



On Multimatrix Models Motivated by Random Noncommutative Geometry I: The Functional Renormalization Group as a Flow in the Free Algebra

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Abstract. Random noncommutative geometry can be seen as a Euclidean path-integral quantization approach to the theory defined by the Spectral Action in noncommutative geometry (NCG). With the aim of investigating phase transitions in random NCG of arbitrary dimension, we study the nonperturbative Functional Renormalization Group for multimatrix models whose action consists of noncommutative polynomials in Hermitian and anti-Hermitian matrices. Such structure is dictated by the Spectral Action for the Dirac operator in Barrett’s spectral triple formulation of fuzzy spaces. The present mathematically rigorous treatment puts forward “coordinate-free” language that might be useful also elsewhere, all the more so because our approach holds for general multimatrix models. The toolkit is a noncommutative calculus on the free algebra that allows to describe the generator of the renormalization group flow—a noncommutative Laplacian introduced here—in terms of Voiculescu’s cyclic gradient and Rota–Sagan–Stein noncommutative derivative. We explore the algebraic structure of the Functional Renormalization Group equation and, as an application of this formalism, we find the β -functions, identify the fixed points in the large- N limit and obtain the critical exponents of two-dimensional geometries in two different signatures.

Contents

1. Introduction	3096
1.1. Organization, Strategy and Results	3099
2. Noncommutative Calculus	3100
2.1. Differential Operators on the Free Algebra	3102

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2.2. The Algebraic Structure	3108
3. Random Noncommutative Geometries and Multimatrix Models	3111
3.1. Fuzzy Geometries as Spectral Triples	3111
3.2. The Spectral Action for Fuzzy Geometries	3113
4. Deriving the Functional Renormalization Group Equation	3117
5. Techniques to Compute the Renormalization Group Flow	3123
5.1. Projection and Truncations	3124
5.2. The FP^{-1} Expansion in the Large- N Limit	3124
6. “Coordinate-Free” Matrix Models	3127
7. Two-Matrix Models from Noncommutative Geometries	3131
7.1. Theory Space	3131
7.2. Compatibility of the RG-Flow with the Spectral Action	3131
7.3. The Truncated Effective Action	3133
7.4. The β -Functions	3133
7.5. Dualities	3137
7.6. Methods and Results for the Geometry $(0, 2)$, or $(-, -)$	3138
7.7. Results for the Geometry $(2, 0)$, or $(+, +)$	3138
8. Conclusion and Discussion	3139
Acknowledgements	3142
Appendix A. Glossary, Conventions, Other Notations	3143
References	3145

1. Introduction

Random Noncommutative Geometry (NCG), initiated by Barrett and Glaser [11], is a path-integral approach to the quantization of noncommutative geometries. This problem is mathematically interesting [20, Sect. 18.4] and has already been addressed by diverse methods in [18, 28, 57]. Also in physics, a satisfactory answer would shed light on the quantum structure of spacetime from a different angle. Namely, what seems to individuate a formulation of quantum of gravity in terms of NCG-structures is that these provide a natural language to treat both pure gravity and gravity coupled with matter at a geometrically indistinguishable footing. This holds for (the classical theory of) established matter sectors like the Standard Model [7, 17] and some theories beyond it [21].

Although the last point evokes rather the mathematical elegance of the NCG-applications, also from a pragmatic viewpoint it is important to stress that the search for a quantum theory of gravity that is capable of incorporating matter is of physical relevance: “matter matters” reads for instance in the asymptotically safe road to quantum gravity [24] (see also [67]). Indeed, a quick argument [31] based on the Renormalization Group (RG) discloses the mutual importance of each sector to the other, concretely

- gravity loops like  appear and influence matter and

- in a similar way, matter modifies the gravity sector 

in the RG-flow. This suggests that both ought to be simultaneously treated and motivates us to develop, as a first step, the *Functional Renormalization Group* in random NCG, where potentially both sectors might harmonically coexist.

The Functional Renormalization Group Equation (FRGE; see the comprehensive up-to-date review [23]) is a modern framework describing the Wilsonian RG-flow [78] that governs the change of a quantum theory with scale. From the technical viewpoint, in order to determine the effective action, the FRGE—derived by Wetterich and Morris [55, 77]—offers an alternative to path-integration by replacing that task with a differential equation.

In this paper, the model of space(time) we focus on is an abstraction of fuzzy spaces [27, 30, 71], whose elements were later assembled into a spectral triple (the spin geometry object in NCG) called *fuzzy geometry* [8, 12]. For the future, in a broader NCG context, it would be desirable to relate the FRGE to the newly investigated truncations in the spectral NCG formalism [22, 42, 43] (see [29] for a preceding related idea), but for initial investigations fuzzy geometries are interesting enough and also in line with them, e.g., for the case of the sphere [76, Sect. 3.3].

One particular advantage of a fuzzy geometry being a spectral triple is the contact with Connes' NCG formalism, in particular, the ability to encode the geometry in a (Dirac) operator D that serves as path integration variable in the quantum theory. Since fuzzy geometries are finite-dimensional, one can provide a mathematically precise definition of the partition function $\mathcal{Z} = \int e^{-\text{Tr } f(D)} dD$ that corresponds to the Spectral Action $\text{Tr } f(D)$, as far as f is a polynomial, in contrast to the bump function f used originally by Connes–Chamseddine [16]. In fact, this way to quantize fuzzy geometries was shown [8, 11] to lead to a certain class of multimatrix models further characterized in [61].

On the physics side, finite-dimensionality should not be seen as a shortage, as this dimension is related to energy or spatial resolution; in fact, rather it is in line with the existence of a minimal or Planck's length. This is intuitively clear for the fuzzy sphere [41] on which—being spanned by finitely many spherical harmonics—it is impossible to separate (i.e. to measure) points lying arbitrarily near.

This discrete-dual picture (Fig. 1) can be interpreted as a pre-geometric phase, analogous to having simplices as building blocks of spacetime in discrete approaches to quantum gravity as Group Field Theory [9], Matrix Models [25, 38] or Tensor Models [46]. For those theories, but also for other approaches (e.g., Causal Dynamical Triangulations [2]), it is important [1, 26, 34, 36, 65] to explore phase transition to a manifold-like phase; in analogous way, the study of a condensation of fuzzy geometries to a continuum is physically relevant [40] (also addressed analytically in dimension-1 by [49]). With this picture in mind, we estimate here candidates for such phase transition.

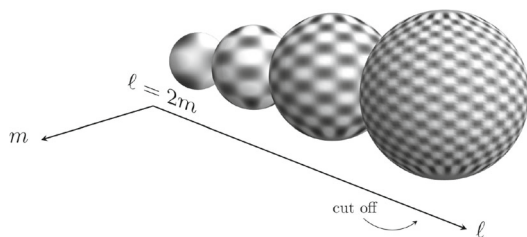


FIGURE 1. A caricature of selected algebra elements of the fuzzy sphere. The real part of spherical harmonics $Y_\ell^{m=\ell/2}$ for $\ell = 4, 10, 20, 40$

The largest part of this paper develops the mathematical formalism that allows such exploration. On top of well-known quantum field theory (QFT) techniques, the nonstandard results this paper bases on can be divided into three classes:

- THE MODELS are originated in Random NCG [11]. Barrett’s characterization of Dirac operators makes contact with certain kind of multimatrix models [8]. Their Spectral Action was systematically computed in [61], organized by chord diagrams, which reappear here.
- THE TOOL is the Functional Renormalization Group. The main idea of the RG-flow parameter being the (logarithm of) the matrix size appeared in [15] and consists in reducing the $N + 1$ square matrix φ to effectively obtain a $N \times N$ matrix field by integrating out the entries $\varphi_{a,N+1}, \varphi_{N+1,a}$ ($a = 1, \dots, N + 1$). Eichhorn and Koslowski provided the nonperturbative, modern formulation of the Brezin–Zinn–Justin idea. They put it forward for Hermitian matrix models in [32] (preceded by a similar approach to scalar field theory on Moyal space [70] and followed by an extension to tensor models [34]). They did not present a proof and in fact it will be convenient to prove for multimatrix models the FRGE, as this equation actually dictates us the algebraic structure (needed for the so-called FP^{-1} -expansion [14, Sect. 2.2.2]) and exonerates us from making any choice.
- Although the Eichhorn–Koslowski approach orients us to find suitable truncations and their scalings to take the large- N limit were auxiliary, the mathematical structure we deal with here is constructed from scratch and does not rely on theirs (which turns out to be entirely replaced). THE LANGUAGE that facilitates this is abstract noncommutative algebra. In order to state the RG-flow in “coordinate-free” fashion, we use Voiculescu’s cyclic derivative [74] and the noncommutative derivative defined by Rota–Sagan–Stein [68].

We do not assume familiarity with any of these references and offer a self-contained approach.

1.1. Organization, Strategy and Results

In Sect. 2 we develop the algebraic language needed for the rest of the paper. We introduce a noncommutative (NC) Hessian and a NC-Laplacian on the free algebra, given in terms of noncommutative differential operators defined by [68, 74, 75]. A graphical method to compute this second-order operator is provided. Section 2 prepares the algebraic structure that will turn out to emerge in the proof of Wetterich–Morris equation for multimatrix models.

Section 3 briefly reviews fuzzy geometries and how their Spectral Action is computed in terms of elements of the free algebra—in mathematics called *words* or *noncommutative polynomials* and in QFT-terminology *operators*—that define a certain class of multimatrix models. For two-dimensional fuzzy geometries, we provide a characterization of allowed terms in the resulting action functional.

In Sect. 4 the FRGE is proven to be governed by the NC-Hessian; in Sect. 5 we introduce truncations and projections in order to compute the β -functions. Also there, the “ FP^{-1} expansion” is developed in the large- N limit, and the *tadpole approximation*, corresponding to order one in that expansion, is restated as a heat equation¹ whose Laplacian is noncommutative (the one of Sect. 2).

Once the formalism is ready, we do not directly proceed with fuzzy geometries, but in Sect. 6 we briefly reconsider the treatment of the FRGE for Hermitian matrix models. A couple of points justify this interlude:

- It serves as a bridge from the index-computations in matrix models to index-free ones proposed in the present paper.
- By using a well-known result to be reproduced by the FRGE, we calibrate the infrared regulator (IR-regulator) that we shall use for the fuzzy geometry matrix models. With a quadratic, instead of the already studied linear IR-regulator, the fixed point is closer to the exact value $-1/12$ for gravity coupled with conformal matter.
- Finally, since the number of flowing operators for the Hermitian matrix model is relatively small, it is helpful for the sake of clearer exposition to present a case whose techniques fit in a couple of pages to prepare the more complex fuzzy two-matrix models.

The actual application of the formalism appears in Sect. 7. We treat there a class of two-matrix models that lies in an orthogonal direction to the well-investigated two-matrix model that describes the Ising model [37, 47, 72], often just referred to as “*the two-matrix model*”, due to its importance. To wit, whilst in the Ising two-matrix model the (trace of) AB appears as the only interaction mixing the two random matrices, A and B , NCG-models forbid this very operator. Instead, these matrices interact via several elements of the free algebra and its tensor powers, i.e. via (traces of) words

$$ABAB, A^2B^2, A^3BAB, \dots \quad \text{and} \quad A \otimes A, A^2 \otimes B^2, A \otimes A^2BAB, \dots$$

¹That a Laplacian plays a role in the Functional Renormalization Group and that this has the form of is not a surprise [69, Sect. 3.3].

whose exact form has been investigated in [61], also for higher dimensions. The RG-flow we analyze does not take place inside the space of Dirac operators—in which coupling constants of the same polynomial degree are correlated—but we consider the general situation in which the symmetry breaking by the IR-regulator kicks the RG-flow out to (couplings indexed by a larger subspace of tensor products of) the free algebra.

For an arbitrary-dimensional fuzzy geometry the *bare action*—the starting point of the RG-scale $t = \log \Lambda$ (or *energy scale* Λ)—is chosen in the space of Dirac operators inside the full *theory space*, the space of running couplings. The exact RG-path *ends* at the precise effective action at RG-scale² $t = -\infty$, which is too hard to determine at present. Making the RG-flow computable introduces two types of errors: on the one hand, deviations caused by projections that consider only operators with unbroken symmetries and, on the other hand, errors due to truncations introduced in order to keep the number of flowing parameters finite. This is depicted in Fig. 2 in a pessimistic scenario, later improved in view of the results of Sect. 7.2.

The large number of the NC-polynomial interactions, on top of the ordinary polynomials in each matrix, makes the projected and truncated RG-flow still computationally demanding³ and at this stage a further simplification is helpful. Namely, we look for critical exponents corresponding to solutions to the fixed point equations that obey the duality $A \leftrightarrow B$, whenever the signature allows it. We find those solutions inside a hypercube in theory space (with coordinates g_i obeying $|g_i| \leq 1$), which, even if it is not the full exploration, it exhausts the scope of the FP^{-1} -expansion. Further improvements are discussed in Sect. 8, together with the conclusion. To ease the reading, some oversized expressions involved in proofs are located outside the main text (see Supplementary Material). Also “Appendix A” serves as a glossary and guide on the notation.

2. Noncommutative Calculus

We address the noncommutative calculus in several (say n) variables. The object of interest is the free algebra spanned by an *alphabet* of n letters x_1, \dots, x_n . The elements of the free algebra are the linear span of words in those n letters, the product being concatenation. Although the physical theories we address are well described by the real version $\mathbb{R}_{\langle n \rangle}$ of it, we consider the complex free algebra $\mathbb{C}_{\langle n \rangle}$. There exists in $\mathbb{C}_{\langle n \rangle}$ an *empty word*, denoted by 1, that behaves as multiplicative neutral. Other than 1, the letters of the alphabet do not commute.

²Actually, it since Λ will be identified with the matrix size the lowest value for the RG-time t is $0 = \log 1 = \log \Lambda_0$. But at this point “we do not know this yet”.

³For instance, at the sixth order the NCG-matrix model includes up to 48 operators in a double-trace even-degree truncation. In contrast, in the same truncation, the Ising two-matrix model would include at most 19 operators, but the RG-flow does not combine letters in the latter case.

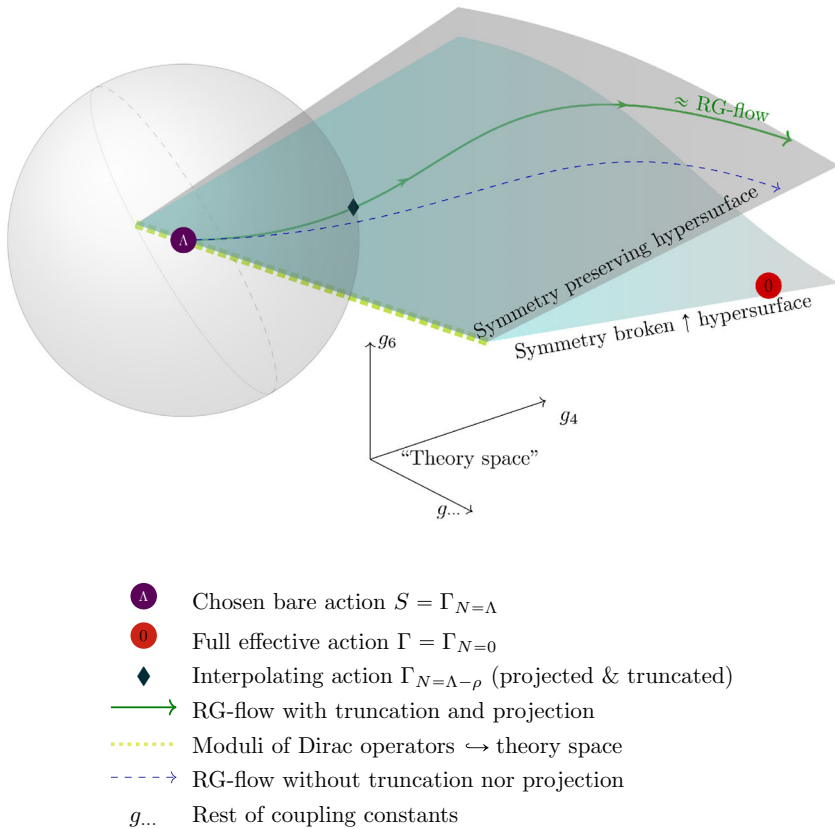


FIGURE 2. Picture of the theory space and two hypersurfaces. The lower one, which considers the modified Ward–Takahashi identity (mWTI), is where the exact flow takes place. The upper one is an approximation with finitely many parameters. If ρ is small, the approximation ignoring the mWTI, together with the truncation and projection for the approximated RG-flow (\approx *RG-flow*) is assumed to not to be far apart from the actual interpolating action

Rather than in the generators x_i in the abstract free algebra, we are interested in their realization as matrices,⁴ $x_i = X_i \in M_N(\mathbb{C})$ for each $i = 1, \dots, n$. In contrast with the convention of taking self-adjoint generators, we have reasons to allow anti-Hermitian generators and set instead

$$X_i^* = \pm X_i \quad \text{if} \quad e_i = \pm 1 \quad (i = 1, \dots, n). \quad (2.1a)$$

⁴This section is the mathematical background of the FRGE. So far, N is still fixed.

In this section the signs e_i are input; later these will be gained from the NCG-structure, which additionally imposes

$$\mathrm{Tr}_N(X_i) := \sum_{a=1}^N (X_i)_{aa} = 0 \quad \text{if} \quad e_i = -1. \quad (2.1b)$$

When the n generators are $N \times N$ matrices, it will be convenient to denote the free algebra by $\mathbb{C}_{\langle n \rangle, N}$. Having fixed signs e_i ($i = 1, \dots, n$), we let

$$\mathcal{M}_N = \{(X_1, \dots, X_n) \mid \text{conditions (2.1) hold for each } X_i \in M_N(\mathbb{C})\}, \quad (2.2)$$

with some abuse of notation concerning the omitted parameters. The tracelessness condition (2.1b) is of no relevance in this section, but important later.

The empty word, which corresponds to the identity matrix $1_N \in M_N(\mathbb{C})$, generates the *constants*. The elements of the free algebra that are not generated by the empty word are referred to as *fields*:

$$\mathbb{C}_{\langle n \rangle, N} = \overbrace{\mathbb{C} \cdot 1_N}^{\text{constants}} \oplus \underbrace{\langle X_{\ell_1} X_{\ell_2} \cdots X_{\ell_k} \mid \ell_j = 1, \dots, n \text{ and } k \neq 0 \rangle}_{\text{fields}}. \quad (2.3)$$

A similar terminology is employed for the analogous splitting of the tensor product:

$$\mathbb{C}_{\langle n \rangle, N} \otimes \mathbb{C}_{\langle n \rangle, N} = \overbrace{\mathbb{C} \cdot 1_N \otimes 1_N}^{\text{constants}} \oplus \text{fields}, \quad (2.4)$$

whose fields in this case are given by

$$\langle X_{\ell_1} X_{\ell_2} \cdots X_{\ell_k} \otimes X_{\ell'_1} X_{\ell'_2} \cdots X_{\ell'_r} \mid \ell'_j, \ell_j = 1, \dots, n \text{ and } r + k \neq 0 \rangle. \quad (2.5)$$

The free algebra is equipped with the trace of $M_N(\mathbb{C})$: $\mathrm{Tr}_N(Q) = \sum_{a=1}^N Q_{aa}$, $Q \in \mathbb{C}_{\langle n \rangle, N}$. Instead of making this trace a state, normalizing it as usual also in probability, $\mathrm{tr}(1_N) = 1$, we still stick to a trace satisfying $\mathrm{Tr}_N(1_N) = N$ in order to make power-counting arguments comparable with other references.

2.1. Differential Operators on the Free Algebra

We now elaborate on the next operators, due to Rota–Sagan–Stein [68] (in one variable to Turnbull [73]) and to Voiculescu [74]. The *noncommutative derivative*—called also the *free difference quotient* [44, 75]—with respect to the j -th variable x_j , denoted by ∂^{x_j} , is defined on generators by

$$\begin{aligned} \partial^{x_j} : \mathbb{C}_{\langle n \rangle} &\rightarrow \mathbb{C}_{\langle n \rangle} \otimes \mathbb{C}_{\langle n \rangle} \\ x_{\ell_1} \cdots x_{\ell_k} &\mapsto \sum_{i=1}^k \delta_{\ell_i}^j \cdot x_{\ell_1} \cdots x_{\ell_{i-1}} \otimes x_{\ell_{i+1}} \cdots x_{\ell_k}. \end{aligned} \quad (2.6)$$

The tensor product keeps track of the spot (in monomial) the derivative acted on. Moreover, the *cyclic derivative* \mathscr{D}^{x_j} with respect to the j -th variable is defined by

$$\mathscr{D}^{x_j} = \tilde{m} \circ \partial^{x_j} \quad \text{where } \tilde{m} : \mathbb{C}_{\langle n \rangle} \otimes \mathbb{C}_{\langle n \rangle} \rightarrow \mathbb{C}_{\langle n \rangle}, \quad \tilde{m}(A \otimes B) = BA. \quad (2.7)$$

Example. In the free algebra generated by the Latin alphabet A, \dots, Z , one has

$$\partial^E(\text{FREENESS}) = \text{FR} \otimes \text{ENESS} + \text{FRE} \otimes \text{NESS} + \text{FREEN} \otimes \text{SS},$$

but notice that (if 1 is the empty word) $\partial^S(\text{FREENESS}) = \text{FRENE} \otimes \text{S} + \text{FREENES} \otimes 1$. For the cyclic derivative it holds:

$$\begin{aligned} \mathcal{D}^E(\text{FREENESS}) &= \tilde{m}(\text{FR} \otimes \text{ENESS} + \text{FRE} \otimes \text{NESS} + \text{FREEN} \otimes \text{SS}) \\ &= \text{ENESSFR} + \text{NESSFRE} + \text{SSFREEN}. \end{aligned}$$

Using the same rules for the abstract derivatives on $\mathbb{C}_{\langle n \rangle}$ for $\mathbb{C}_{\langle n \rangle, N}$, one can make the following

Proposition 2.1. *Let $Y = X_i$ be any of the generators of $\mathbb{C}_{\langle n \rangle, N}$. For any $Q \in \mathbb{C}_{\langle n \rangle, N}$, the derivatives ∂^Y and \mathcal{D}^Y enjoy the following properties:*

1. *the abstract derivative is realized by the derivative with respect to a matrix:*

$$\partial_{ab}^Y = \frac{\delta}{\delta Y_{ba}}, \quad (2.8)$$

that is, letting $(U \otimes V)_{ab;cd} = U_{ab}V_{cd}$ ($U, V \in \mathbb{C}_{\langle n \rangle, N}$), one has

$$[(\partial^Y Q)(X)]_{ab;cd} = \frac{\delta}{\delta Y_{bc}}[Q(X)]_{ad} \quad \text{for } X = (X_1, \dots, X_n) \in \mathcal{M}_N.$$

2. *The cyclic derivative equals the noncommutative derivative of the trace:*

$$\partial^Y \text{Tr } Q = \mathcal{D}^Y Q. \quad (2.9)$$

Proof. Let $Q \in \mathbb{C}_{\langle n \rangle}$. Since the trace is linear, one can verify the property on a monomial $Q(X) = X_{\ell_1} \cdots X_{\ell_k}$ and then obtain

$$\begin{aligned} \frac{\delta}{\delta(X_i)_{bc}} Q(X)_{ad} &= \frac{\delta}{\delta(X_i)_{bc}} (X_{\ell_1} \cdots X_{\ell_k})_{ad} \\ &= \sum_{j=1}^k (X_{\ell_1} \cdots X_{\ell_{j-1}})_{ar} \frac{\delta(X_{\ell_j})_{rs}}{\delta(X_i)_{bc}} (X_{\ell_{j+1}} \cdots X_{\ell_k})_{sd} \\ &= \sum_{i=\ell_j} (X_{\ell_1} \cdots X_{\ell_{j-1}})_{ab} (X_{\ell_{j+1}} \cdots X_{\ell_k})_{cd} \\ &= \sum_{i=\ell_j} (X_{\ell_1} \cdots X_{\ell_{j-1}} \otimes X_{\ell_{j+1}} \cdots X_{\ell_k})_{ab;cd} \\ &= [(\partial^{X_i} Q)(X)]_{ab;cd}. \end{aligned}$$

To obtain the second statement, notice that $\partial_{ab}^{X_i} \text{Tr } Q$ is obtained from the last equation by setting $a = d$ and summing,

$$\begin{aligned} \partial_{cb}^{X_i} \text{Tr}[Q(X)] &= \sum_a \{(\partial^{X_i} Q)(X)\}_{ab;ca} \\ &= \sum_a \sum_{i=\ell_j} (X_{\ell_1} \cdots X_{\ell_{j-1}} \otimes X_{\ell_{j+1}} \cdots X_{\ell_k})_{ab;ca} \end{aligned}$$

$$\begin{aligned}
&= \tilde{m} \left\{ \sum_a \sum_{i=\ell_j} (X_{\ell_1} \cdots X_{\ell_{j-1}} \otimes X_{\ell_{j-1}} \cdots X_{\ell_k})_{ab;ca} \right. \\
&= \{[(\tilde{m} \circ \partial^{X_i})Q](X)\}_{cb} = [(\mathcal{D}^{X_i}Q)(X)]_{cb},
\end{aligned}$$

where the last line follows just by the definition of cyclic derivative. \square

Definition 2.2. The *noncommutative Hessian* (NC-Hessian) is the operator

$$\text{Hess} : \text{im Tr}_N \rightarrow M_n(\mathbb{C}) \otimes \mathbb{C}_{\langle n \rangle, N}^{\otimes 2} \quad (2.10)$$

whose (ij) -entry $(1 \leq i, j \leq n)$ in the first tensor factor is

$$(\text{Hess Tr}_N P)_{ij} := (\partial^{X_i} \circ \partial^{X_j} \text{Tr}_N P) \in \mathbb{C}_{\langle n \rangle, N}^{\otimes 2}. \quad (2.11)$$

Here, $\text{Tr}_N : \mathbb{C}_{\langle n \rangle, N} \rightarrow \mathbb{C}$ is the ordinary trace of $M_N(\mathbb{C}) \supset \mathbb{C}_{\langle n \rangle, N}$. Alternatively,

$$\text{Hess} = \begin{pmatrix} \partial^{X_1} \circ \partial^{X_1} & \partial^{X_1} \circ \partial^{X_2} & \cdots & \partial^{X_1} \circ \partial^{X_n} \\ \partial^{X_2} \circ \partial^{X_1} & \partial^{X_2} \circ \partial^{X_2} & \cdots & \partial^{X_2} \circ \partial^{X_n} \\ \vdots & \vdots & \ddots & \vdots \\ \partial^{X_n} \circ \partial^{X_1} & \partial^{X_n} \circ \partial^{X_2} & \cdots & \partial^{X_n} \circ \partial^{X_n} \end{pmatrix}.$$

It will be convenient to introduce a closely related Hessian, Hess_g , modified by the⁵ “signature” $g = \text{diag}(e_1, \dots, e_n)$,

$$(\text{Hess}_g \text{Tr}_N P)_{ij} := (e_i)^{\delta_{ij}} (\partial^{X_i} \circ \partial^{X_j} \text{Tr}_N P) \in \mathbb{C}_{\langle n \rangle, N}^{\otimes 2}, \quad (2.12)$$

so

$$\text{Hess}_g = \begin{pmatrix} e_1 \partial^{X_1} \circ \partial^{X_1} & \partial^{X_1} \circ \partial^{X_2} & \cdots & \partial^{X_1} \circ \partial^{X_n} \\ \partial^{X_2} \circ \partial^{X_1} & e_2 \partial^{X_2} \circ \partial^{X_2} & \cdots & \partial^{X_2} \circ \partial^{X_n} \\ \vdots & \vdots & \ddots & \vdots \\ \partial^{X_n} \circ \partial^{X_1} & \partial^{X_n} \circ \partial^{X_2} & \cdots & e_n \partial^{X_n} \circ \partial^{X_n} \end{pmatrix}.$$

Tracing the NC-Hessian Hess_g with help of the signature yields the *noncommutative Laplacian* ∇^2 , that is the map

$$\nabla^2 : \text{im Tr}_N \rightarrow \mathbb{C}_{\langle n \rangle, N}^{\otimes 2} \quad \text{given by } \nabla^2 := \sum_{i=1}^n e_i (\partial^{X_i} \circ \partial^{X_i}). \quad (2.13)$$

We abbreviate $\partial^{X_i} \circ \partial^{X_i} = (\partial^{X_j})^2 = \nabla_j^2$, so $\nabla^2 = \sum_{j=1}^n e_j \nabla_j^2$.

We remark that the Hessian matrix (of NC-polynomials in $\mathbb{C}_{\langle n \rangle}^{\otimes 2}$) is not symmetric. Clearly, the NC-Laplacian and the NC-Hessian vanish on degree < 2 . On larger words, we compute them with aid of:

Proposition 2.3. Consider a monomial $Q \in \mathbb{C}_{\langle n \rangle, N}$, $Q = X_{\ell_1} X_{\ell_2} \cdots X_{\ell_k}$ with $k \geq 2$. Then, for $i, j = 1, \dots, n$

$$(\partial^{X_i} \circ \partial^{X_j}) \text{Tr}_N Q = \sum_{\pi=(uv)} \delta_{\ell_u}^j \delta_{\ell_v}^i \pi_1(Q) \otimes \pi_2(Q), \quad (2.14)$$

⁵Later it will be clear this terminology—by now we use quotation marks.

where the sum runs over all (directed) pairings $\pi = (uv)$ between the letters of the word Q distributed on a circle:

$$(2.15)$$

In Eq. (2.14), $\pi_1(Q)$ and $\pi_2(Q)$ are the words between X_{ℓ_u} and X_{ℓ_u} . They fulfill that $\pi_2(Q)X_{\ell_u}\pi_1(Q)X_{\ell_v}$ matches Q up to cyclic reordering.

As a particular case in that definition: for π matching contiguous letters, that is if $v = u \pm 1$, one has the empty word in between, $\pi_{\{1\}}(Q) = 1_N$. Notice that by (2) of Claim 2.1,

$$\begin{aligned} \partial^{X_i} \partial^{X_j} \text{Tr}_N Q &= \partial^{X_i} \mathcal{D}^{X_j} Q \\ &= \partial^{X_i} \left(\sum_u \delta_{\ell_u}^j X_{\ell_{u+1}} X_{\ell_{u+2}} \cdots X_{\ell_k} X_{\ell_1} X_{\ell_2} \cdots X_{\ell_{u-1}} \right) \\ &= \sum_u \delta_{\ell_u}^j \sum_{v \neq u} \delta_{\ell_v}^i X_{\ell_{u+1}} X_{\ell_{u+2}} \cdots X_{\ell_{v-1}} \\ &\quad \otimes X_{\ell_{v+1}} \cdots X_{\ell_k} X_{\ell_1} X_{\ell_2} \cdots X_{\ell_{u-1}} \\ &= \sum_u \delta_{\ell_u}^j \sum_{v \neq u} \delta_{\ell_v}^i \pi_1(Q) \otimes \pi_2(Q), \text{ where } \pi = (uv). \quad \square \end{aligned}$$

Before we give some examples, notice that since for the NC-Laplacian both pairings $\pi = (uv)$ and (vu) appear, one can replace the expression $\nabla^2 \text{Tr}_N Q = \sum_{j=1}^n e_j \sum_{\pi=(uv)} \delta_{\ell_v}^j \delta_{\ell_u}^j \pi_1(Q) \otimes \pi_2(Q)$ by a more symmetric one,

$$\nabla^2 \text{Tr}_N Q = \sum_{j=1}^n e_j \sum_{\substack{\pi=\{uv\} \\ (\text{undirected})}} \delta_{\ell_v}^j \delta_{\ell_u}^j \{ \pi_1(Q) \otimes \pi_2(Q) + \pi_2(Q) \otimes \pi_1(Q) \}. \quad (2.16)$$

These differential operators can be extended to products of traces using the same formulae that defines them in the single-trace case, but they require additional structure. Namely, the NC-Laplacian satisfies the rule

$$\begin{aligned} \nabla^2 \text{Tr}^{\otimes 2}(P \otimes Q) &= \nabla^2(\text{Tr } P \cdot \text{Tr } Q) \\ &= (\nabla^2 \text{Tr } P) \cdot \text{Tr } Q + (\nabla^2 \text{Tr } Q) \cdot \text{Tr } P \\ &\quad + \sum_j e_j \{ \mathcal{D}^{X_j} P \otimes_{\tau} \mathcal{D}^{X_j} Q + \mathcal{D}^{X_j} Q \otimes_{\tau} \mathcal{D}^{X_j} P \}, \quad (2.17) \end{aligned}$$

in terms of a tensor product \otimes_{τ} that does not receive the next natural matrix coordinates for monomials $U, W \in \mathbb{C}_{\langle n \rangle}$,

$$(U \otimes W)_{ab;cd} := U_{ab} W_{cd}, \quad (2.18a)$$

but twisted ones with respect to the transposition $\tau = (13) \in \text{Sym}(4)$ of the four indices,

$$(U \otimes_{\tau} W)_{a_1 a_2; a_3 a_4} := (U \otimes W)_{a_{\tau(1)} a_{\tau(2)}; a_{\tau(3)} a_{\tau(4)}}, \quad (2.18b)$$

or more clearly

$$(U \otimes_{\tau} W)_{ab;cd} = U_{cb} W_{ad}. \quad (2.18c)$$

Before seeing how τ twists the product on $\mathbb{C}_{\langle n \rangle}^{\otimes 2}$, in the next section, notice how expression (2.17) follows directly from a slightly more general one that we do prove:

Proposition 2.4. *The NC-Hessian of a product of traces is*

$$\text{Hess}(\text{Tr } P \text{ Tr } Q) = \text{Tr } Q \text{ Hess}(\text{Tr } P) + \text{Tr } P \text{ Hess } \text{Tr } Q + \Delta(P, Q) \quad (2.19)$$

where the last term is the matrix with entries

$$[\Delta(P, Q)]_{ij} = \mathcal{D}^{X_i} P \otimes_{\tau} \mathcal{D}^{X_j} Q + \mathcal{D}^{X_i} Q \otimes_{\tau} \mathcal{D}^{X_j} P \in \mathbb{C}_{\langle n \rangle}^{\otimes 2}.$$

The matrix just defined satisfies $\Delta(P, Q) = \Delta(Q, P)$ evidently—which is important since P and Q in $\text{Hess}(\text{Tr } P \text{ Tr } Q)$ are interchangeable—but, like the Hessian, it is not symmetric in general, $[\Delta(P, Q)]_{ij} \neq [\Delta(P, Q)]_{ji}$.

It is convenient to split $\mathcal{P} = \partial^{X_i} \partial^{X_j} \text{Tr } P \in \mathbb{C}_{\langle n \rangle} \otimes \mathbb{C}_{\langle n \rangle}$ using (a convenient upper-index version of) Sweedler's notation $\mathcal{P} = \sum \mathcal{P}^{(1)} \otimes \mathcal{P}^{(2)}$. The transition to the index notation can be expressed as⁶

$$\partial_{cb}^{X_i} \partial_{ad}^{X_j} \text{Tr } P = \sum \mathcal{P}_{ab;cd} = \sum \mathcal{P}_{ab}^{(1)} \mathcal{P}_{cd}^{(2)},$$

which follows by direct computation (and is moreover supported by [44, Eq. 4]).

The coordinates of the (i, j) -matrix block of a Hessian are $\text{Hess}(\text{Tr } P)_{ij|ab;cd} := (\text{Hess}(\text{Tr } P)_{ij})_{ab;cd}$. We compute these for the product of traces:

$$\begin{aligned} & \text{Hess}(\text{Tr } P \text{ Tr } Q)_{ij|ab;cd} \\ &= \partial_{cb}^{X_i} \partial_{ad}^{X_j} (\text{Tr } P \text{ Tr } Q) \\ &= (\text{Hess}(\text{Tr } P)_{ij})_{ab;cd} \text{Tr } Q + \text{Tr } Q (\text{Hess}(\text{Tr } P)_{ij})_{ab;cd} \\ &\quad + \mathcal{D}_{cb}^{X_i} P \mathcal{D}_{ad}^{X_j} Q + \mathcal{D}_{cb}^{X_i} Q \mathcal{D}_{ad}^{X_j} P \\ &= (\text{Tr } Q \text{ Hess } \text{Tr } P + \text{Tr } P \text{ Hess } \text{Tr } Q)_{ij|ab;cd} \\ &\quad + (\mathcal{D}^{X_i} P \otimes_{\tau} \mathcal{D}^{X_j} Q + \mathcal{D}^{X_i} Q \otimes_{\tau} \mathcal{D}^{X_j} P)_{ab;cd}. \quad \square \end{aligned}$$

From the last proposition, one can easily show a similar rule holds replacing everywhere by its version Hess_{σ} modified by $\sigma = \text{diag}(e_1, \dots, e_n)$ and the Δ -matrix by $\Delta^{\sigma}(P, Q)$ which has diagonal entries $\Delta_{ii}^{\sigma}(P, Q) = e_i \Delta_{ii}(P, Q)$ and else those of Δ .

In the following, we sketch the action of the operator ∂^{X_j} graphically. The convention is that the first letter of a word is the first after the cut (arrow tail) and the last letter corresponds to the one before the cut (arrow head).

⁶Other choices are possible. However, if one applies these operators to products of traces, as is the case treated here, at least one of the products will show this braiding.

- One can represent the elements of im Tr_N as words on circles. For the NC-derivative $\partial^{X_j} : \text{im Tr}_N \rightarrow \mathbb{C}_{\langle n \rangle}$ one has

$$\begin{aligned}
 & \text{Diagram 1: A circle with vertices labeled } X_{\ell_1}, X_{\ell_2}, \dots, X_{\ell_k} \text{ and arrows pointing clockwise.} \\
 & \mapsto \sum_{j\text{-cuts}} \text{Diagram 2: A circle with vertices labeled } X_{\ell_1}, X_{\ell_2}, \dots, X_{\ell_k} \text{ and arrows pointing clockwise.} \\
 & = \sum_{j\text{-cuts}} \text{Diagram 3: A horizontal line with vertices labeled } X_{\ell_1}, X_{\ell_2}, \dots, X_{\ell_{t-1}}, X_{\ell_{t+1}}, \dots, X_{\ell_{k-1}}, X_{\ell_k} \text{ and arrows pointing right.} \\
 & \hspace{15em} j = \ell_t
 \end{aligned} \tag{2.20}$$

where the ends of the line in the last figure are joined (multiplied).

- In the next representation, each arrow belongs to a different tensor factor. Thus, $\partial^{X_j} : \mathbb{C}_{\langle n \rangle} \rightarrow \mathbb{C}_{\langle n \rangle}^{\otimes 2}$ acts as

$$\text{Diagram 1: A horizontal line with vertices labeled } X_{\ell_1}, X_{\ell_2}, \dots, X_{\ell_k} \text{ and arrows pointing right.} \mapsto \sum_{j\text{-cuts}} \text{Diagram 2: A horizontal line with vertices labeled } X_{\ell_1}, X_{\ell_2}, \dots, X_{\ell_{t-1}}, X_{\ell_{t+1}}, \dots, X_{\ell_k} \text{ and arrows pointing right.} \\
 \hspace{15em} j = \ell_t$$

Together, the two last pictures give the graphical interpretation of the proof of the proposition above.

- Similarly, $\partial^{X_j} : \mathbb{C}_{\langle n \rangle}^{\otimes k} \rightarrow \mathbb{C}_{\langle n \rangle}^{\otimes k+1}$ j -cuts at all places of each tensor-factor (line):

$$\begin{aligned}
 \partial^{X_j} (\text{Diagram 1: A horizontal line with vertices labeled } X_{\ell_1}, X_{\ell_2}, \dots, X_{\ell_k} \text{ and arrows pointing right.}) &= \sum_{r=1}^k \text{Diagram 2: A horizontal line with vertices labeled } X_{\ell_1}, X_{\ell_2}, \dots, X_{\ell_k} \text{ and arrows pointing right.} \\
 &= \sum_{r=1}^k \sum_{j\text{-cuts}} \text{Diagram 3: A horizontal line with vertices labeled } X_{\ell_1}, X_{\ell_2}, \dots, X_{\ell_k} \text{ and arrows pointing right.} \\
 & \hspace{15em} j
 \end{aligned} \tag{2.21}$$

Example. The next examples shall be useful below:

- On $\mathbb{C}_{\langle n \rangle, N}$ it holds $\nabla^2 \text{Tr}(X_i X_j) = \sum_{k=1}^n 2e_i \delta_i^k \delta_j^k 1_N \otimes 1_N = 2e_i \delta_j^i 1_N \otimes 1_N$ from the last statement, since only the empty word is between the two letters.
- On $\mathbb{C}_{\langle 1 \rangle, N} = \mathbb{C}\langle X \rangle$ with $X^* = X$, $\nabla^2 = (\partial^X)^2$ and $(m \geq 2)$

$$\nabla^2 \text{Tr}_N \left(\frac{X^m}{m} \right) = \sum_{\ell=0}^{m-2} X^\ell \otimes X^{m-2-\ell}. \tag{2.22}$$

Now is evident that, even though $\mathbb{C}_{\langle 1 \rangle, N}$ consists of ordinary polynomials, NC-derivatives are not ordinary.

Example. We compute a NC-Hessian and a NC-Laplacian on $\mathbb{C}_{\langle 2 \rangle} = \mathbb{C}\langle A, B \rangle$. With aid of Claim 2.3 and setting $g = \text{diag}(e_1, e_2) = \text{diag}(e_a, e_b)$:

$$\begin{aligned}
 \text{Hess}_g \{ \text{Tr}(ABAB) \} &= \begin{pmatrix} e_a \partial^A \circ \partial^A & \partial^A \circ \partial^B \\ \partial^B \circ \partial^A & e_b \partial^B \circ \partial^B \end{pmatrix} \text{Tr}(ABAB) \\
 &= 2 \begin{pmatrix} e_a B \otimes B & AB \otimes 1 + 1 \otimes BA \\ BA \otimes 1 + 1 \otimes AB & e_b A \otimes A \end{pmatrix} \tag{2.23}
 \end{aligned}$$

which also explicitly shows the asymmetry of the Hessian. To compute, say, the entry (12) of this matrix, which corresponds to the operator $\partial^A \circ \partial^B$, one has four matches: distributing the word $ABAB$ on a circle as in 2.15, with the arrow tail at any letter B , the tip of the arrow can pair the A left (or clockwise) to it or the A to its right (counterclockwise). According to Claim 2.3 these contributions are, respectively, $1 \otimes BA$ and $AB \otimes 1$ for each letter B in the word, hence the factor 2. The Laplacian is the trace of it,

$$\nabla^2 \text{Tr}(ABAB) = \text{Tr}_2 \left(\text{Hess}_g \text{Tr}(ABAB) \right) = 2e_a B \otimes B + 2e_b A \otimes A.$$

2.2. The Algebraic Structure

We consider sums of monomials which either have the form $X \otimes Y$ or $X \otimes_\tau Y$ inside the same algebra:

$$\mathcal{A}_n = \mathbb{C}_{\langle n \rangle}^{\otimes 2} \oplus \mathbb{C}_{\langle n \rangle}^{\otimes \tau 2} \quad \text{and} \quad \mathcal{A}_{n,N} = \mathbb{C}_{\langle n \rangle, N}^{\otimes 2} \oplus \mathbb{C}_{\langle n \rangle, N}^{\otimes \tau 2}, \quad (2.24)$$

where the second symbol emphasizes the matrix realization of the free algebra. On $\mathcal{A}_{n,N}$ there is a product \times defined in coordinates by

$$[(U \otimes_{\vartheta} W) \times (P \otimes_{\varpi} Q)]_{ab;cd} := (U \otimes_{\vartheta} W)_{ax;cy} (P \otimes_{\varpi} Q)_{xb;yd}, \quad (2.25)$$

where ϑ, ϖ represent the twist τ or its absence, and the sums are implicit. The twisted structure modifies the product according to:

Proposition 2.5. *For monomials $U, W, P, Q \in \mathbb{C}_{\langle n \rangle}$ one has*

$$(U \otimes W) \times (P \otimes Q) = UP \otimes WQ, \quad (2.26a)$$

$$(U \otimes W) \times (P \otimes_\tau Q) = WP \otimes_\tau UQ, \quad (2.26b)$$

$$(U \otimes_\tau W) \times (P \otimes Q) = UP \otimes_\tau WQ, \quad (2.26c)$$

$$(U \otimes_\tau W) \times (P \otimes_\tau Q) = WP \otimes UQ. \quad (2.26d)$$

These rules can be remembered by identifying tensor product of monomials $U \otimes W$ with the block diagonal element $\text{diag}(U, W) \in M_2(\mathbb{C}_{\langle n \rangle})$ and each twisted product $U \otimes_\tau W$ with the anti-diagonal $j \text{diag}(U, W) = \begin{pmatrix} 0 & W \\ U & 0 \end{pmatrix}$ for $j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then, the rules (2.26) are just a restatement of matrix multiplication in $M_2(\mathbb{C}_{\langle n \rangle})$, but we do not state it as such since it does not work for polynomials. But in fact Eq. (2.26) can be proven in coordinates:

Proof. We prove the second rule: for $a, \dots, d = 1, \dots, N$, one has

$$\begin{aligned} ((U \otimes W) \times (P \otimes_{\tau} Q))_{ab;cd} &= (U \otimes W)_{am;co} (P \otimes_{\tau} Q)_{mb;od} \\ &= U_{am} W_{co} P_{ob} Q_{md} \quad (\text{implicit } o, m \text{ sum}) \\ &= (U_{am} Q_{md}) (W_{co} P_{ob}) = (WP)_{cb} (UQ)_{ad} \\ &= (WP \otimes_{\tau} UQ)_{ab;cd} \end{aligned}$$

and that rule follows. The first rule (2.26) is obvious, the two left unproven are verified in similar way. \square

As a caveat, notice that

$$(1 \otimes_{\tau} 1) \times (P \otimes Q) = P \otimes_{\tau} Q \quad \text{but} \quad (P \otimes Q) \times (1 \otimes_{\tau} 1) = Q \otimes_{\tau} P.$$

For monomials $P, Q, U, W \in \mathbb{C}_{\langle n \rangle}$, we let also

$$[(U \otimes_{\vartheta} W) \star (P \otimes_{\varpi} Q)]_{ab;cd} := (U \otimes_{\vartheta} W)_{ab;xy} (P \otimes_{\varpi} Q)_{yx;cd},$$

where ϖ, ϑ stand for either τ or an empty label.

Proposition 2.6. *It follows that*

$$(U \otimes_{\tau} W) \star (P \otimes_{\tau} Q) = PU \otimes_{\tau} WQ, \quad (2.27a)$$

$$(U \otimes W) \star (P \otimes_{\tau} Q) = U \otimes PWQ, \quad (2.27b)$$

$$(U \otimes_{\tau} W) \star (P \otimes Q) = WPU \otimes Q, \quad (2.27c)$$

$$(U \otimes W) \star (P \otimes Q) = \text{Tr}(WP)U \otimes Q \quad (2.27d)$$

We prove only the first one, the other proofs being similar:

$$\begin{aligned} ((U \otimes_{\tau} W) \star (P \otimes_{\tau} Q))_{ab;cd} &= (U \otimes_{\tau} W)_{ab;xy} (P \otimes_{\tau} Q)_{yx;cd} \\ &= P_{cx} U_{xb} W_{ay} Q_{yd} = (PU)_{cb} (WQ)_{ad} \\ &= (PU \otimes WQ)_{cb;ad} \\ &= (PU \otimes_{\tau} WQ)_{ab;cd}. \end{aligned} \quad \square$$

One can replace the new product by \times , namely using

$$(U \otimes_{\tau} W) \star (P \otimes_{\tau} Q) = (P \otimes_{\tau} W) \times (U \otimes Q), \quad (2.28)$$

which holds due to

$$\begin{aligned} [(U \otimes_{\tau} W) \star (P \otimes_{\tau} Q)]_{ab;cd} &= (PU \otimes_{\tau} WQ)_{ab;cd} = P_{cx} U_{xb} W_{ay} Q_{yd} \\ &= (U \otimes_{\tau} W)_{ab;xy} (P \otimes_{\tau} Q)_{yx;cd}. \end{aligned}$$

Notice that in Eq. (2.28) τ no longer acts on the matrix indices and it has been transferred to the factors:

$$(Y_1 \otimes_{\tau} Y_2) \star (Y_3 \otimes_{\tau} Y_4) = (Y_{\tau(1)} \otimes_{\tau} Y_{\tau(2)}) \times (Y_{\tau(3)} \otimes Y_{\tau(4)}).$$

Since $(PU \otimes_{\tau} WQ)_{ab;cd} = (PU \otimes WQ)_{cb;ad}$, another useful expression for the sequel is

$$[(U \otimes_{\tau} W) \star (P \otimes_{\tau} Q)]_{ab;cd} = (U \otimes W)_{xb;ay} (P \otimes Q)_{cy;xd}. \quad (2.29)$$

Also, while the product \times loses the twist, $(1 \otimes_\tau 1)^{\times 2} = (1 \otimes 1)$, the \star product preserves it $(1 \otimes_\tau 1)^{\star 2} = (1 \otimes_\tau 1)$ and in fact $(1 \otimes_\tau 1)$ is the unit element:

$$(1 \otimes_\tau 1) \star (P \otimes_\tau Q) = P \otimes_\tau Q, \quad (2.30a)$$

$$(U \otimes_\tau W) \star (1 \otimes_\tau 1) = U \otimes_\tau W, \quad (2.30b)$$

$$(1 \otimes_\tau 1) \star (P \otimes Q) = P \otimes Q, \quad (2.30c)$$

$$(U \otimes W) \star (1 \otimes_\tau 1) = U \otimes W, \quad (2.30d)$$

which follows from Proposition 2.6. Although it might be clear from the definition of \star that on $\mathbb{C}_{(n)}^{\otimes_\tau 2}$ it is associative—since there the first factor right multiplication and in the second ordinary matrix multiplication—it is reassuring to see that it is also associative if untwisted elements are implied:

Proposition 2.7. *The product \star is associative on \mathcal{A}_n .*

Let $A, B, C, D, U, W, P, Q, T, S, X, Y \in \mathbb{C}_{(n)}$. Using Proposition 2.6 one verifies straightforwardly that either bracketing,

$$\{(U \otimes_\tau W + P \otimes Q) \star (T \otimes_\tau S + X \otimes Y)\} \star (A \otimes_\tau B + C \otimes D)$$

or

$$(U \otimes_\tau W + P \otimes Q) \star \{(T \otimes_\tau S + X \otimes Y) \star (A \otimes_\tau B + C \otimes D)\},$$

yields due to the cyclicity of the trace the same result, namely:

$$\begin{aligned} & ATU \otimes_\tau WSB + P \otimes (ATQSB) + WXU \otimes AYB \\ & + \text{Tr}(QX) \cdot (P \otimes AYB) + WSCTU \otimes D + \text{Tr}(TQSC) \cdot (P \otimes D) \\ & + \text{Tr}(YC) \cdot WXU \otimes D + \text{Tr}(XQ) \cdot \text{Tr}(YC) \cdot (P \otimes D). \end{aligned} \quad \square$$

For the sequel, more important than the Hessian is its twisted version

Definition 2.8. The *twisted NC-Hessian* Hess_σ^τ is given by

$$\text{Hess}_\sigma^\tau := (1 \otimes_\tau 1) \times \text{Hess}_\sigma.$$

In other words, by Proposition 2.26, Hess_σ^τ is obtained from Hess_σ after exchanging the products \otimes_τ and \otimes .

Example. We exemplify computing the product of $\text{Hess}_\sigma^\tau(AABB)$, namely

$$\begin{pmatrix} e_a(1 \otimes_\tau BB + BB \otimes_\tau 1) & 1 \otimes_\tau AB + BA \otimes_\tau 1 + A \otimes_\tau B + B \otimes_\tau A \\ 1 \otimes_\tau BA + AB \otimes_\tau 1 + A \otimes_\tau B + B \otimes_\tau A & e_b(1 \otimes_\tau AA + AA \otimes_\tau 1) \end{pmatrix},$$

with $\text{Hess}_\sigma^\tau[\text{Tr } A \text{Tr}(ABB)]$,

$$\begin{pmatrix} e_a(1 \otimes BB + BB \otimes 1) & \text{Tr } A(B \otimes_\tau 1 + 1 \otimes_\tau B) + 1 \otimes AB + 1 \otimes BA \\ \text{Tr } A(B \otimes_\tau 1 + 1 \otimes_\tau B) + AB \otimes 1 + BA \otimes 1 & e_b \text{Tr } A(A \otimes_\tau 1 + 1 \otimes_\tau A) \end{pmatrix}.$$

The diagonal⁷ of $\text{Hess}_\sigma^\tau[\text{Tr } A \text{Tr}(ABB)] \star \text{Hess}_\sigma^\tau(AABB) = \begin{pmatrix} \mathcal{P} & * \\ * & \mathcal{Q} \end{pmatrix}$, which is computed entrywise with \star , is given by (recall $e_a^2 = e_b^2 = 1$)

$$\mathcal{P} = \text{Tr } A\{1 \otimes BBA + ABB \otimes 1 + A \otimes BB + 2B \otimes BA + 2AB \otimes B$$

⁷ The $*$ entries of products of two Hessians are uninteresting in this paper (unless one wants to compute the third order the RG-flow).

$$\begin{aligned}
& + BB \otimes A\} + 1 \otimes_{\tau} AAB B + 2 \cdot 1 \otimes_{\tau} ABAB + 2 \cdot 1 \otimes_{\tau} ABBA \\
& + 2 \cdot 1 \otimes_{\tau} BABA + 1 \otimes_{\tau} BBAA + 2 \cdot 1 \otimes_{\tau} BBBB + 2BB \otimes_{\tau} BB,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{Q} = & \operatorname{Tr} A \{1 \otimes BAB + BAB \otimes 1 + A \otimes BB + B \otimes AB + B \otimes BA \\
& + AB \otimes B + BA \otimes B + BB \otimes A + 1 \otimes AAA + AAA \otimes 1 \\
& + A \otimes AA + AA \otimes A\} + 2AB \otimes_{\tau} AB \\
& + 2AB \otimes_{\tau} BA + 2BA \otimes_{\tau} AB + 2BA \otimes_{\tau} BA.
\end{aligned}$$

The $M_n(\mathbb{C})$ -trace (here for $n = 2$, $\mathcal{P} + \mathcal{Q}$) of products of Hessians—or rather of their anti-commutator—will be shown to be fundamental for the RG-flow. The absolute (not only cyclic) order in the letters of the expressions for the twisted or untwisted Hessians of cyclic NC-polynomials absolutely matters. If one continues taking products of Hessians the order of the matrix factors does matter too (which is why one gets bulky expressions now). Only at the final stage, when we take traces, we can cyclically permute.

3. Random Noncommutative Geometries and Multimatrix Models

We briefly recall the foundations of fuzzy geometries, known to be rephrasable in terms of matrix algebras [66], in Barrett’s matrix geometry setting [8]. The original definition is given in terms of spectral triples, but in that definition the axioms implying the Dirac operator can be directly replaced by a characterization these boil down to.

3.1. Fuzzy Geometries as Spectral Triples

Given a signature $(p, q) \in \mathbb{Z}_{\geq 0}^2$, a fuzzy (p, q) -geometry consists of a quintuple

$$(M_N(\mathbb{C}), V \otimes M_N(\mathbb{C}), D, J, \gamma)$$

whose elements we describe next. The inner product space V is given the structure of $\mathcal{C}\ell(p, q) = \mathcal{C}\ell(\mathbb{R}^{(p, q)})$ -module. The action \underline{c} of the Clifford algebra on the basis elements θ^μ of $\mathbb{R}^{(p, q)} = (\mathbb{R}^{p+q}, \operatorname{diag}(+p, -q))$, where the subindex in each sign means its repetition that many times, yields *gamma-matrices* $\gamma^\mu = \underline{c}(\theta^\mu)$. We assume that they satisfy,

$$(\gamma^\mu)^2 = +1, \quad \mu = 1, \dots, p, \quad \gamma^\mu \text{ Hermitian}, \quad (3.1a)$$

$$(\gamma^\mu)^2 = -1, \quad \mu = p+1, \dots, p+q, \quad \gamma^\mu \text{ anti-Hermitian}, \quad (3.1b)$$

that is, one has Hermitian or anti-Hermitian gamma-matrices according to whether μ is a *time-like* ($1 \leq \mu \leq p$) or a *spatial* index ($p < \mu \leq p+q$). This in turn yields the *chirality* $\gamma = (-i)^{s(s+1)/2} \gamma^1 \dots \gamma^{p+q}$, being $s := q - p$ the *KO-dimension*. The inner product of V together with the Hilbert–Schmidt inner product on $M_N(\mathbb{C})$ endow $\mathcal{H} = V \otimes M_N(\mathbb{C})$ with the structure of Hilbert

space the matrix algebra acts on in the natural way, ignoring V . Moreover, the KO-dimension determines three signs $\epsilon, \epsilon', \epsilon'' \in \{-1, +1\}$ via

$s \equiv q - p \pmod{8}$	0	1	2	3	4	5	6	7
ϵ	+	+	−	−	−	−	+	+
ϵ'	+	−	+	+	+	−	+	+
ϵ''	+	+	−	+	+	+	−	+

The operator $J = C \otimes (\text{complex conjugation})$ on \mathcal{H} defines the *real structure*. Here, $C : V \rightarrow V$ is anti-unitary and satisfies $C^2 = \epsilon$ and $C\gamma^\mu = \epsilon'\gamma^\mu C$ for each $\mu = 1, \dots, p + q$. Last, but most importantly, D , the *Dirac operator*, is a self-adjoint operator on \mathcal{H} that satisfies the *order-one condition* $[[D, Y'], JYJ^{-1}] = 0$ for each $Y, Y' \in M_N(\mathbb{C})$. The signs in the table above imply, as part of the definition,

$$J^2 = \epsilon, \quad JD = \epsilon'DJ, \quad J\Gamma = \epsilon''\Gamma J.$$

After the axioms are solved [8], for an even dimension $q + p$ (thus even KO-dimension), the Dirac operator has the form

$$D = \sum_{\mu} \gamma^{\mu} \otimes k_{\mu} + \sum_{\mu, \nu, \rho} \gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \otimes k_{\mu\nu\rho} + \dots + \sum_{\mu, \nu, \rho} \gamma^{\widehat{\mu\nu\rho}} \otimes k_{\widehat{\mu\nu\rho}} + \sum_{\hat{\mu}} \gamma^{\hat{\mu}} \otimes k_{\hat{\mu}}, \quad (3.2)$$

where

- $\gamma^{\alpha} = \gamma^{\mu_1} \gamma^{\mu_2} \dots \gamma^{\mu_{i_{2r-1}}}$ means the product of all indices included in an increasingly ordered multi-index $\alpha = (\mu_1 \dots \mu_{i_{2r-1}})$. The hatted indices are those omitted from the list $\{1, 2, \dots, p + q\}$. Notice that the sum runs only over multi-indices of odd cardinality; and
- for any $Y \in M_N(\mathbb{C})$, k_{α} are commutators or anti-commutators determined by α via $k_{\alpha}(Y) = X_{\alpha}Y + e_{\alpha}YX_{\alpha}$, being $X_{\alpha} \in M_N(\mathbb{C})$ self-adjoint if $e_{\alpha} = +1$ and traceless anti-Hermitian if $e_{\alpha} = -1$.

For the first $p + q$ values of i , e_i can be read off from $\text{diag}(e_1, \dots, e_{p+q})$, the signature; however, if $p + q \geq 3$, the number n of matrices that parametrize D exceeds $p + q$. This is also true for odd $p + q$, for instance, in signature $(0, 3)$ the Dirac operator can be written as

$$D = \{H, \cdot\} + \iota[L_1, \cdot] + \mathcal{J}[L_2, \cdot] + \mathcal{K}[L_3, \cdot],$$

where $L_i \in \mathfrak{su}(N)$ for each i and $\iota, \mathcal{J}, \mathcal{K}$ are the quaternion units as a realization of the pertaining gamma-matrices. In odd dimensions, the chirality is trivial, which is why the anti-commutator term with a the Hermitian $N \times N$ matrix H has a trivial coefficient, instead of a product of three different gammas matrices.

The complete criterion [61, Appendix A] that fully determines the signs in (2.1a) for even-dimensional fuzzy geometries implies multi-indices α , namely

$$e_{\alpha} = (-1)^{u+r-1} \quad 2r = \#(\alpha) + 1, \quad u = \#\{\text{spatial indices in } \alpha\}. \quad (3.3)$$

After a signature (p, q) and the matrix size N are chosen, notice that the items $(M_N(\mathbb{C}), V \otimes M_N(\mathbb{C}), J, \gamma)$, called also a *fermionic system*, are fixed.

We let $\mathcal{M}_N^{p,q}$ be the space of all the Dirac operators that complete the four objects into a fuzzy geometry. This spectral triple is finite-dimensional but does not fall into the classification made by Krajewski and Paschke–Sitarz [50, 64]. Using Eqs. (3.2) and (3.3) one can obtain $\mathcal{M}_N^{p,q}$ in terms of $\mathfrak{su}(N)$ and \mathbb{H}_N , the Hermitian matrices in $M_N(\mathbb{C})$. For instance $\mathcal{M}_N^{0,4} = \mathbb{H}_N^{\times 4} \times \mathfrak{su}(N)^{\times 4}$ for the Riemannian 4-geometry and $\mathcal{M}_N^{1,3} = \mathbb{H}_N^{\times 2} \times \mathfrak{su}(N)^{\times 6}$ for the Lorentzian case. However, for the formalism below it suffices to know the space of Dirac operators for two-dimensional fuzzy geometries:

$$\mathcal{M}_N^{p,q} = \begin{cases} \mathfrak{su}(N) \times \mathfrak{su}(N) & (p, q) = (0, 2) \\ \mathbb{H}_N \times \mathfrak{su}(N) & (p, q) = (1, 1) \\ \mathbb{H}_N \times \mathbb{H}_N & (p, q) = (2, 0) \end{cases}$$

When we work in fixed signature, we write $\mathcal{M}_N = \mathcal{M}_N^{p,q}$, as above in Eq. (2.2).

3.2. The Spectral Action for Fuzzy Geometries

We review how to compute the Spectral Action $\text{Tr } f(D)$, in order to see its relation with chord diagrams, simultaneously setting the terminology for Sect. 7. We restrict the discussion below to two-dimensional geometry with otherwise arbitrary signature.

As remarked in [11], the computable spectral actions $\text{Tr } f(D)$ require f , which in the original Connes–Chamseddine formulation is a bump function around the origin, rather to be a polynomial, with $f(x) \rightarrow +\infty$ as $|x| \rightarrow \infty$. We thus restrict to positive, even powers of the Dirac operator, $\text{Tr } D^m$, which according to [61], can be computed from *chord diagrams* (C.D.) of m points. A chord diagram consists of a circumference with m marked points and $m/2$ arcs joining them. These diagrams encode traces of products of gamma matrices. For two-dimensional geometries, the description is relative simple, as no multi-indices are required:

$$\frac{1}{2} \text{Tr } D^m = \sum_{\substack{\chi \\ (m\text{-point C.D.)}}} \mathfrak{a}(\chi), \quad (3.4)$$

where the *value* $\mathfrak{a}(\chi)$ of the diagram χ is defined by

$$\begin{aligned} \mathfrak{a}(\chi) = & \sum_{\mu_1, \dots, \mu_m=1,2} \chi^{\mu_1 \dots \mu_m} \left\{ \sum_{\mathcal{T} \in \mathcal{P}_m} \text{sgn}[\mu(\mathcal{T})] \right. \\ & \cdot \text{Tr}_N \left(\prod_{r \in \{1, \dots, m\} \setminus \mathcal{T}}^{\rightarrow} X_{\mu_r} \right) \times \text{Tr}_N \left(\prod_{r \in \mathcal{T}}^{\leftarrow} X_{\mu_r} \right) \Big\}. \end{aligned} \quad (3.5)$$

Herein, for an m -point chord diagram and for each $\mu_1, \dots, \mu_m = 1, 2$, one defines

$$\chi^{\mu_1 \dots \mu_m} = (-1)^{\#\{\text{simple crossings of chords in } \chi\}} \prod_{\substack{v, u=1 \\ v \sim_\chi u}}^m (e_{\mu_v} \delta^{\mu_v \mu_u}), \quad (3.6)$$

where $v \sim_\chi u$ means that the point u and v are joined by a chord of χ , and e_{μ_v} are the signs in the signature $\text{diag}(e_1, e_2)$ of the fuzzy two-dimensional geometry. The rest of the elements is given by:

- \mathcal{P}_m is the power set of $\{1, 2, \dots, m\}$
- for any $\mathcal{I} = \{i, j, \dots, k\} \in \mathcal{P}_m$, $\mu(\mathcal{I})$ is the ordered set $(\mu_i, \mu_j, \dots, \mu_k)$ and $\text{sgn}[\mu(\mathcal{I})] = \prod_{r \in \mathcal{I}} e_{\mu_r}$, which is a sign
- $X_1 = A$, $X_2 = B$ are the (random) matrices
- the arrows on the product indicate the order in which it is performed; the right arrow preserves the order of the set one sums over and the left arrow inverts it.

A quick way to see that the Spectral Action is real, as it should be, bases on the observation that for each word w originated by a chord diagram χ , its adjoint w^* is originated by the mirror image of χ , denoted by χ^* . But this being a chord diagram, it also appears summed in Eq. (3.4).

$$\text{If } \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \text{ originates } w \quad \Rightarrow \quad \left\{ \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right\}^* = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \text{ originates } w^*. \quad (3.7)$$

When the running indices in Eq. (3.5) take a particular value, we color the chords of the corresponding chord diagram: green, if at the ends of the chords there is a matrix A and violet⁸ if it is B . To a fixed word, say $B^4 A B^2 A^3$, generally many diagrams contribute,

$$\begin{array}{ccccccc} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} & + & \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} & + & \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} & + \dots + & \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \\ & & & & & & \\ & + & \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} & + & \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} & & \end{array} \quad (3.8)$$

and we now show that for certain words their sum cannot vanish. (This will lead to the well-definedness of the chosen truncation schemes in the FRGE.)

Lemma 3.1. *Given any word $w \in \mathbb{C}_{\langle 2 \rangle} = \mathbb{C}\langle A, B \rangle$ of even degree in each of its generators, $\deg_A(w), \deg_B(w) \in 2\mathbb{Z}_{\geq 0}$, it holds*

$$[\text{Tr}_N w](\text{Tr } D^m) \neq 0, \quad \text{for } m = \deg_A(w) + \deg_B(w).$$

⁸In the printed version green is just light gray and violet is black.

That is, $\text{Tr}_N w$ has nonzero coefficient in the Spectral Action for a suitable power of the Dirac operator D .

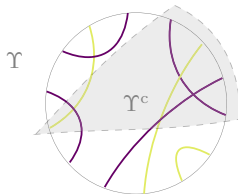
Proof. We use a chord diagram argument. The general situation is that not only one chord diagram gives rise to w . Although the existence of such diagram having m chords is trivial to exhibit, there exists the risk that all those diagrams add up to zero. We now verify that this is impossible.

Suppose that $[\text{Tr}_N w] \mathbf{a}(\chi)$ does not vanish. This does not fix χ but leaves still a freedom of exchanging all green chord ends, corresponding to A , among themselves and the same for and all violet ones, which correspond to B . (As a matter of illustration, for the word $B^4 A B^2 A^3$ above the first line shows such moves among B -chords and the lower the A -chords.) All the diagrams χ' with $[\text{Tr}_N w] \mathbf{a}(\chi') \neq 0$ are obtained by either of these moves applied to the initial χ , hence the number of such χ' is

$$\begin{aligned} \#\{\deg_A(w)\text{-point C.D.'s}\} \times \#\{\deg_B(w)\text{-point C.D.'s}\} \\ = (\deg_A(w) - 1)!! \times (\deg_B(w) - 1)!! \end{aligned}$$

which by assumption is the product of odd numbers, thus itself odd. Notice that the value of the diagrams χ' might only differ from χ by a sign, which is determined by the crossings of the chords. Indeed, in Eq. (3.5) the terms inside the curly bracket are fixed by hypothesis, and in (3.6) the product $\prod_{v,u=1}^m (e_{\mu_v} \delta^{\mu_v \mu_u})$ is the same. This implies that all the diagrams contributing to w never can cancel, as the sum $\sum_{r=1}^{2\ell-1} \varepsilon_r$ never does (for $\ell \in \mathbb{N}, \varepsilon_r \in \{-1, +1\}$). Therefore, the coefficient $[\text{Tr}_N w](\text{Tr } D^m)$ of $\text{Tr}_N w$ in $\text{Tr } D^m$ is by Eq. (3.4) nonzero. \square

Given an m -point C.D., a nontrivial *partition* is a set $\Upsilon \in \mathcal{P}_m$ which is neither empty, nor is complement is, $\Upsilon^c \neq \emptyset$. Such an Υ splits a diagram into product of traces of two words. These words can be read off from the diagram, according to (3.5) one counterclockwise the other clockwise. For instance,



produces $\text{Tr}_N(BAB^2A)$ from Υ^c (denoted by a shaded region) and $\text{Tr}_N(BAB^4A^3)$ from Υ . These nontrivial subsets Υ play the main role in the next

Lemma 3.2. *Let $\mathbb{C}_{\langle 2 \rangle} = \mathbb{C}\langle A, B \rangle$, as before. The coefficient of the double-trace $\text{Tr}_N(w_1) \text{Tr}_N(w_2)$ in the Spectral Action is nontrivial for any word $w = w_1 \otimes w_2 \in \mathbb{C}_{\langle 2 \rangle} \otimes \mathbb{C}_{\langle 2 \rangle}$ satisfying $\deg_A(w_1) + \deg_A(w_2), \deg_B(w_1) + \deg_B(w_2) \in 2\mathbb{Z}_{\geq 0}$:*

$$[\text{Tr}_N^{\otimes 2} w](\text{Tr } D^m) \neq 0, \quad \text{for } m = \sum_{r=1,2} \deg_A(w_r) + \deg_B(w_r).$$

Proof. We use a similar, albeit longer, argument to the single-trace case of Lemma 3.1. Since otherwise the statement reduces to Lemma 3.1 above, we assume that neither w_1 nor w_2 is the trivial word 1_N . This means that only nontrivial partitions ($\mathcal{T}, \mathcal{T}^c \neq \emptyset$) can generate w , that is, if $\mathfrak{a}(\chi) \neq 0$, then w is listed in

$$\sum_{\mu_r=1,2} \chi^{\mu_1 \cdots \mu_m} \left\{ \sum_{\substack{\mathcal{T} \in \mathcal{P}_m \\ \mathcal{T}, \mathcal{T}^c \neq \emptyset}} \text{sgn}[\mu(\mathcal{T})] \cdot \text{Tr}_N \left(\prod_{r \in \{1, \dots, m\} \setminus \mathcal{T}}^{\rightarrow} X_{\mu_r} \right) \cdot \text{Tr}_N \left(\prod_{r \in \mathcal{T}}^{\leftarrow} X_{\mu_r} \right) \right\}, \quad (3.9)$$

with the condition that the first and the second traces yield simultaneously $\text{Tr}_N w_1$ and $\text{Tr}_N w_2$, in either correspondence. To wit, we have the following cases:

- *Case I.* If $\{\text{Tr}_N(w_1^*), \text{Tr}_N(w_2^*)\} = \{\text{Tr}_N(w_1), \text{Tr}_N(w_2)\}$ as sets.
- *Case II.* If the trace of both adjoint words are different, that is if $\text{Tr}_N(w_1^*) \neq \text{Tr}_N(w_1), \text{Tr}_N(w_2)$ as well as $\text{Tr}_N(w_2^*) \neq \text{Tr}_N(w_1), \text{Tr}_N(w_2)$.
- *Case III.* For $\{r, l\} = \{1, 2\}$, if one coincides, $\text{Tr}_N(w_r^*) = \text{Tr}_N(w_v)$, then the other does not, $\text{Tr}_N(w_l^*) \neq \text{Tr}_N(w_u), \{u, v\} = \{1, 2\}$.

In the first case, if $\mathcal{T} \in \mathcal{P}_m$ originates these words, so does \mathcal{T}^c , and their contribution to the previous sum is doubled, for $\text{sgn}[\mu(\mathcal{T})] = \text{sgn}[\mu(\mathcal{T}^c)]$. Hence, in Case I, we can sum over half of the elements encompassed inside the curly brackets in (3.9). Since we excluded the trivial partitions, the total of sets in that sum is $\#(\mathcal{P}_m) - 2 = 2^m - 2 = 2 \cdot (2^{m-1} - 1)$. By hypothesis, the result of (3.9) is twice the sum over half elements, which is $2^{m-1} - 1$. But in Cases II and III, we also can do so, since \mathcal{T}^c does not reproduce the word $w_1 \otimes w_2$, so we can ignore the half of the sets (3.9). In any case, the sum is a multiple of 2 (Case I) of, or directly (Cases II-III), a sum over $2^{m-1} - 1$ elements, which is an odd number, since $w_1 \otimes w_2$ is not the trivial word and thus $m > 1$. Since the three cases are the only possibilities given the two words, the partial conclusion is that the set \mathcal{T} in (3.9) runs over an odd number of independent elements.

Again, finding a C.D. χ that generates w is not hard: one puts together the letters $w_1 w_2^*$ and joins by chords, matching letters. And again, this diagram is ambiguous up to a factor of $(\deg_A(w_1 w_2^*) - 1)!! \times (\deg_B(w_1 w_2^*) - 1)!!$. Considering the initial paragraph, the total number of terms is

$$(2^{m-1} - 1) \cdot (\deg_A(w_1 w_2^*) - 1)!! \times (\deg_B(w_1 w_2^*) - 1)!! \in 2\mathbb{N} + 1, \quad (3.10)$$

since the sum of all such diagrams is the product of (3.10) with the nonredundant odd number. By the same token as before, the sum over all the signs listed in Eq. (3.10) cannot vanish. \square

Concrete expressions for $f(z) = \frac{1}{4}(\frac{z^2}{2} + \frac{z^4}{4} + \frac{z^6}{6})$ are given below.⁹ From now on, we agree to write down rather the operators $\text{Tr}^{\otimes 2}$ should be applied

⁹The common $1/4$ factor results from removing redundant partitions by a set \mathcal{T} and its complement \mathcal{T} in Eq. (3.5), and from $1/\dim_{\mathbb{C}}(V) = 1/2$.

to, in order to get the actual monomials in the action. As for the signs, it is convenient to set $e_a := e_1$ and $e_b := e_2$.

- *Quadratic operators:*

$$1_N \otimes \left(\frac{e_a}{2} A^2 + \frac{e_b}{2} B^2 \right) + \frac{1}{2} (A \otimes A) + \frac{1}{2} (B \otimes B). \quad (3.11)$$

- *Quartic operators:*

$$\begin{aligned} 1_N \otimes \left(\frac{1}{4} A^4 + \frac{1}{4} B^4 + e_a e_b A^2 B^2 - \frac{1}{2} e_a e_b A B A B \right) \\ + A B \otimes A B + 2 e_a e_b A^2 \otimes B^2 + (e_a A^3 + e_b A B^2) \otimes A \\ + (e_a A^2 B + e_b B^3) \otimes B + 3 A^2 \otimes A^2 + 3 B^2 \otimes B^2. \end{aligned} \quad (3.12)$$

- *Sextic operators:* The part bearing a 1_N factor is:

$$\begin{aligned} 1_N \otimes \{ e_a A^6 + 6 e_b A^4 B^2 - 6 e_b A^2 (A B)^2 + 3 e_b (A^2 B)^2 \\ + e_b B^6 + 6 e_a A^2 B^4 - 6 e_a B^2 (B A)^2 + 3 e_a (B^2 A)^2 \}, \end{aligned} \quad (3.13a)$$

and bi-trace terms are:

$$\begin{aligned} A \otimes (2 A^5 + 2 A B^4 + 6 e_a e_b A^3 B^2 - 2 e_a e_b A^2 B A B) \\ + B \otimes (2 B^5 + 2 B A^4 + 6 e_b e_a B^3 A^2 - 2 e_b e_a B^2 A B A) \\ + 8 A B \otimes [e_a A^3 B + e_b B^3 A] \\ + A^2 \otimes \{ e_b [8 A^2 B^2 - 2 B A B A] + e_a [5 A^4 + B^4] \} \\ + B^2 \otimes \{ e_a [8 B^2 A^2 - 2 A B A B] + e_b [5 B^4 + A^4] \} \\ + \frac{10}{3} (A^3 \otimes A^3) + 4 e_a e_b (A B^2 \otimes A^3) + 6 (A^2 B \otimes A^2 B) \\ + \frac{10}{3} (B^3 \otimes B^3) + 4 e_b e_a (B A^2 \otimes B^3) + 6 (B^2 A \otimes B^2 A). \end{aligned} \quad (3.13b)$$

Notice that neither

$$\begin{aligned} A \cdot A \cdot A \cdot A \cdot A \cdot B, & \quad A \cdot A \cdot A \cdot B \cdot B \cdot B, & \quad A \cdot A \cdot B \cdot A \cdot B \cdot B, \\ A \cdot A \cdot B \cdot B \cdot A \cdot B, & \quad A \cdot B \cdot A \cdot B \cdot A \cdot B, & \quad A \cdot B \cdot B \cdot B \cdot B \cdot B. \end{aligned}$$

nor their cyclic permutations are allowed. The same holds for any nontrivial partition of these into two tensor factors (e.g., $A \cdot A \otimes A \cdot A \cdot A \cdot B$), as they are not compatible with chord diagrams, in the sense mentioned at the beginning of this subsection. We also remark that \otimes_τ -products do not appear in the Spectral Action.

4. Deriving the Functional Renormalization Group Equation

We are interested in a nonperturbative approach and pursue the RG-flow governed by Wetterich–Morris equation (or FRGE). Polchinski equation¹⁰ [62, Eq. 27] can be more suitable in a perturbative approach.

¹⁰For Polchinski equation, a review [51] on tensor models might include complex matrix models as a rank-2 case. This can be used as starting point for a perturbative approach for these multimatrix models.

We start with the *bare action* $S[\Phi]$ that describes the model at an “energy” scale $\Lambda \in \mathbb{N}$ (*ultraviolet cutoff*). Let Φ be an n -tuple of matrices $\Phi = (\varphi_1, \dots, \varphi_n) \in \mathcal{M}_\Lambda$, but the following discussion can be easily be made more general taking $\Phi \in M_N(\mathbb{C})^n$. Motivated by fuzzy geometries, the bare action S is assumed to be a functional of the form

$$S[\Phi] = \Gamma_\Lambda[\Phi] = \Lambda \cdot \text{Tr } P + \sum_{\alpha} (\text{Tr} \otimes \text{Tr})(\Psi_{\alpha} \otimes \Upsilon_{\alpha}), \quad (4.1)$$

being P and each Ψ_{α} and Υ_{α} in the finite sum a noncommutative polynomial in the n matrices, $P, \Psi_{\alpha}, \Upsilon_{\alpha} \in \mathbb{R}_{\langle n \rangle} = \mathbb{R}\langle \varphi_1, \dots, \varphi_n \rangle$. The trace $\text{Tr} = \text{Tr}_{\Lambda}$ is that of $M_{\Lambda}(\mathbb{C})$.

Our derivation of Wetterich–Morris equation for multimatrix models is inspired by the ordinary QFT-derivation (e.g., [39]) for the first steps. Let

$$\exp(\mathcal{W}[J]) := \mathcal{Z}[J] := \int_{\mathcal{M}_{\Lambda}} e^{-S[\Phi] + \text{Tr}(J \cdot \Phi)} d\mu_{\Lambda}(\Phi), \quad (4.2)$$

being $J = (J^1, \dots, J^n) \in \mathcal{M}_N$ an n -tuple of matrix sources J^i , and $J \cdot \Phi = \sum_{i=1}^n J^i \varphi_i$ the sum of the n matrix products. Here, $d\mu_{\Lambda}(\Phi)$ is the product Lebesgue measure on \mathcal{M}_{Λ} , for which the notations $\int_{\Lambda} [d\varphi](\cdot)$ and $\int_{\Lambda} \mathcal{D}\varphi(\cdot)$ are also common, mostly in physics.

The fundamental object is the effective action Γ , obtained by the Legendre transform of the free energy $\mathcal{W}[J]$,

$$\Gamma[X] = \sup_J \left(\text{Tr}(J \cdot X) - \mathcal{W}[J] \right). \quad (4.3)$$

Here, X denotes the n -tuple $X = (X_1, \dots, X_n)$ of *classical fields* $X_i := \partial^{J^i} \mathcal{W}[J] = \langle \varphi_i \rangle$. The supremum creates the dependence $J = J[X]$ and yields a functional depending only on X . Notice that since $J \in \mathcal{M}_N$, each source obeys the same (anti)-Hermiticity relation $(J^i)^* = e_i J^i$ as φ_i , for each $i = 1, \dots, n$. As a consequence, $(J \cdot \Phi)^* = (\Phi \cdot J)$ and the classical fields obey the expected rules:

$$X_i^* = (\partial^{J^i} \mathcal{W}[J])^* = \partial^{(J^i)^*} \overline{(\mathcal{W}[J])} = e_i X_i.$$

The effective action $\Gamma[X]$ contains all the quantum fluctuations at all energy scales. In practice, one uses an interpolating average effective action that incorporates only the fluctuations that are stepwise integrated out; the average effective action $\Gamma_N[X]$ results after integration of the modes having an energy larger than N (i.e. matrix indices larger than N), while lower degrees of freedom not yet integrated. The parameter N serves as a threshold splitting the modes in high and low; the latter sit in the $N \times N$ block. Lowering N makes $\Gamma_N[X]$ to approximate the full effective action Γ .

The progressive elimination of degrees of freedom is obtained by adding a mass-like term

$$(\Delta S_N)[\Phi] = \frac{1}{2} \sum_{a,b,c,d=1}^{\Lambda} \sum_{i=1}^n e_i(\varphi_i)_{ba} (R_N^{\tau})_{ab;cd}(\varphi_i)_{dc}, \quad (\Lambda \geq N \in \mathbb{N}). \quad (4.4)$$

This regulator has been adapted from that of Eichhorn–Kosłowski to the multimatrix case.¹¹ Typically the function $R_N^\tau : \{1, \dots, \Lambda\}^4 \rightarrow \mathbb{R}$ restricts the sum to some N -dependent region, but the sum limits in Eq. (4.4) allow for a freedom of regulators R_N^τ . Here, R_N^τ is not meant as a matrix: in particular its k -th power $(R_N^\tau)^k$ does not imply $k - 1$ sums but rather the k -th power pointwise. This can be guaranteed by assuming

$$(R_N^\tau)_{ab;cd} = r_N(a, c)(1 \otimes_\tau 1)_{ab;cd} = r_N(a, c)\delta_c^b \delta_a^d \quad (4.5a)$$

for a \mathbb{R} -valued function r_N , and to satisfy

$$(R_N^\tau)_{ab;cd} = (R_N^\tau)_{ba;dc} \quad \text{and} \quad (R_N^\tau)_{ab;cd} = (R_N^\tau)_{dc;ba}, \quad (4.5b)$$

which hold by imposing $r_N(a, c) = r_N(c, a)$ for all a, c . Since τ implies a twist in the product, we stress that R_N^τ is not a multiple of the identity, only

$$(R_N)_{ab;cd} := r_N(a, c)\delta_{ab}\delta_{cd} = r_N(a, c)(1 \otimes 1)_{ab;cd} = (1 \otimes 1)_{cd;ab} \quad (4.6)$$

is. The choice of R_N is arbitrary up to the following three conditions¹²:

1. $(R_N)_{ab;cd} > 0$ for low modes, i.e. $\max\{a, b, c, d\}/N \rightarrow 0$
2. $(R_N)_{ab;cd} \rightarrow 0$ for high modes, i.e. $N/\min\{a, b, c, d\} \rightarrow 0$
3. $(R_N)_{ab;cd} \rightarrow \infty$ as $N \rightarrow \Lambda \rightarrow \infty$

which have the following effect, respectively:

1. the infrared (IR) regulator suppresses low modes: as a result these are not integrated out, unlike high modes, which do contribute to the average effective action Γ_N
2. is an initial condition for low N , i.e. ensures that one eventually recovers the full quantum effective action by lowering N
3. is an initial condition for large N and ensures that one can recover the bare action S as $N \rightarrow \Lambda \rightarrow \infty$ via the saddle-point approximation.

Thus, incorporating ΔS_N to the action IR-regulates the functional

$$\exp(\mathcal{W}_N[J]) := \mathcal{Z}_N[J] := \int_{\mathcal{M}_\Lambda} e^{-S[\varphi] - \Delta S_N[\varphi] + \text{Tr}(J \cdot \varphi)} d\mu_\Lambda(\Phi) \quad (4.7)$$

in terms of which one can obtain the *interpolating average effective action*

$$\Gamma_N[X] := \sup_J \left(\text{Tr}(J \cdot X) - \mathcal{W}_N[J] \right) - (\Delta S_N)[X]. \quad (4.8)$$

In practice, one uses the FRGE in order to determine it, instead of performing the path-integral. This equation is usually displayed in physics in terms of a *supertrace* STr we next define on the *superspace* $M_n(\mathbb{C}) \otimes \mathcal{A}_{n,\Lambda} = M_n(\mathcal{A}_{n,\Lambda})$. Typical elements there form an $n \times n$ matrix \mathcal{T} with entries

$$(\mathcal{T}_{ij}) = \sum \mathcal{T}_{ij}^{(1)} \otimes \mathcal{T}_{ij}^{(2)}.$$

¹¹The next treatment holds for $1_n \otimes R_N \rightarrow \omega \otimes (1 \otimes_\tau 1)$ with $\omega \in M_n(\mathbb{C})$ diagonal, but we stay with the easiest choice.

¹²This is customary to state in FRGE-papers. This condition deserves a mathematical study itself, in order to find a precise characterization. This is left as a perspective and commented on later.

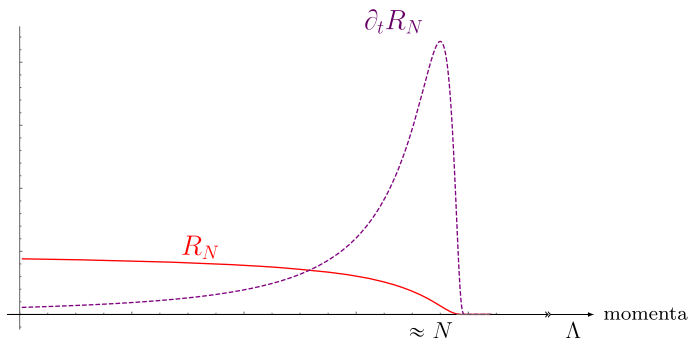


FIGURE 3. The idea behind the regulator R_N and its logarithmic derivative, here illustrated with a ‘bump function’: R_N protects the IR degrees of freedom, while those higher than N are integrated out. Thus, N is the “momentum threshold” that splits modes into high- and low-momenta

for some matrices $\mathcal{T}_{ij}^{(1)}, \mathcal{T}_{ij}^{(2)} \in \mathbb{C}_{\langle n \rangle, \Lambda}$, whose four remaining entries we separate using a vertical bar, to avoid confusion:

$$\mathcal{T} = (\mathcal{T}_{ij|ab;cd})_{\substack{i,j=1,\dots,n \\ a,b,c,d=1,\dots,\Lambda}} \in M_n(\mathbb{C}_{\langle n \rangle, \Lambda}^{\otimes 2}) = M_n(\mathcal{A}_{n, \Lambda}).$$

We let also $\mathbf{1} = 1_n \otimes 1_\Lambda \otimes 1_\Lambda$, lest our notation becomes very loaded (which is a neutral element if \mathcal{A}_n is endowed with \times) but also notice that according to Eq. (2.30) only $\mathbf{1}_\tau = 1_n \otimes 1_\Lambda \otimes_\tau 1_\Lambda$ acts as a unit with respect to the \star -product. The supertrace is given by

$$\text{STr} = \text{Tr}_n \otimes \text{Tr}_{\mathcal{A}_n} : M_n(\mathcal{A}_n) \rightarrow \mathbb{C} \quad (4.9a)$$

$$\text{STr}(\mathcal{Q}) = \sum_{i=1}^n \sum_{a,b=1}^\Lambda \mathcal{Q}_{ii|aa;bb} = \sum_{i=1}^n \sum_{a,b,c,d=1}^\Lambda \mathcal{Q}_{ii|ab;cd} (\delta_a^b \delta_c^d). \quad (4.9b)$$

Since knowing the matrix size will be useful, we use $\text{Tr}_\Lambda^{\otimes 2}$ sometimes instead of $\text{Tr}_{\mathcal{A}_2}$, but as the next $n = 2$ example shows, it is important to be careful with twisted products whose factors are merged inside a same trace:

$$\begin{aligned} \text{STr} \begin{pmatrix} 1 \otimes A^4 & * \\ * & B^2 \otimes_\tau B^2 \end{pmatrix} &= \text{Tr}_{\mathcal{A}_2}(1 \otimes A^4 + B^2 \otimes_\tau B^2) \\ &= \Lambda \text{Tr}(A^4) + \text{Tr}(B^4). \end{aligned}$$

Proposition 4.1. *The interpolating effective action Γ_N of a matrix model with $X = (X_1, \dots, X_n) \in \mathcal{M}_N^{p,q}$ satisfies for each $N \leq \Lambda$ Wetterich–Morris equation, which reads*

$$\partial_t \Gamma_N[X] = \frac{1}{2} \text{STr} \left(\frac{\partial_t R_N^\tau}{\text{Hess}_\sigma^\tau \Gamma_N[X] + R_N^\tau} \right), \quad (\text{FRGE})$$

being $t = \log N$ the RG-flow parameter and $\sigma = \text{diag}(e_1, \dots, e_n)$ with $X_i^* = \pm X_i$ iff $e_i = \pm 1$. These signs are determined by the signature (p, q) of the fuzzy

geometry that originates the matrix model—which for dimensions $p + q \leq 2$ coincides with $g = \text{diag}(e_1, \dots, e_{p+q})$ —and else are given by Eq. (3.3). The quotient of operators is meant with respect to the \times product.

Also $n = 2$ if $p + q = 2$ and $n = 8$ if $p + q = 4$, with general rule $n = 2^{p+q-1}$ as far as $p+q$ is even [61] and R_N is economic notation for $1_n \otimes R_N$. After the proof, we provide the strategy to compute the RHS. The quantity in the “denominator” of the FRGE requires some care; its well-definedness is addressed in Sect. 5.2.

Proof. Directly from the definition of the interpolating action one has

$$\begin{aligned} \partial_t \Gamma_N[X] &= (\partial_t \Gamma_N)[X] = \partial_t \left\{ \sup_J \left(\text{Tr}(J \cdot X) - \mathcal{W}_N[J] \right) - (\Delta S_N)[X] \right\} \\ &= -\partial_t \mathcal{W}_N[J] - \partial_t (\Delta S_N)[X] \\ &= -\frac{1}{\mathcal{Z}_N[J]} \int (-\partial_t \Delta S_N) e^{-S - \Delta S_N + \text{Tr}(J \cdot \varphi)} d\mu_\Lambda(\Phi) \\ &\quad - \frac{1}{2} \sum_{a,b,c,d=1}^A \sum_{i=1}^n e_i(X_i)_{ab} (\partial_t R_N^\tau)_{ab;cd}(X_i)_{cd}. \end{aligned} \quad (4.10)$$

Recalling that $X_i = \mathcal{Z}_N^{-1} \partial^{J^i} \mathcal{Z}_N$, one can use

$$\begin{aligned} \frac{\delta^2 \mathcal{W}_N[J]}{\delta J_{ba}^i \delta J_{dc}^i} &= -\langle (\varphi_i)_{ab} \rangle \langle (\varphi_i)_{cd} \rangle + \frac{1}{\mathcal{Z}_N[J]} \frac{\delta^2 \mathcal{Z}_N[J]}{\delta J_{ba}^i \delta J_{dc}^i} \quad (\text{no } i \text{ sum}) \\ &= -(X_i)_{ab} (X_i)_{cd} \\ &\quad + \frac{1}{\mathcal{Z}_N[J]} \int (\varphi_i)_{ab} (\varphi_i)_{cd} \cdot e^{-S - \Delta S_N + \text{Tr}(J \cdot \varphi)} d\mu_\Lambda(\Phi) \end{aligned}$$

in order to re-express $\partial_t (\Delta S_N)$ appearing in the integrand in the first term, $\mathcal{Z}_N[J]^{-1} \int (-\partial_t \Delta S_N) e^{-S - \Delta S_N + \text{Tr}(J \cdot \varphi)} d\mu_\Lambda(\Phi)$, of Eq. (4.10) to obtain

$$\partial_t \Gamma_N[X] = \frac{1}{2} \sum_{a,b,c,d}^A \sum_{i=1}^n \left(\frac{\delta^2 \mathcal{W}_N[J]}{\delta J_{ba}^i \delta J_{dc}^i} \right) \cdot e_i \cdot (\partial_t R_N^\tau)_{ab;cd}. \quad (4.11)$$

The rest relies on the use of the superspace chain rule

$$\begin{aligned} \delta_{ij}\delta_{ux}\delta_{vy} &= \frac{\delta(X_i)_{uv}}{\delta(X_j)_{xy}} = \sum_{k=1}^n \sum_{l,m} \frac{\delta(X_i)_{uv}}{\delta J_{lm}^k} \frac{\delta J_{lm}^k}{\delta(X_j)_{xy}} \\ &= \sum_{\substack{k=1,\dots,n \\ l,m=1,\dots,\Lambda}} \left\{ \partial_{yx}^{X_j} \partial_{lm}^{X_k} \Gamma_N[X] + e_k \delta_{jk}(R_N^\tau)_{lm;yx} \right\} \cdot \left(\frac{\delta^2 \mathcal{W}_N[J]}{\delta J_{lm}^k \delta J_{vu}^i} \right). \end{aligned} \quad (4.12)$$

Passing from the first to the second line is implied by taking the derivative with respect to X_j of the IR-regulated quantum equation of motion, that is of

$$\begin{aligned} \partial_{ab}^{X_k} \Gamma_N &= \partial_{ab}^{X_k} \left(\text{Tr}(X \cdot J) - \mathcal{W}_N[J] - \Delta S_N[X] \right) \\ &= J_{ab}^k + \text{Tr}(X \cdot \partial_{ab}^{X_k} J) - \partial_{ab}^{X_k} \mathcal{W}_N[J] - e_k \text{Tr}((R_N^\tau)_{ab}; \dots X^k) \\ &= J_{ab}^k - e_k \text{Tr}((R_N^\tau)_{ab}; \dots X^k). \end{aligned}$$

In the second line $\partial_{ab}^{X_k} J$ is a matrix (for fixed a, b) and the trace $\text{Tr}(X \cdot \partial_{ab}^{X_k} J)$, which equals $\partial_{ab}^{X_k} \mathcal{W}_N[J]$ by the chain rule, is taken with respect to those tacit indices of J . In the other trace-term, the shown indices a, b are excluded, so traces are taken for the remaining ones (the dots in R_N^τ); the symmetries (4.5) of R_N^τ have been used too. Hence, indeed

$$\begin{aligned} \frac{\delta J_{pq}^k}{\delta X_{xy}^j} &= \partial_{yx}^{X_j} \partial_{pq}^{X_k} \Gamma_N[X] + e_j \delta_{jk}(R_N^\tau)_{pq;yx} \\ &= (e_k^{\delta_{jk}} \{\text{Hess } \Gamma_N[X]\}_{jk} + e_j \delta_{jk} R_N)_{px; yq}, \end{aligned}$$

after Eq. (4.6) and the index symmetries implied by it. Denoting by \cdot_n the product in the $M_n(\mathbb{C})$ tensor factor (of the superspace), one can moreover replace $(\text{Hess}^J \mathcal{W}_N)_{ki} = \partial^{J^k} \partial^{J^i} \mathcal{W}_N[J]$ by the inverse¹³ of

$$\text{Hess } \Gamma_N + (\sigma \otimes R_N) = \sigma \cdot_n (\text{Hess}_\sigma \Gamma_N + 1_n \otimes R_N),$$

after using $\sigma = \text{diag}(e_1, \dots, e_n)$ and the fact that $1/e_i = e_i$ (since $e_i = \pm 1$). One has

$$\begin{aligned} \partial_t \Gamma_N[X] &= \frac{1}{2} \sum_{a,b,c,d} \sum_{i=1}^n \left(e_i \frac{\delta^2 \mathcal{W}_N[J]}{\delta J_{ba}^i \delta J_{dc}^i} \right) (\partial_t R_N^\tau)_{ab;cd} \\ &= \frac{1}{2} \sum_{a,b,c,d} \sum_{i=1}^n (\text{Hess}_\sigma^J \mathcal{W}_N[J])_{ii|cb;ad} (\partial_t R_N^\tau)_{ab;cd}. \end{aligned} \quad (4.13)$$

The result follows from Eq. (4.11), after realizing that the LHS of (4.12) is $\delta_{ij}\delta_{ux}\delta_{vy} = (1_n \otimes 1 \otimes_\tau 1)_{ij|yx;uv} = (\mathbf{1}_\tau)_{ij|yx;uv}$. In order to invert¹⁴ the Hessian

¹³See discussion after the proof.

¹⁴One could feel tempted to state

$$\left\{ \partial_{ab}^{X_j} \partial_{xy}^{X_k} \Gamma_N[X] + e_j \delta_{jk}(R_N)_{ab;xy} \right\}^{-1} \stackrel{!}{=} \partial_{yx}^{J^k} \partial_{cd}^{J^i} \mathcal{W}_N[J].$$


of \mathcal{W} , we use

$$\begin{aligned} & \{\text{Hess}_\sigma \Gamma_N[X] + R_N\}_{ij|xb;ay} (\text{Hess}_\sigma^J \mathcal{W}[J])_{jk|cx;yd} \\ &= \{\text{Hess}_\sigma \Gamma_N[X] + R_N\}_{ij|\tau(ab;xy)} (\text{Hess}_\sigma^J \mathcal{W}[J])_{jk|\tau(yx;cd)} \\ &= (1_n \otimes 1 \otimes_\tau 1)_{ik|ab;cd} \end{aligned} \quad (4.14)$$

where the \star product and the twisted Hessian can now be recognized. Therefore,

$$\begin{aligned} \partial_t \Gamma_N[X] &= \frac{1}{2} \text{Tr}_n \left\{ \sum_{a,b,c,d}^A ([\text{Hess}_\sigma^\tau \Gamma_N[X] + 1_n \otimes R_N^\tau]^{-1})_{ab;cd} \right. \\ &\quad \left. \times (\partial_t R_N^\tau)_{ab;cd} \right\} \\ &= \frac{1}{2} \text{Tr}_n \otimes \text{Tr}_{\mathcal{A}_n} \{ (\text{Hess}_\sigma^\tau \Gamma_N + R_N^\tau)^{-1} \times (\partial_t R_N^\tau) \}. \end{aligned} \quad (4.15)$$

We renamed indices and we used the symmetry $(R_N^\tau)_{ab;cd} = (R_N^\tau)_{ba;cd}$. \square

The RHS of the FRGE is usually interpreted in terms of a ribbon loop , the thick ribbon being the full propagator. For the present FRGE this picture is obtained by interpreting the ribbon as the supertrace $\text{Tr}_n \otimes \text{Tr}_A^{\otimes 2}$,

$$\partial_t \Gamma_N = \text{Diagram} \quad (4.16)$$

The diagram shows a loop with two rectangular boxes. The left box contains the expression $(\text{Hess}_\sigma^\tau \Gamma + R_N^\tau)^{-1}$ and the right box contains $\frac{1}{2} \partial_t R_N^\tau$. The boxes are connected by two concentric arcs, representing the ribbon loop.

The source marked with a crossed circle is the RG-time derivative term. In order to stress the meaning of the last equation, we consider an ordinary Hermitian n -matrix model. Proposition 4.1 then restricts to signature $(n, 0)$, so each $e_i = 1$, $i = 1, \dots, n$.

Corollary 4.2 (FRGE for Hermitian multimatrix models). *Wetterich–Morris equation for Hermitian n -matrix models is given by*

$$\partial_t \Gamma_N[X] = \frac{1}{2} \text{STr} \left(\frac{\partial_t R_N^\tau}{\text{Hess}^\tau \Gamma_N[X] + R_N^\tau} \right). \quad (4.17)$$

Proof. It is immediate from Proposition 4.1, since for Hermitian matrices one has $\sigma = 1_n$. \square

5. Techniques to Compute the Renormalization Group Flow

The next sections explain how to compute the RHS of the FRGE.

Footnote 14 continued

Although this expression is probably clearer than Eq. (4.14), first one has to invert in superspace, and only thereafter, take the matrix entries.

5.1. Projection and Truncations

The RG-flow generates the infinitely many operators that the symmetries allow. Feasibility forces us first to *project* each matrix X_i to a $N \times N$ matrix $X_i^{(N)}$ and then truncate $\Gamma_N[X^{(N)}]$ to Ansätze implying finitely many operators \mathcal{O}_I indexed by words I of the free algebra. Since this projection will be assumed for the rest of this paper, for the sake of lightness we agree to write $X^{(N)}$ as X . Some truncation schemes are:

- Single trace truncation:

$$\Gamma_N[X] = N \sum_I \bar{g}_I(N) \operatorname{Tr}_N(\mathcal{O}_I(X)).$$

- Bi-tracial truncation:

$$\Gamma_N[X] = N \sum_I \bar{g}_I(N) \operatorname{Tr}_N(\mathcal{O}_I(X)) \quad (5.1)$$

$$+ \sum_{I, I'} \bar{g}_{I|I'}(N) (\operatorname{Tr}_N \otimes \operatorname{Tr}_N) \overbrace{(\mathcal{O}_I(X) \otimes \mathcal{O}_{I'}(X))}^{\mathcal{O}_{I|I'}(X)}. \quad (5.2)$$

- Degree- k truncation:

$$\Gamma_N[X] = \sum_{\substack{I_1, \dots, I_\alpha \\ \sum_\nu \deg \mathcal{O}_{I_\nu}(X) \leq k}} \frac{(\bar{g}_{I_1|I_2|\dots|I_\alpha})(N)}{N^{k-1}} \operatorname{Tr}_N^{\otimes j} \left(\bigotimes_{\nu=1}^{\alpha} \mathcal{O}_{I_\nu}(X) \right),$$

where $\bar{g}_{\dots}(N)$ are the coupling constant, to be later renormalized to $g_{\dots}(N)$, the physical value.

We warn that this choice will be taken together with the assumption of N being large. The price to be paid is the un ability to recover the full effective action (which otherwise would be obtained by $\lim_{N \rightarrow 1} \Gamma_N$) not only because N is large, but also because we compute in a projection.

5.2. The FP^{-1} Expansion in the Large- N Limit

Based on the procedure introduced in [32] for Hermitian matrix models—which soon will be modified—we split the full propagator, for us $\operatorname{Hess}_\sigma \Gamma_N[X] + R_N = P \oplus F[X]$, into field-dependent and field-independent parts. In our multimatrix case, with signs $\sigma = \operatorname{diag}(e_1, \dots, e_n)$ given by Eq. (2.1a), we get $F[X] := \operatorname{Hess}_\sigma \Gamma_N[X] - (\operatorname{Hess}_\sigma \Gamma_N|_{X=0})$ and $P := R_N + (\operatorname{Hess}_\sigma \Gamma_N|_{X=0})$. We now simplify the treatment assuming that

$$Z_i \equiv Z_j =: Z, \text{ when } e_i = e_j := e \text{ for all } i, j,$$

for the rest of the paper. This is not the most general case and particularly excludes for the time being mixed signatures left for later study; however, this simplification has the advantage of leading to a P that is the identity matrix multiplied by a function $\{1, \dots, \Lambda\}^4 \rightarrow \mathbb{C}$ denoted by the same letter, $P = (e^2 Z + R_N) \mathbf{1} = (Z + R_N) \mathbf{1}$ since $e^2 = 1$. Notice that both Z and R_N being always positive P is invertible. In particular, powers P^ℓ of P are meant

pointwise (not as a matrix or tensor). One therefore has the commutation of P with the field part $F[X]$,

$$P \times F[X] = F[X] \times P, \quad \text{for all } X \in \mathcal{M}_N^{p,q}. \quad (5.3a)$$

It is important to realize in which sense the regulated Hessian of the interpolating action is an inverse of the Hessian of \mathcal{W}_N in source space, as this defines the way we have to take the Neumann series to invert $\text{Hess}_\sigma^\tau \Gamma + R_N^\tau$. Although in the $M_n(\mathbb{C})$ factor of superspace this is an ordinary matrix product—see the groupoid property in the indices i, j, k inside the proof of the FRGE, $\{\text{Hess}_\sigma \Gamma_N[X] + R_N\}_{ij|xb;ay}(\text{Hess}_\sigma^J \mathcal{W}[J])_{jk|cx;yd} = (1_n \otimes 1 \otimes_\tau 1)_{ik|ab;cd}$ —each entry of that matrix is multiplied according the product \star ; this product is easier to recognize in Eq. (2.29). That is to say, the way to invert in FRGE the regulated Hessian as dictated by the proof of the FRGE, is the algebra $M_n(\mathcal{A}_n, \star)$ and not $M_n(\mathcal{A}_n, \times)$. The commutation Eq. (5.3) can be replaced by

$$P_\tau \star F_\tau[X] = F_\tau[X] \star P_\tau, \quad \text{for all } X \in \mathcal{M}_N^{p,q}. \quad (5.3b)$$

since for (\mathcal{A}_n, \star) the unit is $1 \otimes_\tau 1$ and P_τ can be treated as a scalar function. We take the Neumann series of the twisted version $(\text{Hess}_\sigma^\tau \Gamma_N[X] + R_N^\tau)^{-1}$. Namely by Eq. (5.3b),

$$\sum_{k=0}^{\infty} (-1)^k P_\tau^{-1} \star \{P_\tau^{-1} F_\tau[X]\}^{\star k} = \sum_{k=0}^{\infty} (-1)^k \{P_\tau^{-(k+1)} F_\tau[X]\}^{\star k}. \quad (5.4)$$

Underlying this structure is the independence of P from the matrices $X = \{X_j\}$. Thus, when evaluated, $(P_\tau)^\ell$ sits in the constant part of $\mathcal{A}_{n,\Lambda}$, so powers of P_τ act on the field part by scalar multiplication. On the other hand, $(F_\tau[X])^{\star k}$ does mean the matrix product in the field part (2.5) of $\mathcal{A}_{n,\Lambda}$. Then, using the associativity of \star (Proposition 2.7), it is routine to check that the series (5.4) serves as inverse of $P_\tau \oplus F_\tau[X]$ in the sense that their product in either order yields $1_\tau = 1_n \otimes 1_\Lambda \otimes_\tau 1_\Lambda$. Therefore,

$$\frac{1}{\text{Hess}_\sigma^\tau \Gamma_N[X] + R_N^\tau} = \sum_{k=0}^{\infty} (-1)^k [P_\tau^{-(k+1)} (F_\tau^{\star k})]. \quad (5.5)$$

Assuming a truncation necessitates a compatible supertrace, STr_N . Since functions $G : \{1, \dots, \Lambda\}^4 \rightarrow \mathbb{C}$ act multiplicatively on the fields, we let

$$\text{STr}_N (G \cdot W[X]) = \left(\sum_{a,b,c,d=1}^{\Lambda} G_{ab;cd} \right) \cdot (\text{Tr}_n \otimes \text{Tr}_N \otimes \text{Tr}_N)(W_N[X]) \quad (5.6)$$

for W a field ($\deg W \neq 0$) in $M_n(\mathbb{C}) \otimes \mathcal{A}_{n,\Lambda}$. Here, W_N is the same matrix of words W projected to $M_n(\mathbb{C}) \otimes \mathcal{A}_{n,N}$. Also, STr is defined to be identically zero on the ‘constants’ of the free algebra (in the terminology of Sect. 2), or

$$\text{STr}_N(L) = 0 \text{ if } L \in \mathbb{C} \cdot (1_n \otimes 1_N \otimes 1_N) \text{ or } L \in \mathbb{C} \cdot (1_n \otimes 1_N \otimes_\tau 1_N). \quad (5.7)$$

This follows from any of the previous Ansätze for Γ_N , but it holds in general on physical grounds, since that constant part in the action corresponds to the vacuum energy [55]. However, the constant part of the algebra cannot be fully

ignored since is the one that regulates the RG-flow and that part appears multiplying the fields.

Remark 5.1. It would be interesting to answer whether the vanishing of $\text{STr}_N(L)$ (here and in the physics literature, as part of the definition) yields constraints on the IR-regulator. Namely, to explore the conditions that the equation $\text{STr}_N(P^{-1}1_N \otimes 1_N) = 0$ imposes on R_N , if one does not automatically include in the definition the condition (5.7).

Proposition 5.2. *The RG-flow is generated by the noncommutative Laplacian scaled by $\varrho := \sum_{a,b,c,d} (\partial_t R_N \cdot P^{-2})_{ab;cd}$. That is, in the ‘tadpole approximation’, the FRGE is given by*

$$\partial_t \Gamma_N[X] = -\frac{1}{2} \varrho \text{Tr}_N \otimes \text{Tr}_N (\nabla^2 \Gamma_N). \quad (5.8)$$

Proof. The tadpole approximation means to cut Eq. (5.5) to $k = 1$. It is immediate that one can undo the twists from the Hessian and R_N^τ altogether, with that of $\partial_t R_N^\tau$ since in this simple case \star is not implied. By Eq. (5.6) this means that

$$\begin{aligned} \partial_t \Gamma_N[X] &= +\frac{1}{2} \text{STr}_N \left\{ \sum_i \frac{\partial_t R_N}{\text{Hess}_\sigma \Gamma_N + R_N} \right\} \\ &= -\frac{1}{2} \left\{ \sum_{a,b,c,d} (\partial_t R_N \cdot P^{-2})_{ab;cd} \right\} \text{Tr}_n \otimes \text{Tr}_N \otimes \text{Tr}_N (F[X]) \\ &= -\frac{1}{2} \varrho \text{Tr}_n \otimes \text{Tr}_N \otimes \text{Tr}_N \{F[X] + F[0]\} \end{aligned}$$

were Eq. (5.7) has been used from the first to the second line, and from there to the third too. Now, $F[X] + F[0] = \text{Hess}_\sigma \Gamma_N[X]$, which traced over the first $M_n(\mathbb{C})$ factor, is by definition the NC-Laplacian. \square

We next justify the approximation given in Eqs. (5.6)–(5.7) and relate it with the definition of STr . Notice that the support of the function $G_k^{(N)} : \{1, \dots, \Lambda\}^4 \rightarrow \mathbb{R}$ given by $G_k^{(N)} = (\partial_t R_N) \cdot P^{-(k+1)}$ becomes an N -dependent region of $\{1, \dots, \Lambda\}^4$. Generally, one cannot find a function $f_n(N)$ such that $\text{STr}(G_k^{(N)} \cdot W[X]) = f_k(N) \cdot \text{Tr}_n \otimes \text{Tr}_{\mathcal{A}_{n,N}}(W_N[X])$, or explicitly such that

$$\sum_{a,b,c,d=1}^{\Lambda} [G_k^{(N)}]_{ba;dc} (W[X])_{ab;cd} = f_k(N) \text{Tr}_n \otimes \text{Tr}_N \otimes \text{Tr}_N (W_N[X]^k)$$

holds for a $W[X] \in M_n(\mathcal{A}_{n,\Lambda})$ in the field part of the free algebra, with $W_N[X] \in M_n(\mathcal{A}_{n,N})$. What is done in practice is to assume this replacement, but in return to let the function $f_k(N)$ be governed by the FRGE. We moreover use a regulator R_N whose support is inside $\{1, \dots, N\}^4$.

In order to exploit the FRGE, one needs to compute the first powers of the expansion (5.4). Defining $\tilde{h}_k(N) = \sum_{a,b,c,d}^{\Lambda} (G_k^{(N)})_{ab;cd}$, which, since neither

$\partial_t R_N$ nor $P^{-(k+1)}$ have field dependence, equals

$$\tilde{h}_k(N) = \sum_{a,b,c,d=1}^A (\partial_t R_N)_{ab;cd} P_{ab;cd}^{-(k+1)}, \quad (5.9)$$

one obtains after projecting

$$\begin{aligned} \partial_t \Gamma_N[X] &\stackrel{\text{(FRGE)}}{=} \frac{1}{2} \text{STr}_N^\tau \left(\sum_{k=0}^{\infty} (-1)^k G_k^{(N)} \cdot \{F_\tau[X]\}^{*k} \right) \\ &\stackrel{\text{(5.6)} \& \text{(5.7)}}{=} \frac{1}{2} \sum_{n=1}^{\infty} (-1)^k \tilde{h}_k(N) (\text{Tr}_n \otimes \text{Tr}_N^{\otimes \tau^2}) \{F_\tau[X]\}^{*k} \\ &= \frac{1}{2} (\text{Tr}_n \otimes \text{Tr}_N^{\otimes \tau^2}) \{ -\tilde{h}_1(N) F_\tau[X] \\ &\quad + \tilde{h}_2(N) (F_\tau[X])^{*2} + \dots \}. \end{aligned} \quad (5.10)$$

where $\text{Tr}_N \otimes_\tau \text{Tr}_N(\mathcal{Q}) = \text{Tr}_{\mathcal{A}_N}((1_N \otimes_\tau 1_N) \times \mathcal{Q})$ in terms of which we STr_N^τ . That twist comes from R_N^τ , whose untwisted part was absorbed in the functions $G_k^{(N)}$. We remark that Eq. (5.6) does not take into account the symmetry breaking caused by the regulator R_N , which is related to ignoring the modified Ward–Takahashi¹⁵ identity [54] caused by R_N .

From this point on, we focus on large- N results and consider the fields as projected matrices of size $N \times N$. Terms of order $O(N^{-1})$ will be often ignored in our computations. Also, since F is not needed again, we rename F_τ to F .

6. “Coordinate-Free” Matrix Models

We cross-check that, notwithstanding the somewhat different statements, our purely algebraic approach yields, for the Hermitian case with $n = 1$, the results that [32] presented in “coordinates” (that is, written with matrix entries). Here, we also calibrate the IR-regulator for later use in Sect. 7.

The interpolating action $\Gamma_N[X]$ is given by (applying $\text{Tr}_N^{\otimes 2}$ to) the next operators that define our truncation:

$$\begin{aligned} &\frac{Z}{2N} 1_N \otimes X^2 + \frac{\bar{\mathbf{g}}_4}{4N} 1_N \otimes X^4 + \frac{\bar{\mathbf{g}}_6}{6N} 1_N \otimes X^6 \\ &\quad + \frac{\bar{\mathbf{g}}_{2|2}}{8} X^2 \otimes X^2 + \frac{\bar{\mathbf{g}}_{2|4}}{8} X^2 \otimes X^4. \end{aligned}$$

Since $n = 1$, the NC-Laplacian equals the NC-Hessian ∂^2 , which on $\text{Tr } \mathcal{O}$ for an operator $\mathcal{O} \in \mathbb{C}_{\langle 1 \rangle}$ equals $(\partial \circ \mathcal{D})\mathcal{O}(X)$ by Claim 2.1. So, by Claim 2.3 and Eq. (2.17) one gets

¹⁵Regarding the Ward–Takahashi identity [59,60] of tensor models, a sister theory of matrix models, the progress of the WTI-constrained RG-flow is reviewed in [13]. See also [52].

$$\begin{aligned}
\frac{1}{2N} \partial^2 \operatorname{Tr}_{\mathcal{A}_{1,N}} (1_N \otimes X^2) &= 1_N \otimes 1_N \\
\frac{1}{4N} \partial^2 \operatorname{Tr}_{\mathcal{A}_{1,N}} (1_N \otimes X^4) &= X \otimes X + 1_N \otimes X^2 + X^2 \otimes 1_N \\
\frac{1}{8} \partial^2 \operatorname{Tr}_{\mathcal{A}_{1,N}} (X^2 \otimes X^2) &= X \otimes_\tau X + 1_N \otimes 1_N \operatorname{Tr}_N \left(\frac{X^2}{2} \right) \\
\frac{1}{6N} \partial^2 \operatorname{Tr}_{\mathcal{A}_{1,N}} (1_N \otimes X^6) &= X \otimes X^3 + 1_N \otimes X^4 + X^2 \otimes X^2 \\
&\quad + X^3 \otimes X + X^4 \otimes 1_N \\
\frac{1}{8} \partial^2 \operatorname{Tr}_{\mathcal{A}_{1,N}} (X^2 \otimes X^4) &= X \otimes_\tau X^3 + X^3 \otimes_\tau X + 1_N \otimes 1_N \operatorname{Tr}_N \left(\frac{X^4}{4} \right) \\
&\quad + \{X^2 \otimes 1_N + X \otimes X + 1_N \otimes X^2\} \operatorname{Tr}_N \left(\frac{X^2}{2} \right).
\end{aligned}$$

One now “twists” these equations. The expression for $F[X] = \operatorname{Hess}^\tau \Gamma[X] - Z(1_N \otimes_\tau 1_N)$ follows from the first equation in this list (after exchange of the tensor product with the twisted version). We keep odd-degree operators in F , even if we first included even-degree ones, since we need powers of F and even-degree operators are generated from odd-degree ones.

By neglecting odd-degree after taking the \star -powers of $F[X]$, as well as truncating them to degree-six operators, the FP^{-1} expansion (5.10) in this setting reads:

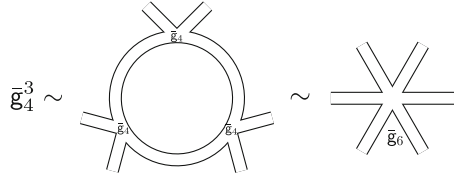
$$\begin{aligned}
\partial_t \Gamma_N[X] &= -\frac{1}{2} \frac{\tilde{h}_1}{N^2} \left\{ (N^2 + 2) \bar{\mathbf{g}}_{2|2} + 4N \bar{\mathbf{g}}_4 \right\} \operatorname{Tr}_N \left(\frac{X^2}{2} \right) \\
&\quad + \left\{ -\frac{\tilde{h}_1}{N^2} \left(\left(4 + \frac{N^2}{2} \right) \bar{\mathbf{g}}_{2|4} + 4N \bar{\mathbf{g}}_6 \right) \right. \\
&\quad + \frac{\tilde{h}_2}{N^2} (12 \bar{\mathbf{g}}_{2|2} \bar{\mathbf{g}}_4 + 4N \bar{\mathbf{g}}_4^2) \left. \right\} \operatorname{Tr}_N \left(\frac{X^4}{4} \right) \\
&\quad + \left\{ \frac{\tilde{h}_2}{N^2} ((8 + N^2) \bar{\mathbf{g}}_{2|2}^2 + 8N \bar{\mathbf{g}}_{2|2} \bar{\mathbf{g}}_4 + 12 \bar{\mathbf{g}}_4^2) \right. \\
&\quad - \frac{\tilde{h}_1}{N^2} (4N \bar{\mathbf{g}}_{2|4} + 4 \bar{\mathbf{g}}_6) \left. \right\} \frac{1}{8} \operatorname{Tr}_N^2(X^2) \\
&\quad + \left\{ \frac{\tilde{h}_2}{N^2} (36 \bar{\mathbf{g}}_{2|4} \bar{\mathbf{g}}_4 + 30 \bar{\mathbf{g}}_{2|2} \bar{\mathbf{g}}_6 + 12N \bar{\mathbf{g}}_4 \bar{\mathbf{g}}_6) \right. \\
&\quad - \frac{\tilde{h}_3}{N^2} (81 \bar{\mathbf{g}}_{2|2} \bar{\mathbf{g}}_4^2 + 6N \bar{\mathbf{g}}_4^3) \left. \right\} \operatorname{Tr}_N \left(\frac{X^6}{6} \right) \\
&\quad + \left\{ \frac{\tilde{h}_2}{N^2} (\bar{\mathbf{g}}_{2|4} ((38 + N^2) \bar{\mathbf{g}}_{2|2} + 12N \bar{\mathbf{g}}_4) + 8N \bar{\mathbf{g}}_{2|2} \bar{\mathbf{g}}_6 + 48 \bar{\mathbf{g}}_4 \bar{\mathbf{g}}_6) \right. \\
&\quad - \frac{\tilde{h}_3}{N^2} (72 \bar{\mathbf{g}}_{2|2}^2 \bar{\mathbf{g}}_4 + 12N \bar{\mathbf{g}}_{2|2} \bar{\mathbf{g}}_4^2 + 48 \bar{\mathbf{g}}_4^3) \left. \right\} \operatorname{Tr}_N \left(\frac{X^2}{2} \right) \operatorname{Tr}_N \left(\frac{X^4}{4} \right),
\end{aligned} \tag{6.1}$$

up to the third nontrivial term ($h_r = 0$ for $r \geq 4$) in the FP^{-1} -expansion. This equation was obtained using the product rules of Proposition 2.26: For

instance, the cubic term in \bar{g}_4 in the fifth line of (6.1) comes from $P^{-4}F^{*3}$, more concretely from

$$-(\tilde{h}_3/2N^2)\bar{g}_4^3 \text{Tr}_N^{\otimes \tau^2} [(X^2 \otimes_\tau 1_N)^{*3} + (1_N \otimes_\tau X^2)^{*3} + \dots],$$

where the dots omit other terms in the cube of F . Graphically, the \bar{g}_4^3 -contribution to \bar{g}_6 is (cf. Eq. (4.16) too)



$$\bar{g}_4^3 \sim \text{diagram} \sim \text{diagram} \quad (6.2)$$

We let $h_k = \lim_{N \rightarrow \infty} Z^k \tilde{h}_k(N)/N^2$, which due to Eq. (5.9) is independent of Z , and choose later an explicit regulator R_N that makes h_k only dependent on k in the large- N limit. Thereafter, the contributions to the β -functions coming from *quantum fluctuations*¹⁶ can be read off from Eq. (6.1). To state the quantum fluctuations in terms of the renormalized quantities (without bar), one needs to find the way these scale with Z and N . We let $\bar{g}_{2k} = Z^{a_k} N^{-b_k} g_{2k}$ and $\bar{g}_{u|2k-u} = Z^{j_k} N^{-i_k} g_{u|2k-u}$ (for even u , with $0 < u < 2k$).

To solve for a_k, b_k, i_k, j_k , one asks the equation $\beta_I = \partial_t g_I$ to remain finite for each operator \mathcal{O}_I as $N \rightarrow \infty$. This leads to

$$\begin{aligned} \bar{g}_4 &= Z^2 N^{-1} g_4, & \bar{g}_6 &= Z^3 N^{-2} g_6, \\ \bar{g}_{2|2} &= Z^2 N^{-2} g_{2|2}, & \text{and} & \quad \bar{g}_{2|4} = Z^3 N^{-3} g_{2|4}. \end{aligned}$$

These scalings, together with the quantum fluctuations from Eq. (6.1), yield for the anomalous dimension $\eta = -\partial_t \log Z$ and the β -functions in the large- N limit:

$$\eta = h_1 \left(\frac{1}{2} g_{2|2} + 2g_4 \right), \quad (6.3a)$$

$$\beta_4 = (1 + 2\eta)g_4 + 4h_2 g_4^2 - h_1 \left(4g_6 + \frac{g_{2|4}}{2} \right), \quad (6.3b)$$

$$\begin{aligned} \beta_{2|2} &= (2 + 2\eta)g_{2|2} - 4h_1 (g_{2|4} + g_6) \\ &\quad + h_2 (g_{2|2}^2 + 8g_{2|2}g_4 + 12g_4^2), \end{aligned} \quad (6.3c)$$

$$\beta_6 = (2 + 3\eta)g_6 + 12g_4 g_6 h_2 - 6g_4^3 h_3, \quad (6.3d)$$

$$\begin{aligned} \beta_{2|4} &= (3 + 3\eta)g_{2|4} + h_2 (g_{2|2}g_{2|4} + 8g_{2|2}g_6 \\ &\quad + 12g_{2|4}g_4 + 48g_4g_6) - h_3 (12g_{2|2}g_4^2 + 48g_4^3). \end{aligned} \quad (6.3e)$$

We only are in debt with the explicit regulator (R_N^r) $_{ab;cd} = r_N(a, c)\delta_a^d \delta_c^b$ for r_N defined on $\{1, \dots, A\}^2$ and given by

$$r_N(a, b) = Z \cdot \left[\frac{N^2}{a^2 + b^2} - 1 \right] \cdot \Theta_{\mathbb{D}_N}(a, b), \quad (6.4)$$

¹⁶These are the coefficients of a $\partial_t \Gamma_N[X]$ in the operator in question.

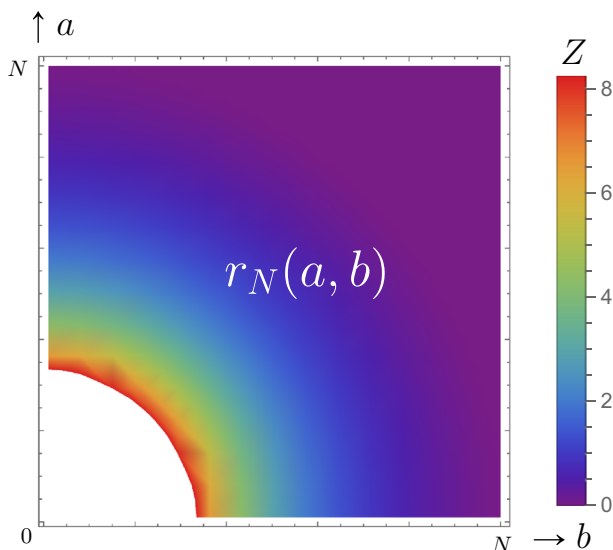


FIGURE 4. The plot shows the support of the chosen IR-regulator $r_N(a, b)$, contained in the square $\mathbb{R}_{\leq N}^+ \times \mathbb{R}_{\leq N}^+$. (The white quarter of disk means a truncation of the graph around the origin.)

being $\Theta_{\mathbb{D}_N}(a, b)$ the indicator function in the disc $a^2 + b^2 \leq N^2$.

It turns out that for this regulator, $Z^k \tilde{h}_k / N^2$ indeed converges to a number h_k independent of N , when this parameter is large. The first values are in fact

$$h_1 = \frac{\pi}{24}(6 - 5\eta), \quad h_2 = \frac{\pi}{48}(8 - 7\eta), \quad h_3 = \frac{\pi}{80}(10 - 9\eta). \quad (6.5)$$

Inserting the four fixed point equations, i.e. $\beta_{g_I^\bullet}|_{\eta^\bullet = \eta(g^\bullet)} = 0$ for $I = 2, 4, 2|2$ and $2|4$, one finds, on top of the Gaussian trivial fixed point ($g_I = 0$ for each I), several fixed points, tagged here with a little black diamond. The interesting one to be reproduced is expected to be $-1/12$, the critical value of g_4 for gravity coupled to conformal matter [25]. The latter has been identified in [32], who report $g_4^\bullet|_{[32]} = -0.056$ using the very same truncation.¹⁷ In contrast, we get

¹⁷The same authors report the possibility to obtain the exact solution in [33] by imposing it and then solving for the regulator (in the tadpole approximation); but our aim here is to compare regulators in the same truncation.

$$\begin{aligned}
\eta^\bullet &= -0.2494, & \mathbf{g}_4^\bullet &= -0.08791, & \mathbf{g}_{2|2}^\bullet &= -0.17415, \\
\mathbf{g}_6^\bullet &= -0.003386, & \mathbf{g}_{2|4}^\bullet &= -0.02423.
\end{aligned} \tag{6.6}$$

This fixed point, obtained with the IR-regulator r_N of Eq. (6.4) gets far closer ($\mathbf{g}_4^\bullet = -0.08791$) to the exact value $\mathbf{g}_c = -1/12 = -0.083\bar{3}$, which suggests that we should stick to our r_N for the two-matrix models treated next.

7. Two-Matrix Models from Noncommutative Geometries

7.1. Theory Space

The conventions for the coupling constants are the following, with numerical factors incorporated later. For

$$n_1, \dots, n_{2t}, l_2, \dots, l_{2t-1}, l_{2t-1} \in \mathbb{Z}_{>0} \quad \text{and} \quad l_1, l_{2t}, k_1, k_{2t} \in \mathbb{Z}_{\geq 0},$$

we associate with each operator the following coupling constants:

$$\mathbf{a}_{2k} \leftrightarrow A^{2k} \quad (k \geq 2) \tag{7.1a}$$

$$\mathbf{b}_{2k} \leftrightarrow B^{2k} \quad (k \geq 2) \tag{7.1b}$$

$$\mathbf{c}_{n_1 n_2 \dots n_{2t}} \leftrightarrow \underbrace{A^{n_1} B^{n_2} \dots A^{n_{2t-1}} B^{n_{2t}}}_{AB\text{-alternating}} \tag{7.1c}$$

$$\mathbf{d}_{l_1 l_2 \dots l_{2s} | k_1 k_2 \dots k_{2t}} \leftrightarrow \underbrace{A^{l_1} B^{l_2} \dots B^{l_{2s}}}_{AB\text{-alternating}} \otimes \underbrace{A^{k_1} B^{k_2} \dots B^{k_{2t}}}_{AB\text{-alternating}} \tag{7.1d}$$

Notice the alternating convention in the letters. For coupling constants of type \mathbf{c} and \mathbf{d} (mnemonics: ‘combined’ and ‘disconnected’) some care is needed. Operators can always begin with the highest power of A , which for \mathbf{c} is never zero—otherwise the respective operator is a pure power of either A or of B —in order to reduce the number of constants. This is due to the possibility to cyclicly reorder (\sim) the words, as these appear inside a trace. Only the first and last parameters can be zero for \mathbf{d} -constants. In order to include an odd number of powers of the letters, the last integer is allowed to be zero. If this is so, we agree to omit the rightmost zero.

Both conventions are illustrated with $ABA \otimes BAB \sim ABAB^0 \otimes AB^2$, whose coupling constant is $\mathbf{d}_{1110|12} = \mathbf{d}_{111|12}$. On the other hand a leftmost zero is important: from the definition $\mathbf{d}_{l_1 l_2 \dots l_t | I} \neq \mathbf{d}_{0 l_1 l_2 \dots l_t | I}$, since $A^{l_1} B^{l_2} \dots B^{l_{2t}} \neq A^0 B^{l_1} A^{l_2} \dots A^{l_{2t}}$. Notice that \mathbf{d} has to satisfy a symmetry condition: $\mathbf{d}_{I|I'} = \mathbf{d}_{I'|I}$ for any integer multi-indices I, I' (since the respective operators do), so we only keep one of the two.

As before, a bar on a coupling constant, $\bar{\mathbf{a}}, \dots, \bar{\mathbf{d}}$, denotes its unrenormalized value, whose N -dependence we do not show, for the sake of keeping the notation compact.

7.2. Compatibility of the RG-Flow with the Spectral Action

We now prove that in the double-trace truncation the RG-flow does not generate more operators than those allowed by the NCG-structure.

Proposition 7.1. *Pick a two-matrix model that includes finitely many single-trace operators $\mathrm{Tr}_N Q$, $Q \in \mathbb{C}_{\langle n \rangle, N}$, and assume that each of them appears (probably with other coefficient) in the Spectral Action for certain fuzzy two-dimensional geometry. Then, the RG-flow generates at any order in the FP^{-1} -expansion exclusively operators that appear again, generally with a different nonzero coefficient, in the Spectral Action $\mathrm{Tr} f(D)$, which in the worst case would require a suitable (generally higher-degree) polynomial f .*

Proof. Suppose that $\mathrm{Tr}_N Q$, with $Q \in \mathbb{C}_{\langle n \rangle, N}$, features in the Spectral Action for a fuzzy geometry. First, we show that the NC-polynomial $(\partial^A \circ \partial^A \mathrm{Tr}_N Q)^{\star k} \in \mathcal{A}_2$ appears for each $k \in \mathbb{Z}_{\geq 1}$ in the Spectral Action for the same fuzzy geometry—we argue later for the most general case containing mixed derivatives. From (2.16), $\partial^A \circ \partial^A \mathrm{Tr}_N Q$ contains two powers of A less than the original NC-polynomial Q , which, since it appears in the Spectral Action, has an even degree $\deg_A(Q), \deg_B(Q) \in 2\mathbb{Z}_{\geq 0}$. Therefore, so does the double derivative, and by Lemmas 3.1 and 3.2, $\partial^A \circ \partial^A \mathrm{Tr}_N Q$ appears in the Spectral Action. The condition holds for any power $(\partial^A \circ \partial^A \mathrm{Tr}_N Q)^{\star k}$ since the even-degree conditions are still satisfied and therefore each monomial $w_1 \otimes w_2$ or $w_1 \otimes_\tau w_2$ in $(\partial^A \circ \partial^A \mathrm{Tr}_N Q)^{\star k}$ appears in the Spectral Action $\mathrm{Tr} f(D)$ for a polynomial f with nonzero coefficient in degree m , being $m = m(w_1, w_2)$ given by Lemma 3.2.

The argument is still true for different NC-polynomials Q_i appearing in the original Spectral Action and the even-degree argument holds not only for powers of double derivatives of these, but can be clearly extended to

$$\begin{aligned} & \sum_{j_1, j_2, \dots, j_r} (\partial^{X_{j_1}} \circ \partial^{X_{j_1}} \mathrm{Tr}_N Q_1) \star (\partial^{X_{j_1}} \circ \partial^{X_{j_2}} \mathrm{Tr}_N Q_2) \star \\ & \quad \dots \star (\partial^{X_{j_r}} \circ \partial^{X_{j_r}} \mathrm{Tr}_N Q_{r+1}) \end{aligned}$$

since in the product the same derivative ∂^{X_k} ($X_k \in \{A, B\}$) appears an even number of times. All the NC-polynomials generated by the supertrace in the FRGE are of this form, and having even degree in both matrices, the argument above leads in this case to the result. \square

Proposition 7.1 says that if the bare action would contain only single-trace operators, then all the operators that the RG-flow generates, including double-trace operators, are compatible with the structure of fuzzy geometries. This implies that for a realistic bare action, which includes only double-trace operators as dictated by the Spectral Action for a fuzzy two-dimensional geometry, the RG-flow generates (up to triple traces excluded in the truncation) exclusively NCG-compatible operators. Both structures can therefore be seen as highly compatible.

TABLE 1. Quadratic and quartic operators and their coupling constants

Degree	Operators	Coupling constant	Scalings
QUADRATIC	$1_N \otimes (A \cdot A)$	$\frac{1}{2} Z_a e_a$	*
	$1_N \otimes (B \cdot B)$	$\frac{1}{2} Z_b e_b$	*
	$A \otimes A$	$\frac{1}{2} \bar{d}_{1 1}$	$1/N$
	$B \otimes B$	$\frac{1}{2} \bar{d}_{01 01}$	$1/N$
QUARTIC	$1_N \otimes (A \cdot A \cdot A \cdot A)$	$\frac{1}{4} \bar{a}_4$	$1/N$
	$1_N \otimes (B \cdot B \cdot B \cdot B)$	$\frac{1}{4} \bar{b}_4$	$1/N$
	$1_N \otimes (A \cdot A \cdot B \cdot B)$	$\bar{c}_{22} e_a e_b$	$1/N$
	$1_N \otimes (A \cdot B \cdot A \cdot B)$	$-\frac{1}{2} \bar{c}_{1111} e_a e_b$	$1/N$
	$(A \cdot B) \otimes (A \cdot B)$	$\bar{d}_{11 11}$	$1/N^2$
	$(A \cdot A) \otimes (B \cdot B)$	$2 \bar{d}_{2 02} e_a e_b$	$1/N^2$
	$A \otimes (A \cdot A \cdot A)$	$\bar{d}_{1 3} e_a$	$1/N^2$
	$A \otimes (A \cdot B \cdot B)$	$\bar{d}_{1 12} e_b$	$1/N^2$
	$B \otimes (A \cdot A \cdot B)$	$\bar{d}_{01 21} e_a$	$1/N^2$
	$B \otimes (B \cdot B \cdot B)$	$\bar{d}_{01 03} e_b$	$1/N^2$
	$(A \cdot A) \otimes (A \cdot A)$	$3 \bar{d}_{2 2}$	$1/N^2$
	$(B \cdot B) \otimes (B \cdot B)$	$3 \bar{d}_{02 02}$	$1/N^2$

Notice that $A \otimes B$ and $1_N \otimes A \cdot B$ (the latter appearing in the Ising two-matrix model) are forbidden. The scalings corresponding to the quadratic connected operators are in the wave function renormalization Z_a, Z_b , which are in each case determined by the RG-flow

7.3. The Truncated Effective Action

The model we adopt includes all the operators appearing in the Spectral Action for fuzzy geometries computed in [61] up to the sixth degree. For two-dimensional fuzzy geometries,

$$\Gamma_N[A, B] = \text{Tr}_{\mathcal{A}_2} \left\{ 1_N \otimes P(A, B) + \sum_{\alpha} \Psi_{\alpha}(A, B) \otimes \Upsilon_{\alpha}(A, B) \right\},$$

where $P, \Psi_{\alpha}, \Upsilon_{\alpha} \in \mathbb{C}_{\langle 2 \rangle} = \mathbb{C}\langle A, B \rangle$ are given, degree by degree by Tables 1 and 2. There, a dot means the usual matrix product. The number of running coupling constants turns out to depend not only on the dimension, but also on the signature of the fuzzy geometry, see Table 3. We stress that for the quartic and quadratic operators we do take the coupling constants with the symmetry factors and signs present in the NCG-action. For the sextic operators we drop the numerical normalization factors, in order to avoid rational coefficients.

7.4. The β -Functions

We present now the set of equations satisfied by the fixed points $\mathbf{g}_{\bullet} = \{a_{\bullet}, b_{\bullet}, c_{\bullet}, d_{\bullet}\}$, determined by the vanishing of all β -functions $\beta_{\bullet} =$

TABLE 2. Sextic operators, with their running coupling constants and scalings

Sextic Operators	NCG coefficient value	Coupling constant	Scalings
$1_N \otimes (A \cdot A \cdot A \cdot A \cdot A \cdot A)$	e_a	\bar{a}_6	$1/N^2$
$1_N \otimes (A \cdot A \cdot A \cdot A \cdot B \cdot B)$	$6e_b$	\bar{c}_{42}	$1/N^2$
$1_N \otimes (A \cdot A \cdot A \cdot B \cdot A \cdot B)$	$-6e_b$	\bar{c}_{3111}	$1/N^2$
$1_N \otimes (A \cdot A \cdot B \cdot A \cdot A \cdot B)$	$3e_b$	\bar{c}_{2121}	$1/N^2$
$1_N \otimes (B \cdot B \cdot B \cdot B \cdot B \cdot B)$	e_b	\bar{b}_6	$1/N^2$
$1_N \otimes (A \cdot A \cdot B \cdot B \cdot B \cdot B)$	$6e_a$	\bar{c}_{24}	$1/N^2$
$1_N \otimes (A \cdot B \cdot B \cdot B \cdot A \cdot B)$	$-6e_a$	\bar{c}_{1311}	$1/N^2$
$1_N \otimes (A \cdot B \cdot B \cdot A \cdot B \cdot B)$	$3e_a$	\bar{c}_{1212}	$1/N^2$
$A \otimes (A \cdot A \cdot A \cdot A \cdot A)$	2	$\bar{d}_{1 5}$	$1/N^3$
$A \otimes (A \cdot B \cdot B \cdot B \cdot B)$	2	$\bar{d}_{1 14}$	$1/N^3$
$A \otimes (A \cdot A \cdot A \cdot B \cdot B)$	$6e_a e_b$	$\bar{d}_{1 32}$	$1/N^3$
$A \otimes (A \cdot A \cdot B \cdot A \cdot B)$	$-2e_a e_b$	$\bar{d}_{1 2111}$	$1/N^3$
$B \otimes (A \cdot A \cdot A \cdot A \cdot B)$	2	$\bar{d}_{01 41}$	$1/N^3$
$B \otimes (A \cdot A \cdot A \cdot B \cdot B)$	$6e_a e_b$	$\bar{d}_{1 23}$	$1/N^3$
$B \otimes (A \cdot B \cdot B \cdot A \cdot B)$	$-2e_a e_b$	$\bar{d}_{01 1211}$	$1/N^3$
$B \otimes (B \cdot B \cdot B \cdot B \cdot B)$	2	$\bar{d}_{01 05}$	$1/N^3$
$(A \cdot B) \otimes (A \cdot A \cdot A \cdot B)$	$8e_a$	$\bar{d}_{11 31}$	$1/N^3$
$(A \cdot B) \otimes (A \cdot B \cdot B \cdot B)$	$8e_b$	$\bar{d}_{11 13}$	$1/N^3$
$(A \cdot A) \otimes (A \cdot A \cdot B \cdot B)$	$8e_b$	$\bar{d}_{2 22}$	$1/N^3$
$(A \cdot A) \otimes (A \cdot B \cdot A \cdot B)$	$-2e_b$	$\bar{d}_{2 1111}$	$1/N^3$
$(A \cdot A) \otimes (A \cdot A \cdot A \cdot A)$	$5e_a$	$\bar{d}_{2 4}$	$1/N^3$
$(A \cdot A) \otimes (B \cdot B \cdot B \cdot B)$	e_a	$\bar{d}_{2 04}$	$1/N^3$
$(B \cdot B) \otimes (A \cdot A \cdot B \cdot B)$	$8e_a$	$\bar{d}_{02 22}$	$1/N^3$
$(B \cdot B) \otimes (A \cdot B \cdot A \cdot B)$	$-2e_a$	$\bar{d}_{02 1111}$	$1/N^3$
$(B \cdot B) \otimes (B \cdot B \cdot B \cdot B)$	$5e_b$	$\bar{d}_{02 04}$	$1/N^3$
$(B \cdot B) \otimes (A \cdot A \cdot A \cdot A)$	e_b	$\bar{d}_{02 4}$	$1/N^3$
$(A \cdot A \cdot A) \otimes (A \cdot A \cdot A)$	$\frac{10}{3}$	$\bar{d}_{3 3}$	$1/N^3$
$(A \cdot B \cdot B) \otimes (A \cdot A \cdot A)$	$4e_a e_b$	$\bar{d}_{12 3}$	$1/N^3$
$(A \cdot A \cdot B) \otimes (A \cdot A \cdot B)$	6	$\bar{d}_{21 21}$	$1/N^3$
$(B \cdot B \cdot B) \otimes (B \cdot B \cdot B)$	$\frac{10}{3}$	$\bar{d}_{03 03}$	$1/N^3$
$(A \cdot A \cdot B) \otimes (B \cdot B \cdot B)$	$4e_a e_b$	$\bar{d}_{21 03}$	$1/N^3$
$(A \cdot B \cdot B) \otimes (A \cdot B \cdot B)$	6	$\bar{d}_{12 12}$	$1/N^3$

TABLE 3. Number of operators for each signature

Geometry	Signature	KO-dim.	# Operators in the RG-flow	# Operators with duality
'Double time'	(+, +)	6	48	26
Lorentzian	(+, -)	0	41	-
Riemannian	(-, -)	2	34	19

There is no duality for the (1,1) geometry

$\partial_t g_{\cdot}$. We recall that

$$h_k = \lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{a,b,c,d=1}^{\Lambda} \frac{(\partial_t R_N)_{ab;cd}}{P_{ab;cd}^{(k+1)}},$$

which are real numbers in the case of the quadratic regulator of Sect. 6 and whose values are given by Eq. (6.5). The next result is more transparent if one does not specify these coefficients yet (and holds for any R_N verifying that these h_k are all independent of N).

Theorem 7.2. *Assuming $Z_a = Z_b =: Z$, to the second order in the FP^{-1} expansion ($h_r = 0, r \geq 3$), in the double-trace and sixth-degree truncation, the β -functions of the two-matrix model corresponding to a two-dimensional fuzzy geometry with signature $\text{diag}(e_a, e_b)$ are given in the large- N limit by the following blocks of equations:*

First, the degree-2 operators yield the anomalous dimension and following relations:

$$\begin{aligned} 2h_1(a_4 + c_{22} + 2d_{2|02} + 6d_{2|2}) &= \eta_a \\ 2h_1(b_4 + c_{22} + 6d_{02|02} + 2d_{2|02}) &= \eta_b \\ -h_1[e_a(a_4 - c_{1111}) + 2d_{1|12} + 6d_{1|3}] + d_{1|1}(\eta + 1) &= \beta(d_{1|1}) \\ -h_1[e_b(b_4 - c_{1111}) + 6d_{01|03} + 2d_{01|21}] + d_{01|01}(\eta + 1) &= \beta(d_{01|01}) \end{aligned}$$

The next block encompasses the connected quartic couplings:

$$\begin{aligned} h_2(4a_4^2 + 4c_{22}^2) + a_4(2\eta + 1) \\ -h_1(24a_6e_a + 4c_{42}e_b + 4d_{02|4}e_b + 4d_{2|4}e_a) &= \beta(a_4) \\ h_2(4b_4^2 + 4c_{22}^2) + b_4(2\eta + 1) \\ -h_1(24b_6e_b + 4c_{24}e_a + 4d_{02|04}e_b + 4d_{2|04}e_a) &= \beta(b_4) \\ -h_1(2e_ac_{1212} + e_b2c_{2121} + 3e_ac_{24} + 3e_bc_{42} + e_ad_{02|22} + e_bd_{2|22}) \\ +h_2(2a_4c_{22} + 2b_4c_{22} + 2e_ae_bc_{1111}^2 + 2e_ae_bc_{22}^2) + c_{22}(2\eta + 1) &= \beta(c_{22}) \\ 8e_ae_bc_{1111}c_{22}h_2 + c_{1111}(2\eta + 1) \\ +h_1(4e_ac_{1311} + 4e_bc_{3111} + 2e_ad_{02|1111} + 2e_bd_{2|1111}) &= \beta(c_{1111}) \end{aligned}$$

The β -functions for the connected sextic couplings are

$$\begin{aligned} 2h_2(6a_4a_6 + e_ae_bc_{22}c_{42}) + a_6(3\eta + 2) &= \beta(a_6) \\ 2h_2(6b_4b_6 + e_ae_bc_{22}c_{24}) + b_6(3\eta + 2) &= \beta(b_6) \\ 4h_2\{a_4c_{3111} + e_ae_b[c_{22}(c_{1311} + 2c_{3111}) \\ -c_{1111}(2c_{2121} + c_{42})]\} + c_{3111}(3\eta + 2) &= \beta(c_{3111}) \\ 2h_2[2a_4c_{2121} + e_ae_b(-2c_{1111}c_{3111} \\ + 4c_{2121}c_{22} + c_{22}c_{24})] + c_{2121}(3\eta + 2) &= \beta(c_{2121}) \\ 2h_2[a_4c_{24} + 3b_4c_{24} + 2e_ae_b(c_{22}(3b_6 + c_{2121} + c_{24} + c_{42}) \\ -c_{1111}c_{1311})] + c_{24}(3\eta + 2) &= \beta(c_{24}) \\ 4h_2\{b_4c_{1311} + e_ae_b[c_{22}(2c_{1311} + c_{3111}) \\ -c_{1111}(2c_{1212} + c_{24})]\} + c_{1311}(3\eta + 2) &= \beta(c_{1311}) \\ 2h_2[2b_4c_{1212} + e_ae_b(c_{22}(4c_{1212} + c_{42}) \\ -2c_{1111}c_{1311})] + c_{1212}(3\eta + 2) &= \beta(c_{1212}) \\ 2h_2[3a_4c_{42} + 2e_ae_b(3a_6c_{22} - c_{1111}c_{3111} + c_{1212}c_{22} \\ + c_{22}c_{24} + c_{22}c_{42}) + b_4c_{42}] + c_{42}(3\eta + 2) &= \beta(c_{42}) \end{aligned}$$

TABLE 4. Hessians of some second- and fourth-order operators

Operator	Its Hess _{σ}
$\text{Tr}(A^4)$	$\begin{pmatrix} 4e_a(1 \otimes A^2 + A^2 \otimes 1 + A \otimes A) & 0 \\ 0 & 0 \end{pmatrix}$
$\text{Tr}^2 B$	$\begin{pmatrix} 0 & 0 \\ 0 & 2e_b 1 \otimes_\tau 1 \end{pmatrix}$
$\text{Tr}(ABAB)$	$\begin{pmatrix} 2e_a B \otimes B & 2(1 \otimes BA + AB \otimes 1) \\ 2(1 \otimes AB + BA \otimes 1) & 2e_b A \otimes A \end{pmatrix}$
$\text{Tr}(A) \text{Tr}(A^3)$	$\begin{pmatrix} 3e_a[\text{Tr}(A)(A \otimes 1 + 1 \otimes A) & 0 \\ +1 \otimes_\tau A^2 + A^2 \otimes_\tau 1] & \\ 0 & 0 \end{pmatrix}$
$\text{Tr} A^2 \text{Tr} B^2$	$\begin{pmatrix} 2e_a 1 \otimes 1 \text{Tr} B^2 & 4A \otimes_\tau B \\ 4B \otimes_\tau A & 2e_b 1 \otimes 1 \text{Tr} A^2 \end{pmatrix}$
$\text{Tr}^2 A^2$	$\begin{pmatrix} 4e_a(1 \otimes 1 \text{Tr} A^2 + 2A \otimes_\tau A) & 0 \\ 0 & 0 \end{pmatrix}$

And a last block of β -functions for the disconnected couplings is located in Section III of Supplementary Material.

Proof. We address first the first order in the FP^{-1} -expansion. This part of the proof consists of the following steps:

- *Step 1.* The computation of the second-order derivatives of all the operators, determine the NC-Hessians to insert in the FP^{-1} -expansion. We now give the NC-Hessians computed using Claim 2.3 as well as their trace, the NC-Laplacian using Eq. (2.17). We write some of them down up in Table 4 to quartic operators; those omitted might be obtained by the exchange $A \leftrightarrow B$, $e_a \leftrightarrow e_b$ (and adjusting the matrix structure).

The expressions for Hess _{σ} Tr($AABB$) and Hess _{σ} {Tr(A) Tr(ABB)}, the quartic operators missing in Table 4 were already given in Example 2.2 and show that the complexity rapidly grows. For the sake of readability, the bulkier sixth-degree operators completing the running 34 or 48 involved in the flow, are located in Section II of Supplementary Material.

- *Step 2.* To the first order, one computes the regularized NC-Laplacian $F = e_a F_{aa} + e_b F_{bb}$ of the effective action in terms of

$$F_{aa} = (\partial^A \circ \partial^A) \Gamma_N[A, B] - (\partial^A \circ \partial^A) \Gamma_N[A, B]|_{A=B=0}$$

and of

$$F_{bb} = (\partial^B \circ \partial^B) \Gamma_N[A, B] - (\partial^B \circ \partial^B) \Gamma_N[A, B]|_{A=B=0}.$$

- *Step 3.* One takes the double traces of the resulting expression. The h_1 -terms, i.e. the trace $\text{Tr}_N^{\otimes 2}$ of the NC-Laplacian, read as in Section IV of the Supplementary Material. From that expression one can deduce

some of the quantum fluctuations. In the large- N limit, according to the scalings given in Tables 1 and 2, the matching of the h_1 coefficients in the fixed point equations given in the statement can be verified. The scaling $N^{-m(Q)}$ of the coupling constant g_Q that corresponds with the operator $Q \in \mathbb{C}_{(2)} \otimes \mathbb{C}_{(2)}$ determines the coefficient of the form

$$\left(\frac{\deg_A(Q) + \deg_B(Q)}{2} \eta + m(Q) \right) \times g_Q,$$

appearing in the β_Q -function.

We now sketch the second order: Having computed in Step 1 the 48 NC-Hessians, one \star -squares the $(48 - 2)$ Hessians appearing in F (the two subtracted operators are A^2 and B^2 whose Hessian is absorbed in P). The $\sim 10^3$ matrices of size 2×2 with NC-polynomial entries are omitted, but each of these was computed as in Example 2.2. Then, one traces $F^{\star 2}$ in superspace to collect quantum fluctuations for each operator. Taking the large- N limit of these leads to the results. \square

7.5. Dualities

It is convenient to look for dual solutions while aiming at determining the fixed points from the vanishing β -functions. The duality is meant in the following sense. To reduce the number of fixed-point equations, one makes some of them redundant by imposing the $A \leftrightarrow B$ duality for couples of operators that allow it. Thus, e.g.,

$$\mathrm{Tr}_N(A^3BAB) \leftrightarrow \mathrm{Tr}_N(B^3ABA).$$

is reflected in the duality $c_{1311} \leftrightarrow c_{3111}$. Imposing dualities does not halve the number of running constants, since some operators, e.g., $\mathrm{Tr}_N(ABAB)$, are invariant under the $A \leftrightarrow B$ exchange (self-dual). With this in mind, we have the following list:

Remark 7.3. For the geometries $(2, 0)$ and $(0, 2)$ a duality in the effective action is manifest. Therefore, the β -functions together with the equations for the anomalous dimensions η_b and η_a are invariant under the following exchange for the $(2, 0)$ -geometry:

$$\begin{array}{lll} \eta_b \leftrightarrow \eta_a, & d_{1|1} \leftrightarrow d_{01|01}, & d_{01|03} \leftrightarrow d_{1|3}, \\ d_{01|21} \leftrightarrow d_{1|12}, & d_{02|02} \leftrightarrow d_{2|2}, & b_6 \leftrightarrow a_6, \\ c_{1311} \leftrightarrow c_{3111}, & c_{1212} \leftrightarrow c_{2121}, & d_{01|05} \leftrightarrow d_{1|5}, \\ d_{01|41} \leftrightarrow d_{1|14}, & d_{01|1211} \leftrightarrow d_{1|2111}, & d_{11|13} \leftrightarrow d_{11|31}, \\ d_{02|4} d_{02|1111} \leftrightarrow d_{2|1111}, & d_{02|04} \leftrightarrow d_{2|4}, & d_{03|03} \leftrightarrow d_{3|3}, \\ d_{21|03} \leftrightarrow d_{12|3}, & d_{12|12} \leftrightarrow d_{21|21}, & d_{01|23} \leftrightarrow d_{1|32}, \\ b_4 \leftrightarrow a_4, & c_{24} \leftrightarrow c_{42}, & \leftrightarrow d_{2|04}, \\ d_{02|22} \leftrightarrow d_{2|22}, \end{array}$$

For the $(0, 2)$ -geometry, one excludes from this list the exchanges implying

$$d_{01|01}, d_{01|03}, d_{01|21}, d_{01|05}, d_{01|23}, d_{01|41}, \text{ and } d_{01|1211}. \quad (7.2)$$

Proof. For both geometries $e_a = e_b$ holds. The duality is straightforwardly verified by inspecting the 48 equations. For the $(0, 2)$ -geometry, $\text{Tr}_N B = 0$ which means that we make $d_{01|I} \equiv 0$, where I stands for any index combination, which is the list (7.2). (Also $\text{Tr}_N A = 0$ but excluding all $d_{01|}$'s automatically excludes all $d_{1|}$'s.) \square

7.6. Methods and Results for the Geometry $(0, 2)$, or $(-, -)$

The fixed-point equations are the simultaneous zeros of all the β -functions listed in Theorem 7.2 (and the two first equations there for the anomalous dimension). These are the eigenvalues of the stability matrix

$$-\text{Eig}\left\{\left(\frac{\partial\beta_I(\eta^\bullet, \mathbf{g}^\bullet)}{\partial\mathbf{g}_{I'}}\right)\bigg|_{\mathbf{g}^\bullet}\right\}_{I, I'}, \quad (7.3)$$

where I, I' run over the flowing coupling constants. While analyzing the solutions:

- We exclude the Gaussian fixed point $\mathbf{g}^\bullet = 0$ with critical exponents determined by the scalings.
- We report fixed points with at least one nonvanishing connected coupling ($\mathbf{a}, \mathbf{b}, \mathbf{c}$ types; but solutions with only nonvanishing \mathbf{d} -type do exist).
- We do not report solutions that lead to imaginary critical exponents. That is, the reported solutions correspond all to solely real eigenvalues of the stability matrix Eq. (7.3).
- We only report solutions with coupling constants inside the $|\mathbf{g}^\bullet| \leq 1$ hypercube. This restriction is due to our approach, which uses the FP^{-1} -expansion. Without this restriction, the operators kept in the truncation would be less important than those we dropped.

Under these criteria, from the ~ 600 fixed point solutions for the $(0, 2)$ geometry, we obtain a unique solution with a single positive eigenvalue, or in other words, a single relevant direction:

$$\theta = +0.2749. \quad (7.4)$$

The values of the coupling constants corresponding to it read

$$\begin{array}{llll} \eta^\bullet = -0.3625 & \mathbf{a}_4^\bullet = -0.07972 & \mathbf{a}_6^\bullet = 0 & \mathbf{c}_{1111}^\bullet = 0 \\ \mathbf{c}_{2121}^\bullet = 0 & \mathbf{c}_{22}^\bullet = -0.03986 & \mathbf{c}_{3111}^\bullet = 0 & \mathbf{c}_{42}^\bullet = 0 \\ \mathbf{d}_{2|02}^\bullet = -0.01337 & \mathbf{d}_{2|04}^\bullet = 0 & \mathbf{d}_{2|1111}^\bullet = 0 & \mathbf{d}_{12|3}^\bullet = 0 \\ \mathbf{d}_{11|11}^\bullet = -0.004201 & \mathbf{d}_{2|4}^\bullet = 0 & \mathbf{d}_{2|22}^\bullet = 0 & \mathbf{d}_{11|31}^\bullet = 0 \\ \mathbf{d}_{2|2}^\bullet = -0.005156 & \mathbf{d}_{21|21}^\bullet = 0 & \mathbf{d}_{3|3}^\bullet = 0. & \end{array}$$

7.7. Results for the Geometry $(2, 0)$, or $(+, +)$

We report now the fixed points under the same criteria listed for the $(0, 2)$ geometry (Sect. 7.6), which restricts the ~ 600 real solutions to a few we

now describe. If we further impose that the solution has precisely one relevant direction, then that critical exponent is unique and given by

$$\theta = +0.2749 \quad (7.5)$$

and the corresponding fixed point has the coupling constants:

$$\begin{aligned} \eta^\bullet &= -0.3625 & \mathbf{a}_4^\bullet &= -0.07972 & \mathbf{a}_6^\bullet &= 0 & \mathbf{c}_{1111}^\bullet &= 0 \\ \mathbf{c}_{22}^\bullet &= -0.03986 & \mathbf{c}_{2121}^\bullet &= 0 & \mathbf{c}_{3111}^\bullet &= 0 & \mathbf{c}_{42}^\bullet &= 0 \\ \mathbf{d}_{2|02}^\bullet &= -0.01337 & \mathbf{d}_{2|04}^\bullet &= 0 & \mathbf{d}_{2|1111}^\bullet &= 0 & \mathbf{d}_{1|5}^\bullet &= 0 \\ \mathbf{d}_{2|2}^\bullet &= -0.005156 & \mathbf{d}_{2|22}^\bullet &= 0 & \mathbf{d}_{2|4}^\bullet &= 0 & \mathbf{d}_{12|3}^\bullet &= 0 \\ \mathbf{d}_{1|12}^\bullet &= -0.00985 & \mathbf{d}_{3|3}^\bullet &= 0 & \mathbf{d}_{21|21}^\bullet &= 0 & \mathbf{d}_{1|14}^\bullet &= 0 \\ \mathbf{d}_{1|3}^\bullet &= -0.00985 & \mathbf{d}_{1|2111}^\bullet &= 0 & \mathbf{d}_{1|32}^\bullet &= 0 & & \\ \mathbf{d}_{01|01}^\bullet &= -0.2543 & \mathbf{d}_{11|11}^\bullet &= -0.004201 & \mathbf{d}_{11|31}^\bullet &= 0. & & \end{aligned} \quad (7.6)$$

Solutions with more connected nonvanishing coupling constants exist (e.g., $\mathbf{c}_{1111} \neq 0$ relevant for the *ABAB*-model [3, 4, 36, 48]), but they require two relevant directions (in this truncation). These are located in Table A and B of the Supplementary Material . In particular, the agreement with the result of [48] for the A^4 -coupling is remarkable:

$$\mathbf{a}_4^\bullet = -0.07972 \approx -\frac{1}{4\pi} = (\mathbf{a}_4^\bullet)|_{[48]} (= -0.079577...) \quad (7.7)$$

if one takes into account the flipped sign convention for $(\mathbf{a}_4^\bullet)|_{[48]}$ (called $-\alpha$ there). Also notice that

$$\mathbf{c}_{22}^\bullet = -0.03986 \approx -\frac{1}{8\pi} (= -0.039788...)$$

8. Conclusion and Discussion

Fuzzy geometry has elsewhere [10, 40] motivated intrinsically random noncommutative geometric, numerical methods and statistical tools. Here, we use the fact that random NCG is in line with (Euclidean) QFT in order to explore fuzzy geometries via the Functional Renormalization Group for the multimatrix models these boil down to.

Using differential operators based on abstract algebra, noncommutative calculus was useful to describe the Functional Renormalization Group for general multimatrix models. This paper focused on those derived from fuzzy spectral triples, which therefore allow both Hermitian and anti-Hermitian random matrices. We introduced a NC-Hessian—a nonsymmetric(!) matrix of noncommutative derivatives—and a NC-Laplacian¹⁸ on the free algebra. The latter is given by

$$\nabla^2 = \nabla \circ \mathcal{D} = \text{noncommutative divergence} \circ \text{cyclic gradient},$$

¹⁸These differential operators are treated more in detail in a companion paper [58].

wherein the noncommutative divergence is the operator

$$\nabla Q = \sum_{i=1}^n e_i \partial^{X_i} Q_i \text{ for } Q = (Q_1, \dots, Q_n) \in \mathbb{C}_{\langle n \rangle}^n$$

and the cyclic gradient $\mathcal{D}\Phi = (\mathcal{D}^{X_1}\Phi, \dots, \mathcal{D}^{X_n}\Phi)$ for $\Phi \in \mathbb{C}_{\langle n \rangle} = \mathbb{C}\langle X_1, \dots, X_n \rangle$. The NC-Hessian governs the exact Wetterich–Morris FRGE and ∇^2 does so in the tadpole approximation, where it has the form of a noncommutative heat equation (Proposition 5.2). One advantage of the present analysis is the ability to drop the assumption made by [32] that P commutes with $F[X]$ —supposed there to hold in a certain approximation scheme. This turns out to be a consequence of the structure of the free algebra.

The coordinate-free setting common to algebrists speeds up computations and facilitates writing proofs, which can be taken as a tool for more mathematical works implying the functional RG. Introducing that elegant language was “priced” at introducing \otimes_τ , a new (twisted) product additional to Kronecker’s. In fact,

the RG-flow for n -matrix models takes place in the algebra $M_n(\mathcal{A}_{n,N})$ of matrices over $\mathcal{A}_n = (\mathbb{C}_{\langle n \rangle, N})^{\otimes 2} \oplus (\mathbb{C}_{\langle n \rangle, N})^{\otimes \tau 2}$ with \star -product¹⁹ given by Proposition 2.6,

where $\mathbb{C}_{\langle n \rangle, N}$ is the free algebra generated by n matrices of size $N \times N$, and the RG-time²⁰ is $\log N$. Importantly, this \star -multiplication is not chosen by us here just because it satisfies nice mathematical properties, rather the FRGE *dictates* it. In that sense, to present the proof of a “standard result” sometimes pays off. Since many of the operators that appear in free algebra were originated in matrix theory (see [44, 68, 75]), we remark that the \star -product given in Proposition 2.6 (which, concretely for matrices had to be proven here) can be taken as a definition in abstract algebra, as no reference to matrix size or entries is made, replacing the trace Tr_N by a state $\varphi : \mathcal{A}_n \rightarrow \mathbb{C}$, whose cyclicity renders \star associative (by Proposition 2.7):

$$(U \otimes_\tau W) \star (P \otimes_\tau Q) = PU \otimes_\tau WQ, \quad (8.1a)$$

$$(U \otimes W) \star (P \otimes_\tau Q) = U \otimes PWQ, \quad (8.1b)$$

$$(U \otimes_\tau W) \star (P \otimes Q) = WPU \otimes Q, \quad (8.1c)$$

$$(U \otimes W) \star (P \otimes Q) = \varphi(WP)U \otimes Q. \quad (8.1d)$$

Similarly, the “obvious” product in Proposition 2.26, which resembles (only for monomials though) matrix multiplication on $M_2(\mathbb{C}_{\langle n \rangle})$, suggests that the algebra $M_n(M_2(\mathbb{C}_{\langle n \rangle}))$ could be relevant²¹ for an additional description of the FRGE, if one trades the product \star by \times using relations like Eq. (2.28).

¹⁹This similar *notation* is otherwise, also in noncommutative field theory, a known product. But our here does **not** refer to Moyal product.

²⁰The right RG-parameter was not discovered here, but it was long known since [6, 15].

²¹If the FRGE were not a second-order NC-differential equation, the number 2 would not appear in $M_n(M_2(\mathbb{C}_{\langle n \rangle}))$. The number 2 should not be confused with the two of two-matrix models, or the number of products of traces allowed here, which is also two.

Most of our findings rely on the algebraic structure of the RG-flow but important part of the conclusion are the critical exponents for each geometry. For matrix models corresponding to two-dimensional fuzzy geometries, the β -functions were extracted (Theorem 7.2) and the fixed point equations were numerically solved. The critical exponents found here—for the $(0, 2)$ and $(2, 0)$ geometries $\theta = +0.27491$ —were obtained from all the fixed point solutions as the unique solution that featured a single relevant direction. The fixed-point coupling constants do require a matrix mix, e.g., the coupling c_{22} corresponding to the operator $ABBA$ is nonvanishing (see Sect. I in Supplementary Material, where we report fixed points with two relevant directions for where more nonvanishing mixed operators in the flow, e.g., $ABAB$).

It is also remarkable that the operators that appear here in the $(2, 0)$ geometry (of $(+, +)$ signature) are all generated by the RG-flow of the Hermitian two-matrix $ABAB$ -model, whose exact solution by Kazakov–Zinn–Justin [48] predicts a critical value $1/4\pi$ for the common coupling constant of the operators²² $-\frac{1}{4} \text{Tr}(A^4 + B^4)$ and $-\frac{1}{2} \text{Tr}(ABAB)$. In view of Eq. (7.7), we obtained for the coupling of A^4 and B^4 a strikingly close value

$$\text{our prediction} = 1.00179 \times \text{exact solution}.$$

However, the prediction of the other coupling does not enjoy the same success.

Concerning the NCG-structure, we showed in Sect. 7.2 that a truncation by operator-degree and by number of traces was consistent with the structure of the Spectral Action for fuzzy two-dimensional geometries. Due the complexity²³ of the free algebra $\mathbb{C}_{\langle 2 \rangle}$, it is not obvious that the RG-flow should respect this structure. For example, recall that in the Hermitian random matrix model the operators $X^m \otimes X^l$, with m and l odd, are generated by the RG-flow; these are removed by hand (in the truncations used in Sect. 6 and [32]). In contrast, truncations for fuzzy geometries do not require to drop other operators than triple traces and operators that exceed a maximum degree. Notwithstanding this high compatibility, as perspective, it remains to improve the precision of the present results. We identify possible error sources in the computation of the fixed points as well as improvements to our approach:

- Extending the exploration from the examined hypercubes to a larger region and estimation of residues in order to look for fixed points that correspond to Dirac operators (i.e. obey a relation between the coefficients similar to that of Tables 1, 2). This would allow to compare with Monte-Carlo simulations for the true Dirac operator of fuzzy geometries [40].
- The exact RG-flow should consider operators that are not pure traces of elements in the free algebra, but that are smeared with functions resulting from the IR-regulator.

²²Mind the flipped sign convention. Also that the couplings of the operators A^4 and B^4 have to coincide.

²³Thinking of words in $\mathbb{C}_{\langle 2 \rangle}$ as sequences of 0's and 1's, this algebra has enough “memory space” for any digitizable data.

- Addressing the solution removing the duality imposed here; otherwise we might miss important fixed points for which the $A \leftrightarrow B$ symmetry is broken.
- Another improvement that might lead to accuracy is to consider more terms ($h_3 \neq 0$) in the FP^{-1} -expansion. With 48 running operators, this analysis requires time.
- The arbitrariness in the definition of the IR-regulator R_N might affect the numerical results. For this paper, this regulator has been calibrated by imposing on it to lead to a good approximation to the expected solution for Hermitian matrix models, but the lack of “uniqueness” of R_N is unsatisfying. More constrictions on R_N should be thoroughly investigated. An important guide²⁴ in order to achieve the optimization of the matrix IR-regulator is [53]. The adaptation of that idea from the bosonic QFT-case to the matrix case might be might sound straightforward, but the different sort of propagators should be taken into account—this actually requires some care.)

Further related directions are:

- The NC-differential operators that we employed here govern also the Schwinger–Dyson of entirely general multimatrix models [45, 56]. Based on it, one can continue the investigations using Topological Recursion [19, 35] to address a solution of the models treated here. For one-dimensional geometries [5] report progress in this topic, using different analytic methods. Also, multimatrix models are known to be related to free probability whose tools might be helpful for this task. This paper puts a common language forward, at least.
- In order to obtain the present results, we studied geometries whose effective action was manifestly symmetric in both random matrices and for which the theory space was reducible to nonredundant couplings. The search for fixed points in the absence of the dualities, which for instance for the (1, 1)-geometry means 41 flowing operators in the present truncation, was postponed. However, the formalism is appropriate for these and higher-dimensional ones.
- Adding matter fields to these models can be accomplished by random almost-commutative geometries (in progress [63]). With the FRGE developed here, one has a tool to delve into fuzzy geometries coupled to simplified matter sectors, e.g., Maxwell or Yang–Mills(–Higgs) theories. This brings us even closer to the original motivation (Sect. 1).

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²⁴I thank an anonymous referee for this reference

be more specific, I thank an anonymous referee. The author was supported by the TEAM programme of the Foundation for Polish Science co-financed by the European Union under the European Regional Development Fund (POIR.04.04.00-00-5C55/17-00).

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Appendix A. Glossary, Conventions, Other Notations

- 1_N Identity matrix in $M_N(\mathbb{C})$, corresponding to the empty word of $\mathbb{C}_{\langle n \rangle, N}$
- \cdot Sum of matrix products $X \cdot J = \sum_{i=1}^n X_i J^i$; also sometimes an ordinary matrix product
- ◆ Tags a fixed point (RG-context)
- ◆ $_k$ k -th fixed point
- \star Product on \mathcal{A}_n
- \times (When not applied to scalars) is obvious product on \mathcal{A}_n ($A \otimes B$) \times ($C \otimes D$) = $AC \times BD$ (see Proposition 2.26)
- \otimes_τ Twisted tensor product
- $*$ Adjoint of a matrix, i.e. mainly dagger in physics
- ∇^2 Noncommutative Laplacian
- ∇_τ^2 Twisted NC-Laplacian (trace of the twisted NC-Hessian)
- ∇_i^2 Abbreviates $\partial^{X_i} \circ \partial^{X_i}$



- A, B random matrices in the two-dimensional fuzzy geometries
- a, b, c, d indices corresponding to matrix entries
- \mathcal{A}_n $\mathbb{C}_{\langle n \rangle}^{\otimes 2} \oplus \mathbb{C}_{\langle n \rangle}^{\otimes \tau 2}$
- a_I renormalized coupling constant associated with an operator $\mathcal{O}_I(A)$
- b_I coupling constant associated with $\mathcal{O}_I(B)$
- c_I coupling constant associated with $\mathcal{O}_I(A, B)$
- $d_{I|I'}$ coupling constant associated a disconnected operator $\mathcal{O}_I(A, B) \otimes \mathcal{O}_{I'}(A, B)$ on either matrix
- $\mathbb{C}_{\langle n \rangle}$ free algebra in n generators

- $\mathbb{C}_{\langle n \rangle, N}$ free algebra in n generators in $M_N(\mathbb{C})$
- D Dirac operator
- $\mathcal{D}^j, \mathcal{D}^{X_j}$ cyclic derivative with respect to X_j
- $\partial^j, \partial^{X_j}$ noncommutative derivative with respect to X_j
- e_i signs; $e_i = +1$ if X_i is Hermitian, and $e_i = -1$ if it is anti-Hermitian
- field a nontrivial word in the free algebra, or in $\mathbb{C}_{\langle n \rangle, N}^{\otimes k}$
- $\bar{g}_I, \bar{g}_\bullet$ coupling constants (not yet renormalized)
- g_I, g_\bullet renormalized coupling constants
- $\tilde{h}_k(N)$ corresponds with $[\hat{R}P^{k+1}]$ of [32] only before an IR-regulator is specified (mind the shift)
- Hess_σ noncommutative Hessian with diagonal entries scaled by $\sigma = \text{diag}(e_1, \dots, e_n)$
- Hess_σ^τ twisted NC-Hessian
- $h_k(N)$ corresponds to $\tilde{h}_k(N)/N^2 = [\hat{R}P^{k+1}]/N^2$ (cf [32])
- I generic index corresponding to (allowed) elements of $\mathbb{C}_{\langle n \rangle}$
- i, j indices corresponding typically to $i, j = 1, \dots, n$
- J sources (QFT-context)
- Λ is a large integer that serves as (globally in this paper, absolute) UV-cutoff that verifies $\Lambda \geq N$ (Λ corresponds to N' in [33])
- \mathcal{M}_N the space of matrices parametrizing the space of Dirac operators, shorthand for $\mathcal{M}_N^{p,q}$
- n the number of (random) matrices; number of generators of the free algebra. Caveat: in general n does not coincide with the dimension $p + q$ of the fuzzy geometry that originates the matrix model
- N is the “energy scale”, here an integer that verifies $\Lambda \geq N$. Often here, N is assumed also large
- operator in QFT-slang for monomial in the effective/bare action. Thus, in our setting, an operator is a NC-polynomial
- $\mathcal{O}_I(X)$ operator in the random matrix (or matrices) X
- p number of + signs in the signature of a fuzzy geometry
- q number of minus signs in the same context
- $q \pm p$ dimension / KO-dimension of a fuzzy geometry
- R_N IR-regulator (cutoff function)
- STr supertrace (no reference to supersymmetry)
- STr_N supertrace in the truncation scheme
- t t is the logarithm of the scale, here $t = \log N$
- Tr, Tr_N traces on $M_\Lambda(\mathbb{C})$ and $M_N(\mathbb{C})$ respectively
- Tr_N^k Q bracket-saving notation for $[\text{Tr}_N(Q)]^k$
- τ permutation $\tau = (13) \in \text{Sym}(4)$ or “twist”
- ΔS_N mass-like (quadratic in the fields) IR-regulator term
- $\mathcal{W}[J], \mathcal{W}_N[J]$ free energy (logarithm of the partition function)
- X n -tuple of matrices, $X = (X_1, \dots, X_n)$
- X_i random matrix obtained by $X_i = \langle \varphi_i \rangle$; the averaged field φ_i , Sect. 4

Z wave function renormalization constant
 $\mathcal{Z}, \mathcal{Z}_N$ partition functions; the second one IR-regulated

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