Cwebs in multiparton scattering amplitude: Structures at four loops

Neelima Agarwal[1], *Sourav* Pal[2], *Aditya* Srivastav[3,*], and *Anurag* Tripathi[3]

[1]Chaitanya Bharathi Institute of Technology, Hyderabad, India
[2]Physical Research Laboratory, Ahmedabad, India
[3]Indian Institute of Technology Hyderabad, India

Abstract. Soft function exponentiates in terms of the soft anomalous dimension Γ_S; the Feynman diagrams contributing to it are called Cwebs. The colour and kinematics of a Cweb mix via a web mixing matrix – calculation of web mixing matrices at higher loop orders is a nontrivial task using replica trick. Here I discuss a recent development of an efficient algorithm to find the Cwebs that are present at any loop order, and the result of mixing matrices, and exponentiated colour factors associated with Boomerang Cweb at four loops connecting three and four Wilson lines.

1 Introduction

The study of infrared (IR) singularities in non-abelian gauge theories is essential for understanding high-energy scattering processes in collider physics. These universal singularities cancel when real emissions are added to virtual corrections but produce large logarithms in specific regions. Subtraction schemes address these issues, while the QCD factorization theorem isolates IR divergences via soft functions constructed from Wilson lines. These renormalized functions follow a renormalization group equation, with solutions involving the soft anomalous dimension.

In processes with multipartons, the soft anomalous dimension becomes a matrix, a central focus of the research discussed in this article. Calculations of the anomalous dimension have been carried out to one, two, and even three loops, with ongoing efforts to extend these to four loops. The article explores an alternative method for determining the soft function's exponentiation as diagrammatic exponentiation, which introduces the concept of Cwebs [1, 2]. The logarithm of the Soft function is a sum over all the Cwebs at each perturbative order:

$$\mathcal{S} = \exp\left[\sum_{W}\sum_{d,d'\in W} K(d)\, R_W(d,d')\, C(d')\right]. \tag{1}$$

The d here denotes a diagram in a Cweb W and its corresponding kinematic and colour factor are denoted by $K(d)$ and $C(d)$. The action of web mixing matrix R_W on the colour of a diagram $C(d)$ generates its exponentiated colour factor $\widetilde{C}(d)$,

$$\widetilde{C}(d) = \sum_{d'\in W} R_W(d,d')\, C(d'). \tag{2}$$

*e-mail: shrivastavadi333@gmail.com

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Cwebs are sets of Feynman diagrams whose contributions are summed over gluon attachments on Wilson lines, with recent extensions introducing Boomerang Cwebs [3, 4] for massive particle scattering and at five loops in massless case [5].

In this paper, the results on the mixing matrices for Boomerang Cwebs, which describe the interactions of multiple massive Wilson lines, are provided. Modifications to existing algorithms for enumerating Cwebs have been made and implemented in an updated Mathematica code, `CwebGen 2.0`, which is used to perform the calculations presented in this study. The structure of the paper includes a review of Cweb properties, a description of the modified algorithm, and explicit results for Boomerang Cwebs at four loops. These findings contribute to a deeper understanding of the soft anomalous dimension.

2 Cwebs and web mixing matrices

For the convenience of the reader, we collect here some of the definitions that are used in this article.

Definitions

Web: A set of diagrams closed under shuffles of the gluon attachments — originating from different gluon propagators or three-point or four-point gluon vertices — on each Wilson line.

Cweb: A set of diagrams closed under shuffles of the gluon attachments — originating from different gluon correlators

Multi-pronged-boomerang correlator: An m-point gluon correlator ($m \geq 2$) whose all ends attach to one Wilson line. The contributions of these for massless Wilson lines vanish identically.

Boomerang Cweb: A Cweb that contains at least one multi-pronged-boomerang correlator with $m = 2$ and none with $m > 2$.

Weight factor (*s*-factor): The weight factor $s(d)$ for a diagram d is defined as the number of ways in which the gluon correlators can be *sequentially* shrunk to their common origin.

Column weight vector: We can construct a column weight vector out of the s-factors for a Cweb with n diagrams as

$$S = \{s(d_1), s(d_2), \ldots, s(d_n)\}. \tag{3}$$

$\mathbf{W_n^{(c_2,\ldots,c_p)}(k_1, \ldots, k_n)}$: A Cweb constructed out of c_m m-point connected gluon correlators ($m = 2, \ldots, p$), where k_i denotes number of attachments on i^{th} Wilson line.

Note that the perturbative expansion for an m-point connected gluon correlator starts at $O(g^{m-2})$, while each attachment to a Wilson line carries a further power of g, the perturbative expansion for a Cweb can be written as

$$W_n^{(c_2,\ldots,c_p)}(k_1, \ldots, k_n) = g^{\sum_{i=1}^n k_i + \sum_{r=2}^p c_r(r-2)} \sum_{j=0}^{\infty} W_{n,j}^{(c_2,\ldots,c_p)}(k_1, \ldots, k_n)\, g^{2j}, \tag{4}$$

which defines the perturbative coefficients $W_{n,j}^{(c_2,\ldots,c_p)}(k_1, \ldots, k_n)$. Here the perturbative order of this Cweb is $g^{\sum_{i=1}^n k_i + \sum_{r=2}^p c_r(r-2)}$.

Cwebs are the proper building blocks of the logarithm of the Soft function — they are useful in the organization and counting of diagrammatic contributions at higher perturbative orders [1, 2]. The contribution of colour and kinematic factors to a web W can be arranged in a more transparent manner if we diagonalize the mixing matrix R:

$$W = \left(K^T Y^{-1}\right) YRY^{-1} (YC) = \sum_{j=1}^{r} \left(K^T Y^{-1}\right)_j (YC)_j \,, \tag{5}$$

where Y is the diagonalizing matrix and $YRY^{-1} \equiv \mathcal{D}$ is the diagonal matrix that we get; furthermore we have arranged $\mathcal{D}_{jj} = 1$ for $1 \leq j \leq r$. $(YC)_j$ are also referred to as exponentiated colour factors and the corresponding kinematic factors are $\left(K^T Y^{-1}\right)_j$. In this article we will present the exponentiated colour factors in terms of $(YC)_j$.

Properties of mixing matrices

1. *Idempotence*: A Cweb mixing matrix is idempotent, that is, $R^2 = R$
2. *Zero row sum rule*: The entries of R obey the zero row sum rule $\sum_{d'} R(d, d') = 0$.
3. *Column sum rule*: Along with these general properties, the mixing matrices also obey a conjectured column-sum rule of the form
$$\sum_{d} s(d) R(d, d') = 0 \,. \tag{6}$$
4. *Uniqueness*: For a given column weight vector $S = \{s(d_1), s(d_2), \ldots, s(d_k)\}$ with all $s(d_i) \neq 0$, the mixing matrix is unique.

In section 5, we will verify these properties for four loop Boomerang Cwebs. In the next section, we describe a recursive algorithm that generates Cwebs present at $\mathcal{O}(g^{2l+2})$ using Cwebs at $\mathcal{O}(g^{2l})$.

3 An improved algorithm to generate Cwebs

We find that it is sufficient to use Cwebs at l loops connecting m Wilson lines to generate all Cwebs at $l + 1$ loops connecting the same number of Wilson lines. Based on these facts, an improved and efficient algorithm to generate Cwebs recursively is given below:

1. To generate Cwebs at $l + 1$ loops connecting at most $l + 1$ Wilson lines starting from a Cweb at l loops connecting m $(1 \leq m \leq l + 1)$ lines
   (a) Connect any two existing Wilson lines by introducing a two-point gluon correlator (for massive Wilson lines a two point correlator can also be attached to a single line),
   (b) Connect any existing k-point gluon correlator to an existing Wilson line.
2. To generate Cwebs at $l + 1$ loops connecting the highest number $(l + 2)$ of Wilson lines, the following steps need to be applied to Cwebs at l loops connecting the highest number $(l + 1)$ of lines allowed at this order:
   (a) Connect a new Wilson line to any of the existing lines by introducing a two-point gluon correlator.
   (b) Connect any existing k-point gluon correlator to a new Wilson line.
3. Discard the duplicate Cwebs.

4 A brief description of `CwebGen 2.0`

We present below the algorithm of current version of the in-house Mathematica code `CwebGen 2.0`, that incorporates replica trick algorithm and is a significantly improved version of the codes that were used in [1, 2]. This code has been used in this work to obtain the Cweb mixing matrices.

- The code begins by generating the colour of one of the Cweb diagrams; however, it is different from the usual colour assignment as we also assign replica variables $\{i, j, k, \ldots\}$ to each of the gluon correlators that are present in the diagram.
- A subroutine then shuffles the attachments on each of the Wilson lines belonging to different correlators, and correspondingly different replica variables, to generate all the diagrams of the Cweb.
- Next hierarchies between the replica variables are generated. For example, for a Cweb with two correlators, with replica variables i and j, the code generates hierarchies $\{h\} = \{i = j, i > j, i < j\}$. Then number of distinct replica variables N_r for each hierarchy is calculated, for example, for hierarchy $i = j$, $N_r = 1$, whereas, for $i > j$, and $i < j$, $N_r = 2$. Using this, $M_{N_r}(h)$ is obtained.

  Note that previous versions of the code could generate all the possible hierarchies only if the number of replica variables was less than or equal to 4. `CwebGen 2.0` can generate the hierarchies for any number of replica variables.
- Another subroutine then generates replica ordered colour factors $\mathbf{R}[C(d)\big|h]$ for each of the hierarchies for every diagram of the Cweb. A diagonalizing matrix Y_W is constructed to diagonalize R_W using its right eigenvectors. Y_W then acts on the column vector containing the colour factors of each of the diagrams of the Cweb, and then gives the corresponding independent exponentiate colour factors.

The algorithm and its implementation in two older versions of the same code can be found in [1, 2]. To make a comparison of the present version with the previous versions, we have calculated the mixing matrix of $\mathrm{W}_5^{(4)}(1, 1, 1, 1, 4)$ appearing at four loops [1]. The largest dimension mixing matrix is 24×24 and it took 7 days in the first version [1], upon subsequent improvement it took 6.4 hours [2]. The same calculation by `CwebGen 2.0` takes only 1.54 seconds. Thus, the latest version of code is almost 15000 times faster as compared to the previous versions.

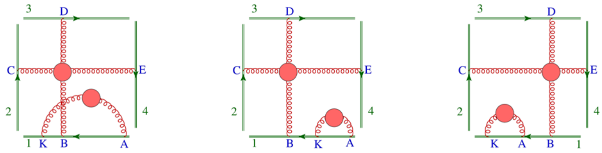
5 Boomerang Cwebs at Four loops

Using `CwebGen 2.0` and discarding the duplicates we get 8 and 20 Boomerang Cwebs connecting four and three Wilson lines respectively. In [3], we had obtained the diagonal blocks of mixing matrices for all the 8 Boomerang Cwebs that are present at four loops using the formalism introduced in [8]. Of course, we could not obtain the complete mixing matrices and exponentiated colour factors in that work which is the subject matter of this paper.

Before going into the details of the calculation, reader is advised to review the concepts used here such as Normal ordering, Fused-Web, and Uniqueness theorem, which were introduced in [8]. We now present the results for one Cweb connecting four Wilson lines. The calculations for all such Cwebs is provided in [4].

$W_4^{(1,0,1)}(3, 1, 1, 1)$*: a four-line Boomerang Cweb*

Boomerang Cweb $\mathrm{W}_4^{(1,0,1)}(3, 1, 1, 1)$, shown in fig (1), connects four Wilson lines with one multi-pronged-boomerang correlator (m=2) and a four point gluon correlator. This Cweb



C_1 C_2 C_3

Figure 1. Diagrams of Boomerang Cweb $W_4^{(1,0,1)}(3, 1, 1, 1)$.

has three possible shuffles on Wilson line 1. The shuffle of gluon attachments from the two correlators generates three diagrams as shown in fig. (1); thus the order of mixing matrix for this Cweb is three. To proceed we choose the order of the diagrams as in the fig. (1). This order is labeled using the order of attachments on line 1: $C_1 = \{ABK\}$, $C_2 = \{AKB\}$ and $C_3 = \{BAK\}$. The mixing matrix R and the diagonalizing matrix Y has been calculated using `CwebGen 2.0` and we get,

$$R = \frac{1}{2}\begin{pmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \qquad Y = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix}. \tag{7}$$

This matches with the result obtained in [3].

The column weight vector S for this Cweb can be calculated by determining the s-factors for each diagram. The s-factor for C_1 is zero as one can not shrink any of the two correlators to the origin of Wilson lines. In the diagram C_2 one can shrink the multi-pronged-boomerang correlator first then the four point correlator, thus there is only one way to shrink all the correlators to origin, making $s = 1$. The s-factor for C_3 is also one except the fact that here the multi-pronged-boomerang correlator has to be shrunk after the four point correlator. Thus the column weight vector becomes $S = \{0, 1, 1\}$.

The matrix R satisfies all the known properties of mixing matrix: it is idempotent, $R^2 = R$, satisfies the zero row sum rule as in each row of matrix sum of entries is zero, and, the conjectured column sum rule. The rank of this mixing matrix is two as there are only two independent rows in R. We obtain the independent exponentiated colour factors given as,

$$(YC)_1 = 0, \qquad (YC)_2 = -i\, f^{abn} f^{bch} f^{deh}\, \mathbf{T}_1^a \mathbf{T}_1^n \mathbf{T}_2^c \mathbf{T}_3^d \mathbf{T}_4^e. \tag{8}$$

The vanishing of one of the independent ECFs can be understood in terms of $\widetilde{C}$. Applying R on the C we get,

$$\widetilde{C}_1 = \frac{1}{2}(2C_1 - C_2 - C_3), \qquad \widetilde{C}_2 = \frac{1}{2}(C_2 - C_3), \qquad \widetilde{C}_3 = \frac{1}{2}(C_3 - C_2). \tag{9}$$

Recall that when a Cweb is Normal Ordered, the block D gives mixing between the reducible diagrams. All these diagrams contain Boomerang that give C_A factors, and thus their ECFs vanish.

6 Conclusions

In this article we study Boomerang Cwebs that connect three and four Wilson lines at four loops which are important ingredients in the studies of scattering involving massive particles.

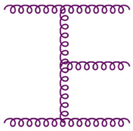


Figure 2. Exponentiated colour factor obtained from all the Boomerang Cwebs.

To enumerate Cwebs at a given perturbative order, we have introduced an improved version of the algorithm that was developed in [1]. We presented some details of the current version of the in-house Mathematica code `CwebGen 2.0`, that incorporates this new algorithm, and the replica trick, and it is a significantly faster version of the codes that were used in [1, 2].

We have computed the mixing matrices, the diagonalizing matrices and the exponentiated colour factors for all these Cwebs. We have verified that our results match with the predictions of the diagonal blocks presented in [3]. We found that the general structure of exponentiated colour factor of all the twenty eight Boomerang Cwebs, shown in fig. (2), is same as the general structure of ECFs for the Cwebs without the multi-pronged-boomerang correlators at four loops [1, 2]. This is an artifact of the Cwebs with multi-pronged-boomerang correlator, for which quadratic Casimir C_A (the colour of a Boomerang) is absent from all the exponentiated colour factors. This is in agreement with the calculations of Boomerang Cwebs at three loops [9]. We found that the ECFs corresponding to the D block of mixing matrices vanish which is consistent with [9].

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