots I_4}$
$\hat{K}^{ab}$	$-\,\hat{K}_{ab}$

where $\star R^{I_1\ldots I_4} = \frac{1}{4!}\,\varepsilon^{I_1\ldots I_4 J_1\ldots J_4}\,R^{J_1\ldots J_4}$.

We now give the action of E_{11} on the l_1 representation generators. The commutation relations with level 0 generators of E_{11} are given by

$$
\begin{aligned}
[K^a{}_b,\, P_c] &= -\,\delta^a_c\,P_b + \frac{1}{2}\,\delta^a_b\,P_c,\\
\big[K^a{}_b,\, Z^{I_1I_2}\big] &= \frac{1}{2}\,\delta^a_b\,Z^{I_1I_2}, \quad [K^a{}_b,\, Z_{I_1I_2}] = \frac{1}{2}\,\delta^a_b\,Z_{I_1I_2},\\
[K^a{}_b,\, Z^c] &= \delta^c_b\,Z^a + \frac{1}{2}\,\delta^a_b\,Z^c, \quad \big[K^a{}_b,\, Z^{cI}{}_J\big] = \delta^c_b\,Z^{aI}{}_J + \frac{1}{2}\,\delta^a_b\,Z^{cI}{}_J,\\
\big[K^a{}_b,\, Z^{cI_1\ldots I_4}\big] &= \delta^c_b\,Z^{aI_1\ldots I_4} + \frac{1}{2}\,\delta^a_b\,Z^{cI_1\ldots I_4}, \quad \big[R^I{}_J,\, P_c\big] = 0,\\
\big[R^I{}_J,\, Z^{I_1I_2}\big] &= 2\,\delta^{[I_1}_J\,Z^{|I|I_2]} - \frac{1}{4}\,\delta^I_J\,Z^{I_1I_2},\\
\big[R^I{}_J,\, Z_{I_1I_2}\big] &= -\,2\,\delta^I_{[I_1}\,Z_{|J|I_2]} + \frac{1}{4}\,\delta^I_J\,Z_{I_1I_2},\\
\big[R^I{}_J,\, Z^a\big] &= 0, \quad \big[R^I{}_J,\, Z^{aK}{}_L\big] = \delta^K_J\,Z^{aI}{}_L - \delta^I_L\,Z^{aK}{}_J,\\
\big[R^I{}_J,\, Z^{aI_1\ldots I_4}\big] &= 4\,\delta^{[I_1}_J\,Z^{a|I|I_2\ldots I_4]} - \frac{1}{2}\,\delta^I_J\,Z^{aI_1\ldots I_4},\\
\big[R^{I_1\ldots I_4},\, P_a\big] &= 0, \quad \big[R^{I_1\ldots I_4},\, Z^{J_1J_2}\big] = \frac{1}{24}\,\varepsilon^{I_1\ldots I_4J_1\ldots J_4}\,Z_{J_3J_4},\\
\big[R^{I_1\ldots I_4},\, Z_{J_1J_2}\big] &= \delta^{[I_1I_2}_{J_1J_2}\,Z^{I_3I_4]}, \quad \big[R^{I_1\ldots I_4},\, Z^a\big] = 0,\\
\big[R^{I_1\ldots I_4},\, Z^{aI}{}_J\big] &= -\,\frac{4}{3}\,\delta^{[I_1}_J\,Z^{a|I|I_2I_3I_4]} + \frac{1}{6}\,\delta^I_J\,Z^{aI_1\ldots I_4},\\
\big[R^{I_1\ldots I_4},\, Z^{aJ_1\ldots J_4}\big] &= \frac{1}{12}\,\varepsilon^{J_1\ldots J_4[I_1I_2I_3|K|}\,Z^{aI_4]}{}_K.
\end{aligned}
\tag{B.9}
$$

The commutators with the E_{11} level 1 generators are given by

$$
\begin{aligned}
\big[R^{aI_1I_2},\, P_b\big] &= \delta^a_b\,Z^{I_1I_2},\\
[R^a{}_{I_1I_2},\, P_b] &= \delta^a_b\,Z_{I_1I_2},\\
\big[R^{aI_1I_2},\, Z^{J_1J_2}\big] &= -\,Z^{aI_1I_2J_1J_2},\\
\big[R^a{}_{I_1I_2},\, Z^{J_1J_2}\big] &= \delta^{[J_1}_{[I_1}\,Z^{aJ_2]}{}_{I_2]} + \delta^{J_1J_2}_{I_1I_2}\,Z^a,\\
\big[R^{aI_1I_2},\, Z_{J_1J_2}\big] &= \delta^{[I_1}_{[J_1}\,Z^{aI_2]}{}_{J_2]} - \delta^{I_1I_2}_{J_1J_2}\,Z^a,\\
[R^a{}_{I_1I_2},\, Z_{J_1J_2}] &= \frac{1}{24}\,\varepsilon_{I_1I_2J_1J_2K_1\ldots K_4}\,Z^{aK_1\ldots K_4}.
\end{aligned}
\tag{B.10}
$$

The commutators with the E_{11} level 2 generators are given by

$$\begin{aligned}
\left[\hat{K}^{(a_1a_2)},\, P_a\right] &= \delta_a^{(a_1}\, Z^{a_2)},\\
\left[R^{a_1a_2I}{}_J,\, P_a\right] &= -\frac{1}{2}\,\delta_a^{[a_1}\, Z^{a_2]I}{}_J,\\
\left[R^{a_1a_2I_1\ldots I_4},\, P_a\right] &= -\frac{1}{6}\,\delta_a^{[a_1}\, Z^{a_2]I_1\ldots I_4}.
\end{aligned} \tag{B.11}$$

The commutators with the E_{11} level -1 generators are

$$\begin{aligned}
&[R_{aI_1I_2},\, P_b] = 0, && \left[R_{aI_1I_2},\, Z^{J_1J_2}\right] = 2\,\delta^{I_1I_2}_{J_1J_2}\, P_a,\\
&[R_{aI_1I_2},\, Z_{J_1J_2}] = 0, && \left[R_a^{I_1I_2},\, P_b\right] = 0,\\
&\left[R_a{}^{I_1I_2},\, Z^{J_1J_2}\right] = 0, && \left[R_a{}^{I_1I_2},\, Z_{J_1J_2}\right] = -\,2\,\delta^{J_1J_2}_{I_1I_2}\, P_a,\\
&\left[R_{aI_1I_2},\, Z^b\right] = -\,2\,\delta^b_a\, Z_{I_1I_2}, && \left[R_{aI_1I_2},\, Z^{bI}{}_J\right] = -\,8\,\delta^b_a\,\delta^I_{[I_1}\, Z_{I_2]J},\\
&\left[R_{aI_1I_2},\, Z^{bJ_1\ldots J_4}\right] = -\,12\,\delta^b_a\,\delta^{[J_1J_2}_{I_1I_2}\, Z^{J_3J_4]}, && \left[R_a{}^{I_1I_2},\, Z^b\right] = -\,2\,\delta^b_a\, Z^{I_1I_2},\\
&\left[R_a{}^{I_1I_2},\, Z^{bI}{}_J\right] = 8\,\delta^b_a\,\delta^{[I_1}_J\, Z^{I_2]I}, && \left[R_a{}^{I_1I_2},\, Z^{bJ_1\ldots J_4}\right] = -\,\frac{1}{2}\,\delta^b_a\,\varepsilon^{J_1\ldots J_2I_1\ldots I_4}\, Z_{I_3I_4}.
\end{aligned} \tag{B.12}$$

C Author's publication list

This is the list of all papers co-written by the author. Papers [1] - [5] form the basis of this thesis, while paper [6] is on unrelated topic.

[1] A. G. Tumanov and P. West *"E_{11}, Romans theory and higher level duality relations"* [arXiv:1611.03369 [hep-th]]

[2] A. G. Tumanov and P. West *"E_{11} in $11D$"* Phys.Lett.B 758:278–285, 2016 [arXiv:1601.03974v3 [hep-th]]

[3] A. G. Tumanov and P. West *"E_{11} must be a symmetry of strings and branes"* Phys.Lett.B 759:663–671, 2016 [arXiv:1512.01644v3 [hep-th]]

[4] A. G. Tumanov and P. West *"E_{11} and exceptional field theory"* Int. J. Mod. Phys. A 31, 1650066 (2016) [arXiv:1507.08912 [hep-th]]

[5] A. G. Tumanov and P. West *"Generalised vielbeins and non-linear realisations"* J. High Energ. Phys. (2014) 2014: 9 [arXiv:1405.7894 [hep-th]]

[6] D. Diakonov, A. G. Tumanov and A. A. Vladimirov *"Low-energy general relativity with torsion: a systematic derivative expansion"* Phys.Rev.D84:124042, 2011 [arXiv:1104.2432 [hep-th]]

References

[1] E. Cremmer, B. Julia and J. Scherk *"Supergravity Theory in Eleven Dimensions"* Phys. Lett. 76B (1978) 409.

[2] C. Campbell and P. West *"$N = 2$ $D = 10$ nonchiral supergravity and its spontaneous compactification"* Nucl. Phys. B243 (1984) 112.

[3] M. Huq and M. Namazie *"Kaluza–Klein supergravity in ten dimensions"* Class. Q. Grav. 2 (1985).

[4] F. Giani and M. Pernici *"$N = 2$ supergravity in ten dimensions"* Phys. Rev. D30 (1984) 325.

[5] J. Schwarz and P. West *"Symmetries and Transformation of Chiral $N = 2$ $D = 10$ Supergravity"* Phys. Lett. 126B (1983) 301.

[6] P. Howe and P. West *"The Complete $N = 2$, $D = 10$ Supergravity"* Nucl.Phys. B238 (1984) 181-220

[7] J. Schwarz *"Covariant Field Equations of Chiral $N = 2$ $D = 10$ Supergravity"* Nucl.Phys. B226 (1983) 269

[8] B. Julia *"Group Disintegrations"* Conf.Proc. C8006162 (1980) 331-350

[9] B. Julia *"Dualities in the classical supergravity limits"* LPTENS-98/07 [arXiv:9805083 [hep-th]]

[10] E. Cremmer, B. Julia *"The $N = 8$ Supergravity Theory. 1. The Lagrangian"* Phys.Lett. B80 (1978) 48

[11] E. Cremmer, B. Julia *"The $SO(8)$ Supergravity"* Nucl.Phys. B159 (1979) 141

[12] N. Marcus, J. H. Schwarz *"Three-dimensional supergravity theories"* Nucl.Phys. B228 (1983) 145

[13] H. Nicolai *"The Integrability of $N = 16$ Supergravity"* Phys.Lett. B194 (1987) 402

[14] H. Nicolai, N. P. Warner *"The Structure of $N = 16$ Supergravity in two dimensions"* Commun.Math.Phys. 125 (1989) 369

[15] E. Cremmer, B. Julia, H. Lu and C. Pope *"Dualisation of dualities II: Twisted selfduality of doubled fields and superdualities"* Nucl.Phys.B535:242-292,1998 [arXiv:9806106 [hep-th]]

[16] P. West *"Hidden Superconformal Symmetries of M theory"* JHEP 0008:007, 2000 [arXiv:0005270 [hep-th]]

[17] P. West *"E_{11} and M Theory"* Class.Quant.Grav. 18 (2001) 4443-4460 [arXiv:0104081v2 [hep-th]]

[18] P. West *"E_{11}, $SL(32)$ and Central Charges"* Phys.Lett.B 575:333-342, 2003 [arXiv:0307098v2 [hep-th]]

[19] I. Schnakenburg and P. West *"Kac-Moody Symmetries of IIB Supergravity"* Phys.Lett. 517B (2001) 421-428 [arXiv:0107181 [hep-th]]

[20] P. West *"E_{11} origin of Brane charges and U-duality multiplets"* JHEP 0408 (2004) 052 [arXiv:0406150 [hep-th]]

[21] F. Riccioni and P. West *"The E_{11} origin of all maximal supergravities"* JHEP 0707(2007) 063 [arXiv:0705.0752 [hep-th]]

[22] F. Riccioni and P. West, *"E_{11}-extended spacetime and gauged supergravities"* JHEP 0802 (2008) 039 [arXiv:0712.1795 [hep-th]]

[23] F. Riccioni and P. West *"Local E_{11}"* JHEP 0904 (2009) 051 [arXiv:0902.4678 [hep-th]]

[24] F. Riccioni, D. Steele and P. West *"The E_{11} origin of all maximal supergravities - the hierarchy of field-strengths"* JHEP 0909 (2009) 095 [arXiv:0906.1177 [hep-th]]

[25] A. G. Tumanov and P. West *"Generalised vielbeins and non-linear realisations"* J. High Energ. Phys. (2014) 2014: 9 [arXiv:1405.7894 [hep-th]]

[26] A. G. Tumanov and P. West *"E_{11} and exceptional field theory"* Int. J. Mod. Phys. A 31, 1650066 (2016) [arXiv:1507.08912 [hep-th]]

[27] A. G. Tumanov and P. West *"E_{11} must be a symmetry of strings and branes"* Phys.Lett.B 759:663–671, 2016 [arXiv:1512.01644v3 [hep-th]]

[28] A. G. Tumanov and P. West *"E_{11} in $11D$"* Phys.Lett.B 758:278–285, 2016 [arXiv:1601.03974v3 [hep-th]]

[29] A. G. Tumanov and P. West *"E_{11}, Romans theory and higher level duality relations"* [arXiv:1611.03369 [hep-th]]

[30] N. Lambert and P. West *"Coset Symmetries in Dimensionally Reduced Bosonic String Theory"* Nucl.Phys. B615 (2001) 117-132 [arXiv:0107209 [hep-th]]

[31] T. Nutma *SimpLie, a simple program for Lie algebras* https://code.google.com/p/simplie/

[32] P. West *"Introduction to Strings and Branes"* Cambridge University Press (2012) ISBN: 978-0521817479

[33] S. Coleman, J. Wess and B. Zumino *"Structure of phenomenological Lagrangians 1"* Phys.Rev., 177 (1969) 2239

[34] C. Callan, S. Coleman, J. Wess and B. Zumino *"Structure of phenomenological Lagrangians 2"* Phys. Rev., 177 (1969) 2247

[35] P. West *"Generalised Space-time and Gauge Transformations"* KCL-MTH-14-05 [arXiv:1403.6395 [hep-th]]

[36] A. Borisov and V. Ogievetsky *"Theory of dynamical affine and conformal symmetries as gravity theory of the gravitational field"* Theor. Math. Phys., 21 (1975) 1179

[37] E. Cremmer *"Supergravities in 5 dimensions"*
In Cambridge 1980, Proceedings, Superspace and supergravity 267-282

[38] P. West *"E_{11}, ten forms and supergravity"* KCL-MTH-05-13
[arXiv:0511153 [hep-th]]

[39] N. Gromov, A. Tumanov and P. West *to appear*

[40] M. Duff *"Duality Rotations In String Theory"* Nucl. Phys. B 335 (1990) 610

[41] M. Duff and J. Lu *"Duality rotations in membrane theory"*
Nucl. Phys. B347 (1990) 394

[42] A. Tseytlin *"Duality Symmetric Formulation Of String World Sheet Dynamics"*
Phys.Lett. B242 (1990) 163

[43] A. Tseytlin *"Duality Symmetric Closed String Theory And Interacting Chiral Scalars"* Nucl. Phys. B 350, 395 (1991)

[44] M. Duff *"Duality rotations in membrane theory"* Nucl. Phys. B347 (1990) 394

[45] W. Siegel *"Superspace duality in low-energy superstrings"*
Phys. Rev. D 48 (1993) 2826 [arXiv:9305073 [hep-th]].

[46] W. Siegel *"Two vierbein formalism for string inspired axionic gravity"*
Phys. Rev. D 47 (1993) 5453 [arXiv:9302036 [hep-th]]

[47] C. Hull and B. Zwiebach *"Double Field Theory"* JHEP 0909 (2009) 099
[arXiv:0904.4664 [hep-th]]

[48] P. West *"E_{11}, Generalised space-time and equations of motion in four dimensions"*
JHEP 1212 (2012) 068 [arXiv:1206.7045 [hep-th]]

[49] P. West *"Generalised Geometry, eleven dimensions and E_{11}"*
JHEP 1202 (2012) 018 [arXiv:1111.1642 [hep-th]]

[50] P. West *"Dual gravity and E_{11}"* [arXiv:1411.0920 [hep-th]]

[51] N. Boulanger, P. Sundell and P. West *"Gauge fields and infinite chains of dualities"* JHEP 1509 (2015) 192 [arXiv:1502.07909 [hep-th]]

[52] L. J. Romans *"Massive $N = 2a$ Supergravity in Ten Dimensions"* Phys. Lett. 169B (1986) 374

[53] A. Kleinschmidt, I. Schnakenburg and P. West *"Very-extended Kac-Moody algebras and their interpretation at low levels"* Class.Quant.Grav.21:2493-2525,2004 [arXiv:0309198 [hep-th]]

[54] A. Kleinschmidt and P. West *"Representations of G+++ and the role of space-time"* JHEP0402: 033, 2004 [arXiv:0312247 [hep-th]]