THE SIMPLE BEAM-LOADING THEORY FOR CONSTANT-GRADIENT LINACS
(A Supplement to TN-63-9)

1. We have given in TN-63-9 the functional form of $f(t + \tau)$ for constant energy operation under the condition that $\tau \geq t'_F$. Now we wish to discuss the other case where $\tau \leq t'_F$.

The condition for attaining constant energy gain is given by Eq. (23) in TN-63-9, namely,

$$\int_{t+\tau-t'_F}^{t+\tau} \alpha_o \psi'(t + \tau - \xi)\, f'(\xi) U(\xi) d\xi = U\left(t'_F - t\right) \cdot \chi \alpha_o \psi'(t) \left\{ \alpha_o \psi(t'_F) - \alpha_o \psi(t) \right\} . \tag{1}$$

As discussed therein, the integrand $\alpha_o \psi'(t + \tau - \xi)\, f'(\xi)$ must either vanish or be an oscillating function of period t'_F when $\xi \geq \tau$, so that the integral over any whole period may vanish when $t \geq t'_F$.

The zeroth-order solution of Eq.(1) is not difficult to find. If the time variable is restricted to $0 \leq t \leq T \equiv M t'_F$, M being an integer, this solution is as follows:

$$\begin{aligned} f'_o(\xi) = {} & \chi \alpha_o v_{go} \exp\left(2\alpha_o v_{go} \tau\right) \cdot \psi'_o(2\xi) \\ & \cdot \left[\frac{\psi_o\left(t'_F\right)}{\psi_o(\tau)} \sum_{m=0}^{M} \left\{ U\left(\xi - m t'_F\right) - U\left(\xi - \tau - m t'_F\right) \right\} \cdot \left\{ 1/\psi'_o\left(m t'_F\right) \right\} \right. \\ & \left. - \frac{\psi'_o\left(t'_F\right)}{\psi'_o(\tau)} \sum_{m=1}^{M} \left\{ U\left(\xi - \tau - m t'_F + t'_F\right) - U\left(\xi - \tau - m t'_F\right) \right\} \cdot \left\{ 1/\psi'_o\left(m t'_F\right) \right\} \right] \end{aligned} \tag{2}$$

Here we may note the similarity between this equation and Eq. (2) in

TN-63-7 for constant-α linacs. When $\tau \to t'_F$,

$$f'_o(\xi) \to \left\{U(\xi) - U(\xi - \tau)\right\} \cdot \chi\alpha_o \exp\left(2\alpha_o v_{go}\tau\right) \cdot \psi'_o(2\xi).$$

This agrees with Eq.(30) in TN-63-9.

By integrating Eq.(2) with respect to ξ and inserting proper integration constants, we obtain

$$\begin{aligned} f_o(\xi) = {} & (\chi|2)\alpha_o v_{go} \exp(2\alpha_o v_{go}\tau) \cdot \left\{\psi_o(t'_F)/\psi_o(\tau)\right\} \\ & \cdot \left[\sum_{m=0}^{M} \left\{U\left(\xi - mt'_F\right) - U\left(\xi - \tau - mt'_F\right)\right\} \cdot \left\{\psi_o(2\xi) - \psi_o\left(2mt'_F\right)\right\} \Big/ \psi'_o(mt'_F) \right. \\ & \left. + \sum_{m=0}^{M} U\left(\xi - \tau - mt'_F\right) \cdot \left\{\psi_o\left(2\tau + 2mt'_F\right) - \psi_o\left(2mt'_F\right)\right\} \Big/ \psi'_o\left(mt'_F\right)\right] \\ & - (\chi|2)\alpha_o v_{go} \exp\left(2\alpha_o v_{go}\tau\right) \cdot \left\{\psi'_o\left(t'_F\right) \Big/ \psi'_o(\tau)\right\} \\ & \cdot \left[\sum_{m=1}^{M} \left\{U\left(\xi - \tau - mt'_F + t'_F\right) - U\left(\xi - \tau - mt'_F\right)\right\} \cdot \left\{\psi_o(2\xi) \right.\right. \\ & \left. - \psi_o\left(2\tau + 2mt'_F - 2t'_F\right)\right\} \Big/ \psi'_o\left(mt'_F\right) \\ & \left. + \sum_{m=1}^{M} U\left(\xi - \tau - mt'_F\right) \cdot \left\{\psi_o\left(2\tau + 2mt'_F\right) - \psi_o\left(2\tau + 2mt'_F - 2t'_F\right)\right\} \Big/ \psi'_o\left(mt'_F\right)\right]. \end{aligned} \tag{3}$$

When $\tau \to t'_F$,

$$f_o(\xi) \to \left(\chi\alpha_o | 2\right) \exp\left(2\alpha_o v_{go}\tau\right) \cdot \left[\left\{U(\xi) - U(\xi - \tau)\right\} \psi_o(2\xi) + U(\xi - \tau)\, \psi_o(2\tau)\right]$$

This is in agreement with Eq. (33) in TN-63-9.

The first-order equation satisfied by $f_1'(\xi)$ is obtained from Eq. (29) in TN-63-9 by cancelling out the zeroth-order terms. This is

$$\int_{t+\tau-t_F'}^{t+\tau} d\xi\, U(\xi) \exp\left\{2\alpha_o v_{go}(\xi - \tau)\right\} \cdot f_1'(\xi) = \Phi(t)$$

$$+ U\left(t_F' - t\right) \cdot \chi\alpha_o \cdot \left[\left\{\psi_o(t) - \psi_o\left(t_F'\right)\right\} + 2\left\{\psi_o\left(t + t_F'\right) - \psi_o\left(2t_F'\right)\right\} + 3\left\{\psi_o\left(2t_F'\right) - \psi_o(2t)\right\}\right], \tag{4}$$

where

$$\Phi(t) = -\exp\left(-2\alpha_o v_{go} t\right) \cdot \int_{t+\tau-t_F'}^{t+\tau} d\xi\, U(\xi) \exp\left\{4\alpha_o v_{go}(\xi - \tau)\right\} \cdot 2f_o'(\xi) \tag{5}$$

Now we introduce the following notations:

$$I_{t\xi}\left\{f_1'(\xi)\right\} = \int_{t+\tau-t_F'}^{t+\tau} d\xi\, U(\xi) \exp\left\{2\alpha_o v_{go}(\xi - \tau)\right\} \cdot f_1'(\xi) \tag{6}$$

and

$$f_1'(\xi) = u_a'(\xi) + u_b'(\xi) + u_c'(\xi) + u_d'(\xi), \tag{7}$$

where

$$I_{t\xi}\left\{u_a'(\xi)\right\} = U\left(t_F' - t\right) \cdot \chi\alpha_o \cdot \left\{\psi_o(t) - \psi_o\left(t_F'\right)\right\}, \tag{8a}$$

$$I_{t\xi}\left\{u_b'(\xi)\right\} = U\left(t_F' - t\right) \cdot 2\chi\alpha_o \cdot \left\{\psi_o\left(t + t_F'\right) - \psi_o\left(2t_F'\right)\right\}, \tag{8b}$$

$$I_{t\xi}\left\{u_c'(\xi)\right\} = U\left(t_F' - t\right) \cdot 3\chi\alpha_o \cdot \left\{\psi_o\left(2t_F'\right) - \psi_o(2t)\right\}, \tag{8c}$$

and

$$I_{t\xi}\left\{u_d'(\xi)\right\} = \Phi(t). \tag{8d}$$

Obviously,

$$u_a'(\xi) = - f_o'(\xi). \tag{9}$$

Thus there are three unknown functions to be determined in order to obtain the solution

$$f'(\xi) = f_o'(\xi) + \left(v_{go}/c\right) f_1'(\xi).$$

Two of these functions, $u_b'(\xi)$ and $u_c'(\xi)$, are not difficult to obtain. They are as follows:

$$\begin{aligned} u_b'(\xi) = & - 2\chi\alpha_o v_{go} \exp\left(2\alpha_o v_{go}\tau\right) \cdot \psi_o'\left(2\xi + t_F'\right) \\ & \cdot \left[\frac{\psi_o(t_F')}{\psi_o(\tau)} \sum_{m=0}^{M} \left\{U\left(\xi - mt_F'\right) - U\left(\xi - \tau - mt_F'\right)\right\} \cdot \left\{1/\psi_o'\left(mt_F'\right)\right\}\right. \\ & \left. - \frac{\psi_o'(t_F')}{\psi_o'(\tau)} \sum_{m=1}^{M} \left\{U\left(\xi - \tau - mt_F' + t_F'\right) - U\left(\xi - \tau - mt_F'\right)\right\} \cdot \left\{1/\psi_o'\left(mt_F'\right)\right\}\right] ; \end{aligned} \tag{10}$$

$$\begin{aligned} u_c'(\xi) = & \; 6\chi\alpha_o v_{go} \exp\left(2\alpha_o v_{go}\tau\right) \cdot \psi_o'(3\xi) \\ & \cdot \left[\frac{\psi_o(2t_F')}{\psi_o(2\tau)} \sum_{m=0}^{M} \left\{U\left(\xi - mt_F'\right) - U\left(\xi - \tau - mt_F'\right)\right\} \cdot \left\{1/\psi_o'\left(2mt_F'\right)\right\}\right. \\ & \left. - \frac{\psi_o'(2t_F')}{\psi_o'(2\tau)} \sum_{m=1}^{M} \left\{U\left(\xi - \tau - mt_F' + t_F'\right) - U\left(\xi - \tau - mt_F'\right)\right\} \cdot \left\{1/\psi_o'\left(2mt_F'\right)\right\}\right]. \end{aligned} \tag{11}$$

The solution of Eq. (8d) for the remaining function $u_d'(\xi)$ is quite tedious, because $\Phi(t)$ is a complicated function. Before describing

$\Phi(t)$ and $u_d'(\xi)$ we further introduce the following notations:

$$C_1 = \psi_o'(t_F')/\psi_o'(0), \tag{12a}$$

$$C_2 = \frac{\psi_o(t_F')}{\psi_o(\tau)} \cdot \frac{\psi_o'(\tau)}{\psi_o'(0)}, \tag{12b}$$

$$C_3 = \frac{\psi_o(t_F')}{\psi_o(\tau)} \cdot \frac{\psi_o'(\tau)}{\psi_o'(t_F')} = C_2/C_1, \tag{12c}$$

$$C_4 = \psi_o'(2t_F')/\psi_o'(0) = (C_1)^2, \tag{12d}$$

$$C_5 = \frac{\psi_o(t_F')}{\psi_o(\tau)} \cdot \frac{\psi_o'(\tau)}{\psi_o'(0)} - \frac{\psi_o'(t_F')}{\psi_o'(\tau)} \cdot \frac{\psi_o'(t_F')}{\psi_o'(0)}, \tag{12e}$$

and

$$C_6 = \frac{\psi_o(t_F')}{\psi_o(\tau)} \cdot \left\{\frac{\psi_o'(\tau)}{\psi_o'(0)} - \frac{\psi_o'(t_F')}{\psi_o'(0)}\right\}. \tag{12f}$$

Using these abbreviation constants we obtain

$$\begin{aligned}
- \Phi(t) \Big/ \left\{2\chi\alpha_o v_{go} \exp\left(-2\alpha_o v_{go} t\right)\right\} &= \left\{U(t) - U\left(t + \tau - t_F'\right)\right\} \cdot \left(C_2\tau - t\right) \\
&+ \left\{U\left(t + \tau - t_F'\right) - U\left(t - t_F'\right)\right\} \cdot \left\{C_2\left(t_F' - t\right) + C_3\left(t + \tau - t_F'\right) - t\right\} \\
&+ \sum_{m=1}^{M} \left[U\left(t - mt_F'\right) - U\left\{t + \tau - (m+1)t_F'\right\}\right] \cdot \left\{\psi_o'(0)/\psi_o'\left(mt_F'\right)\right\} \\
&\cdot \left[C_2\tau - C_1\left\{(m+1)t_F' - t\right\} - \left(t - mt_F'\right)\right] \\
&+ \sum_{m=1}^{M-1} \left[U\left\{t + \tau - (m+1)t_F'\right\} - U\left\{t - (m+1)t_F'\right\}\right] \cdot \left\{\psi_o'(0) \Big/ \psi_o'\left(mt_F'\right)\right\} \\
&\cdot \Big[C_2\left\{(m+1)t_F' - t\right\} + C_3\left\{t + \tau - (m+1)t_F'\right\} \\
&- C_1\left\{(m+1)t_F' - t\right\} - \left(t - mt_F'\right)\Big]
\end{aligned} \tag{13}$$

and

$$- u_d'(\xi)/2\chi\alpha_o v_{go} = \left\{U(\xi) - U(\xi - \tau)\right\} \cdot C_4 \left\{1 + 2\alpha_o v_{go}(\tau - \xi)\right\} \cdot \left\{\psi_o'(2\xi - 2\tau)\Big/\psi_o'(0)\right\}$$

$$+ \sum_{m=0}^{M} \left\{U\left(\xi - mt_F'\right) - U\left(\xi - \tau - mt_F'\right)\right\} \cdot C_5 \left\{\psi_o'(\xi - \tau)\Big/\psi_o'(0)\right\}$$

$$+ \sum_{m=1}^{M} \left\{U\left(\xi - mt_F'\right) - U\left(\xi - \tau - mt_F'\right)\right\} \cdot \left\{\psi_o'(2\xi - 2\tau)\Big/\psi_o'\left(mt_F'\right)\right\}$$

$$\cdot \left[C_6 m \left\{1 + 2\alpha_o v_{go}\left(mt_F' + \tau - \xi\right)\right\} + C_1 2\alpha_o v_{go} mt_F' - C_2 2\alpha_o v_{go} m\tau\right]$$

$$+ \sum_{m=0}^{M-1} \left[U\left(\xi - \tau - mt_F'\right) - U\left\{\xi - (m+1)t_F'\right\}\right] \cdot \left\{\psi_o'(2\xi - 2\tau)\Big/\psi_o'\left(mt_F'\right)\right\}$$

$$\cdot \left[\left\{C_1 m - (m+1)\right\} \cdot \left\{1 + 2\alpha_o v_{go}\left(mt_F' + \tau - \xi\right)\right\}\right.$$

$$\left. + C_1 2\alpha_o v_{go} mt_F' - C_2 2\alpha_o v_{go}(m+1)\tau\right]. \qquad (14)$$

When $\tau \to t_F'$,

$$u_b'(\xi) \to - \left\{U(\xi) - U(\xi - \tau)\right\} \cdot 2\chi\alpha_o \exp\left(2\alpha_o v_{go} t_F'\right) \psi_o'\left(2\xi + t_F'\right);$$

$$u_c'(\xi) \to + \left\{U(\xi) - U(\xi - \tau)\right\} \cdot 6\chi\alpha_o \exp\left(2\alpha_o v_{go} t_F'\right) \psi_o'(3\xi),$$

and

$$u_d'(\xi) \to - \left\{U(\xi) - U(\xi - \tau)\right\} \cdot 2\chi\alpha_o \exp\left(2\alpha_o v_{go} t_F'\right)$$

$$\cdot \left\{1 + 2\alpha_o v_{go}(\tau - \xi)\right\} \cdot \psi_o'\left(2\xi + t_F'\right).$$

Hence,

$$f_1'(\xi) \to \left\{U(\xi) - U(\xi - \tau)\right\} \cdot \chi\alpha_o \cdot \exp\left(2\alpha_o v_{go} t_F'\right) \cdot \left[6\psi_o'(3\xi) - 2\psi_o'\left(2\xi + t_F'\right)\right.$$

$$\left. - \psi_o'(2\xi) - 2\left\{1 + 2\alpha_o v_{go}(\tau - \xi)\right\} \cdot \psi_o'\left(2\xi + t_F'\right)\right]. \qquad (15)$$

This is in agreement with Eq. (32) in TN-63-9.

Having obtained $f_1'(\xi)$ we may then integrate to obtain $f_1(\xi)$ in a straight-forward manner. The expression of $f_1(\xi)$ will not be given here because of its considerable length.

2. Up to this point it has been assumed that $f(\xi)$ is a continuous function and $f(0) = 0$. When $f(0) \neq 0$, we may proceed as in TN-63-7 to separate $f(\xi)$ into two parts,

$$f(\xi) = f(0) + h(\xi), \tag{16}$$

so that $h(0) = 0$.

Let

$$\alpha_o L_{a1}(t) = f(0) \int_{t+\tau-t_F'}^{t+\tau} d\xi \; U(\xi)\alpha_o \psi'(t + \tau - \xi) \tag{17a}$$

and

$$\alpha_o L_{a2}(t) = \int_{t+\tau-t_F'}^{t+\tau} d\xi \; U(\xi) \; \alpha_o \psi'(t + \tau - \xi) \; h(\xi) \; . \tag{17b}$$

Then the contribution to the electron energy gain from $f(\xi)$ is

$$\alpha_o L_a(t) = \alpha_o L_{a1}(t) + \alpha_o L_{a2}(t). \tag{18}$$

In case $\tau \geq t_F'$, $\alpha_o L_{a1}(t) = \left\{f(0)|2\right\}\cdot\left(1 - C_1\right)\cdot\left\{1 + \left(v_{go}/c\right)C_1\right\}$. Here C_1 is given by Eq. (12a). Since $\alpha_o L_{a1}(t)$ is independent of t, no further discussion is needed.

In case $\tau \leq t_F'$,

$$\begin{aligned}\alpha_o L_{a1}(t) = {} & f(0)\alpha_o \cdot \left[\psi_o(t + \tau) - U\left(t + \tau - t_F'\right)\left\{\psi_o(t + \tau) - \psi_o\left(t_F'\right)\right\}\right] \\ & + \left(v_{go}/c\right)f(0)\alpha_o \cdot \left[\left\{\psi_o(2t + 2\tau) - \psi_o(t + \tau)\right\} - U\left(t + \tau - t_F'\right)\right. \\ & \left. \cdot\left\{\psi_o(2t + 2\tau) - \psi_o\left(2t_F'\right) - \psi_o(t + \tau) + \psi_o\left(t_F'\right)\right\}\right].\end{aligned} \tag{19}$$

This case may be treated by the same procedure as discussed in TN-63-7. Thus, we consider $h(\xi)$ to be the solution which satisfies $\alpha_o L_{a2}(t) + \alpha_o L_b(t) = \text{const.}$, and introduce one additional part $w(\xi)$ into $f(\xi)$,

$$f(\xi) = f(0) + h(\xi) + w(\xi), \tag{20}$$

so that the sum of $\alpha_o L_{a1}(t)$ and

$$\alpha_o L_{a3}(t) = \int_{t+\tau-t'_F}^{t+\tau} d\xi \; U(\xi) \, \alpha_o \psi'(t + \tau - \xi) \, w(\xi) \tag{21}$$

is a constant. From this condition it follows that

$$\alpha_o L_b(t) + \sum_{\mu=1}^{3} \alpha_o L_{a\mu}(t) = \text{const.},$$

i.e., the total gain of electron energy is constant.

The remaining problem is to determine $w(\xi)$ from the following equation:

$$\frac{d}{dt}\left\{\alpha_o L_{a1}(t) + \alpha_o L_{a3}(t)\right\} = 0. \tag{22}$$

As may be inferred from the case of constant-α linacs discussed in TN-63-7, Eq. (22) cannot be solved for large intervals of time unless $w(\xi)$ is allowed to have discontinuities at points $\xi = t'_F, 2t'_F, 3t'_F, \ldots$ The solution of Eq. (22) can most conveniently be found step by step. The first step is to consider the small interval of time, $0 \leq t < t'_F - \tau$. In this interval we have, according to Eq. (19) and Eq. (21),

$$\frac{d}{dt}\left\{\alpha_o L_{a1}(t)\right\} = f(0)\alpha_o \cdot \left[\psi'_o(t + \tau) - \left(v_{go}/c\right)\left\{\psi'_o(t + \tau) - 2\psi'_o(2t + 2\tau)\right\}\right] \tag{23}$$

and

$$\frac{d}{dt}\left\{\alpha_o L_{a3}(t)\right\} = \int_0^{t+\tau} d\xi \; \alpha_o \psi'(t + \tau - \xi) \, w'(\xi). \tag{24}$$

In Eq. (24), $\psi'(\xi) = \psi_o'(\xi) - (v_{go}/c)\left\{\psi_o'(\xi) - 2\psi_o'(2\xi)\right\}$ and $w'(\xi) = w_o'(\xi) + (v_{go}/c)\ w_1'(\xi)$. By substituting Eqs. (23) and (24) into Eq. (22) and then separating the zeroth- and the first-order terms, we obtain

$$\int_0^{t+\tau} d\xi \alpha_o \psi_o'(t + \tau - \xi)\ w_o'(\xi) = -\ f(0)\ \alpha_o \psi_o'(t + \tau) \tag{25a}$$

and

$$\int_0^{t+\tau} d\xi \alpha_o \psi_o'(t + \tau - \xi) w_1'(\xi) = -\ 2f(0)\alpha_o \psi_o'(2t + 2\tau) - \int_0^{t+\tau} d\xi\ 2\alpha_o \psi_o'\left(2t + 2\tau - 2\xi\right)\ w_o'(\xi). \tag{25b}$$

The zeroth-order equation may be solved at once. The solution is

$$w_o'(\xi) = -\left\{U(\xi) - U(\xi - \tau)\right\} \cdot \frac{f(0)}{\tau} \times \frac{\psi_o'(\xi)}{\psi_o'(0)} . \tag{26}$$

On substituting this expression of $w_o'(\xi)$, Eq. (25b) becomes

$$\int_0^{t+\tau} d\xi \alpha_o \psi_o'(t + \tau - \xi) w_1'(\xi) = 2f(0)\alpha_o \psi_o'(2t + 2\tau) \cdot \left\{\frac{1}{\tau} \cdot \frac{\psi_o(\tau)}{\psi_o'(\tau)} - 1\right\} . \tag{27}$$

From this equation $w_1'(\xi)$ can easily be obtained.

$$w_1'(\xi) = 2\,\frac{f(0)}{\tau} \left\{\frac{1}{\tau}\,\frac{\psi_o(\tau)}{\psi_o'(\tau)} - 1\right\} \cdot \left[\left\{U(\xi) - U(\xi - \tau)\right\} \cdot \left\{\frac{\psi_o'(\xi)}{\psi_o'(0)} - 2\alpha_o v_{go} \tau\, \frac{\psi_o'(2\xi)}{\psi_o'(0)}\right\} - \left\{U(\xi - \tau) - U\left(\xi - t_F'\right)\right\} \cdot 2\alpha_o v_{go} \tau\ \frac{\psi_o'(2\xi)}{\psi_o'(0)}\right] . \tag{28}$$

Hence,

$$w_o(\xi) = -\frac{f(0)}{\tau}\cdot\left[\left\{U(\xi) - U(\xi - \tau)\right\}\frac{\psi_o(\xi)}{\psi_o'(0)} + U(\xi - \tau)\frac{\psi_o(\tau)}{\psi_o'(0)}\right] \tag{29a}$$

and

$$w_1(\xi) = 2\frac{f(0)}{\tau}\cdot\left\{\frac{1}{\tau}\frac{\psi_o(\tau)}{\psi_o'(\tau)} - 1\right\}\cdot\left[\left\{U(\xi) - U(\xi - \tau)\right\}\left\{\frac{\psi_o(\xi)}{\psi_o'(0)} - \alpha_o v_{go}\tau\frac{\psi_o(2\xi)}{\psi_o'(0)}\right\} + U(\xi - \tau)\frac{\psi_o(\tau)}{\psi_o'(0)} - \left\{U(\xi - \tau) - U\left(\xi - t_F'\right)\right\}\alpha_o v_{go}\tau\frac{\psi_o(2\xi)}{\psi_o'(0)}\right] \tag{29b}$$

Having obtained $w_o(\xi)$ and $w_1(\xi)$ we may then calculate $\alpha_o L_{a_3}(t)$, using Eq. (21).

$$\alpha_o L_{a_3}(t) = -f(0)\alpha_o\psi_o(t+\tau) + \frac{f(0)}{2}\cdot\left\{1 - \frac{1}{\tau}\frac{\psi_o(\tau)}{\psi_o'(0)}\right\} + \left(v_{go}/c\right)\cdot\left[f(0)\alpha_o\left\{\psi_o(t+\tau) - \psi_o(2t+2\tau)\right\} + f(0)\cdot\left\{\frac{1}{\tau}\frac{\psi_o(\tau)}{\psi_o'(\tau)} - 1\right\}\left\{\frac{1}{\tau}\frac{\psi_o(\tau)}{\psi_o'(0)} - \frac{1}{2}\right\}\right]. \tag{30}$$

Therefore, with reference to Eq. (19),

$$\alpha_o L_{a_1}(t) + \alpha_o L_{a_3}(t) = \frac{f(0)}{2}\cdot\left\{1 - \frac{1}{\tau}\cdot\frac{\psi_o(\tau)}{\psi_o'(0)}\right\} + \left(v_{go}/c\right) f(0)\cdot\left\{\frac{1}{\tau}\frac{\psi_o(\tau)}{\psi_o'(\tau)} - 1\right\}\cdot\left\{\frac{1}{\tau}\frac{\psi_o(\tau)}{\psi_o'(0)} - \frac{1}{2}\right\}. \tag{31}$$

As expected, this is independent of t.

In the next interval of time, $t_F' - \tau \leq t < t_F'$, the two equations derived from Eq. (22) are as follows:

$$\int_{t+\tau-t_F'}^{t+\tau} d\xi \alpha_o \psi_o'(t + \tau - \xi)\, w_o'(\xi) + \alpha_o \psi_o'\left(t + \tau - t_F'\right)\left\{ w_o\left(t_F' + 0\right) - w_o\left(t_F' - 0\right)\right\} = 0; \tag{32a}$$

$$\begin{aligned} \int_{t+\tau-t_F'}^{t+\tau} d\xi \alpha_o \psi_o'(t + \tau - \xi)\, w_1'(\xi) + \alpha_o \psi_o'\left(t + \tau - t_F'\right)\left\{ w_1\left(t_F' + 0\right) - w_1\left(t_F' - 0\right)\right\} \\ = - \int_{t+\tau-t_F'}^{t+\tau} d\xi 2\alpha_o \psi_o'\left(2t + 2\tau - 2\xi\right) w_o'(\xi) \\ - 2\alpha_o \psi_o'\left(2t + 2\tau - 2t_F'\right)\left\{ w_o\left(t_F' + 0\right) - w_o\left(t_F' - 0\right)\right\}. \end{aligned} \tag{32b}$$

Eq. (25a) and Eq. (32a) must be consistent with each other when $t + \tau \to t_F'$. From this requirement we obtain

$$w_o\left(t_F' + 0\right) - w_o\left(t_F' - 0\right) = f(0) \cdot C_1 . \tag{33a}$$

Similarly, Eq. (25b) and Eq. (32b) must be consistent with each other when $t + \tau \to t_F'$. Thus,

$$w_1\left(t_F' + 0\right) - w_1\left(t_F' - 0\right) = 2f(0) \cdot \left(C_4 - C_1\right). \tag{33b}$$

Because of these results Eqs.(32a) and (32b) become, respectively,

$$\int_{t+\tau-t_F'}^{t+\tau} d\xi \alpha_o \psi_o'(t + \tau - \xi)\ \ w_o'(\xi) = - C_1 f(0) \alpha_o \psi_o'\left(t + \tau - t_F'\right) \tag{34a}$$

and

$$\int_{t+\tau-t_F'}^{t+\tau} d\xi \alpha_o \psi_o'(t+\tau-\xi)\, w_1'(\xi) = -2f(0)\alpha_o \left\{\left(C_4 - C_1\right) \psi_o'\left(t+\tau-t_F'\right) + C_1\psi_o'\left(2t+2\tau-2t_F'\right)\right\} - \int_{t+\tau-t_F'}^{t+\tau} d\xi 2\alpha_o \psi_o'(2t+2\tau-2\xi)\, w_o'(\xi). \tag{34b}$$

Eq. (34a) can be solved easily. The solution is

$$w_o'(\xi) = -\frac{f(0)}{\tau}\cdot\frac{\psi_o'(\xi)}{\psi_o'(0)}\cdot\left[\left\{U(\xi) - U(\xi-\tau)\right\} + \left\{U\left(\xi - t_F'\right) - U\left(\xi - \tau - t_F'\right)\right\}\right]. \tag{35}$$

Having determined $w_o'(\xi)$ we may then solve Eq. (34b) to obtain

$$w_1'(\xi) = 2\,\frac{f(0)}{\tau}\left\{\frac{1}{\tau}\,\frac{\psi_o(\tau)}{\psi_o'(\tau)} - 1\right\}\cdot\left[\left\{U(\xi) - U(\xi-\tau)\right\}\cdot\left\{\frac{\psi_o'(\xi)}{\psi_o'(0)} - 2\alpha_o v_{go}\tau\,\frac{\psi_o'(2\xi)}{\psi_o'(0)}\right\} - \left\{U(\xi-\tau) - U\left(\xi - t_F'\right)\right\}\cdot 2\alpha_o v_{go}\tau\,\frac{\psi_o'(2\xi)}{\psi_o'(0)} + \left\{U\left(\xi - t_F'\right) - U\left(\xi-\tau-t_F'\right)\right\}\cdot\left\{\frac{\psi_o'(\xi)}{\psi_o'(0)} - 2\alpha_o v_{go}\tau\,\frac{\psi_o'(2\xi)}{\psi_o'\left(t_F'\right)}\right\}\right] + 2\,\frac{f(0)}{\tau}\cdot\left\{U\left(\xi - t_F'\right) - U\left(\xi - \tau - t_F'\right)\right\}\cdot\left\{1 + 2\alpha_o v_{go}\tau - \frac{\psi_o'\left(t_F'\right)}{\psi_o'(\tau)}\right\}\cdot\frac{\psi_o'(2\xi)}{\psi_o'\left(t_F'\right)}. \tag{36}$$

Thus,

$$w_o(\xi) = -\frac{f(0)}{\tau}\cdot\left[\left\{U(\xi) - U(\xi - \tau)\right\}\frac{\psi_o(\xi)}{\psi_o'(0)} + U(\xi - \tau)\frac{\psi_o(\tau)}{\psi_o'(0)} + \left\{U\left(\xi - t_F'\right) - U\left(\xi - \tau - t_F'\right)\right\}\cdot C_1\left\{\frac{\psi_o\left(\xi - t_F'\right)}{\psi_o'(0)} - \tau\right\}\right] \tag{37}$$

and

$$\begin{aligned} w_1(\xi) = 2\,\frac{f(0)}{\tau}\cdot\left\{\frac{1}{\tau}\,\frac{\psi_o(\tau)}{\psi_o'(\tau)} - 1\right\} \\ \cdot\Bigg[\left\{U(\xi) - U(\xi - \tau)\right\}\cdot\left\{\frac{\psi_o(\xi)}{\psi_o'(0)} - \alpha_o v_{go}\tau\,\frac{\psi_o(2\xi)}{\psi_o'(0)}\right\} \\ + U(\xi - \tau)\,\frac{\psi_o(\tau)}{\psi_o'(0)} - \left\{U(\xi - \tau) - U\left(\xi - t_F'\right)\right\}\alpha_o v_{go}\tau\,\frac{\psi_o(2\xi)}{\psi_o'(0)} \\ + \left\{U\left(\xi - t_F'\right) - U\left(\xi - \tau - t_F'\right)\right\}\cdot\Bigg\{C_1\,\frac{\psi_o\left(\xi - t_F'\right)}{\psi_o'(0)} - \alpha_o v_{go}\tau\,\frac{\psi_o(2\tau)}{\psi_o'(0)} \\ - \alpha_o v_{go}\tau\cdot C_1\,\frac{\psi_o\left(2\xi - 2t_F'\right)}{\psi_o'(0)}\Bigg\}\Bigg] \\ + \frac{f(0)}{\tau}\cdot\left\{U\left(\xi - t_F'\right) - U\left(\xi - \tau - t_F'\right)\right\}\cdot\Bigg[\left\{1 + 2\alpha_o v_{go}\tau - \frac{\psi_o\left(t_F'\right)}{\psi_o'(\tau)}\right\} \\ \cdot C_1\,\frac{\psi_o\left(2\xi - 2t_F'\right)}{\psi_o'(0)} + 2\tau\left(C_4 - C_1\right)\Bigg]. \end{aligned} \tag{38}$$

These expressions of $w_o(\xi)$ and $w_1(\xi)$ are applicable for the time interval $0 \leq t < t_F'$. This is by far the interval of most practical interest. Clearly, we may extend these functions, if required, to later time intervals by the same procedure as described above.