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Experimental implementation of a four-player quantum game

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Abstract. Game theory is central to the understanding of competitive interactions arising in many fields, from the social and physical sciences to economics. Recently, as the definition of information is generalized to include entangled quantum systems, quantum game theory has emerged as a framework for understanding the competitive flow of quantum information. Up till now, only two- and three-player quantum games have been demonstrated with restricted strategy sets. Here, we report the first experiment that implements a four-player quantum minority game over tunable four-partite entangled states encoded in the polarization of single photons. Experimental application of appropriate player strategies gives equilibrium payoff values well above those achievable in the classical game. These results are in excellent quantitative agreement with our theoretical analysis of the symmetric Pareto optimal strategies. Our results demonstrate for the first time how nontrivial equilibria can arise in a competitive situation involving quantum agents.

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Contents

1. Introduction **2**

2. Theory **6**

3. Experimental implementation **8**

3.1. State observation 8

3.2. Implementation of the game 8

3.3. Results 9

4. Conclusion **12**

Acknowledgments **12**

References **12**

1. Introduction

Originally developed for economics, the impact of game theory in this field is now pervasive and has spread to many other disciplines as diverse as evolutionary biology [1], the social sciences [2] and international conflict [3]. Two important ideas in game theory are the *Nash equilibrium* (NE), the result from which no player can improve their payoff by a unilateral change in strategy, and the *Pareto optimal* (PO) outcome, the one from which no player can improve their payoff without another player being worse off. The former can be considered as the equilibrium that is best for each player individually, while the latter is generally the best result for the players as a group.

In the search for new applications of quantum information processing based on competitive interactions between agents, the appropriate language is quantum game theory. Applications can be based on exchanges between several agents, who share entanglement, in contrast to direct classical communication. The sharing of an entangled resource permits competitive von Neumann-type games, with applications such as quantum auctions [4], quantum voting [5] and quantum communication [6]. This is distinct from cooperative games that arise when agents are permitted to communicate either directly or through a third party. In quantum games, entanglement is used as a resource that can increase the payoffs of well-known game-theoretic equilibria [7].

There have been nuclear magnetic resonance implementations of the two- and three-player quantum prisoner's dilemma [8, 9] and an optical implementation of a two-player quantum prisoners' dilemma [10]. However, for these games the players' strategic spaces were limited to a subset of unitary operators. While the rules of the game may impose a restriction on the available strategies, and much study of such games has been made, it is most general to consider the full space of unitary strategies. Here, we report for the first time the implementation of a multiplayer quantum game, with the full unitary space of strategies, driven by different four-particle entangled states [11, 12]. When played with the full set of unitary strategies, two-player quantum games do not see the enhancement of the pure strategy equilibrium payoffs [13, 14]; however, these can arise in multiplayer settings [7]. The practical exploration of quantum games allows us to learn about the nature of quantum information in a real situation. Figure 1 compares the classical and quantum versions of a four-player game.

We choose to implement a four-player minority game. The minority game arises as a simple multi-agent model for studying strategic decision making within a group of agents [15].

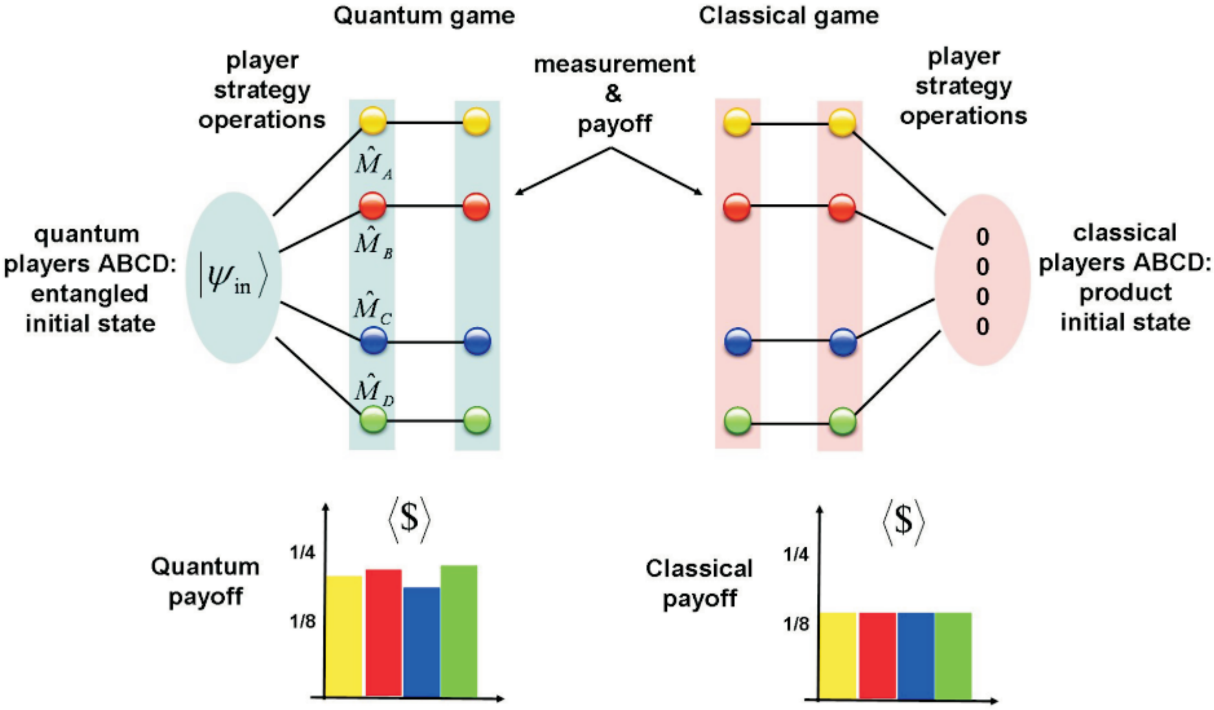

Figure 1. Schematic picture showing the flow of information in a four-player classical game (right) and a four-player quantum game (left). In the quantum version, the players utilize an entangled state as a resource; however, in both versions there is no communication between the players. For the particular case of the minority game, the equilibrium payoffs are indicated in the bar graph, with those of the quantum case being the values achieved experimentally with our setup.

Its classical version has been used as an iterative model of buying and selling in a stockmarket [16]–[18]. In each play, the agents independently select between one of two options, labeled '0' and '1' ('buy' and 'sell'). Those that choose the minority are awarded a payoff of one unit, while the others receive no payoff. Players can utilize knowledge of past successful choices to optimize their strategy. In the quantum situation, we can represent each player's binary choice by the polarization of a photon as in figure 2. Quantum versions of a one-shot minority game have generated some theoretical interest [7], [19]–[22]. In this paper, we report on the first experimental implementation of the quantum minority game (QMG).

We begin with a theoretical exploration of the QMG. With the particular experiment in mind, we choose the initial state to belong to a continuous set of four-partite entangled states involving the GHZ state and products of Bell pairs. We calculate the NE and PO strategy profiles as a function of the initial state. Our method is a general one that can be applied to any quantum game that has a similar protocol. Our quantization of the minority game is described as follows and is shown schematically in figure 1. Each of four players is given one qubit from a known four-partite entangled state. This state is an element of the subspace spanned by the four-qubit GHZ state and a product state of two Bell pairs. The players are permitted to act on their qubit with any local unitary operation. The choice of such an operator is the player's

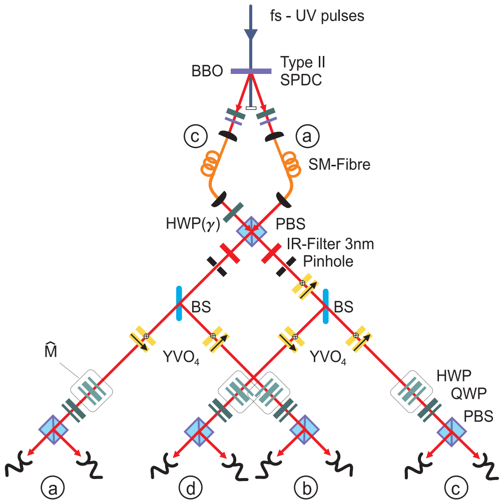

Figure 2. Setup for the playing four-player QMG. The photons of second-order emission of type II SPDC interfere at a PBS and are further split at non-polarizing beam splitters (BS). The family of four-photon entangled states is observed in case a photon is detected in each spatial mode a, b, c, d. A particular state of the family is determined by the setting γ of an HWP prior to the interference. Phases are set using pairs of YVO_4 crystals. The photons are spectrally and spatially filtered by interference filters (IR) and pinholes, respectively, to increase the interference contrast. The operator $\hat{M}$ corresponding to the strategy of each player is realized by a series of quarter-, half- and quarter-waveplates (QWP). HWP and QWP in conjunction with a PBS are used for polarization analysis of the photons.

strategy. During this stage, coherence is maintained and no communication between the players is permitted. After the players' actions, a referee measures the qubits in the computational basis and awards payoffs using the classical payoff scheme. In the dominant protocol of quantum games due to Eisert *et al* [7, 23], an entangling gate is used to produce a GHZ state from an initial state of $|00\ldots0\rangle$. After the players' actions, the inverse operator is applied to the multi-partite state. For the minority game, this last operator only has the effect of interchanging between states where the same players win and so can be omitted without changing the expected payoffs. Our arrangement is consistent with the generalized quantum game formalisms of Lee and Johnson [24] and Gutoski and Watrous [25], and is particularly suited to an implementation using entangled photon states and linear optics quantum logic.

A four-player QMG was first examined by Benjamin and Hayden [7], and later generalized to multiple players [19] and to the consideration of decoherence [20]. Formally, the QMG is played by computing the state

$$\rho_{\mathrm{final}} = (\hat{A}\otimes\hat{B}\otimes\hat{C}\otimes\hat{D})\rho_{\mathrm{in}}(\hat{A}\otimes\hat{B}\otimes\hat{C}\otimes\hat{D})^{\dagger}, \qquad (1)$$

where $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$ are the operators representing the strategies of the four players and ρ_{in} is some four-qubit input state. The qubits in ρ_{final} are subsequently measured in the computation basis and payoffs are awarded using the classical payoff matrix. Work to date has concentrated on the consideration of an initial GHZ state [7, 19, 20]. In this paper, we instead consider an initial state that is a superposition of the GHZ state with products of Bell pairs:

$$|\Psi(\alpha)\rangle = \frac{\alpha}{\sqrt{2}}(|HHHH\rangle + |VVVV\rangle) + \frac{\sqrt{1-\alpha^2}}{2}\Big(|HVHV\rangle + |HVVH\rangle + |VHHV\rangle + |VHVH\rangle\Big), \quad (2)$$

where $\alpha \in [0, 1]$ is an adjustable parameter. This particular tunable state has been made possible by a flexible utilization of a spontaneous parametric down-conversion (SPDC) photon source combined with multiphoton interference [11, 12]. Using it enables us to explore the particular role of four-partite entanglement, as opposed to pairs of two-party entanglement, in determining the optimal strategies and maximal payoffs in the game. Generally in the past, only four-partite entanglement has been considered. We show later that some four-partite entanglement is necessary to produce an enhancement in the equilibrium payoffs, but the optimal payoff does not scale linearly with the extent of this entanglement. The qubits are encoded in the polarization of single photons propagating in well-defined spatial modes. The computational basis states $|0\rangle$ and $|1\rangle$ are represented by the states $|H\rangle_i$ and $|V\rangle_i$, denoting the state of a single photon in the spatial mode i with linear horizontal and linear vertical polarization, respectively. The spatial mode will be evident from the context and hence the subscript will be omitted.

To make allowances for some loss of fidelity in the preparation of this state, the initial state will be written as the mixed state

$$\rho_{\text{in}} = f|\Psi_\alpha\rangle\langle\Psi_\alpha| + \frac{1-f}{16}\Sigma_{ijk\ell=0,1}|ijk\ell\rangle\langle ijk\ell|, \quad (3)$$

where $f \in [0, 1]$ is a measure of the fidelity of production of the desired initial state. The second term in (3) represents completely random noise [26]. The players' strategies are single-qubit unitary operators that can be parametrized in the form

$$\hat{M}(\theta, \beta_1, \beta_2) = \begin{pmatrix} e^{i\beta_1}\cos(\theta/2) & ie^{i\beta_2}\sin(\theta/2) \\ ie^{-i\beta_2}\sin(\theta/2) & e^{-i\beta_1}\cos(\theta/2) \end{pmatrix}, \quad (4)$$

where $\theta \in [0, \pi]$, $\beta_1, \beta_2 \in [-\pi, \pi]$ and $\hat{M}$ is applied independently by the players A, B, C and D. In our construction of the QMG, only the difference in the phases is relevant to the expected payoff so it suffices to use a restricted set of operators where $\beta \equiv \beta_1 = -\beta_2$.

In the case where $\alpha = 1$, i.e. for GHZ states, it is known that a symmetric NE occurs when all players choose the strategy [7]

$$\hat{M}\left(\frac{\pi}{2}, \frac{\pi}{8}, \frac{-\pi}{8}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\pi/8} & ie^{-i\pi/8} \\ ie^{i\pi/8} & e^{-i\pi/8} \end{pmatrix}. \quad (5)$$

Although the value $\beta_1 = -\beta_2 = \pi/8$ is not the unique optimum, it is a focal point [27] that attracts the attention of the players since it is the simplest optimum value, and therefore there is no great difficulty in arriving at this NE. Given that at most one player of the four can be in the minority, $\frac{1}{4}$ is the greatest average payoff that can be expected. This is realized with the above strategy when the initial state has maximum fidelity. As $f \to 0$, the payoff decreases to $\frac{1}{8}$, the optimal payoff in a one-shot classical minority game, where the players can do no better than choose 0 or 1 with equal probability. This is expected, since in the absence of entanglement the QMG cannot give any advantage over its classical counterpart.

2. Theory

We can search for a NE in the case of general α as follows. If there exists θ^*, β^* such that

$$\left\langle \$_{\rm D}(\hat{M}(\theta^*, \beta^*, -\beta^*)^{\otimes 4})\right\rangle \geqslant \left\langle \$_{\rm D}\left(\hat{M}(\theta^*, \beta^*, -\beta^*)^{\otimes 3} \otimes \hat{M}(\theta, \beta, -\beta)\right)\right\rangle \quad \forall\, \theta, \beta, \tag{6}$$

where $\$_{\rm D}(\cdot)$ denotes the payoff to the fourth player (Debra) for the indicated strategy profile, then $\hat{M}(\theta^*, \beta^*, -\beta^*)$ is a symmetric NE strategy. There is, in principle, no objection to asymmetric NE strategy profiles, where the players choose different strategies. However, such equilibria are necessarily non-unique: an interchange of strategies between players results in a new NE strategy profile. Hence, in practice these equilibria cannot be reliably achieved in the absence of communication between the players since the players do not know which equilibrium to choose. In contrast, although the symmetric equilibria that are studied in this paper are also not unique,[8] there is a simplest strategy, with the mimimum value of the phase angles β_1 and β_2 and with $\beta_1 = -\beta_2$, that acts as a focal point, attracting the players' attention and making it likely that all players will choose this strategy.

Necessary, but not sufficient, conditions for the existence of a symmetric NE are

$$\left.\frac{{\rm d}\langle \$_{\rm D}\rangle}{{\rm d}\theta}\right|_{\theta=\theta^*,\,\beta=\beta^*} = 0, \qquad \frac{{\rm d}\langle \$_{\rm D}\rangle}{{\rm d}\beta}|_{\theta=\theta^*,\,\beta=\beta^*} = 0, \tag{7a}$$

$$\left.\frac{{\rm d}^2\langle \$_{\rm D}\rangle}{{\rm d}\theta^2}\right|_{\theta=\theta^*,\,\beta=\beta^*} \leqslant 0, \qquad \frac{{\rm d}^2\langle \$_{\rm D}\rangle}{{\rm d}\beta^2}|_{\theta=\theta^*,\,\beta=\beta^*} \leqslant 0, \tag{7b}$$

where $\langle \$_{\rm D}\rangle$ is here the payoff on the right-hand side of (6). Inequalities in (7*b*) indicate a local maximum in the payoff to Debra; however, this may not be a global maximum. An equality in (7*b*) may mean a local maximum, minimum or an inflection point in the payoff. We shall now enumerate all the symmetric NE strategies by considering equations (7*a*)–(7*b*) over the range of $\alpha \in [0, 1]$.

If Alice, Bob and Charles use the strategy $\hat{M}(\theta, \beta, -\beta)$ while Debra plays $\hat{M}(\theta', \beta', -\beta')$, we can calculate the payoff to Debra using the payoff operator,

$$\hat{\$}_{\rm D} = |1110\rangle\langle 1110| + |0001\rangle\langle 0001|, \tag{8}$$

take derivatives with respect to Debra's parameters, θ' and β', and then set $\theta' = \theta$ and $\beta' = \beta$ since we are interested in symmetric strategy profiles:

$$\begin{aligned}\left.\frac{{\rm d}\langle \$_{\rm D}\rangle}{{\rm d}\theta'}\right|_{\theta'=\theta,\,\beta'=\beta} &= \frac{\sin 2\theta}{8}\Big[2\alpha^2 + 2\alpha\sqrt{2-2\alpha^2}\,\cos 4\beta \\ &\quad + (2\alpha^2 - 2 - 2\alpha\sqrt{2-2\alpha^2}\,\cos 4\beta - \alpha^2\cos^2 4\beta)\sin^2\theta\Big], \\ \frac{{\rm d}\langle \$_{\rm D}\rangle}{{\rm d}\beta'}|_{\theta'=\theta,\,\beta'=\beta} &= \frac{\alpha}{2}\sin 4\beta\,\sin^2\theta \\ &\quad \times [(\sqrt{2-2\alpha^2} + \alpha\cos 4\beta)\sin^2\theta - 2\sqrt{2-2\alpha^2}].\end{aligned} \tag{9}$$

[8] For example, for $\alpha = 1$, if for a given integer n all players choose the operator $\hat{M}(\frac{\pi}{2}, \beta, -\beta)$, with $\beta = (2n+1)\frac{\pi}{8}$, the result is a NE equivalent to (5).

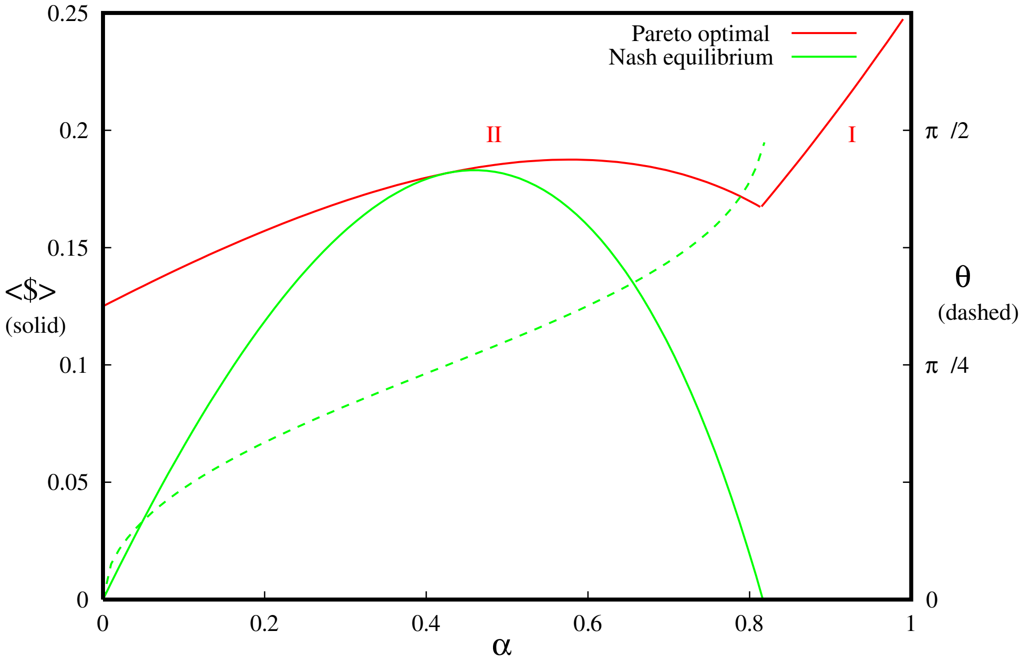

Figure 3. Left scale, solid lines: the PO (red) and NE (green) payoffs as a function of the initial state parameter α. The PO payoff curve is labeled I or II for strategy $\hat{M}(\frac{\pi}{2}, \frac{\pi}{8}, -\frac{\pi}{8})$ or $\hat{M}(\frac{\pi}{4}, 0, 0)$, respectively. Right scale, dashed line: the value of θ given by equation (10) that, along with $\beta = 0$, yields a symmetric NE.

We want to find θ, β for which these derivatives are simultaneously zero. Apart from the known NE for $\alpha = 1$, we find that the only other symmetric NE occurs for $\alpha \leqslant \sqrt{\frac{2}{3}}$ when $\cos 4\beta = 1$ and

$$\cos\theta = \sqrt{\frac{2-3\alpha^2}{2-\alpha^2+2\alpha\sqrt{2-2\alpha^2}}}. \qquad (10)$$

The expected payoff to each player for this equilibrium is

$$\langle \$ \rangle = \frac{\alpha(2-3\alpha^2)(\alpha+\sqrt{2-2\alpha^2})}{4-2\alpha^2+4\alpha\sqrt{2-2\alpha^2}}, \qquad (11)$$

which reaches a maximum value of $(3+2\sqrt{3})/(18+10\sqrt{3}) \approx 0.183$ at $\alpha = \sqrt{\frac{1}{6}(3-\sqrt{3})}$. Figure 3 gives the value of θ and the resulting payoff for this solution.

We now consider the PO strategy profile. Again we will only consider symmetric strategy profiles. That is, we are interested in θ^*, β^* for which

$$\left\langle \$(\hat{M}(\theta^*, \beta^*, -\beta^*)^{\otimes 4}) \right\rangle \geqslant \left\langle \$\left(\hat{M}(\theta, \beta, -\beta)^{\otimes 4}\right) \right\rangle \quad \forall\, \theta, \beta, \qquad (12)$$

where \$ represents the payoff to any one of the four players for the indicated strategy profile. Suppose all players select the strategy $\hat{M}(\theta, \beta, -\beta)$ for some θ, β to be determined. We proceed as before by finding the stationary points of the payoff to each player as a function of the parameters θ and β. We find that for $\alpha > \sqrt{\frac{2}{3}}$, where the initial state is dominated by the GHZ contribution to the superposition, the optimal strategy is $\hat{M}(\frac{\pi}{2}, \frac{\pi}{8}, -\frac{\pi}{8})$, while for the Bell pair-dominated region, $\alpha < \sqrt{\frac{2}{3}}$, the optimal strategy is $\hat{M}(\frac{\pi}{4}, 0, 0)$. At $\alpha = \sqrt{\frac{2}{3}}$ the components in the initial state are equally weighted and both strategies yield the same results. The payoffs for the two regions are

$$\langle \$ \rangle_{\mathrm{I}} = \frac{1}{8} + \frac{f}{16}\alpha(2\sqrt{2-2\alpha^2} - \alpha), \quad \alpha \leqslant \sqrt{\frac{2}{3}}, \tag{13}$$

$$\langle \$ \rangle_{\mathrm{II}} = \frac{1}{8} + \frac{f}{8}\alpha(2\alpha^2 - 1), \quad \alpha \geqslant \sqrt{\frac{2}{3}}. \tag{14}$$

Figure 3 shows the payoffs for these cases for the expected states, i.e. $f = 1$.

There has been some recent interest in the correspondence between equilibria in quantum game theory and the violation of Bell inequalities [28, 29]. In our case, we note that the curve for the symmetric PO payoff is the same as that for the maximal violation of the Mermin–Ardehali–Belinski–Klyshko (MABK) inequality [30].

3. Experimental implementation

3.1. State observation

The states are obtained with a new linear optics network that enables the observation of a whole family of states in a single setup by the tuning of one experimental parameter, which allows for the utilization of the observed states for a wealth of quantum information applications [11, 12]. The setup relies on the interference of the second-order emission of type II non-collinear SPDC and is depicted in figure 2. The down-conversion emission yields four photons, two horizontally and two vertically polarized, in two spatial modes a and c. The photons are overlapped on a polarizing beam splitter (PBS) and afterwards symmetrically split by two polarization-independent beam splitters (BS) into the four spatial modes $\{a, b, c, d\}$. Prior to interference at the PBS, the polarization of the photons in mode c is rotated by a half-waveplate (HWP). Under the condition of detecting one photon in each spatial mode, the desired state is observed. The weighting coefficient α is thereby determined by the orientation of the principal axis γ of the HWP according to

$$\alpha(\gamma) = \frac{2\sqrt{2}\sin^2(2\gamma)}{\sqrt{5 - 4\cos(4\gamma) + 3\cos(8\gamma)}}, \tag{15}$$

with $\gamma \in [0, \frac{\pi}{8}]$.

3.2. Implementation of the game

The implementation of the four-player QMG consists of acting with the strategy operator on each qubit and afterwards measuring the resulting output state in different bases. The unitary transformation corresponding to the players' choice of strategy is realized by a series of quarter-, half- and quarter-waveplates ($\hat{M}$ in figure 2). In order to play strategy I and to act with $\hat{M}_{\mathrm{I}} \equiv \hat{M}(\frac{\pi}{2}, \frac{\pi}{8}, -\frac{\pi}{8})$, the angles of the waveplates are chosen as $\{-\frac{\pi}{8}, \frac{5\pi}{16}, 0\}$, and for strategy II and $\hat{M}_{\mathrm{II}} \equiv \hat{M}(\frac{\pi}{4}, 0, 0)$ as $\{\frac{\pi}{2}, \frac{\pi}{16}, \frac{\pi}{2}\}$. The payoff is evaluated for a chosen strategy by averaging over a given period of time.

3.3. Results

We start the measurement with $\alpha = 1$, as in this case a NE solution is expected. The expected state after the application of strategy I, which is here the optimal one, is of the form

$$(\hat{M}_{\mathrm{I}}^{\otimes 4})|\mathrm{GHZ}\rangle = -\frac{1}{2\sqrt{2}}\Big(|HHHV\rangle + |HHVH\rangle + |HVHH\rangle + |VHHH\rangle - |VVVH\rangle - |VVHV\rangle - |VHVV\rangle - |HVVV\rangle\Big) = \frac{\mathrm{i}}{\sqrt{2}}(|RRRR\rangle - |LLLL\rangle), \qquad (16)$$

with $|R/L\rangle = \frac{1}{\sqrt{2}}(|H\rangle \pm \mathrm{i}|V\rangle)$ being the right and left circular polarization states. The transformed state is also of GHZ type, and thus the fidelity $F = \langle \mathrm{GHZ}|M_{\mathrm{I}}^{\dagger\otimes 4}\rho_{\mathrm{exp}}^{\alpha=1}\hat{M}_{\mathrm{I}}^{\otimes 4}|\mathrm{GHZ}\rangle$ of the experimental state $\rho_{\mathrm{exp}}^{\alpha=1}$ equals the average expectation value of the GHZ stabilizer operators [31]. A measurement of the respective correlations is hence sufficient to evaluate the state fidelity. In our case, this requires nine measurement settings and we obtain a value of $F = 0.746 \pm 0.019$. For comparison, the fidelity of the untransformed initial GHZ state obtained with our setup is $F = 0.745 \pm 0.022$, and thus did not decrease due to the transformation. Rather, its value is due to the well-known higher order SPDC noise and imperfect interference at the PBS.

The payoff awarded on average to each player can be evaluated from the correlation measurement in the computational basis, $\sigma_z^{\otimes 4}$. Averaged over all four players, the payoff obtained for $\alpha = 1$ is $\langle \$_{\mathrm{I}}\rangle_{\alpha=1} = 0.206 \pm 0.009$, which is well above the classical limit of $\frac{1}{8}$.

As was shown earlier, the maximal achievable payoff depends on α and, moreover, there is an interesting change in the PO strategy at $\alpha = \sqrt{\frac{2}{3}}$. In order to test this experimentally, we chose to perform measurements for other distinguished states, $|\Psi(0)\rangle$, $|\Psi(\sqrt{\frac{2}{11}})\rangle$ and $|\Psi(\sqrt{\frac{2}{3}})\rangle$. The results obtained are summarized in figure 4, which gives the average PO payoffs for the four values of α for strategies I and II, along with theoretical curves. As can be seen in the boxes at some example points, the payoffs differ slightly for each player but are generally comparable within typical measurement errors of around 9–12%.[9] The only exception occurs for player C and player A for $\alpha = \sqrt{2/11}$ and $\sqrt{2/3}$, respectively. The cause of this could not be rigorously identified. The average payoffs follow the expected dependence for both strategies, once imperfect state quality is taken into account. To allow for loss in the states' fidelity, the data in figure 4 are fitted assuming a mixed input state of the form (3). Although the assumptions of an admixture of white noise and constant state quality are only approximations, the value of $f = 0.71 \pm 0.03$ obtained from the fit is in good agreement with the measured GHZ fidelity, where $F = (1 + 15f)/16 \approx 0.73$. Let us discuss the results for each measured point in more detail.

For $\alpha = 0$, a product of two Bell states is expected. As this state is not four-qubit entangled, it should yield at best the classical payoff for strategy II. This is nicely reproduced in the experiment with an average payoff of $\langle \$_{\mathrm{II}}\rangle_{\alpha=0} = 0.125 \pm 0.006$. If the players choose to play strategy I, zero payoff is expected for ideal, pure input states. However, for an increasing mixedness of the input states, the payoff approaches the classical limit, with equality for $f \to 0$.

[9] The errors are derived from the photon count rates and their associated Poissonian fluctuations.

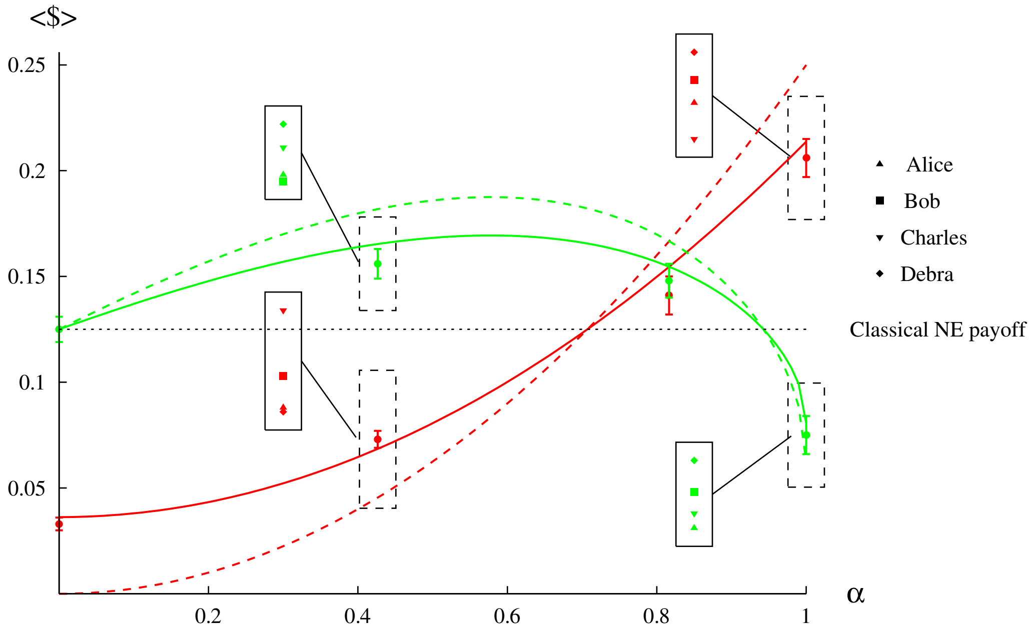

Figure 4. Measured payoff $\langle \$_{\mathrm{I,II}} \rangle$ averaged over all four players as a function of α for strategy I (red) and strategy II (green). The data are fitted assuming a mixed input state of the form $f\,|\Psi(\alpha)\rangle\langle\Psi(\alpha)|+(1-f)/16\,\mathbb{1}^{\otimes 4}$ (see equation (3)). Dashed lines correspond to the ideal case $f=1$. The boxes show the individual values for the four players for some sample points. Typical errors for individual measurements are 9–12%. The maximum equilibrium payoff in a classical game is $\frac{1}{8}$, as indicated by the dotted line.

This fact leads to a nonzero value for the experimentally measured average payoff for strategy I, $\langle \$_{\mathrm{I}} \rangle_{\alpha=0} = 0.033 \pm 0.003$.

Similar behavior is obtained for $\alpha = \sqrt{2/11}$, where the measured payoffs for strategy II, $\langle \$_{\mathrm{II}} \rangle_{\alpha=\sqrt{2/11}} = 0.156 \pm 0.007$, and strategy I, $\langle \$_{\mathrm{I}} \rangle_{\alpha=\sqrt{2/11}} = 0.073 \pm 0.004$, are lower and higher than expected, respectively. Nevertheless, the experimentally reached state quality is high enough to ensure, for the proper strategy, a payoff that exceeds the one maximally achievable in the equivalent classical game. As mentioned before, $\alpha = \sqrt{2/3}$ is special as both strategies lead to the same payoff. It is the point about which the players have to switch between the different strategies. The corresponding state, $|\Psi(\sqrt{2/3})\rangle \equiv |\Psi_4\rangle$, is well known from the literature as it can be directly obtained from SPDC [26, 32, 33] and has several interesting applications in quantum information (see, for example, [34, 35]). The point's feature of being a quantum fulcrum is nicely reproduced in the experiment. Within measurement errors, the average payoffs, $\langle \$_{\mathrm{I}} \rangle_{\alpha=\sqrt{2/3}} = 0.141 \pm 0.009$ and $\langle \$_{\mathrm{II}} \rangle_{\alpha=\sqrt{2/3}} = 0.148 \pm 0.008$, are the same for strategies I and II (see figure 4).

Finally, the values obtained in the GHZ case fit well in the dependence prescribed by the previous points. Like for the other states, due to imperfect state quality the average payoff is slightly lower (higher) than the ideal value of $\frac{1}{4}$ ($\frac{1}{16}$) for strategy I (II).

In summary, we can state that, for all values of α, the players are awarded payoffs above the classical NE value if they choose the appropriate strategy. The values are consistent with $f = 0.73$ for the initial production of the four-partite entangled state.

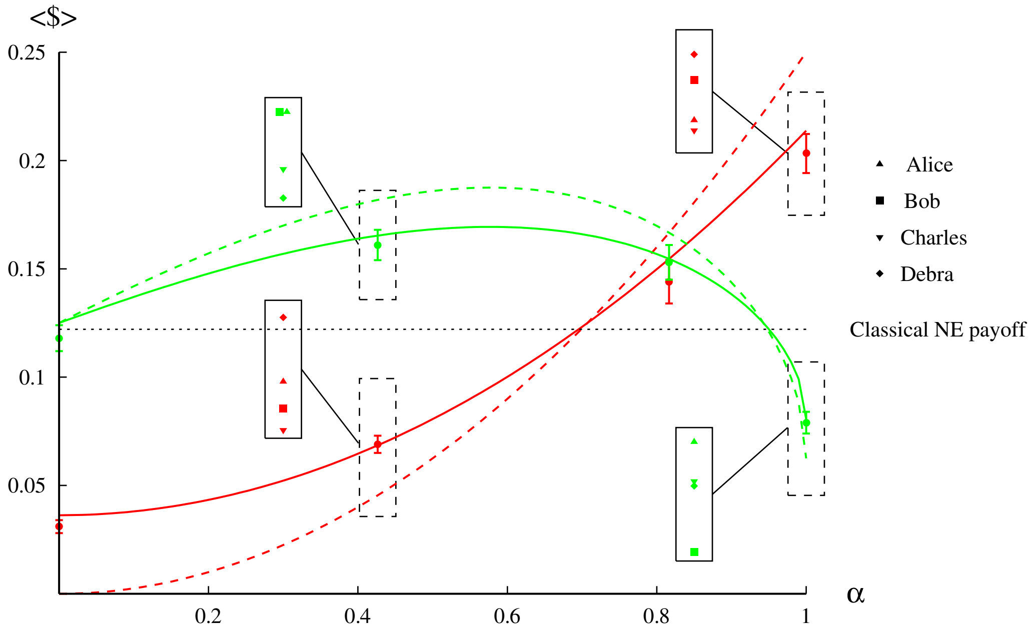

Figure 5. Measured payoff $\langle \$_{\mathrm{I,II}} \rangle$ averaged over all four players as a function of α in the basis $\sigma_x^{\otimes 4}$ for strategy I (red) and in the basis $\sigma_y^{\otimes 4}$ for strategy II (green). Compare with figure 4.

In our realization of the minority game, we do not use an unentangling gate. Thus a measurement in the computational basis alone does not prove that the higher than classical payoff values have their origins in the four-qubit entanglement of $|\Psi(\alpha)\rangle$. In principle, they could be caused by the separable state

$$\rho_{\mathrm{sep}} = \frac{1}{8}\Big(|HHHV\rangle\langle HHHV| + \text{permutations} + |VVVH\rangle\langle VVVH| + \text{permutations}\Big), \quad (17)$$

which yields a maximal 'payoff' of $\frac{1}{4}$ when measured in the computational basis. However, measured in other bases, the state ρ_{sep} will not give a payoff above the classical limit.

In contrast, the state $|\Psi(\alpha)\rangle$ has the extraordinary property that it exhibits the same term structure in the bases $\sigma_z^{\otimes 4}$ and $\sigma_x^{\otimes 4}$ when transformed by $\hat{M}_{\mathrm{I}}^{\otimes 4}$, and in the bases $\sigma_z^{\otimes 4}$ and $\sigma_y^{\otimes 4}$ when transformed by $\hat{M}_{\mathrm{II}}^{\otimes 4}$. Consequently, if the payoff is evaluated analogously in these bases, the same dependence on α is expected. In order to prove this, we have performed the same measurements as before in the bases $\sigma_x^{\otimes 4}$ and $\sigma_y^{\otimes 4}$. The result is shown in figure 5.

For each basis and appropriate strategy, we find very similar curves to those for the computational basis. We applied the same fitting procedure as before. The resulting values for f are slightly different for both bases ($f_x = 0.736 \pm 0.019$ and $f_y = 0.706 \pm 0.053$), but comparable with the one found for the computational basis. The small asymmetry between the bases indicates that, contrary to our assumption, the experimental noise is not purely white noise.

4. Conclusion

We have studied the four-player QMG where the initial state is drawn from a continuous set of four-partite entangled states consisting of a superposition of the GHZ state and a product of Bell pairs. We experimentally implemented this game using four-photon entangled states observed via SPDC. The results for four different initial states were obtained, with results consistent with our theoretical predictions. The average equilibrium payoffs were above those of a classical minority game, thus demonstrating the usefulness of entanglement. This is the first implementation of a quantum game with the full set of unitary strategies.

While we focused on a particular quantum implementation of a classical game, quantum game theory is the natural extension to consider other competitive situations in quantum information settings. For example, eavesdropping in quantum communication (see [6, 36]) and optimal cloning [37] can be conceived as strategic games between two or more players.

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