FERMILAB-SLIDES-24-0030-SQMS

Qutrit Circuits and Algebraic Relations: A Pathway to Efficient Spin-1 Hamiltonian Simulation

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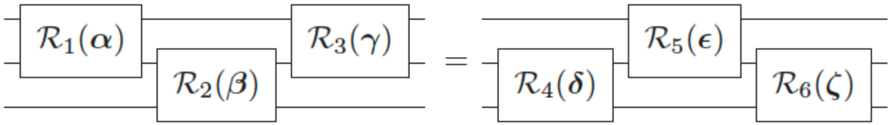
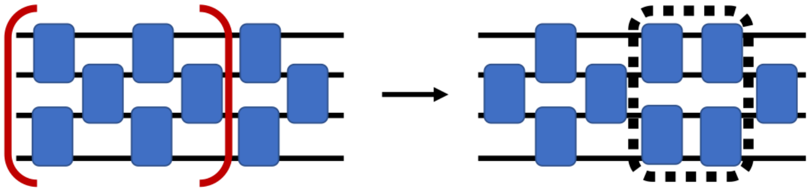
Yang-Baxter-type equations and Qubit Circuit Compression

- *Circuit compression by applying the Yang-Baxter type equation on a two-qubit time propagator*
  $R_\theta = e^{-i\theta(A \otimes B)}$, *where* $A, B \in SU(2)$. - *[Phys. Rev. A 106, 012412 (2022), B. Peng, S. Gulania, Y. Alexeev, and N.Govind.]*





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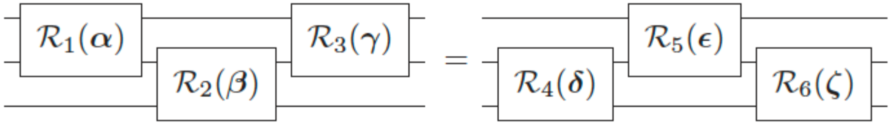

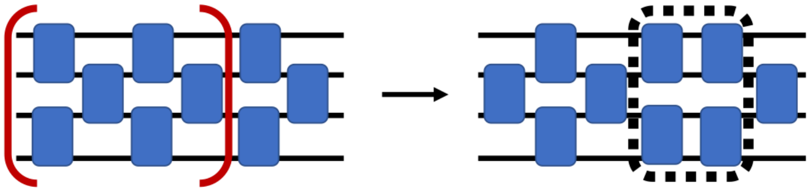

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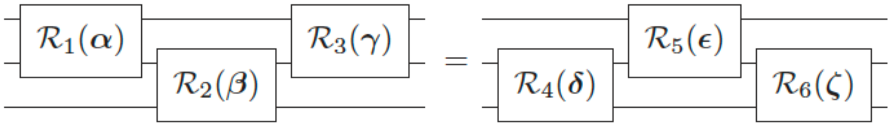

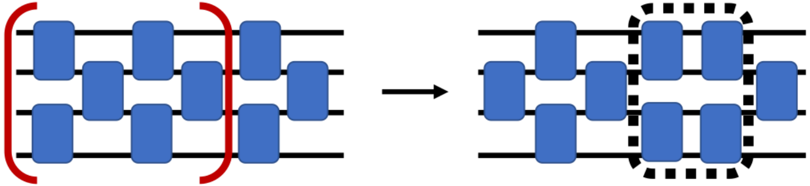

- Remarkably, turn-over relation compresses time evolution circuit to a depth that scales linearly with respect to the number of qubits.

- **Question:** Can we generalize idea to quantum circuits with higher dimensions?

  Exploratory effort in this direction, - quantum time dynamics of the one-dimensional spin-1 Heisenberg model.

- *Map Spin-1 system states onto qutrit states for efficient algebraic relations.*

Notation and Spin-Algebra

Treat qudits as spins quantum states via $s = (d-1)/2$

$d = 3 \Rightarrow s = 1$, with z-basis representation

$$S^x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}[(X \oplus 0) + (0 \oplus X)],$$

$$S^y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}[(Y \oplus 0) + (0 \oplus Y)]$$

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where X, Y, Z are the Pauli gates.

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where X, Y, Z are the Pauli gates.

Rewrite in a simpler way for numerical ease via Permutation matrices

$$\tilde{S}^x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{pmatrix}, \qquad \tilde{S}^y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} = P_y \tilde{S}^x P_y^\dagger, \qquad \tilde{S}^z = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = P_z \tilde{S}^x P_z^\dagger$$

$$P_y = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Similar to the SU(2) cases, $\{\tilde{S}^x,\ \tilde{S}^y,\ \tilde{S}^z\}$ satisfy commutation algebra

$$P_z = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

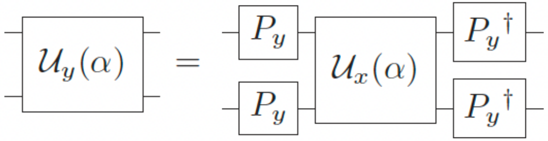
$$[\tilde{S}^x, \tilde{S}^y] = i\tilde{S}^z, \qquad [\tilde{S}^y, \tilde{S}^z] = i\tilde{S}^x, \qquad [\tilde{S}^z, \tilde{S}^x] = i\tilde{S}^y$$

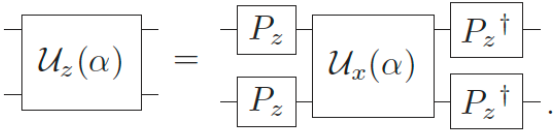
Note: Basis change $\{S^x,\ S^y,\ S^z\} \leftrightarrow \{\tilde{S}^x,\ \tilde{S}^y,\ \tilde{S}^z\}$ does not affect algebraic/circuit relations

Algebraic and Numerical turn-overs

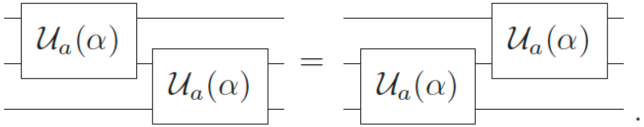
Two Qutrit time propagators: $\mathcal{U}_x(\alpha) = \exp(-i\alpha\, \tilde{S}^x \otimes \tilde{S}^x)\,, \quad \mathcal{U}_y(\alpha) = \exp(-i\alpha\, \tilde{S}^y \otimes \tilde{S}^y)\,, \quad \mathcal{U}_z(\alpha) = \exp(-i\alpha\, \tilde{S}^z \otimes \tilde{S}^z)$

From the Permutation matrices,





A simple feature is that $\mathcal{U}_a(\alpha),\ a \in \{x, y, z\}$ satisfies



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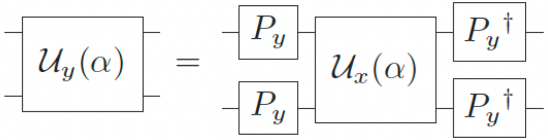
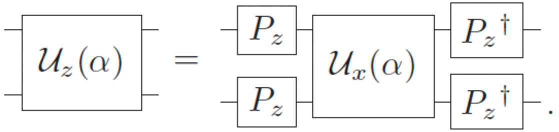
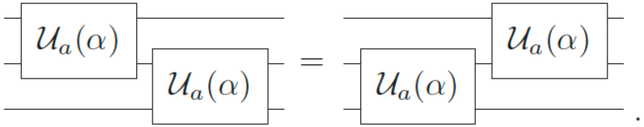


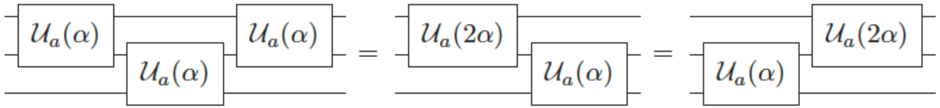


An interesting feature is that $\mathcal{U}_a(\alpha)$, $a \in \{x, y, z\}$ satisfies

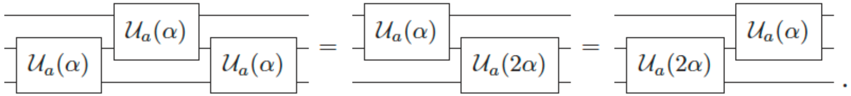


<u>Case 1</u>: Same rotation (algebraic proof)

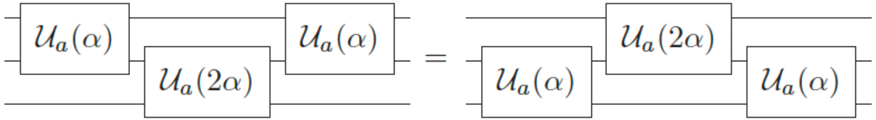
(1) LHS:



(2) RHS:

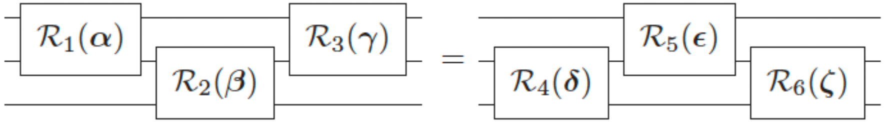


(3) LHS = RHS:



Numerically approximate turn-over identities

<u>Case 2:</u> Different rotations (*algebraic proof still searching?*).

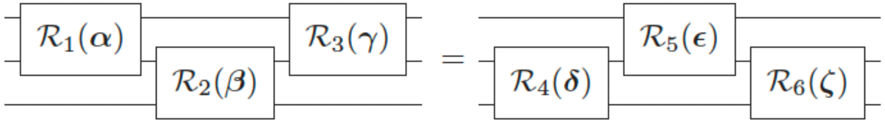


Establish $(3^3 \times 3^3)$ matrix relation:

$$[\mathcal{R}_1(\alpha) \otimes I_3[I_3 \otimes \mathcal{R}_2(\beta)][\mathcal{R}_3(\gamma) \otimes I_3] = [I_3 \otimes \mathcal{R}_4(\delta)][\mathcal{R}_5(\epsilon) \otimes I_3][I_3 \otimes \mathcal{R}_6(\zeta)]$$

i.e. Denoting LHS as $W_L(\vec{\theta}_L)$ and RHS as $W_R(\vec{\theta}_R)$, fix $\vec{\theta}_L$ while optimizing for $\vec{\theta}_R$.

Numerically approximate turn-over identities



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i.e. Denoting LHS as $W_L(\vec{\theta}_L)$ and RHS as $W_R(\vec{\theta}_R)$, fix $\vec{\theta}_L$ while optimizing for $\vec{\theta}_R$.

- Explore Spin-1 XY Hamiltonian on 3 qutrits:

$$\mathrm{H_{XY}} = -\mathrm{J} \sum_{i=0}^{2} \tilde{S}_i^x \tilde{S}_{i+1}^x + \tilde{S}_i^y \tilde{S}_{i+1}^y ,$$

$$e^{-itH_{XY}} = e^{itJ \sum_{i=0}^{1} \tilde{S}_i^x \tilde{S}_{i+1}^x + \tilde{S}_i^y \tilde{S}_{i+1}^y}$$

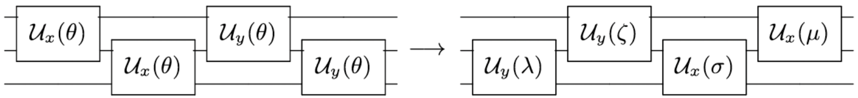
- Consider possible trotterizations, and explore approximate identities by minimizing gate infidelity :

$$C(\vec{\theta}_L, \vec{\theta}_R) = 1 - \frac{1}{(3^3)^2} \left\| tr\left(W_L(\vec{\theta}_L) W_R^{\dagger}(\vec{\theta}_R)\right) \right\|^2$$

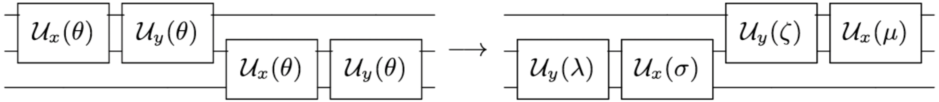
Numerically approximate turn-over identities

Possible Trotterizations of the 3-qutrits isotropic XY Hamiltonian

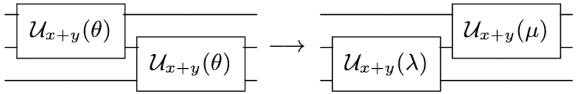
T_1:



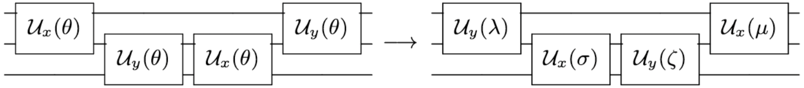
T_2:



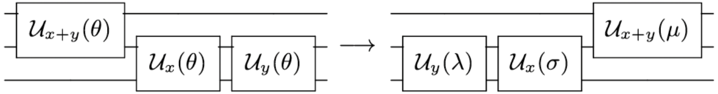
T_3:



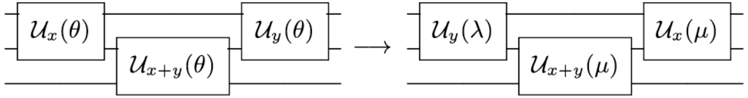
T_4:



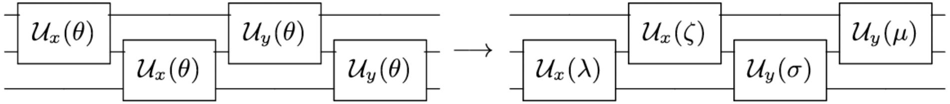
T_5:



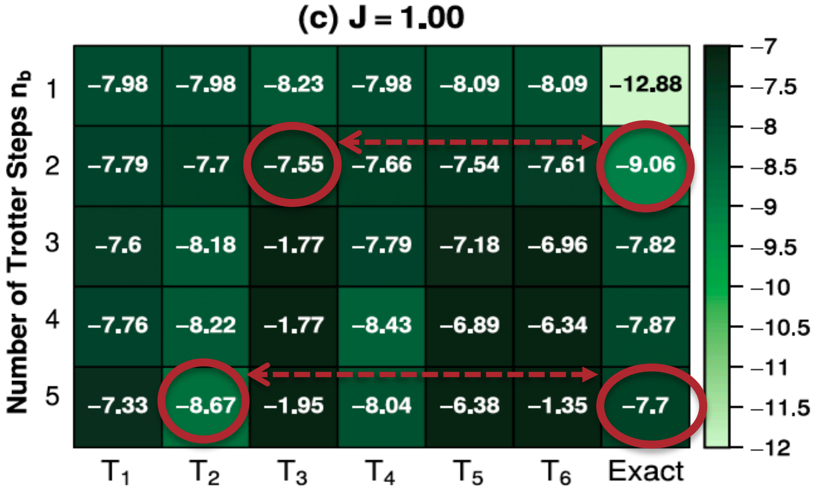
T_6:



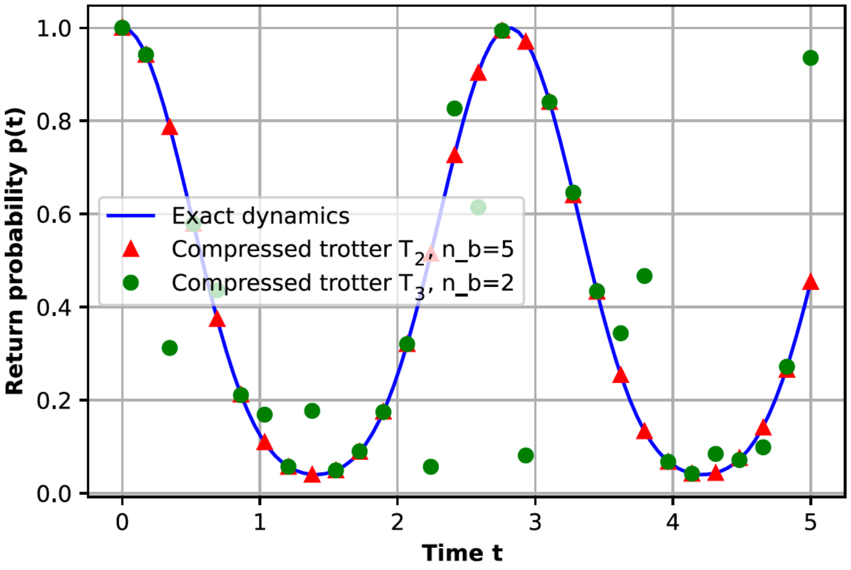
$$\mathcal{U}_x^{i,j}(\theta) = \exp\left(-i\theta \tilde{S}_i^x \otimes \tilde{S}_j^x\right),$$
$$\mathcal{U}_y^{i,j}(\theta) = \exp\left(-i\theta \tilde{S}_i^y \otimes \tilde{S}_j^y\right),$$
$$\mathcal{U}_{x+y}^{i,j}(\theta) = \exp\left(-i\theta\left(\tilde{S}_i^x \otimes \tilde{S}_j^x + \tilde{S}_i^y \otimes \tilde{S}_j^y\right)\right).$$



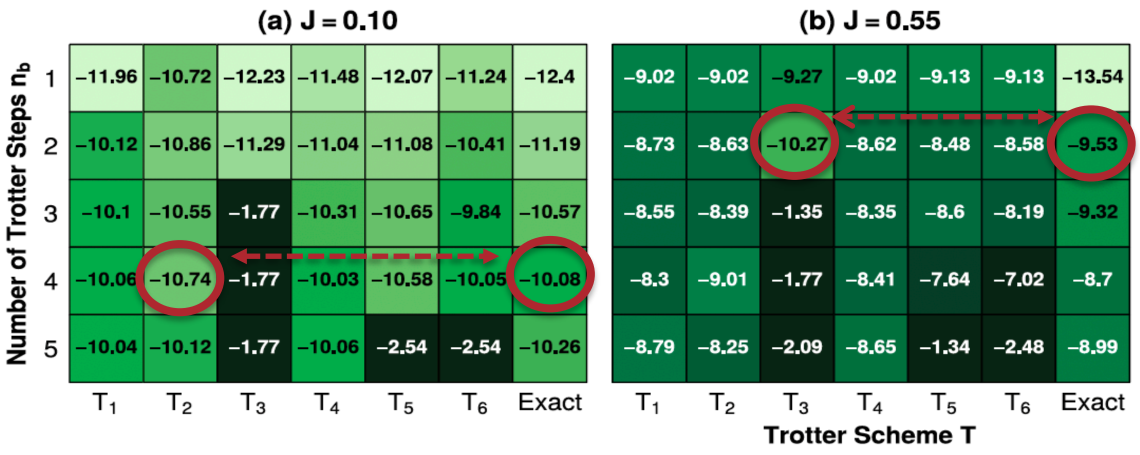
An exact-in-principle identity used for setting up the lower bounds in the numerical tests.



$p(t) = \left\|\langle 202|e^{itH_{XY}}|202\rangle\right\|^2$ as a function of time.



Infidelity, $\log_{10}[\min_{\vec{\theta}_R} C(\vec{\theta}_L, \vec{\theta}_R)]$, for Trotter forms 1,…,6. Infidelity to be relatively low as compared to numerical values of the exact identity.



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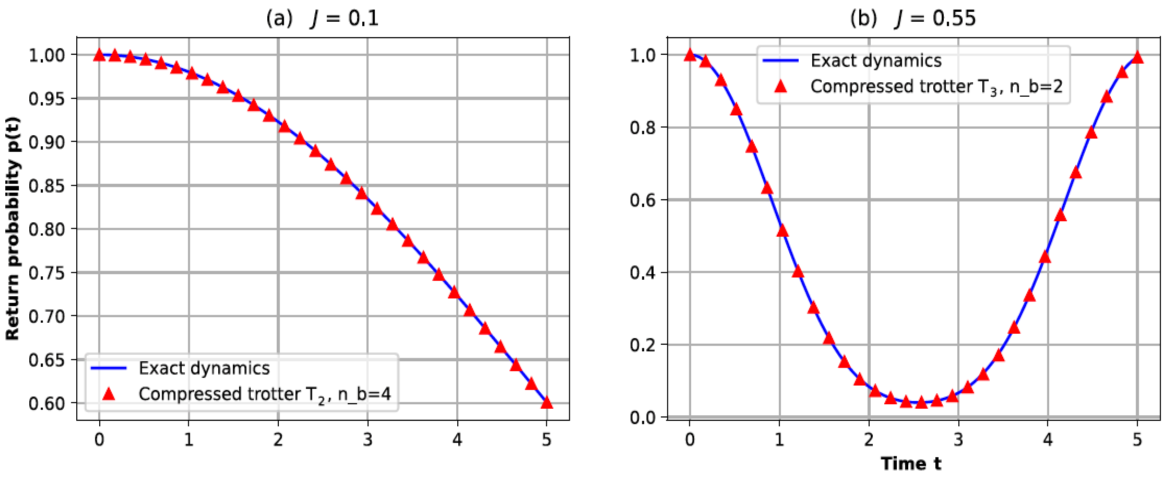

$p(t) = \left\|\langle 202|e^{itH_{XY}}|202\rangle\right\|^2$ as a function of time. For $T_3, n_b = 2$, approximately $\frac{2n}{3}$ gates are removed, while $T_2, n_b = 4$, approximately $\left\lfloor\frac{2n}{5}\right\rfloor - 1$ gates are removed.

THANK YOU

Additional SQMS Presentations

This material is based upon work supported by the U.S. Department of Energy, Office of Science, National Quantum Information Science Research Centers, Superconducting Quantum Materials and Systems Center (SQMS) under contract number DE-AC02-07CH11359