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Heaps of Linear Connections and Their Endomorphism Truss

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Abstract: We examine the heap of linear connections on anchored vector bundles and Lie algebroids. Naturally, this covers the example of affine connections on a manifold. We present some new interpretations of classical results via this ternary structure of connections. Endomorphisms of linear connections are studied, and their ternary structure, in particular the endomorphism truss, is explicitly presented. We remark that the use of ternary structures in differential geometry is novel and that the endomorphism truss of linear connections provides a concrete geometric example of a truss.

Keywords: heaps; trusses; linear connections; Lie algebroids

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1. Introduction and Preliminaries

1.1. Introduction

We need hardly mention that the notion of a connection in its various forms is of vital importance in differential geometry and geometric approaches to physics (see, for example, [1]). As an important example of the role of connections in modern mathematics, we point to the construction of characteristic classes of principal bundles via Chern–Weil theory (for an introduction, see [2], Chapter 11). In physics, connections are related to gauge fields and are vital in general relativity and other geometric approaches to gravity such as metric-affine gravity. Connections are also found in geometric approaches to relativistic mechanics, deformation quantisation and the BRST formalism. In short, connections are found throughout modern geometry and physics.

In this note, we study linear connections on anchored vector bundles, especially Lie algebroids, in the category of smooth real supermanifolds (in the sense of Berezin and Leĭtes; see [3]). Such connections, here formulated following Koszul as covariant derivatives, generalise affine connections on manifolds, so our results are directly restricted to the classical setting.

Heaps were first defined and studied by Prüfer [4] and Baer [5] as a set equipped with a ternary operation satisfying simple axioms, including a ternary generalisation of associativity. A heap can, loosely, be thought of as a group in which the identity has been discarded. Given some group, the ternary operation $(a, b, c) \mapsto ab^{-1}c$ defines a heap. For example, in an affine space, one can construct a heap operation as $(u, v, w) \mapsto u - v + w$. In the other direction, by selecting any element in a heap, one can reduce the ternary operation to a group operation such that the chosen element is the identity element. Our main reference for heaps and related structures is the book by Hollings and Lawson [6], which presents translations of Wagner's original works on heaps and related ternary structures.

Via the above paragraph, it is clear that, as it forms an affine space, the set of affine/linear connections on a (super)manifold/vector bundle forms a heap. Informally, you cannot add two connections together, but you can combine three of them. We investigate the immediate consequences of this heap structure. An interesting observation is that the set of endomorphisms of linear connections forms a truss. The latter structures are ring-like algebraic structures in which the binary addition is replaced with a heap operation, together with some natural distributivity axioms (see [7–11]). We explicitly construct the endomorphism truss of linear connections on a Lie algebroid (or just an anchored vector bundle) in this note. We comment that the notion of endomorphisms of linear connections does not seem to be a widely studied or used concept in mathematical physics. Presumably, this is due to the fact that, in general relativity and very closely related theories, the connection of interest is the Levi-Civita connection. However, in theories such as metric-affine gravity and affine gauge theory, more general affine connections are considered (see [12] for a review of metric-affine theories). Thus, some of the ideas presented here may be of relevance in novel theories of gravity (see [13] for an introduction to gravity theories other than general relativity).

We remark that although we are working in supergeometry, none of the results of this note hinges on that fact. All the statements and results cover the classical setting of affine connections on manifolds. We assume the reader has some familiarity with supermanifolds and supervector bundles. However, we will work in a coordinate-free way with globally defined objects; thus, by forgetting the spurious sign factors, one will obtain the classical results. For details on supermanifolds, the reader may consult Carmeli et al. [14], for example. For details on (super) vector bundles, one may consult [15]. Local expressions for Lie algebroids can be found in the introductory section of [16]. We will often neglect the prefix 'super', and, unless explicitly stated, everything will be $\mathbb{Z}_2$-graded. In the examples we give, the Riemmanian manifold can be replaced with a pseudo-Riemannian manifold.

Arrangement

We continue this section with a brief description of the algebraic structures needed throughout this paper. In particular, heaps are recalled, as are Brzeziński trusses. The main content of this paper is found in Section 2. In that section, we recall the notion of linear connections on anchored vector bundles and Lie algebroids and present their heap structure. From there, we discuss metric compatibility, torsion and curvature, Lie derivatives and the dual connection from a heap perspective. The main result of this paper, the construction of the endomorphism truss of linear connections, is given at the end of that section. We end this paper in Section 3 with a few concluding remarks.

1.2. Algebraic Preliminaries

A *semiheap* (H) is a (possibly empty) set equipped with a ternary operation $((a,b,c) \mapsto [a,b,c])$ that is para-associative, i.e.,

$$[[a,b,c],d,e] = [a,[d,c,b],e] = [a,b,[c,d,e]]\,.$$

A semiheap is said to be *abelian* if $[a,b,c] = [c,b,a]$ for all a, b and $c \in H$. If all the elements are *bi-unitary*, that is, $[a,b,b] = a$ and $[b,b,a] = a$ for all a and $b \in H$, then we have a *heap*. We recall that a *left truss* is an abelian heap together with an associative binary operation that distributes over the ternary operation (from the left), i.e.,

$$a \cdot [b,c,d] = [a \cdot b, a \cdot c, a \cdot d]\,.$$

Similarly, a *right truss* can be defined. If we have both left and right distributivity, then we speak of a *truss*. We divert the reader to the original literature on trusses for more details (see [7–11]). A video lecture outlining the theory of trusses, including motivation and a historical perspective, is available in [17].

2. The Ternary Structure of Connections

2.1. Connections on Anchored Vector Bundles

Let us recall the definition of an anchored vector bundle [18], here adapted to the setting of supergeometry.

Definition 1. *A vector bundle (in the category of supermanifolds) $\pi : A \to M$, where π is a smooth, subjectivesubmersion; see Section 5.2 in [14] for details on this notion in supergeometry) is said to be an* anchored vector bundle *if it is equipped with a vector bundle homomorphism (over the identity) ($\rho : A \to \mathrm{T}M$), which is referred to as the anchor.*

The $C^\infty(M)$ module of sections of a vector bundle is $\mathbb{Z}_2$-graded, i.e.,

$$\underline{\mathrm{Sec}}(A) = \mathrm{Sec}_0(A) \oplus_{C^\infty(M)} \mathrm{Sec}_1(A).$$

We generally consider homogeneous sections, i.e., either even or odd, in definitions and proofs. The statements extend to inhomogeneous sections via linearity. By minor abuse of notation, we also write

$$\rho : \underline{\mathrm{Sec}}(A) \longrightarrow \mathrm{Vect}(M)\,, \tag{1}$$

for the associated (even) homomorphism of $C^\infty(M)$ modules. When convenient, we write $\rho_u := \rho(u)$, where $u \in \underline{\mathrm{Sec}}(A)$. We denote the Grassmann parity (or degree) of sections (as well as functions, tensors, etc.) using "tildes", i.e., $\widetilde{u} \in \mathbb{Z}_2 = \{0,1\}$.

Example 1. *The tangent bundle ($\mathrm{T}M$) of a supermanifold (M), is an anchored vector bundle, with the anchor being the identity map. The sections of the tangent bundle are vector fields on the supermanifold (M).*

Example 2. *Let $\tau : E \to M$ be a vector bundle over the supermanifold (M). Then, as fibre products exist in the category of supermanifolds, $A := \mathrm{T}M \times_M E$ is also a vector bundle over M. The anchor is the projection onto the first factor (a little care is needed, as we have a locally ringed space, but this can be made sense of using local coordinates, for example). Sections are clearly $\underline{\mathrm{Sec}}(A) = \mathrm{Vect}(M) \otimes_{C^\infty(M)} \underline{\mathrm{Sec}}(E)$. Then, on pure tensor products, $\rho(X \otimes u) = X$.*

Definition 2. *A linear connection on an anchored vector bundle (A, ρ) is an $\mathbb{R}$-bilinear map, i.e.,*

$$\nabla : \underline{\mathrm{Sec}}(A) \times \underline{\mathrm{Sec}}(A) \longrightarrow \underline{\mathrm{Sec}}(A)\,,$$

such that

1. $\widetilde{\nabla_u v} = \widetilde{u} + \widetilde{v}$;
2. $\nabla_{fu} v = f\, \nabla_u v$; *and*
3. $\nabla_u(fv) = \rho_u(f)v + (-1)^{\widetilde{u}\widetilde{f}} f \nabla_u v$,
   for all $f \in C^\infty(M)$, and $u, v \in \underline{\mathrm{Sec}}(A)$.

We will denote the set (or affine space) of linear connections on (A, ρ) as $\mathcal{C}(A)$.

Remark 1.

1. *In this note, we only consider Grassmann even connections. Odd connections are not a truly separate notion, as uncovered in [19].*
2. *The existence of Lie algebroid connections and, therefore, anchored vector bundles is established in [20], for example, using a partition of unity. For real supermanifolds (as locally ringed spaces), we always have partitions of unity.*
3. *There is the related notion of an A-valued connection ($\hat{\nabla} : \underline{\mathrm{Sec}}(A) \times \underline{\mathrm{Sec}}(E) \to \underline{\mathrm{Sec}}(E)$). As we will want to discuss torsion, linear connections, as defined above, are needed.*
4. *Linear connections can be reformulated as odd vector fields on a particular bi-graded supermanifold built from the initial anchored vector bundle (see [21] for details). We avoid graded/weighted geometry in this note and stick to a more classical presentation.*

Example 3. *A real vector space (V; non-super for simplicity) can be considered as an anchored vector bundle over a single point, where the anchor is the zero map. A linear connection on (V) is simply an $\mathbb{R}$-bilinear map ($\nabla : V \times V \to V$).*

Example 4. *Let $\pi : A \to M$ be an arbitrary vector bundle (in the category of supermanifolds). This vector bundle can be considered an anchored vector bundle by setting the anchor to be the zero map. As the zero vector field can be considered as both even and odd, this choice is consistent. We refer to such structures as zero-anchored vector bundles. Then, a linear connection on a zero vector bundle is an even $\mathbb{R}$-bilinear map such that*

$$\nabla_{fu}v = (-1)^{\widetilde{u}\widetilde{f}}\,\nabla_u(fv) = f\nabla_u v\,,$$

that is, a linear connection in this context is a bilinear form on the $C^\infty(M)$ module ($\underline{\mathrm{Sec}}(A)$).

Example 5. *Given a vector bundle connection ($\bar{\nabla} : \mathrm{Vect}(M) \times \underline{\mathrm{Sec}}(A) \to \underline{\mathrm{Sec}}(A)$ on (A, ρ)), one has a canonically associated linear connection by setting $\nabla_u v := \bar{\nabla}_{\rho(u)}v$.*

Definition 3. *The ternary operation on the affine space of linear connections on an anchored bundle is defined as*

$$[\nabla^{(1)}, \nabla^{(2)}, \nabla^{(3)}] := \nabla^{(1)} - \nabla^{(2)} + \nabla^{(3)}\,, \tag{2}$$

for arbitrary $\nabla^{(i)} \in \mathcal{C}(A)$ (here, $i = 1, 2, 3$).

The ternary operation does, indeed, produce another linear connection. Note, of course, that the sum or difference of two linear connections is not a linear connection. The reader can easily verify the following proposition (see Section 1.2).

Proposition 1. *Let (A, ρ) be an anchored vector bundle. Then, the set of linear connections ($\mathcal{C}(A)$) is an abelian heap, with the ternary operation being defined by (2).*

From the general theory of heaps, we know that if we fix some connection ($\nabla^{(0)} \in \mathcal{C}(A)$), then we have an associated abelian group structure on $\mathcal{C}(A)$ given by

$$\nabla^{(1)} \bullet_{\nabla^{(0)}} \nabla^{(2)} := [\nabla^{(1)}, \nabla^{(0)}, \nabla^{(2)}]\,,$$

and the inverse operation is given by

$$(\nabla)^{-1} := [\nabla^{(0)}, \nabla, \nabla^{(0)}]\,.$$

The identity element is the chosen connection ($\nabla^{(0)}$). All such abelian groups associated with a different choice of reference connection are isomorphic.

Example 6. *Let (M, g) be a Riemannian manifold. Then, we have the canonical Levi-Civita connection on the tangent bundle ($\mathrm{T}M$), which we denote as ∇^0. We consider the tangent bundle an anchored vector bundle where the anchor is the identity map. Thus, the set of affine connections on (M, g) is canonically an abelian group, with the group product and inverse being*

$$\nabla^{(1)} \cdot \nabla^{(2)} := [\nabla^{(1)}, \nabla^0, \nabla^{(2)}]\,,$$
$$(\nabla)^{-1} := [\nabla^0, \nabla, \nabla^0]\,.$$

If $\nabla^{(i)}$ are metric connections, i.e., $\nabla g = 0$, then, due to the linearity of the metric, $[\nabla^{(1)}, \nabla^{(2)}, \nabla^{(3)}]$ is also a metric connection, that is, metric connections form a subheap of the heap of all affine connections. In turn, we also have a subgroup of metric connections. We will discuss this in a little more detail in Section 2.2. Moreover, given an arbitrary affine connection (∇), we note that

$$\nabla = [\nabla, \nabla^0, \nabla^0] = \nabla^0 + (\nabla - \nabla^0)\,,$$

which gives a "heapy" origin to the well-known fact that any affine connection on a Riemannian manifold is the Levi-Civita connection plus a tensor of type $(1, 2)$.

Remark 2. *The previous example directly generalises to even and odd Riemannian supermanifolds, as we, again, have a canonical Levi-Civita connection.*

A section ($u \in \underline{\mathrm{Sec}}(A)$) is said to be *auto-parallel* if there exists a linear connection ($\nabla \in \mathcal{C}(A)$) such that $\nabla_u u = 0$. We denote the set of all such linear connections as $\mathcal{C}(A, u)$. The following proposition is evident.

Proposition 2. *Let (A, ρ) be an anchored vector bundle. The set ($\mathcal{C}(A, u)$) for any section ($u \in \underline{\mathrm{Sec}}(A)$) is closed under ternary operation (2).*

2.2. Metric Compatibility

A *Riemannian metric* on an anchored vector bundle ($\pi : A \to M$) is a smooth assignment of an (even) inner product to the fibres.

$$\langle - | - \rangle : \underline{\mathrm{Sec}}(A) \times \underline{\mathrm{Sec}}(A) \longrightarrow C^\infty(M)\,.$$

By even, we mean that $\widetilde{\langle u|v\rangle} = \widetilde{u} + \widetilde{v}$. It is possible to consider odd Riemannian metrics, but in order to keep close to the classical Riemannian geometry, we restrict our attention to even structures. Note that the existence of an even Riemannian metric requires that $\mathrm{Rank}(A) = n|2m$. We refer to an anchored vector bundle equipped with a Riemannian metric as a *Riemannian anchored bundle*.

A linear connection ($\nabla \in \mathcal{C}(A)$) is said to be *metric-compatible* if

$$\rho_u \langle v|w\rangle = \langle \nabla_u v|w\rangle + (-1)^{\widetilde{u}\,\widetilde{v}} \langle v|\nabla_u w\rangle \tag{3}$$

for all u, v and $w \in \mathcal{C}(A)$ (see [22]).

Example 7. *Consider a Riemmanian manifold (M, g). The tangent bundle (TM) is then a Riemannian anchored bundle (the anchor is the identity map). The metric compatibility condition reduces to the 'classical' metric compatibility condition.*

Proposition 3. *Let $\nabla^{(1)}, \nabla^{(2)}$ and $\nabla^{(3)} \in \mathcal{C}(A)$ be metric-compatible connections on a Riemannian anchored bundle. Then, $[\nabla^{(1)}, \nabla^{(2)}, \nabla^{(3)}]$ is also a metric-compatible connection.*

Proof. Let $\nabla^{(1)}, \nabla^{(2)}$ and $\nabla^{(3)} \in \mathcal{C}(A)$ be metric-compatible connections on a Riemannian anchored bundle, and let u, v and $w \in \underline{\mathrm{Sec}}(A)$ be arbitrary sections. Using the linear nature of the inner product (metric), together with the metric compatibility condition (3), we observe that

$$\begin{aligned}
\langle [\nabla^{(1)}, \nabla^{(2)}, \nabla^{(3)}]_u v|w\rangle + (-1)^{\widetilde{u}\,\widetilde{v}} \langle v|[\nabla^{(1)}, \nabla^{(2)}, \nabla^{(3)}]_u w\rangle &= \langle \nabla^{(1)}_u v|w\rangle + (-1)^{\widetilde{u}\,\widetilde{v}} \langle v|\nabla^{(1)}_u w\rangle \\
&\quad - \langle \nabla^{(2)}_u v|w\rangle - (-1)^{\widetilde{u}\,\widetilde{v}} \langle v|\nabla^{(2)}_u w\rangle \\
&\quad + \langle \nabla^{(3)}_u v|w\rangle + (-1)^{\widetilde{u}\,\widetilde{v}} \langle v|\nabla^{(3)}_u w\rangle \\
&= \rho_u \langle v|w\rangle - \rho_u \langle v|w\rangle + \rho_u \langle v|w\rangle \\
&= \rho_u \langle v|w\rangle.
\end{aligned}$$

This is precisely the metric compatibility condition for $[\nabla^{(1)}, \nabla^{(2)}, \nabla^{(3)}]$. □

Corollary 1. *The subset of metric-compatible connections ($\mathcal{C}_{MC}(A) \subset \mathcal{C}(A)$) on a Riemannian anchored bundle forms an abelian subheap of $(\mathcal{C}(A), [-, -, -])$.*

2.3. Torsion and Curvature

To discuss torsion and curvature in the setting of anchored vector bundles, we require a bracket on the space of sections. We focus on the situation of Lie algebroids, though one can relax the Jacobi identity and the compatibility of the anchor with the bracket, if desired. For completeness, we define a Lie algebroid following Pradines (see [18]), modified to the setting of supergeometry. The reader may also consult Mackenzie for further details [23].

Definition 4. *An anchored vector bundle (A, ρ) is a* Lie algebroid *if the space of sections $\underline{\mathrm{Sec}}(A)$ comes equipped with an $\mathbb{R}$-bilinear map, i.e., a Lie bracket,*

$$[-,-] : \underline{\mathrm{Sec}}(A) \times \underline{\mathrm{Sec}}(A) \longrightarrow \underline{\mathrm{Sec}}(A)\,,$$

that satisfies the following:

1. $\widetilde{[u,v]} = \widetilde{u} + \widetilde{v}$;
2. $[u,v] = -(-1)^{\widetilde{u}\widetilde{v}}\,[v,u]$;
3. $[u, fv] = \rho_u(f)v + (-1)^{\widetilde{u}\widetilde{f}}\, f[u,v]$; *and*
4. $[u,[v,w]] = [[u,v],w] + (-1)^{\widetilde{u}\widetilde{v}}\,[v,[u,w]]$

   for all u, v and $w \in \underline{\mathrm{Sec}}(A)$ and $f \in C^\infty(M)$.

The above conditions imply that the anchor is a homomorphism of Lie algebras, i.e.,

$$\rho_{[u,v]} = [\rho(u), \rho(v)]\,.$$

Example 8. *The tangent bundle of a supermanifold* (T*M) is a Lie algebroid, with the anchor being the identity map and the Lie bracket being the standard commutator of vector fields.*

Example 9. *Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a supervector space. Then, associated with this via the 'manifoldcation' functor, is the linear supermanifold (here, thought of as a vector bundle over a point $(\mathfrak{g}^{\mathrm{man}} \to \star)$). The important aspect of the construction is that $\mathfrak{g} \cong \underline{\mathrm{Sec}}(\mathfrak{g}^{\mathrm{man}})$. If $\mathfrak{g}$ is a Lie algebra, then $\mathfrak{g}^{\mathrm{man}}$ is a Lie algebroid with zero anchor.*

General Lie algebroids are, loosely, a mixture of the two above examples. For further examples, the reader may consult [23]. The mantra here is that whatever can be done with the tangent bundle can be done in the setting of Lie algebroids. In particular, we have the notion of torsion. The *torsion tensor* of a linear connection on a Lie algebroid $(A, \rho, [-,-])$ is given by

$$T_\nabla(u,v) := \nabla_u v - (-1)^{\widetilde{u}\widetilde{v}}\,\nabla_v u - [u,v]\,, \tag{4}$$

for all $u, v \in \underline{\mathrm{Sec}}(A)$.

Remark 3. *The Fundamental Theorem of Riemannian Lie Algebroids states that there is a unique connection on a Riemannian Lie algebroid characterised by the two properties of vanishing torsion and metric compatibility. Such connections, following the classical nomenclature, are known as Levi–Civita connections.*

Proposition 4. *Let $\nabla^{(1)}, \nabla^{(2)}$ and $\nabla^{(3)} \in \mathcal{C}(A)$ be connections on a Lie algebroid $(A, \rho, [-,-])$. Then,*

$$T_{[\nabla^{(1)},\nabla^{(2)},\nabla^{(3)}]} = T_{\nabla^{(1)}} - T_{\nabla^{(2)}} + T_{\nabla^{(3)}}\,.$$

Proof. Directly, given any $u, v \in \underline{\mathrm{Sec}}(A)$,

$$\begin{aligned} T_{[\nabla^{(1)},\nabla^{(2)},\nabla^{(3)}]}(u,v) &= [\nabla^{(1)},\nabla^{(2)},\nabla^{(3)}]_u v - (-1)^{\widetilde{u}\widetilde{v}}[\nabla^{(1)},\nabla^{(2)},\nabla^{(3)}]_v u - [u,v] \\ &= \nabla^{(1)}_u v - (-1)^{\widetilde{u}\widetilde{v}}\nabla^{(1)}_v u - [u,v] - \nabla^{(2)}_u v + (-1)^{\widetilde{u}\widetilde{v}}\nabla^{(2)}_v u + [u,v] \\ &+ \nabla^{(3)}_u v - (-1)^{\widetilde{u}\widetilde{v}}\nabla^{(3)}_u v - [u,v] = T_{\nabla^{(1)}}(u,v) - T_{\nabla^{(2)}}(u,v) + T_{\nabla^{(3)}}(u,v)\,. \end{aligned}$$

□

Remark 4. *The above proposition shows that torsion can be considered a heap homomorphism from the heap of connections to the heap of vector-valued two forms.*

Corollary 2.

1. *The subset of torsion-free connections ($\mathcal{C}_{TF}(A) \subset \mathcal{C}(A)$) forms an abelian subheap of $(\mathcal{C}(A), [-,-,-])$.*
2. $T_{[\nabla^{(1)},\nabla^{(2)},\nabla^{(3)}]} + T_{[\nabla^{(3)},\nabla^{(1)},\nabla^{(2)}]} + T_{[\nabla^{(2)},\nabla^{(3)},\nabla^{(1)}]} = T_{\nabla^{(1)}} + T_{\nabla^{(2)}} + T_{\nabla^{(3)}}$.

Example 10. *Continuing Example 6, the torsion-free connections on a Riemannian manifold (M, g) form an abelian subheap of the abelian heap of all affine connections. Moreover, the set of torsion-free connections canonically comes with an abelian group structure.*

Following Brzeziński, from Definition 2.9 and Proposition 2.10 of [9], we know that there is a subheap relation ($\sim_{\mathcal{C}_{TF}(A)}$ on $\mathcal{C}(A)$) defined by $\nabla^{(1)} \sim_{\mathcal{C}_{TF}(A)} \nabla^{(2)}$ if there exists a $\nabla \in \mathcal{C}_{TF}(A)$ such that $[\nabla^{(1)},\nabla^{(2)},\nabla] \in \mathcal{C}_{TF}(A)$. In fact, if two connections are equivalent, then $[\nabla^{(1)},\nabla^{(2)},\nabla'] \in \mathcal{C}_{TF}(A)$ for all $\nabla' \in \mathcal{C}_{TF}(A)$. It is known that such a relation defines an equivalence relation.

We then observe from Proposition 4, assuming we have equivalent connections, that $T_{[\nabla^{(1)},\nabla^{(2)},\nabla]} = T_{\nabla^{(1)}} - T_{\nabla^{(2)}} = 0$. Thus, the torsion tensors for two equivalent connections must be equal. In other words, connections with the same torsion are representatives of the same equivalence class within the heap of linear connections. As a standard, we will denote an equivalence class via a chosen representative as $[\nabla]$.

Example 11. *Let (M, g) be a Riemannian manifold. We have a canonical torsion-free affine connection ($\nabla^{(0)}$), i.e., the Levi-Civita connection. Then, we can canonically choose $\nabla^{(0)} + T$, where T is an even skew-symmetric $(1,2)$ tensor, as a representative of any equivalence class of affine connections.*

The *curvature tensor* of a connection on a Lie algebroid is given by

$$R_{\nabla}(u,v)w := \nabla_u \nabla_v w - (-1)^{\widetilde{u}\widetilde{v}}\nabla_v \nabla_u w - \nabla_{[u,v]} w\,, \tag{5}$$

for arbitrary u, v and $w \in \underline{\mathrm{Sec}}(A)$.

Proposition 5. *Let $\nabla^{(1)}, \nabla^{(2)}$ and $\nabla^{(3)} \in \mathcal{C}(A)$ be connections on a Lie algebroid $(A, \rho, [-,-])$. Then,*

$$\begin{aligned} R_{[\nabla^{(1)},\nabla^{(2)},\nabla^{(3)}]}(u,v)w &= R_{\nabla^{(1)}}(u,v)w + R_{\nabla^{(2)}}(u,v)w + R_{\nabla^{(3)}}(u,v)w \\ &+ 2\nabla^{(2)}_{[u,v]} w + \sum_{i \neq j} (-1)^{ij}[\nabla^{(i)}_u, \nabla^{(j)}_v] w\,, \end{aligned}$$

for arbitrary u, v and $w \in \underline{\mathrm{Sec}}(A)$.

Proof. Via direct calculation,

$$\begin{aligned}
R_{[\nabla^{(1)},\nabla^{(2)},\nabla^{(3)}]}(u,v)w &= [\nabla^{(1)}_u,\nabla^{(2)}_u,\nabla^{(3)}_u][\nabla^{(1)}_v,\nabla^{(2)}_v,\nabla^{(3)}_v]w \\
&- (-1)^{\widetilde{u}\widetilde{v}}[\nabla^{(1)}_v,\nabla^{(2)}_v,\nabla^{(3)}_v][\nabla^{(1)}_u,\nabla^{(2)}_u,\nabla^{(3)}_u]w - [\nabla^{(1)}_{[u,v]},\nabla^{(2)}_{[u,v]},\nabla^{(3)}_{[u,v]}]w \\
&= [\nabla^{(1)}_u,\nabla^{(1)}_v]w - \nabla^{(1)}_{[u,v]}w + [\nabla^{(2)}_u,\nabla^{(2)}_v]w + \nabla^{(2)}_{[u,v]}w\nabla^{(2)}_{[u,v]}w - \nabla^{(2)}_{[u,v]}w \\
&+ [\nabla^{(3)}_u,\nabla^{(3)}_v]w - \nabla^{(3)}_{[u,v]}w + \sum_{i\neq j}(-1)^{ij}[\nabla^{(i)}_u,\nabla^{(j)}_v]w \\
&= R_{\nabla^{(1)}}(u,v)w + R_{\nabla^{(2)}}(u,v)w + R_{\nabla^{(3)}}(u,v)w \\
&+ 2\nabla^{(2)}_{[u,v]}w + \sum_{i\neq j}(-1)^{ij}[\nabla^{(i)}_u,\nabla^{(j)}_v]w\,.
\end{aligned}$$

□

Setting $\nabla = \nabla^{(1)} = \nabla^{(2)} = \nabla^{(3)}$ allows for a quick check.

$$R_{\nabla}(u,v)w = 3R\nabla(u,v)w + 2\nabla_{[u,v]}w - 2[\nabla_u,\nabla_v]w = 3R_{\nabla}(u,v)w - 2R_{\nabla}(u,v)w = R_{\nabla}(u,v)w\,,$$

so the proposition is consistent.

Corollary 3. *Let $\nabla^{(i)} \in \mathcal{C}(A)$ be flat connections; then,*

$$R_{[\nabla^{(1)},\nabla^{(2)},\nabla^{(3)}]}(u,v)w = 2\nabla^{(2)}_{[u,v]}w + \sum_{i\neq j}(-1)^{ij}[\nabla^{(i)}_u,\nabla^{(j)}_v]w\,.$$

Specifically, the connection $[\nabla^{(1)},\nabla^{(2)},\nabla^{(3)}]$ need not be flat.

The above corollary tells us that, in general, the heap structure on connections on a Lie algebroid does *not* close on the subset of flat connections. This is not surprising, as the curvature tensor is not first-order in the connection but second-order. Thus, we do not have the analogue of the particularly nice result for torsion (see Proposition 4).

Example 12. *Consider a manifold (M) equipped with three affine connections ($\nabla^{(i)} \in \mathcal{C}(\mathrm{T}M) =: \mathcal{C}(M)$). Then, the Riemannian curvature can be written with respect to a coordinate basis as*

$${}^{(1,2,3)}R^{\rho}{}_{\sigma\mu\nu} = \sum_{i=1}^{3} {}^{(i)}R^{\rho}{}_{\sigma\mu\nu} + \sum_{i,j=1,i\neq j}^{3}(-1)^{ij}\left(\partial_\mu {}^{(i)}\Gamma^{\rho}_{\nu\sigma} - \partial_\nu {}^{(j)}\Gamma^{\rho}_{\mu\sigma} + {}^{(i)}\Gamma^{\rho}_{\mu\lambda}\,{}^{(j)}\Gamma^{\lambda}_{\nu\sigma} - {}^{(j)}\Gamma^{\rho}_{\nu\lambda}\,{}^{(i)}\Gamma^{\lambda}_{\mu\sigma}\right),$$

where Γ represents the Christoffel symbols of the affine connections. The second term is interpreted as a mixed curvature built from all three connections.

2.4. Lie Derivatives of Connections

The Lie derivative of a linear connection on a Lie algebroid is defined as

$$(L_u\nabla)(v,w) = L_u(\nabla_v w) - \nabla_{L_u v}w - (-1)^{\widetilde{u}\widetilde{v}}\,\nabla_v L_u w\,,$$

where $L_u v := [u,v]$. Note that the Lie derivative of a connection is, itself, not a connection but a tensor. The linear property of the ternary product means that the following is evident.

Proposition 6. *Let $\nabla^{(1)},\nabla^{(2)}$ and $\nabla^{(3)} \in \mathcal{C}(A)$ be linear connections on a Lie algebroid $(A,\rho,[-,-])$. Then,*

$$(L_u[\nabla^{(1)},\nabla^{(2)},\nabla^{(3)}])(v,w) = (L_u\nabla^{(1)})(v,w) - (L_u\nabla^{(2)})(v,w) + (L_u\nabla^{(3)})(v,w)$$

for all u,v and $w \in \underline{\mathrm{Sec}}(A)$.

Remark 5. *The above proposition shows that the Lie derivative with respect to a given section can be considered as a heap homomorphism from the heap of connections to the heap of vector-valued two forms.*

2.5. The Dual Connection

Given a linear connection ($\nabla \in \mathcal{C}(A)$) on a Lie algebroid, the *dual connection* is defined as

$$\bar{\nabla}_u v := (-1)^{\widetilde{u}\widetilde{v}}\, \nabla_v u + [u, v]\,.$$

The reader can quickly check that this defines another linear connection on A.

Example 13. *Consider a Lie superalgebra ($\mathfrak{g}$), in particular $\mathfrak{g}^{\text{man}} \to \star$, as a Lie algebroid, with the anchor being the zero map and the Lie bracket as the bracket. Then, $\nabla_u v := [u, v]$ defines a linear connection.*

Proposition 7. *The dual map* $\bar{\ }: \mathcal{C}(A) \to \mathcal{C}(A)$ *is a heap homomorphism, i.e.,*

$$\overline{[\nabla^{(1)}, \nabla^{(2)}, \nabla^{(3)}]} = [\bar{\nabla}^{(1)}, \bar{\nabla}^{(2)}, \bar{\nabla}^{(3)}]\,.$$

Proof. Let $u, v \in \underline{\mathrm{Sec}} A$ be arbitrary (but homogeneous) sections of a Lie algebroid; then,

$$\begin{aligned}
\overline{[\nabla^{(1)}, \nabla^{(2)}, \nabla^{(3)}]}_u v &= (-1)^{\widetilde{u}\widetilde{v}}\, \nabla^{(1)}_v u - (-1)^{\widetilde{u}\widetilde{v}}\, \nabla^{(2)}_v u + (-1)^{\widetilde{u}\widetilde{v}}\, \nabla^{(3)}_v u + [u, v] \\
&= \big((-1)^{\widetilde{u}\widetilde{v}}\, \nabla^{(1)}_v u + [u, v]\big) - \big((-1)^{\widetilde{u}\widetilde{v}}\, \nabla^{(2)}_v u + [u, v]\big) \\
&+ \big((-1)^{\widetilde{u}\widetilde{v}}\, \nabla^{(3)}_v u + [u, v]\big) \\
&= \bar{\nabla}^{(1)}_u v - \bar{\nabla}^{(2)}_u v + \bar{\nabla}^{(3)}_u v = [\bar{\nabla}^{(1)}, \bar{\nabla}^{(2)}, \bar{\nabla}^{(3)}]_u v\,.
\end{aligned}$$

□

2.6. The Endomorphism Truss of Connections

Definition 5. *Let (A, ρ) be an anchored vector bundle. Then, an* anchored vector bundle endomorphism *is a vector bundle map (over the identity) ($\phi : A \to A$) that preserves the anchor, i.e., $\rho = \rho \circ \phi$. Furthermore, if $(A, \rho, [-, -])$ is a Lie algebroid, then an anchored vector bundle endomorphism is a* Lie algebroid endomorphism *if $\phi[u, v] = [\phi(u), \phi(v)]$ for all $u, v \in \underline{\mathrm{Sec}}(A)$.*

Example 14. *Consider the tangent bundle of a supermanifold ($\mathrm{T}M$) equipped with an idempotent endomorphism, i.e., a vector bundle map over the identity $\phi : \mathrm{T}M \to \mathrm{T}M$ such that $\phi \circ \phi = \phi$. Then, we consider $\mathrm{T}M_\phi := (\mathrm{T}M, \phi)$ as an anchored vector bundle. Due to the idempotent condition, ϕ is also an endomorphism of $\mathrm{T}M_\phi$.*

For this subsection, we need only discuss anchored vector bundles; there is no real change at all when extending to Lie algebroids. In particular, the Lie bracket plays no role in the following.

Definition 6. *Let (A, ρ) be an anchored vector bundle. An* endomorphism of the set of connections *($\Phi = (\phi, \omega) : \mathcal{C}(A) \to \mathcal{C}(A)$) consists of the following two parts:*

1. *An anchored vector bundle endomorphism ($\phi : A \to A$);*
2. *An even $(1, 2)$ tensor, thought of as a $C^\infty(M)$-linear map ($\omega : \underline{\mathrm{Sec}}(A) \times \underline{\mathrm{Sec}}(A) \to \underline{\mathrm{Sec}}(A)$)*

and is defined as

$$(\Phi\nabla)_u v := \nabla_{\phi(u)} v + \omega(u, v)$$

for arbitrary $u, v \in \underline{\mathrm{Sec}}(A)$. Composition is defined in the obvious way, i.e., if $\Phi = (\phi, \omega)$ and $\Phi' = (\phi', \omega')$, then $\Phi \circ \Phi' := (\phi \circ \phi', \omega + \omega')$. The identity endomorphism is $\mathbf{1} = (\mathbb{1}, 0)$, where 0 is the zero map on sections, i.e., the map sends any pair of sections to the zero section. We denote the monoid of endomorphisms of connections as $\mathrm{End}(\mathcal{C}(A))$.

For completeness, we explicitly check that $\Phi\nabla$ is, indeed, a connection on (A, ρ). Firstly, $\Phi\nabla$ is (Grassmann) even, i.e.,

$$\widetilde{(\Phi\nabla)_u v} = \widetilde{u} + \widetilde{v}\,.$$

Secondly,

$$(\Phi\nabla)_{fu} v = \nabla_{\phi(fu)} v + \omega(fu, v) = \nabla_{f\phi(u)} v + f\,\omega(u, v) = f\big(\nabla_{\phi(u)} v + \omega(u, v)\big)\,.$$

Thirdly,

$$\begin{aligned}(\Phi\nabla)_u f v &= \nabla_{\phi(u)} f v + \omega(u, f v) = \rho_{\phi(u)}(f)\, v + (-1)^{\widetilde{u}\widetilde{f}} f \left(\nabla_{\phi(u)} v + \omega(u, v)\right) \\ &= \rho_u(f)\, v + (-1)^{\widetilde{u}\widetilde{f}} f \left(\nabla_{\phi(u)} v + \omega(u, v)\right).\end{aligned}$$

Thus, compared with Definition 2, we see that $\Phi\nabla$ is a linear connection. Moreover, quick calculations show we do, indeed, have a heap morphism, i.e.,

$$\begin{aligned}\Phi[\nabla^{(1)}, \nabla^{(2)}, \nabla^{(3)}]_u v &= \nabla^{(1)}_{\phi(u)} v + \omega(u, v) - \nabla^{(2)}_{\phi(u)} v - \omega(u, v) + \nabla^{(3)}_{\phi(u)} v + \omega(u, v) \\ &= [\Phi\nabla^{(1)}, \Phi\nabla^{(2)}, \Phi\nabla^{(3)}]_u v\,.\end{aligned}$$

Remark 6. *We can also consider automorphisms by insisting that $\phi : A \to A$ is invertible. Then, $\Phi^{-1} := (\phi^{-1}, -\omega)$. Also note that the gauge group* ($\mathrm{Gau}(A)$) *is the group of vertical automorphisms of the vector bundle; there is no compatibility condition with the anchor or brackets. The action is defined a $(\psi\nabla)_u v := \psi(\nabla_u \psi^{-1}(v))$, where $\psi \in \mathrm{Gau}(A)$.*

Proposition 8. *Let (A, ρ) be an anchored vector bundle. Then, the $\mathrm{End}(\mathcal{C}(A))$ set is a truss.*

Proof. This follows from general facts about endomorphisms of abelian heaps (see Section 3.7 in [9]). We explicitly construct $[\Phi^{(1)}, \Phi^{(2)}, \Phi^{(3)}]$ via its evaluation. Specifically,

$$\begin{aligned}\big([\Phi^{(1)}, \Phi^{(2)}, \Phi^{(3)}]\nabla\big)_u v &:= [\Phi^{(1)}\nabla, \Phi^{(2)}\nabla, \Phi^{(3)}\nabla]_u v \\ &= \nabla_{\phi^{(1)}u} v - \nabla_{\phi^{(2)}u} v + \nabla_{\phi^{(3)}u} v \\ &\quad + \omega^{(1)}(u, v) - \omega^{(2)}(u, v) + \omega^{(3)}(u, v)\,.\end{aligned}$$

Thus, we have the structure of an abelian heap on the $\mathrm{End}(\mathcal{C}(A))$ set. The binary product is given as

$$\big(\Phi \circ \Phi'\nabla\big)_u v := \nabla_{\phi\phi' u} v + \omega(u, v) + \omega'(u, v)\,.$$

Defining $\Phi \circ [\Phi^{(1)}, \Phi^{(2)}, \Phi^{(3)}] := [\Phi \circ \Phi^{(1)}, \Phi \circ \Phi^{(2)}, \Phi \circ \Phi^{(3)}]$, we have the left distributivity property. Explicitly,

$$\begin{aligned}\big(\Phi \circ [\Phi^{(1)}, \Phi^{(2)}, \Phi^{(3)}]\nabla\big)_u v &:= [\Phi\Phi^{(1)}\nabla, \Phi\Phi^{(2)}\nabla, \Phi\Phi^{(3)}\nabla]_u v \\ &= \nabla_{\phi\phi^{(1)}u} v - \nabla_{\phi\phi^{(2)}u} v + \nabla_{\phi\phi^{(3)}u} v \\ &\quad + \big(\omega^{(1)}(u, v) + \omega(u, v)\big) - \big(\omega^{(2)}(u, v) + \omega(u, v)\big) \\ &\quad + \big(\omega^{(3)}(u, v) + \omega(u, v)\big)\,.\end{aligned}$$

The right distributivity property follows similarly. □

For completeness, given a Lie algebroid, the torsion transform is expressed as

$$T_{\Phi\nabla}(u,v) = T_\nabla(\phi(u),v) + T_\nabla(u,\phi(v)) - \nabla_u\phi(v) + (-1)^{\widetilde{u}\widetilde{v}}\,\nabla_v\phi(u) + [\phi(u),v] + [u,\phi(v)] - [u,v] + \omega(u,v) - (-1)^{\widetilde{u}\widetilde{v}}\,\omega(v,u)\,,$$

and the curvature transforms as

$$R_{\Phi\nabla}(u,v)w = R_\nabla(\phi(u),\phi(v))w + \nabla_{\phi(u)}\omega(v,w) - (-1)^{\widetilde{u}\widetilde{v}}\,\omega(v,\nabla_{\phi(u)}w) - (-1)^{\widetilde{u}\widetilde{v}}\,\nabla_{\phi(v)}\omega(u,w) + \omega(u,\nabla_{\phi(v)}w) + \omega(u,\omega(v,w)) - (-1)^{\widetilde{u}\widetilde{v}}\,\omega(v,\omega(u,w)) - \omega([u,v],w)\,.$$

Note that if $\omega = 0$, i.e., we are only considering bundle endomorphisms (over the identity), then $R_{\Phi\nabla}(u,v)w = R_\nabla(\phi(u),\phi(v))w$.

Example 15. *Consider the tangent bundle* (T*M) of a supermanifold. As a Lie algebroid, the anchor is the trivial map, and the Lie bracket is the standard Lie bracket of vector fields. We denote the set of affine connections as $\mathcal{C}(M)$. We fix $\phi : \mathrm{T}M \to \mathrm{T}M$ to be the identity map. Then, any even $(1,2)$ tensor defines a "restricted" endomorphism of $\mathcal{C}(M)$ or a "general shift" via*

$$\nabla_X Y \longmapsto \nabla_X Y + \omega(X,Y)\,.$$

The truss structure is defined as

$$[\omega^{(1)},\omega^{(2)},\omega^{(3)}] := \omega^{(1)} - \omega^{(2)} + \omega^{(3)}\,,$$

the binary product being the addition of tensors, i.e., $\omega \circ \omega' = \omega + \omega'$. The distributivity of the binary product over the ternary product is evident, i.e.,

$$\omega \circ [\omega^{(1)},\omega^{(2)},\omega^{(3)}] := \left(\omega^{(1)} + \omega\right) - \left(\omega^{(2)} + \omega\right) + \left(\omega^{(3)} + \omega\right),$$

and similarly for the right distributivity. The torsion transforms as

$$T_{\Phi\nabla}(X,Y) = T_\nabla(X,Y) + \omega(X,Y) - (-1)^{\widetilde{X}\widetilde{Y}}\,\omega(Y,X)\,.$$

The Riemannian curvature transforms as

$$R_{\Phi\nabla}(X,Y)Z = R_\nabla(X,Y)Z + \nabla_X\omega(Y,Z) - (-1)^{\widetilde{X}\widetilde{Y}}\,\omega(Y,\nabla_X Z) - (-1)^{\widetilde{X}\widetilde{Y}}\,\nabla_Y\omega(X,Z) + \omega(X,\nabla_Y Z) + \omega(X,\omega(Y,Z)) - (-1)^{\widetilde{X}\widetilde{Y}}\,\omega(Y,\omega(X,Z)) - \omega([X,Y],Z)\,.$$

Remark 7. *General shifts in affine connections (see Example 15) are well-known in the literature and are studied in the context of metric-affine gravity (MAG) (see, for example, [24] and references therein). The new aspect here is the realisation that there is a truss behind these shifts.*

3. Concluding Remarks

In this note, we have presented the heap structure on a set of linear connections on a Lie algebroid. Some preliminary consequences, such as the torsion and curvature of a triple product of connections, have been presented. A key result here is the explicit construction of the endomorphism truss of linear connections.

We remark that the results of this note extend verbatim to the algebraic setting of (left) connections on anchored modules and Lie–Rinehart pairs over associative supercommutative, unital superalgebras. Indeed, we have avoided using local descriptions in any calculations. With a little effort, we expect the results presented here to generalise to the setting of almost commutative Lie algebroids (see [25]).

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