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PREFACE

This course of lectures was presented during the Third International School on **Symmetry in Integrable Systems and Nuclear Physics** (SISNP-2013), organized by the International Center for Advanced Studies at Yerevan State University and the Joint Institute for Nuclear Research (Dubna, Russia). It was held in Tsakhkadzor from 3 to 13 July, 2013. The first School of this series was organized in Dubna in 1999 and the previous one was held in Tsakhkadzor in 2011.

The program of the School including five/six 60 minutes duration lectures was scheduled for eight working days. More than 40 students and young researchers participated in the School. Twenty-five Professors from ten countries presented the lectures on the following topics: integrable and superintegrable systems, supersymmetries in quantum mechanics and field theories, symmetries in atomic and nuclear physics, multiparticle dynamics, Lie group and algebra contraction and its applications in high energy physics.

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Editors

Invariant Differential Operators for Non-compact Lie Groups: The Reduced SU(4,4) Multiplets

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Abstract

In the present paper we continue the project of systematic construction of invariant differential operators on the example of the non-compact algebras $su(n,n)$. Earlier were given the main multiplets of indecomposable elementary representations for $n \leq 4$, and the reduced ones for $n = 2, 3$. Here we give all reduced multiplets containing physically relevant representations including the minimal ones for $n = 4$. Due to the recently established parabolic relations the results are valid also for the algebras $sl(8, \mathbb{R})$ and $su^*(8)$ with suitably chosen maximal parabolic subalgebras.

1 Introduction

Invariant differential operators play very important role in the description of physical symmetries. In a recent paper [1] we started the systematic explicit construction of invariant differential operators. We gave an explicit description of the building blocks, namely, the parabolic subgroups and subalgebras from which the necessary representations are induced. Thus we have set the stage for study of different non-compact groups.

In the present paper we focus on the algebra $su(n,n)$. These algebras belong to a narrow class of algebras, which we call 'conformal Lie algebras', which have very similar properties to the canonical conformal algebras of Minkowski space-time. This class was identified from our point of view in [2]. The same class was identified independently from different considerations and under different names in [3,4].

This paper is a sequel of [5], and due to the lack of space we refer to it and to [6] for motivations and extensive list of literature on the subject.

2 Preliminaries

Let G be a semisimple non-compact Lie group, and K a maximal compact subgroup of G. Then we have an Iwasawa decomposition $G = KA_0N_0$, where A_0 is abelian simply connected vector subgroup of G, N_0 is a nilpotent simply connected subgroup of G preserved by the action of A_0. Further, let M_0 be the centralizer of A_0 in K. Then the

subgroup $P_0 = M_0 A_0 N_0$ is a minimal parabolic subgroup of G. A parabolic subgroup $P = MAN$ is any subgroup of G which contains a minimal parabolic subgroup.

The importance of the parabolic subgroups comes from the fact that the representations induced from them generate all (admissible) irreducible representations of G [7–9].

Let ν be a (non-unitary) character of A, $\nu \in \mathcal{A}^*$, let μ fix an irreducible representation D^μ of M on a vector space V_μ.

We call the induced representation $\chi = \mathrm{Ind}_P^G(\mu \otimes \nu \otimes 1)$ an *elementary representation* of G [10]. Their spaces of functions are:

$$\mathcal{C}_\chi = \{\mathcal{F} \in C^\infty(G, V_\mu) \mid \mathcal{F}(gman) = e^{-\nu(H)} \cdot D^\mu(m^{-1})\, \mathcal{F}(g)\} \tag{2.1}$$

where $a = \exp(H) \in A$, $H \in \mathcal{A}$, $m \in M$, $n \in N$. The representation action is the *left* regular action:

$$(\mathcal{T}^\chi(g)\mathcal{F})(g') = \mathcal{F}(g^{-1}g')\,, \quad g, g' \in G\,. \tag{2.2}$$

For our purposes we need to restrict to *maximal* parabolic subgroups P, so that $\mathrm{rank}\, A = 1$. Thus, for our representations the character ν is parameterized by a real number d, called the conformal weight or energy.

An important ingredient in our considerations are the *highest/lowest weight representations* of $\mathcal{G}$. These can be realized as (factor-modules of) Verma modules V^Λ over $\mathcal{G}^{\mathbf{C}}$, where $\Lambda \in (\mathcal{H}^{\mathbf{C}})^*$, $\mathcal{H}^{\mathbf{C}}$ is a Cartan subalgebra of $\mathcal{G}^{\mathbf{C}}$, weight $\Lambda = \Lambda(\chi)$ is determined uniquely from χ [11, 12].

Actually, since our ERs will be induced from finite-dimensional representations of $\mathcal{M}$ (or their limits) the Verma modules are always reducible. Thus, it is more convenient to use *generalized Verma modules* $\tilde{V}^\Lambda$ such that the role of the highest/lowest weight vector v_0 is taken by the space $V_\mu v_0$. For the generalized Verma modules (GVMs) the reducibility is controlled only by the value of the conformal weight d. Relatedly, for the intertwining differential operators only the reducibility w.r.t. non-compact roots is essential.

One main ingredient of our approach is as follows. We group the (reducible) ERs with the same Casimirs in sets called *multiplets* [12, 13]. The multiplet corresponding to fixed values of the Casimirs may be depicted as a connected graph, the vertices of which correspond to the reducible ERs and the lines between the vertices correspond to intertwining operators. The explicit parametrization of the multiplets and of their ERs is important for understanding of the situation.

In fact, the multiplets contain explicitly all the data necessary to construct the intertwining differential operators. Actually, the data for each intertwining differential operator consists of the pair (β, m), where β is a (non-compact) positive root of $\mathcal{G}^{\mathbf{C}}$, $m \in \mathbb{N}$, such that the BGG [14] Verma module reducibility condition (for highest weight modules) is fulfilled:

$$(\Lambda + \rho, \beta^\vee) = m\,, \quad \beta^\vee \equiv 2\beta/(\beta, \beta)\,. \tag{2.3}$$

When (2.3) holds then the Verma module with shifted weight $V^{\Lambda - m\beta}$ (or $\tilde{V}^{\Lambda - m\beta}$ for GVM and β non-compact) is embedded in the Verma module V^Λ (or $\tilde{V}^\Lambda$). This embedding is realized by a singular vector v_s determined by a polynomial $\mathcal{P}_\beta^m(\mathcal{G}^-)$ in the universal enveloping algebra $(U(\mathcal{G}_-))\, v_0$, $\mathcal{G}^-$ is the subalgebra of $\mathcal{G}^{\mathbf{C}}$ generated by the

negative root generators [15]. More explicitly, [12], $v^s_{m,\beta} = \mathcal{P}^m_\beta v_0$ (or $v^s_{m,\beta} = \mathcal{P}^m_\beta V_\mu v_0$ for GVMs). Then there exists [12] an intertwining differential operator

$$\mathcal{D}^m_\beta \ : \ \mathcal{C}_{\chi(\Lambda)} \ \longrightarrow \ \mathcal{C}_{\chi(\Lambda - m\beta)} \tag{2.4}$$

given explicitly by:

$$\mathcal{D}^m_\beta \ = \ \mathcal{P}^m_\beta(\widehat{\mathcal{G}^-}) \tag{2.5}$$

where $\widehat{\mathcal{G}^-}$ denotes the *right* action on the functions $\mathcal{F}$, cf. (2.1).

3 The non-compact Lie algebra $su(4,4)$

Let $\mathcal{G} = su(4,4)$. This algebra has discrete series representations and highest/lowest weight representations since the maximal compact subalgebra is $\mathcal{K} \cong u(1) \oplus su(4) \oplus su(4)$.

We choose a *maximal* parabolic $\mathcal{P} = \mathcal{MAN}$ such that $\mathcal{A} \cong so(1,1)$, while the factor $\mathcal{M}$ has the same finite-dimensional (nonunitary) representations as the finite-dimensional (unitary) representations of the semi-simple subalgebra of $\mathcal{K}$, i.e., $\mathcal{M} = sl(4,\mathbb{C})_{\mathbb{R}}$, cf. [1]. Thus, these induced representations are representations of finite $\mathcal{K}$-type [16]. Relatedly, the number of ERs in the corresponding multiplets is equal to $|W(\mathcal{G}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})| \, / \, |W(\mathcal{K}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})| \ = \ 70$, cf. [6], where $\mathcal{H}$ is a Cartan subalgebra of both $\mathcal{G}$ and $\mathcal{K}$. Note also that $\mathcal{K}^{\mathbb{C}} \cong u(1)^{\mathbb{C}} \oplus sl(4,\mathbb{C}) \oplus sl(4,\mathbb{C}) \cong \mathcal{M}^{\mathbb{C}} \oplus \mathcal{A}^{\mathbb{C}}$. Finally, note that $\dim_{\mathbb{R}} \mathcal{N} = 16$.

We label the signature of the ERs of $\mathcal{G}$ as follows:

$$\chi \ = \ \{ n_1 , n_2 , n_3 , n_5 , n_6 , n_7 \, ; \, c \} \ , \qquad n_j \in \mathbb{Z}_+ \ , \quad c = d - 4 \tag{3.6}$$

where the last entry of χ labels the characters of $\mathcal{A}$, and the first 6 entries are labels of the finite-dimensional nonunitary irreps of $\mathcal{M}$ when all $n_j > 0$ or limits of the latter when some $n_j = 0$.

Below we shall use the following conjugation on the finite-dimensional entries of the signature:

$$(n_1, n_2, n_3, n_5, n_6, n_7)^* \ \doteq \ (n_5, n_6, n_7, n_1, n_2, n_3) \ . \tag{3.7}$$

The ERs in the multiplet are related also by intertwining integral operators introduced in [17]. These operators are defined for any ER, the general action being:

$$\begin{aligned} & G_{KS} \ : \ \mathcal{C}_\chi \ \longrightarrow \ \mathcal{C}_{\chi'} \ , \\ & \chi \ = \ \{ n_1, n_2, n_3, n_5, n_6, n_7 \, ; \, c \} \ , \qquad \chi' \ = \ \{ (n_1, n_2, n_4, n_5)^* \, ; \, -c \} . \end{aligned} \tag{3.8}$$

The above action on the signatures is also called restricted Weyl reflection, since it represents the nontrivial element of the 2-element restricted Weyl group which arises canonically with every maximal parabolic subalgebra.

For the classification of the multiplets we shall need one more conjugation for the entries of the $\mathcal{M}$ representations:

$$(n_1, n_2, n_3, n_5, n_6, n_7)^\spadesuit \ \doteq \ (n_7, n_6, n_5, n_3, n_2, n_1) \tag{3.9}$$

involving full reordering of the entries (unlike the conjugation (3.7) which just exchanges the two $su(4)$ sets of indices).

Further, we need the root system of the complexification $\mathcal{G}^{\mathbb{C}} = sl(8,\mathbb{C})$. The positive roots in terms of the simple roots are given standardly as:

$$\begin{aligned} \alpha_{ij} &= \alpha_i + \cdots + \alpha_j \ , \quad 1 \leq i < j \leq 5 \ , \\ \alpha_{jj} &= \alpha_j \ , \quad 1 \leq j \leq 7 \end{aligned} \tag{3.10}$$

From these the compact roots are those that form (by restriction) the root system of the semisimple part of $\mathcal{K}^{\mathbb{C}}$, the rest are noncompact, i.e.,

$$\text{noncompact}: \qquad \alpha_{ij} \ , \quad 1 \leq i \leq 4 \ , \qquad 4 \leq j \leq 7 \ . \tag{3.11}$$

Further, we give the correspondence between the signatures χ and the highest weight Λ. The connection is through the Dynkin labels:

$$m_i \ \equiv \ (\Lambda + \rho, \alpha_i^\vee) \ = \ (\Lambda + \rho, \alpha_i) \ , \quad i = 1, \ldots, 7, \tag{3.12}$$

where $\Lambda = \Lambda(\chi)$, ρ is half the sum of the positive roots of $\mathcal{G}^{\mathbb{C}}$. The explicit connection is:

$$n_i = m_i \ , \quad c \ = \ -\tfrac{1}{2}(m_{\tilde{\alpha}} + m_4) \ = \ -\tfrac{1}{2}(m_1 + m_2 + m_3 + 2m_4 + m_5 + m_6 + m_7) \tag{3.13}$$

where $\tilde{\alpha} \ = \ \alpha_1 + \cdots + \alpha_7$ is the highest root.

We shall use also the so-called Harish-Chandra parameters:

$$m_{jk} \equiv (\Lambda + \rho, \alpha_{jk}) \ = \ m_j + \cdots + m_k \ , \quad j < k \ , \qquad m_{jj} \equiv m_j \ . \tag{3.14}$$

Note that according to [6] all results about the classification of invariant operators are valid also for the algebra $sl(8,\mathbb{R})$ with maximal parabolic $\mathcal{P}' = \mathcal{M}'\mathcal{A}'\mathcal{N}'$, where $\mathcal{M}' = sl(4,\mathbb{R}) \oplus sl(4,\mathbb{R})$. This is due to the fact that $\mathcal{P}'^{\mathbb{C}} = \mathcal{P}^{\mathbb{C}}$, $\mathcal{M}'^{\mathbb{C}} = \mathcal{M}^{\mathbb{C}} \cong sl(4,\mathbb{C}) \oplus sl(4,\mathbb{C})$. Furthermore, the results are valid also for the algebra $su^*(8)$ with maximal parabolic $\mathcal{P}'' = \mathcal{M}''\mathcal{A}''\mathcal{N}''$, where $\mathcal{M}'' = su^*(4) \oplus su^*(4)$ (noting that $\mathcal{P}''^{\mathbb{C}} = \mathcal{P}^{\mathbb{C}}$, $\mathcal{M}''^{\mathbb{C}} \cong sl(4,\mathbb{C}) \oplus sl(4,\mathbb{C})$).

4 Multiplets of SU(4,4)

4.1 Main multiplets

There are two types of multiplets: main and reduced. The multiplets of the main type are in 1-to-1 correspondence with the finite-dimensional irreps of $su(4,4)$, i.e., they are labelled by the seven positive Dynkin labels $m_i \in \mathbb{N}$. In [5] we have given explicitly the main multiplets for $n = 2, 3, 4$, and the reduced for $n = 2$. In [18] we have given explicitly the reduced for $n = 3$.

The main multiplet R^4 contains 70 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$\chi_0^{\pm} \ = \ \{ \, (m_1, m_2, m_3, m_5, m_6, m_7)^{\pm} \, ; \, \pm\tfrac{1}{2}(m_{\tilde{\alpha}} + m_4) \, \}$$

$$
\begin{aligned}
\chi_{00}^{\pm} &= \{\,(m_1,m_2,m_{34},m_{45},m_6,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{\bar{5}}-m_4)\,\}\\
\chi_{10}^{\pm} &= \{\,(m_1,m_{23},m_4,m_{35},m_6,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{12,57}\,\}\\
\chi_{01}^{\pm} &= \{\,(m_1,m_2,m_{35},m_4,m_{56},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{13,67}\,\}\\
\chi_{20}^{\pm} &= \{\,(m_{12},m_3,m_4,m_{25},m_6,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,57}\,\}\\
\chi_{11}^{\pm} &= \{\,(m_1,m_{23},m_{45},m_{34},m_{56},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{12,67}\,\}\\
\chi_{02}^{\pm} &= \{\,(m_1,m_2,m_{36},m_4,m_5,m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}m_{13,7}\,\}\\
\chi_{30}^{\pm} &= \{\,(m_2,m_3,m_4,m_{15},m_6,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{57}-m_1)\,\}\\
\chi_{21}^{\pm} &= \{\,(m_{12},m_3,m_{45},m_{24},m_{56},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,67}\,\}\\
\chi_{12}^{\pm} &= \{\,(m_1,m_{23},m_{46},m_{34},m_5,m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}m_{12,7}\,\}\\
\chi_{03}^{\pm} &= \{\,(m_1,m_2,m_{37},m_4,m_5,m_6)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{13}-m_7)\,\}\\
\chi_{31}^{\pm} &= \{\,(m_2,m_3,m_{45},m_{14},m_{56},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{67}-m_1)\,\}\\
\chi_{22}^{\pm} &= \{\,(m_{12},m_3,m_{46},m_{24},m_5,m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,7}\,\}\\
\chi_{13}^{\pm} &= \{\,(m_1,m_{23},m_{47},m_{34},m_5,m_6)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{12}-m_7)\,\}\\
\chi_{32}^{\pm} &= \{\,(m_2,m_3,m_{46},m_{14},m_5,m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}(m_7-m_1)\,\}\\
\chi_{23}^{\pm} &= \{\,(m_{12},m_3,m_{47},m_{24},m_5,m_6)^{\pm}\,;\,\pm\tfrac{1}{2}(m_1-m_7)\,\}\\
\chi_{33}^{\pm} &= \{\,(m_2,m_3,m_{47},m_{14},m_5,m_6)^{\pm}\,;\,\mp\tfrac{1}{2}(m_{1,7})\,\}\\
\chi_{00}^{\prime\pm} &= \{\,(m_1,m_{24},m_5,m_3,m_{46},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{12,67}\,\}\\
\chi_{10}^{\prime\pm} &= \{\,(m_{12},m_{34},m_5,m_{23},m_{46},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,67}\,\}\\
\chi_{01}^{\prime\pm} &= \{\,(m_1,m_{24},m_{56},m_3,m_{45},m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}m_{12,7}\,\}\\
\chi_{20}^{\prime\pm} &= \{\,(m_2,m_{34},m_5,m_{13},m_{46},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{67}-m_1)\,\}\\
\chi_{11}^{\prime\pm} &= \{\,(m_{12},m_{34},m_{56},m_{23},m_{45},m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,7}\,\}\\
\chi_{02}^{\prime\pm} &= \{\,(m_1,m_{24},m_{57},m_3,m_{45},m_6)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{12}-m_7)\,\}\\
\chi_{20}^{\prime\prime\pm} &= \{\,(m_{13},m_4,m_5,m_2,m_{36},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,67}\,\}\\
\chi_{21}^{\prime\prime\pm} &= \{\,(m_{13},m_4,m_{56},m_2,m_{35},m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,7}\,\}\\
\chi_{12}^{\prime\prime\pm} &= \{\,(m_{12},m_{35},m_6,m_{23},m_4,m_{57})^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,7}\,\}\\
\chi_{02}^{\prime\prime\pm} &= \{\,(m_1,m_{25},m_6,m_3,m_4,m_{57})^{\pm}\,;\,\pm\tfrac{1}{2}m_{12,7}\,\}\\
\chi_{30}^{\prime\pm} &= \{\,(m_{23},m_4,m_5,m_{12},m_{36},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{67}-m_1)\,\}\\
\chi_{21}^{\prime\pm} &= \{\,(m_2,m_{34},m_{56},m_{13},m_{45},m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}(m_7-m_1)\,\}\\
\chi_{12}^{\prime\pm} &= \{\,(m_{12},m_{34},m_{57},m_{23},m_{45},m_6)^{\pm}\,;\,\pm\tfrac{1}{2}(m_1-m_7)\,\}\\
\chi_{03}^{\prime\pm} &= \{\,(m_1,m_{25},m_{67},m_3,m_4,m_{56})^{\pm}\,;\,\pm\tfrac{1}{2}(m_{12}-m_7)\,\}\\
\chi_{40}^{\prime\pm} &= \{\,(m_3,m_4,m_5,m_1,m_{26},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{67}-m_{12})\,\}\\
\chi_{31}^{\prime\pm} &= \{\,(m_{23},m_4,m_{56},m_{12},m_{35},m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}(m_7-m_1)\,\}\\
\chi_{22}^{\prime\pm} &= \{\,(m_2,m_{34},m_{57},m_{13},m_{45},m_6)^{\pm}\,;\,\mp\tfrac{1}{2}m_{1,7}\,\}\\
\chi_{22}^{\prime\prime\pm} &= \{\,(m_{13},m_4,m_{57},m_2,m_{35},m_6)^{\pm}\,;\,\pm\tfrac{1}{2}(m_1-m_7)\,\}
\end{aligned}
\tag{4.15}
$$

The multiplets are given explicitly in Fig. 1 (first in [5]). The pairs $\Lambda^{\pm}$ are symmetric w.r.t. to the bullet in the middle of the figure - this represents the Weyl symmetry realized by the Knapp-Stein operators (3.8): $G_{KS} \,:\, \mathcal{C}_{\chi^{\mp}} \longleftrightarrow \mathcal{C}_{\chi^{\pm}}$.

Matters are arranged so that in every multiplet only the ER with signature χ_0^- contains a finite-dimensional nonunitary subrepresentation in a finite-dimensional subspace $\mathcal{E}$. The latter corresponds to the finite-dimensional irrep of $su(3,3)$ with signature $\{m_1,\ldots,m_7\}$. The subspace $\mathcal{E}$ is annihilated by the operator G^+, and is the image of the operator G^-. The subspace $\mathcal{E}$ is annihilated also by the intertwining differential operator acting from χ_0^- to χ_a^-. When all $m_i = 1$ then $\dim \mathcal{E} = 1$, and in that case $\mathcal{E}$ is also the trivial one-dimensional UIR of the whole algebra $\mathcal{G}$. Furthermore in that

case the conformal weight is zero: $d = 3 + c = 3 - \frac{1}{2}(m_1 + m_2 + 2m_3 + m_4 + m_5)_{|_{m_i=1}} = 0$.

Analogously, in every multiplet only the ER with signature χ_0^+ contains holomorphic discrete series representation. This is guaranteed by the criterion [11] that for such an ER all Harish-Chandra parameters for non-compact roots must be negative, i.e., in our situation, $m_\alpha < 0$. [That this holds for our χ^+ can be easily checked using the signatures (??).]

Note that the ER χ_0^+ contains also the conjugate anti-holomorphic discrete series. The direct sum of the holomorphic and the antiholomorphic representations are realized in an invariant subspace $\mathcal{D}$ of the ER χ_0^+. That subspace is annihilated by the operator G^-, and is the image of the operator G^+. Note that the corresponding lowest weight GVM is infinitesimally equivalent only to the holomorphic discrete series, while the conjugate highest weight GVM is infinitesimally equivalent to the anti-holomorphic discrete series. The conformal weight of the ER χ_0^+ has the restriction $d = 3 + c = 3 + \frac{1}{2}(m_1 + m_2 + 2m_3 + m_4 + m_5) \geq 6$.

In Fig. 1 and below we use the notation: $\Lambda^\pm = \Lambda(\chi^\pm)$. Each intertwining differential operator is represented by an arrow accompanied by a symbol i_{jk} encoding the root α_{jk} and the number $m_{\alpha_{jk}}$ which is involved in the BGG criterion. This notation is used to save space, but it can be used due to the fact that only intertwining differential operators which are non-composite are displayed, and that the data β, m_β, which is involved in the embedding $V^\Lambda \longleftarrow V^{\Lambda - m_\beta, \beta}$ turns out to involve only the m_i corresponding to simple roots, i.e., for each β, m_β there exists $i = i(\beta, m_\beta, \Lambda) \in \{1, \ldots, 7\}$, such that $m_\beta = m_i$. Hence the data α_{jk}, $m_{\alpha_{jk}}$ is represented by i_{jk} on the arrows.

4.2 Main reduced multiplets

There are seven types of main reduced multiplets, R_a^4, $a = 1, \ldots, 7$, which may be obtained from the main multiplet by setting formally $m_a = 0$. Multiplets of type R_5^4, R_6^4, R_7^4, are conjugate to the multiplets of type R_3^4, R_2^4, R_1^4, resp., as follows. First we make the conjugation on the roots and exchange all indices: $1 \longleftrightarrow 7$, $2 \longleftrightarrow 6$, $3 \longleftrightarrow 5$. With this operation we obtain the diagrams of the conjugated cases from one another. For the entries of the $\mathcal{M}$ representation we have further to employ the conjugation (3.9). Then we obtain the signatures of the conjugated cases from one another. Thus, we give explicitly only first four types.

The reduced multiplets of type R_4^4 contain 50 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$
\begin{aligned}
\chi_0^\pm &= \{(m_1, m_2, m_3, m_5, m_6, m_7)^\pm\,;\, \pm\tfrac{1}{2} m_{\bar{a}}\} \\
\chi_{10}^\pm &= \{(m_1, m_{23}, 0, m_{3,5}, m_6, m_7)^\pm\,;\, \pm\tfrac{1}{2} m_{12,57}\} \\
\chi_{01}^\pm &= \{(m_1, m_2, m_{3,5}, 0, m_{56}, m_7)^\pm\,;\, \pm\tfrac{1}{2} m_{13,67}\} \\
\chi_{20}^\pm &= \{(m_{12}, m_3, 0, m_{23,5}, m_6, m_7)^\pm\,;\, \pm\tfrac{1}{2} m_{1,57}\} \\
\chi_{11}^\pm &= \{(m_1, m_{23}, m_5, m_3, m_{56}, m_7)^\pm\,;\, \pm\tfrac{1}{2} m_{12,67}\} \\
\chi_{02}^\pm &= \{(m_1, m_2, m_{3,56}, 0, m_5, m_{67})^\pm\,;\, \pm\tfrac{1}{2} m_{13,7}\} \\
\chi_{30}^\pm &= \{(m_2, m_3, 0, m_{13,5}, m_6, m_7)^\pm\,;\, \pm\tfrac{1}{2}(m_{57} - m_1)\} \\
\chi_{21}^\pm &= \{(m_{12}, m_3, m_5, m_{23}, m_{56}, m_7)^\pm\,;\, \pm\tfrac{1}{2} m_{1,67}\} \\
\chi_{12}^\pm &= \{(m_1, m_{23}, m_{56}, m_3, m_5, m_{67})^\pm\,;\, \pm\tfrac{1}{2} m_{12,7}\}
\end{aligned}
$$

$$
\begin{aligned}
\chi^{\pm}_{03} &= \{\,(m_1,m_2,m_{3,57},0,m_5,m_6)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{13}-m_7)\,\}\\
\chi^{\pm}_{31} &= \{\,(m_2,m_3,m_5,m_{13},m_{56},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{67}-m_1)\,\}\\
\chi^{\pm}_{22} &= \{\,(m_{12},m_3,m_{56},m_{23},m_5,m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,7}\,\}\\
\chi^{\pm}_{13} &= \{\,(m_1,m_{23},m_{57},m_3,m_5,m_6)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{12}-m_7)\,\}\\
\chi^{\pm}_{32} &= \{\,(m_2,m_3,m_{56},m_{13},m_5,m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}(m_7-m_1)\,\}\\
\chi^{\pm}_{23} &= \{\,(m_{12},m_3,m_{57},m_{23},m_5,m_6)^{\pm}\,;\,\pm\tfrac{1}{2}(m_1-m_7)\,\}\\
\chi^{\pm}_{33} &= \{\,(m_2,m_3,m_{57},m_{13},m_5,m_6)^{\pm}\,;\,\mp\tfrac{1}{2}(m_{1,7})\,\}\\
\chi''^{\pm}_{20} &= \{\,(m_{13},0,m_5,m_2,m_{3,56},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,67}\,\}\\
\chi''^{\pm}_{21} &= \{\,(m_{13},0,m_{56},m_2,m_{3,5},m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,7}\,\}\\
\chi''^{\pm}_{12} &= \{\,(m_{12},m_{3,5},m_6,m_{23},0,m_{57})^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,7}\,\}\\
\chi''^{\pm}_{02} &= \{\,(m_1,m_{23,5},m_6,m_3,0,m_{57})^{\pm}\,;\,\pm\tfrac{1}{2}m_{12,7}\,\}\\
\chi'^{\pm}_{30} &= \{\,(m_{23},0,m_5,m_{12},m_{3,56},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{67}-m_1)\,\}\\
\chi'^{\pm}_{03} &= \{\,(m_1,m_{23,5},m_{67},m_3,0,m_{56})^{\pm}\,;\,\pm\tfrac{1}{2}(m_{12}-m_7)\,\}\\
\chi'^{\pm}_{40} &= \{\,(m_3,0,m_5,m_1,m_{23,56},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{67}-m_{12})\,\}\\
\chi'^{\pm}_{31} &= \{\,(m_{23},0,m_{56},m_{12},m_{35},m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}(m_7-m_1)\,\}\\
\chi''^{\pm}_{22} &= \{\,(m_{13},0,m_{57},m_2,m_{3,5},m_6)^{\pm}\,;\,\pm\tfrac{1}{2}(m_1-m_7)\,\}
\end{aligned}
\tag{4.16}
$$

This may be called the main type of reduced multiplets since here in χ^{+}_{0} are contained the limits of the (anti)holomorphic discrete series. The multiplets are given in Fig. 2a.

The reduced multiplets of type R^4_3 contain 50 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$
\begin{aligned}
\chi^{\pm}_{0} &= \{\,(m_1,m_2,0,m_5,m_6,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_5+m_4)\,\}\\
\chi^{\pm}_{10} &= \{\,(m_1,m_2,m_4,m_{45},m_6,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{12,57}\,\}\\
\chi^{\pm}_{20} &= \{\,(m_{12},0,m_4,m_{2,45},m_6,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,57}\,\}\\
\chi^{\pm}_{11} &= \{\,(m_1,m_2,m_{45},m_4,m_{56},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{12,67}\,\}\\
\chi^{\pm}_{21} &= \{\,(m_{12},0,m_{45},m_{2,4},m_{56},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,67}\,\}\\
\chi^{\pm}_{12} &= \{\,(m_1,m_2,m_{46},m_4,m_5,m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}m_{12,7}\,\}\\
\chi^{\pm}_{30} &= \{\,(m_2,0,m_4,m_{12,45},m_6,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{57}-m_1)\,\}\\
\chi^{\pm}_{31} &= \{\,(m_2,0,m_{45},m_{12,4},m_{56},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{67}-m_1)\,\}\\
\chi^{\pm}_{22} &= \{\,(m_{12},0,m_{46},m_{2,4},m_5,m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,7}\,\}\\
\chi^{\pm}_{13} &= \{\,(m_1,m_2,m_{47},m_4,m_5,m_6)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{12}-m_7)\,\}\\
\chi^{\pm}_{32} &= \{\,(m_2,0,m_{46},m_{12,4},m_5,m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}(m_7-m_1)\,\}\\
\chi^{\pm}_{23} &= \{\,(m_{12},0,m_{47},m_{2,4},m_5,m_6)^{\pm}\,;\,\pm\tfrac{1}{2}(m_1-m_7)\,\}\\
\chi^{\pm}_{33} &= \{\,(m_2,0,m_{47},m_{12,4},m_5,m_6)^{\pm}\,;\,\mp\tfrac{1}{2}(m_{1,7})\,\}\\
\chi'^{\pm}_{00} &= \{\,(m_1,m_{2,4},m_5,0,m_{46},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{12,67}\,\}\\
\chi'^{\pm}_{10} &= \{\,(m_{12},m_4,m_5,m_2,m_{46},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,67}\,\}\\
\chi'^{\pm}_{01} &= \{\,(m_1,m_{2,4},m_{56},0,m_{45},m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}m_{12,7}\,\}\\
\chi'^{\pm}_{20} &= \{\,(m_2,m_4,m_5,m_{12},m_{46},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{67}-m_1)\,\}\\
\chi'^{\pm}_{11} &= \{\,(m_{12},m_4,m_{56},m_2,m_{45},m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,7}\,\}\\
\chi'^{\pm}_{02} &= \{\,(m_1,m_{2,4},m_{57},0,m_{45},m_6)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{12}-m_7)\,\}\\
\chi''^{\pm}_{12} &= \{\,(m_{12},m_{45},m_6,m_2,m_4,m_{57})^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,7}\,\}\\
\chi''^{\pm}_{02} &= \{\,(m_1,m_{2,45},m_6,0,m_4,m_{57})^{\pm}\,;\,\pm\tfrac{1}{2}m_{12,7}\,\}\\
\chi'^{\pm}_{21} &= \{\,(m_2,m_4,m_{56},m_{12},m_{45},m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}(m_7-m_1)\,\}\\
\chi'^{\pm}_{12} &= \{\,(m_{12},m_4,m_{57},m_2,m_{45},m_6)^{\pm}\,;\,\pm\tfrac{1}{2}(m_1-m_7)\,\}\\
\chi'^{\pm}_{03} &= \{\,(m_1,m_{2,45},m_{67},0,m_4,m_{56})^{\pm}\,;\,\pm\tfrac{1}{2}(m_{12}-m_7)\,\}
\end{aligned}
$$

$$\chi_{40}'^{\pm} = \{ (0, m_4, m_5, m_1, m_{2,46}, m_7)^{\pm} ; \pm\tfrac{1}{2}(m_{67} - m_{12}) \} \qquad (4.17)$$

The multiplets are given in Fig. 2b.

The reduced multiplets of type R_2^4 contain 50 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$\begin{aligned}
\chi_0^{\pm} &= \{ (m_1, 0, m_3, m_5, m_6, m_7)^{\pm} ; \pm\tfrac{1}{2}(m_{\bar{a}} + m_4) \} \\
\chi_{00}^{\pm} &= \{ (m_1, 0, m_{34}, m_{45}, m_6, m_7)^{\pm} ; \pm\tfrac{1}{2}(m_{\bar{a}} - m_4) \} \\
\chi_{01}^{\pm} &= \{ (m_1, 0, m_{35}, m_4, m_{56}, m_7)^{\pm} ; \pm\tfrac{1}{2} m_{1,3,67} \} \\
\chi_{20}^{\pm} &= \{ (m_1, m_3, m_4, m_{35}, m_6, m_7)^{\pm} ; \pm\tfrac{1}{2} m_{1,57} \} \\
\chi_{02}^{\pm} &= \{ (m_1, 0, m_{36}, m_4, m_5, m_{67})^{\pm} ; \pm\tfrac{1}{2} m_{1,3,7} \} \\
\chi_{30}^{\pm} &= \{ (0, m_3, m_4, m_{1,35}, m_6, m_7)^{\pm} ; \pm\tfrac{1}{2}(m_{57} - m_1) \} \\
\chi_{21}^{\pm} &= \{ (m_1, m_3, m_{45}, m_{34}, m_{56}, m_7)^{\pm} ; \pm\tfrac{1}{2} m_{1,67} \} \\
\chi_{03}^{\pm} &= \{ (m_1, 0, m_{37}, m_4, m_5, m_6)^{\pm} ; \pm\tfrac{1}{2}(m_{1,3} - m_7) \} \\
\chi_{31}^{\pm} &= \{ (0, m_3, m_{45}, m_{1,34}, m_{56}, m_7)^{\pm} ; \pm\tfrac{1}{2}(m_{67} - m_1) \} \\
\chi_{22}^{\pm} &= \{ (m_1, m_3, m_{46}, m_{34}, m_5, m_{67})^{\pm} ; \pm\tfrac{1}{2} m_{1,7} \} \\
\chi_{32}^{\pm} &= \{ (0, m_3, m_{46}, m_{1,34}, m_5, m_{67})^{\pm} ; \pm\tfrac{1}{2}(m_7 - m_1) \} \\
\chi_{23}^{\pm} &= \{ (m_1, m_3, m_{47}, m_{34}, m_5, m_6)^{\pm} ; \pm\tfrac{1}{2}(m_1 - m_7) \} \\
\chi_{33}^{\pm} &= \{ (0, m_3, m_{47}, m_{1,34}, m_5, m_6)^{\pm} ; \mp\tfrac{1}{2}(m_{1,7}) \} \\
\chi_{10}'^{\pm} &= \{ (m_1, m_{34}, m_5, m_3, m_{46}, m_7)^{\pm} ; \pm\tfrac{1}{2} m_{1,67} \} \\
\chi_{20}'^{\pm} &= \{ (0, m_{34}, m_5, m_{1,3}, m_{46}, m_7)^{\pm} ; \pm\tfrac{1}{2}(m_{67} - m_1) \} \\
\chi_{11}'^{\pm} &= \{ (m_1, m_{34}, m_{56}, m_3, m_{45}, m_{67})^{\pm} ; \pm\tfrac{1}{2} m_{1,7} \} \\
\chi_{20}''^{\pm} &= \{ (m_{1,3}, m_4, m_5, 0, m_{36}, m_7)^{\pm} ; \pm\tfrac{1}{2} m_{1,67} \} \\
\chi_{21}''^{\pm} &= \{ (m_{1,3}, m_4, m_{56}, 0, m_{35}, m_{67})^{\pm} ; \pm\tfrac{1}{2} m_{1,7} \} \\
\chi_{12}''^{\pm} &= \{ (m_1, m_{35}, m_6, m_3, m_4, m_{57})^{\pm} ; \pm\tfrac{1}{2} m_{1,7} \} \\
\chi_{30}'^{\pm} &= \{ (m_3, m_4, m_5, m_1, m_{36}, m_7)^{\pm} ; \pm\tfrac{1}{2}(m_{67} - m_1) \} \\
\chi_{21}'^{\pm} &= \{ (0, m_{34}, m_{56}, m_{1,3}, m_{45}, m_{67})^{\pm} ; \pm\tfrac{1}{2}(m_7 - m_1) \} \\
\chi_{12}'^{\pm} &= \{ (m_1, m_{34}, m_{57}, m_3, m_{45}, m_6)^{\pm} ; \pm\tfrac{1}{2}(m_1 - m_7) \} \\
\chi_{31}'^{\pm} &= \{ (m_3, m_4, m_{56}, m_1, m_{35}, m_{67})^{\pm} ; \pm\tfrac{1}{2}(m_7 - m_1) \} \\
\chi_{22}'^{\pm} &= \{ (0, m_{34}, m_{57}, m_{1,3}, m_{45}, m_6)^{\pm} ; \mp\tfrac{1}{2} m_{1,7} \} \\
\chi_{22}''^{\pm} &= \{ (m_{1,3}, m_4, m_{57}, 0, m_{35}, m_6)^{\pm} ; \pm\tfrac{1}{2}(m_1 - m_7) \} \qquad (4.18)
\end{aligned}$$

The multiplets are given in Fig. 2c.

The reduced multiplets of type R_1^4 contain 50 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$\begin{aligned}
\chi_0^{\pm} &= \{ (0, m_2, m_3, m_5, m_6, m_7)^{\pm} ; \pm\tfrac{1}{2}(m_{\bar{a}} + m_4) \} \\
\chi_{00}^{\pm} &= \{ (0, m_2, m_{34}, m_{45}, m_6, m_7)^{\pm} ; \pm\tfrac{1}{2}(m_{\bar{a}} - m_4) \} \\
\chi_{10}^{\pm} &= \{ (0, m_{23}, m_4, m_{35}, m_6, m_7)^{\pm} ; \pm\tfrac{1}{2} m_{2,57} \} \\
\chi_{01}^{\pm} &= \{ (0, m_2, m_{35}, m_4, m_{56}, m_7)^{\pm} ; \pm\tfrac{1}{2} m_{23,67} \} \\
\chi_{20}^{\pm} &= \{ (m_2, m_3, m_4, m_{25}, m_6, m_7)^{\pm} ; \pm\tfrac{1}{2} m_{57} \} \\
\chi_{11}^{\pm} &= \{ (0, m_{23}, m_{45}, m_{34}, m_{56}, m_7)^{\pm} ; \pm\tfrac{1}{2} m_{2,67} \} \\
\chi_{02}^{\pm} &= \{ (0, m_2, m_{36}, m_4, m_5, m_{67})^{\pm} ; \pm\tfrac{1}{2} m_{23,7} \} \\
\chi_{21}^{\pm} &= \{ (m_2, m_3, m_{45}, m_{24}, m_{56}, m_7)^{\pm} ; \pm\tfrac{1}{2} m_{67} \} \\
\chi_{12}^{\pm} &= \{ (0, m_{23}, m_{46}, m_{34}, m_5, m_{67})^{\pm} ; \pm\tfrac{1}{2} m_{2,7} \} \\
\chi_{03}^{\pm} &= \{ (0, m_2, m_{37}, m_4, m_5, m_6)^{\pm} ; \pm\tfrac{1}{2}(m_{23} - m_7) \}
\end{aligned}$$

$$
\begin{aligned}
\chi^{\pm}_{22} &= \{\,(m_2,m_3,m_{46},m_{24},m_5,m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}m_7\,\}\\
\chi^{\pm}_{13} &= \{\,(0,m_{23},m_{47},m_{34},m_5,m_6)^{\pm}\,;\,\pm\tfrac{1}{2}(m_2-m_7)\,\}\\
\chi^{\pm}_{23} &= \{\,(m_2,m_3,m_{47},m_{24},m_5,m_6)^{\pm}\,;\,\mp\tfrac{1}{2}m_7\,\}\\
\chi'^{\pm}_{00} &= \{\,(0,m_{24},m_5,m_3,m_{46},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{2,67}\,\}\\
\chi'^{\pm}_{10} &= \{\,(m_2,m_{34},m_5,m_{23},m_{46},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{67}\,\}\\
\chi'^{\pm}_{01} &= \{\,(0,m_{24},m_{56},m_3,m_{45},m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}m_{2,7}\,\}\\
\chi'^{\pm}_{11} &= \{\,(m_2,m_{34},m_{56},m_{23},m_{45},m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}m_7\,\}\\
\chi'^{\pm}_{02} &= \{\,(0,m_{24},m_{57},m_3,m_{45},m_6)^{\pm}\,;\,\pm\tfrac{1}{2}(m_2-m_7)\,\}\\
\chi''^{\pm}_{02} &= \{\,(0,m_{25},m_6,m_3,m_4,m_{57})^{\pm}\,;\,\pm\tfrac{1}{2}m_{2,7}\,\}\\
\chi'^{\pm}_{30} &= \{\,(m_{23},m_4,m_5,m_2,m_{36},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{67}\,\}\\
\chi'^{\pm}_{12} &= \{\,(m_2,m_{34},m_{57},m_{23},m_{45},m_6)^{\pm}\,;\,\mp\tfrac{1}{2}m_7\,\}\\
\chi'^{\pm}_{03} &= \{\,(0,m_{25},m_{67},m_3,m_4,m_{56})^{\pm}\,;\,\pm\tfrac{1}{2}(m_2-m_7)\,\}\\
\chi'^{\pm}_{40} &= \{\,(m_3,m_4,m_5,0,m_{26},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{67}-m_2)\,\}\\
\chi'^{\pm}_{31} &= \{\,(m_{23},m_4,m_{56},m_2,m_{35},m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}m_7\,\}\\
\chi''^{\pm}_{22} &= \{\,(m_{23},m_4,m_{57},m_2,m_{35},m_6)^{\pm}\,;\,\mp\tfrac{1}{2}m_7\,\}
\end{aligned}
\tag{4.19}
$$

The multiplets are given in Fig. 2d.

4.3 Further reduction of multiplets

There are further reductions of the multiplets denoted by R^4_{ab}, $a,b=1,\ldots,7$, $a<b$, which may be obtained from the main multiplet by setting formally $m_a=m_b=0$. From these 21 reductions 9 are conjugate to others under the used above conjugation. From the remaining three do not contain representations of physical interest, i.e., induced from finite-dimensional irreps of the $\mathcal{M}$ subalgebra. Thus, we present 9 multiplets.

The reduced multiplets of type R^4_{13} contain 36 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$
\begin{aligned}
\chi^{\pm}_{0} &= \{\,(0,m_2,0,m_5,m_6,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_5+m_4)\,\}\\
\chi^{\pm}_{10} &= \{\,(0,m_2,m_4,m_{45},m_6,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{2,57}\,\}\\
\chi^{\pm}_{20} &= \{\,(m_2,0,m_4,m_{2,45},m_6,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{57}\,\}\\
\chi^{\pm}_{11} &= \{\,(0,m_2,m_{45},m_4,m_{56},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{2,67}\,\}\\
\chi^{\pm}_{21} &= \{\,(m_2,0,m_{45},m_{2,4},m_{56},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{67}\,\}\\
\chi^{\pm}_{12} &= \{\,(0,m_2,m_{46},m_4,m_5,m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}m_{2,7}\,\}\\
\chi^{\pm}_{22} &= \{\,(m_2,0,m_{46},m_{2,4},m_5,m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}m_7\,\}\\
\chi^{\pm}_{13} &= \{\,(0,m_2,m_{47},m_4,m_5,m_6)^{\pm}\,;\,\pm\tfrac{1}{2}(m_2-m_7)\,\}\\
\chi^{\pm}_{23} &= \{\,(m_2,0,m_{47},m_{2,4},m_5,m_6)^{\pm}\,;\,\mp\tfrac{1}{2}m_7\,\}\\
\chi'^{\pm}_{00} &= \{\,(0,m_{2,4},m_5,0,m_{46},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{2,67}\,\}\\
\chi'^{\pm}_{10} &= \{\,(m_2,m_4,m_5,m_2,m_{46},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{67}\,\}\\
\chi'^{\pm}_{01} &= \{\,(0,m_{2,4},m_{56},0,m_{45},m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}m_{2,7}\,\}\\
\chi'^{\pm}_{11} &= \{\,(m_2,m_4,m_{56},m_2,m_{45},m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}m_7\,\}\\
\chi'^{\pm}_{02} &= \{\,(0,m_{2,4},m_{57},0,m_{45},m_6)^{\pm}\,;\,\pm\tfrac{1}{2}(m_2-m_7)\,\}\\
\chi''^{\pm}_{02} &= \{\,(0,m_{2,45},m_6,0,m_4,m_{57})^{\pm}\,;\,\pm\tfrac{1}{2}m_{2,7}\,\}\\
\chi'^{\pm}_{12} &= \{\,(m_2,m_4,m_{57},m_2,m_{45},m_6)^{\pm}\,;\,\mp\tfrac{1}{2}m_7\,\}\\
\chi'^{\pm}_{03} &= \{\,(0,m_{2,45},m_{67},0,m_4,m_{56})^{\pm}\,;\,\pm\tfrac{1}{2}(m_2-m_7)\,\}\\
\chi'^{\pm}_{40} &= \{\,(0,m_4,m_5,0,m_{2,46},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{67}-m_2)\,\}
\end{aligned}
\tag{4.20}
$$

The multiplets are given in Fig. 3-13.

The reduced multiplets of type R^4_{14} contain 36 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$\begin{aligned}
\chi^{\pm}_{00} &= \{\,(0, m_2, m_3, m_5, m_6, m_7)^{\pm}\,;\, \pm\tfrac{1}{2}m_{23,57}\,\}\\
\chi^{\pm}_{10} &= \{\,(0, m_{23}, 0, m_{3,5}, m_6, m_7)^{\pm}\,;\, \pm\tfrac{1}{2}m_{2,57}\,\}\\
\chi^{\pm}_{01} &= \{\,(0, m_2, m_{3,5}, 0, m_{56}, m_7)^{\pm}\,;\, \pm\tfrac{1}{2}m_{23,67}\,\}\\
\chi^{\pm}_{20} &= \{\,(m_2, m_3, 0, m_{23,5}, m_6, m_7)^{\pm}\,;\, \pm\tfrac{1}{2}m_{57}\,\}\\
\chi^{\pm}_{11} &= \{\,(0, m_{23}, m_5, m_3, m_{56}, m_7)^{\pm}\,;\, \pm\tfrac{1}{2}m_{2,67}\,\}\\
\chi^{\pm}_{02} &= \{\,(0, m_2, m_{3,56}, 0, m_5, m_{67})^{\pm}\,;\, \pm\tfrac{1}{2}m_{23,7}\,\}\\
\chi^{\pm}_{21} &= \{\,(m_2, m_3, m_5, m_{23}, m_{56}, m_7)^{\pm}\,;\, \pm\tfrac{1}{2}m_{67}\,\}\\
\chi^{\pm}_{12} &= \{\,(0, m_{23}, m_{56}, m_3, m_5, m_{67})^{\pm}\,;\, \pm\tfrac{1}{2}m_{2,7}\,\}\\
\chi^{\pm}_{03} &= \{\,(0, m_2, m_{3,57}, 0, m_5, m_6)^{\pm}\,;\, \pm\tfrac{1}{2}(m_{23} - m_7)\,\}\\
\chi^{\pm}_{22} &= \{\,(m_2, m_3, m_{56}, m_{23}, m_5, m_{67})^{\pm}\,;\, \pm\tfrac{1}{2}m_7\,\}\\
\chi^{\pm}_{13} &= \{\,(0, m_{23}, m_{57}, m_3, m_5, m_6)^{\pm}\,;\, \pm\tfrac{1}{2}(m_2 - m_7)\,\}\\
\chi^{\pm}_{23} &= \{\,(m_2, m_3, m_{57}, m_{23}, m_5, m_6)^{\pm}\,;\, \mp\tfrac{1}{2}m_7\,\}\\
\chi''^{\pm}_{02} &= \{\,(0, m_{23,5}, m_6, m_3, 0, m_{57})^{\pm}\,;\, \pm\tfrac{1}{2}m_{2,7}\,\}\\
\chi'^{\pm}_{30} &= \{\,(m_{23}, 0, m_5, m_2, m_{3,56}, m_7)^{\pm}\,;\, \pm\tfrac{1}{2}m_{67}\,\}\\
\chi'^{\pm}_{03} &= \{\,(0, m_{23,5}, m_{67}, m_3, 0, m_{56})^{\pm}\,;\, \pm\tfrac{1}{2}(m_2 - m_7)\,\}\\
\chi'^{\pm}_{40} &= \{\,(m_3, 0, m_5, 0, m_{23,56}, m_7)^{\pm}\,;\, \pm\tfrac{1}{2}(m_{67} - m_2)\,\}\\
\chi'^{\pm}_{31} &= \{\,(m_{23}, 0, m_{56}, m_2, m_{3,5}, m_{67})^{\pm}\,;\, \pm\tfrac{1}{2}m_7\,\}\\
\chi''^{\pm}_{22} &= \{\,(m_{23}, 0, m_{57}, m_2, m_{3,5}, m_6)^{\pm}\,;\, \mp\tfrac{1}{2}m_7\,\}
\end{aligned} \tag{4.21}$$

The multiplets are given in Fig. 3-14.

The reduced multiplets of type R^4_{15} contain 36 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$\begin{aligned}
\chi^{\pm}_{0} &= \{\,(0, m_2, m_3, 0, m_6, m_7)^{\pm}\,;\, \pm\tfrac{1}{2}(m_{\hat{a}} + m_4)\,\}\\
\chi^{\pm}_{01} &= \{\,(0, m_2, m_{34}, m_4, m_6, m_7)^{\pm}\,;\, \pm\tfrac{1}{2}m_{23,67}\,\}\\
\chi^{\pm}_{11} &= \{\,(0, m_{23}, m_4, m_{34}, m_6, m_7)^{\pm}\,;\, \pm\tfrac{1}{2}m_{2,67}\,\}\\
\chi^{\pm}_{02} &= \{\,(0, m_2, m_{34,6}, m_4, 0, m_{67})^{\pm}\,;\, \pm\tfrac{1}{2}m_{23,7}\,\}\\
\chi^{\pm}_{21} &= \{\,(m_2, m_3, m_4, m_{24}, m_6, m_7)^{\pm}\,;\, \pm\tfrac{1}{2}m_{67}\,\}\\
\chi^{\pm}_{12} &= \{\,(0, m_{23}, m_{4,6}, m_{34}, 0, m_{67})^{\pm}\,;\, \pm\tfrac{1}{2}m_{2,7}\,\}\\
\chi^{\pm}_{03} &= \{\,(0, m_2, m_{34,67}, m_4, 0, m_6)^{\pm}\,;\, \pm\tfrac{1}{2}(m_{23} - m_7)\,\}\\
\chi^{\pm}_{22} &= \{\,(m_2, m_3, m_{4,6}, m_{24}, 0, m_{67})^{\pm}\,;\, \pm\tfrac{1}{2}m_7\,\}\\
\chi^{\pm}_{13} &= \{\,(0, m_{23}, m_{4,67}, m_{34}, 0, m_6)^{\pm}\,;\, \pm\tfrac{1}{2}(m_2 - m_7)\,\}\\
\chi^{\pm}_{23} &= \{\,(m_2, m_3, m_{4,67}, m_{24}, 0, m_6)^{\pm}\,;\, \mp\tfrac{1}{2}m_7\,\}\\
\chi'^{\pm}_{00} &= \{\,(0, m_{24}, 0, m_3, m_{4,6}, m_7)^{\pm}\,;\, \pm\tfrac{1}{2}m_{2,67}\,\}\\
\chi'^{\pm}_{10} &= \{\,(m_2, m_{34}, 0, m_{23}, m_{4,6}, m_7)^{\pm}\,;\, \pm\tfrac{1}{2}m_{67}\,\}\\
\chi'^{\pm}_{01} &= \{\,(0, m_{24}, m_6, m_3, m_4, m_{67})^{\pm}\,;\, \pm\tfrac{1}{2}m_{2,7}\,\}\\
\chi'^{\pm}_{11} &= \{\,(m_2, m_{34}, m_6, m_{23}, m_4, m_{67})^{\pm}\,;\, \pm\tfrac{1}{2}m_7\,\}\\
\chi'^{\pm}_{02} &= \{\,(0, m_{24}, m_{67}, m_3, m_4, m_6)^{\pm}\,;\, \pm\tfrac{1}{2}(m_2 - m_7)\,\}\\
\chi'^{\pm}_{30} &= \{\,(m_{23}, m_4, 0, m_2, m_{34,6}, m_7)^{\pm}\,;\, \pm\tfrac{1}{2}m_{67}\,\}\\
\chi'^{\pm}_{40} &= \{\,(m_3, m_4, 0, 0, m_{26}, m_7)^{\pm}\,;\, \pm\tfrac{1}{2}(m_{67} - m_2)\,\}\\
\chi'^{\pm}_{31} &= \{\,(m_{23}, m_4, m_6, m_2, m_{34}, m_{67})^{\pm}\,;\, \pm\tfrac{1}{2}m_7\,\}
\end{aligned} \tag{4.22}$$

The multiplets are given in Fig. 3-15.

The reduced multiplets of type R^4_{16} contain 36 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$\begin{aligned}
\chi_0^{\pm} &= \{\,(0,m_2,m_3,m_5,0,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{\bar{a}}+m_4)\,\}\\
\chi_{00}^{\pm} &= \{\,(0,m_2,m_{34},m_{45},0,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{\bar{a}}-m_4)\,\}\\
\chi_{10}^{\pm} &= \{\,(0,m_{23},m_4,m_{35},0,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{2,5,7}\,\}\\
\chi_{20}^{\pm} &= \{\,(m_2,m_3,m_4,m_{25},0,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{5,7}\,\}\\
\chi_{02}^{\pm} &= \{\,(0,m_2,m_{35},m_4,m_5,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{23,7}\,\}\\
\chi_{12}^{\pm} &= \{\,(0,m_{23},m_{45},m_{34},m_5,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{2,7}\,\}\\
\chi_{03}^{\pm} &= \{\,(0,m_2,m_{37},m_4,m_5,0)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{23}-m_7)\,\}\\
\chi_{22}^{\pm} &= \{\,(m_2,m_3,m_{45},m_{24},m_5,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_7\,\}\\
\chi_{13}^{\pm} &= \{\,(0,m_{23},m_{47},m_{34},m_5,0)^{\pm}\,;\,\pm\tfrac{1}{2}(m_2-m_7)\,\}\\
\chi_{23}^{\pm} &= \{\,(m_2,m_3,m_{47},m_{24},m_5,0)^{\pm}\,;\,\mp\tfrac{1}{2}m_7\,\}\\
\chi_{01}^{\prime\pm} &= \{\,(0,m_{24},m_5,m_3,m_{45},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{2,7}\,\}\\
\chi_{11}^{\prime\pm} &= \{\,(m_2,m_{34},m_5,m_{23},m_{45},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_7\,\}\\
\chi_{02}^{\prime\pm} &= \{\,(0,m_{24},m_{5,7},m_3,m_{45},0)^{\pm}\,;\,\pm\tfrac{1}{2}(m_2-m_7)\,\}\\
\chi_{02}^{\prime\prime\pm} &= \{\,(0,m_{25},0,m_3,m_4,m_{5,7})^{\pm}\,;\,\pm\tfrac{1}{2}m_{2,7}\,\}\\
\chi_{12}^{\prime\pm} &= \{\,(m_2,m_{34},m_{5,7},m_{23},m_{45},0)^{\pm}\,;\,\mp\tfrac{1}{2}m_7\,\}\\
\chi_{03}^{\prime\pm} &= \{\,(0,m_{25},m_7,m_3,m_4,m_5)^{\pm}\,;\,\pm\tfrac{1}{2}(m_2-m_7)\,\}\\
\chi_{31}^{\prime\pm} &= \{\,(m_{23},m_4,m_5,m_2,m_{35},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_7\,\}\\
\chi_{22}^{\prime\prime\pm} &= \{\,(m_{23},m_4,m_{5,7},m_2,m_{35},0)^{\pm}\,;\,\mp\tfrac{1}{2}m_7\,\}
\end{aligned}\tag{4.23}$$

The multiplets are given in Fig. 3-16.

The reduced multiplets of type R^4_{17} contain 36 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$\begin{aligned}
\chi_0^{\pm} &= \{\,(0,m_2,m_3,m_5,m_6,0)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{\bar{a}}+m_4)\,\}\\
\chi_{00}^{\pm} &= \{\,(0,m_2,m_{34},m_{45},m_6,0)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{\bar{a}}-m_4)\,\}\\
\chi_{10}^{\pm} &= \{\,(0,m_{23},m_4,m_{35},m_6,0)^{\pm}\,;\,\pm\tfrac{1}{2}m_{2,56}\,\}\\
\chi_{01}^{\pm} &= \{\,(0,m_2,m_{35},m_4,m_{56},0)^{\pm}\,;\,\pm\tfrac{1}{2}m_{23,6}\,\}\\
\chi_{20}^{\pm} &= \{\,(m_2,m_3,m_4,m_{25},m_6,0)^{\pm}\,;\,\pm\tfrac{1}{2}m_{56}\,\}\\
\chi_{11}^{\pm} &= \{\,(0,m_{23},m_{45},m_{34},m_{56},0)^{\pm}\,;\,\pm\tfrac{1}{2}m_{2,6}\,\}\\
\chi_{02}^{\pm} &= \{\,(0,m_2,m_{36},m_4,m_5,m_6)^{\pm}\,;\,\pm\tfrac{1}{2}m_{23}\,\}\\
\chi_{21}^{\pm} &= \{\,(m_2,m_3,m_{45},m_{24},m_{56},0)^{\pm}\,;\,\pm\tfrac{1}{2}m_6\,\}\\
\chi_{12}^{\pm} &= \{\,(0,m_{23},m_{46},m_{34},m_5,m_6)^{\pm}\,;\,\pm\tfrac{1}{2}m_2\,\}\\
\chi_{22}^{\pm} &= \{\,(m_2,m_3,m_{46},m_{24},m_5,m_6)^{\pm}\,;\,0\,\}\\
\chi_{00}^{\prime\pm} &= \{\,(0,m_{24},m_5,m_3,m_{46},0)^{\pm}\,;\,\pm\tfrac{1}{2}m_{2,6}\,\}\\
\chi_{10}^{\prime\pm} &= \{\,(m_2,m_{34},m_5,m_{23},m_{46},0)^{\pm}\,;\,\pm\tfrac{1}{2}m_6\,\}\\
\chi_{01}^{\prime\pm} &= \{\,(0,m_{24},m_{56},m_3,m_{45},m_6)^{\pm}\,;\,\pm\tfrac{1}{2}m_2\,\}\\
\chi_{11}^{\prime\pm} &= \{\,(m_2,m_{34},m_{56},m_{23},m_{45},m_6)^{\pm}\,;\,0\,\}\\
\chi_{30}^{\prime\pm} &= \{\,(m_{23},m_4,m_5,m_2,m_{36},0)^{\pm}\,;\,\pm\tfrac{1}{2}m_6\,\}\\
\chi_{03}^{\prime\pm} &= \{\,(0,m_{25},m_6,m_3,m_4,m_{56})^{\pm}\,;\,\pm\tfrac{1}{2}m_2\,\}\\
\chi_{40}^{\prime\pm} &= \{\,(m_3,m_4,m_5,0,m_{26},0)^{\pm}\,;\,\pm\tfrac{1}{2}(m_6-m_2)\,\}\\
\chi_{31}^{\prime\pm} &= \{\,(m_{23},m_4,m_{56},m_2,m_{35},m_6)^{\pm}\,;\,0\,\}
\end{aligned}\tag{4.24}$$

The multiplets are given in Fig. 3-17.

The reduced multiplets of type R^4_{24} contain 36 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$
\begin{aligned}
\chi_0^{\pm} &= \{\,(m_1,0,m_3,m_5,m_6,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{\bar{6}}\,\}\\
\chi_{01}^{\pm} &= \{\,(m_1,0,m_{3,5},0,m_{56},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,3,67}\,\}\\
\chi_{20}^{\pm} &= \{\,(m_1,m_3,0,m_{3,5},m_6,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,57}\,\}\\
\chi_{02}^{\pm} &= \{\,(m_1,0,m_{3,56},0,m_5,m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,3,7}\,\}\\
\chi_{30}^{\pm} &= \{\,(0,m_3,0,m_{1,3,5},m_6,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{57}-m_1)\,\}\\
\chi_{21}^{\pm} &= \{\,(m_1,m_3,m_5,m_3,m_{56},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,67}\,\}\\
\chi_{03}^{\pm} &= \{\,(m_1,0,m_{3,57},0,m_5,m_6)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{1,3}-m_7)\,\}\\
\chi_{31}^{\pm} &= \{\,(0,m_3,m_5,m_{1,3},m_{56},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{67}-m_1)\,\}\\
\chi_{22}^{\pm} &= \{\,(m_1,m_3,m_{56},m_3,m_5,m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,7}\,\}\\
\chi_{32}^{\pm} &= \{\,(0,m_3,m_{56},m_{1,3},m_5,m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}(m_7-m_1)\,\}\\
\chi_{23}^{\pm} &= \{\,(m_1,m_3,m_{57},m_3,m_5,m_6)^{\pm}\,;\,\pm\tfrac{1}{2}(m_1-m_7)\,\}\\
\chi_{33}^{\pm} &= \{\,(0,m_3,m_{57},m_{1,3},m_5,m_6)^{\pm}\,;\,\mp\tfrac{1}{2}(m_{1,7})\,\}\\
\chi_{20}^{\prime\prime\pm} &= \{\,(m_{1,3},0,m_5,0,m_{3,56},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,67}\,\}\\
\chi_{21}^{\prime\prime\pm} &= \{\,(m_{1,3},0,m_{56},0,m_{3,5},m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,7}\,\}\\
\chi_{12}^{\prime\prime\pm} &= \{\,(m_1,m_{3,5},m_6,m_3,0,m_{57})^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,7}\,\}\\
\chi_{30}^{\prime\pm} &= \{\,(m_3,0,m_5,m_1,m_{3,56},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{67}-m_1)\,\}\\
\chi_{31}^{\prime\pm} &= \{\,(m_3,0,m_{56},m_1,m_{35},m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}(m_7-m_1)\,\}\\
\chi_{22}^{\prime\prime\pm} &= \{\,(m_{1,3},0,m_{57},0,m_{3,5},m_6)^{\pm}\,;\,\pm\tfrac{1}{2}(m_1-m_7)\,\}
\end{aligned}
\tag{4.25}
$$

The multiplets are given in Fig. 3-24.

The reduced multiplets of type R^4_{25} contain 36 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$
\begin{aligned}
\chi_0^{\pm} &= \{\,(m_1,0,m_3,0,m_6,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{1,3,67}+2m_4)\,\}\\
\chi_{01}^{\pm} &= \{\,(m_1,0,m_{34},m_4,m_6,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,3,67}\,\}\\
\chi_{02}^{\pm} &= \{\,(m_1,0,m_{34,6},m_4,0,m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,3,7}\,\}\\
\chi_{21}^{\pm} &= \{\,(m_1,m_3,m_4,m_{34},m_6,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,67}\,\}\\
\chi_{03}^{\pm} &= \{\,(m_1,0,m_{34,67},m_4,0,m_6)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{1,3}-m_7)\,\}\\
\chi_{31}^{\pm} &= \{\,(0,m_3,m_4,m_{14},m_6,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{67}-m_1)\,\}\\
\chi_{22}^{\pm} &= \{\,(m_1,m_3,m_{4,6},m_{34},0,m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,7}\,\}\\
\chi_{32}^{\pm} &= \{\,(0,m_3,m_{4,6},m_{1,34},0,m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}(m_7-m_1)\,\}\\
\chi_{23}^{\pm} &= \{\,(m_1,m_3,m_{4,67},m_{34},0,m_6)^{\pm}\,;\,\pm\tfrac{1}{2}(m_1-m_7)\,\}\\
\chi_{33}^{\pm} &= \{\,(0,m_3,m_{4,67},m_{1,34},0,m_6)^{\pm}\,;\,\mp\tfrac{1}{2}(m_{1,7})\,\}\\
\chi_{10}^{\prime\pm} &= \{\,(m_1,m_{34},0,m_3,m_{4,6},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,67}\,\}\\
\chi_{20}^{\prime\pm} &= \{\,(0,m_{34},0,m_{1,3},m_{4,6},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{67}-m_1)\,\}\\
\chi_{11}^{\prime\pm} &= \{\,(m_1,m_{34},m_6,m_3,m_4,m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,7}\,\}\\
\chi_{20}^{\prime\prime\pm} &= \{\,(m_{1,3},m_4,0,0,m_{34,6},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,67}\,\}\\
\chi_{21}^{\prime\prime\pm} &= \{\,(m_{1,3},m_4,m_6,0,m_{34},m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,7}\,\}\\
\chi_{30}^{\prime\pm} &= \{\,(m_3,m_4,0,m_1,m_{34,6},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{67}-m_1)\,\}\\
\chi_{21}^{\prime\pm} &= \{\,(0,m_{34},m_6,m_{1,3},m_4,m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}(m_7-m_1)\,\}\\
\chi_{31}^{\prime\pm} &= \{\,(m_3,m_4,m_6,m_1,m_{34},m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}(m_7-m_1)\,\}
\end{aligned}
\tag{4.26}
$$

The multiplets are given in Fig. 3-25.

The reduced multiplets of type R^4_{26} contain 36 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$\begin{aligned}
\chi^{\pm}_{0} &= \{\,(m_1,0,m_3,m_5,0,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{\bar{a}}+m_4)\,\}\\
\chi^{\pm}_{00} &= \{\,(m_1,0,m_{34},m_{45},0,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{\bar{a}}-m_4)\,\}\\
\chi^{\pm}_{20} &= \{\,(m_1,m_3,m_4,m_{35},0,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,5,7}\,\}\\
\chi^{\pm}_{02} &= \{\,(m_1,0,m_{35},m_4,m_5,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,3,7}\,\}\\
\chi^{\pm}_{30} &= \{\,(0,m_3,m_4,m_{1,35},0,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{5,7}-m_1)\,\}\\
\chi^{\pm}_{03} &= \{\,(m_1,0,m_{35,7},m_4,m_5,0)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{1,3}-m_7)\,\}\\
\chi^{\pm}_{22} &= \{\,(m_1,m_3,m_{45},m_{34},m_5,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,7}\,\}\\
\chi^{\pm}_{32} &= \{\,(0,m_3,m_{45},m_{1,34},m_5,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_7-m_1)\,\}\\
\chi^{\pm}_{23} &= \{\,(m_1,m_3,m_{45,7},m_{34},m_5,0)^{\pm}\,;\,\pm\tfrac{1}{2}(m_1-m_7)\,\}\\
\chi^{\pm}_{33} &= \{\,(0,m_3,m_{45,7},m_{1,34},m_5,0)^{\pm}\,;\,\mp\tfrac{1}{2}(m_{1,7})\,\}\\
\chi'^{\pm}_{11} &= \{\,(m_1,m_{34},m_5,m_3,m_{45},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,7}\,\}\\
\chi''^{\pm}_{21} &= \{\,(m_{1,3},m_4,m_5,0,m_{35},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,7}\,\}\\
\chi''^{\pm}_{12} &= \{\,(m_1,m_{35},0,m_3,m_4,m_{5,7})^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,7}\,\}\\
\chi'^{\pm}_{21} &= \{\,(0,m_{34},m_5,m_{1,3},m_{45},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_7-m_1)\,\}\\
\chi'^{\pm}_{12} &= \{\,(m_1,m_{34},m_{5,7},m_3,m_{45},0)^{\pm}\,;\,\pm\tfrac{1}{2}(m_1-m_7)\,\}\\
\chi'^{\pm}_{31} &= \{\,(m_3,m_4,m_5,m_1,m_{35},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_7-m_1)\,\}\\
\chi'^{\pm}_{22} &= \{\,(0,m_{34},m_{5,7},m_{1,3},m_{45},0)^{\pm}\,;\,\mp\tfrac{1}{2}m_{1,7}\,\}\\
\chi''^{\pm}_{22} &= \{\,(m_{1,3},m_4,m_{5,7},0,m_{35},0)^{\pm}\,;\,\pm\tfrac{1}{2}(m_1-m_7)\,\}
\end{aligned}\tag{4.27}$$

The multiplets are given in Fig. 3-26.

The reduced multiplets of type R^4_{35} contain 36 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$\begin{aligned}
\chi^{\pm}_{0} &= \{\,(m_1,m_2,0,0,m_6,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{\bar{a}}+m_4)\,\}\\
\chi^{\pm}_{11} &= \{\,(m_1,m_2,m_4,m_4,m_6,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{12,67}\,\}\\
\chi^{\pm}_{21} &= \{\,(m_{12},0,m_4,m_{2,4},m_6,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,67}\,\}\\
\chi^{\pm}_{12} &= \{\,(m_1,m_2,m_{4,6},m_4,0,m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}m_{12,7}\,\}\\
\chi^{\pm}_{31} &= \{\,(m_2,0,m_4,m_{12,4},m_6,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{67}-m_1)\,\}\\
\chi^{\pm}_{22} &= \{\,(m_{12},0,m_{4,6},m_{2,4},0,m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,7}\,\}\\
\chi^{\pm}_{13} &= \{\,(m_1,m_2,m_{4,67},m_4,0,m_6)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{12}-m_7)\,\}\\
\chi^{\pm}_{32} &= \{\,(m_2,0,m_{4,6},m_{12,4},0,m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}(m_7-m_1)\,\}\\
\chi^{\pm}_{23} &= \{\,(m_{12},0,m_{4,67},m_{2,4},0,m_6)^{\pm}\,;\,\pm\tfrac{1}{2}(m_1-m_7)\,\}\\
\chi^{\pm}_{33} &= \{\,(m_2,0,m_{4,67},m_{12,4},0,m_6)^{\pm}\,;\,\mp\tfrac{1}{2}(m_{1,7})\,\}\\
\chi'^{\pm}_{00} &= \{\,(m_1,m_{2,4},0,0,m_{4,6},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{12,67}\,\}\\
\chi'^{\pm}_{10} &= \{\,(m_{12},m_4,0,m_2,m_{4,6},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,67}\,\}\\
\chi'^{\pm}_{01} &= \{\,(m_1,m_{2,4},m_6,0,m_4,m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}m_{12,7}\,\}\\
\chi'^{\pm}_{20} &= \{\,(m_2,m_4,0,m_{12},m_{4,6},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{67}-m_1)\,\}\\
\chi'^{\pm}_{11} &= \{\,(m_{12},m_4,m_6,m_2,m_4,m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,7}\,\}\\
\chi'^{\pm}_{21} &= \{\,(m_2,m_4,m_6,m_{12},m_4,m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}(m_7-m_1)\,\}\\
\chi'^{\pm}_{03} &= \{\,(m_1,m_{2,4},m_{67},0,m_4,m_6)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{12}-m_7)\,\}\\
\chi'^{\pm}_{40} &= \{\,(0,m_4,0,m_1,m_{2,4,6},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{67}-m_{12})\,\}
\end{aligned}\tag{4.28}$$

The multiplets are given in Fig. 3-35.

4.4 Yet further reduction of multiplets

There are further reductions of the multiplets denoted by R^4_{abc}, $a,b,c = 1,\ldots,7$, $a<b<c$, which may be obtained from the main multiplet by setting formally $m_a = m_b = m_c = 0$. From these reductions only six are non nonconjugate and contain representations of physical interest.

The reduced multiplets of type R^4_{135} contain 26 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$\begin{aligned}
\chi^{\pm}_{0} &= \{\,(0,m_2,0,0,m_6,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{\bar{a}}+m_4)\,\}\\
\chi^{\pm}_{11} &= \{\,(0,m_2,m_4,m_4,m_6,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{2,67}\,\}\\
\chi^{\pm}_{21} &= \{\,(m_2,0,m_4,m_{2,4},m_6,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{67}\,\}\\
\chi^{\pm}_{12} &= \{\,(0,m_2,m_{4,6},m_4,0,m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}m_{2,7}\,\}\\
\chi^{\pm}_{22} &= \{\,(m_2,0,m_{4,6},m_{2,4},0,m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}m_{7}\,\}\\
\chi^{\pm}_{13} &= \{\,(0,m_2,m_{4,67},m_4,0,m_6)^{\pm}\,;\,\pm\tfrac{1}{2}(m_2-m_7)\,\}\\
\chi^{\pm}_{23} &= \{\,(m_2,0,m_{4,67},m_{2,4},0,m_6)^{\pm}\,;\,\mp\tfrac{1}{2}m_{7}\,\}\\
\chi'^{\pm}_{00} &= \{\,(0,m_{2,4},0,0,m_{4,6},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{2,67}\,\}\\
\chi'^{\pm}_{01} &= \{\,(0,m_{2,4},m_6,0,m_4,m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}m_{2,7}\,\}\\
\chi'^{\pm}_{02} &= \{\,(0,m_{2,4},m_{67},0,m_4,m_6)^{\pm}\,;\,\pm\tfrac{1}{2}(m_2-m_7)\,\}\\
\chi'^{\pm}_{30} &= \{\,(m_2,m_4,0,m_2,m_{4,6},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{67}\,\}\\
\chi'^{\pm}_{40} &= \{\,(0,m_4,0,0,m_{26},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{67}-m_2)\,\}\\
\chi'^{\pm}_{31} &= \{\,(m_2,m_4,m_6,m_2,m_4,m_{67})^{\pm}\,;\,\pm\tfrac{1}{2}m_{7}\,\}
\end{aligned}\tag{4.29}$$

The multiplets are given in Fig. 3-135.

The reduced multiplets of type R^4_{136} contain 26 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$\begin{aligned}
\chi^{\pm}_{0} &= \{\,(0,m_2,0,m_5,0,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{\bar{a}}+m_4)\,\}\\
\chi^{\pm}_{10} &= \{\,(0,m_2,m_4,m_{45},0,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{2,5,7}\,\}\\
\chi^{\pm}_{20} &= \{\,(m_2,0,m_4,m_{2,45},0,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{5,7}\,\}\\
\chi^{\pm}_{12} &= \{\,(0,m_2,m_{45},m_4,m_5,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{2,7}\,\}\\
\chi^{\pm}_{22} &= \{\,(m_2,0,m_{45},m_{2,4},m_5,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{7}\,\}\\
\chi^{\pm}_{13} &= \{\,(0,m_2,m_{45,7},m_4,m_5,0)^{\pm}\,;\,\pm\tfrac{1}{2}(m_2-m_7)\,\}\\
\chi^{\pm}_{23} &= \{\,(m_2,0,m_{45,7},m_{2,4},m_5,0)^{\pm}\,;\,\mp\tfrac{1}{2}m_{7}\,\}\\
\chi'^{\pm}_{01} &= \{\,(0,m_{2,4},m_5,0,m_{45},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{2,7}\,\}\\
\chi'^{\pm}_{11} &= \{\,(m_2,m_4,m_5,m_2,m_{45},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{7}\,\}\\
\chi'^{\pm}_{02} &= \{\,(0,m_{2,4},m_{5,7},0,m_{45},0)^{\pm}\,;\,\pm\tfrac{1}{2}(m_2-m_7)\,\}\\
\chi''^{\pm}_{02} &= \{\,(0,m_{2,45},0,0,m_4,m_{5,7})^{\pm}\,;\,\pm\tfrac{1}{2}m_{2,7}\,\}\\
\chi'^{\pm}_{12} &= \{\,(m_2,m_4,m_{5,7},m_2,m_{45},0)^{\pm}\,;\,\mp\tfrac{1}{2}m_{7}\,\}\\
\chi'^{\pm}_{03} &= \{\,(0,m_{2,45},m_7,0,m_4,m_5)^{\pm}\,;\,\pm\tfrac{1}{2}(m_2-m_7)\,\}
\end{aligned}\tag{4.30}$$

The multiplets are given in Fig. 3-136.

The reduced multiplets of type R^4_{137} contain 26 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$\begin{aligned}
\chi^{\pm}_{0} &= \{\,(0,m_2,0,m_5,m_6,0)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{\bar{a}}+m_4)\,\}\\
\chi^{\pm}_{10} &= \{\,(0,m_2,m_4,m_{45},m_6,0)^{\pm}\,;\,\pm\tfrac{1}{2}m_{2,56}\,\}
\end{aligned}$$

$$
\begin{aligned}
\chi^{\pm}_{20} &= \{\,(m_2,0,m_4,m_{2,45},m_6,0)^{\pm}\,;\,\pm\tfrac{1}{2}m_{56}\,\}\\
\chi^{\pm}_{11} &= \{\,(0,m_2,m_{45},m_4,m_{56},0)^{\pm}\,;\,\pm\tfrac{1}{2}m_{2,6}\,\}\\
\chi^{\pm}_{21} &= \{\,(m_2,0,m_{45},m_{2,4},m_{56},0)^{\pm}\,;\,\pm\tfrac{1}{2}m_{6}\,\}\\
\chi^{\pm}_{12} &= \{\,(0,m_2,m_{46},m_4,m_5,m_6)^{\pm}\,;\,\pm\tfrac{1}{2}m_{2}\,\}\\
\chi^{\pm}_{22} &= \{\,(m_2,0,m_{46},m_{2,4},m_5,m_6)^{\pm}\,;\,0\,\}\\
\chi'^{\pm}_{00} &= \{\,(0,m_{2,4},m_5,0,m_{46},0)^{\pm}\,;\,\pm\tfrac{1}{2}m_{2,6}\,\}\\
\chi'^{\pm}_{01} &= \{\,(0,m_{2,4},m_{56},0,m_{45},m_6)^{\pm}\,;\,\pm\tfrac{1}{2}m_{2}\,\}\\
\chi'^{\pm}_{30} &= \{\,(m_2,m_4,m_5,m_2,m_{45},0)^{\pm}\,;\,\pm\tfrac{1}{2}m_{6}\,\}\\
\chi'^{\pm}_{03} &= \{\,(0,m_{2,45},m_6,0,m_4,m_{56})^{\pm}\,;\,\pm\tfrac{1}{2}m_{2}\,\}\\
\chi'^{\pm}_{40} &= \{\,(0,m_4,m_5,0,m_{2,46},0)^{\pm}\,;\,\pm\tfrac{1}{2}(m_6-m_2)\,\}\\
\chi'^{\pm}_{31} &= \{\,(m_2,m_4,m_{56},m_2,m_{45},m_6)^{\pm}\,;\,0\,\}
\end{aligned}
\tag{4.31}
$$

The multiplets are given in Fig. 3-137.

The reduced multiplets of type R^4_{147} contain 26 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$
\begin{aligned}
\chi^{\pm}_{00} &= \{\,(0,m_2,m_3,m_5,m_6,0)^{\pm}\,;\,\pm\tfrac{1}{2}m_{23,56}\,\}\\
\chi^{\pm}_{10} &= \{\,(0,m_{23},0,m_{3,5},m_6,0)^{\pm}\,;\,\pm\tfrac{1}{2}m_{2,56}\,\}\\
\chi^{\pm}_{01} &= \{\,(0,m_2,m_{3,5},0,m_{56},0)^{\pm}\,;\,\pm\tfrac{1}{2}m_{23,6}\,\}\\
\chi^{\pm}_{20} &= \{\,(m_2,m_3,0,m_{23,5},m_6,0)^{\pm}\,;\,\pm\tfrac{1}{2}m_{56}\,\}\\
\chi^{\pm}_{11} &= \{\,(0,m_{23},m_5,m_3,m_{56},0)^{\pm}\,;\,\pm\tfrac{1}{2}m_{2,6}\,\}\\
\chi^{\pm}_{02} &= \{\,(0,m_2,m_{3,56},0,m_5,m_6)^{\pm}\,;\,\pm\tfrac{1}{2}m_{23}\,\}\\
\chi^{\pm}_{21} &= \{\,(m_2,m_3,m_5,m_{23},m_{56},0)^{\pm}\,;\,\pm\tfrac{1}{2}m_{6}\,\}\\
\chi^{\pm}_{12} &= \{\,(0,m_{23},m_{56},m_3,m_5,m_6)^{\pm}\,;\,\pm\tfrac{1}{2}m_{2}\,\}\\
\chi^{\pm}_{22} &= \{\,(m_2,m_3,m_{56},m_{23},m_5,m_6)^{\pm}\,;\,0\,\}\\
\chi'^{\pm}_{30} &= \{\,(m_{23},0,m_5,m_2,m_{3,56},0)^{\pm}\,;\,\pm\tfrac{1}{2}m_{6}\,\}\\
\chi'^{\pm}_{03} &= \{\,(0,m_{23,5},m_6,m_3,0,m_{56})^{\pm}\,;\,\pm\tfrac{1}{2}m_{2}\,\}\\
\chi'^{\pm}_{40} &= \{\,(m_3,0,m_5,0,m_{23,56},0)^{\pm}\,;\,\pm\tfrac{1}{2}(m_6-m_2)\,\}\\
\chi'^{\pm}_{31} &= \{\,(m_{23},0,m_{56},m_2,m_{3,5},m_6)^{\pm}\,;\,0\,\}
\end{aligned}
\tag{4.32}
$$

The multiplets are given in Fig. 3-147.

The reduced multiplets of type R^4_{246} contain 26 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$
\begin{aligned}
\chi^{\pm}_{0} &= \{\,(m_1,0,m_3,m_5,0,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,3,5,7}\,\}\\
\chi^{\pm}_{20} &= \{\,(m_1,m_3,0,m_{3,5},0,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,5,7}\,\}\\
\chi^{\pm}_{02} &= \{\,(m_1,0,m_{3,5},0,m_5,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,3,7}\,\}\\
\chi^{\pm}_{30} &= \{\,(0,m_3,0,m_{1,3,5},0,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{5,7}-m_1)\,\}\\
\chi^{\pm}_{03} &= \{\,(m_1,0,m_{3,5,7},0,m_5,0)^{\pm}\,;\,\pm\tfrac{1}{2}(m_{1,3}-m_7)\,\}\\
\chi^{\pm}_{22} &= \{\,(m_1,m_3,m_5,m_3,m_5,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,7}\,\}\\
\chi^{\pm}_{32} &= \{\,(0,m_3,m_5,m_{1,3},m_5,m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_7-m_1)\,\}\\
\chi^{\pm}_{23} &= \{\,(m_1,m_3,m_{5,7},m_3,m_5,0)^{\pm}\,;\,\pm\tfrac{1}{2}(m_1-m_7)\,\}\\
\chi^{\pm}_{33} &= \{\,(0,m_3,m_{5,7},m_{1,3},m_5,0)^{\pm}\,;\,\mp\tfrac{1}{2}(m_{1,7})\,\}\\
\chi''^{\pm}_{21} &= \{\,(m_{1,3},0,m_5,0,m_{3,5},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,7}\,\}\\
\chi''^{\pm}_{12} &= \{\,(m_1,m_{3,5},0,m_3,0,m_{5,7})^{\pm}\,;\,\pm\tfrac{1}{2}m_{1,7}\,\}\\
\chi'^{\pm}_{31} &= \{\,(m_3,0,m_5,m_1,m_{3,5},m_7)^{\pm}\,;\,\pm\tfrac{1}{2}(m_7-m_1)\,\}
\end{aligned}
$$

$$\chi''^{\pm}_{22} = \{ (m_{1,3}, 0, m_{5,7}, 0, m_{3,5}, 0)^{\pm} ; \pm\tfrac{1}{2}(m_1 - m_7) \} \tag{4.33}$$

The multiplets are given in Fig. 3-246.

4.5 Last reduction of multiplets

There are further reductions of the multiplets - quadruple, etc., but only one quadruple reduction contains representations of physical interest. Namely, this is the multiplet R^4_{1357}, which may be obtained from the main multiplet by setting formally $m_1 = m_3 = m_5 = m_7 = 0$.

The reduced multiplets of type R^4_{1357} contain 19 ERs/GVMs whose signatures can be given in the following manner:

$$\begin{aligned}
\chi^{\pm}_0 &= \{ (0, m_2, 0, 0, m_6, 0)^{\pm} ; \pm\tfrac{1}{2}(m_6 + m_4) \} \\
\chi^{\pm}_{11} &= \{ (0, m_2, m_4, m_4, m_6, 0)^{\pm} ; \pm\tfrac{1}{2} m_{2,6} \} \\
\chi^{\pm}_{21} &= \{ (m_2, 0, m_4, m_{2,4}, m_6, 0)^{\pm} ; \pm\tfrac{1}{2} m_6 \} \\
\chi^{\pm}_{12} &= \{ (0, m_2, m_{46}, m_4, 0, m_6)^{\pm} ; \pm\tfrac{1}{2} m_2 \} \\
\chi^{\pm}_{22} &= \{ (m_2, 0, m_{46}, m_{2,4}, 0, m_6)^{\pm} ; 0 \} \\
\chi'^{\pm}_{00} &= \{ (0, m_{2,4}, 0, 0, m_{46}, 0)^{\pm} ; \pm\tfrac{1}{2} m_{2,6} \} \\
\chi'^{\pm}_{01} &= \{ (0, m_{2,4}, m_6, 0, m_4, m_6)^{\pm} ; \pm\tfrac{1}{2} m_2 \} \\
\chi'^{\pm}_{30} &= \{ (m_2, m_4, 0, m_2, m_{46}, 0)^{\pm} ; \pm\tfrac{1}{2} m_6 \} \\
\chi'^{\pm}_{40} &= \{ (0, m_4, 0, 0, m_{2,46}, 0)^{\pm} ; \pm\tfrac{1}{2}(m_6 - m_2) \} \\
\chi_{31} &= \{ (m_2, m_4, m_6, m_2, m_4, m_6) ; 0 \}
\end{aligned} \tag{4.34}$$

The multiplets are given in Fig. 3-1357. Note that the ER χ_{31} is not in a pair and is placed in the middle of the figure as the bullet. That ER contains the *minimal irreps* of $SU(4,4)$ characterized by three positive integers which are denoted in this context as m_2, m_4, m_6. Each such irrep is the kernel of the three invariant differential operators $\mathcal{D}^{m_2}_{15}$, $\mathcal{D}^{m_4}_{26}$, $\mathcal{D}^{m_6}_{37}$, which are of order m_2, m_4, m_6, resp., and correspond to the noncompact roots α_{15}, α_{26}, α_{37}, resp., cf. (2.5).

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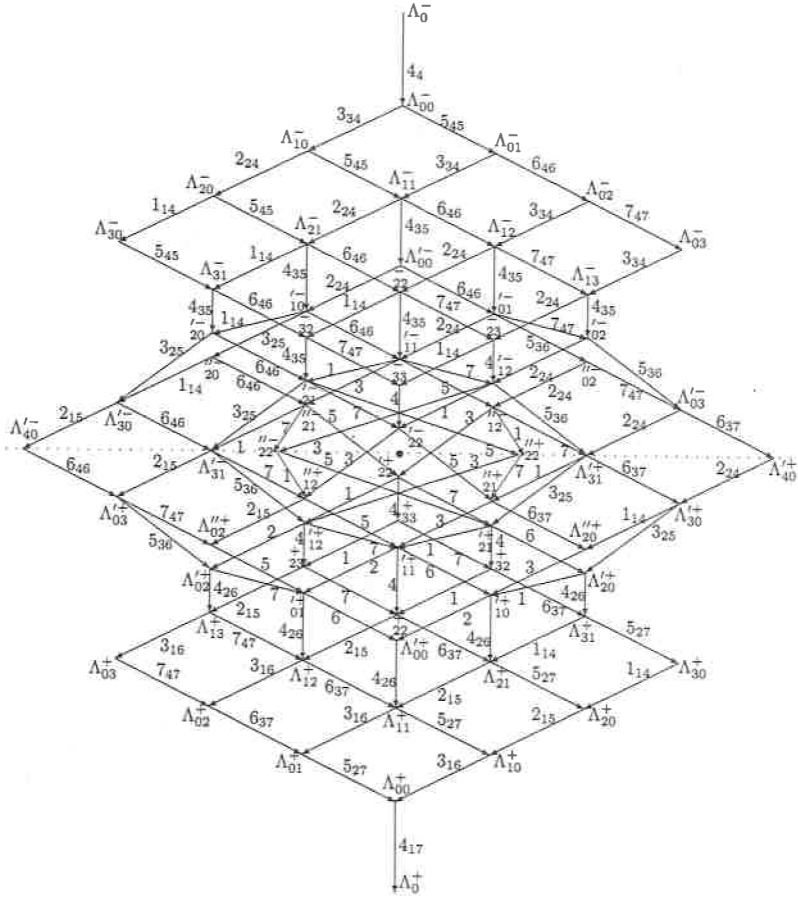

Fig. 1. Main multiplets for $SU(4,4)$

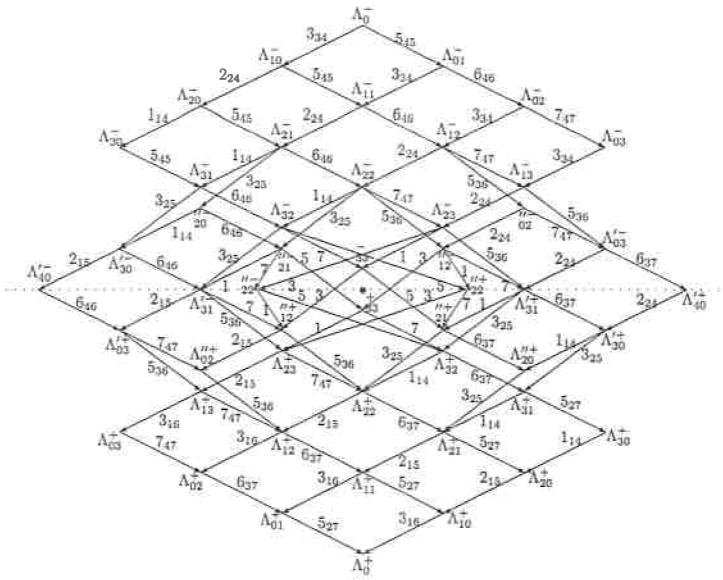

Fig. 2a. $SU(4,4)$ reduced multiplets of type R_4^1

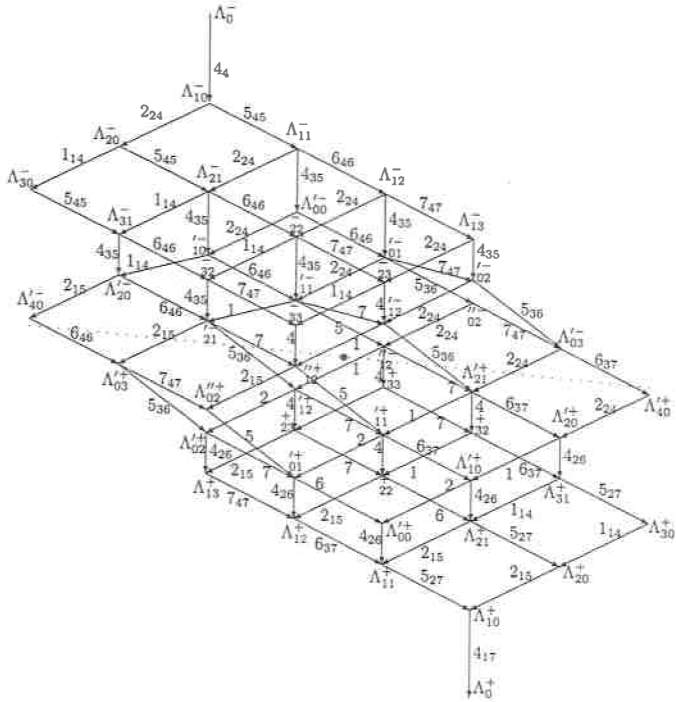

Fig. 2b. $SU(4,4)$ reduced multiplets of type R_3^4

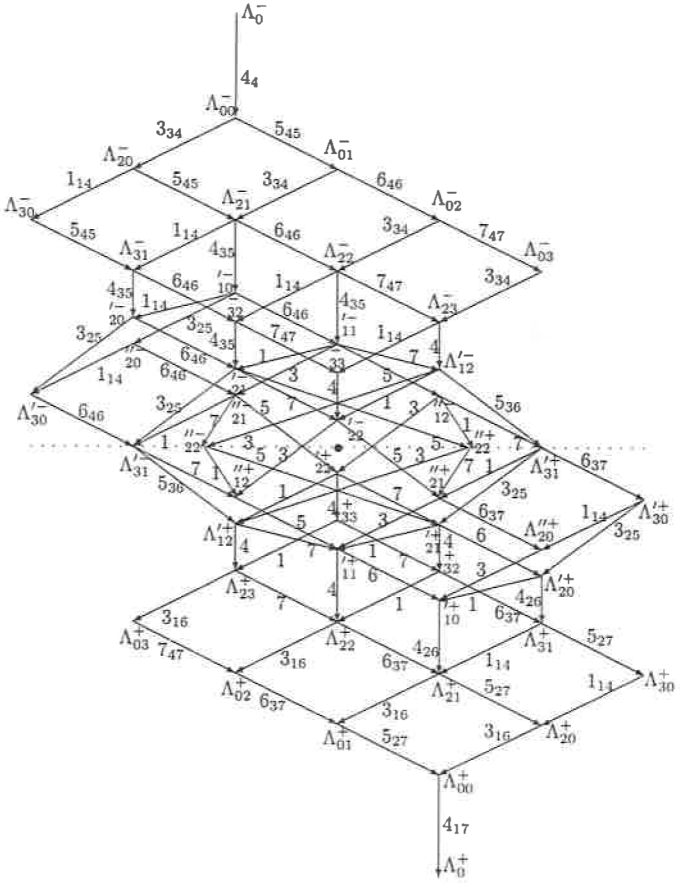

Fig. 2c. $SU(4,4)$ reduced multiplets of type R_2^4

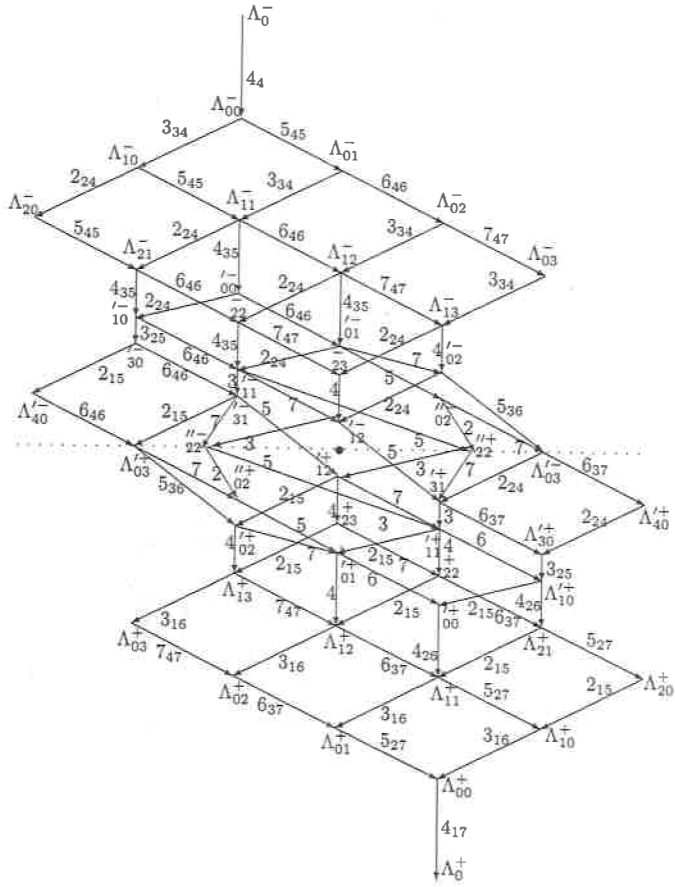

Fig. 2d. $SU(4,4)$ reduced multiplets of type R_1^4

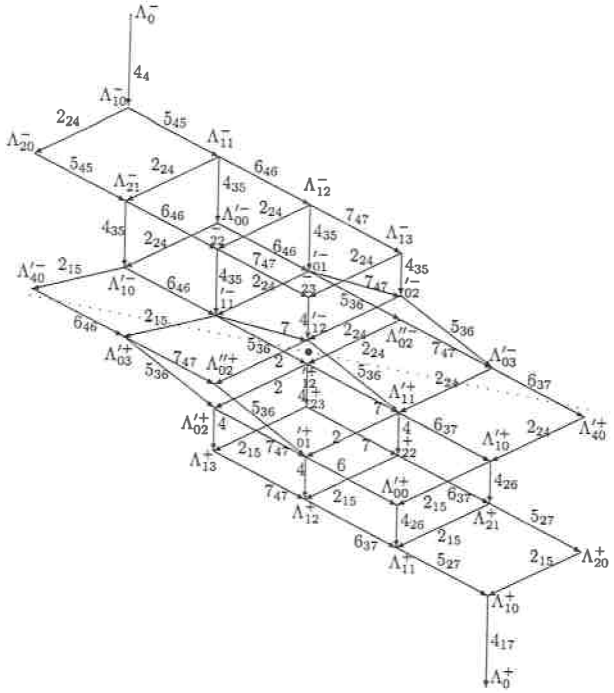

Fig. 3-13. $SU(4,4)$ reduced multiplets of type R^4_{13}

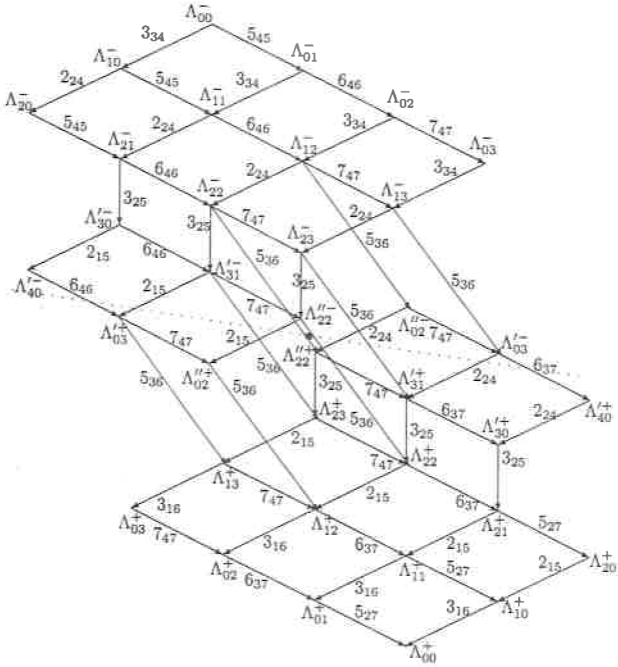

Fig. 3-14. $SU(4,4)$ reduced multiplets of type R^4_{14}

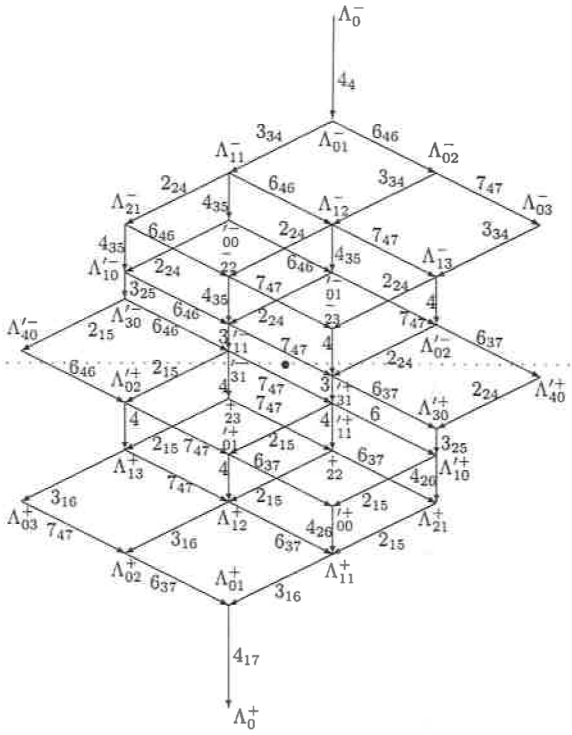

Fig. 3-15. $SU(4,4)$ reduced multiplets of type R^4_{15}

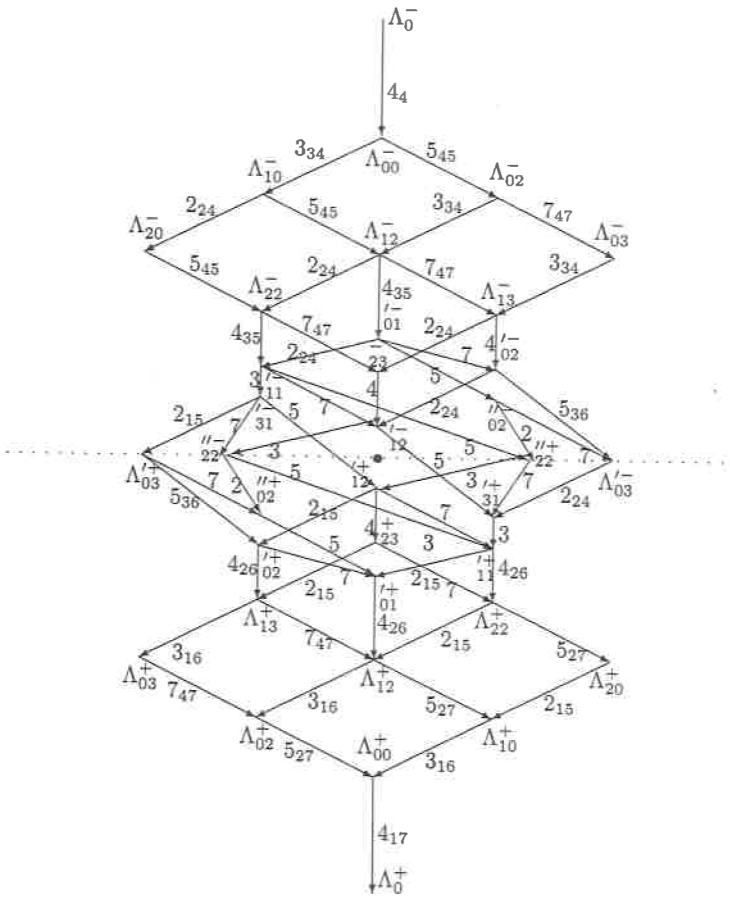

Fig. 3-16. $SU(4,4)$ reduced multiplets of type R^4_{16}

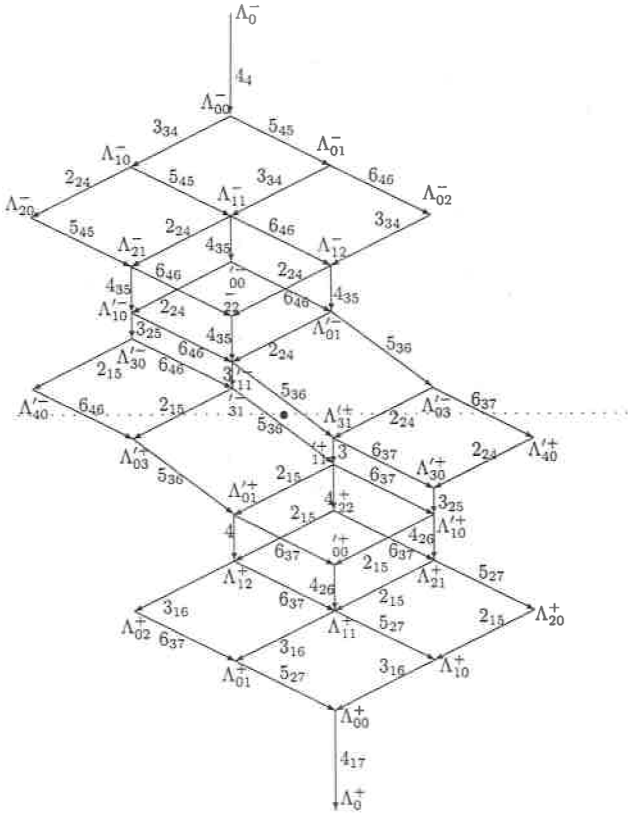

Fig. 3-17. $SU(4,4)$ reduced multiplets of type R^4_{17}

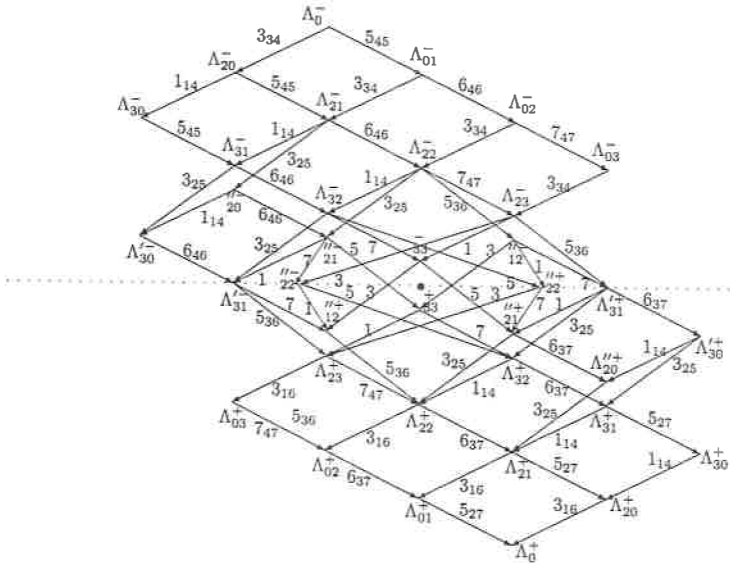

Fig. 3-24. $SU(4,4)$ reduced multiplets of type R^A_{24}

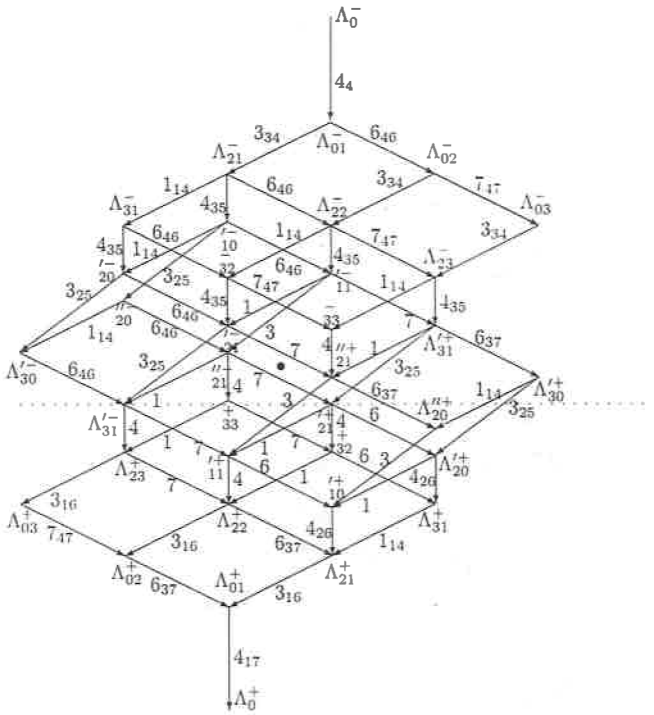

Fig. 3-25. $SU(4,4)$ reduced multiplets of type R^4_{25}

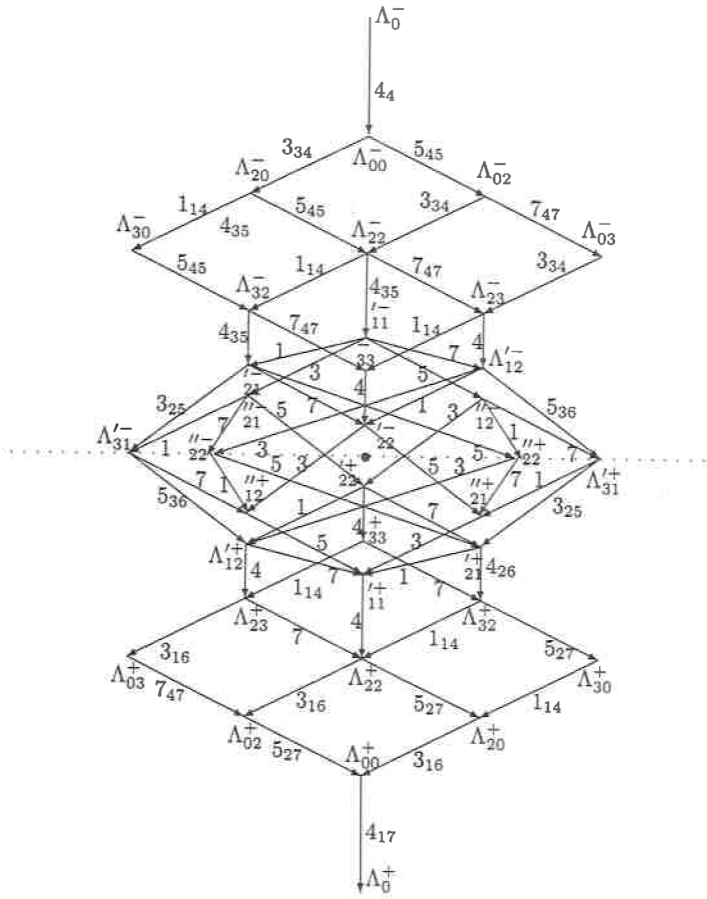

Fig. 3-26. $SU(4,4)$ reduced multiplets of type R^4_{26}

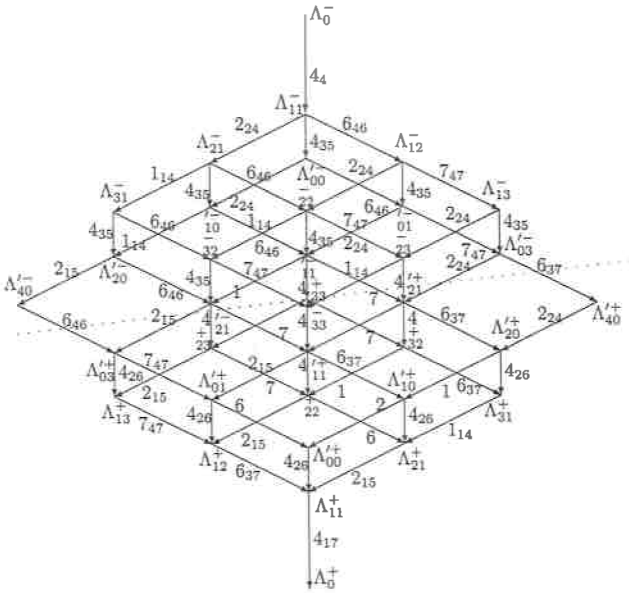

Fig. 3-35. $SU(4,4)$ reduced multiplets of type R^4_{35}

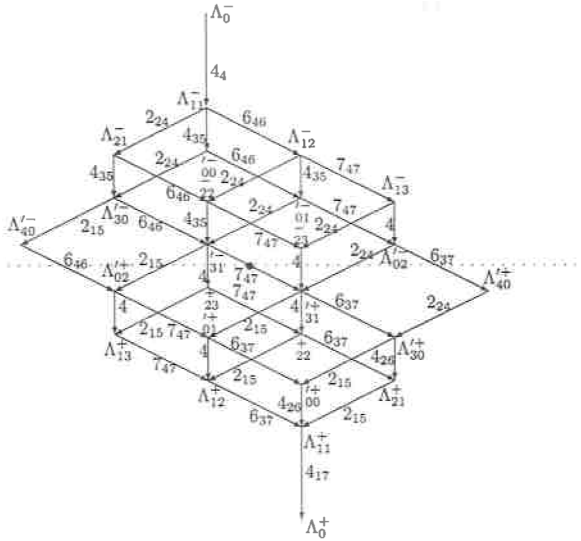

Fig. 3-135. $SU(4,4)$ reduced multiplets of type R^4_{135}

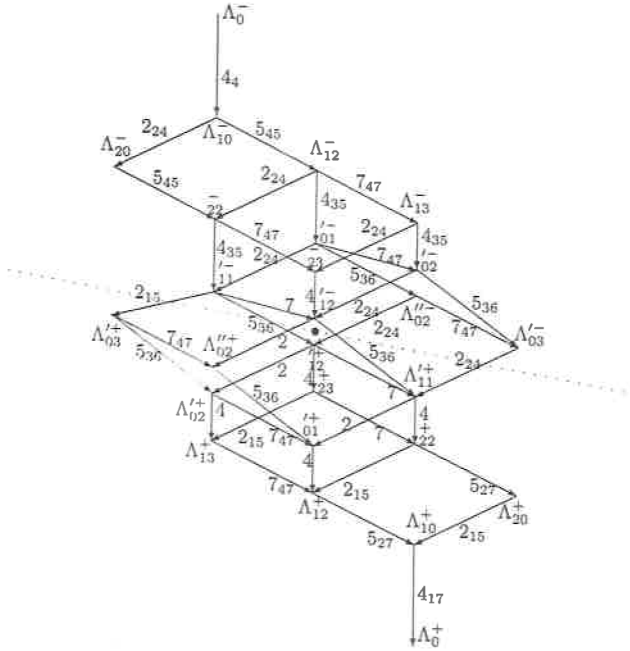

Fig. 3-136. $SU(4,4)$ reduced multiplets of type R^4_{136}

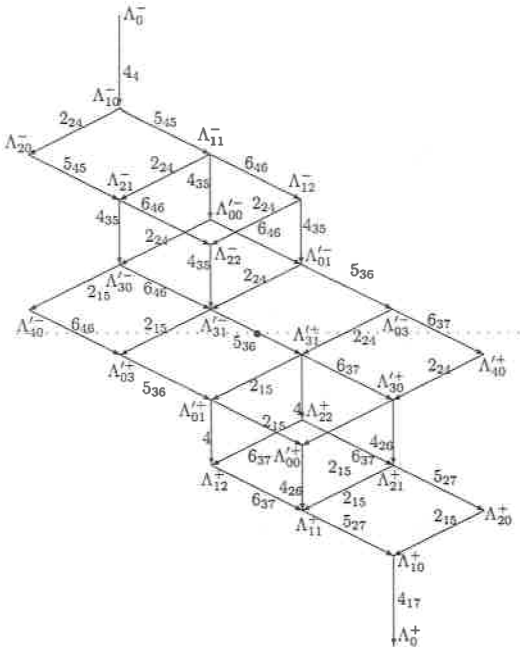

Fig. 3-137. $SU(4,4)$ reduced multiplets of type R^4_{137}

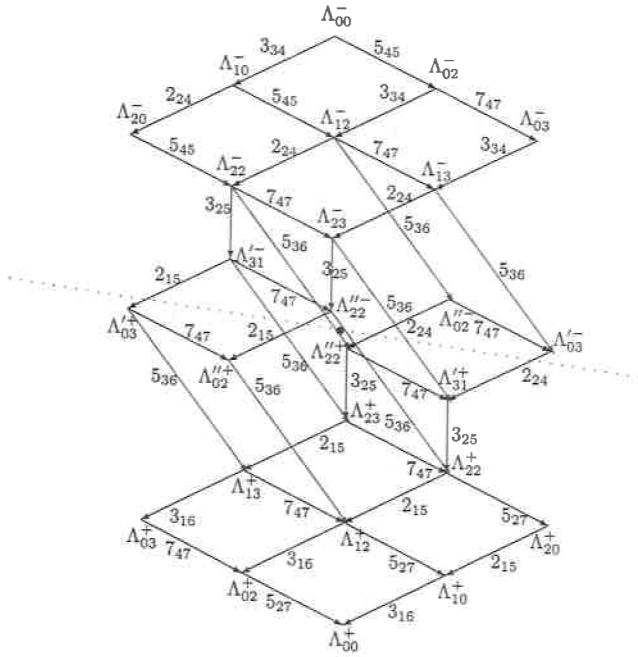

Fig. 3-146. $SU(4,4)$ reduced multiplets of type R^4_{146}

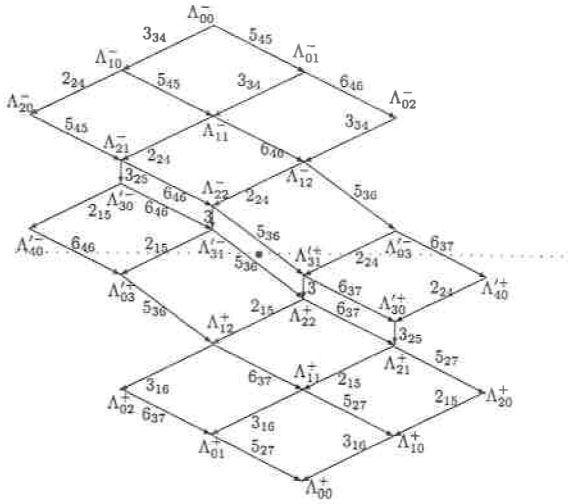

Fig. 3-147. $SU(4,4)$ reduced multiplets of type R^4_{147}

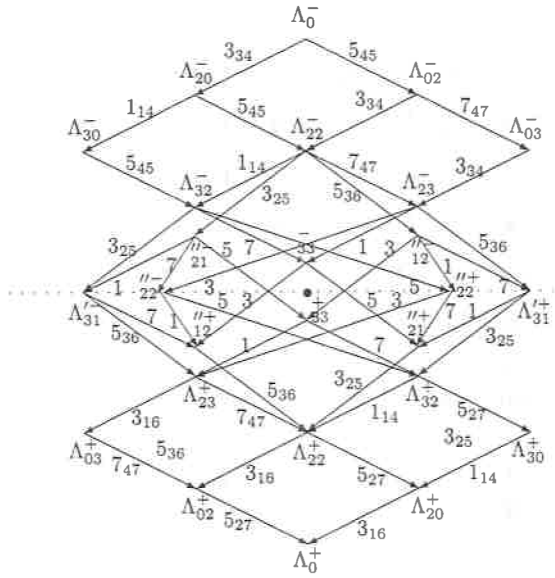

Fig. 3-246. $SU(4,4)$ reduced multiplets of type R^4_{246}

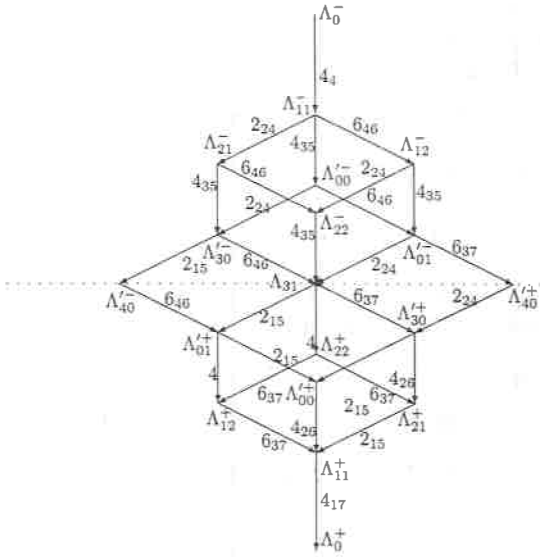

Fig. 3-1357. $SU(4,4)$ reduced multiplets of type R^4_{1357}

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The Nuclear Pseudo-spin Symmetry in Covariant Energy-Density Functional Theory

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Abstract

The pseudo-spin symmetry (PSS) in nuclei is the name for the quasi-degeneracy of a pair of single-particle states with quantum numbers (n, l, j) and $(n' = n-1, l' = l+2, j' = j+1)$. The PSS is known to exist in many cases, in spherical as well as deformed nuclei. In these lectures, we will discuss this property of atomic nuclei in the framework of a covariant energy-density functional theory (CEDFT) which can describe quantitatively well the properties of spherical as well as deformed nuclei. In particular, we show how perturbation theory can be used to investigate the PSS by relating the nuclear Dirac Hamiltonian of CEDFT to relativistic harmonic oscillator (RHO) potentials.

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1 Introduction

The atomic nucleus is a rather complex many-body system which is nowadays well explored, and whose dynamical properties can be understood to a large extent in terms of effective interactions between its constituents, the nucleons. In the long quest for modelling the structure of atomic nuclei which started over a century ago, some fundamental concepts emerged at very early stages. For the purpose of the present study, we shall concentrate on two of these concepts: 1) the existence of an average nuclear and electric mean field resulting from the two-body interactions among the constituent nucleons; 2) the existence of certain symmetries which are inherent to the nature of the nuclear mean field.

One of the earliest nuclear mean field models is the 3-dimensional harmonic oscillator potential. Very soon it turned out that the empirical data required that an additional spin-orbit potential - of the form $V_{LS}(r)\mathbf{l}.\mathbf{s}$ - should be added to the central harmonic oscillator potential. In this way, it became possible to account for the observed energy splitting between the two members $j_> = l + \frac{1}{2}$ and $j_< = l - \frac{1}{2}$ of a spin-orbit doublet.

states are labelled by the principal quantum number $n(n = 1, 2, ...)$, the orbital angular momentum $l(l = 0, 1, 2, ...)$, the total angular momentum $j(j = l \pm \frac{1}{2})$ and its third component $m(m = -j, -j + 1, ...j - 1, j)$. The spin-orbit potential lifts the degeneracy between the states having the same principal quantum number n and orbital angular momentum l but different total angular momentum, $j_> = l + \frac{1}{2}$ and $j_< = l - \frac{1}{2}$. This situation is very familiar in the mean field picture of atomic nuclei. For example, if one looks at the empirical single-proton spectrum of a nucleus like ^{16}O, it can be seen that the $1p\frac{3}{2}$ occupied level is located about 6.3 MeV below the $1p\frac{1}{2}$ occupied level, a direct evidence of the effect of the proton spin-orbit potential.

The first suggestion of a possible near-degeneracy of the two members of a pseudo-spin doublet was made as early as 1969, by Arima, Harvey and Shimizu [1] and by Hecht and Adler [2]. In a single-particle picture, a pseudo-spin doublet is composed of the two single-particle states (nlj) and $(n' = n - 1, l' = l + 2, j' = j + 1)$. Sometimes, one introduces an extra quantum number, the pseudo-orbital angular momentum $\tilde{l} = l + 1 = l' - 1$ to characterize a pseudo-spin doublet.

In fact, looking back at the first single-particle energy schemes suggested by Goeppert Mayer[3] and by Haxel et al.[4] one can see many occurrences of empirical near-degeneracies of pseudo-spin doublets: $1d\frac{3}{2} - 2s\frac{1}{2}, 1f\frac{5}{2} - 2p\frac{3}{2}, 1g\frac{7}{2} - 2d\frac{5}{2}, 2d\frac{3}{2} - 3s\frac{1}{2}, 1h\frac{9}{2} - 2f\frac{7}{2}, 2f\frac{5}{2} - 3p\frac{3}{2}$, etc... There was no systematic explanation or predictions of such near-degeneracies. These cases are also mentioned by J.N. Ginocchio[5, 6]. In his comprehensive article of 2005 on relativistic symmetry in nuclei[7] he showed the systematic trend of the measured pseudo-spin splitting of the $3s\frac{1}{2} - 2d\frac{3}{2}$ and $2d\frac{5}{2} - 1g\frac{7}{2}$ neutron levels throughout the Sb isotopic chain, and of the $3s\frac{1}{2} - 2d\frac{3}{2}$ neutron levels in the Tl isotopic chain. In the Sb chain those splittings vary monotonously from $+10\%$ to -10% of $\hbar\omega$ while in the Tl chain they are between 3% and 6% of $\hbar\omega$. Thus, it seems that the pseudo-spin partners have a tendency to stay not too far apart although their degeneracy is far from quantitative.

It is, however, difficult to find a general explanation to the fact that sometimes the degeneracies seem well obeyed and sometimes somewhat violated. In the 1990's the relativistic mean field (RMF) models became very popular, and they gave the hope to provide a theoretical framework for understanding the occurrence or breaking of the pseudo-spin symmetry (PSS) in nuclei [8]. In the rest of this contribution we shall use the relativistic framework to discuss the PSS and its breaking in atomic nuclei.

2 General properties of pseudo-spin symmetry in nuclei

The natural framework for discussing the PSS in atomic nuclei is to start from a microscopic independent particle model where the nucleons - neutrons and protons - are moving in a mean field created self-consistently by their mutual 2-body interactions. In this paper we adopt the covariant approach in which the nucleon wave functions are solutions of a Dirac equation rather than of a Schrödinger equation, and where the nucleon-nucleon effective interactions are mediated by the coupling of nucleon fields to meson-like fields. At this point, it is worthwhile to recall that there is a situation where the PSS is well obeyed,

it is when the single-particle states are eigenstates of a Dirac equation with Lorentz scalar and vector potentials, U_s and U_v, such that $U_s + U_v$ = constant at all points in space. At large distance, both U_s and U_v must vanish, therefore this constant value of $U_s + U_v$ should be zero. However, in order to have bound single-particle states $U_s + U_v$ must be negative in the interior region of the nucleus, hence the impossibility to realize strictly the condition $U_s + U_v$ = constant. This is clearly illustrated by Fig. 8 of Ref.[7] which shows typical U_s and U_v potentials calculated in ^{16}O and ^{208}Pb.

The covariant and self-consistent approach to nuclear systems is widely used since the pioneering work of Walecka and of Serot [9], mostly in its Dirac-Hartree version (called the Relativistic Mean Field (RMF) approach in the literature [10]), and pairing was added for non-closed shell nuclei. In parallel, methods for treating the exchange (Fock) terms were developed [11] and successful Relativistic Hartree-Fock (RHF) Lagrangians were proposed for incorporating effects coming from pion-nucleon or ρ(tensor)-nucleon couplings [12] which are absent in the RMF models.

The general method is to start from a covariant Lagrangian density containing nucleonic fields, mesonic fields and meson-nucleon couplings. By applying the variational principle one can write down a nucleonic Hamiltonian containing one- and two-body terms, and where the two-body nucleonic interactions are mediated by effective mesonic fields. This method has been carried out to determine the interaction parameters of the model and to calculate extensively the properties of atomic nuclei throughout the periodic table [13]. It has been extended to treat the case of nuclei with pairing correlations, by adding a Gogny-type pairing force and solving the full RHF-Bogoliubov (RHFB) set of equations[14].

In any case, the common feature of all RMF and RHF models is that they lead to self-consistent mean field scalar and vector potentials (U_s and U_v) such that $U_s + U_v$ is about -80 to -60 MeV in the nuclear interior, and going to zero beyond the nuclear surface. Thus, the strict PSS condition $U_s + U_v = 0$ cannot be fulfilled everywhere. Nevertheless, the covariant framework is quite suitable for studying the PSS and its breaking, as we shall see in the remaining part of this contribution.

In the RMF as well as in the RHF model, a very systematic feature emerges if one compares the wave functions of two pseudo-spin partner states, e.g., $a = (nlj)$ and $a' = (n' = n-1, l' = l+2, j' = j+1)$. This is illustrated by Fig.1 where are shown the upper components G and lower components F of the neutron states ($a = 2s1/2, a' = 1d3/2$) (left panels). The right panels of Fig.1 show a similar comparison for the pair of neutron states ($a = 3s1/2, a' = 2d3/2$). The calculations are done [15] for the doubly closed-shell nucleus ^{132}Sn using a density-dependent relativistic Hartree-Fock (DDRHF) model with a typical interaction PKO1. It is remarkable that the lower components F of the wave functions of the two pseudospin partners are very similar, whereas the upper components G show marked differences, especially in the case of the $2s1/2 - 1d3/2$ pair of states. The fact that the lower components are very close to each other is somehow related to the PSS. Since the single-particle energies depend more on the upper than on the lower components of the wave functions, the net effect on the single-particle energies is that the $2s1/2$ level is at some 3 MeV above the $1d3/2$ level, whereas the energy difference between the $3s1/2$ and $2d3/2$ levels is less than 200 keV.

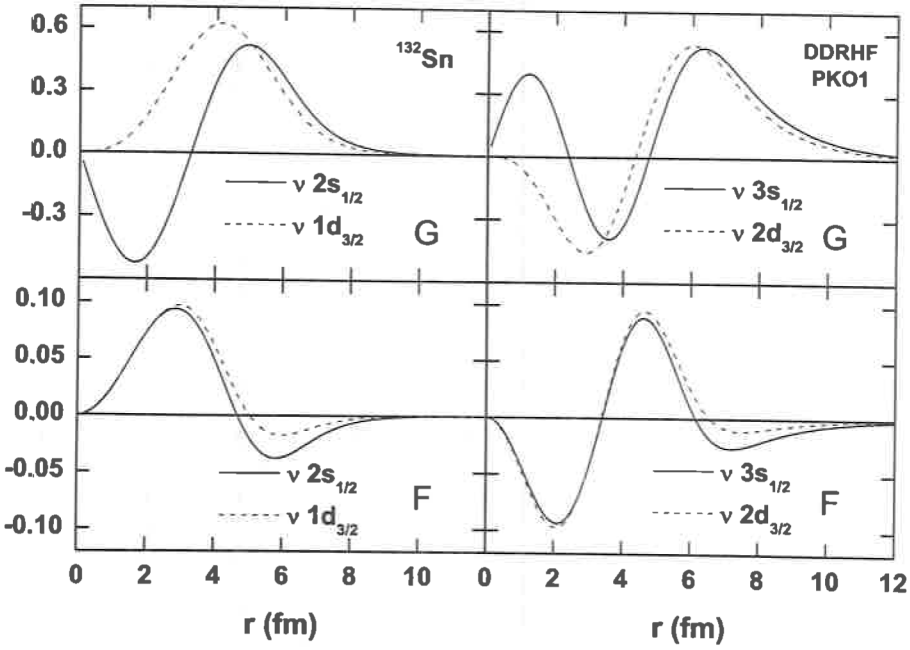

Figure 1: (Color online) The upper and lower components, G and F, of the radial wave functions for the neutron pseudospin partners $2s1/2-1d3/2$ (left panels) and $3s1/2-2d3/2$ (right panels), in the nucleus ^{132}Sn. The DDRHF calculations are done with the PKO1 parameter set.

3 Perturbation theory approach to PSS

Recently, the perturbation theory approach was used to investigate the symmetries of the Dirac Hamiltonian and their breaking in atomic nuclei[16]. The PSS as well as the spin symmetry (SS) were discussed in that work. It will be illuminating to consider here the case of PSS to understand the mechanism of its breaking. We will follow the method introduced in Ref.[16]. The actual numerical applications presented are done in RMF approximation for the representative nucleus ^{132}Sn with interaction PK1[17].

In spherical symmetry, the radial Dirac equations of the RMF model can be cast in the form

$$H\Psi = E\Psi \tag{1}$$

with

$$H = \begin{pmatrix} \Sigma(r) + M & -\frac{d}{dr} + \frac{\kappa}{r} \\ \frac{d}{dr} + \frac{\kappa}{r} & -\Delta(r) - M \end{pmatrix}, \text{ and } \Psi = \begin{pmatrix} G(r) \\ F(r) \end{pmatrix}, \tag{2}$$

where $\Sigma(r) = S(r) + V(r)$ and $\Delta(r) = S(r) - V(r)$ denote the combinations of the scalar and vector potentials, and κ is defined as $\kappa = (l - j)(2j + 1)$. Taking the nucleus ^{132}Sn as an example, the potentials $\Sigma(r)$ and $\Delta(r)$ for neutrons calculated by self-consistent RMF theory are shown in Fig. 2. The typical values of $\Sigma(r)$ and $\Delta(r)$ in the nuclear interior are about 70 MeV and 700 MeV, respectively.

In order to apply the Rayleigh-Schrödinger perturbation theory, we split the Dirac Hamiltonian H in Eq. (2) as

$$H = H_0 + W, \tag{3}$$

where H_0 leads to the exact spin (pseudospin) symmetry and W is identified as the corresponding symmetry breaking potential. In the case of the spin and pseudospin SU(2) symmetry limits [18] the Dirac Hamiltonians with exact symmetries read

$$H_0^{\rm SS} = \begin{pmatrix} \Sigma + M & -\frac{d}{dr} + \frac{\kappa}{r} \\ \frac{d}{dr} + \frac{\kappa}{r} & -\Delta_0 - M \end{pmatrix}, \qquad H_0^{\rm PSS} = \begin{pmatrix} \Sigma_0 + M & -\frac{d}{dr} + \frac{\kappa}{r} \\ \frac{d}{dr} + \frac{\kappa}{r} & -\Delta - M \end{pmatrix}, \tag{4}$$

whereas their spin- and pseudo-spin symmetry breaking potentials are

$$W^{\rm SS} = \begin{pmatrix} 0 & 0 \\ 0 & \Delta_0 - \Delta \end{pmatrix}, \qquad W^{\rm PSS} = \begin{pmatrix} \Sigma - \Sigma_0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{5}$$

where Δ_0, Σ_0 are constant whereas Δ, Σ are position-dependent.

For studying the relationship between the eigenstates of H and H_0 by perturbation theory, it is equivalent to use the definitions $H = H_0 + W$ and $H_0 = H - W$. Since the spectrum of H has discrete eigenstates which have no counterparts in H_0, the eigen-energies and wave functions (E_i, ψ_i) of H_0 are expanded on the complete set of eigenstates Ψ_m) of H for all the calculations discussed here below, i.e., the ground state eigen-energy E_0 and wave function ψ_0 of H_0 are expanded as

$$E_0 = E^{(0)} + E^{(1)} + E^{(2)} + \cdots, \tag{6a}$$

$$\psi_0 = \sum_m a_m \Psi_m \quad \text{with} \quad a_m = a_m^{(0)} + a_m^{(1)} + a_m^{(2)} + \cdots. \tag{6b}$$

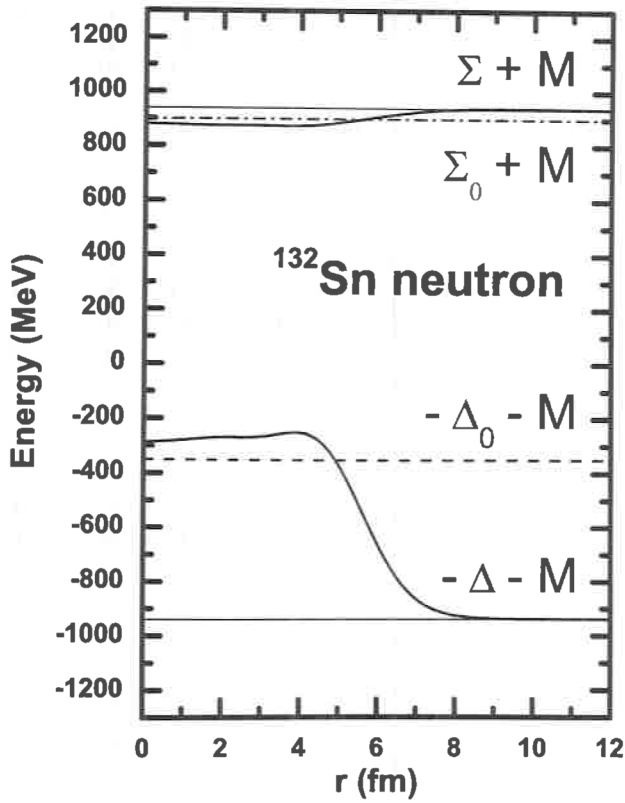

Figure 2: (Color online) Single-particle potentials for neutrons in the nucleus ^{132}Sn. The self-consistent potentials calculated by RMF theory with PK1 [17] are shown as solid lines. The potentials $-\Delta_0 - M$ in $H_0^{\rm SS}$ and $\Sigma_0 + M$ in $H_0^{\rm PSS}$ are illustrated as dashed, and dash-dotted lines, respectively.

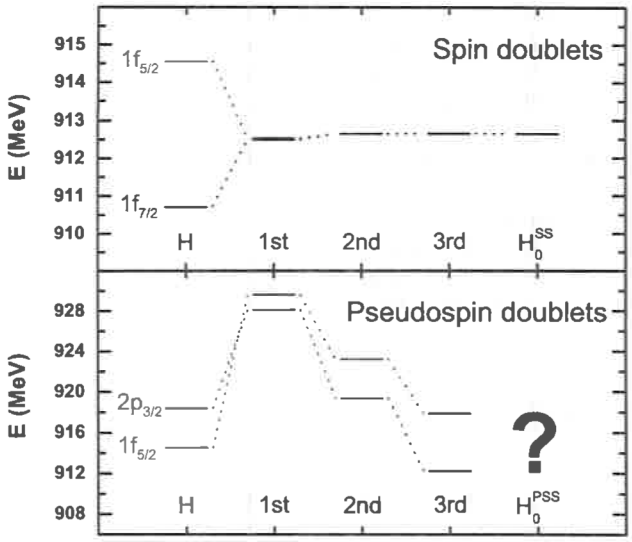

Figure 3: (Color online) Single-particle energies of spin doublets $1f$ (upper panel) and pseudospin doublets $1\tilde{d}$ (lower panel) obtained by RMF theory, and by the first, second, and third order perturbation calculations, as well as those at the exact symmetry limits.

In contrast to adopting the Schrödinger-like equations as done in the previous studies [5, 8], it is clearly shown that the operators H, H_0 and W used in the present calculations are all Hermitian, and they do not contain any singularity. This allows us to perform the order-by-order perturbation calculations. In addition, it should also be noticed that only W corresponds to the symmetry breaking potential within the present decomposition, thus the ambiguity caused by the strong cancellations among the different terms in the Schrödinger-like equations can also be avoided. Therefore, the present method can provide a clear and quantitative way for investigating the perturbative nature of SS and PSS. This method can be universally applied to the cases where the nature of the symmetry is either perturbative or non-perturbative. When the nature of the symmetry is perturbative, the link between the single-particle states in realistic nuclei and their counterparts in the symmetry limits can be constructed. If the symmetry is non-perturbative, the divergence of the perturbation series can be found explicitly.

In the present calculations, as illustrated with dashed and dash-dotted lines in Fig. 2, the constant potentials in Eqs. (4-5) are chosen as $-\Delta_0 - M = -350$ MeV and $\Sigma_0 + M = 900$ MeV. We have checked that the convergence of the perturbation calculations are not sensitive to these values.

Let us then use the model of Figure 2 to examine as a specific example the perturbation corrections to the single-particle energies of the spin doublets $1f$ and pseudospin doublets $1\tilde{d}$. In Fig. 3, from left to right, the single-particle energies obtained by self-consistent RMF theory, and their counterparts obtained by the first, second, and third

order perturbation calculations, as well as those obtained at the exact symmetry limits are shown. It can be seen clearly that the energy degeneracy of the spin doublets is well restored by the second order perturbation calculations. However, the energy degeneracy of the pseudospin doublets cannot be restored up to the third order perturbation calculations. Moreover, there exist no discrete eigenstates of $H_0^{\rm PSS}$. Thus, the link between the pseudospin doublets in realistic nuclei and their counterparts in the $S+V=$ Const limit is still unclear.

Thus, from the perturbative point of view, a path can be established between the Dirac Hamiltonian in realistic nuclei and the symmetry limit of $S-V=$ Const. This is not the case for the limit $S+V=$ Const. This confirms in an explicit way that the nature of PSS is non-perturbative, if the Dirac Hamiltonian with $S+V=$ Const is regarded as the symmetry limit.

4 Conclusion

In these notes, we have introduced the issue of PSS in nuclear systems. This question originates from the empirical observation that pairs of pseudo-spin partners sometimes seem to be nearly degenerate. This degeneracy may appear in empirical observations, or in predictions of simple potential models. On the other hand, there are also numerous situations where the PSS is not obeyed.

An analogous issue is offered by the question of the observed splitting of spin-orbit partners, which is the opposite situation. In this latter case, it is known for a long time that this splitting is caused by a spin-orbit component in the nuclear mean field.

This second issue can be handled by considering in the nuclear Hamiltonian a spin-symmetric part (which keeps the spin-orbit partner states degenerate) an a spin-symmetry breaking component (the spin-orbit potential) which causes the non-degeneracy of the spin-orbit partner states. We have seen that, using a relativistic nuclear model, one can understand by perturbation theory how to relate the non-degenerate situation (the realistic case) to the degenerate situation (the limiting case).

Then, coming back to the first issue one can try the same perturbative method, again with the same total Hamiltonian but separated into a pseudo-spin conserving part and a pseudo-spin non-conserving part. Then, one finds that the limiting case has no solution. Thus, the degeneracy of PSS partners does not happen - at least not systematically - in the realistic case, nor in the limiting case.

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Group Contractions and Physical Applications

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Abstract

The paper is devoted to the description of the contraction (or limit transition) method in application to classical Lie groups and Lie algebras of orthogonal and unitary series. As opposed to the standard Wigner-Inönü contractions based on insertion of one or several zero tending parameters in group (algebra) structure the alternative approach, which is connected with consideration of algebraic structures over Pimenov algebra with commutative nilpotent generators is used. The definitions of orthogonal and unitary Cayley–Klein groups are given. It is shown that the basic algebraic constructions, characterizing Cayley–Klein groups can be found using simple transformations from the corresponding constructions for classical groups. The theorem on the classifications of transitions is proved, which shows that all Cayley–Klein groups can be obtained not only from simple classical groups. As a starting point one can choose any pseudogroup as well. As applications of the developed approach to physics the kinematics groups and contractions of the Electroweak Model at the level of classical gauge fields are regarded. The interpretations of kinematics as spaces of constant curvature are given. Two possible contractions of the Electroweak Model are discussed and are interpreted as zero and infinite energy limits of the modified Electroweak Model with the contracted gauge group.

1 Introduction

Group-Theoretical Methods are essential part of modern theoretical and mathematical physics. It is enough to remind that the most advanced theory of fundamental interactions, namely Standard Electroweak Model, is a gauge theory with gauge group $SU(2) \times U(1)$. All types of classical groups of infinite series: orthogonal, unitary and symplectic as well as inhomogeneous groups, which are semidirect products of their subgroups, are used in different areas of physics. Euclidean, Lobachevsky, Galilei, Lorentz, Poincaré, (anti) de Sitter groups are the bases for space and space-time symmetries. Supergroups and supersymmetric models in the field theory predict the existence of new supersymmetric partners of known elementary particles. Quantum deformations of Lie groups and Lie algebras lead to non-commutative space-time models (or kinematics).

Contractions of Lie groups is the method for receiving new Lie groups from the initial ones. In the standard E. Wigner and E. Inönü approach [34] continuous parameter ϵ is introduced in such a way that in the limit $\epsilon \to 0$ group operation is changed but Lie group structure and its dimension are conserved. It is well known that studying non-degenerate structures is easier then the degenerate ones. So one represent a general Lie group as semidirect product of semisimple and solvable groups and reduce the problem of Lie groups classification to the classifications of semisimple and solvable groups. But, while the classification of semisimple groups was established long ago there is no hope to find the classification of solvable groups [42]. In general, a contracted group is a semidirect product of its subgroups. In particular, a contraction of semisimple groups gives non-semisimple ones. Therefore, the contraction method is a tool for studying of non-semisimple groups starting from the well known semisimple (or simple) Lie groups.

The method of contractions (limit transitions) was extended later to other types of groups and algebras. Graded contractions [43, 44] additionally conserve grading of Lie algebra. Lie bialgebra contractions [3] conserve both Lie algebra structure and cocommutator. Contractions of Hopf algebras (or quantum groups) are introduced in such a way [8, 9] that in the limit $\epsilon \to 0$ new expressions for coproduct, counit and antipode appear which satisfies Hopf algebra axioms. All this gives rise to the following generalization of the notion of group contraction on contraction of algebraic structures [25].

Definition. Contraction of algebraic structure $(M, *)$ is the map ϕ_ϵ dependent on parameter ϵ

$$\psi_\epsilon : (M, *) \to (N, *'), \tag{1}$$

where $(N, *')$ is an algebraic structure of the same type, which is isomorphic $(M, *)$ when $\epsilon \neq 0$ and non-isomorphic when $\epsilon = 0$.

There is another approach [23] to the description of non-semisimple Lie groups (algebras) and corresponding quantum groups based on their consideration over Pimenov algebra $P_n(\iota)$ with nilpotent commutative generators. In this approach the motion groups of constant curvature spaces (or Cayley–Klein groups) are realized as matrix groups of special form over $P_n(\iota)$ and can be obtained from the simple classical orthogonal group by substitution of its matrix elements for Pimenov algebra elements. It turns out that such substitution coincides with the introduction of Wigner–Inönü contraction parameter ϵ [34]. So our approach demonstrates that the existence of the corresponding structures over algebra $P_n(\iota)$ is the mathematical base of the contraction method.

It should be noted that both approaches supplement each other and in the final analysis give the same results. Nilpotent generators are more suitable in the mathematical consideration of contractions whereas the contraction parameter continuously tending to zero more corresponds to physical intuition according to which a physical system continuously changes its state and smoothly goes into its limit state.

It is well known in geometry (see, for example, review [58]) that there are 3^n different geometries of dimension n, which admit the motion group of maximal order. R.I. Pimenov suggested [48, 51] a unified axiomatic description of all 3^n geometries of constant curvature (or Cayley–Klein geometries) and demonstrated that all these geometries can be locally simulated in some region of n-dimension spherical space with named coordinates, which can be real, imaginary and nilpotent ones. According to Erlanger programm by F. Klein the main content in geometry is its motion group whereas the properties of transforming

objects are secondary. The motion group of n-dimensional spherical space is isomorphic to the orthogonal group $SO(n+1)$. In their turn the groups obtained from $SO(n+1)$ by contractions and analytical continuations are isomorphic to the motion groups of Cayley–Klein spaces. This correspondence provides the geometrical interpretation of Cayley–Klein contraction scheme. By analogy this interpretation is transferred to the contractions of other algebraic structures.

The method for achieving this goal is the method of transitions, which has clear geometrical meaning, and is based on the introduction of a set of contraction parameters $j = (j_1, \ldots, j_n)$, each of them taking three values: a real unit, an imaginary unit and a nilpotent unit.

The method of transitions between groups apart from being of interest for group theory itself is of interest for theoretical physics too. If there is a group-theoretical description of a physical system then the contraction of its symmetry group corresponds to some limit case of the system under consideration. So the reformulation of the system description in terms of the transition method and the subsequent physical interpretations of contraction parameters j gives an opportunity to study different limit behaviours of the physical system. An example of such approach is given for the Electroweak Model of elementary particle interactions.

It is likely that developed formalism is an essential tool to construct "general theory of physical systems" according to which "it is necessary to turn from group-invariant study of a single physical theory in Klein understanding (i.e. characterized by symmetry group) to a simultaneous study of a set of limit theories. Then some physical and geometrical properties will be the invariant properties of all set of theories and they should be considered in the first place. Other properties will be relevant only for the particular representatives and will be changed under limit transition from one theory to another" [59].

2 Dual Numbers and Pimenov Algebra

2.1 Dual numbers

Dual numbers were introduced by W.K. Clifford [10] as far back as in the XIX century. They were used by A.P. Kotel'nikov [39] for constructing his theory of screws in three-dimensional spaces of Euclid, Lobachevsky and Riemann, by B.A. Rosenfeld [53, 54], for description of non-Euclidean spaces, by R.I. Pimenov [48, 49, 51] for axiomatic study of spaces with a constant curvature. Some applications of dual numbers in kinematics can be found in the work by I.M. Yaglom [57]. The applications of dual numbers in geometry and in theory of group representations were discussed by V.V. Kisil [36]. Fine distinctions between the quantum and classical mechanics were investigated with the help of dual numbers [37, 38]. The theory of dual numbers as number systems is exposed in monographs by D.N. Zeiliger [60] and A.Sh. Bloch [6]. Nevertheless, it is impossible to say that dual numbers are well-known, so we start with their description.

Definition. By the associative algebra of rank n over the real numbers field $\mathbf{R}$ we mean n-dimensional vector space over this field, on which the operation of multiplication is defined, associative $a(bc) = (ab)c$, distributive in respect to addition $(a+b)c = ac + bc$

and related with the multiplication of elements by real numbers as follows $(ka)b = k(ab) = a(kb)$, where a, b, c are the elements of algebra; k is a real number. If there is such element e of algebra that for any element a of algebra the relations $ae = ea = a$ are valid, then the element e is called a unit.

Definition. Dual numbers $a = a_0e_0 + a_1e_1$, $a_0, a_1 \in \mathbf{R}$ are the elements of associative algebra of rank 2 with the unit and the generators satisfying the following conditions: $e_0^2 = e_0$, $e_0e_1 = e_1e_0$, $e_1^2 = 0$.

This associative algebra is commutative and e_0 is its unit. Therefore, further we shall write 1 instead of e_0 and denote generator e_1 by ι_1 (the Greek letter "*iota*") and call it a (purely) dual unit.

For a sum, a product and a quotient of dual numbers a and b we have

$$a + b = (a_0 + \iota_1 a_1) + (b_0 + \iota_1 b_1) = a_0 + b_0 + \iota_1(a_1 + b_1),$$

$$ab = (a_0 + \iota_1 a_1)(b_0 + \iota_1 b_1) = a_0 b_0 + \iota_1(a_1 b_0 + a_0 b_1),$$

$$\frac{a}{b} = \frac{a_0 + \iota_1 a_1}{b_0 + \iota_1 b_1} = \frac{a_0}{b_0} + \iota_1 \left(\frac{a_1}{b_0} - a_0 \frac{b_1}{b_0^2} \right). \tag{2}$$

Division can not always be carried out. Purely dual numbers $a_1\iota_1$ do not have an inverse element. Therefore dual numbers do not form a number field. As an algebraic structure they perform a ring. Dual numbers are equal $a = b$, if their real parts are equal $a_0 = b_0$ and their purely dual parts are equal $a_1 = b_1$. Thus, the equation $a_1\iota_1 = b_1\iota_1$ has the unique solution $a_1 = b_1$ for $a_1, b_1 \neq 0$. This fact can be written formally as $\iota_1/\iota_1 = 1$ and this is how the last relation has to be interpreted, because division $1/\iota_1$ is not defined.

Functions of dual variable $x = x_0 + \iota_1 x_1$ are defined by their Taylor expansion

$$f(x) = f(x_0) + \iota_1 x_1 \frac{\partial f(x_0)}{\partial x_0}, \tag{3}$$

where all terms with coefficients $\iota_1^2, \iota_1^3, \ldots$ are omitted. In particular, for dual x we have

$$\sin x = \sin x_0 + \iota_1 x_1 \cos x_0, \quad \sin(\iota_1 x_1) = \iota_1 x_1,$$

$$\cos x = \cos x_0 - \iota_1 x_1 \sin x_0, \quad \cos(\iota_1 x_1) = 1. \tag{4}$$

According to (3), the difference of two functions of dual variable can be presented as

$$f(x) - h(x) = f(x_0) - h(x_0) + \iota_1 x_1 \left(\frac{\partial f(x_0)}{\partial x_0} - \frac{\partial h(x_0)}{\partial x_0} \right), \tag{5}$$

therefore, if real parts $f(x_0)$ and $h(x_0)$ of functions coincide, then functions $f(x)$ and $h(x)$ also coincide. Using this fact, D.N. Zeiliger shows [60] that in the domain of dual numbers all identities of algebra and trigonometry, all theorems of differential and integral calculus remain valid. In particular, the derivative of a function of a dual variable over a dual variable can be found as

$$\frac{df(x)}{dx} = \frac{\partial f(x_0)}{\partial x_0} + \iota_1 x_1 \left(\frac{\partial^2 f(x_0)}{\partial x_0^2} \right). \tag{6}$$

2.2 Pimenov algebra

Let us consider now a more general situation, where several nilpotent units are taken as generators of associative algebra with a unit. (Further on we will use the name *nilpotent unit* instead of *dual unit*). R.I. Pimenov was the first who introduced [48, 49, 51] several nilpotent commutative units and used them for the unified axiomatic description of spaces with constant curvature. Therefore we name such algebra as a Pimenov algebra and denote it as $\mathbf{P}_n(\iota)$.

Definition. Pimenov algebra $\mathbf{P}_n(\iota)$ is an associative algebra with a unit and n nilpotent generators $\iota_1, \iota_2, \ldots, \iota_n$ with properties: $\iota_k\iota_p = \iota_p\iota_k \neq 0,\ k \neq p,\ \iota_k^2 = 0,\ p, k = 1, 2, \ldots, n$.

Any element of $\mathbf{P}_n(\iota)$ is a linear combination of monomials $\iota_{k_1}\iota_{k_2}\ldots\iota_{k_r},\ k_1 < k_2 < \ldots < k_r$, which together with a unit element make a basis in algebra as in a linear space of dimension 2^n:

$$a = a_0 + \sum_{r=1}^{n} \sum_{k_1,\ldots,k_r=1}^{n} a_{k_1\ldots k_r}\iota_{k_1}\ldots\iota_{k_r}. \tag{7}$$

This notation becomes unique, if we put an additional requirement $k_1 < k_2 < \ldots < k_r$ or condition of symmetry of coefficients $a_{k_1\ldots k_r}$ in respect to indices $k_1, \ldots k_r$. Two elements a, b of algebra $\mathbf{P}_n(\iota)$ coincide, if their coefficients in the expansion (7) are equal, i.e. $a_0 = b_0,\ a_{k_1\ldots k_r} = b_{k_1\ldots k_r}$. As in the case of dual numbers, this definition of equality of the elements of algebra $\mathbf{P}_n(\iota)$ is expressed in the possibility of cancellation of equal (with the same index) nilpotent units $\iota_k/\iota_k = 1,\ k = 1, 2, \ldots, n$ (but not ι_k/ι_m or $\iota_m/\iota_k,\ k \neq m$, as far as such expressions are not defined).

Here it is appropriate to compare Pimenov algebra $\mathbf{P}_n(\iota)$ with Grassmann algebra $\Gamma_{2n}(\epsilon)$, i.e. associative algebra with a unit, where a set of nilpotent generators $\epsilon_1, \epsilon_2, \ldots, \epsilon_{2n},\ \epsilon_0$ exhibits the properties of anticommutativity $\epsilon_k\epsilon_p = -\epsilon_p\epsilon_k \neq 0,\ p \neq k,\ p, k = 1, \ldots, 2n$. Any element f of Grassmann algebra $\Gamma_{2n}(\epsilon)$ can be expressed [5] as

$$f(\epsilon) = f(0) + \sum_{r=1}^{2n} \sum_{k_1,\ldots,k_r=1}^{2n} f_{k_1\ldots k_r}\epsilon_{k_1}\ldots\epsilon_{k_1}. \tag{8}$$

The representation is unique, if one requires $k_1 < k_2 < \ldots < k_r$ or puts on condition of skew-symmetry $f_{k_1\ldots k_r}$ in respect to indices $k_1, \ldots, k_r$. If in the expansion (8) only the terms with an even r differ from zero, then the element f is called even in respect to the set of canonical generators ϵ_k, if in the expansion (8) only the terms with an odd r differ from zero, then f is called an odd element. As a linear space, Grassmann algebra splits into even Γ_{2n}^0 and odd Γ_{2n}' subspaces: $\Gamma_{2n}(\epsilon) = \Gamma_{2n}^0 + \Gamma_{2n}^1$, where Γ_{2n}^0 is not only subspace, but also a subalgebra.

Let us consider nonzero products $\epsilon_{2k-1}\epsilon_{2k},\ k = 1, 2, \ldots, n$ of the generators of Grassmann algebra $\Gamma_{2n}(\epsilon)$. It is easy to see that these products possess the same properties as generators $\iota_k = \epsilon_{2k-1}\epsilon_{2k},\ k = 1, 2, \ldots, n$. Thus Pimenov algebra $\mathbf{P}_n(\iota)$ is a subalgebra of the even part Γ_{2n}^0 of Grassmann algebra $\Gamma_{2n}(\epsilon)$. It is worth mentioning that even products of Grassmannian anticommuting generators are also called para-Grassmannian variables. The latter are employed for classical and quantum descriptions of massive and massless particlies with an integer spin [11, 14, 15] and in theory of strings [61].

3 Cayley–Klein Orthogonal Groups and Algebras

3.1 Three fundamental geometries on a line

Let us introduce elliptic geometry on a line. Let us consider a circle $\mathbf{S}_1^* = \{x_0^{*2} + x_1^{*2} = 1\}$ on the Euclid plane $\mathbf{R}_2$. The rotations $x'^* = g(\varphi^*)x^*$, i.e.

$$\begin{aligned} x_0^{*'} &= x_0^* \cos\varphi^* - x_1^* \sin\varphi^*, \\ x_1^{*'} &= x_0^* \sin\varphi^* + x_1^* \cos\varphi^* \end{aligned} \tag{9}$$

of group $SO(2)$ bring the circle into itself. Let us identify diametrically the opposite points of the circle and introduce an internal coordinate $w^* = x_1^*/x_0^*$. Then the following transformations correspond to the rotations (9) in $\mathbf{R}_2$ for $\varphi^* \in (-\pi/2, \pi/2)$:

$$w^{*'} = \frac{w^* - a^*}{1 + w^* a^*}, \quad a^* = \tan\varphi^*, \quad a^* \in \mathbf{R}. \tag{10}$$

These transformations make a group of translations (motions) G_1 of an elliptic line with the rule of composition

$$a^{*'} = \frac{a^* + a_1^*}{1 - a^* a_1^*}. \tag{11}$$

Let us consider the representation of the group $SO(2)$ in the space of differentiable functions on $\mathbf{R}_2$, defined by the left shifts

$$T(g(\varphi^*))f(x^*) = f(g^{-1}(\varphi^*)x^*). \tag{12}$$

The generator of the representation

$$X^* f(x^*) = \frac{d(T(g(\varphi^*))f(x^*))}{d\varphi^*}|_{\varphi^*=0}, \tag{13}$$

corresponding to the transformation (9), can be easily found:

$$X^*(x_0^*, x_1^*) = x_1^* \frac{\partial}{\partial x_0^*} - x_0^* \frac{\partial}{\partial x_1^*}. \tag{14}$$

For the representation of group G_1 by the left shifts in space of differentiable functions on elliptic line the generator Z^*, corresponding to the transformation (10), can be written as

$$Z^*(w^*) = (1 + w^{*2}) \frac{\partial}{\partial w^*}. \tag{15}$$

It is worth mentioning that matrix generator

$$Y^* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{16}$$

corresponds to rotations $g(\varphi^*) \in SO(2)$.

The transformation of Euclidean plane $\mathbf{R}_2$, consisting of multiplication of Cartesian coordinate x_1 by parameter j_1, namely

$$\phi : \mathbf{R}_2 \to \mathbf{R}_2(j_1)$$

$$\phi x_0^* = x_0, \quad \phi x_1^* = j_1 x_1, \tag{17}$$

where $j_1 = 1, \iota_1, i$, brings $\mathbf{R}_2$ into plane $\mathbf{R}_2(j_1)$; the geometry of the latter is defined by metrics $x^2(j_1) = x_0^2 + j_1^2 x_1^2$. It is easy to see that $R_2(j_1 = i)$ is Minkowski plane and $R_2(j_1 = \iota_1)$ is Galilean plane.

Our main idea is that the transformation of geometries (17) induces the transformation of the corresponding motion groups and their algebras. Let us show how to derive these transformations.

The definition of angle measure in Euclidean plane R_2 is determined by the ratio x_1^*/x_0^*, which under the transformation (17) turns into $j_1 x_1/x_0$, i.e. angles are transformed according to the rule $\phi\varphi^* = j_1\varphi$. The asterisk marks the initial quantities (coordinates, angles, generators and so on). The transformed quantities are denoted by the same symbols without asterisk. Changing the coordinates in (9) according to (17) and the angles according to the derived transformation rule and multiplying both sides of the second equation by j_1^{-1}, we get the rotations in the plane $\mathbf{R}_2(j_1)$:

$$\begin{aligned} x_0' &= x_0 \cos j_1\varphi - x_1 j_1 \sin j_1\varphi, \\ x_1' &= x_0 \tfrac{1}{j_1} \sin j_1\varphi + x_1 \cos j_1\varphi, \end{aligned} \tag{18}$$

which make group $SO(2; j_1)$. According to (4), $\cos \iota_1\varphi = 1$, $\sin \iota_1\varphi = \iota_1\varphi$, therefore the transformations of group $SO(2; \iota_1)$ are Galilean transformations and the elements of group $SO(2; i)$ are Lorentz transformations, if x_0 is interpreted as time, and x_1 as a spatial coordinate. The domain of definition $\Phi(j_1)$ of the group parameter φ is $\Phi(1) = (-\pi/2, \pi/2)$, $\Phi(\iota_1) = \Phi(i) = \mathbf{R}$.

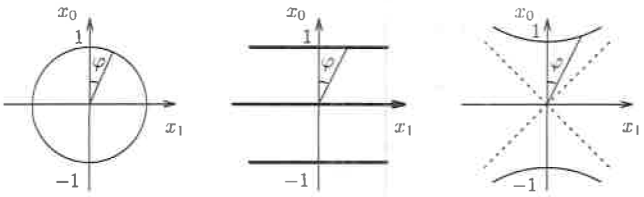
The rotations (18) preserve the circle $\mathbf{S}_1(j_1) = \{x_0^2 + j_1^2 x_1^2 = 1\}$ (Fig. 1) in the plane $\mathbf{R}_2(j_1)$. the identification of diametrically opposite points gives the upper semicircle (for $j_1 = 1$) and the connected component of the sphere (circle), passing through the point $(x_0 = 1, x_1 = 0)$, for $j_1 = \iota_1, i$. The internal coordinate on the circle w^* is transformed according to the rule $\phi w^* = j_1 w$. Substituting in (10) and canceling j_1 out of both sides we get the formula for translations on a line:

$$w' = \frac{w - a}{1 + j_1^2 w a}, \quad a = \frac{1}{j_1} \tan j_1\varphi \in R, \tag{19}$$

which make group $G_1(j_1)$, i.e. the group of motions of the elliptic line $\mathbf{S}_1(1)$ for $j_1 = 1$, the parabolic line $\mathbf{S}_1(\iota_1)$ for $j_1 = \iota_1$, and the hyperbolic line $\mathbf{S}_1(i)$ for $j_1 = i$.

In the space of differentiable functions on $\mathbf{R}_2(j_1)$ the generator $X(x)$ of the representation of group $SO(2; j_1)$ is defined by the relation (13), where all quantities are taken without asterisks. Under the transformation (17) derivative $d/d\varphi^*$ turns $j_1^{-1}(d/d\varphi)$, therefore, to obtain derivative $d/d\varphi$ the generator X^* must be multiplied by j_1, i.e. the generators $X^*(\phi x)$ and $X(x)$ are interrelated by the transformation

$$X(x) = j_1 X^*(\phi x^*) = j_1^2 x_1 \frac{\partial}{\partial x_0} - x_0 \frac{\partial}{\partial x_1}. \tag{20}$$



a) $j_1 = 1$ b) $j_1 = \iota_1$ c) $j_1 = i$

Figure 1: The circles of unit radius on the planes $\mathbf{R}_2(j_1)$

The generator Z is transformed according to the same rule:

$$Z(w) = j_1 Z^*(\phi w^*) = (1 + j_1^2 w^2)\frac{\partial}{\partial w}. \tag{21}$$

The transformation rule for the matrix generator of the rotation Y is as follows:

$$Y = j_1 Y^*(\rightarrow) = j_1 \begin{pmatrix} 0 & -j_1 \\ j_1^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -j_1^2 \\ 1 & 0 \end{pmatrix}. \tag{22}$$

Expressions (18)–(22) describe Cayley–Klein space and group in the traditional way with the help of real coordinates, generators and so on. Such approach was used in [23]. There is another way of describing Cayley–Klein spaces with the help of the named (i.e. having one of names: real, nilpotent, imaginary) coordinates of the form $j_1 x_1$, when under transformation (17) and the substitution $\phi\varphi^* = j_1\varphi$ in (9) both sides of the second equation are not multiplied by j_1^{-1}. Then the rotations on the plane $\mathbf{R}_2(j_1)$ with coordinates x_0, $j_1 x_1$ are written in the form

$$\begin{pmatrix} x_0' \\ j_1 x_1' \end{pmatrix} = \begin{pmatrix} \cos j_1\varphi & -\sin j_1\varphi \\ \sin j_1\varphi & \cos j_1\varphi \end{pmatrix} \begin{pmatrix} x_0 \\ j_1 x_1 \end{pmatrix}. \tag{23}$$

These rotations form group $SO(2; j_1)$, whose matrix generator is as follows

$$Y = j_1 Y^* = \begin{pmatrix} 0 & -j_1 \\ j_1 & 0 \end{pmatrix}. \tag{24}$$

The symbol Y^* instead of $Y^*(\rightarrow)$ in (22) means that the generator Y^* (16) is not transformed. It is the the second approach that we shall use in this book. One of its advantages is that for $j_1 = \iota_1$ the rotation matrix (23) from group $SO(2; \iota_1)$

$$\begin{pmatrix} 1 & -\iota_1\varphi \\ \iota_1\varphi & 1 \end{pmatrix}, \tag{25}$$

depend on group parameter φ, whereas for $j_1 \to 0$ it is equal to the unit matrix.

The group of motions $G_1(j_1)$ of one-dimensional Cayley–Klein space $\mathbf{S}_1(j_1)$ is closely connected with rotation group $SO(2; j_1)$ in space $\mathbf{R}_2(j_1)$. Therefore, under Cayley–Klein

space we shall further mean both $\mathbf{S}_1(j_1)$ and $\mathbf{R}_2(j_1)$, and under their groups of motion — both $G_1(j_1)$ and $SO(2;j_1)$. We shall follow the same rule in the case of higher dimensions.

We have studied comprehensively the simplest case of groups $SO(2;j_1)$, $G_1(j_1)$ because here the main ideas of methods of transitions reveal themselves in the most clear way, not aggravated with mathematical calculations. These ideas are as follows: (a) to define the transformation (17) from Euclidean space to arbitrary Cayley–Klein space; (b) to find the rules of transformations of motion,generators etc. of the group; (c) using the approach exposed in (b), to derive motion,generators etc. of Cayley–Klein group from the corresponding quantities of classical orthogonal group. The method of transitions, in spite of its simplicity, enables us to describe all Cayley–Klein groups, being aware of only classical orthogonal ones.

3.2 Nine Cayley–Klein groups

Mapping

$$\phi : \mathbf{R}_3 \to \mathbf{R}_3(j)$$

$$\phi x_0^* = x_0, \quad \phi x_1^* = j_1 x_1, \quad \phi x_2^* = j_1 j_2 x_2, \tag{26}$$

where $j = (j_1, j_2)$, $j_1 = 1, \iota_1, i$, $j_2 = 1, \iota_2, i$, turns three-dimensional Euclidean space into spaces $\mathbf{R}_3(j)$, on the spheres (or connected components of spheres) of which

$$\mathbf{S}_2(j) = \{x_0^2 + j_1^2 x_1^2 + j_1^2 j_2^2 x_2^2 = 1\} \tag{27}$$

nine geometries of Cayley–Klein planes are realized. The interrelation of the geometries and values of parameters j is clear from Fig. 2.

Rotation angle φ_{rs} in the coordinate plane $\{x_r, x_s\}$, $r < s$, $r, s = 0, 1, 2$, is determined by the ratio x_s/x_r and under the mapping (26) is transformed as $\varphi_{rs}^* \to \varphi_{rs}(r, s)$, where

$$(i, k) = \prod_{l=\min(i,k)+1}^{\max i,k} j_l, \quad (k, k) = 1. \tag{28}$$

Therefore for one-parametric rotations in the plane $\{x_r, x_s\}$ of space $\mathbf{R}_3(j)$ the following relations are valid

$$\begin{aligned}(0, r)x_r' &= x_r(0, r)\cos(\varphi_{rs}(r, s)) - x_s(0, s)\sin(\varphi_{rs}(r, s)), \\ (0, s)x_s' &= x_r(0, r)\sin(\varphi_{rs}(r, s)) + x_s(0, s)\cos(\varphi_{rs}(r, s)).\end{aligned} \tag{29}$$

The rest of the coordinates is not changed $x_p' = x_p$, $p \neq r, s$.

It is easy to find the matrix generators of the rotations (29)

$$Y_{01} = j_1 Y_{01}^* = \begin{pmatrix} 0 & -j_1 & 0 \\ j_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; Y_{12} = j_2 Y_{12}^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -j_2 \\ 0 & j_2 & 0 \end{pmatrix},$$

$$Y_{02} = j_1 j_2 Y_{02}^* = \begin{pmatrix} 0 & 0 & -j_1 j_2 \\ 0 & 0 & 0 \\ j_1 j_2 & 0 & 0 \end{pmatrix}. \tag{30}$$

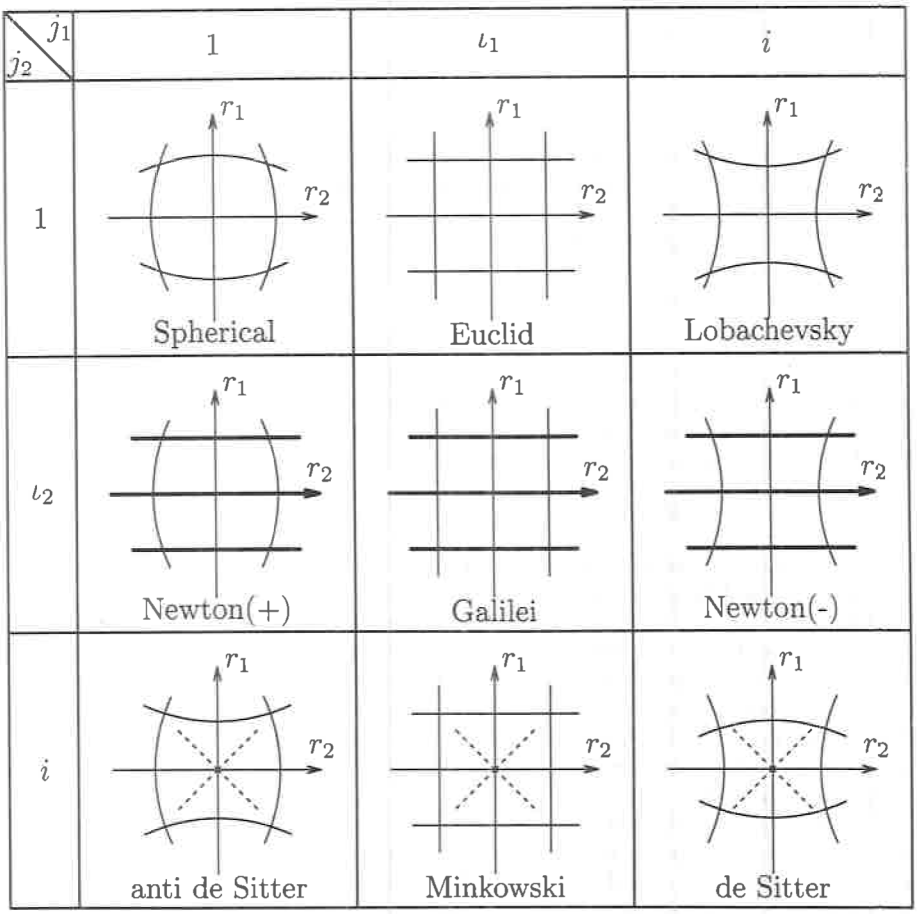


Figure 2: Cayley-Klein planes. The fibers are shown by thick lines and the light cone in $(1+1)$ kinematics are shown by dashed lines. Internal coordinates take values $r_1 = x_1/x_0$, $r_2 = x_2/x_0$

They make a basis of Lie algebra $so(3;j)$. The rule of transformations for the generators of representation of group $SO(3;j)$ in the space of differentiable functions on $\mathbf{R}_3(j)$ by left shifts coincides with the rule of transformations for parameters φ_{rs} and can be written as follows [18, 20]:

$$X_{rs}(x) = (r,s)X^*_{rs}(\phi x^*), \tag{31}$$

and the generators themselves as

$$X_{rs}(x) = (r,s)^2 x_s \frac{\partial}{\partial x_r} - x_r \frac{\partial}{\partial x_s}. \tag{32}$$

Knowing the generators, one can evaluate their commutators. But we shall derive the commutators from the commutation relations of algebra $so(3)$. Let us introduce new notations for the generators $X^*_{01} = H^*$, $X^*_{02} = P^*$, $X^*_{12} = K^*$. As it is well-known, the commutators of Lie algebra $so(3)$ can be written as follows:

$$[H^*, P^*] = K^*, \quad [P^*, K^*] = H^*, \quad [H^*, K^*] = -P^*. \tag{33}$$

Generators of algebra $so(3)$ are transformed according to the rule $H = j_1 H^*, P = j_1 j_2 P^*$, $K = j_2 K^*$, i.e. $H^* = j_1^{-1} H$, $P^* = j_1^{-1} j_2^{-1} P$, $K^* = j_2^{-1} K$. Substituting these expressions in (33) and multiplying each commutator by a factor, equal to the denominator on the left side of each equation, i.e. the first — by $j_1^2 j_2$, the second — by $j_1 j_2^2$, and the third — by $j_1 j_2$, we get commutators of algebra Lie for group $SO(3;j)$:

$$[H, P] = j_1^2 K, \quad [P, K] = j_2^2 H, \quad [H, K] = -P. \tag{34}$$

Cayley–Klein spaces $\mathbf{S}_2(j)$ (or spaces of constant curvature) for $j_1 = 1, \iota_1, i$, $j_2 = \iota_2, i$ can serve as models of kinematics, i.e. space–time geometries. In this case internal coordinate $t = x_1/x_0$ can be interpreted as the temporal axis, and internal coordinate $r = x_2/x_0$ as the spatial one. Then H is the generator of the temporal shift, P is the generator of the spatial shift, and K is the generator of Galilean transformation for $j_2 = \iota_2$ or Lorentz transformation for $j_2 = i$. The semispherical group $SO(3;1,\iota_2)$ (or Newton group) is isomorphic to the cylindrical group, which describes movement of a point on a cylindrical surface. This group is interpreted as the $E(2)$-like little group for massless particles [35].

The final relations should not involve division by a nilpotent number. This requirement suggests the way of finding the rule of transformations for algebraic constructions. Let an algebraic quantity $Q^* = Q^*(A^*_1, \ldots, A^*_k)$ be expressed in terms of quantities $A^*_1, \ldots, A^*_k$ with a known rule of transformation under mapping ϕ, for example, $A_1 = J_1 A^*_1, \ldots A_k = J_k A^*_k$, where coefficients $J_1, \ldots, J_k$ are some products of parameters j. Substituting $A^*_1 = J_1^{-1} A_1, \ldots, A^*_k = J_k^{-1} A_k$ in the relation for Q^*, we get the formula $Q^*(J_1^{-1} A_1, \ldots, J_k^{-1} A_k)$, involving, in general, indeterminate expressions, when parameters j are equal to the nilpotent units. For this reason the last formula should be multiplied by such minimal coefficient J that the final formula would not involve indeterminate expressions:

$$Q = JQ^*(J_1^{-1} A_1, \ldots, J_k^{-1} A_k). \tag{35}$$

Then (35) is the rule of transformation for quantity Q under mapping ϕ.

Such method, stemmed out directly from the definition of coincidence of elements of Pimenov algebra $\mathbf{P}_n(\iota)$ turns out to be very useful and further will be widely employed. The rule of transformation (35) for algebraic quantity Q, derived from the requirement of absence of indeterminate expressions for nilpotent values of parameters j, is automatically satisfied for imaginary values of these parameters.

Let us exemplify this rule by Casimir operator. The only Casimir operator for algebra $so(3)$ is

$$C_2^*(H^*,\ldots) = H^{*2} + P^{*2} + K^{*2}. \tag{36}$$

Substituting $H^* = j_1^{-1}H$, $P^* = j_1^{-1}j_2^{-1}P$, $K^* = j_2^{-1}K$ in (36), we get

$$C_2^*(j_1^{-1}H,\ldots) = j_1^{-2}H^2 + j_1^{-2}j_2^{-2}P^2 + j_2^{-2}K^2. \tag{37}$$

The most singular factor for $j_1 = \iota_1$ and $j_2 = \iota_2$ is coefficient $(j_1 j_2)^{-2}$ of the term P^2. Multiplying both sides of the equation (37) by $(j_1 j_2)^2$, we get rid of the indeterminate expressions and derive the rule of transformation and Casimir operator for algebra $so(3;j)$:

$$C_2(j;H,\ldots) = j_1^2 j_2^2 C_2^*(j_1^{-1}H,\ldots) = j_2^2 H^2 + P^2 + j_1^2 K^2. \tag{38}$$

As it is known, Casimir operator for two dimensional Galilean algebra $so(3;\iota_1,\iota_2)$ is $C_2(\iota_1,\iota_2) = P^2$ (see, for example, [40]), for Poincaré algebra $so(3;\iota_1,i)$ is $C(\iota_1,i) = P^2 - H^2$, for algebra $so(3;i;1) = so(2,1)$ is $C_2(i,1) = H^2 + P^2 - K^2$ (see [45]). All these Casimir operators can be obtained from (38) for the corresponding values of parameters j.

The matrix generators (30) make the basis of fundamental representation of Lie algebra $so(3;j)$ of group $SO(3;j)$. Using exponential mapping one can put in correspondence to the general element

$$Y(\mathbf{r};j) = r_1 Y_{01} + r_2 Y_{02} + r_3 Y_{12} = \begin{pmatrix} 0 & -j_1 r_1 & -j_1 j_2 r_2 \\ j_1 r_1 & 0 & -j_2 r_3 \\ j_1 j_2 r_2 & j_2 r_3 & 0 \end{pmatrix} \tag{39}$$

of algebra $so(3;j)$ the finite rotation $g(\mathbf{r};j) = \exp Y(\mathbf{r};j)$:

$$g(\mathbf{r};j) = E\cos(r) + Y(\mathbf{r};j)\frac{\sin r}{r} + Y'(\mathbf{r},j)\frac{1-\cos r}{r^2},$$

$$Y'(\mathbf{r};j) = \begin{pmatrix} j_2^2 r_3^2 & -j_1 j_2^2 r_2 r_3 & j_1 j_2 r_1 r_3 \\ -j_1 j_2^2 r_2 r_3 & j_1^2 j_2^2 r_2^2 & -j_1^2 j_2 r_1 r_2 \\ j_1 j_2 r_1 r_3 & -j_1^2 j_2 r_1 r_2 & j_1^2 r_1^2 \end{pmatrix},$$

$$r^2 = j_1^2 r_1^2 + j_1^2 j_2^2 r_2^2 + j_2^2 r_3^2, \tag{40}$$

acting on vector $(x_0, j_1 x_1, j_1 j_2 x_2)^t \in \mathbf{R_3(j)}$ with the named components.

The disadvantage of the general parametrization (39), (40) is the complexity of the composition rule for parameters $\mathbf{r}$ under group multiplication. F.I. Fedorov [12] has proposed parametrization of rotation group $SO(3)$ for which the group composition law is especially simple. It turns out that it is possible to construct analogues of such parametrization for all groups $SO(3;j)$ [21]. The matrix of the finite rotations of group $SO(3;j)$ can be written as follows

$$g(\mathbf{c};j) = \frac{1+c^*(j)}{1-c^*(j)} = 1 + 2\frac{c^*(j) + c^{*2}(j)}{1+c^2(j)},$$

$$c^*(j) = \begin{pmatrix} 0 & -j_1^2 c_3 & j_1^2 j_2^2 c_2 \\ c_3 & 0 & -j_2^2 c_1 \\ -c_2 & c_1 & 0 \end{pmatrix},$$

$$c^2(j) = j_2^2 c_1^2 + j_1^2 j_2^2 c_2^2 + j_1^2 c_3^2, \tag{41}$$

and parameters $\mathbf{c}''$ correspond to matrix $g(\mathbf{c}''; j) = g(\mathbf{c}; j)g(\mathbf{c}'; j)$. These parameters can be expressed in terms of $\mathbf{c}$ and $\mathbf{c}'$ as follows

$$\mathbf{c}'' = \frac{\mathbf{c} + \mathbf{c}' + [\mathbf{c}, \mathbf{c}']_j}{1 - (\mathbf{c}, \mathbf{c}')_j}. \tag{42}$$

Here the scalar product of vectors $\mathbf{c}$ and $\mathbf{c}'$ is given by (41), and the vector product is given by

$$[\mathbf{c}, \mathbf{c}']_j = (j_1^2[\mathbf{c}, \mathbf{c}']_1,\ [\mathbf{c}, \mathbf{c}']_2,\ j_2^2[\mathbf{c}, \mathbf{c}']_3), \tag{43}$$

where $[\mathrm{c}, \mathrm{c}']_k$ are components of usual vector product in $\mathbf{R_3}$.

E.P. Wigner and E. Inönü [34] have introduced the operation of contraction (limit transition) of groups, algebras and their representations. Under this operation the generators of the initial group (algebra) undergo transformation, depending on a parameter ϵ, so that for $\epsilon \neq 0$ this transformation is non-singular and for $\epsilon \to 0$ it becomes singular. If the limits of the transformed generators exist for $\epsilon \to 0$, then they are the generators of a new (contracted) group (algebra), non isomorphic to the initial one. It is worth mentioning that the transformation (31) of the generators of algebra $so(3)$ for the nilpotent values of parameters j is Wigner–Inönü contraction. Really, $X^*_{rs}(\phi x^*)$ is the singularly transformed generator of initial algebra $so(3)$, the product (r, s) plays the role of parameter ϵ, tending to zero, and the resulted generators $X_{rs}(\mathbf{x})$ are the generators of the contracted algebra $so(3; j)$.

Comparing the rule of transformation for generators (31) and the expression (39) for a general element of algebra $so(3)$, we find that for the imaginary values of parameters j some of the real group parameters r_k become imaginary, i.e. they are analytically continued from the field of real numbers to the field of complex numbers. In this case orthogonal group $SO(3)$ is transformed into pseudoorthogonal group $SO(p, q)$, $p + q = 3$. When parameters j take nilpotent values, real group parameters r_k become elements of Pimenov algebra $\mathbf{P}(\iota)$ of the special form and we get the contraction of group $SO(3)$. Thus, from the point of view of the group transformation under mapping ϕ, both operations — analytical continuation of groups and contraction of groups different at first sight — have the same nature: the continuation of real group parameters to the complex numbers field or to Pimenov algebra $\mathbf{P}(\iota)$.

3.3 Extension to higher dimensions

Cayley–klein geometries of the dimension n are realized on spheres

$$\mathbf{S}_n(j) = \{(x, x) = x_0^2 + \sum_{k=1}^{n} (0, k)^2 x_k^2 = 1\} \tag{44}$$

in the spaces $\mathbf{R}_{n+1}(j)$ resulting from Euclidean space $\mathbf{R}_{n+1}$ under mapping

$$\phi : \mathbf{R}_{n+1} \to \mathbf{R}_{n+1}(j)$$

$$\phi x_0^* = x_0, \quad \phi x_k^* = (0,k)x_k, \quad k = 1,2,\ldots,n, \tag{45}$$

where $j = (j_1,\ldots,j_n)$, $j_k = 1, \iota_k, i$, $k = 1,2,\ldots,n$. If all parameters are equal to one $j_k = 1$, then ϕ is identical mapping, if all or some parameters are imaginary $j_k = i$ and the other are equal to 1, then we obtain pseudoeuclidean spaces of different signature. The space $\mathbf{R}_{n+1}(j)$ is called non-fiber, if no of the parameters $j_1,\ldots,j_n$ take nilpotent value.

Definition. The space $\mathbf{R}_{n+1}(j)$ is called $(k_1,k_2,\ldots,k_p)$-fiber space, if $1 \le k_1 < k_2 < \ldots < k_p \le n$ and $j_{k_1} = \iota_{k_1},\ldots,j_{k_p} = \iota_{k_p}$ and other $j_k = 1,i$.

These fiberings are trivial [7] and can be characterized by a set of consequently nested projections $pr_1, pr_2, \ldots, pr_p$, where for pr_1 the base is the subspace, spanned over the basis vector $\{x_0, x_1, \ldots, x_{k_1-1}\}$, and the fiber is the subspace, spanned over $\{x_{k_1}, x_{k_1+1}, \ldots x_n\}$; for pr_2 the base is the subspace $\{x_{k_1}, x_{k_1+1}, \ldots, x_{k_2-1}\}$, and the fiber is the subspace $\{x_{k_2}, x_{k_1+1}, \ldots, x_n\}$ and so on.

From the mathematical point of view the fibering in the space $\mathbf{R}_{n+1}(j)$ is trivial, i.e. its global and local structures are the same. From the physical point of view the fibering gives an opportunity to model quantities of different physical dimensions. For example, in Galilean space, which is realized on the sphere $\mathbf{S}_4(\iota_1,\iota_2,1,1)$, there are time $t = x_1$, $[t] = \sec$ and space $\mathbf{R}_3 = \{x_2,x_3,x_4\}$, $[x_k] = \mathrm{sm}$, $k = 2,3,4$ variables.

Definition. Group $SO(n+1;j)$ consists of all the transformations of the space $\mathbf{R}_{n+1}(j)$ with unit determinant, keeping invariant the quadratic form (44).

The totality of all possible values of parameters j gives 3^n different Cayley–Klein spaces $\mathbf{R}_{n+1}(j)$ and geometries $\mathbf{S}_n(j)$. It is customary to identify the spaces (and their group of motions), if their metrics have the same signature, i.e., for example, space $\mathbf{R}_3(1,i)$ with metric $x_0^2+x_1^2-x_2^2$ and space $\mathbf{R}_3(i,i)$ with metric $x_0^2-x_1^2+x_2^2$. But we have fixed Cartesian coordinate axes in $\mathbf{R}_{n+1}(j)$ ascribing to them fixed numbers, and for this reason in our case spaces $\mathbf{R}_3(1,i)$ and $\mathbf{R}_3(i,i)$ (and, correspondingly groups $SO(3;1,i)$ and $SO(3;i,i)$) are different. Groups $SO(3;1,i) \equiv SO(2,1)$ and $SO(3;i,1) \equiv SO(1,2)$ are also considered to be different. This was made for convenience of applications of method being developed.

Really, the application of some mathematical formalism in a concrete science means first of all substantial interpretation of base mathematical constructions. For example, if we interpret in space $\mathbf{R}_4(i,1,1)$ with metric $x_0^2 - x_1^2 - x_2^2 - x_3^2$ the first Cartesian coordinate x_0 as the time axis and the other x_1,x_2,x_3 as the space axes, then we get relativistic kinematic (space-time model). In this example the substantial interpretation of coordinates is the numbers of Cartesian coordinate axes: axis number one, axis number two etc.

The rotations in the two-dimensional plane $\{x_r,x_s\}$, the rule of transformation for representation generators and the generators themselves are given, correspondingly, by (29), (31), (32), where $r,s = 0,1,\ldots,n$, $r < s$. For the non-zero elements of the matrix generators of rotations the following relations are valid: $(Y_{rs})_{sr} = -(Y_{rs})_{rs} = (r,s)$. The commutation relations for Lie algebra $so(n+1;j)$ can be most simply derived from the commutators of algebra $so(n+1)$. as it has been done in section 3.2. The non-zero commutators are

$$[X_{r_1s_1}, X_{r_2s_2}] = \begin{cases} (r_1,s_1)^2 X_{s_1s_2}, & r_1 = r_2,\ s_1 < s_2, \\ (r_2,s_2)^2 X_{r_1r_2}, & r_1 < r_2,\ s_1 = s_2, \\ -X_{r_1s_2}, & r_1 < r_2 = s_1 < s_2. \end{cases} \tag{46}$$

Algebra $so(n+1)$ has $[(n+1)/2]$ independent Casimir operators, where $[x]$ is the integer part of a number x. As it is known [4], for even $n = 2k$ Casimir operators are given by

$$\hat{C}^*_{2p}(X^*_{rs}) = \sum_{a_1,\ldots,a_p=0}^{n} X^*_{a_1a_2} X^*_{a_2a_3} \ldots X^*_{a_{2p}a_1}, \tag{47}$$

where $p = 1, 2, \ldots, k$. For odd $n = 2k+1$ the operator

$$C'^*_n(X^*_{rs}) = \sum_{a_1,\ldots,a_n=0}^{n} \epsilon_{a_1a_2\ldots a_n} X^*_{a_1a_2} X^*_{a_3a_4} \ldots X^*_{a_na_{n+1}}, \tag{48}$$

where $\epsilon_{a_1\ldots a_n}$ is a completely antisymmetric unit tensor, must be added to the operators (47).

Casimir operators C^*_{2p} can be defined in another way [13] as a sum of principal minors of order $2p$ for antisymmetric matrix A, composed of generators X^*_{rs}, i.e. $(A)_{rs} = X^*_{rs}, (A)_{sr} = -X^*_{rs}$. To obtain Casimir operators of algebra $so(n+1;j)$ we use the method of section 3.2. We find $X^*_{rs} = (r,s)^{-1} X_{rs}$ from (31) and substitute in (47). The most singular coefficient $(0,n)^{-2p}$ is that of the term $X_{0n}X_{n0}\ldots X_{n0}$ in (47). To eliminate it in the minimal manner we multiply $\hat{C}^*_{2p}$ by $(0,n)^{2p}$. Thus, the rule of transformation for Casimir operators $\hat{C}_{2p}$ is

$$\hat{C}_{2p}(j; X_{rs}) = (0,n)^{2p} \hat{C}^*_{2p}((r,s)^{-1} X_{rs}), \tag{49}$$

and Casimir operators themselves turn out to be

$$\hat{C}_{2p}(j) = \sum_{a_1,\ldots,a_{2p}=0}^{n} (0,n)^{2p} \prod_{v=1}^{2p} (r_v, s_v)^{-1} X_{a_1a_2} \ldots X_{a_{2p}a_1}, \tag{50}$$

where $r_v = \min(a_v, a_{v+1})$, $s_v = \max(a_v, a_{v+1})$, $v = 1, 2, \ldots, 2p-1$, $r_{2p} = \min(a_1, a_{2p})$, $s_{2p} = \max(a_1, a_{2p})$.

For operators C_{2p} and C'_n the expression without singular terms can be obtained, multiplying them by factor q, equal to the least common denominator of coefficients of terms, arising after the substitution of generators X for X^*. This least common denominator can be found by induction [19]. We restrict ourselves with the final expressions for the rule of transformations for these Casimir operators:

$$C_{2p}(j; X_{rs}) = \left(\prod_{m=1}^{p-1} j_m^{2m} j_{n-m+1}^{2m} \prod_{l=p}^{n-p+1} j_l^{2p} \right) C^*_{2p}(X_{rs}(r,s)^{-1}),$$

$$p = 1, 2, \ldots, k,$$

$$C'_n(j; X_{rs}) = \left(j_{(n+1)/2}^{(n+1)/2} \prod_{m=1}^{(n-1)/2} j_m^m j_{n-m+1}^m \right) C'^*_n(X_{rs}(r,s)^{-1}). \tag{51}$$

Operator $C_{2p}(j)$ (or $C'(j)$) commutes with all generators X_{rs} of algebra $so(n+1;j)$. Really, evaluating zero commutator $[C^*_{2p}, X^*_{rs}]$, we get the same terms with the opposite signs. Under the transformations (31), (49) both terms are multiplied by the same combination of parameters, which is a product of even powers of parameters. Therefore, both

terms either change their sign, or vanish, or do not change their sign, but in all cases their sum is equal to zero. Moreover, operators $C_{2p}(j)$ for $p = 1, 2, \ldots, k$ are linearly independent because they consist of the different powers of generators X_{rs}.

The next question to be cleared up is as follows: do $[(n+1)/2]$ Casimir operators (51) exhaust all the invariant operators of algebra $so(n+1;j)$? The answer is given by the following theorem.

Theorem. For any set of values of parameters j the number of invariant operators of algebra $so(n+1;j)$ is $[(n+1)/2]$.

The proof is given in [23]. Thus, all invariant operators of algebra $so(n+1;j)$ are polynomial and are given by (51).

4 Cayley–Klein Unitary Groups and Algebras

4.1 Definitions, generators, commutators

Special unitary groups $SU(n+1;j)$ are connected with complex Cayley–Klein spaces $\mathbf{C}_{n+1}(j)$ which come out from $(n+1)$-dimensional complex space $\mathbf{C}_{n+1}$ under the mapping

$$\phi : \mathbf{C}_{n+1} \to \mathbf{C}_{n+1}(j)$$

$$\phi z_0^* = z_0^*, \quad \phi z_k^* = (0,k) z_k, \quad k = 1, 2, \ldots, n, \tag{52}$$

where $z_0^*, z_k^* \in \mathbf{C}_{n+1}$, $z_0, z_k \in \mathbf{C}_{n+1}(j)$ are complex Cartesian coordinates; $j = (j_1, \ldots, j_n)$, each of parameters j_k takes three values: $j_k = 1, \iota_k, i$. Quadratic form $(z^*, z^*) = \sum_{m=0}^{n} |z_m^*|^2$ of the space $\mathbf{C}_{n+1}$ turns into quadratic form

$$(z,z) = |z_0|^2 + \sum_{k=1}^{n} (0,k)^2 |z_k|^2 \tag{53}$$

of the space $\mathbf{C}_{n+1}(j)$ under the mapping (52). Here $|z_k| = (x_k^2 + y_k^2)^{1/2}$ is absolute value (modulus) of complex number $z_k = x_k + jy_k$, and z is complex vector: $z = (z_0, z_1, \ldots, z_n)$.

Definition of complex fiber space is similar to the real fiber space in section 3.3.

Definition. Group $SU(n+1;j)$ consists of all transformations of space $\mathbf{C}_{n+1}(j)$ with unit determinant, keeping invariant the quadratic form (53).

In the $(k_1, k_2, \ldots, k_p)$-fiber space $\mathbf{C}_{n+1}(j)$ we have $p+1$ quadratic forms, which remains invariant under transformations of group $SU(n+1;j)$. Under transformations of group $SU(n+1;j)$, which do not affect coordinates $z_0, z_1, \ldots, z_{k_s-1}$, the form

$$(z,z)_{s+1} = \sum_{a=k_s}^{k_{s+1}-1} (k_s, a)^2 |z_a|^2, \tag{54}$$

where $s = 0, 1, \ldots, p$, $k_0 = 0$, remains invariant. For $s = p$ the summation over a goes up to n.

The mapping (52) induces the transition of classical group $SU(n+1)$ into group $SU(n+1;j)$, correspondingly, of algebra $su(n+1)$ into algebra $su(n+1;j)$. All $(n+1)^2 - 1$

generators of algebra $su(n+1)$ are Hermitian matrices. However, because the commutators for Hermitian generators are not symmetric, usually one prefers matrix generators A^*_{km}, $k,m = 0,1,2,\ldots,n$ of general linear algebra $gl_{n+1}(R)$, such that $(A^*_{km})_{km} = 1$ and all other matrix elements vanish. (The asterisk means that A^* is a generator of a classical algebra.) The commutators of generators A^* satisfy the following relations

$$[A^*_{km}, A^*_{pq}] = \delta_{mp}A^*_{kq} - \delta_{kq}A^*_{pm}, \tag{55}$$

where δ_{mp} is Kronecker symbol. Independent Hermitian generators of algebra $su(n+1)$ are given by the equations

$$Q^*_{rs} = \frac{i}{2}(A^*_{rs} + A^*_{sr}), \quad L^*_{rs} = \frac{1}{2}(A^*_{sr} - A^*_{rs}),$$

$$P^*_k = \frac{i}{2}(A^*_{k-1,k-1} - A^*_{kk}), \tag{56}$$

where $r = 0,1,\ldots,n-1$, $s = r+1, r+2,\ldots,n$, $k = 1,2,\ldots,n$.

Matrix generators A^* are transformed under the mapping (52) as follows:

$$A_{rs}(j) = (r,s)A^*_{rs}, \quad A_{kk}(j) = A^*_{kk}. \tag{57}$$

The commutators of generators $A(j)$ can be easily found [31]:

$$[A_{km}, A_{pq}] = (k,m)(p,q)\left(\delta_{mp}A_{kq}(k,q)^{-1} - \delta_{kp}A_{pm}(m,p)^{-1}\right). \tag{58}$$

Hermitian generators (56) are transformed in the same way under transition from algebra $su(n+1)$ to algebra $su(n+1;j)$. This enables to find matrix generators of algebra $su(n+1;j)$ for the case, when group $SU(n+1;j)$ acts in the space $C_{n+1}(j)$ with named coordinates

$$Q_{rs}(j) = (r,s)Q^*_{rs}, \quad L_{rs}(j) = (r,s)L^*_{rs}, \quad P_k(j) = P^*_k. \tag{59}$$

We do not cite the commutation relations for these generators because they are cumbersome. They can be found, using (58).

Let us cite one more realization of generators for unitary group. If group GL_{n+1} acts via left translations in the space of analytic functions on $\mathbf{C}_{n+1}$, then the generators of its algebra are $X^*_{ab} = z^{*b}\partial^*_a$, where $\partial^*_a = \frac{\partial}{\partial z^{*a}}$. Hermitian generators of algebra $su(n+1)$ can be expressed in terms of X^*_{ab} using (56), in which A^* must be changed for X^*. Under the mapping ψ they are transformed according to the rule

$$Z_{ab} = (a,b)Z^*_{ab}(\psi z^*), \tag{60}$$

where $Z_{ab} = Q_{rs}$, L_{rs}, $P_k = P_{kk}$. Generators X^*_{ab} are transformed in a similar way, and this gives us

$$X_{kk} = z_k\partial_k, \; X_{sr} = z_r\partial_s, \; X_{rs} = (r,s)^2 z_s\partial_r, \tag{61}$$

where $k = 1,2,\ldots,n$, $r,s = 0,1,\ldots,n$, $r < s$.

The matrix generators (59) make a basis of Lie algebra $su(n+1;j)$. To the general element of the algebra

$$Z(\mathbf{u},\mathbf{v},\mathbf{w};j) = \sum_{t=1}^{n(n+1)/2}(u_tQ_t(j) + v_tL_t(j)) + \sum_{k=1}^{n} w_kP_k, \tag{62}$$

where index t is connected with the indices r, s, $r < s$, by relation

$$t = s + r(n-1) - \frac{r(r-1)}{2}, \tag{63}$$

and the group parameters u_t, v_t, w_k are real, corresponds a finite group transformation of group $SU(n+1;j)$

$$W(\mathbf{u},\mathbf{v},\mathbf{w};j) = \exp\{Z(\mathbf{u},\mathbf{v},\mathbf{w};j)\}. \tag{64}$$

According to Cayley–Hamilton theorem, matrix W can be algebraically expressed in terms of matrices Z^m, $m = 0, 1, 2, \ldots, n$, but one can to derive it explicitly only for groups $SU(2;j_1)$ and $SU(3;j_1,j_2)$, which will be discussed in the next sections.

4.2 Unitary group $SU(2;j_1)$

The group $SU(2;j_1)$ is the simplest one from unitary Cayley–Klein groups. **Definition.** The set of all transformations of the space $\mathbf{C}_2(j_1)$, leaving invariant the quadratic form $|z_0|^2 + j_1^2|z_1|^2$, make up the special unitary Cayley-Klein group $SU(2;j_1)$.

The group $SU(2;j_1)$ acts on the space $\mathbf{C}_2(j)$

$$z'(j_1) = \begin{pmatrix} z'_0 \\ j_1 z'_1 \end{pmatrix} = \begin{pmatrix} \alpha & j_1\beta \\ -j_1\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} z_0 \\ j_1 z_1 \end{pmatrix} = u(j_1)z(j_1),$$

$$\det u(j_1) = |\alpha|^2 + j_1^2|\beta|^2 = 1, \quad u(j_1)u^\dagger(j_1) = 1, \tag{65}$$

Here the bar notes complex conjugation. Constructing generators of algebra $su(2;j_1)$ according to (59), we get

$$P_1 = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q_{01} = \frac{i}{2}\begin{pmatrix} 0 & j_1 \\ j_1 & 0 \end{pmatrix}, \quad L_{01} = \frac{1}{2}\begin{pmatrix} 0 & -j_1 \\ j_1 & 0 \end{pmatrix}, \tag{66}$$

and find commutation relations

$$[P_1, Q_{01}] = L_{01}, \quad [L_{01}, P_1] = Q_{01}, \quad [Q_{01}, L_{01}] = j_1^2 P_1. \tag{67}$$

The generators (66) for $j_1 = 1$ up to factors coincide with Pauli matrices. It is also worth mentioning that if under contraction $j_1 = \imath_1$ the dimension (number of linearly independent generators) of general linear group $GL(2;j_1)$ (or its algebra) diminishes, because the generator $A_{01}(\imath_1)$ vanishes, then for special unitary groups (algebras) in complex Cayley–Klein spaces the dimension of the groups (algebras) for any (including nilpotent) values of parameters remains unchanged.

One-dimensional subgroup, corresponding to the generators (66), are as follows:

$$g_1(r;j_1) = \exp rQ_{01}(j_1) = \begin{pmatrix} \cos\frac{1}{2}j_1 r & i\sin\frac{1}{2}j_1 r \\ i\sin\frac{1}{2}j_1 r & \cos\frac{1}{2}j_1 r \end{pmatrix},$$

$$g_2(s;j_1) = \exp sL_{01}(j_1) = \begin{pmatrix} \cos\frac{1}{2}j_1 s & -\sin\frac{1}{2}j_1 s \\ \sin\frac{1}{2}j_1 s & \cos\frac{1}{2}j_1 s \end{pmatrix},$$

$$g_3(w) = \exp wP_1 = \begin{pmatrix} e^{iw/2} & 0 \\ 0 & e^{-iw/2} \end{pmatrix}, \tag{68}$$

and to general element $Z = rQ_{01} + sL_{01} + wP_1$ of algebra $su(2; j_1)$ we, using exponential mapping, put in correspondence the matrix of finite transformation of group $SU(2; j_1)$, which can be easily found

$$g(\zeta, w; j_1) = \exp Z = \begin{pmatrix} \cos\frac{v}{2} + i\frac{w}{v}\sin\frac{v}{2} & -j_1\frac{\zeta}{v}\sin\frac{v}{2} \\ j_1\frac{\zeta}{v}\sin\frac{v}{2} & \cos\frac{v}{2} - i\frac{w}{v}\sin\frac{v}{2} \end{pmatrix},$$

$$v^2(j_1) = w^2 + j_1^2|\zeta|^2, \quad \zeta = s + ir. \tag{69}$$

In Euler parametrization [56] transformations from group $SU(2; j_1)$ can be written as

$$g(\varphi, \theta, \omega; j_1) = g_3(\varphi; j_1)g_1(\theta; j_1)g_3(\omega; j_1) =$$

$$= \begin{pmatrix} e^{i\frac{\omega+\varphi}{2}}\cos j_1\frac{\theta}{2} & e^{-i\frac{\omega-\varphi}{2}}\, i\sin j_1\frac{\theta}{2} \\ e^{i\frac{\omega-\varphi}{2}}\, i\sin j_1\frac{\theta}{2} & e^{-i\frac{\omega+\varphi}{2}}\cos j_1\frac{\theta}{2} \end{pmatrix}, \tag{70}$$

where group parameters (Euler angles) are in the bounds

$$0 \le \varphi < 2\pi, \quad -2\pi \le \omega \le 2\pi, \quad \theta \in \Theta(j) = \begin{cases} (0, \pi), & j_1 = 1 \\ (0, \infty), & j_1 = \iota \\ (-\infty, 0), & j_1 = i. \end{cases} \tag{71}$$

Let us note, that for $j_1 = 1$ matrices $g(\varphi, \theta, \omega; j_1)$ coincide with matrices (1.1.3–4), ch. III in [56], for $j_1 = i$ they coincide with the matrices (1.3.4–5), ch. VI in [56], and for $j_1 = \iota_1$ they describe Euclidean group $SU(2; \iota_1)$ in Euler parametrization.

5 Classification of Transitions between Cayley–Klein Spaces and Groups

In the previous sections we have found orthogonal and unitary groups in Cayley–Klein spaces and shown that their generators, Casimir operators and other algebraic constructions can be obtained by transformation of the corresponding constructions for classical groups. Such approach is natural and is justified by the fact that classical groups and their characteristic algebraical constructions are well studied. But is such approach the only one? Is it possible to take one of the groups in Cayley–Klein space as the initial one? The positive answer to this question is given by the following theorem on the structure of transitions between groups.

Let us define (formally) the transition from the space $\mathbf{C}_{n+1}(j)$ and the generators $Z_{ab}(\mathbf{z}; j)$ of unitary group $SU(n+1; j)$ to the space $\mathbf{C}_{n+1}(j')$ and the generators $Z_{ab}(\mathbf{z}'; j')$ via transformations, which can be obtained from the transformations (52) and (60), substituting in the latter the parameters j_k for $j'_k j_k^{-1}$:

$$\phi' : \mathbf{C}_{n+1}(j) \to \mathbf{C}_{n+1}(j')$$

$$\phi' z_0 = z'_0, \; \phi' z_k = z'_k \prod_{m=1}^{k} j'_m j_m^{-1}, \quad k = 1, 2, \ldots, n,$$

$$Z_{ab}(\mathbf{z}';j') = (\prod_{i=1+\min(a,b)}^{\max(a,b)} j'_i j_i^{-1}) Z_{ab}(\phi'\mathbf{z};j). \tag{72}$$

The inverse transitions can be obtained from (72) by the change of the dashed parameters j' for the undashed parameters j and vice versa. Applying (72) to the quadratic form (53) and the generators (61), we obtain

$$(z',z') = |z'_0|^2 + \sum_{k=1}^{n} |z'_k|^2 \prod_{m=1}^{k} j'^2_m,$$

$$X_{kk} = z'_k\partial'_k,\ X_{sr} = z'_r\partial'_s,\ X_{rs} = (\prod_{l=1+r}^{s} j'^2_l) z'_s\partial'_r, \tag{73}$$

i.e. quadratic form in space $C_{n+1}(j')$ and generators of group $SU(n+1;j')$.

However, the constructed transitions do not make sense for all groups and spaces, because for the nilpotent values of parameters j the expressions ι_k^{-1}, $\iota_m \cdot \iota_k^{-1}$ for $k \neq m$ are not defined. We have defined in section 2 only the expressions $\iota_k \cdot \iota_k^{-1} = 1$, $k = 1, 2, \ldots, n$. So if for some k we put $j_k = \iota_k$, then the transformations (72) will be defined and give us (73) only in the case when the dashed parameter with the same number is equal to the same nilpotent number, i.e. $j'_k = \iota_k$.

The transitions from space $\mathbf{R}_{n+1}(j)$ to space $\mathbf{R}_{n+1}(j')$, and from groups $SO(n+1;j)$, $Sp(n;j)$ to groups $SO(n+1;j')$, $Sp(n;j')$ as well, can be correspondingly, obtained from the transition (45), (31) by the same substitution of parameters j_k for $j'_k j_k^{-1}$. Similarly can be justified the permissibility of these relations. Let us introduce the notations: $G(j) = SO(n+1;j)$, $SU(n+1;j)$, $Sp(n;j)$, $\mathbf{R}(j) = \mathbf{R}_{n+1}(j)$, $\mathbf{C}_{n+1}(j)$, $\mathbf{R}_n(j) \rtimes \mathbf{R}_n(j)$ and denote the transformation of group generators by the symbol $\Phi G(j) = G(j')$. Easy analysis of the transformations (72) and their inverse transformations from the point of view of admissibility of the transitions [24] implies the following theorem.

Transitions classification theorem. I. Let $G(j)$ be a group in non-fiber space $\mathbf{R}(j)$ and $G(j')$ be a group in arbitrary space $\mathbf{R}(j')$, then $G(j') = \Psi G(j)$. If $\mathbf{R}(j')$ is a non-fiber space, then Ψ is one-to-one mapping, and $G(j) = \Psi^{-1}G(j')$.

II. Let $G(j)$ be a group in $(k_1, k_2, \ldots, k_p)$-fiber space $\mathbf{R}(j)$ and $G(j')$ be a group in $(m_1, m_2, \ldots, m_q)$-fiber space $\mathbf{R}(j')$, then $G(j') = \Psi G(j)$, if the set of integers $(k_1, \ldots, k_p)$ is involved in the set of numbers $(m_1, \ldots, m_q)$. The inverse transition $G(j) = \Phi^{-1}G(j')$ is valid if and only if $p = q$, $k_1 = m_1, \ldots, k_p = m_p$.

It follows from the theorem that the group $G(j)$ for any set of values of the parameters j can be obtained not only from a classical group, but from a group in an arbitrary non-fiber Cayley–Klein space, i.e. from pseudoorthogonal, pseudounitary or pseudosymplectic groups. It is naturally that the transitions between other algebraic constructions, in particular between Casimir operators, are described by this theorem as well.

6 Kinematics as Spaces of Constant Curvature

Possible kinematic groups, i.e. groups of motion for four-dimensional models of space-time (kinematics), satisfying natural physical postulates: 1) space is isotropic; 2) spatial

property of being even inversion of time are automorphisms of kinematic groups; 3) boosts (rotations in spatial-temporal plane) make a non-compact subgroups are described by H. Bacry and J.-M. Levy-Leblond [1]. In [2] H. Bacry and J. Nuyts rejected postulates 2) and 3) and obtained a more wider set of groups with spatial isotropy. Now we shall bring the geometric interpretation of kinematics [22, 33].

All kinematic groups are 10-dimensional; for this reason kinematics from the geometrical point of view, should be among four-dimensional maximally homogeneous spaces — spaces of constant curvature, which groups of motions are of dimension 10. These spaces are realized on the connected components of the sphere

$$S_4(j) = \{x_0^2 + \sum_{k=1}^{4}(0,k)x_k^2 = 1\}. \tag{74}$$

Let us introduce internal (Beltramian) coordinates $\xi_k = x_k/x_0,\ k = 1,2,3,4$ on $S_4(j)$. The generators (32) of group $SO(4;j)$ can be expressed in terms of the internal coordinates ξ via formulas

$$X_{0s}(\xi) = -\partial_1 - (0,s)^2\xi_s\sum_{k=1}^{4}\xi_k\partial_k, \quad \partial_k = \partial/\partial\xi_k,$$

$$X_{rs}(u) = -\xi_r\partial_s + (r,s)^2\xi_s\partial_r, \quad r < s,\ r,s = 1,2,3,4 \tag{75}$$

and satisfy the commutation relations (46). The generator $X_{0s}(u)$ has a meaning of generator for translation along the s-th Beltrami axis, and $X_{rs}(u)$ is generator of rotation in two-dimensional plane $\{\xi_r, \xi_s\}$.

Physical postulates 1)–3) can be expressed in terms of parameters j. Postulate 1) means that under the transformations (45) three Beltrami coordinates should be multiplied by the same quantity and interpreted as a temporal axis of kinematics. It is possible in two cases:

A) for $j_3 = j_4 = 1$, when coordinates ξ_2, ξ_3, ξ_4 are multiplied by the product $j_1 j_2$ and called spatial and ξ_1 is multiplied by j_1 and called temporal;

B) for $j_2 = j_3 = 1$, when the spatial coordinates $\xi_k = r_k,\ k = 1,2,3$ are multiplied by j_1, and temporal coordinate $\xi_4 = t$ is multiplied by the product $j_1 j_4$.

Postulate 3) imposes restrictions on the character of rotations in two-dimensional planes, spanned over temporal and spatial axes of kinematics, requiring these rotations to be Lorentzian and Galilean. In terms of parameters j this gives $j_2 = \iota_2, i$ in the case A) and $j_4 = \iota_4, i$ in the case B). The requirements of postulate 2) can be taken into account by the definition of space with the constant curvature as a connected component of the sphere (74).

In the case A) the kinematic generators H, $\mathbf{P} = (P_1, P_2, P_3)$ (spatial-temporal translations), $\mathbf{J} = (J_1, J_2, J_3)$ (rotations), $\mathbf{K} = (K_1, K_2, K_3)$ (boosts) are expressed in terms of generators (75) in accordance with above mentioned interpretation by the relations $H = -X_{01},\ P_k = -X_{0,k+1},\ K_k = -X_{1,k+1},\ J_1 = X_{34},\ J_2 = -X_{24},\ J_3 = X_{23},\ k = 1,2,3$, and satisfy the commutation relations

$$[H, \mathbf{J}] = 0, \quad [H, \mathbf{K}] = \mathbf{P}, \quad [H, \mathbf{P}] = -j_1^2\mathbf{K}$$

$$[\mathbf{P}, \mathbf{P}] = j_1^2 j_2^2\mathbf{J}, \quad [\mathbf{K}, \mathbf{K}] = j_2^2\mathbf{J}, \quad [P_k, K_l] = -j_2^2\delta_{kl}H. \tag{76}$$

Here $[\mathbf{X},\mathbf{Y}]=\mathbf{Z}$ means $[X_k,Y_l]=e_{klm}Z_m$, where e_{klm} is the antisymmetric unit tensor. The spaces of constant curvature $S_4(j_1,j_2,1,1)\equiv S_4(j_1,j_2)$, $j_1=1,\iota_1,i$, $j_2=\iota_2,i$ are shown at Fig. 2 (see section 3.2), where the spatial axis r should be imagined as a three-dimensional space. Semispherical group $SO(5;1,\iota_2)$ and semihyperbolic group $SO(5;i,\iota_2)$ correspond to Newton groups $N_\pm$ (sometimes the latter are called Hooke groups). The interpretation of other groups is well-known.

In the case of B) the temporal and spatial axes of kinematics are expressed in another way in terms of Beltramian coordinates of space with the constant curvature; correspondingly, the geometrical generators $X(\xi)$ obtain another kinematic interpretation: $H=X_{04}$, $P_k=-X_{0k}$, $K_k=X_{k4}$, $J_1=X_{23}$, $J_2=-X_{13}$, $J_3=X_{12}$ and satisfy the commutation relations

$$\begin{gathered}[\mathbf{J},\mathbf{J}]=\mathbf{J},\quad [\mathbf{J},\mathbf{P}]=\mathbf{P},\quad [\mathbf{J},\mathbf{K}]=\mathbf{K},\\ [H,\mathbf{J}]=0,\quad [H,\mathbf{K}]=-j_4^2\mathbf{P},\quad [H,\mathbf{P}]=j_1^2\mathbf{K},\\ [\mathbf{P},\mathbf{P}]=j_1^2\mathbf{J},\quad [\mathbf{K},\mathbf{K}]=j_4^2\mathbf{J},\quad [P_k,K_l]=\delta_{kl}H.\end{gathered}\tag{77}$$

The value of parameter $j_4=i$, as it can be readily understood, does not lead to new kinematics, because $SO(5;j_1,1,1,i)$ for $j_1=1,\iota_1,i$ is, correspondingly, de Sitter group, Poincaré group and anti-de Sitter group.

Kinematic Carroll group [19] of motions of the flat Carroll space, first described in physical terms by J.-M. Levy-Leblond [40] corresponds to the values of parameters $j_1=\iota_1$, $j_4=\iota_4$. Comparing the commutators (77) with the commutators in the paper [1] by H. Bacry and J.-M. Levy-Leblond, we find that group $SO(5;1,1,1,\iota_4)$ coincides with kinematic group $ISO(4)$, and group $SO(5;i,1,1,\iota_4)$ is "para-Poincaré" group P'. As parameter j_1 determines the sign of the space curvature (curvature is positive for $j_1=1$, zero for $j_1=\iota_1$ and negative for $j_1=i$) we conclude that group $SO(5;1,1,1,\iota_4)$ (or $ISO(4)$) is the group of motions of Carroll kinematics with a positive curvature, group $SO(5;1,1,1,\iota_4)$ (or P') is the group of motions of Carroll kinematics with a negative curvature. Such interpretation of kinematic groups $ISO(4)$ and P', as far as it can be seen, was not recognized by the authors of [1], and this fact, by the way, is reflected in the names and notations of these groups. Further Carroll kinematics will be denoted as $\mathcal{C}_4(j_1)$, and their kinematic groups as $G(j_1)=SO(5;j_1,1,1,\iota_4)$.

H. Bacry and J.-M. Levy-Leblond [1] have described 11 kinematical groups. Nine of them have obtained geometrical interpretation as spaces of constant curvature. The rest two kinematics — "para-Galilean" and static — can not be identified with any of the spaces of constant curvature. For example, kinematic "para-Galilean" group is obtained from Galilean group $SO(5;\iota_1,\iota_2)$ by substitution $\mathbf{P}\to\mathbf{K}$, $\mathbf{K}\to\mathbf{P}$, i.e. under the new interpretation of generators , in which the generators of spatial translations of Galilean kinematics are claimed to be the generators of boosts of "para-Galilean" kinematics, and the generators of Galilean boosts — to be the generators of spatial "para-Galilean" translations.

7 Standard Electroweak Model

The standard Electroweak Model (Weinberg-Glashow-Salam theory) is a gauge theory based on the group $SU(2)\times U(1)$ and gives a good description of electroweak processes

[46, 47, 55]. Mathematically this theory is very complicated with nonlinear dynamics of the involved fields.

The Electroweak Model involve particles with integer spins: photon, responsible for electromagnetic interactions, neutral Z^0 and charged $W^{\pm}$ bosons, which are week interaction carriers. For each subgroup $SU(2)$ and $U(1)$ of the gauge group its own coupling constants g and g' are introduced. Complex space $\mathbf{C}_2$ of the fundamental representation of the group $SU(2)$ is interpreted as the space of matter fields $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in \mathbf{C}_2$. Gauge fields $A_\mu(x)$ for the group $SU(2)$ take their values in Lie algebra $su(2)$

$$A_\mu(x) = -ig \sum_{k=1}^{3} T_k A_\mu^k(x), \tag{78}$$

where matrices T_k, connected with Pauli matrices τ^k by the following relations

$$T_1 = \frac{1}{2}\tau^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T_2 = \frac{1}{2}\tau^2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$T_3 = \frac{1}{2}\tau^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{79}$$

submit commutation relations $[T_k, T_p] = i\epsilon_{kpr}T_r$ and represent the algebra $su(2)$ with structure constants $C_{kpr} = \epsilon_{kpr}$. The gauge fields (78) are as follows in the matrix form:

$$A_\mu(x) = -i\frac{g}{2}\begin{pmatrix} A_\mu^3 & A_\mu^1 - iA_\mu^2 \\ A_\mu^1 + iA_\mu^2 & -A_\mu^3 \end{pmatrix}. \tag{80}$$

For the group $U(1)$ with generator $Y = \frac{1}{2}1$ the gauge field takes the form

$$B_\mu(x) = -i\frac{g'}{2}\begin{pmatrix} B_\mu & 0 \\ 0 & B_\mu \end{pmatrix} \tag{81}$$

and has stress tensor $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. For the field $A_\mu(x)$ its stress tensor is given by

$$F_{\mu\nu}(x) = \mathcal{F}_{\mu\nu}(x) + [A_\mu(x), A_\nu(x)] \tag{82}$$

and has the components

$$F_{\mu\nu}^1 = \mathcal{F}_{\mu\nu}^1 + g(A_\mu^2 A_\nu^3 - A_\mu^3 A_\nu^2) = \mathcal{F}_{\mu\nu}^1 + g\sum_{k,m=1}^{3} \epsilon_{1km} A_\mu^k A_\nu^m,$$

$$F_{\mu\nu}^2 = \mathcal{F}_{\mu\nu}^2 + g(A_\mu^3 A_\nu^1 - A_\mu^1 A_\nu^3) = \mathcal{F}_{\mu\nu}^2 + g\sum_{k,m=1}^{3} \epsilon_{2km} A_\mu^k A_\nu^m,$$

$$F_{\mu\nu}^3 = \mathcal{F}_{\mu\nu}^3 + g(A_\mu^1 A_\nu^2 - A_\mu^2 A_\nu^1) = \mathcal{F}_{\mu\nu}^3 + g\sum_{k,m=1}^{3} \epsilon_{3km} A_\mu^k A_\nu^m, \tag{83}$$

where $\mathcal{F}^k_{\mu\nu} = \partial_\mu A^k_\nu - \partial_\nu A^k_\mu$ is the stress tensor for Abel group. Boson sector of Electroweak Model is characterized by Lagrangian

$$L_B = L_A + L_\phi, \tag{84}$$

which comprises two parts: the gauge fields Lagrangian

$$L_A = \frac{1}{8g^2}\mathrm{Tr}(F_{\mu\nu})^2 - \frac{1}{4}(B_{\mu\nu})^2 =$$

$$= -\frac{1}{4}[(F^1_{\mu\nu})^2 + (F^2_{\mu\nu})^2 + (F^3_{\mu\nu})^2] - \frac{1}{4}(B_{\mu\nu})^2, \tag{85}$$

and the matter fields Lagrangian

$$L_\phi = \frac{1}{2}(D_\mu\phi)^\dagger D_\mu\phi - V(\phi). \tag{86}$$

The potential is taken in the special form

$$V(\phi) = \frac{\lambda}{4}\left(\phi^\dagger\phi - v^2\right)^2, \tag{87}$$

where λ, v are constants. Covariant derivative

$$D_\mu\phi = \partial_\mu\phi - ig\left(\sum_{k=1}^{3} T_k A^k_\mu\right)\phi - ig'YB_\mu\phi \tag{88}$$

for the matter fields ϕ_1, ϕ_2 is given by

$$D_\mu\phi_1 = \partial_\mu\phi_1 - \frac{i}{2}(gA^3_\mu + g'B_\mu)\phi_1 - \frac{ig}{2}(A^1_\mu - iA^2_\mu)\phi_2,$$

$$D_\mu\phi_2 = \partial_\mu\phi_2 + \frac{i}{2}(gA^3_\mu - g'B_\mu)\phi_2 - \frac{ig}{2}(A^1_\mu + iA^2_\mu)\phi_1. \tag{89}$$

Space-time variables are numbered by Greek indexes $\mu, \nu, \ldots = 0, 1, 2, 3$.

To obtain vector boson masses the special mechanism of spontaneous symmetry breaking (or Higgs mechanism) is used. One of Lagrangian L_B ground states

$$\phi^{vac} = \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad A^k_\mu = B_\mu = 0 \tag{90}$$

is taken as vacuum of the model, and then small field excitations

$$\phi_1(x), \quad \phi_2(x) = v + \chi(x), \quad A^a_\mu(x), \quad B_\mu(x) \tag{91}$$

with respect to this vacuum are regarded. Matrix $Q = Y + T_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, which annihilates the ground state $Q\phi^{vac} = 0$, is the generator of electromagnetic subgroup $U(1)_{em}$. New gauge fields are introduced

$$W^\pm_\mu = \frac{1}{\sqrt{2}}\left(A^1_\mu \mp iA^2_\mu\right), \quad Z_\mu = \frac{1}{\sqrt{g^2 + g'^2}}\left(gA^3_\mu - g'B_\mu\right),$$

$$A_\mu = \frac{1}{\sqrt{g^2+g'^2}}\left(g'A_\mu^3 + gB_\mu\right), \tag{92}$$

where $W_\mu^\pm$ are complex fields $\bar{W}_\mu^- = W_\mu^+$, and Z_μ, A_μ are real fields.

Boson Lagrangian (84) can be rewritten in the form

$$L_B = L_B^{(2)} + L_B^{int}. \tag{93}$$

As usual, the second order terms

$$L_B^{(2)} = \frac{1}{2}(\partial_\mu\chi)^2 - \frac{1}{2}m_\chi^2\chi^2 - \frac{1}{4}\mathcal{Z}_{\mu\nu}\mathcal{Z}_{\mu\nu} + \frac{1}{2}m_Z^2 Z_\mu Z_\mu - \\ -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu} - \frac{1}{2}\mathcal{W}_{\mu\nu}^+\mathcal{W}_{\mu\nu}^- + m_W^2 W_\mu^+ W_\mu^-, \tag{94}$$

where $\mathcal{Z}_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu$, $\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\mathcal{W}_{\mu\nu}^\pm = \partial_\mu W_\nu^\pm - \partial_\nu W_\mu^\pm$, describe the spectrum of boson particles, and higher-order terms L_B^{int} are interpreted as their interactions. So, Lagrangian (94) describes charged W-bosons with identical mass $m_W = \frac{1}{2}gv$, massless photon A_μ, $m_A = 0$, neutral Z-boson with mass $m_Z = \frac{v}{2}\sqrt{g^2+g'^2}$ and Higgs boson χ, $m_\chi = \sqrt{2\lambda}v$. W- and Z-bosons were observed experimentally and have masses $m_W = 80\,\text{GeV}$, $m_Z = 91\,\text{GeV}$. Higgs boson with the mass of 125 GeV was detected at LHC in 2012.

Besides gauge bosons, there are fermions in the Electroweak Model. The fermion sector is represented by leptons and quarks. Leptons are fermions, which do not interact strongly. There are three types of charged leptons in Nature: electron e^-, muon μ^-, τ-muon τ^- and three types of neutrinos ν_e, ν_μ, ν_τ, as well as the corresponding antiparticles. Neutrino masses, if they exist, are extremely small, therefore in the Electroweak Model neutrinos are considered as massless particles. Neutrinos are fermions of left chirality, i.e. their spin projection is opposite to the direction of movement. The name "left fermion" , is used in this case. Pairs (or generations) of leptons (ν_e, e^-), (ν_μ, μ^-), (ν_τ, τ^-) have identical properties with respect to all interactions. Therefore it is sufficient to discuss only one generation, for example, (ν_e, e^-).

The lepton Lagrangian is taken in the form

$$L_L = L_l^\dagger i\tilde{\tau}_\mu D_\mu L_l + e_r^\dagger i\tau_\mu D_\mu e_r - h_e[e_r^\dagger(\phi^\dagger L_l) + (L_l^\dagger\phi)e_r], \tag{95}$$

where $L_l = \begin{pmatrix}\nu_l \\ e_l\end{pmatrix}$ is $SU(2)$-doublet (vector in the space $\mathbf{C}_2$), e_r is $SU(2)$-singlet (scalar with respect of $SU(2)$), h_e is a constant. All fields e_r, e_l, ν_l are in their turn two-component Lorentz spinors. Here τ_μ are Pauli matrixes, $\tau_0 = \tilde{\tau}_0 = \mathbf{1}$, $\tilde{\tau}_k = -\tau_k$. The above mentioned division of the fields on doublets and singlets is based on the experimental fact that only the *left* components of an electron and a neutrino interact with $W^\pm$-boson fields, but the right components do not interacted with $W^\pm$-boson.

The covariant derivatives of the lepton fields $D_\mu L_l$ in (95) are given by the formula (88) for $Y = -\frac{1}{2}$ with L_l instead of ϕ, and $D_\mu e_r = (\partial_\mu + ig'B_\mu)e_r$. For the new fields (92) these derivatives are as follows

$$D_\mu e_r = \partial_\mu e_r + ig'A_\mu e_r\cos\theta_w - ig'Z_\mu e_r\sin\theta_w,$$

$$D_\mu = \partial_\mu - i\frac{g}{\sqrt{2}}\left(W_\mu^+ T_+ + W_\mu^- T_-\right) -$$

$$-i\frac{g}{\cos\theta_w} Z_\mu \left(T_3 - Q\sin^2\theta_w\right) - ieA_\mu Q, \tag{96}$$

where $T_\pm = T_1 \pm iT_2$, and e is electron charge

$$O = Y + T_3|_{Y=-\frac{1}{2}} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \frac{gg'}{\sqrt{g^2+g'^2}},$$

$$g = \frac{e}{\sin\theta_w}, \quad \cos\theta_w = \frac{g}{\sqrt{g^2+g'^2}}, \quad \sin\theta_w = \frac{g'}{\sqrt{g^2+g'^2}}. \tag{97}$$

Then lepton Lagrangian (95) can be rewritten in the form

$$L_L = e_l^\dagger i\tilde{\tau}_\mu \partial_\mu e_l + \nu_l^\dagger i\tilde{\tau}_\mu \partial_\mu \nu_l + e_r^\dagger i\tau_\mu \partial_\mu e_r + \frac{g}{2\cos\theta_w}\nu_l^\dagger \tilde{\tau}_\mu Z_\mu \nu_l +$$

$$+\frac{g}{\sqrt{2}} e_l^\dagger \tilde{\tau}_\mu W_\mu^- \nu_l - e e_l^\dagger \tilde{\tau}_\mu A_\mu e_l + \frac{g\cos 2\theta_w}{2\cos\theta_w} e_l^\dagger \tilde{\tau}_\mu Z_\mu e_l +$$

$$+\frac{g}{\sqrt{2}}\nu_l^\dagger \tilde{\tau}_\mu W_\mu^+ e_l - g'\cos\theta_w e_r^\dagger \tau_\mu A_\mu e_r + g'\sin\theta_w e_r^\dagger \tau_\mu Z_\mu e_r -$$

$$-h_e[e_r^\dagger \phi_2^\dagger e_l + e_l^\dagger \phi_2 e_r + e_r^\dagger \phi_1^\dagger \nu_l + \nu_l^\dagger \phi_1 e_r]. \tag{98}$$

The first three terms are kinetic terms of the left electron, the left neutrino and the right electron. The last four terms with the multiplier h_e are mass terms of the electron. The rest of the terms describe the electron and neutrino interactions with the gauge bosons $A_\mu, Z_\mu, W_\mu^\pm$.

The next two lepton generations are introduced in the same way. They are left $SU(2)$-doublets

$$\begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_l, \quad \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_l, \quad Y = -\frac{1}{2} \tag{99}$$

and right $SU(2)$-singlets

$$\mu_r, \quad \tau_r, \quad Y = -1. \tag{100}$$

The complete lepton Lagrangian is given by the sum

$$L_L = L_{L,e} + L_{L,\mu} + L_{L,\tau}, \tag{101}$$

where each summand has the structure (98) with its own constants h_e, h_μ, h_τ.

Quarks are strong interacting fermions. Six types of quarks are known. From the viewpoint of electroweak interactions all known quarks are divided into three generations: (u,d), (c,s) and (t,b). Electroweak interactions of all quark generations are identical, therefore we discuss quarks of the first generation in the beginning. The quark Lagrangian is given by

$$L_Q = Q_l^\dagger i\tilde{\tau}_\mu D_\mu Q_l + u_r^\dagger i\tau_\mu D_\mu d_r -$$

$$-h_d[d_r^\dagger(\phi^\dagger Q_l) + (Q_l^\dagger \phi)d_r] - h_u[u_r^\dagger(\tilde{\phi}^\dagger Q_l) + (Q_l^\dagger \tilde{\phi})u_r], \tag{102}$$

where the left fields u- and d-quark of the first generation form doublet $Q_l = \begin{pmatrix} u_l \\ d_l \end{pmatrix}$ relative to the electroweak group $SU(2)$, and the right fields u_r, d_r are $SU(2)$-singlets, $\tilde{\phi}_i = \epsilon_{ik}\bar{\phi}_k, \epsilon_{00} = 1, \epsilon_{ii} = -1$ are conjugate representation of the group $SU(2)$, at last , h_u, h_d are constant multipliers for mass terms. All fields u_l, d_l, u_r, d_r are two-component Lorentz spinors.

The left fields of the next quark generations

$$\begin{pmatrix} c_l \\ s_l \end{pmatrix}, \quad \begin{pmatrix} t_l \\ b_l \end{pmatrix}, \quad Y = \frac{1}{6}, \tag{103}$$

are described by $SU(2)$-doublets, and the right fields are $SU(2)$-singlets

$$c_r,\ t_r, \quad Y = \frac{2}{3}; \quad s_r,\ b_r, \quad Y = -\frac{1}{3}. \tag{104}$$

The covariant derivatives are given by the formulae

$$D_\mu Q_l = \left(\partial_\mu - ig\sum_{k=1}^{3}\frac{\tau_k}{2}A_\mu^k - ig'\frac{1}{6}B_\mu\right)Q_l,$$

$$D_\mu a_r = \left(\partial_\mu - ig'\frac{2}{3}B_\mu\right)a_r, \quad D_\mu f_r = \left(\partial_\mu + ig'\frac{1}{3}B_\mu\right)f_r, \tag{105}$$

where $a = u, c, t$ and $f = d, s, b$, but Q_l now denote the left fields of all three quark generations. The complete quark Lagrangian is the sum

$$L_Q = L_{Q,(u,d)} + L_{Q,(c,s)} + L_{Q,(t,b)}, \tag{106}$$

where each term has the structure (102) with its own constants $h_u, h_d, h_c, h_s,\ h_t,\ h_b$.

Lagrangian of the Standard Electroweak Model is the sum

$$L = L_B + L_L + L_Q, \tag{107}$$

of boson L_B (84), (93), lepton L_L (98), (101) and quark L_Q (102), (106) Lagrangians.

8 The Electroweak Model with Contracted Gauge Group

As far as all three lepton and quark generations behave in the same way, we shall further consider only the first generations. Contracted gauge group $SU(2;j) \times U(1)$ acts in the boson, lepton and quark sectors. The contracted group $SU(2;j)$ is obtained by the consistent rescaling of the fundamental representation of the group $SU(2)$ and the space $\mathbf{C}_2$ [28, 29]:

$$z'(j) = \begin{pmatrix} jz_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} \alpha & j\beta \\ -j\bar{\beta} & \bar{\alpha} \end{pmatrix}\begin{pmatrix} jz_1 \\ z_2 \end{pmatrix} = u(j)z(j),$$
$$\det u(j) = |\alpha|^2 + j^2|\beta|^2 = 1, \quad u(j)u^\dagger(j) = 1, \tag{108}$$

where contraction parameter $j \to 0$ or is equal to the nilpotent unit $j = \iota$. The hermitian form $z^\dagger z(j) = j^2|z_1|^2 + |z_2|^2$. remains invariant under this rescaling. The actions of the unitary group $U(1)$ and the electromagnetic subgroup $U(1)_{em}$ in the fiber space $\mathbf{C}_2(\iota)$ with the base $\{z_2\}$ and the fiber $\{z_1\}$ are given by the same matrices as in the space $\mathbf{C}_2$.

The space $\mathbf{C}_2(j)$ of the fundamental representation of $SU(2;j)$ group can be obtained from $\mathbf{C}_2$ substituting z_1 by jz_1. The substitution $z_1 \to jz_1$ induces the substitution of the Lie algebra generators

$$T_1 \to jT_1, \quad T_2 \to jT_2, \quad T_3 \to T_3. \tag{109}$$

As far as the gauge fields take their values in Lie algebra, we can substitute the gauge fields instead of transforming the generators (109), namely:

$$A^1_\mu \to jA^1_\mu, \quad A^2_\mu \to jA^2_\mu, \quad A^3_\mu \to A^3_\mu, \quad B_\mu \to B_\mu. \tag{110}$$

Indeed, due to commutativity and associativity of multiplication by j we have

$$\begin{aligned} su(2;j) \ni \left\{A^1_\mu(jT_1) + A^2_\mu(jT_2) + A^3_\mu T_3\right\} \\ = \left\{(jA^1_\mu)T_1 + (jA^2_\mu)T_2 + A^3_\mu T_3\right\}. \end{aligned} \tag{111}$$

For the gauge fields (92) the substitutions (110) are as follows:

$$W^\pm_\mu \to jW^\pm_\mu, \quad Z_\mu \to Z_\mu, \quad A_\mu \to A_\mu. \tag{112}$$

The left lepton $L_l = \begin{pmatrix} \nu_l \\ e_l \end{pmatrix}$ and quark $Q_l = \begin{pmatrix} u_l \\ d_l \end{pmatrix}$ fields are $SU(2)$-doublets, so their components are transformed in the similar way as the components of the vector z, namely:

$$\nu_l \to j\nu_l, \quad e_l \to e_l, \quad u_l \to ju_l. \quad d_l \to d_l. \tag{113}$$

The right lepton and quark fields are $SU(2)$-singlets and therefore are not changed.

After the transformations (112), (113) and spontaneous symmetry breaking (90) the boson Lagrangian (84)–(86) can be represented in the form [27, 29]:

$$\begin{aligned} L_B(j) = L^{(2)}_B(j) + L^{int}_B(j) = \\ = \frac{1}{2}\left(\partial_\mu \chi\right)^2 - \frac{1}{2}m^2_\chi \chi^2 - \frac{1}{4}\mathcal{Z}_{\mu\nu}\mathcal{Z}_{\mu\nu} + \frac{1}{2}m^2_Z Z_\mu Z_\mu - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu} + \\ + j^2\left\{-\frac{1}{2}\mathcal{W}^+_{\mu\nu}\mathcal{W}^-_{\mu\nu} + m^2_W W^+_\mu W^-_\mu\right\} + L^{int}_B(j), \end{aligned} \tag{114}$$

where as usual the second order terms describe the boson particles content of the model. Higher order terms

$$L^{int}_B(j) = \frac{gm_z}{2\cos\theta_W}\left(Z_\mu\right)^2 \chi - \lambda v\chi^3 + \frac{g^2}{8\cos^2\theta_W}\left(Z_\mu\right)^2 \chi^2 - \frac{\lambda}{4}\chi^4 +$$

$$+ j^2\left\{-2ig\left(W^+_\mu W^-_\nu - W^-_\mu W^+_\nu\right)\left(\mathcal{F}_{\mu\nu}\sin\theta_W + \mathcal{Z}_{\mu\nu}\cos\theta_W\right) + gW^+_\mu W^-_\mu \chi -\right.$$

$$-\frac{i}{2}e\left[A_\mu\left(\mathcal{W}^+_{\mu\nu}W^-_\nu-\mathcal{W}^-_{\mu\nu}W^+_\nu\right)-A_\nu\left(\mathcal{W}^+_{\mu\nu}W^-_\mu-\mathcal{W}^-_{\mu\nu}W^+_\mu\right)\right]-$$

$$-\frac{i}{2}g\cos\theta_W\left[Z_\mu\left(\mathcal{W}^+_{\mu\nu}W^-_\nu-\mathcal{W}^-_{\mu\nu}W^+_\nu\right)-Z_\nu\left(\mathcal{W}^+_{\mu\nu}W^-_\mu-\mathcal{W}^-_{\mu\nu}W^+_\mu\right)\right]+$$

$$+\frac{g^2}{4}\left[\left(W^+_\mu W^-_\nu-W^-_\mu W^+_\nu\right)^2+W^+_\mu W^-_\nu\chi^2\right]-\frac{e^2}{4}\left\{\left[\left(W^+_\mu\right)^2+\left(W^-_\mu\right)^2\right](A_\nu)^2-\right.$$

$$-2\left(W^+_\mu W^+_\nu+W^-_\mu W^-_\nu\right)A_\mu A_\nu+\left[\left(W^+_\nu\right)^2+\left(W^-_\nu\right)^2\right](A_\mu)^2\Big\}-$$

$$-\frac{g^2}{4}\cos\theta_W\left\{\left[\left(W^+_\mu\right)^2+\left(W^-_\mu\right)^2\right](Z_\nu)^2-2\left(W^+_\mu W^+_\nu+W^-_\mu W^-_\nu\right)Z_\mu Z_\nu+\right.$$

$$+\left[\left(W^+_\nu\right)^2+\left(W^-_\nu\right)^2\right](Z_\mu)^2\Big\}-eg\cos\theta_W\Big\{W^+_\mu W^-_\mu A_\nu Z_\nu+W^+_\nu W^-_\nu A_\mu Z_\mu-$$

$$-\frac{1}{2}\left(W^+_\mu W^-_\nu+W^+_\nu W^-_\mu\right)\left(A_\mu Z_\nu+A_\nu Z_\mu\right)\Big\}\Big\} \tag{115}$$

are regarded as their interactions. The lepton Lagrangian (98) in terms of electron and neutrino fields takes the form [30]

$$L_L(j)=e^\dagger_l i\tilde{\tau}_\mu\partial_\mu e_l+e^\dagger_r i\tau_\mu\partial_\mu e_r-m_e(e^\dagger_r e_l+e^\dagger_l e_r)+$$

$$+\frac{g\cos 2\theta_w}{2\cos\theta_w}e^\dagger_l\tilde{\tau}_\mu Z_\mu e_l-ee^\dagger_l\tilde{\tau}_\mu A_\mu e_l-g'\cos\theta_w e^\dagger_r\tau_\mu A_\mu e_r+$$

$$+g'\sin\theta_w e^\dagger_r\tau_\mu Z_\mu e_r+j^2\Big\{\nu^\dagger_l i\tilde{\tau}_\mu\partial_\mu\nu_l+\frac{g}{2\cos\theta_w}\nu^\dagger_l\tilde{\tau}_\mu Z_\mu\nu_l+$$

$$+\frac{g}{\sqrt{2}}\left[\nu^\dagger_l\tilde{\tau}_\mu W^+_\mu e_l+e^\dagger_l\tilde{\tau}_\mu W^-_\mu\nu_l\right]\Big\}=L_{L,b}+j^2L_{L,f}. \tag{116}$$

The quark Lagrangian (102) in terms of u- and d-quarks fields can be written as

$$L_Q(j)=d^\dagger i\tilde{\tau}_\mu\partial_\mu d+d^\dagger_r i\tau_\mu\partial_\mu d_r-m_d(d^\dagger_r d+d^\dagger d_r)-\frac{e}{3}d^\dagger\tilde{\tau}_\mu A_\mu d-$$

$$-\frac{g}{\cos\theta_w}\left(\frac{1}{2}-\frac{2}{3}\sin^2\theta_w\right)d^\dagger\tilde{\tau}_\mu Z_\mu d-\frac{1}{3}g'\cos\theta_w d^\dagger_r\tau_\mu A_\mu d_r+$$

$$+\frac{1}{3}g'\sin\theta_w d^\dagger_r\tau_\mu Z_\mu d_r+j^2\Big\{u^\dagger i\tilde{\tau}_\mu\partial_\mu u+u^\dagger_r i\tau_\mu\partial_\mu u_r-$$

$$-m_u(u^\dagger_r u+u^\dagger u_r)+\frac{g}{\cos\theta_w}\left(\frac{1}{2}-\frac{2}{3}\sin^2\theta_w\right)u^\dagger\tilde{\tau}_\mu Z_\mu u+$$

$$+\frac{2e}{3}u^\dagger\tilde{\tau}_\mu A_\mu u+\frac{g}{\sqrt{2}}\left[u^\dagger\tilde{\tau}_\mu W^+_\mu d+d^\dagger\tilde{\tau}_\mu W^-_\mu u\right]+$$

$$+\frac{2}{3}g'\cos\theta_w u^\dagger_r\tau_\mu A_\mu u_r-\frac{2}{3}g'\sin\theta_w u^\dagger_r\tau_\mu Z_\mu u_r\Big\}=$$

$$=L_{Q,b}+j^2L_{Q,f}, \tag{117}$$

where $m_e=h_e v/\sqrt{2}$ and $m_u=h_u v/\sqrt{2}$, $m_d=h_d v/\sqrt{2}$ represent electron and quark masses.

The complete Lagrangian of the modified model is the sum

$$L(j) = L_B(j) + L_Q(j) + L_L(j) = L_b + j^2 L_f. \tag{118}$$

The boson Lagrangian $L_B(j)$ was discussed in [27, 29], where it was shown that masses of all particles involved in the Electroweak Model remain the same under the contraction $j^2 \to 0$ in both formulations: the standard one [27] and without Higgs boson [29]. In this limit the contribution $j^2 L_f$ of neutrino, W-boson and u-quark fields as well as their interactions with the other fields to the Lagrangian (118) become vanishingly small in comparison with the contribution L_b of electron, d-quark and the remaining boson fields. So Lagrangian (118) describes a very rare interaction of neutrino fields with the matter, which consists of quarks and leptons in the Standard Electroweak Model. On the other hand, the contribution of the neutrino part $j^2 L_f$ to the complete Lagrangian is risen when the parameter j^2 is increased, which corresponds to the experimental facts. It follows from this that the contraction parameter is connected with neutrino energy and this dependence can be obtained from the experimental data.

9 Description of Physical Systems and Group Contractions

The standard way of describing a physical system in the field theory is its decomposition on independent more or less simple subsystems, which can be exactly described, and then introducing interactions between them. In Lagrangian formalism this implies that some terms describe independent subsystems (free fields) and the rest of the terms correspond to interactions between the fields. When the subsystems do not interact with each other the composed system is a formal unification of the subsystems and symmetry group of the whole system is the direct product $G = G_1 \times G_2$, where G_1 and G_2 are symmetry groups of the subsystems. The Electroweak Model gives a nice example of such approach. Indeed, there are free boson, lepton and quark fields in Lagrangian and the terms which describe interactions between these fields.

The operation of group contraction transforms a simple or semisimple group G to a non-semisimple one with the structure of a semidirect product $G = A \circledS G_1$, where A is Abel and $G_1 \subset G$ is an untouched subgroup. At the same time the fundamental representation space of the group G is fibered under the contraction in such a way that the subgroup G_1 acts in the fiber. The gauge theory with a contracted gauge group describes a physical system, which is divided on two subsystems S_b and S_f. One subsystem S_b includes all fields from the base and the other subsystem S_f is built from fiber fields. S_b forms a closed system since according to semi-Riemannian geometry [50, 26] the properties of the base do not depend on the points of the fiber, which physically means that the fields from the fiber do not interact with the fields from the base. On the contrary the properties of the fiber depend on the points of the base, therefore the subsystem S_b exerts influence upon S_f. More precisely, the fields from the base are outer (or background) fields for the subsystem S_f and specify outer conditions in every fiber.

In particular, the simple group $SU(2)$ is contracted to the non-semisimple group $SU(2;\iota)$, which is isomorphic to the Euclid group $E(2) = A_2 \circledS SO(1)$, where Abel sub-

group A_2 is generated by the translations [27, 28, 29]. The fields space of the Standard Electroweak Model is fibered after the contraction in such a way that neutrino, W-boson and u-quark fields are in the fiber, whereas all the other fields are in the base.

The simple and the best known example of fiber space is the non-relativistic space–time with one-dimensional base, which is interpreted as time, and three-dimensional fiber, which is interpreted as proper space. It is well known, that in non-relativistic physics the time is absolute and does not depend on the space coordinates, while the space properties can be changed in time. The simplest demonstration of this fact is Galilei transformation $t' = t$, $x' = x + vt$. The space–time of the special relativity is transformed to the non-relativistic space–time when a dimensional parameter — the velocity of light c — tends to the infinity and a dimensionless parameter tends to zero $\frac{v}{c} \to 0$.

10 Rarely Neutrino-Matter Interactions

To discover the connection of gauge group contraction with the limiting case of the Electroweak Model and to establish the physical meaning of the contraction parameter we consider neutrino elastic scattering on electrons and quarks. The corresponding diagrams for the neutral and charged currents interactions are represented in Fig. 3 and Fig. 4.

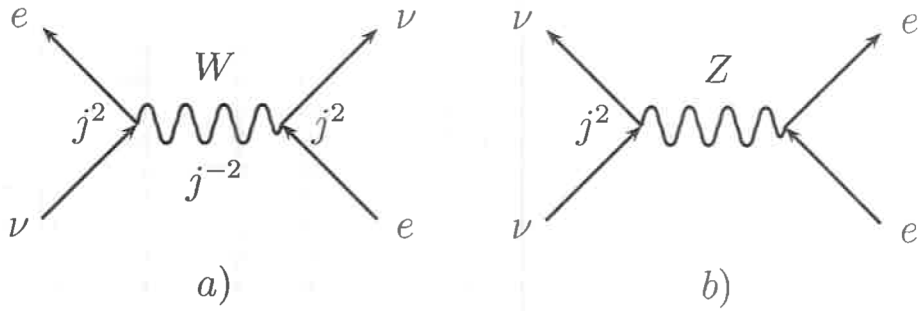


Figure 3: Neutrino elastic scattering on electron

Under substitutions (112), (113) both vertices of diagram in Fig. 3, a) are multiplied by j^2, as it follows from lepton Lagrangian (116). The propagator of virtual fields W according to boson Lagrangian (114) is multiplied by j^{-2}. Indeed, a propagator is an inverse operator toan operator of a free field, but the later for W-fields is multiplied by j^2.

So on the whole the probability amplitude for charged weak current interactions is transformed as $\mathcal{M}_W \to j^2\mathcal{M}_W$. For the diagram in Fig. 3, b) only one vertex is multiplied by j^2, whereas the second vertex and the propagator of Z virtual field do not change, so the corresponding amplitude for neutral weak current interactions is transformed in a similar way $\mathcal{M}_Z \to j^2\mathcal{M}_Z$. Cross-section is proportionate to squared amplitude, so neutrino-electron scattering cross-section is proportionate to j^4. For low energies $s \ll m_W^2$ this cross-section makes a principal contribution to the electron-neutrino interaction and is as

follows [46]

$$\sigma_{\nu e} = G_F^2 s f(\xi) = \frac{g^4}{m_w^4} \tilde{f}(\xi), \tag{119}$$

where $G_F = 10^{-5} \frac{1}{m_p^2} = 1,17 \cdot 10^{-5}$ GeV^{-2} is Fermi constant, s is squared energy in center-of-mass system, $\xi = \sin\theta_w$, $\tilde{f}(\xi) = f(\xi)/32$ is the function of Weinberg angle. The cross-section in the laboratory system for neutrino energy $m_e \ll E_\nu \ll m_W$ is given by [52]

$$\sigma_{\nu e} = G_F^2 m_e E_\nu \tilde{g}(\xi). \tag{120}$$

On the other hand, taking into account that the contraction parameter j is dimensionless, we can write down

$$\sigma_{\nu e} = j^4 \sigma_0 = (G_F s)(G_F f(\xi)) \tag{121}$$

and obtain

$$j^2(s) = \sqrt{G_F s} \approx \frac{g\sqrt{s}}{m_W}. \tag{122}$$

So the contraction parameter is expressed in terms of Fermi constant and the fundamental parameters of the Electroweak Model.

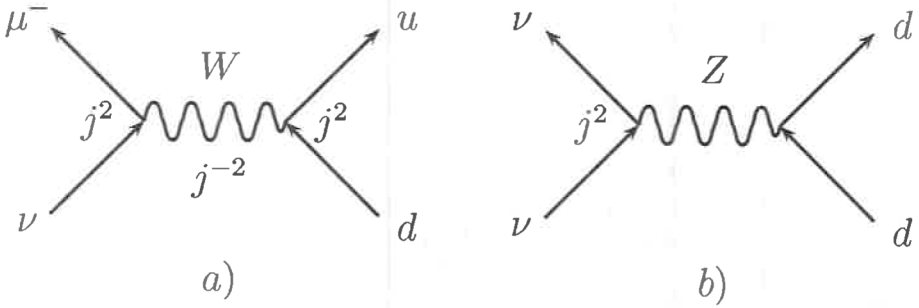


Figure 4: Neutrino elastic scattering on quarks

Neutrino elastic scattering on quarks by means of neutral and charged currents is pictured in Fig. 4. Cross-sections for neutrino-quarks scattering are obtained in a way similar to the lepton case and are as follows [46]

$$\sigma_\nu^W = G_F^2 s \hat{f}(\xi)), \quad \sigma_\nu^Z = G_F^2 s\, h(\xi). \tag{123}$$

Nucleons are some composite constructions of quarks, therefore some form-factors appear in the expressions for neutrino-nucleons scattering cross-sections. The final expression

$$\sigma_{\nu n} = G_F^2 s \hat{F}(\xi) \tag{124}$$

coincides with (119), i.e. this cross-section is transformed as (121) with the contraction parameter (122). At low energies scattering interactions make the leading contribution to the total neutrino-matter cross-section, therefore it has the same properties (121), (122) with respect to contraction of the gauge group.

We have shown that contraction of the gauge group of the Standard Electroweak Model corresponds to its low-energy limit. The zero tending contraction parameter depends on neutrino energy and determines the energy dependence of the neutrino-matter interaction cross-section.

The limit transition $c \to \infty$ in special relativity resulted in the notion of group contraction [34]. In the Electroweak Model the notion of group contraction is used on the contrary to explain the experimentally verified fundamental limit process of nature: a decrease of the neutrinos-matter cross-section when neutrino energy tends to zero.

11 Electroweak Model at Infinite Energy

In the previous section we have shown that contraction of the gauge group of the Standard Electroweak Model corresponds to its low-energy limit. In this limit the first components of the lepton and quark doublets become infinitely small in comparison with their second components. On the contrary, when energy increases the first components of the doublets become greater then their second ones. In the infinite energy limit the second components of the lepton and quark doublets will be infinitely small as compared with their second components. To describe this limit we introduce instead of (108) new contraction parameter ϵ and *new consistent rescaling* of the group $SU(2)$ and the space C_2 as follows

$$z'(\epsilon) = \begin{pmatrix} z'_1 \\ \epsilon z'_2 \end{pmatrix} = \begin{pmatrix} \alpha & \epsilon\beta \\ -\epsilon\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} z_1 \\ \epsilon z_2 \end{pmatrix} = u(\epsilon) z(\epsilon),$$

$$\det u(\epsilon) = |\alpha|^2 + \epsilon^2 |\beta|^2 = 1, \quad u(\epsilon) u^\dagger(\epsilon) = 1, \tag{125}$$

where $\epsilon \to 0$. Both contracted groups $SU(2;j)$ (108) and $SU(2;\epsilon)$ (125) are the same and are isomorphic to Euclid group $E(2)$, but the space $C_2(\epsilon)$ is splited in the limit $\epsilon \to 0$ on the one-dimension base $\{z_1\}$ and the one-dimension fiber $\{z_2\}$. From the mathematical point of view it is not important if the first or the second Cartesian axis forms the base of fibering and in this sence constructions (108) and (125) are equivalent. But the doublet components are interpreted as certain physical fields, therefore the fundamental representations (108) and (125) of the same contracted unitary group lead to different limit cases of the Electroweak Model, namely, its zero energy and infinite energy limits.

In the second contraction scheme (125) all gauge bosons are transformed according to the rules (112) with the natural substitution of j by ϵ. Instead of (113) the lepton and quark fields are transformed now as follows

$$e_l \to \epsilon e_l, \quad d_l \to \epsilon d_l, \quad \nu_l \to \nu_l, \quad u_l \to u_l. \tag{126}$$

The next reason for inequality of the first and second doublet components is the special mechanism of spontaneous symmetry breaking, which is used to generate mass of vector bosons and other elementary particles of the model. In this mechanism one of Lagrangian ground states $\phi^{vac} = \begin{pmatrix} 0 \\ v \end{pmatrix}$ is taken as vacuum of the model and then small field excitations $v + \chi(x)$ with respect to this vacuum are regarded. So Higgs boson field χ and

the constant v are multiplied by ϵ. As far as masses of all particlies are proportionate to v we obtain the following transformation rule for contraction (125)

$$\chi \to \epsilon\chi, \quad v \to \epsilon v, \quad m_p \to \epsilon m_p, \tag{127}$$

where $p = \chi, W, Z, e, u, d$.

After transformations (112), (126)-(127) the boson Lagrangian of the Electroweak Model can be represented in the form

$$L_B(\epsilon) = -\frac{1}{4}\mathcal{Z}_{\mu\nu}^2 - \frac{1}{4}\mathcal{F}_{\mu\nu}^2 + \epsilon^2 L_{B,2} + \epsilon^3 g W_\mu^+ W_\mu^- \chi + \epsilon^4 L_{B,4},$$

$$L_{B,4} = m_W^2 W_\mu^+ W_\mu^- - \frac{1}{2}m_\chi^2\chi^2 - \lambda v\chi^3 - \frac{\lambda}{4}\chi^4 + \frac{g^2}{4}W_\mu^+ W_\nu^-\chi^2 +$$

$$+\frac{g^2}{4}\left(W_\mu^+ W_\nu^- - W_\mu^- W_\nu^+\right)^2,$$

$$L_{B,2} = \frac{1}{2}\left(\partial_\mu\chi\right)^2 + \frac{1}{2}m_Z^2\left(Z_\mu\right)^2 -$$

$$-\frac{1}{2}\mathcal{W}_{\mu\nu}^+\mathcal{W}_{\mu\nu}^- + \frac{g m_z}{2\cos\theta_W}\left(Z_\mu\right)^2\chi + \frac{g^2}{8\cos^2\theta_W}\left(Z_\mu\right)^2\chi^2 -$$

$$-2ig\left(W_\mu^+ W_\nu^- - W_\mu^- W_\nu^+\right)\left(\mathcal{F}_{\mu\nu}\sin\theta_W + \mathcal{Z}_{\mu\nu}\cos\theta_W\right) -$$

$$-\frac{i}{2}e\left[A_\mu\left(\mathcal{W}_{\mu\nu}^+ W_\nu^- - \mathcal{W}_{\mu\nu}^- W_\nu^+\right) + \frac{i}{2}eA_\nu\left(\mathcal{W}_{\mu\nu}^+ W_\mu^- - \mathcal{W}_{\mu\nu}^- W_\mu^+\right)\right] -$$

$$-\frac{i}{2}g\cos\theta_W\left[Z_\mu\left(\mathcal{W}_{\mu\nu}^+ W_\nu^- - \mathcal{W}_{\mu\nu}^- W_\nu^+\right) - Z_\nu\left(\mathcal{W}_{\mu\nu}^+ W_\mu^- - \mathcal{W}_{\mu\nu}^- W_\mu^+\right)\right] -$$

$$-\frac{e^2}{4}\left\{\left[\left(W_\mu^+\right)^2 + \left(W_\mu^-\right)^2\right]\left(A_\nu\right)^2 - 2\left(W_\mu^+ W_\nu^+ + W_\mu^- W_\nu^-\right)A_\mu A_\nu +\right.$$

$$\left.\left[\left(W_\nu^+\right)^2 + \left(W_\nu^-\right)^2\right]\left(A_\mu\right)^2\right\} - \frac{g^2}{4}\cos\theta_W\left\{\left[\left(W_\mu^+\right)^2 + \left(W_\mu^-\right)^2\right]\left(Z_\nu\right)^2 -\right.$$

$$\left.-2\left(W_\mu^+ W_\nu^+ + W_\mu^- W_\nu^-\right)Z_\mu Z_\nu + \left[\left(W_\nu^+\right)^2 + \left(W_\nu^-\right)^2\right]\left(Z_\mu\right)^2\right\} -$$

$$-eg\cos\theta_W\left[W_\mu^+ W_\mu^- A_\nu Z_\nu + W_\nu^+ W_\nu^- A_\mu Z_\mu -\right.$$

$$\left.-\frac{1}{2}\left(W_\mu^+ W_\nu^- + W_\nu^+ W_\mu^-\right)\left(A_\mu Z_\nu + A_\nu Z_\mu\right)\right]. \tag{128}$$

In terms of electron and neutrino fields the lepton Lagrangian takes the form

$$L_L(\epsilon) = L_{L,0} + \epsilon^2 L_{L,2} = \nu_l^\dagger i\tilde{\tau}_\mu\partial_\mu\nu_l + e_\tau^\dagger i\tau_\mu\partial_\mu e_\tau + g'\sin\theta_w e_\tau^\dagger\tau_\mu Z_\mu e_\tau -$$

$$-g'\cos\theta_w e_\tau^\dagger\tau_\mu A_\mu e_\tau + \frac{g}{2\cos\theta_w}\nu_l^\dagger\tilde{\tau}_\mu Z_\mu\nu_l + \epsilon^2\left\{e_l^\dagger i\tilde{\tau}_\mu\partial_\mu e_l - m_e(e_\tau^\dagger e_l + e_l^\dagger e_\tau) +\right.$$

$$\left.+\frac{g\cos 2\theta_w}{2\cos\theta_w}e_l^\dagger\tilde{\tau}_\mu Z_\mu e_l - e e_l^\dagger\tilde{\tau}_\mu A_\mu e_l + \frac{g}{\sqrt{2}}\left(\nu_l^\dagger\tilde{\tau}_\mu W_\mu^+ e_l + e_l^\dagger\tilde{\tau}_\mu W_\mu^-\nu_l\right)\right\}. \tag{129}$$

In terms of u- and d-quarks fields the quark Lagrangian can be written as

$$L_Q(\epsilon) = L_{Q,0} - \epsilon\, m_u(u_r^\dagger u_l + u_l^\dagger u_r) + \epsilon^2 L_{Q,2},$$

$$L_{Q,0} = d_r^\dagger i\tau_\mu \partial_\mu d_r + u_l^\dagger i\tilde{\tau}_\mu \partial_\mu u_l + u_r^\dagger i\tau_\mu \partial_\mu u_r - \frac{1}{3}g' \cos\theta_w d_r^\dagger \tau_\mu A_\mu d_r +$$

$$+\frac{1}{3}g' \sin\theta_w d_r^\dagger \tau_\mu Z_\mu d_r + \frac{2e}{3}u_l^\dagger \tilde{\tau}_\mu A_\mu u_l + \frac{g}{\cos\theta_w}\left(\frac{1}{2} - \frac{2}{3}\sin^2\theta_w\right) u_l^\dagger \tilde{\tau}_\mu Z_\mu u_l +$$

$$+\frac{2}{3}g' \cos\theta_w u_r^\dagger \tau_\mu A_\mu u_r - \frac{2}{3}g' \sin\theta_w u_r^\dagger \tau_\mu Z_\mu u_r,$$

$$L_{Q,2} = d_l^\dagger i\tilde{\tau}_\mu \partial_\mu d_l - m_d(d_r^\dagger d_l + d_l^\dagger d_r) - \frac{e}{3}d_l^\dagger \tilde{\tau}_\mu A_\mu d_l -$$

$$-\frac{g}{\cos\theta_w}\left(\frac{1}{2} - \frac{2}{3}\sin^2\theta_w\right) d_l^\dagger \tilde{\tau}_\mu Z_\mu d_l + \frac{g}{\sqrt{2}}\left[u_l^\dagger \tilde{\tau}_\mu W_\mu^+ d_l + d_l^\dagger \tilde{\tau}_\mu W_\mu^- u_l\right]. \qquad (130)$$

The complete Lagrangian of the modified model is given by the sum $L(\epsilon) = L_B(\epsilon) + L_L(\epsilon) + L_Q(\epsilon)$ and for the infinite energy (for $\epsilon = 0$) is equal to

$$L_\infty = -\frac{1}{4}\mathcal{Z}_{\mu\nu}^2 - \frac{1}{4}\mathcal{F}_{\mu\nu}^2 + \nu_l^\dagger i\tilde{\tau}_\mu \partial_\mu \nu_l + u_l^\dagger i\tilde{\tau}_\mu \partial_\mu u_l + e_r^\dagger i\tau_\mu \partial_\mu e_r +$$

$$+d_r^\dagger i\tau_\mu \partial_\mu d_r + u_r^\dagger i\tau_\mu \partial_\mu u_r + L_\infty^{int}(A_\mu, Z_\mu),$$

$$L_\infty^{int}(A_\mu, Z_\mu) = \frac{g}{2\cos\theta_w}\nu_l^\dagger \tilde{\tau}_\mu Z_\mu \nu_l + \frac{g}{\cos\theta_w}\left(\frac{1}{2} - \frac{2}{3}\sin^2\theta_w\right) u_l^\dagger \tilde{\tau}_\mu Z_\mu u_l +$$

$$+\frac{2e}{3}u_l^\dagger \tilde{\tau}_\mu A_\mu u_l + g' \sin\theta_w e_r^\dagger \tau_\mu Z_\mu e_r - g' \cos\theta_w e_r^\dagger \tau_\mu A_\mu e_r - \frac{1}{3}g' \cos\theta_w d_r^\dagger \tau_\mu A_\mu d_r +$$

$$+\frac{1}{3}g' \sin\theta_w d_r^\dagger \tau_\mu Z_\mu d_r + \frac{2}{3}g' \cos\theta_w u_r^\dagger \tau_\mu A_\mu u_r - \frac{2}{3}g' \sin\theta_w u_r^\dagger \tau_\mu Z_\mu u_r. \qquad (131)$$

The limit model includes only massless particles: neutral massless Z-bosons Z_μ and photons A_μ, massless right electrons e_r and neutrinos ν_l, and massless left and right quarks u_l, u_r, d_r. The electroweak interactions become long-range because they are mediated by the massless neutral Z-bosons and photons. There are no interactions between particles of different kind, for example neutrinos interact only with each other by neutral currents. Similar higher energies can exist in the early Universe after inflation and reheating on the first stages of the Hot Big Bang [17, 41]. The electroweak phase transition and neutrino decoupling which take place during the first second after the Big Bang [16] are apparently in correspondence with the infinity energy limit of the Electroweak Model (131). The mass term of u-quark in the complete Lagrangian is proportional to ϵ whereas the mass terms of electron and d-quark are multiplied by ϵ^2, so u-quark first restores its mass in the evolution of the Universe.

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A Short Introduction to Geometric Intrinsic Symmetries in Nuclear Physics

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1 Introduction

A possibility of existence of high-rank nuclear symmetries related to the geometric properties, usually understood as some deformations of nuclei, has been shown several years ago [1]. One of the most reach symmetry is the tetrahedral/octahedral symmetry which can produce large shell gaps in the single particle spectra because of the characteristic for these point groups four-fold degeneracy. Large degeneracy of the energy spectrum increases the average level spacing [2, 3]. This leads to the specific tetrahedral-magic shell-closures for nucleon numbers 32, 40, 56, 64, 70, 90-94, 112, and 136-138.

There were several experiments related to the problem of 'tetrahedral' nuclei performed. For example see Ref.[4, 5]. In the Rare Earth nuclei such as ^{152,156}Gd, ^{154,156}Dy, ^{164}Er, ^{164}Yb, but also in the Actinides in $^{230-234}$U, there were found some interesting properties suggesting existence of searched symmetries, however, the results are not unique [6, 7].

The word $\sigma\upsilon\mu\mu\varepsilon\tau\rho\iota\alpha$ (symmetry) comes from Greek language: $\sigma\upsilon\mu$ ('together') and $\mu\varepsilon\tau\rho\omega\nu$ ('measure'). Before the contemporary physics the symmetry was rather related to such notions as beauty, perfectness, harmony or 'proper proportions'. The contemporary meaning of the symmetry concept was invented more or less in Renaissance.

2 Space–time versus intrinsic symmetries

The most striking property of the space-time is its symmetry. The space-time symmetry group G_{ST} corresponds to a relativistic group as, for example Poincarè group in the theory of special relativity. In numerous relativistic and non-relativistic models of the space-time considered in physics, one considers the various space-time symmetry groups. The second kind of important physical symmetries are intrinsic symmetries G_{int} which commute with the space time symmetries G_{ST}. The intrinsic symmetries can have two origins:

Figure 1: Platonic solids: tetrahedron, cube, octahedron, icosahedron, dodecahedron.

- the first type describes these intrinsic properties of the physical body which are independent of the space and time structure, e.g. the symmetries related the isospin, conservation of the electric charge, conservation of the particle number and so on. The corresponding symetry group we denote here as G_c,
- the second type is determined by the geometric properties of the physical body. One of the most important geometric feature is shape of the body. These kinds of symmetries leads to the so called intrinsic groups consisted of the geometric transformations constructed in the intrinsic frame of the body. In this paper the intrinsic groups are labelled by the bar symbol over the group name, e.g. $\overline{G}$.

In this lecture we are interested only in the second kind of the intrinsic symmetries.

In case of a nucleus (non–relativistic description), let us assume, that this nucleus is considered in the the coordinate frame in which center of mass is fixed in the position space. The remaining non-relativistic space–time symmetry is the orthogonal group O(3). Every nuclear collective Hamiltonian has to be invariant in respect to this orthogonal group O(3). However, the nucleus can have additional geometric intrinsic symmetry group which is a subgroup of the corresponding intrinsic orthogonal group $\overline{G} \subset \overline{O(3)}$. It implies that, in the case of non-relativistic description of a nucleus the general intrinsic symmetries collected in the group G_{int} can be considered as the direct product:

$$G_{int} = \overline{G} \times G_c. \tag{1}$$

Historically, the most known symmetries are related to the geometric symmetries of some solids invented by Platon (428 - 347 BC). In three dimensional space there is known 5 Platonic solids which are the regular, convex polyhedrons. They are constructed from the faces which are congruent, regular polygons: triangles, squares or pentagons. These 5 Platonic solids are called: tetrahedron, cube, octahedron, dodecahedron and icosahedron, see Fig. 1

The proof of existence of only five Platonic solids is based on the Euler's formula:

$$V + F = E + 2, \tag{2}$$

where V, F, E denote the total number of V = vertices, F = faces and E = edges. There is an open question: do exist the nuclear Platonic solids in the Universe? We will have this problem in our mind in the following text.

Many scientists was and still is fascinated by the notion of symmetry. One of the first was Johannes Kepler who believed in symmetry and proposed the palnetary model

Figure 2: The Kepler's planetary model, http:// en.wikipedia.org/wiki/ File:Kepler-solar-system-2.png.

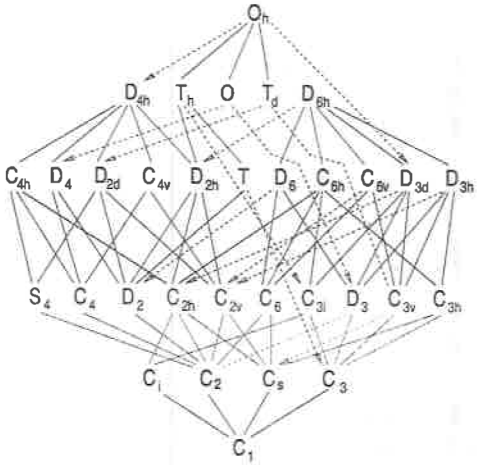

Figure 3: Point groups chains.

built from the Paltonic solids Similarly, in the contemporary physics we are searching for elementary particles, nuclear magic numbers, universal properties of matter etc., using the symmetry building block called the irreducible representations of the symmetry groups.

Above we have mentioned that there is an open problem about existence of nuclei having symmetries of Platonic solids. These symmetries are related to the so called point groups consisted of transformations which leave one or more points of the three dimensional space unchanged. The most important is a set of 32 point groups shown on the Fig. 3. The dashed line denote not-invariant subgroup. Adding translations to point groups one gets 230 crystalographic space groups, 14 Bravais' lattices and 7 crystal lattices.

Because of relatively large degeneration of the energy spectra of the Hamiltonians invariant in respect to the tetrahedral and octahedral symmetries, both the tetrahedral and octahedral groups are the first candidates for analysis of nuclear point symmetries.

Figure 4: The tetrahedral surfaces for three different values of the deformation parameters $\alpha_{32} = 0.1, 0.2$ and 0.3, respectively.

In the simplest case, the tetrahedrally invariant shapes are generated by the deformation tensor α_{32}, where the deformation parameters are identified with the expansion coefficients of the nuclear surface:

$$R(\alpha;\theta,\phi) = R_0 \left(1 + \sum_{\lambda\mu} \alpha^*_{\lambda\mu} Y_{\lambda\mu}(\theta,\phi)\right). \tag{3}$$

The examples of the simplest tetrahedrally invariant surfaces are determined, eg. by the following equation

$$R(\alpha;\theta,\phi) = R_0 \left(1 + \alpha_{32}(Y_{32}(\theta,\phi) + Y_{3,-2}(\theta,\phi))\right). \tag{4}$$

The equation (3) allows to write down equation for different shapes of a nucleus classified in respect to the multipolarity λ.

Usually it is assumed that the dipole parameters $\alpha_{1\mu}$ describe a shift of the surface. It is only an approximation which has to be always verified in a given application. In Fig. (5) there is presented an effect of the dipole deformation on the quadrupole shape. In the right figure it is seen that the dipole deformation not only shifts the surface but it also change its shape. In the figures below only the non-zero parameters are explicitely written in their captions. One of the problems related to the above parametrization of the nuclear surface is that for larger deformations one can get quite unphysical surfaces, an example of such pure quadrupole surface are presented in Fig. (6). On the other hand, the regular quadrupole shapes are of the expected form, see Fig. (7).

3 Collective variables

The deformation parameters of the nuclear surface can be used as the collective variables, like in the Bohr type collective models. However, one can obtained the more general description assuming q_1, q_2 and q_3 are curvelinear coordinates in R^3. Then the most general equation of the nuclear surface can be written as

$$q_k = q_k(u,v) \text{ where } k = 1,2,3, \tag{5}$$

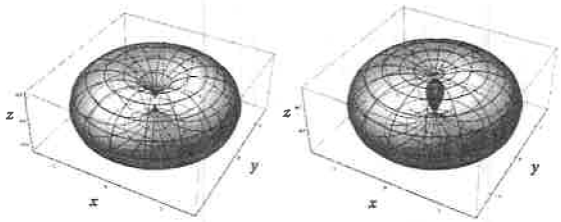

Figure 5: The shape for $\alpha_{20} = -1.50$ (left) and the shape for $\alpha_{10} = 1.50$, $\alpha_{20} = -1.50$ (right).

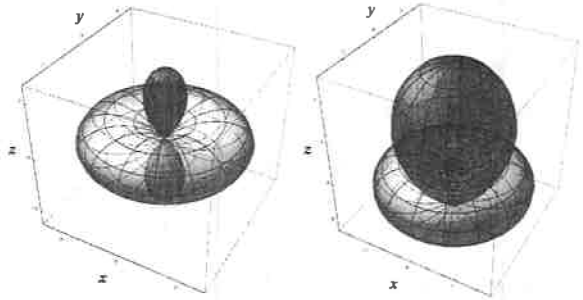

Figure 6: The monster quadrupole shapes, $\alpha_{20} = -5.50$ (left), $\alpha_{10} = 9.0$, $\alpha_{20} = -5.50$ (right)

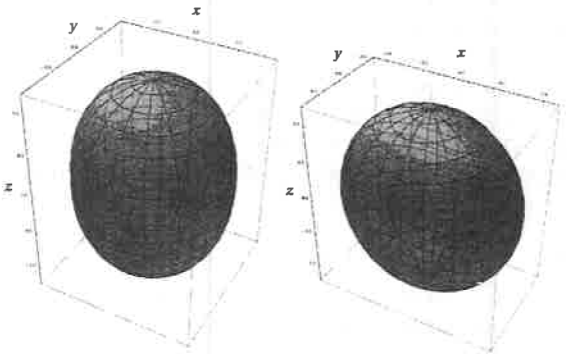

Figure 7: The regular quadrupole shapes, $\alpha_{20} = 0.30$ (left), $\alpha_{20} = -0.3$ $\alpha_{22} = 0.3$ (right).

where $(u, v) \in S \subset R^2$ are two real continuous parameters.

Assume the functions $q_k \in L^2(S)$ are square integrable functions, where the compact subset $S \subset R^2$ of variables parametrizes the surface in the space of a single-nucleon.

Let the set of the three vectors $\{e_n(u, v)\}$ gives the orthonormal basis in the space $L^2_\rho(S)$

$$\langle e_n | e_m \rangle = \int_S du; dv; \rho(u, v) e_n(u, v)^* e_m(u, v) = \delta_{nm}, \tag{6}$$

where $\rho(u, v) \geq 0$ is the appropriate weight function.

Using of this basis allows for expansion of the surface (5) in the following form

$$q_k(u, v) = \sum_n \alpha_{n,k}\, e_n(u, v). \tag{7}$$

The basis should be chosen to have a physical meaning determined by a set of commuting physical observables $\hat{A}_l$, where $l = 1, 2, \ldots, r$ i.e.

$$A_l e_n(u, v) = a_{ln} e_n(u, v), \text{ for all } l = 1, 2, \ldots, r. \tag{8}$$

In this case, the expansion coefficients

$$\alpha_{n,k} = \int_S du dv\, \rho(u, v)\, e_n(u, v)^* q_k(u, v) \tag{9}$$

can be used as new variables describing the nuclear surface in terms of the observables $\{\hat{A}_l\}$.

The very well known example of this procedure is the description of the nuclear surface with the expansion (3). In this case one needs to identified the variables in the three dimensional space with the spherical variables $\{q_1 = r, q_2 = \theta, q_3 = \phi\}$ and asume $u = \theta, v = \phi$. The equation of the surface $r = R(\theta, \phi) \in L^2(SO(2))$ can be expanded into eigenfunctions of the angular momentum observables $\hat{A}_1 = \hat{J}^2$ and $\hat{A}_2 = \hat{J}_z$, where $\hat{J}^2$ is square of the total angular momentum and $\hat{J}_z$ denotes its third component. In this case the basis $e_n(u, v) = Y_{lm}(\theta, \phi)$ consists of the spherical harmonic functions. As the result one obtains the equation (3). In practice, in the nuclear physics, the equation of the nuclear surface written in the laboratory frame (in this case we label the deformation parameters with the superscript (lab), $\alpha^{(lab)}_{\lambda,\mu}$) usually has the additional coefficient $c(\alpha^{(lab)})$ in front of the equation (3), which allows to satisfy the volume conservation condition for the nuclear matter. The reality of the radius $R(\theta, \phi)$, its invariance in respect to the space rotations $\hat{R}(\Omega)$ and the space inversion $\hat{C}_i$ leads to the standard relations for the expansion coefficients $\alpha^{(lab)}_{\lambda,\mu}$:

- Reality of the surface: $(\alpha^{(lab)}_{\lambda\mu})^* = \alpha^{(lab)}_{\lambda,-\mu}$.

  This condition can be obtained by making use of the reality of the radius $r = R(\theta, \phi)$ and properties of the spherical harmonic functions

$$\begin{aligned} &R(\alpha^{(lab)}; \theta, \phi) = R^*(\alpha^{(lab)}; \theta, \phi) \\ &\sum_{\lambda,\,\mu} \alpha^{(lab)*}_{\lambda,\mu} Y_{\lambda\mu}(\theta, \phi) = \sum_{\lambda,\,\mu} \alpha^{(lab)}_{\lambda,\mu} (-1)^\mu Y_{\lambda,-\mu}(\theta, \phi). \end{aligned} \tag{10}$$

- Rotational properties of the surface: $\hat{R}(\Omega)\alpha^{(lab)}_{\lambda\mu} = \sum_{\mu'} D^{\lambda}_{\mu'\mu}(\Omega)\alpha^{(lab)}_{\lambda\mu'}$, where $D^{\lambda}_{\mu'\mu}(\Omega)$ denotes the Wigner functions of the rotation group and the operator $\hat{R}(\Omega)$ represents the rotation operator parametrized with the Euler angles $\Omega = (\Omega_1, \Omega_2, \Omega_3)$.

  This condition follows from the transformation properties of the spherical harmonic functions and invariance of the radius $r = R(\theta, \phi)$ of the surface in respect to the space inversion:

$$\begin{aligned}
&\hat{R}(\Omega)R(\alpha^{(lab)};\theta,\phi) = R(\alpha^{(lab)};\theta,\phi)\\
&\hat{R}(\Omega)R(\alpha^{(lab)};\theta,\phi) = R\left(\hat{R}(\Omega)\alpha^{(lab)};\hat{R}(\Omega)\{\theta,\phi\}\right)\\
&= R_0\left(1+\sum_{\lambda,\,\mu}\left(\hat{R}(\Omega)\alpha^{(lab)}_{\lambda,\mu}\right)^{*}\hat{R}(\Omega)Y_{\lambda,\mu}(\theta,\phi)\right)\\
&= R_0\left(1+\sum_{\lambda,\,\mu}\left(\hat{R}(\Omega)\alpha^{(lab)}_{\lambda,\mu}\right)^{*}\sum_{\eta}D^{\lambda}_{\eta\mu}(\Omega)Y_{\lambda,\eta}(\theta,\phi)\right). \qquad (11)
\end{aligned}$$

  Comparing the equation of the surface before and after rotation results in the transformation properties of the deformation parameters in respect to the space rotation.

- Space inversion transformation: $C_i\alpha^{(lab)}_{\lambda\mu} = (-1)^{\lambda}\alpha^{(lab)}_{\lambda\mu}$.

  This property follows directly from the properties of the spherical harmonics and invariance of the radius :

$$\begin{aligned}
&\hat{C}_i R(\alpha^{(lab)};\theta,\phi) = R(\alpha^{(lab)};\theta,\phi)\\
&\hat{C}_i R(\alpha^{(lab)};\theta,\phi) = R_0(1+\sum_{\lambda,\,\mu}\alpha^{(lab)*}_{\lambda,\mu}(-1)^{\lambda}Y_{\lambda\mu}(\theta,\phi)). \qquad (12)
\end{aligned}$$

  As above, comparing both expressions before and after the transformation of the surface gives the transformation properties of the deformation parameters.

These properties show that the deformation parameters (collective variables) $\alpha^{(lab)}_{\lambda\mu}$ are the covariant components of the spherical tensor of the rank λ (tensor in respect to the rotation group SO(3)). The important property of these tensors is existence of the scalar product of two tensors. Let ξ_λ and η_λ be the tensor of the same multipolarity, then the scalar product is defined as

$$\xi_\lambda \cdot \eta_\lambda = \sum_{\mu\nu} g^{\mu\nu}\xi_{\lambda\mu}\eta_{\lambda\nu}, \qquad (13)$$

where the metric tensor is generated by the Clebsch–Gordan coefficients of the rotation group $(\lambda_1\mu_1\lambda_2\mu_2|\lambda\mu)$

$$g^{\mu\nu} = \sqrt{2\lambda+1}(\lambda\mu\lambda\nu|00) = (-1)^{\mu}\delta^{-\nu}_{\mu}. \qquad (14)$$

The scalars (rotational invariants) obtained in this way play an important role in description of nuclear collective motion. For example, the total multipole deformation of a nucleus is proportional to the multiplication operator $\hat{\beta}_\lambda$

$$\hat{\beta}_\lambda\psi(\alpha^{(lab)}) = \beta_\lambda\psi(\alpha^{(lab)}), \qquad (15)$$

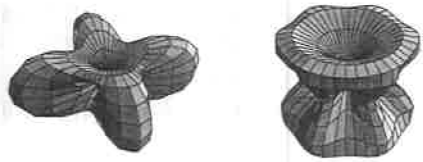

Figure 8: The spin orientation probability for a rotating system. The chosen wave functions are proportional to some combinations of the Wigner functions: $\psi \sim D^5_{M2}(\Omega) - D^5_{M,-2}(\Omega)$ (left) and $\psi \sim D^5_{M3}(\Omega) - D^5_{M,-3}(\Omega)$ (right).

where

$$\beta^2_\lambda = \alpha^{(lab)}_\lambda \cdot \alpha^{(lab)}_\lambda = \sum_{\mu\nu} \alpha^{(lab)}_{\lambda\mu} [(-1)^\mu \alpha^{(lab)}_{\lambda,-\mu}] = \sum_\mu |\alpha^{(lab)}_{\lambda\mu}|^2 \in \mathbb{R}. \tag{16}$$

This kind of invariants is important in construction of the collective Hamiltonians. For example, the classical harmonic oscillator Hamiltonian

$$H_{ho} = \sum_\lambda \left[\frac{1}{2B_\lambda} \dot{\alpha}^{(lab)}_\lambda \cdot \dot{\alpha}^{(lab)}_\lambda + \frac{1}{2} B_\lambda \omega^2_\lambda \beta^2_\lambda \right] \tag{17}$$

is constructed from such invariants and finally it is invariant in respect to the rotation group as it is required in physics.

4 Intrinsic frame

The classical rotation is well understood phenomenon in which the orientation of a body is changing with time. Contrary, the quantum rotation allows to determine only the probability of a given orientation and there is no time variable in the wave function. The quantum rotation can be presented in the graphical form as the surface drawn by the end of the vector pointing out in the same direction as the spin of a rotating body and its lentgh equal to the probability of finding a given orientation of the spin, see Fig. (8) .

The notion of the quantum rotational motion allows to define the rotating intrinsic frame, e.g. the body fixed frame, for the collective variables $\{\alpha^{(lab)}_{\lambda\mu}\}$. The corresponding colective variables in the intrinsic frame we denote by $\{\alpha_{\lambda\mu}\}$. They can be obtained by the quantum rotation of the laboratory collective variables $\{\alpha^{(lab)}_{\lambda\mu}\}$ with the rotation operator $\hat{R}(\Omega)$

$$\alpha_\lambda = \hat{R}(\Omega) \alpha^{(lab)}_\lambda \tag{18}$$

assuming, in addition, that the rotation group $SO(3) \ni \hat{R}(\Omega)$ parameters, repesented by the Euler angles $\Omega = (\Omega_1, \Omega_2, \Omega_3)$, are considered as a part of intrinsic variables. The intrinsic variables α_λ are invariant in respect to the laboratory rotations $\hat{R}(\Omega)$. It is

important to notice that inclusion of the Euler angles into the set of intrinsic variables makes this set of variables redundant, 3 variable more than needed. It implies that, the definition of the intrinsic frame requires three additional conditions which recover the same number of variables in both frames

$$F_k(\alpha, \Omega) = 0, \qquad \text{where } k = 1, 2, 3. \tag{19}$$

In this way one can obtain a new description of a physical system, e.g. a nucleus, in which the rotational motion can be directly described by the Euler angles.

5 Intrinsic groups

There is an intersting question: how to investigate symmetries of a nucleus in the intrinsic frame. A part of symmetries, eg. trnaslational symmetry are not seen in the intrinsic frame. Due to the general principles, the nuclear Hamiltonian has to be invariant in respect to the orthogonal group O(3) defined in the laboratory frame. On the other hand, it is obvious, that the nucleus should have some geometrical symmetries related to its shape. The transformations furnishing an intrinsic symmetry group have to be defined in the intrinsic frame. In group theory, there is known the notion of left and right shift on the group magnifold. This idea was used to define the so called intrinsic groups which, in fact, act in the intrinsic frame.

A convenient definition was formulated in [15] in the following form:
for each element g of the group G, one can define a corresponding operator $\hat{\bar{g}}$ in the group linear space $\mathcal{L}_G$ as:

$$\hat{\bar{g}}|S\rangle = |Sg\rangle, \qquad \text{for all } |S\rangle \in \mathcal{L}_G, \tag{20}$$

where all elements inside the ket vectors $S = \sum_{g\in G} c_g g$, here c_g are the complex numbers, form a group algebra of the group G.

In this definition the notion of the group linear space $\mathcal{L}_G$ is used. This space is defined as the linear space spanned by all possible formal linear combinations of the elements of the group G

$$\mathcal{L}_G = \left\{ |S\rangle : |S\rangle = \sum_{g\in \mathrm{G}} c_g g, \text{ where } c_g \in \mathbb{C} \right\}. \tag{21}$$

It looks like the group algebra mentioned above, but, it is important that the elements of $\mathcal{L}_G$ have to be considered only as vectors, not as the elements of the group algebra.

The group formed by the collection of the operators $\hat{\bar{g}}$ is called the intrinsic group $\overline{\mathrm{G}}$ related to the group G.

One of the most important property of the intrinsic group $\overline{\mathrm{G}}$ is that this group commutes with its partner group G

$$[\mathrm{G}, \overline{\mathrm{G}}] = 0. \tag{22}$$

The groups G and $\overline{\mathrm{G}}$ are antyisomorphic. The required anti-isomorphism between the partner groups G and $\overline{\mathrm{G}}$ is given by

$$\phi_G : \overline{\mathrm{G}} \to \mathrm{G}, \text{ where } \quad \phi_G(\bar{g}) = g \text{ and } \phi_G(\bar{g}\bar{g}') = \phi_G(\bar{g}')\phi_G(\bar{g}). \tag{23}$$

This property suggests that the partner groups G and $\overline{\mathrm{G}}$ have a lot of common properties as e.g. similar structure of representations, decompositions of the Kronecker products, the Clebsch-Gordan coefficients and many others.

As an example let us consider a relation among representations of both groups. Because the partner groups commute one can find common basis $|\Gamma mk\rangle$ for representations of the group G and the group $\overline{\mathrm{G}}$. The representations are defined as:

$$g|\Gamma mk\rangle = \sum_{m'} \Delta^{(\Gamma)}_{m'm}(g)|\Gamma m'k\rangle, \tag{24}$$

$$\overline{g}|\Gamma mk\rangle = \sum_{k'} \bar{\Delta}^{(\Gamma)}_{k'k}(\overline{g})|\Gamma mk'\rangle. \tag{25}$$

To compare both representation one can use as the basis the generalized projection operators (elements of the group linear space $\mathcal{L}_G$)

$$|\Gamma mk\rangle = \frac{\dim[\Gamma]}{\mathrm{card(G)}} \sum_{g\in \mathrm{G}} \Delta^{(\Gamma)}_{mk}(g)^* g, \tag{26}$$

where $\dim[\Gamma]$ denotes the dimension of the representation Γ and $\mathrm{card(G)}$ is the number of elements in the group G. This allows to calculate (25)

$$\begin{aligned}
\overline{g}|\Gamma mk\rangle &= \frac{\dim[\Gamma]}{\mathrm{card(G)}} \sum_{g'\in \mathrm{G}} \Delta^{(\Gamma)}_{mk}(g')^* g'g \\
&= \frac{\dim[\Gamma]}{\mathrm{card(G)}} \sum_{g'\in \mathrm{G}} \Delta^{(\Gamma)}_{mk}(g'g^{-1})^* g' \\
&= \frac{\dim[\Gamma]}{\mathrm{card(G)}} \sum_{g'\in \mathrm{G}} \sum_{k'} \Delta^{(\Gamma)}_{mk'}(g')^* \Delta^{(\Gamma)}_{kk'}(g) g' = \sum_{k'} \Delta^{(\Gamma)}_{kk'}(g)|\Gamma mk'\rangle,
\end{aligned} \tag{27}$$

where $\Delta^{(\Gamma)}_{mm'}(g)$ are matrix elements of the representation Γ of the group G. Comparing both expression one can see that the matrices of both representations are related. The representations of the intrinsic group are transposed representations of the partner group

$$\bar{\Delta}^{(\Gamma)}_{mk}(\overline{g}) = \Delta^{(\Gamma)}_{km}(g). \tag{28}$$

A bit different are definitions of irreducible tensors in respect to the laboratory group G and the intrinsic group $\overline{\mathrm{G}}$. By definition the irreducible tensors in respect to the laboratory group G transform as

$$\hat{g} T^{(\Gamma)}_m \hat{g}^{-1} = \sum_l \Delta^{(\Gamma)}_{lm}(g) T^{(\Gamma)}_l. \tag{29}$$

The tensors in respect to the intrinsic group $\overline{\mathrm{G}}$, due to the anti-isomorphism between both groups, have to be defined in the following way

$$\hat{\overline{g}} \overline{T}^{(\Gamma)}_k \hat{\overline{g}}^{-1} = \sum_l \Delta^{(\Gamma)}_{lk}(g^{-1}) \overline{T}^{(\Gamma)}_l. \tag{30}$$

As an example, let us consider the action of the intrinsic group in the collective space consisted of the square integrable functions of the deformation parameters and the Euler angles. The intrinsic rotation operators $\hat{R}(\bar{g}_1, \bar{g}_2) \in \overline{\mathrm{SO}(3)}_\alpha \times \overline{\mathrm{SO}(3)}_\Omega$ (the indices α and Ω show the variables which are affected by the corresponding group) are defined as follows

$$\hat{R}(\bar{g}_1, \bar{g}_2) f(\alpha, \Omega) = f(\{\hat{\bar{g}}_1 \alpha\}, \Omega \phi_G(\bar{g}_2)^{-1}), \tag{31}$$

where $\bar{g}_1 \in \overline{\mathrm{SO}(3)}_\alpha$ and $\bar{g}_2 \in \overline{\mathrm{SO}(3)}_\Omega$. The action of the group $\overline{\mathrm{SO}(3)}_\alpha$ onto the deformation variables is a bit non-standard and is given by the following equation

$$\hat{\bar{g}}_1 \alpha_{\lambda\mu} = \sum_{\mu'} D^{\lambda}_{\mu'\mu}(\phi_G(\bar{g}_1)^{-1}) \alpha_{\lambda\mu'}. \tag{32}$$

The intrinsic group $\overline{\mathrm{SO}(3)}$ corresponding to the 'laboratory' rotation group SO(3) defined in the laboratory frame consists of all rotations $\hat{R}(\bar{g}, \bar{g})$ for which the deformation parameters and the Euler angles are rotated with the same angles.

The required anti-isomorphism between the partner groups SO(3) and $\overline{\mathrm{SO}(3)}$ is given by (23).

It is important to notice that, in general, not all transformations $(\bar{g}_1, \bar{g}_2) \in \overline{\mathrm{SO}(3)}_\alpha \times \overline{\mathrm{SO}(3)}_\Omega$

$$(\bar{g}_1, \bar{g}_2) \colon \ (\alpha, \Omega) \to (\alpha', \Omega') \tag{33}$$

are allowed in the intrinsic frame. They are allowed if they do not break the conditions which define the intrinsic frame (19)

$$(\hat{\bar{g}}_1, \hat{\bar{g}}_2) F_k(\alpha, \Omega) = F_k(\hat{\bar{g}}_1 \alpha, \Omega \bar{g}_2^{-1}) = 0, \ \text{where } k = 1, 2, 3. \tag{34}$$

For example, in the case of the quadrupole colective variables α_2 with the standard Bohr condition which define the intrinsic frame: $\alpha_{2,\pm 1} = 0$ and $\alpha_{22} = \alpha_{2-2}$, the allowed intrinsic rotations $\hat{R}(\bar{g}_1, \bar{g}_2) \in \overline{\mathrm{SO}(3)}_\alpha \times \overline{\mathrm{SO}(3)}_\Omega$ have to fulfil the following conditions

$$\hat{R}(\bar{g}_1, e_G)\alpha_{2\pm 1} = 0 \text{ and } \hat{R}(\bar{g}_1, e_G)\alpha_{22} = \hat{R}(\bar{g}_1, e_G)\alpha_{2-2}, \tag{35}$$

where the second argument represents the unit element of the group $\overline{\mathrm{SO}(3)}_\Omega$. The Bohr conditions allow for the arbitrary rotations $\bar{g}_2 \in \overline{\mathrm{SO}(3)}_\Omega$.

Using the conditions (35) the allowed rotations of the deformation parameters α have to satisfy the following equations

$$\begin{aligned} D^2_{0,\pm 1}(\bar{g}_1^{-1}) &= 0, \\ D^2_{-2\pm 1}(\bar{g}_1^{-1}) + D^2_{2\pm 1}(\bar{g}_1^{-1}) &= 0, \\ D^2_{02}(\bar{g}_1^{-1}) - D^2_{0,-2}(\bar{g}_1^{-1}) &= 0, \\ D^2_{-2,-2}(\bar{g}_1^{-1}) + D^2_{2,-2}(\bar{g}_1^{-1}) &= D^2_{-22}(\bar{g}_1^{-1}) + D^2_{22}(\bar{g}_1^{-1}). \end{aligned} \tag{36}$$

In this case, the octahedral point group $\overline{\mathrm{O}}_\alpha \subset \overline{\mathrm{SO}(3)}_\alpha$ acting only on the variables α provide the solution of the set of equations (36).

6 Uniqueness of quantum states

In practice, the transformation to the intrinsic frame is not a one-to-one function. For the further purpose it is useful to define a group of intrinsic transformations $\bar{h} \in \overline{\mathrm{G}}_s$:

$$(\alpha, \Omega) \xrightarrow{\bar{h}} (\alpha', \Omega'), \tag{37}$$

where $\alpha = \{\alpha_{\lambda\mu}\}$ and which leave invariant the corresponding laboratory coordinates:

$$\begin{aligned} &\alpha^{(lab)}(\alpha', \Omega') = \alpha^{(lab)}(\alpha, \Omega), \\ &F_k(\alpha', \Omega') = F_k(\alpha, \Omega) = 0, \text{ for } k = 1, 2, 3, \end{aligned} \tag{38}$$

where $\alpha^{(lab)}(\alpha, \Omega) = \hat{R}(\Omega^{-1})\alpha$, see (18).

The group $\overline{\mathrm{G}}_s$ we call the symmetrization group.

The symmetrization group decomposes the collective manifold into orbits of physically equivalent points. Let the function $\Psi^{(lab)}$ denotes a state vector of a nucleus in the laboratory frame. The corresponding state vector in the intrinsic frame has to fulfil the obvious equation

$$\Psi(\alpha, \Omega) = \Psi^{(lab)}(\alpha^{(lab)}) \tag{39}$$

which represents the fact that the wave function of the physical system written in the laboratory frame has to be a well and uniquelly defined function.

However, after the transformation of the intrinsic variable with the elements of the symmetrization group we do not change the laboratory state vector

$$\Psi(\alpha', \Omega') = \Psi^{(lab)}(\alpha^{(lab)}). \tag{40}$$

This implies the uniqueness condition for the states in the intrinsic frame

$$\Psi(\alpha', \Omega') = \Psi(\alpha, \Omega). \tag{41}$$

This is a very well known but not fully solved problem in the collective models of the Bohr type.

In principle, there are two possibilities to achieve uniqueness of transformation from the laboratory to the intrinsic frame:

- first, one can define the appropriate region of the intrinsic collective variables in which the transformation from the laboratory to intrinsic frame is a one-to-one function,
- second, one can allow for the whole range of collective variables but then one needs to fulfil the symmetrization condition for physical states. The symmetrization condition can be expressed as invariance of the intrinsic state vectors in respect to all transformations $\bar{h} \in \overline{\mathrm{G}}_s$,

$$\bar{h}\Psi(\alpha, \Omega) = \Psi(\alpha, \Omega), \tag{42}$$

where the group $\overline{\mathrm{G}}_s$ is the symmetrization group.

As an example, let us came back to the very well know example of the quadrupole variables $(\alpha_{20}, \alpha_{22}, \Omega)$ with the Bohr conditions which define the intrinsic frame (35).

Using the conditions (35) one can see that the allowed rotations of the deformation parameters α have to satisfy Eqs. (36) which are fulfilled by the rotations belonging to the octahedral point group $\overline{\mathrm{O}}_\alpha \subset \overline{\mathrm{SO}(3)}_\alpha$. The required invariance of the transformation formula from laboratory to the intrinsic frame (38) implies that both rotations $(\bar{g}_1, \bar{g}_2) \in \overline{\mathrm{O}}_\alpha \times \overline{\mathrm{O}}_\Omega$ have to be rotations about the same angles $\theta = (\theta_1, \theta_2, \theta_3)$, i.e. $\bar{g}_1 = \bar{g}_1(\theta)$ and $\bar{g}_2 = \bar{g}_2(\theta)$. This considerations suggest that the symmetrization group is equal to the octahedral group $\overline{\mathrm{G}}_s = \overline{\mathrm{O}} \subset \overline{\mathrm{O}}_\alpha \times \overline{\mathrm{O}}_\Omega$ transforming simultaneously the deformation parameters and the Euler angles by the same rotation. Because the quadrupole variables are invariant in respect to the space inversion, this transformation should be formally added to the symmetrization group, in this way one obtains $\overline{\mathrm{G}}_s = \overline{\mathrm{O}}_h$.

Obviously, instead of the standard Bohr conditions the following alternative definition of the intrinsic frame can be used:

- the collective variables are now chosen as $(\alpha_{20}, \alpha_{21}, \Omega)$,
- the conditions which define the intrinsic frame (variables) are now assumed as

$$F_{1,2}(\alpha, \Omega) = \alpha_{2\pm 2} = 0 \text{ and } F_3(\alpha, \Omega) = \alpha_{21} + \alpha_{2-1} = 0. \tag{43}$$

These definitions lead to the equations for allowed rotations and the symmetrization group:

$$\begin{aligned} D^2_{\pm 20}(g) &= 0 \\ D^2_{\pm 2,1}(g) - D^2_{\pm 2,-1}(g) &= 0 \\ D^2_{10}(g) + D^2_{-1,0}(g) &= 0 \\ D^2_{11}(g) - D^2_{1,-1}(g) &= D^2_{-1-1}(g) - D^2_{-11}(g). \end{aligned} \tag{44}$$

The allowed rotations are now given by $\overline{\mathrm{D}}_{2;\alpha} \times \overline{\mathrm{SO}(3)}_\Omega$. The symmetrization group, in turn, is given by much smaller group $\overline{\mathrm{D}}_2 \subset \overline{\mathrm{D}}_{2;\alpha} \times \overline{\mathrm{D}}_{2;\Omega}$ than in the previous case.

We see that using of different condition defining the intrinsic frame lead to formally different structure of the colective spaces.

This considerations born an interesting question. Do both sets of collective variables

$$\text{set 1.:} \quad \alpha_{20}, \quad \alpha_{22} = \alpha_{2-2}, \quad \alpha_{2\pm 1} = 0, \tag{45}$$

$$\text{set 2.:} \quad \alpha'_{20}, \quad \alpha'_{21} = -\alpha'_{2-1}, \quad \alpha'_{2\pm 2} = 0 \tag{46}$$

describe the same set of shapes? Do are they physically equivalent?

To answer these questions one needs to check if there exists the one-to-one relation between both frames. The required transformation is given by

$$\alpha_{20} = -\frac{1}{2}\alpha'_{20}, \tag{47}$$

$$\alpha_{22} = \exp(-2i\theta_1)\left(\frac{1}{2}\sqrt{\frac{3}{2}}\alpha'_{20} + i\alpha'_{21}\right), \tag{48}$$

where the rotation angle θ_1 can be calculated from the following formula

$$\alpha'_{21}\cos(2\theta_1) = \frac{1}{2}\sqrt{\frac{3}{2}}\alpha'_{20}\sin(2\theta_1),$$
$$\theta_2 = \theta_3 = \frac{\pi}{2}. \tag{49}$$

In fact, the angles $\theta_1, \theta_2, \theta_3$ parametrize the rotation which transform the second set of variables into the first one.

7 An example of a symmetry structure of the collective configuration space

Let us denote by $X_{\alpha^{(lab)}}, X_\alpha$ and $X_{\alpha\Omega}$ the configuration spaces consisted of: a) the laboratory variables $\alpha^{(lab)}$, b) the intrinsic deformation parameters α and c) the full intrinsic configuration space, respectively.

Let us consider again the case of the collective space consisted of Bohr variables $(\alpha_{20}, \alpha_{22})$ which are equivalent to the popular polar parametrization of nuclear shapes (β, γ), where

$$\alpha_{20} = \beta\cos\gamma \quad \text{and} \quad \alpha_{22} = \frac{\beta}{\sqrt{2}}\sin\gamma. \tag{50}$$

The symmetrization group $\overline{\mathrm{O}}$ (inversion omitted) is generated by the following rotations $\bar{R}_1 = \bar{C}_{2y}, \bar{R}_2 = \bar{C}_{4z}, \bar{R}_3 = \bar{R}(\pi/2, \pi/2, \pi)$, where C_{nq} denote the rotation by the angle $2\pi/n$ around the q axis.

To find the region of uniquenes of transformation from the laboratory to the intrinsic frame one needs to construct the orbits obtained from the action of the symmetrization group $\mathrm{G}_s = \overline{\mathrm{O}}$ onto the full intrinsic configuration space $X_{\alpha\Omega}$. In our case the orbits are represented by the following sets

$$\mathrm{orb}(\overline{\mathrm{O}}; \beta_0, \gamma_0, \Omega_0) = \left\{(\beta, \gamma, \Omega) : (\beta, \gamma, \Omega) = \hat{\bar{g}}(\beta_0, \gamma_0, \Omega_0),\ \bar{g} \in \overline{\mathrm{O}}\right\}. \tag{51}$$

Every orbit consists of 24 elements of the configuration space $X_{\alpha\Omega}$ which correspond to the same laboratory deformation. Here, we have used the polar parametrization of the quadrupole variable due to the simpler action of the octahedral group on these variable than on the α_{20}, α_{22} themselves

$$\hat{\bar{g}}\beta = \beta, \quad \hat{\bar{g}}\gamma \in \{\pm\gamma, \pm(\gamma - k\frac{2\pi}{3})\},\ k = 1, 2, 3, \quad \hat{\bar{g}}\Omega = \Omega g. \tag{52}$$

Formally, to have one-to-one transformation from the laboratory to the intrinsic frame one needs to construct the following quotient of the collective configuration space $X_{\alpha\Omega}$:

$$X^C_{\alpha\Omega} = X_{\alpha\Omega}/\mathrm{orb}(\overline{\mathrm{O}}), \tag{53}$$

where two points of the collective magnifold $(\beta', \gamma', \Omega')$ and $(\beta'', \gamma'', \Omega'')$ belong to the same equivalent class of intrinsic points if both points belong to the same orbit, i.e.

$$(\beta', \gamma', \Omega') = (\beta'', \gamma'', \Omega'') \mod(\mathrm{orb}(\overline{\mathrm{O}}))$$
$$\text{iff there exists the point } (\beta_0, \gamma_0, \Omega_0) \in X_{\alpha\Omega}$$
$$\text{such that } (\beta', \gamma', \Omega'), (\beta'', \gamma'', \Omega'') \in \mathrm{orb}(\overline{\mathrm{O}}; \beta_0, \gamma_0, \Omega_0). \tag{54}$$

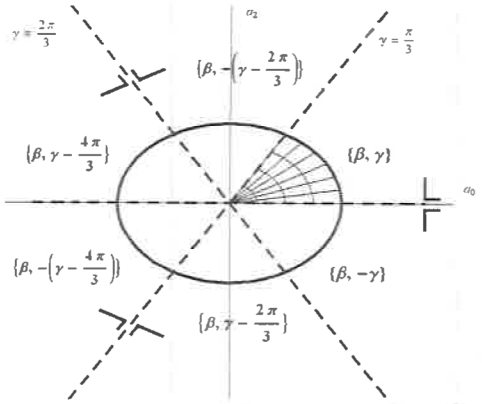

Figure 9: 6 equivalent regions, each region consists of 4 orbits of the symmetrization group $\overline{O}$

The above construction leads to a problem with the notion of the angular momentum operators because in the configuration space $X^C_{\alpha\Omega}$ for the fixed shape the Euler angles are restricted to a subset of the full range of the angles, e.g. the points (β, γ, Ω) and $(\beta, \gamma, \Omega C_{2q})$, where $q = x, y, z$ represent a nucleus of the same shape and the same space orientation in respect to the laboratory frame, though the Euler angles are different.

To recover the angular momentum as the physical observable one needs to join some orbits in such a way to obtain the full range of angles. This can be achieved by the appropriate restriction of the symmetrization group. The restricted symmetrization group $\overline{O}_\alpha \times \bar{1}_\Omega$, where the symbol $\bar{1}_\Omega$ denotes the trivial group consisted of the unit element only, allows to construct the new 6 elements orbits

$$\begin{aligned} &\mathrm{orb}(\overline{O}_\alpha \times \bar{1}_\Omega; \beta_0, \gamma_0, \Omega_0) = \\ &\{(\beta_0, \gamma, \Omega_0) : \gamma = \pm\gamma_0, \pm(\gamma_0 - \frac{2\pi}{3}), \pm(\gamma_0 - \frac{4\pi}{3})\} \end{aligned} \tag{55}$$

and subsequently the collective configuration space in which the Euler angles have the required physical range

$$X^{Bohr}_{\alpha\Omega} = X_{\alpha\Omega}/\mathrm{orb}(\overline{O}_\alpha \times \bar{1}_\Omega). \tag{56}$$

However, in this way we again obtain not invertible transformation from the laboratory to the intrinsic frame. On the other hand, in this case only six (not 24) points in the intrinsic frame correspond to one point $\alpha_2^{(lab)}$ in the laboratory frame, see Fig. (9).

This means that we loose the uniqueness of transformations between the laboratory and the intrinsic frames which was the most important feature we wanted to obtain by our construction.

The same considerations one can performe for the example of the alternative choice of the collective variable $(\alpha_{20}, \alpha_{21})$. In this case the symmetrization group $\overline{D}_2$ (inversion

ommited) consists of the following rotations $\{e_G, \bar{C}_{2x}, \bar{C}_{2y}, \bar{C}_{2z}\}$. The orbits (4 elements each) can be written as

$$\begin{aligned}\mathrm{orb}(\overline{\mathrm{D}}_2; \breve{\alpha}_{20}, \breve{\alpha}_{21}, \breve{\Omega}) = \{&(\breve{\alpha}_{20}, \breve{\alpha}_{21}, \breve{\Omega}), (\breve{\alpha}_{20}, \breve{\alpha}_{21}, \breve{\Omega}\bar{C}_{2y}\bar{C}_{2z}),\\ &(\breve{\alpha}_{20}, -\breve{\alpha}_{21}, \breve{\Omega}\bar{C}_{2y}), (\breve{\alpha}_{20}, -\breve{\alpha}_{21}, \breve{\Omega}\bar{C}_{2z})\}.\end{aligned} \tag{57}$$

In this case we have a very simple action of the group operations onto the collective magnifold

$$\begin{aligned}&\hat{\bar{C}}_{2;a}\alpha_{20} = \alpha_{20}, \quad a = x, y, z; \quad \hat{\bar{C}}_{2;a}\alpha_{21} = -\alpha_{21}, \quad a = y, z;\\ &\hat{\bar{C}}_{2;a}\Omega = \Omega C_{2;a}.\end{aligned} \tag{58}$$

Despite of this, again, one needs to join orbits in such a way to recover full range of angles, to have well defined angular momentum quantum numbers. The restricted group $\overline{\mathrm{D}}_{2;a} \times \bar{\mathrm{I}}_\Omega$ leads to a set of two elements orbits

$$\mathrm{orb}(\overline{\mathrm{D}}_{2;a} \times \bar{\mathrm{I}}_\Omega; \breve{\alpha}_{20}, \breve{\alpha}_{21}, \breve{\Omega}) = \{(\breve{\alpha}_{20}, \alpha_{21}, \breve{\Omega}) : \alpha_{21} = \pm\breve{\alpha}_{21}\}. \tag{59}$$

And the corresponding collective configuration space is given by

$$\begin{aligned}&X_{a\Omega}^{Alter} = X_{a\Omega}/\mathrm{orb}(\overline{\mathrm{D}}_{2;a} \times \bar{\mathrm{I}}) =\\ &\bigcup_{\alpha_{20}\in\mathbb{R}} \bigcup_{\alpha_{21}\in\mathbb{R}_+} \bigcup_{\Omega\in SO(3)} \{(\alpha_{20}, \alpha_{21}, \Omega), (\alpha_{20}, -\alpha_{21}, \Omega)\}.\end{aligned} \tag{60}$$

Finally we get NOT INVERTIBLE (1 to 2)-transformation from the laboratory to the intrinsic frame. This is a typical situation in practical applications.

An alternative way to describe the space of quantum states is to use the space of square integrable functions $\psi\colon X_{a\Omega} \to \mathbb{C}$ with symmetrization condition for quantum states ψ. However, it is important to notice that, in this case, the arguments of the quantum states (collective functions) run over the full configuration space $X_{a\Omega}$.

8 Symmetrization

An idea expressed in the last sentences of the previous section requires a bit more detailed analysis of a structure of the space of states. The physical state space consists of all the functions $\phi\colon X_{a\Omega} \to \mathbb{C}$ which fulfil the symmetrization condition

$$\mathcal{K} = \{\phi(\alpha, \Omega) : \hat{\bar{g}}\phi = \phi, \text{ for all } \bar{g} \in \overline{\mathrm{G}}_s\}. \tag{61}$$

The collective Hamiltonians $\hat{\mathcal{H}}$ are generally defined in the wider space $\mathcal{K}_{coll}$ consisted of all square integrable functions, not only symmetrized. In fact, to have physical solutions one needs to restrict, in some way, the Hamiltonian $\hat{\mathcal{H}}$ to the physical subspace $\mathcal{K}_{coll}$. There are two possible procedures:

1. **Projection.** First, the Hamiltonian $\hat{\mathcal{H}}$ is projected onto the physical space $\mathcal{K}$: $\hat{\mathcal{H}}_1 = \hat{P}_{\mathcal{K}}\hat{\mathcal{H}}\hat{P}_{\mathcal{K}}$. Second, one needs to solve it in the space of symmetrized functions $\mathcal{K}$. An important notice: in this case the Hamiltonian $\hat{\mathcal{H}}_1 = \hat{P}_{\mathcal{K}}\hat{\mathcal{H}}\hat{P}_{\mathcal{K}}$ has the symmetry provided by the symmetrization group $\overline{\mathrm{G}}_s$.

2. **Selection.** First, one can solve the Hamiltonian $\hat{\mathcal{H}}$ in the full (in general not symmetrized) space of states $\mathcal{K}_{coll}$ and afterward one needs to choose the solutions belonging to the space of symmetrized states $\mathcal{K}$.

An open question is which procedure is physical?

To show differences and similarities between both approaches one needs to define the projection operator onto the scalar representation of the symmetrization group $\overline{\mathrm{G}}_s$) in the space $\mathcal{K}$:

$$\hat{P}_{\mathcal{K}} = \frac{1}{\mathrm{card}(\overline{\mathrm{G}}_s)} \sum_{\hat{g} \in \overline{\mathrm{G}}_s} \hat{\hat{g}}. \tag{62}$$

The first procedure 'Projection' creates a new Hamiltonian from the original one

$$\hat{\mathcal{H}}_1 \equiv \hat{P}_{\mathcal{K}} \hat{\mathcal{H}} \hat{P}_{\mathcal{K}} \tag{63}$$

$$\hat{\mathcal{H}}_1 |\Psi_{1;\nu}\rangle = E_{1;\nu} |\Psi_{1;\nu}\rangle. \tag{64}$$

In this case the action of the projection operator $\hat{P}_{\mathcal{K}} |\Psi_{1;\nu}\rangle = |\Psi_{1;\nu}\rangle \in \mathcal{K}$ is closed within the physical state space.

The Hamiltonian $\hat{\mathcal{H}}_1$ can be expressed in terms of its eigenvectors and eigenvalues by making use of the spectral theorem

$$\hat{\mathcal{H}}_1 = \sum_{\nu} E_{1;\nu} |\Psi_{1;\nu}\rangle\langle\Psi_{1;\nu}|. \tag{65}$$

As it was mentioned earlier, the Hamiltonian $\hat{\mathcal{H}}_1$ has the intrinsic symmetry which is not smaller than the symmetrization group $\overline{\mathrm{G}}_s$. Sometimes it can have even a larger symmetry group. It happens independently of the symmetry of the original Hamiltonian $\hat{\mathcal{H}}$.

The second procedure 'Selection' requires first to solve the original Hamiltonian $\hat{\mathcal{H}}$ in the full (in general not physical) space of states $\mathcal{K}_{coll}$

$$\hat{\mathcal{H}} |\Psi_n\rangle = E_n |\Psi_n\rangle. \tag{66}$$

The next step is to choose the solutions which fulfil the symmetrization condition (42). Let us dentoe these eigenstates of (66) by $|\Psi_{2;n}\rangle$ and the corresponding eigenenergies E_n by $E_{2;n}$,

$$\hat{P}_{\mathcal{K}} |\Psi_{2;n}\rangle = |\Psi_{2;n}\rangle \equiv |\Psi_{2;n}\rangle_{\mathcal{K}}. \tag{67}$$

This set of the symmetrized states and the corresponding eigenenergies allow to construct (by the spectral theorem) the effective Hamiltonian which fulfiles the required conditions: its action is closed within the physical subspace $\mathcal{K}$ and it is invariant in respect to the symmetrization group. This effective Hamiltonian H_2 can be written down as

$$\hat{\mathcal{H}}_2 = \sum_{n} E_{2;n} |\Psi_{2;n}\rangle_{\mathcal{K}\mathcal{K}}\langle\Psi_{2;n}|. \tag{68}$$

Both Hamiltonians $\hat{\mathcal{H}}_1$ and $\hat{\mathcal{H}}_2$ can be related. Let us assume that the kets $|\Psi_{2;n}\rangle \in \mathcal{K}$ are the symmetrized eigenvectors of the full Hamiltonian $\hat{\mathcal{H}}$, then

$$\hat{\mathcal{H}} |\Psi_{2;n}\rangle_{\mathcal{K}} = E_{2;n} |\Psi_{2;n}\rangle_{\mathcal{K}} \;\Rightarrow \hat{\mathcal{H}}_1 |\Psi_{2;n}\rangle_{\mathcal{K}} = E_{2;n} |\Psi_{2;n}\rangle_{\mathcal{K}} \tag{69}$$

and the solutions obtained from the second procedure are also the solutions which we obtain from the first procedure.

However, the OPPOSITE property is not TRUE.

To show this conjecture let us consider the eigenstates of the effective Hamiltonian $\hat{\mathcal{H}}_1$

$$\hat{\mathcal{H}}_1|\Psi_{1;\nu}\rangle = E_{1;\nu}|\Psi_{1;\nu}\rangle. \tag{70}$$

Then, in general, putting the projection operator $\hat{P}_{\mathcal{K}}$ (it projects onto the physical subspace) and $\hat{Q}_{\mathcal{K}} = 1 - \hat{P}_{\mathcal{K}}$, into Eq. (70) one obtains

$$\hat{\mathcal{H}}|\Psi_{1;\nu}\rangle = \left\{\hat{\mathcal{H}}_1 + (\hat{P}_{\mathcal{K}}\hat{\mathcal{H}}\hat{Q}_{\mathcal{K}} + \hat{Q}_{\mathcal{K}}\hat{\mathcal{H}}\hat{P}_{\mathcal{K}}) + \hat{Q}_{\mathcal{K}}\hat{\mathcal{H}}\hat{Q}_{\mathcal{K}}\right\}|\Psi_{1;\nu}\rangle =$$
$$E_{1;\nu}|\Psi_{1;\nu}\rangle + \hat{Q}_{\mathcal{K}}\hat{\mathcal{H}}|\Psi_{1;\nu}\rangle \neq c|\Psi_{1;\nu}\rangle, \tag{71}$$

where c is the proportionality coefficient.

We see that the projected hamiltonian $\hat{\mathcal{H}}_1$ can provide more solutions than the 'generating' Hamiltonian $\hat{\mathcal{H}}$ used with the second procedure. It means that both symmetrization procedures are not equivalent and can lead to different physical quantum models.

One needs to notice that the 'Selection' procedure is used in the standard Bohr–like colective nuclear models.

As a pattern/example let us consider the Bohr Hamiltonian in the case of quadrupole variables β, γ, Ω:

$$\hat{\mathcal{H}}_{Bohr} = \hat{\mathcal{H}}_{vib}(\beta,\gamma) + \hat{\mathcal{H}}_{rot}(\Omega) + \hat{\mathcal{H}}_{vr}(\beta,\gamma,\Omega), \tag{72}$$

where the vibrational part of the Hamiltonian is

$$\hat{\mathcal{H}}_{vib} = -\frac{\hbar^2}{2B}\left\{\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} - \frac{1}{\beta^2\sin(3\gamma)}\frac{\partial}{\partial\gamma}\sin(3\gamma)\frac{\partial}{\partial\gamma} + \beta^2\right\} + V_B, \tag{73}$$

the "rigid" rotation part is given by

$$\hat{\mathcal{H}}_{rot} = \frac{1}{2}\sum_{k=1,2,3}\frac{J_k^2}{\mathcal{J}_k}. \tag{74}$$

and the coupling part which describes discrepances beetween the terms with the constant moment of inertia and the hydrodinamical moment of inertia which depends on the vibrational variables β, γ is of the following form

$$\hat{\mathcal{H}}_{vr} = \frac{1}{8\beta^4}\sum_{k=1,2,3}\frac{J_k^2}{\sin^2(\gamma - (2\pi/3)k)} - \hat{\mathcal{H}}_{rot}. \tag{75}$$

This is not the difficult exercise to check that the vibrational sub-Hamiltonian has an octahedral symmetry:

$$\mathrm{Sym}(\hat{\mathcal{H}}_{vib}) = \overline{\mathrm{O}}_{h;\alpha}. \tag{76}$$

It is sufficient to check invariance of the vibrational sub-Hamiltonian with respect to the generators of the group $\overline{\mathrm{O}}_{h;\alpha}$, represented by the following rotations

$$R_1 \equiv \bar{R}(0,\pi,0): \quad (\beta,\gamma) \to (\beta,\gamma),$$
$$R_2 \equiv \bar{R}(0,0,\pi/2): \quad (\beta,\gamma) \to (\beta,-\gamma),$$
$$R_3 \equiv \bar{R}(\pi/2,\pi/2,\pi): \quad (\beta,\gamma) \to (\beta,\gamma-\pi/3). \tag{77}$$

The easiest way to proceed is to notice that the sub-Hamiltonian $\hat{\mathcal{H}}_{vib}(\beta,\gamma) = \hat{\mathcal{H}}_{vib}(\partial/\partial\beta, \partial/\partial$ is a function of invariants of the group $\overline{\mathrm{O}}_{h;\alpha}$.

In a similar way one can find the symmetry of the rotational sub-Hamiltonian. It has simple, dihedral symmetry acting on the Euler angles of the system

$$\mathrm{Sym}(\hat{\mathcal{H}}_{rot}) = \overline{\mathrm{D}}_{2h;\Omega}. \tag{78}$$

This group has two generators which transform the collective variables and the angular momenta operators in the following way

$$\begin{aligned} \bar{C}_{2y}: &\quad (\beta,\gamma) \to (\beta,\gamma), \quad J_k^2 \to J_k^2, \\ \bar{C}_{2z}: &\quad (\beta,\gamma) \to (\beta,\gamma), \quad J_k^2 \to J_k^2. \end{aligned} \tag{79}$$

Similarly as in the previous case the rotational sub-Hamiltonian $\hat{\mathcal{H}}_{rot}(\Omega) = \hat{\mathcal{H}}_{rot}(J_x, J_y, J_z)$ is a function of the invariants of the dihedral group $\overline{\mathrm{D}}_{2h;\Omega}$. The coupling term $\hat{\mathcal{H}}_{vr}$ has a bit more complicated symmetry group represented by the direct product of two groups which, in fact does not contain the symmetrization groups as a subgroup:

$$\overline{\mathrm{O}}_{h,\alpha} \times \overline{\mathrm{D}}_{2h,\Omega} \not\supset \overline{\mathrm{O}}_h. \tag{80}$$

The last property, that the symmetrization group is not a symmetry of the Bohr Hamiltonian shows that the Bohr Hamiltonian can be treated only as the generating Hamiltonian which after either the 'Projection' or 'Selection' symmetrization procedure can be converted into the physical quantum Hamiltonian in the intrinsic frame. Traditionally, the 'Selection' symmetrization procedure is used.

8.1 Summary

In this short lecture we wanted to show the main ingredients which allow to prepare description of a physical system in the intrinsic frame. In this introduction to the problem of physics in the intrinsic frames, to make the lecture as simple as possible, we have used only the rotation intrinsic frame. However, a generalization to other kinds of the intrinsic frames is traightforward.

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Five Lectures on Supersymmetry: Elementary Introduction

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Abstract

These five lectures collect elementary facts about $4D$ supersymmetric theories with emphasis on $\mathcal{N} = 1$ supersymmetry, as well as the basic notions of supersymmetric quantum mechanics. Contents: **I.** From symmetries to supersymmetry; **II.** Basic features of supersymmetry; **III.** Representations of supersymmetry; **IV.** Superspace and superfields; **V.** Supersymmetric quantum mechanics.

1 Lecture I: From symmetries to supersymmetry

1.1 Groups and symmetries

Symmetries play the central role in physics: They underlie all the theories of interest known to date. Their basis is the *Group Theory*.

- Gravity: Based on the local diffeomorphism group of the space-time, $Diff\, R^4$, $x^m \Rightarrow x'^m(x)$.

- Maxwell theory and its non-abelian generalization, Yang-Mills theory: Based on the gauge groups $U(1)$ and $SU(n)$, with group parameters being arbitrary functions of the space-time point.

- Standard model, the unification of the electro-week theory and quantum chromodynamics: [Gauge $U(2)_{e.w.} \otimes SU(3)_c$] $\otimes$ [Global Flavor $SU(N)_f$ (broken)].

- String theory: Diffeomorphisms of the worldsheet $(z\,, \bar{z})$.

- Supergravity, Superstrings, Superbranes: Supersymmetry (local, global, conformal, ...).

Group: Some manifold $G = \{g_n\}$, $n = 1, 2, \ldots$, such that the following axioms are valid:

1. Closedness under the appropriate product:

$$g_1 \cdot g_2 = g_3 \in G;$$

2. The existence of the unit element $I \in G$:

$$g \cdot I = I \cdot g = g;$$

3. The existence of the inverse element for any $g_n \in G$:

$$g \cdot g^{-1} = g^{-1} \cdot g = I;$$

4. Associativity of the product:

$$(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3).$$

Simplest examples: 1) $(1, -1)$ with respect to the standard multiplication; 2)integer numbers, with respect to the summation, with 0 as the unit element, etc.

Types of groups: 1) finite groups; 2) infinite countable groups; 3) continuous or topological groups (Lie groups). We will be interested in the third type.

- **Lie groups**:

$$G = \{g(x)\} \quad x := (x^1, x^2, \ldots, x^r), \quad r(\text{rank}) = \text{Dim}\, G,$$
$$g(x) \cdot g(y) = g(z(x,y)) \in G, \quad g(0) = I, \quad z(0,y) = y, \; z(x,0) = x.$$

For Lie groups, one can always parametrize their elements, in a vicinity of the unit element, as

$$g(x) = \exp\{x^i T_i\}, \quad [T_i, T_k] = c^l_{ik} T_l, \quad c^l_{ik} = -c^l_{ki},$$

where T_i are *generators* and c^l_{ik} are *structure constants*.

The generators T_i span the algebra called *Lie algebra*. The Lie algebra is specified by its structure constants which, in virtue of the *Jacobi* identity

$$[T_l, [T_k, T_i]] + [T_i, [T_l, T_k]] + [T_k, [T_i, T_l]] = 0,$$

satisfy the fundamental relation

$$c^m_{ki} c^p_{lm} + c^m_{lk} c^p_{im} + c^m_{il} c^p_{km} = 0.$$

Example: The group $SU(2)$:

$$g = \exp\{i\lambda_a T_a\}\,,\ (T_a)^\dagger = T_a\,,\quad [T_a, T_b] = i\varepsilon_{abc}T_c\,,\quad a,b,c = 1,2,3\,,$$
$$\varepsilon_{abc}\varepsilon_{dce} + \varepsilon_{eac}\varepsilon_{bcd} + \varepsilon_{dec}\varepsilon_{acb} = 0\,.$$

There are two vast classes of symmetries in the Nature:

- **I. Internal symmetries**: Isotopic $SU(2)$, flavor $SU(n)$, etc. Their main feature: They are realized as transformations of fields without affecting the space-time coordinates. The generators are matrices acting on some external indices of fields, no any x-derivatives are present.
  Example: Realization of $SU(2)$ on the doublet of fields $\psi_i(x)$ ("neutron - proton")

  $$\delta\psi_i(x) = i\lambda_a\frac{1}{2}(\sigma_a)^k_i\psi_k(x)\,,\quad [\frac{1}{2}\sigma_a, \frac{1}{2}\sigma_b] = i\varepsilon_{abc}\frac{1}{2}\sigma_c\,,$$
  $$\sigma_a\sigma_b = \delta_{ab}\mathbf{I} + i\varepsilon_{abc}\sigma_c\,,$$

  σ_a are Pauli matrices:

  $$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- **II. Space-time symmetries**: Lorentz, Poincaré and conformal groups. Generators in the realization on fields involve x-derivatives.
  Example: Transformation of the scalar field $\varphi(x)$ in the Poincaré group:

  $$\delta\varphi(x) := -ic^m P_m - i\omega^{[mn]}L_{mn}\varphi(x) = -c^m\partial_m\varphi(x) - \omega^{[mn]}\frac{1}{2}(x_m\partial_n - x_n\partial_m)\varphi(x)\,,$$
  $$P_m = \frac{1}{i}\partial_m\,,\quad L_{mn} = \frac{1}{2i}(x_m\partial_n - x_n\partial_m)\,,\quad m,n = 0,1,2,3\,.$$

1.2 Invariant Lagrangians

The primary fundamental symmetry principle is the invariance of the action:

$$S = \int d^4x\mathcal{L}(\phi_A, \partial\phi_A, \psi_\alpha, ...)\,,\quad \delta S = \frac{\delta S}{\delta\phi_A}\delta\phi_A = 0 \leftrightarrow \delta\mathcal{L} = \partial_m\mathcal{A}^m\,.$$

Example: The free Lagrangian of the scalar field

$$\mathcal{L}^{(1)}_{free} = \frac{1}{2}\partial^m\phi(x)\partial_m\phi(x)$$

transforms under the Poincaré group as

$$\delta_\omega\mathcal{L}^{(1)}_{free} = -\frac{1}{2}\partial_n(\omega^{mn}x_m\partial^s\phi\partial_s\phi)\,,\quad \delta_c\mathcal{L}^{(1)}_{free} = -\frac{1}{2}c^m\partial_m(\partial^n\phi\partial_n\phi),$$

whence the invariance of the relevant action follows.

In the systems with few scalar fields one can realize internal symmetries. The free Lagrangian of one complex field

$$\mathcal{L}^{(2)}_{free} = \partial^m \phi(x) \partial_m \bar{\phi}(x)$$

is invariant under $U(1)$ symmetry

$$\delta\phi = i\lambda\phi\,, \quad \delta\bar{\phi} = -i\lambda\bar{\phi},$$

three real scalar fields can be joined into a triplet of the group $SU(2)$:

$$\mathcal{L}^{(3)}_{free} = \frac{1}{2}\partial^m \phi_a(x)\partial_m\phi_a(x)\,, \quad \delta\phi_a = \varepsilon_{abc}\lambda_b\phi_c \;\Rightarrow\; \delta\mathcal{L}^{(3)}_{free} = 0\,.$$

One more possibility to construct $SU(2)$ invariant Lagrangian is to join two complex scalar fields into $SU(2)$ doublet

$$\mathcal{L}^{(4)}_{free} = \partial^m \phi_\alpha(x)\partial_m\bar{\phi}^\alpha(x)\,,$$
$$\delta\phi_\alpha = \frac{i}{2}\lambda_a(\sigma_a)^\beta_\alpha\phi_\beta\,, \quad \delta\bar{\phi}^\alpha = -\frac{i}{2}\lambda_a(\sigma_a)^\alpha_\beta\bar{\phi}^\beta\,, \;\Rightarrow\; \delta\mathcal{L}^{(4)}_{free} = 0\,.$$

Extending the sets of fields (and adding interaction terms), we can further enlarge internal symmetries.

The characteristic feature of all these symmetries is that the group parameters are ordinary commuting numbers, and so the group transformations do not mix *bosonic fields* (*Bose-Einstein* statistics, integer spins $0, 1, \ldots$) with *fermionic fields* (*Fermi-Dirac* statistics, half-integer spins $1/2, 3/2, \ldots$). The bosonic and fermionic parts of the Lagrangian are invariant *separately*.

1.3 Supersymmetry as symmetry between bosons and fermions

Let us now consider a sum of the free Lagrangians of the massless complex scalar field $\varphi(x)$ and the Weyl fermionic field $\psi^\alpha(x)$

$$\mathcal{L}_{\phi+\psi} = \partial^m\varphi\partial_m\bar{\varphi} - \frac{i}{4}\Big[\psi^\alpha(\sigma^m)_{\alpha\dot{\alpha}}\partial_m\bar{\psi}^{\dot{\alpha}} - \partial_m\psi^\alpha(\sigma^m)_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}\Big]\,,$$

where $(\sigma^m)_{\alpha\dot{\alpha}} = (\delta_{\alpha\dot{\alpha}}, (\sigma^a)_{\alpha\dot{\alpha}})$ are the so called sigma matrices, the basic object of the spinor two-component formalism of the Lorentz group (they are invariant under simultaneous Lorentz transformation of the vector $m = 0, 1, 2, 3$, and spinor $\alpha, \dot{\alpha} - 1, 2$ indices).

The evident symmetries of this Lagrangian are Poincaré and phase $U(1)$ symmetries which separately act on $\varphi(x)$ and $\psi^\alpha(x)$.

However, there is a new much less obvious symmetry. Namely, this Lagrangian transforms by a total derivative under the following transformations mixing bosonic and fermionic fields

$$\delta\varphi = -\epsilon^\alpha \psi_\alpha\,, \quad \delta\bar{\varphi} = -\bar{\psi}_{\dot\alpha}\bar{\epsilon}^{\dot\alpha}\,, \quad \delta\psi_\alpha = 2i(\sigma^m)_{\alpha\dot\alpha}\bar{\epsilon}^{\dot\alpha}\partial_m\varphi\,.$$

One sees that the transformation parameters $\epsilon^\alpha, \bar{\epsilon}^{\dot\alpha}$ have the dimension $cm^{1/2}$, so these transformations *do not* define an internal symmetry (the relevant group parameters would be *dimensionless*). Moreover, for the action to be invariant, these parameters should *anti-commute* among themselves and with the fermionic fields, $\{\epsilon,\epsilon\} = \{\epsilon,\bar{\epsilon}\} = \{\epsilon(\bar{\epsilon}),\psi\} = 0$, and *commute* with the scalar field, $[\epsilon(\bar{\epsilon}),\varphi] = 0$, and with the parameters of the ordinary symmetries, e.g., $[\epsilon(\bar{\epsilon}), c^m] = 0$.

To see which kind of algebraic structure is behind this invariance one needs to consider the *Lie bracket* of two successive transformations on the scalar $\varphi(x)$:

$$(\delta_1\delta_2 - \delta_2\delta_1)\varphi = -(\epsilon_2^\alpha\delta_1\psi_\alpha) - (\epsilon_1^\alpha\delta_2\psi_\alpha) = 2\,(\epsilon_1\sigma^m\bar{\epsilon}_2 - \epsilon_2\sigma^m\bar{\epsilon}_1)\,(\frac{1}{i}\partial_m\varphi).$$

Thus the result is an infinitesimal 4-translation with the parameter $i\,(\epsilon_1\sigma^m\bar{\epsilon}_2 - \epsilon_2\sigma^m\bar{\epsilon}_1)$.

Rewriting the c variation in the form

$$\delta\varphi = i\left(\epsilon^\alpha Q_\alpha + \bar{\epsilon}_{\dot\alpha}\bar{Q}^{\dot\alpha}\right)\varphi\,,$$

and taking into account that the spinor parameters *anticommute* with $Q_\alpha, \bar{Q}^{\dot\alpha}$, we find that the above *Lie bracket* structure is equivalent to the following *anticommutation* relations for the supergenerators

$$\begin{aligned}
&\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\,(\sigma^m)_{\alpha\dot\beta}P_m\,, \quad P_m = \frac{1}{i}\frac{\partial}{\partial x^m}\,,\\
&\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0\,,\\
&[P_m, Q_\alpha] = [P_m, \bar{Q}_{\dot\alpha}] = 0\,.
\end{aligned}$$

This is what is called $\mathcal{N} = 1$ *Poincaré superalgebra.*

2 Lecture II: Basic features of supersymmetry

The full set of the (anti)commutation relations of the $\mathcal{N} = 1$ Poincaré superalgebra reads

$$\begin{aligned}
&\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\,(\sigma^m)_{\alpha\dot\beta}P_m\,,\\
&\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0\,,\\
&[P_m, Q_\alpha] = [P_m, \bar{Q}_{\dot\alpha}] = 0\,,\\
&[J_{mn}, Q_\alpha] = -\frac{1}{2}\,(\sigma_{mn})_\alpha^{\ \beta}Q_\beta\,, \quad [J_{mn}, \bar{Q}_{\dot\alpha}] = \frac{1}{2}\,(\bar{\sigma}_{mn})_{\dot\alpha}^{\ \dot\beta}\bar{Q}_{\dot\beta}\,,\\
&[J_{mn}, P_s] = i\,(\eta_{ns}P_m - \eta_{ms}P_n)\,,\\
&[J_{mn}, J_{sq}] = i\,(\eta_{ns}J_{mq} - \eta_{ms}J_{nq} + \eta_{nq}J_{sm} - \eta_{mq}J_{sn})\,,\\
&[R, Q_\alpha] = Q_\alpha\,, \quad [R, \bar{Q}_{\dot\alpha}] = -\bar{Q}_{\dot\alpha} \quad [R, P_m] = [R, J_{mn}] = 0\,.
\end{aligned} \tag{2.1}$$

Here $J_{mn} = L_{mn} + S_{mn}$ are the full Lorentz group generators (S_{mn} is the *spin* part acting on the external vector and spinor indices) and R is a generator of an extra internal $U(1)$ symmetry (the so-called R symmetry). Also,

$$\eta_{mn} = \mathrm{diag}(1,-1,-1,-1)\,, \quad (\sigma_{mn})^{\beta}_{\alpha} = \frac{i}{2}\,(\sigma^m\tilde{\sigma}^n - \sigma^n\tilde{\sigma}^m)^{\beta}_{\alpha}\,,$$
$$(\tilde{\sigma}^{mn})^{\dot{\beta}}_{\dot{\alpha}} = \frac{i}{2}\,(\tilde{\sigma}^m\sigma^n - \tilde{\sigma}^n\sigma^m)^{\dot{\beta}}_{\dot{\alpha}}\,, \quad \tilde{\sigma}^{m\dot{\alpha}\alpha} = (\delta^{\dot{\alpha}\alpha}, -\sigma^{a\dot{\alpha}\alpha})\,.$$

Some important common features and consequences of supersymmetry can be figured out just from these (anti)commutation relations.

- The Poincaré superalgebra is an example of Z_2-graded algebra. The latter is defined in the following way: one ascribes parities ± 1 to all its elements, calling them, respectively, *even* (parity $+1$) and *odd* (parity -1) elements, and requires the structure relations to respect these parities:

  $$[\mathrm{odd},\mathrm{odd}] \sim \mathrm{even}\,,\ [\mathrm{even},\mathrm{odd}] \sim \mathrm{odd}\,,\ [\mathrm{even},\mathrm{even}] \sim \mathrm{even}\,.$$

  From the above (anti)commutation relations we observe that the spinor generators $Q_\alpha, \bar{Q}_{\dot{\alpha}}$ can be assigned the parity -1 and so they are *odd*; all bosonic generators can be assigned the parity $+1$ and so they are *even*.

- Lie superalgebras satisfy the same axioms as the Lie algebras, the difference is that the relevant generators satisfy the *graded* Jacobi identities because the fermionic generators are subject to the *anticommutation* relations. E.g.,

  $$\{[B_1,F_2],F_3\} - \{[F_3,B_1],F_2\} + [\{F_2,F_3\},B_1] = 0\,,$$
  $$[\{F_1,F_2\},F_3] + [\{F_3,F_1\},F_2] + [\{F_2,F_3\},F_1] = 0\,,$$

  where B_1 is a bosonic generator and F_1, F_2, F_3 are fermionic ones.

- Since the generators $Q_\alpha, \bar{Q}_{\dot{\alpha}}$ are fermionic, irreducible multiplets of supersymmetry (*supermultiplets*) should unify bosons with fermions. Action of the spinor generators on the bosonic state yields a fermionic state and vice versa.

- Since the translation operator P_m is non-vanishing on *any* field given on the Minkowski space, the same should be true for the spinor generators as well. So any field should belong to a non-trivial supermultiplet.

- It follows from the relations $[P_m, Q_\alpha] = [P_m, \bar{Q}_{\dot{\alpha}}] = 0$ that $[P^2, Q_\alpha] = [P^2, \bar{Q}_{\dot{\alpha}}] = 0$. The operator P^2 is a Casimir of the Poincaré group, $P^2 = m^2$. So it is also a Casimir of the Poincaré supergroup. Hence all components of the irreducible supermultiplet should have the same mass. No mass degeneracy between bosons and fermions is observed in Nature, so supersymmetry should be *broken* in one or another way.

- In any representation of supersymmetry, such that the operator P_m is invertible, there should be *equal numbers* of bosons and fermions.

- In any supersymmetric theory the energy P_0 should be non-negative. Indeed, from the basic anticommutator it follows

$$\sum_{\alpha=1,2} \left(|Q_\alpha|^2 + |\bar{Q}_{\dot{\alpha}}|^2\right) = 4P_0 \geq 0\,.$$

- Rigid supersymmetry, with constant parameters, implies the translation invariance. Gauge supersymmetry, with the parameters being arbitrary functions of the space-time point, implies the invariance under arbitrary *diffeomorphisms* of the Minkowski space. Hence the theory of gauged supersymmetry necessarily contains *gravity*. The theory of gauged supersymmetry is *supergravity*. Its basic gauge fields are *graviton* (spin 2) and *gravitino* (spin 3/2).

2.1 Extended supersymmetry

Supersymmetry allows one to evade the famous *Coleman-Mandula* theorem about impossibility of non-trivial unification of the space-time symmetries with the internal ones. It states that any symmetry of such type (in dimensions ≥ 3), under the standard assumptions about the spin-statistics relation, is inevitably reduced to the direct product of the Poincaré group and the internal symmetry group.

The arguments of this theorem do not apply to superalgebras, when one deals with both commutation and anticommutation relations. **Haag, Lopushanski**, and **Sohnius** showed that the most general superextension of the Poincaré group algebra is given by the following relations

$$\begin{aligned}
&\{Q^i_\alpha, \bar{Q}_{\dot{\beta}k}\} = 2\delta^i_k\,(\sigma^m)_{\alpha\dot{\beta}} P_m\,,\\
&\{Q^i_\alpha, Q^j_\beta\} = \epsilon_{\alpha\beta} Z^{ij}\,, \quad \{\bar{Q}_{\dot{\alpha}i}, \bar{Q}_{\dot{\beta}j}\} = \epsilon_{\dot{\alpha}\dot{\beta}} \bar{Z}_{ij}\,,\\
&[T^i_j, Q^k_\alpha] = -i\left(\delta^k_j Q^i_\alpha - \frac{1}{\mathcal{N}}\delta^i_j Q^k_\alpha\right), \quad [T^i_j, \bar{Q}_{\dot{\alpha}k}] = i\left(\delta^i_k \bar{Q}_{\dot{\alpha}j} - \frac{1}{\mathcal{N}}\delta^i_j \bar{Q}_{\dot{\alpha}k}\right),\\
&[T^i_j, T^k_l] = i\left(\delta^i_l T^k_j - \delta^k_j T^i_l\right),
\end{aligned}$$

where T^i_j ($(T^i_j)^\dagger = -T^j_i$, $T^i_i = 0$) are generators of the group $SU(\mathcal{N})$. The generators $Z^{ij} = -Z^{ji}$, $\bar{Z}_{ij} = -\bar{Z}_{ji}$ are central charges, they commute with all generators except the $SU(\mathcal{N})$ ones

$$[Z, Z] = [Z, \bar{Z}] = [Z, P] = [Z, J] = [Z, Q] = [Z, \bar{Q}] = 0\,.$$

The relevant supergroup is called *$\mathcal{N}$-extended Poincaré supergroup.*

Due to the property that the spinor generators $Q^i_\alpha, \bar{Q}_{\dot{\beta}k}$ carry the internal symmetry indices, the supermultiplets of extended supersymmetries join fields having not only different statistics and spins, but also belonging to different representations of the internal symmetry group $U(\mathcal{N})$. In other words, in the framework of extended supersymmetry the *actual unification* of the space-time and internal symmetries comes about. The relevant supergravities involve, as a subsector, gauge theories of internal symmetries, i.e. they yield non-trivial unifications of *Einstein* gravity with *Yang-Mills* theories.

2.2 Auxiliary fields

An important ingredient of supersymmetric theories is the *auxiliary fields*. They ensure the closedness of the supersymmetry transformations *off mass shell*.

Let us come back to the realization of $\mathcal{N} = 1$ supersymmetry on the fields $\varphi(x), \psi_\alpha(x)$ and calculate Lie bracket of the odd transformations on $\psi_\alpha(x)$:

$$(\delta_1\delta_2 - \delta_2\delta_1)\psi_\alpha = -2i\left(\epsilon_1\sigma^m\bar{\epsilon}_2 - \epsilon_2\sigma^m\bar{\epsilon}_1\right)\partial_m\psi_\alpha + 2i\left[\epsilon_{1\alpha}\bar{\epsilon}_{2\dot{\alpha}}(\tilde{\sigma}^m)^{\dot{\alpha}\beta}\partial_m\psi_\beta - (1 \leftrightarrow 2)\right].$$

The first term in the r.h.s. is the translation one, as for $\varphi(x)$. However, there is one extra term. It is clear that the Lie bracket should have the same form on all members of the supermultiplet, i.e. reduce to translations. The condition of vanishing of the second term is

$$\tilde{\sigma}^m\partial_m\psi = \sigma^m\partial_m\bar{\psi} = 0\,.$$

But this is just the free equation of motion for $\psi_\alpha(x)$. Thus $\mathcal{N} = 1$ supersymmetry is closed only on-shell, i.e. modulo equations of motion.

How to secure the off-shell closure? The way out is to introduce a new field $F(x)$ of non-canonical dimension cm^{-2} and to extend the free action of φ, ψ_α as

$$\mathcal{L}_{\phi+\psi+F} = \partial^m\varphi\partial_m\bar{\varphi} - \frac{i}{4}\left[\psi^\alpha(\sigma^m)_{\alpha\dot{\alpha}}\partial_m\bar{\psi}^{\dot{\alpha}} - \partial_m\psi^\alpha(\sigma^m)_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}\right] + F\bar{F}\,.$$

It is invariant, up to a total derivative, under the modified transformations having the correct closure for all fields:

$$\delta\phi = -\epsilon^\alpha\psi_\alpha\,,\ \delta\psi_\alpha = -2i(\sigma^m)_{\alpha\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}}\partial_m\phi - 2\epsilon_\alpha F\,,\ \delta F = -i\bar{\epsilon}^{\dot{\alpha}}(\sigma^m)_{\alpha\dot{\alpha}}\partial_m\psi^\alpha\,. \tag{2.2}$$

The auxiliary fields satisfy the *algebraic* equations of motion

$$F = \bar{F} = 0\,.$$

After substitution of this solution back in the Lagrangian and supersymmetry transformations, we reproduce the previous on-shell realization. The auxiliary fields do not propagate also in the quantum case, possessing delta-function propagators.

The only (but very important!) role of the auxiliary fields is just to ensure the correct off-shell realization of supersymmetry, such that it does not depend on the precise choice of the invariant Lagrangian, like in the cases of ordinary symmetries.

The simplest non-trivial choice is

$$\mathcal{L}_{\rm WZ} = \partial^m\phi\partial_m\bar{\phi} - \frac{i}{4}\left[\psi^\alpha(\sigma^m)_{\alpha\dot{\alpha}}\partial_m\bar{\psi}^{\dot{\alpha}} - \partial_m\psi^\alpha(\sigma^m)_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}\right] + F\bar{F}$$
$$+ \left[m\left(\phi F - \frac{1}{4}\psi\psi\right) + g\left(\phi^2 F - \frac{1}{2}\phi\psi\psi\right) + c.c.\right].$$

This model was the first example of *renormalizable* supersymmetric quantum field theory and it is called the **Wess-Zumino** model, after names of its discoverers. The Lagrangian

$\mathcal{L}_{\rm WZ}$ is invariant under the same transformations as the free Lagrangian we have considered before.

The Wess-Zumino model Lagrangian was originally found by the "trying and error" method. The systematic way of constructing invariant off-shell Lagrangians is the *superfield* method which we will discuss in the Lectures IV and V.

Using this systematic method, one can equally construct more general Lagrangians of the fields (ϕ, ψ_α, F), invariant under the same linear *off-shell* $\mathcal{N} = 1$ supersymmetry transformations (2.2). After eliminating the auxiliary fields from these Lagrangians by their equations of motion, we will obtain the Lagrangians in terms of the physical fields (ϕ, ψ_α) only. These physical Lagrangians are invariant under the nonlinear *on-shell* $\mathcal{N} = 1$ supersymmetry transformations the precise form of which depends on the form of the on-shell Lagrangian, though it is uniquely specified by the off-shell Lagrangian.

To summarize, the fields (ϕ, ψ_α, F) form the set *closed* under the off-shell $\mathcal{N} = 1$ supersymmetry transformations, and it is impossible to select any lesser closed set of fields in it. Thus these fields constitute the simplest irreducible multiplet of $\mathcal{N} = 1$ supersymmetry. It is called *scalar* $\mathcal{N} = 1$ *supermultiplet.*

3 Lecture III: Representations of supersymmetry

The fields on Minkowski space are distributed over the irreducible multiplets of the Poincaré group according to the eigenvalues of two Casimirs of this group: the square of P_m (which is m^2) and the square of the Pauli-Lubanski vector (which $\propto s(s+1)$, where s is the spin of the field). For the case of zero mass the diverse Poincaré group multiplets are characterized by the *helicity*, the projection of spin on the direction of motion. What about irreps of supersymmetry? Once again, the contents of the supermultiplets are different for massive and massless cases.

3.1 Massive case

Choose the rest frame

$$P_m = (m, 0, 0, 0)\,.$$

In this frame

$$\text{(a)}\ \{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0\,; \quad \text{(b)}\ \{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2m\delta_{\alpha\dot\beta}\,.$$

i.e. $\mathcal{N} = 1$ superalgebra becomes the Clifford algebra of two mutually conjugated fermionic creation and destruction operators. $\bar{Q}_{\dot\alpha}$ and Q_α. Define the "Clifford vacuum" $|s>$ as the irrep of the Poincaré group with mass m and spin s:

$$Q_\alpha |s>= 0\,.$$

An irrep of the full supersymmetry can be then produced by the successive action of $\bar{Q}_{\dot{\alpha}}$ on the vacuum

State	Spin	# of components
$\|s\rangle$	s	$2s+1$
$\bar{Q}_{\dot{\alpha}}\|s\rangle$	$s \pm 1/2$	$4s+2$
$(\bar{Q})^2\|s\rangle$	s	$2s+1$

Here $(\bar{Q})^2 \equiv \bar{Q}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}$. Further acting by $\bar{Q}_{\dot{\alpha}}$ yields zero. Thus the full number of states is $2^2(2s+1)$, one half being fermions and the second one bosons. The dimensionality of the Clifford vacuum (the number of independent states in it) is just $d_{|s>} = 2s+1$.

Since off shell $P^2 \neq 0$, this spin contents characterizes any off-shell supermultiplet. E.g., the scalar multiplet corresponds to $s = 0$: In this case $s + 1/2 = 1/2$ and we are left just with two complex scalars and one Weyl fermion.

Thus massive $\mathcal{N} = 1$ supermultiplets are entirely specified by the spin s of their Clifford vacua. This spin is called *superspin* Y of the given $\mathcal{N} = 1$ supermultiplet. Each multiplet with $P^2 \neq 0$ and superspin Y involves the following set of spins

$$Y\,,\; Y + \frac{1}{2}\,,\; Y - \frac{1}{2}\,,\; Y\,.$$

The *scalar* supermultiplet ($Y = 0$) contains spins $1/2, (0)^2$ and describes $\mathcal{N} = 1$ matter. The supermultiplet with $Y = 1/2$ involves states with spins $1, (1/2)^2, 0$ and stands for the *gauge* supermultiplet. The supermultiplet with $Y = 3/2$ has the spin content $(3/2)^2, 2, 1$. It is the so-called $\mathcal{N} = 1$ *Weyl* supermultiplet. It corresponds to conformal $\mathcal{N} = 1$ supergravity.

3.2 Massless case

We can choose the frame

$$P_m = (p, 0, 0, p)\,, \quad P^m P_m = 0\,.$$

The only non-zero anticommutator in this frame is

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2p\,(I + \sigma^3)_{\alpha\dot{\alpha}}\,,$$

The full set of the antcommutation relations is

$$\{Q_1, \bar{Q}_{\dot{1}}\} = 4p\,, \quad \{Q_1, \bar{Q}_{\dot{2}}\} = \{Q_2, \bar{Q}_{\dot{2}}\} = \{Q_2, \bar{Q}_{\dot{1}}\} = 0\,,$$
$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0\,.$$

Then one can define the Clifford vacuum $|\lambda>$ with the helicity λ by the conditions

$$Q_1|\lambda> = Q_2|\lambda> = \bar{Q}_{\dot{2}}|\lambda> = 0\,.$$

The only creation operator is $\bar{Q}_{\dot{1}}$. Due to its nilpotency, $(\bar{Q}_{\dot{1}})^2 = 0$, the procedure of constructing the irreducible set of states terminates at the 1st step:

State	Helicity	# of components
$\|\lambda\rangle$	λ	1
$\bar{Q}_{\dot{1}}\|\lambda\rangle$	$\lambda - 1/2$	1

Thus in $\mathcal{N}=1$ supersymmetry the massless supermultiplets are formed by pairs of states with the adjacent helicities, $|\lambda\rangle$, $|\lambda-1/2\rangle$. In particular, massless particle with zero helicity should be accompanied by a particle with the helicity $-1/2$, a particle with $\lambda=1/2$ should be paired with a particle having $\lambda=0$, helicities ±1 can be embedded either into the multiplets $(1,1/2)$, $(-1/2,-1)$, or $(-1,-3/2)$, $(3/2,1)$, the minimal embeddings for the helicities ±2 are into the multiplets $(2,3/2)$ and $(-3/2,-2)$, etc. The multiplets with the opposite helicities are related through CPT conjugation.

3.3 Massless multiplets of $\mathcal{N}$ extended supersymmetry

In this case (without central charges) the only non-vanishing anticommutator is

$$\{Q_1^i, \bar{Q}_{\dot{1}j}\} = 4\delta_j^i p\,, \tag{3.3}$$

The Clifford vacuum $|\lambda>$ is defined by

$$Q_1^i|\lambda\rangle = Q_2^i|\lambda\rangle = \bar{Q}_{\dot{2}i}|\lambda\rangle = 0\,, \tag{3.4}$$

and the irreducible tower of states is constructed by acting on the vacuum by $\mathcal{N}$ independent creation operators $\bar{Q}_{\dot{1}i}$:

State	Helicity	# of components
$\|\lambda\rangle$	λ	1
$\bar{Q}_{\dot{1}i}\|\lambda\rangle$	$\lambda-1/2$	$\mathcal{N}$
$\bar{Q}_{\dot{1}i}\bar{Q}_{\dot{1}j}\|\lambda\rangle$	$\lambda-1$	$\mathcal{N}(\mathcal{N}-1)/2$
$\vdots$	$\vdots$	$\vdots$
$(\bar{Q})^{\mathcal{N}}\|\lambda\rangle$	$\lambda-\mathcal{N}/2$	1

For $\mathcal{N}=2$ supersymmetry, irreps are formed by the states $|\lambda\rangle$, $|\lambda-1/2\rangle^2$, $|\lambda-1\rangle$, etc.

Recall that the multiplets with opposite helicities can be obtained via CPT conjugation. Of special interest are the so-called *"self-conjugated"* multiplets which, from the very beginning, involve the full spectrum of helicities from λ to $-\lambda$. Equating

$$\lambda - \mathcal{N}/2 = -\lambda \;\Rightarrow \lambda = \mathcal{N}/4\,, \tag{3.5}$$

we find that, up to $\mathcal{N}=8$, there exist the following self-conjugated massless supermultiplets

$$\begin{aligned}
&\mathcal{N}=2 \text{ matter multiplet:} && 1/2, (0)^2, -1/2;\\
&\mathcal{N}=4 \text{ gauge multiplet:} && 1, (1/2)^4, (0)^6, (-1/2)^4, -1;\\
&\mathcal{N}=8 \text{ supergravity multiplet:} && 2, (3/2)^8, (1)^{28}, (1/2)^{56}, (0)^{70},\\
& && (-1/2)^{56}, (-1)^{28}, (-3/2)^8, -2.
\end{aligned}$$

Note that for $\mathcal{N}>8$ the massless supermultiplets would include helicities >2. The relevant theories are called "higher-spin theories" and, for self-consistency at the full interaction level, they should include the whole infinite set of such spins (helicities). Such complicated theories are under intensive study at present, but their consideration is beyond the scope of my lectures.

4 Lecture IV: Superspace and superfields

4.1 Superspace

When considering one or another symmetry and constructing physical models invariant with respect to it, it is very important to find out the proper space and/or the fundamental multiplet on which this symmetry is realized in the most natural and simplest way.

The Poincaré group has a natural realization in the Minkowski space $x^m, m = 0, 1, 2, 3$, as the group of linear rotations and shifts of x^m preserving the flat invariant interval $ds^2 = \eta_{mn} dx^m dx^n$. Analogously, *supersymmetry* has a natural realization in the Minkowski *superspace*.

The translation generators P_m can be realized as shifts of $x^m, x^{m\prime} = x^m + c^m$. In the case of $\mathcal{N} = 1$ supersymmetry we have additional spinor generators $Q_\alpha, \bar{Q}_{\dot{\alpha}}$ and anticommuting parameters $\epsilon^\alpha, \bar{\epsilon}^{\dot{\alpha}}$. Then it is natural to introduce new spinor coordinates $\theta^\alpha, \bar{\theta}^{\dot{\alpha}}$ having the same dimension $cm^{1/2}$ as the spinor parameters and to realize the spinorial generators as shifts of these new coordinates

$$\theta^{\alpha\prime} = \theta^\alpha + \epsilon^\alpha\,, \quad \bar{\theta}^{\dot{\alpha}\prime} = \bar{\theta}^{\dot{\alpha}} + \bar{\epsilon}^{\dot{\alpha}}\,.$$

The extended manifold

$$\mathcal{M}^{(4|4)} = \left(x^m\,,\ \theta^\alpha\,,\ \bar{\theta}^{\dot{\alpha}}\right),$$

is called $\mathcal{N} = 1$ **Minkowski superspace.**

Its natural generalization is

$$\mathcal{M}^{(4|4\mathcal{N})} = \left(x^m\,,\ \theta^\alpha_i\,,\ \bar{\theta}^{\dot{\alpha}\,i}\right),$$

and it is called $\mathcal{N}$ **extended Minkowski superspace.**

The spinor coordinates are called odd or Grassmann coordinates and have the Grassmann parity -1, while x^m are even coordinates having the Grassmann parity $+1$

$$[\theta^\alpha_i, x^m] = [\bar{\theta}^{\dot{\alpha}\,i}, x^m] = 0\,, \quad \{\theta^\alpha_i, \theta^\beta_k\} = \{\theta^\alpha_i, \bar{\theta}^{\beta\,k}\} = 0\,.$$

The spinor coordinates also anticommute with the parameters $\epsilon^\alpha, \bar{\epsilon}^{\dot{\alpha}}$.

Since two supertranslations yield a shift of x^m, they should be non-trivially realized on x^m. In the $\mathcal{N} = 1$ case:

$$x^{m\prime} = x^m - i(\epsilon\sigma^m\bar{\theta} - \theta\sigma^m\bar{\epsilon})\,, \quad (\delta_1\delta_2 - \delta_2\delta_1)x^m = 2i(\epsilon_1\sigma^m\bar{\epsilon}_2 - \epsilon_2\sigma^m\bar{\epsilon}_1),$$

(an analogous transformation takes place in the general case of $\mathcal{N}$ extended supersymmetry).

4.2 Superfields

Superfields are functions on superspace, such that they have definite transformation properties under supersymmetry. The general scalar $\mathcal{N}=1$ superfield is $\Phi(x,\theta,\bar{\theta})$ with the following transformation law

$$\Phi'(x',\theta',\bar{\theta}') = \Phi(x,\theta,\bar{\theta})\,.$$

The most important property of superfield is that its series expansion in Grassmann coordinates terminates at the finite step. The reason is that these coordinates are *nilpotent*, because they anticommute. E.g., $\{\theta_\alpha,\theta_\beta\}=0 \Rightarrow \theta_1\theta_1=\theta_2\theta_2=0$. Then

$$\begin{aligned}\Phi(x,\theta,\bar{\theta}) &= \phi(x)+\theta^\alpha\psi_\alpha(x)+\bar{\theta}_{\dot\alpha}\bar{\chi}^{\dot\alpha}(x)+\theta^2 M(x)+\bar{\theta}^2 N(x)\\ &\quad +\theta\sigma^m\bar{\theta}\,A_m(x)+\bar{\theta}^2\theta^\alpha\rho_\alpha(x)+\theta^2\bar{\theta}_{\dot\alpha}\bar{\lambda}^{\dot\alpha}(x)+\theta^2\bar{\theta}^2 D(x)\,,\end{aligned}$$

where $\theta^2 := \theta^\alpha\theta_\alpha = \epsilon_{\alpha\beta}\theta^\alpha\theta^\beta$, $\bar{\theta}^2 = \bar{\theta}_{\dot\alpha}\bar{\theta}^{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar{\theta}^{\dot\beta}\bar{\theta}^{\dot\alpha}$, $\epsilon_{12}=\epsilon_{\dot1\dot2}=1$.

Here one deals with the set of 8 bosonic and 8 fermionic independent complex component fields. The reality condition

$$\overline{(\Phi)} = \Phi$$

implies the following reality conditions for the component fields

$$\begin{aligned}&\phi(x)=\overline{\phi(x)}\,,\ \bar{\chi}_{\dot\alpha}(x)=\overline{\psi_\alpha(x)}\,,\ M(x)=\overline{N(x)}\,,\ A_m(x)=\overline{A_m(x)}\,,\\ &\bar{\lambda}^{\dot\alpha}(x)=\overline{\rho^\alpha(x)}\,,\ D(x)=\overline{D(x)}\,.\end{aligned}$$

They leave in Φ just $(8+8)$ independent real components.

The transformation law $\Phi'(x,\theta,\bar{\theta})=\Phi(x-\delta x,\theta-\epsilon,\bar{\theta}-\bar{\epsilon})$ implies

$$\begin{aligned}&\delta\Phi = -\epsilon^\alpha\frac{\partial\Phi}{\partial\theta^\alpha}-\bar{\epsilon}_{\dot\alpha}\frac{\partial\Phi}{\partial\bar{\theta}_{\dot\alpha}}-\delta x^m\frac{\partial\Phi}{\partial x^m}\equiv i\left(\epsilon^\alpha Q_\alpha+\bar{\epsilon}_{\dot\alpha}\bar{Q}^{\dot\alpha}\right)\Phi\,,\\ &Q_\alpha = i\frac{\partial}{\partial\theta^\alpha}+\bar{\theta}^{\dot\alpha}(\sigma^m)_{\alpha\dot\alpha}\frac{\partial}{\partial x^m}\,,\quad \bar{Q}_{\dot\alpha}=-i\frac{\partial}{\partial\bar{\theta}^{\dot\alpha}}-\theta^\alpha(\sigma^m)_{\alpha\dot\alpha}\frac{\partial}{\partial x^m}\,,\\ &\{Q_\alpha,\bar{Q}_{\dot\alpha}\}=2P_m\,,\ \{Q_\alpha,Q_\beta\}=\{\bar{Q}_{\dot\alpha},\bar{Q}_{\dot\beta}\}=0\,,\quad P_m=\frac{1}{i}\frac{\partial}{\partial x^m}\,.\end{aligned}$$

The relevant component transformations are read off from the formula $\delta\Phi=\delta\phi+\theta^\alpha\delta\psi_\alpha+\ldots+\theta^2\bar{\theta}^2\delta D$. They are

$$\begin{aligned}&\delta\phi=-c\psi-\bar{c}\bar{\chi}\,,\quad \delta\psi_\alpha=-i(\sigma^m\bar{c})_\alpha\partial_m\phi-2c_\alpha M-(\sigma^m\bar{c})_\alpha A_m,\ldots\,,\\ &\delta D=\frac{i}{2}\partial_m\rho\sigma^m\bar{c}-\frac{i}{2}c\sigma^m\partial_m\bar{\lambda}\,.\end{aligned}$$

These transformations uniformly close on x^m translations without use of any dynamical equations. However, the supermultiplet of fields encompassed by $\Phi(x,\theta,\bar{\theta})$ is *reducible*: it contains in fact both the *scalar* and *gauge* $\mathcal{N}=1$ supermultiplets (superspins $Y=0$ and

$Y = 1/2$). How to describe *irreducible* supermultiplets in the superfield language?

An important element of the superspace formalism are spinor covariant derivatives

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\bar\theta^{\dot\alpha}(\sigma^m)_{\alpha\dot\alpha}\frac{\partial}{\partial x^m}, \quad \bar D_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha(\sigma^m)_{\alpha\dot\alpha}\frac{\partial}{\partial x^m},$$
$$\{D_\alpha, \bar D_{\dot\alpha}\} = -2i(\sigma^m)_{\alpha\dot\alpha}\partial_m, \quad \{D_\alpha, D_\beta\} = \{\bar D_{\dot\alpha}, \bar D_{\dot\beta}\} = 0.$$

The covariant spinor derivatives *anticommute* with supercharges, $\{D, Q\} = \{D, \bar Q\} = 0$, so $D_\alpha\Phi$ and $\bar D_{\dot\alpha}\Phi$ are again superfields, e.g.,

$$\delta D_\alpha\Phi = D_\alpha\delta\Phi = D_\alpha i\left(\epsilon^\alpha Q_\alpha + \bar\epsilon_{\dot\alpha}\bar Q^{\dot\alpha}\right)\Phi = i\left(\epsilon^\alpha Q_\alpha + \bar\epsilon_{\dot\alpha}\bar Q^{\dot\alpha}\right)D_\alpha\Phi.$$

Now, it becomes possible to define the "irreducible" superfields. (**Analogy**: In Minkowski space the vector field A_m is known to carry two Poincaré spins 1 and 0. The irreducible components are distinguished by imposing on A_m the supplementary differential conditions

$$\partial^m A_m = 0 \leftrightarrow \text{spin } 1, \quad \partial_m A_n - \partial_n A_m = 0 \leftrightarrow \text{spin } 0.)$$

Analogous conditions can be imposed on the superfield Φ in order to single out the irreducible multiplets with the superspins 0 and 1/2. These conditions are defined with the help of the covariant spinor derivatives.

The simplest condition of this type is the *chirality* or *anti-chirality* conditions

$$(a)\ \bar D_{\dot\alpha}\Phi_L(x,\theta,\bar\theta) = 0, \quad \text{or} \quad (b)\ D_\alpha\Phi_R(x,\theta,\bar\theta) = 0.$$

Eq. (a), e.g., implies

$$\Phi_L(x,\theta,\bar\theta) = \varphi_L(x_L,\theta) = \phi(x_L) + \theta^\alpha\psi_\alpha(x_L) + \theta\theta F(x_L),$$
$$x_L^m = x^m + i\theta\sigma^m\bar\theta,$$

i.e. we are left with the independent fields ϕ, ψ_α, F.

From the general transformation laws of the component fields it follows that this set is closed under $\mathcal{N} = 1$ supersymmetry:

$$\delta\phi = -\epsilon\psi, \quad \delta\psi_\alpha = -2i(\sigma^m\bar\epsilon)_\alpha\partial_m\phi - 2\epsilon_\alpha F, \quad \delta F = -i\bar\epsilon\bar\sigma^m\partial_m\psi.$$

These are just the transformation laws of the scalar $\mathcal{N} = 1$ supermultiplet.

The geometric interpretation: the coordinate set (x_L^m, θ^α) is closed under $\mathcal{N} = 1$ supersymmetry:

$$\delta x_L^m = 2i\theta\sigma^m\bar\epsilon, \quad \delta\theta^\alpha = \epsilon^\alpha. \tag{4.6}$$

It is called *left-chiral* $\mathcal{N} = 1$ *superspace*.

In the basis $(x_L^m, \theta^\alpha, \bar\theta^{\dot\alpha})$ the chirality condition (a) is reduced to the *Grassmann Cauchy-Riemann* conditions:

$$\bar D_{\dot\alpha}\Phi_L(x_L,\theta,\bar\theta) = 0 \ \Rightarrow\ \frac{\partial}{\partial\bar\theta^{\dot\alpha}}\Phi_L = 0 \ \Rightarrow\ \Phi_L = \varphi_L(x_L,\theta). \tag{4.7}$$

4.3 Superfield actions

Having superfields, one can construct out of them, as well as their vector and covariant spinor derivatives, scalar superfield Lagrangians. Any local product of superfields is again a superfield:

$$\mathcal{L} = \mathcal{L}(\Phi, D_\alpha\Phi, \bar{D}_{\dot{\alpha}}\Phi, \partial_m\Phi, \ldots)\,, \quad \delta\mathcal{L} = i\left(\epsilon^\alpha Q_\alpha + \bar{\epsilon}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}\right)\mathcal{L}\,.$$

It is easy to see that the variation of the highest component in the θ expansion of any superfield is a total derivative. Then one takes the highest component field in the θ expansion of the superfield Lagrangian and integrates it over Minkowski space. It will be just an action invariant under $\mathcal{N} = 1$ supersymmetry!

A manifestly covariant way to write supersymmmetric actions is to use the Berezin integral. It is equivalent to differentiation in Grassmann coordinates. In the considered case of $\mathcal{N} = 1$ superspace it is defined by the rules

$$\int d^2\theta\,(\theta)^2 = 1\,, \quad \int d^2\bar{\theta}\,(\bar{\theta})^2 = 1\,, \quad \int d^2\theta d^2\bar{\theta}\,(\theta)^4 = 1\,, \quad (\theta)^4 \equiv (\theta)^2(\bar{\theta})^2\,.$$

Hence the Berezin integral yields an efficient and manifestly supersymmetric way of singling out the coefficients of the highest-order θ monomials in the superfield Lagrangians.

The simplest invariant action of chiral superfields producing the kinetic terms of the scalar multiplet is as follows

$$S_{kin} = \int d^4x d^4\theta\,\varphi(x_L, \theta)\bar{\varphi}(x_R, \bar{\theta})\,, \quad x_R^m = \overline{(x_L^m)} = x^m - i\theta\sigma^m\bar{\theta}\,.$$

After performing integration over Grassmann coordinates, one obtains

$$S \sim \int d^4x \left(\partial^m\bar{\phi}\partial_m\phi - \frac{i}{2}\psi\sigma^m\partial_m\bar{\psi} + F\bar{F}\right).$$

The total Wess-Zumino model action is reproduced by adding, to this kinetic term, also potential superfield term

$$S_{pot} = \int d^4x_L d^2\theta\left(\frac{g}{3}\varphi^3 + \frac{m}{2}\varphi^2\right) + \text{c.c.}\,.$$

This action is the only renormalizable action of the scalar $\mathcal{N} = 1$ multiplet. In principle, one can construct more general actions, e.g., the action of Kähler sigma model and the generalized potential terms,

$$\tilde{S}_{kin} = \int d^4x d^4\theta\,K\left[\varphi(x_L, \theta), \bar{\varphi}(x_R, \bar{\theta})\right]\,, \quad \tilde{S}_{pot} = \int d^4x_L d^2\theta\,P(\varphi) + \text{c.c.}\,.$$

The multiplet with the superspin $Y = 1/2$ is described by the gauge superfield $V(x, \theta, \bar{\theta})$ possessing the gauge freedom

$$\delta V(x, \theta, \bar{\theta}) = i[\bar{\lambda}(x^m - i\theta\sigma^m\bar{\theta}, \bar{\theta}) - \lambda(x^m + i\theta\sigma^m\bar{\theta}, \theta)],$$

where $\lambda(x_L, \theta)$ is an arbitrary chiral superfield parameter.

Using this freedom, one can fix the so called Wess-Zumino gauge

$$V_{WZ}(x, \theta, \bar{\theta}) = 2\,\theta\sigma^m\bar{\theta}\,A_m(x) + 2i\bar{\theta}^2\theta^\alpha\,\psi_\alpha(x) - 2i\theta^2\bar{\theta}_{\dot{\alpha}}\,\bar{\psi}^{\dot{\alpha}}(x) + \theta^2\bar{\theta}^2\,D(x)\,.$$

Thus in the WZ gauge we are left with the irreducible set of fields forming the gauge (or vector) off-shell supermultiplet: The gauge field $A_m(x)$, $A'_m(x) = A_m + \partial_m\lambda(x)$, the fermionic field of gaugino $\psi_\alpha(x)$, $\bar{\psi}_{\dot{\alpha}}(x)$ and the auxiliary field $D(x)$.

The invariant action is written as an integral over the chiral superspace

$$S^{N=1}_{gauge} = \frac{1}{16}\int d\zeta_L\,(W^\alpha W_\alpha) + \text{c.c.}\,, \quad W_\alpha = -\frac{1}{2}\bar{D}^2 D_\alpha V\,,\ \bar{D}_{\dot{\alpha}} W_\alpha = 0\,.$$

Everything is easily generalized to the non-abelian case. The corresponding component off-shell action reads

$$S = \int d^4x\,\mathrm{Tr}\left[-\frac{1}{4}F^{mn}F_{mn} - i\bar{\psi}\sigma^m\mathcal{D}_m\bar{\psi} + \frac{1}{2}D^2\right].$$

What about superfield approach to higher $\mathcal{N}$ supersymmetries? The difficulties arise because the relevant superspaces contain too many θ coordinates and it is a very complicated problem to define the superfields which would correctly describe the relevant irreps.

For $\mathcal{N} = 2$, the off-shell gauge multiplet contains the vector gauge field $A_m(x)$, the complex scalar physical field $\varphi(x)$, the $SU(2)$ doublet of Weyl fermions $\psi^i_\alpha(x), \bar{\psi}_{\dot{\alpha}\,i}(x)$ and the auxiliary real $SU(2)$ triplet $D^{(ik)}(x)$.

There is no simple way to define $\mathcal{N} = 2$ analog of the $\mathcal{N} = 1$ gauge prepotential V (unless we apply to $\mathcal{N} = 2$ *harmonic superspace*). However, one can define the appropriate covariant superfield strength W. In the abelian case, it is defined by the off-shell constraints

$$(a)\,\bar{D}^i_{\dot{\alpha}} W = 0\,, \quad (b)\,D^{\alpha i}D^k_\alpha W = \bar{D}^i_{\dot{\alpha}}\bar{D}^{\dot{\alpha}k}\bar{W}\,,$$

which, in particular, imply the Bianchi identity for the gauge field strength. The invariant action is an integral over chiral $\mathcal{N} = 2$ superspace

$$S \sim \int d^4x_L d^4\theta W^2 + \text{c.c.}\,.$$

What about maximally extended $\mathcal{N} = 4$ super Yang-Mills? It has no superfield formulation with all $\mathcal{N} = 4$ supersymmetries being manifest and off-shell. There is $\mathcal{N} = 1$ superfield formulation with one gauge superfield and three chiral superfields; $\mathcal{N} = 2$ formulation in terms of $\mathcal{N} = 2$ gauge superfield and one massless matter *hypermultiplet*. The latter possesses an off-shell formulation only in the $\mathcal{N} = 2$ *harmonic superspace*. At last, exists a formulation with three manifest off-shell supersymmetries - in $\mathcal{N} = 3$ *harmonic superspace*. It involves gauge superfields only.

5 Lecture V: Supersymmetric quantum mechanics

5.1 Supersymmetry in one dimension

Quantum mechanics can be treated as one-dimensional field theory. Correspondingly, the relevant supersymmetry can be understood as the $d = 1$ reduction of higher-dimensional Poincaré supersymmetry. More generally, the $\mathcal{N}$-extended $d = 1$ "Poincaré" supersymmetry can be defined by the (anti)commutation relations

$$\{Q^m, Q^n\} = 2\delta^{mn} H\,, \quad [H, Q^m] = 0\,, \overline{Q^m} = Q^m\,,\ m = 1, \ldots \mathcal{N}\,.$$

The associated systems are models of supersymmetric quantum mechanics (SQM) with H as the relevant Hamiltonian. The SQM models have a lot of applications in various physical and mathematical domains.

We will deal with the simplest non-trivial $\mathcal{N} = 2, d = 1$ supersymmetry

$$Q = \frac{1}{\sqrt{2}}(Q^1 + iQ^2)\,, \quad \bar{Q} = \frac{1}{\sqrt{2}}(Q^1 - iQ^2)\,,$$

$$\{Q, \bar{Q}\} = 2H\,, \quad Q^2 = \bar{Q}^2 = 0, \quad [H, Q] = [H, \bar{Q}] = 0\,.$$

It is also instructive to add the commutators with the generator J of the group $O(2) \sim U(1)$ which is the automorphism group of the $\mathcal{N} = 2$ superalgebra:

$$[J, Q] = Q\,, \quad [J, \bar{Q}] = -\bar{Q}\,, \quad [H. J] = 0\,.$$

$\mathcal{N} = 2, d = 1$ superspace is defined as:

$$\mathcal{M}^{(1|2)} = (t, \theta, \bar{\theta})\,, \quad \delta\theta = \epsilon\,, \quad \delta\bar{\theta} = \bar{\epsilon}\,, \quad \delta t = i(\epsilon\bar{\theta} + \bar{\epsilon}\theta)\,.$$

One can also define the $\mathcal{N}$=2 covariant spinor derivatives:

$$D = \partial_\theta - i\bar{\theta}\partial_t\,, \qquad \bar{D} = -\partial_{\bar{\theta}} + i\theta\partial_t\,, \qquad \{D, \bar{D}\} = 2i\partial_t\,.$$

The simplest superfield is the real one, $\Phi(t, \theta, \bar{\theta})$,

$$\Phi'(t', \theta', \bar{\theta}') = \Phi(t, \theta, \bar{\theta}) \;\Rightarrow\; \delta\Phi = -\delta t\partial_t\Phi - \epsilon\partial_\theta\Phi - \bar{\epsilon}\partial_{\bar{\theta}}\Phi\,.$$

On the component fields appearing in the θ expansion of Φ,

$$\Phi(t, \theta, \bar{\theta}) = x(t) + \theta\psi(t) - \bar{\theta}\bar{\psi}(t) + \theta\bar{\theta}y(t)\,,$$

$\mathcal{N}$= 2 supersymmetry is realized as

$$\delta x = \bar{\epsilon}\bar{\psi} - \epsilon\psi\,,\ \delta\psi = \bar{\epsilon}(i\dot{x} - y)\,,\ \delta\bar{\psi} = -\epsilon(i\dot{x} + y)\,,\ \delta y = i(\epsilon\dot{\psi} + \bar{\epsilon}\dot{\bar{\psi}})\,.$$

The superfield $\Phi(t, \theta, \bar{\theta})$ comprises the irreducible $\mathcal{N} = 2, d = 1$ multiplet $(\mathbf{1}, \mathbf{2}, \mathbf{1})$. Other $\mathcal{N} = 2, d = 1$ multiplets exist as well, e.g., $(\mathbf{2}, \mathbf{2}, \mathbf{0})$ which is described by a chiral

$\mathcal{N}=2, d=1$ superfield.

The simplest invariant superfield action containing interaction reads

$$S^{(\mathcal{N}=2)} = \int dt d^2\theta \left[\bar{D}\Phi\, D\Phi + W(\Phi) \right].$$

Here $W(\Phi)$ is the superpotential. After integrating over Grassmann coordinates, we obtain

$$S^{(\mathcal{N}=2)} = \int dt \left[\dot{x}^2 - i\left(\dot{\bar{\psi}}\psi - \bar{\psi}\dot{\psi}\right) + y^2 + y\partial_x W(x) + (\psi\bar{\psi})\partial_x^2 W(x) \right].$$

The next step is to eliminate the auxiliary field y by its algebraic equation of motion

$$y = -\frac{1}{2}\partial_x W\,.$$

The on-shell action is then

$$S^{(\mathcal{N}=2)} = \int dt \left[\dot{x}^2 - i\left(\dot{\bar{\psi}}\psi - \bar{\psi}\dot{\psi}\right) - \frac{1}{4}(\partial_x W)^2 + (\psi\bar{\psi})\partial_x^2 W(x) \right].$$

The action is invariant under the transformations

$$\delta x = \bar{\epsilon}\bar{\psi} - \epsilon\psi\,,\ \delta\psi = \bar{\epsilon}(i\dot{x} + \frac{1}{2}\partial_x W)\,,\ \delta\bar{\psi} = -\epsilon(i\dot{x} - \frac{1}{2}\partial_x W)\,.$$

5.2 Hamiltonian formalism and quantization

The quantum Hamiltonian obtained in a standard way from the canonical one reads

$$H = \frac{1}{4}\left[\hat{p}^2 + \left(\frac{dW}{d\hat{x}}\right)^2\right] - \frac{1}{2}\frac{d^2W}{d\hat{x}^2}\left(\hat{\psi}\hat{\bar{\psi}} - \hat{\bar{\psi}}\hat{\psi}\right),$$

where we have Weyl-ordered the fermionic term. The supercharges calculated by the Noether procedure and then brought into the quantum form through passing to the operators are

$$Q = \hat{\psi}\left(\hat{p} + i\frac{dW}{d\hat{x}}\right), \qquad \bar{Q} = \hat{\bar{\psi}}\left(\hat{p} - i\frac{dW}{d\hat{x}}\right).$$

The algebra of the basic quantum operators is

$$[\hat{x}, \hat{p}] = i\,, \qquad \{\hat{\psi}, \hat{\bar{\psi}}\} = \tfrac{1}{2}\,.$$

Using it, we can calculate the anticommutators of the quantum supercharges and check that they form $\mathcal{N}=2, d=1$ superalgebra

$$\{Q, \bar{Q}\} = 2H\,, \qquad \{Q, Q\} = \{\bar{Q}, \bar{Q}\} = 0\,. \tag{5.8}$$

By the graded Jacobi identities, one also derives

$$[Q, H] = [\bar{Q}, H] = 0\,.$$

We use the standard realization for $\hat{p}$, $\hat{p} = \frac{1}{i}\frac{\partial}{\partial x}$, and the Pauli-matrix realization for the fermionic operators

$$\hat{\psi} = \frac{1}{2\sqrt{2}}\,(\sigma_1 + i\sigma_2)\,, \quad \hat{\bar{\psi}} = \frac{1}{2\sqrt{2}}\,(\sigma_1 - i\sigma_2)\,, \quad \hat{\psi}\hat{\bar{\psi}} - \hat{\bar{\psi}}\hat{\psi} = \tfrac{1}{2}\,\sigma_3\,.$$

Then the Hamiltonian and supercharges are represented by 2×2 matrices

$$H = \frac{1}{4}\,[-\partial_x{}^2 + (W_x)^2] \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{4} W_{xx} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Q = -\frac{i}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} (\partial_x - W_x)\,, \quad \bar{Q} = -\frac{i}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} (\partial_x + W_x)\,.$$

Thus the wave functions form a doublet and, taking into account the conditions $[Q, H] = [\bar{Q}, H] = 0$, the relevant matrix spectral problem is

$$H \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \lambda \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}.$$

It is equivalent to the two ordinary problems

$$H_\pm \psi_\pm = \lambda\,\psi_\pm\,, \qquad H_\pm = -\frac{1}{4}(\partial_x \mp W_x)(\partial_x \pm W_x).$$

Using the intertwining property

$$H_-(\partial_x + W_x) = (\partial_x + W_x)H_+\,, \quad H_+(\partial_x - W_x) = (\partial_x - W_x)H_-\,,$$

now it easy to show that the states

$$Q \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} -i(\partial_x - W_x)\psi_- \\ 0 \end{pmatrix}, \quad \bar{Q} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} 0 \\ -i(\partial_x + W_x)\psi_+ \end{pmatrix}$$

are the eigenfunctions of H_+ and H_- with the same eigenvalue λ as ψ_+ and ψ_-. Thus we observe the double degeneracy of the spectrum. This double degeneracy is the most characteristic feature of the $\mathcal{N} = 2$ supersymmetry in $d = 1$ (and of any higher $\mathcal{N}$ supersymmetry in $d = 1$).

In general, the Hilbert space of quantum states of $\mathcal{N} = 2$ SQM is divided into the following three sectors

$$\begin{aligned} &(a)\ \text{Ground state}: \ Q\Psi_0 = \bar{Q}\Psi_0 = H\Psi_0 = 0\,, \\ &(b)\ H\Psi_1 = E\Psi_1\,,\ Q\Psi_1 \neq 0\,,\ \bar{Q}\Psi_1 = 0\,, \\ &(c)\ H\Psi_2 = E\Psi_2\,,\ \bar{Q}\Psi_2 \neq 0\,,\ Q\Psi_2 = 0\,. \end{aligned}$$

Based on this consideration, one can conclude that many QM models with the double degeneracy of the energy spectrum can be identified with some $\mathcal{N} = 2$ SQM model.

6 Summary

- Supersymmetry between fermions and bosons is a new unusual concept in the mathematical physics. It allowed to construct a lot of new theories with remarkable and surprising features: supergravities, superstrings, superbranes, $\mathcal{N} = 4$ super Yang-Mills theory (the first example of the ultraviolet-finite quantum field theory). etc. It also allowed to establish unexpected relations between these theories, e.g., the AdS/CFT (or "gravity/gauge") correspondence, AGT correspondence, etc.

- It predicts new particles (superpartners) which still await their experimental discovery.

- The natural approach to supersymmetric theories is the superfield methods.

For those who wish to get deeper insights into the subjects sketched in these lectures, I may recommend the text-books and the review papers in the list of references below.

Acknowledgements

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The 1/N Expansion Method in Quantum Field Theory

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Abstract

The motivations of the $1/N$ expansion method in quantum field theory are explained in an introductory part. The method is first illustrated with the $O(N)$ model of scalar fields. A second example is considered with the two-dimensional Gross-Neveu model of fermion fields with global $U(N)$ and discrete chiral symmetries. The case of QCD is briefly sketched.

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Keywords: Large number of components, Global symmetries, Spontaneous symmetry breaking, Asymptotic freedom, Dynamical mass generation, Mass transmutation.

1 Introduction and motivations

Methods of resolution of quantum field theory equations are very rare. Usually one uses perturbation theory with respect to the coupling constant g, starting from free field theory. The size of (the dimensionless) g gives an estimate of the strength of the interaction. The presence in the theory of other parameters than the coupling constant may allow the use of perturbation theory with respect to those parameters. This may enlarge the possibilities of approximate resolutions of the theory.

A new parameter may emerge if the system under consideration satisfies symmetry properties with respect to a group of internal transformations. For example, constituents of the system (particles, nucleons, nuclei, energy excitations, etc.) belonging to different species, may have the same masses and possess the same dynamical properties with respect to the interaction. In such a case interchanges between these constituents would not modify the physical properties of the system. The latter interchanges might also be considered in infinitesimal or continuous forms as in the case of the rotations in ordinary space. The system then satisfies an invariance property under continuous symmetry transformations. In many cases, the invariance might also be only approximate.

Among the continuous symmetry groups, two play an important role in physical problems. The first is $O(N)$, the orthogonal group, generated by the $N \times N$ orthogonal matrices. It has $N(N-1)/2$ parameters and generators. The second is $SU(N)$, the unitary group, generated by the $N \times N$ unitary complex matrices, with determinant equal to 1. It has (N^2-1) parameters and generators.

In particle and nuclear physics, one has the approximate isospin symmetry group $SU(2)$ and the approximate flavor symmetry group $SU(3)$. Using these approximations, one can establish relations between masses and physical parameters of various particles, prior to solving the dynamics of the system under consideration.

In the cases of the groups $O(N)$ and $SU(N)$ mentioned above, N has a well defined fixed value in each physical problem, $N = 2, 3, \ldots$, etc. It is however tempting to consider the case where N is a free parameter which can be varied at will. In particular, large values of N, with the limit $N \to \infty$, seem to be of interest. At first sight, it might seem that taking large values of N would lead to more complicated situations, since the number of parameters increases and the group representations become intricate. However, it has been noticed that when the limit is taken in an appropriate way, in conjunction with the coupling constant of the theory, it may lead to simpler results than the cases of finite N. If this happens, then an interesting perspective of resolution is opened.

One may solve the problem in the simplified situation of the limit $N \to \infty$ and then, to improve the predictions, consider the contributions of the terms of order $1/N$ as a perturbation. If the true N of the physical problem is sufficiently large, then the zeroth-order calculation done in the limit $N \to \infty$ would already provide the main dominant aspects of the solution of the physical problem under consideration. This is the spirit of the $1/N$ expansion method in quantum field theory. We emphasize that generally the solution thus obtained contains nonperturbative effects when expressed in terms of the coupling constant g of the theory and therefore it provides nontrivial insight into the dynamics of the theory, which otherwise would be unattainable with the use of ordinary perturbation theory with respect to g.

The interest of the large-N limit was first noticed by Stanley [1] in statistical physics, who used it in the framework of the Heisenberg model (spin-spin interactions). He showed that in that limit the model reduces to the spherical approximation of the Ising model, which was soluble.

The method was introduced in quantum field theory by Wilson [2], who applied it to the $O(N)$ model of scalar and fermion fields.

In 1974, 't Hooft [3, 4] applied it to Quantum Chromodynamics (QCD), the newly born theory of the strong interaction, which is a gauge theory with the non-Abelian gauge group $SU(3)_c$ in the internal space of color quantum numbers. (Not to be mixed up with the global flavor symmetry group $SU(3)$ met previously.) This theory cannot be solved with the only large-N limit, but many simplifications occur. In two space-time dimensions, it is almost soluble.

We shall illustrate the method by two explicit examples: 1) The $O(N)$ model of scalar fields; 2) The two-dimensional model of Gross-Neveu with global $U(N)$ symmetry of fermion fields and discrete chiral symmetry.

Reviews and lectures on the $1/N$ expansion method can be found in Refs. [5, 6, 7, 8] (the list is not exhaustive).

2 The scalar O(N) model

The $O(N)$ model is a theory of N real scalar fields ϕ^a $(a = 1, 2, \ldots, N)$, with a quartic interaction, invariant under the $O(N)$ group of transformations. This group is similar in structure to the rotation group, but acts in the internal space of N species of the fields.

The Lagrangian density is

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi^a\partial^\mu\phi^a - \frac{1}{2}\,\mu_0^2\phi^a\phi^a - \frac{\lambda_0}{8N}(\phi^a\phi^a)^2. \tag{1}$$

(Summation on repeated indices is understood. $\partial_\mu = \frac{\partial}{\partial x^\mu}$, etc.) μ_0^2 and λ_0 are real parameters, representing the bare mass squared and the bare coupling constant; the physical mass and coupling constant could be defined only after quantum radiative corrections are taken into account. In four space-time dimensions the coupling constant is dimensionless. The factor $1/N$ has been explicitly introduced in the interaction term for future convenience. When the limit $N \to \infty$ is taken, the coupling constant λ_0 will be assumed to be independent of N. Had we defined a coupling constant $\tilde{\lambda}_0$ as being equal to λ_0/N, we would be obliged later, to maintain a physical content for the theory, to assume that the product $N\tilde{\lambda}_0$ remains finite in the above limit, which leads back to our initial choice.

A detailed study of this model can be found in Ref. [9].

2.1 Classical approximation

We search for the ground state of the theory at the classical level. The lagrangian density is composed of the kinetic energy density minus the potential energy density. The kinetic energy term gives generally a positive contribution to the total energy of the system. Its minimum value is zero, corresponding to constant fields. We stick to that situation. The potential energy density is then

$$U = \frac{1}{2}\,\mu_0^2\phi^a\phi^a + \frac{\lambda_0}{8N}(\phi^a\phi^a)^2. \tag{2}$$

This is a quartic function of the ϕs. It is evident that the energy is not bounded from below if $\lambda_0 < 0$. In this case there is no a stable ground state. We assume henceforth that $\lambda_0 > 0$.

Two cases have to be distinguished: 1) $\mu_0^2 > 0$; 2) $\mu_0^2 < 0$.

1) $\mu_0^2 > 0$.

The shape of the function U is presented in Fig. 1.

The minimum of the energy corresponds to the values $\phi_0^a = 0$, $(a = 1, \ldots, N)$. The ϕs in this case can be considered as excitations from the ground state.

Considering the lagrangian density (1), we can interpret it as describing the dynamics of N particles with degenerate masses, equal to μ_0, interacting by means of the quartic interaction. We have complete $O(N)$ symmetry between the particles.

This mode of symmetry realization is called the Wigner mode or the normal mode.

2) $\mu_0^2 < 0$.

The shape of the function U is presented in Fig. 2.

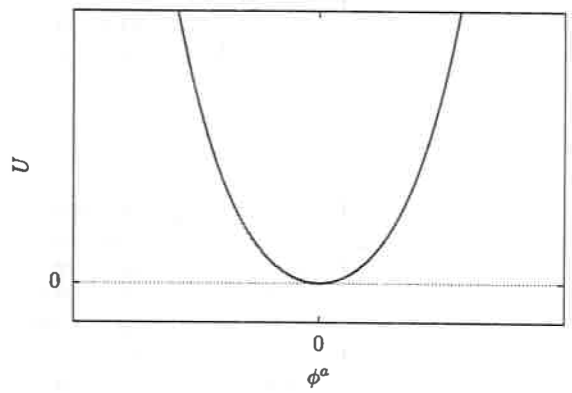

Figure 1: The potential energy density for $\mu_0^2 > 0$.

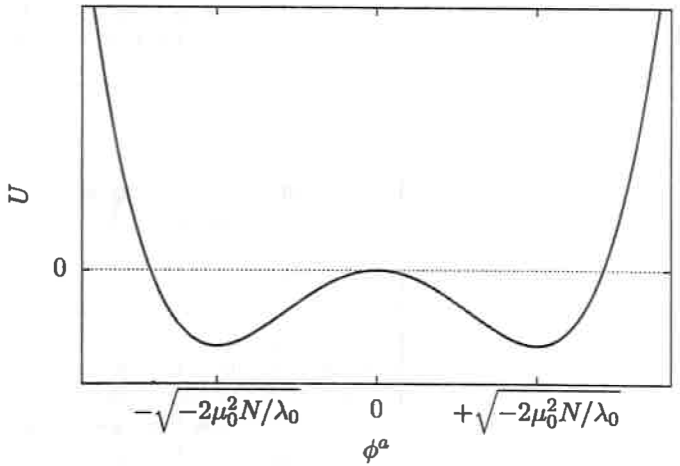

Figure 2: The potential energy density for $\mu_0^2 < 0$.

The minimum of the energy is no longer $\phi_0^a = 0$, but a shifted value at $\phi_0^a = \pm\sqrt{-\frac{2\mu_0^2 N}{\lambda_0}} \equiv < \phi >$ for one of the as. There is a degeneracy between the different ϕ^as. For definiteness, we choose the ground state at the minimum obtained with ϕ^N.

We redefine the fields around the new ground state:

$$\chi = \phi^N - < \phi >, \qquad \pi^a = \phi^a, \quad a = 1, \ldots, N-1. \tag{3}$$

The potential energy density becomes

$$U = -\mu_0^2 \chi^2 + \frac{\lambda_0}{2N} < \phi > \chi^3 + \frac{\lambda_0}{8N}(\pi^a \pi^a + \chi^2)^2. \tag{4}$$

The fields π^a have no longer mass terms, while the field χ has a mass term. We have now the masses:

$$m_{\pi^a}^2 = 0, \quad a = 1, \ldots, N-1, \qquad m_\chi^2 = -2\mu_0^2. \tag{5}$$

The $O(N)$ symmetry that we had initially has partially disappeared. There is now $O(N-1)$ symmetry in the space of the fields π^a. It is said that the $O(N)$ symmetry has been spontaneously broken. This happened because the ground state of the theory is not symmetric, while the lagrangian density is.

This phenomenon is accompanied with the appearance of $(N-1)$ massless fields. These are called Goldstone bosons. This way of realization of the symmetry is called the Goldstone mode.

In particle and nuclear physics, the isospin $SU(2)$ symmetry and the quark flavor $SU(3)$ symmetry are realized with the Wigner mode. The chiral $SU(3)_R \times SU(3)_L$ symmetry is realized with the Goldstone mode.

2.2 Quantum effects

We want now to take into account the quantum effects of the model that we are considering. For this, it is necessary to compute the quantum corrections that contribute to the potential energy. In quantum field theory, these are represented by the radiative corrections.

The definition of the potential energy density U is enlarged. The new potential energy is called the effective potential, which is composed of the classical part, U_{class}, that we met before [Eq. (2)], and of a part, U_{rad}, coming from the radiative corrections:

$$U_{\text{eff}} = U_{\text{class}} + U_{\text{rad}} \ . \tag{6}$$

U_{rad} is best defined in the path integral formalism. We refer the reader to Ref. [10] and to the many textbooks that exist on the subject. The key object is the generating functional of one-particle irreducible diagrams or proper vertices. The effective potential is obtained from the latter by considering only external constant fields in x-space or external lines with zero momenta in momentum space. Diagramatically, U_{rad} is given by the sum of all loop diagrams with such external lines. These are also accompanied with appropriate combinatorial factors due to the existing symmetry properties under exchanges among the external lines [11]. Another method of calculation hinges on a direct evaluation of the path

integral contribution, avoiding explicit summation of diagrams, [12]. However, whatever the method of evaluation is, the exact calculation of the effective potential is almost impossible, since it involves an infinity of many complicated contributions. Nevertheless, the use of the $1/N$ expansion method considerably simplifies the situation. The leading terms of this expansion are calculable.

Prior to the evaluation of the effective potential, we shall introduce the propagators, vertices and loops, that are needed for our calculations.

2.3 Propagators, vertices and loops

The radiative corrections can be calculated starting from the situation where $\mu_0^2 > 0$. Radiative corrections will modify the value of μ_0^2, bringing it into a new value μ^2. It is then sufficient to consider at the end the analytic continuation of μ^2 into negative values to complete the study. We therefore consider the initial Lagrangian density (1):

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi^a\partial^\mu\phi^a - \frac{1}{2}\,\mu_0^2\phi^a\phi^a - \frac{\lambda_0}{8N}(\phi^a\phi^a)^2. \tag{7}$$

The inverse of the free propagator of the field ϕ^a is essentially represented by the coefficients of the quadratic parts of $\mathcal{L}$. In momentum space, the free propagator is

$$D_0^{ab}(p) = \delta_{ab}D_0(p) = \int d^4x e^{ip.x} < 0|T(\phi^a(x)\phi^b(0))|0 >, \tag{8}$$

where the last term represents the vacuum expectation value of the chronological product of the field operators, and D_0 has the expression

$$D_0(p) = \frac{i}{p^2 - \mu_0^2 + i\varepsilon} \qquad \Longleftrightarrow \qquad \text{———————} \tag{9}$$

We have associated with the propagator a graphical representation in the form of a full straight line.

The bare vertex is equal to the coefficient of the four-field interaction term (contact interaction), with a multiplicative i factor (Fig. 3). Its order in N, for large values of N, is N^{-1}.

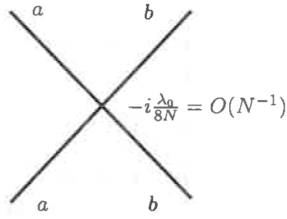

Figure 3: The bare vertex. We have explicitly indicated its order in N.

Radiative corrections are represented by loops, made of closed lines (propagators). Figs. 4 and 5 represent examples of one-loop and two-loop diagrams, respectively. The

order in N of a loop diagram is calculated by taking into account that of the vertex at the contact point of the loop with the external lines and that of the possibly existing summation of indices of the loop. Thus, if the index b of the propagators of a loop is independent of the index a of external lines, then it is summed over the N values the index b can take; b in this case is a dummy index; therefore it produces a multiplicative factor of N. External lines are not counted.

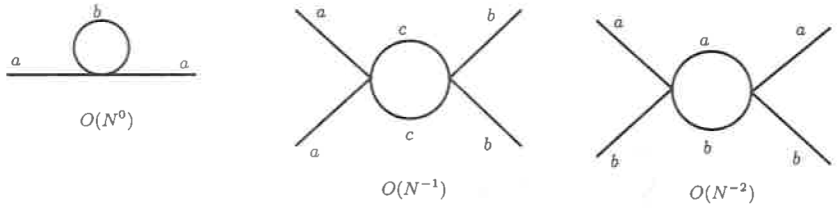

Figure 4: Examples of one-loop diagrams with their order in N.

2.4 The auxiliary field method

We observe on Fig. 4 that the last two diagrams, which have the same topological structure, have different behaviors for large N, depending on the values of indices the loop propagators have. This is an annoying situation, because at higher orders (many loops) it becomes more and more difficult to continue the analysis, since the number of possibilities the indices can have rapidly increases with a corresponding increase of different categories of behavior in N. The ideal situation is the one in which all diagrams with the same topological structure have the same behavior in N; in such a case, we do not need to consider the detailed values of indices of each line.

To remedy the present difficulty, we shall resort to a method, called the auxiliary field method, often used in quantum field theory, which consists of replacing composite fields by a new, nonpropagating field, without changing the physical content of the theory. The validity of the latter property is rather easily shown in the path integral formalism; when an operator formalism is used, a simple hint is provided with the use of the equations of motion.

We wish to replace the composite field $\phi^a\phi^a$ appearing in the interaction term by a single field, which we shall designate by σ. (Since in $\phi^a\phi^a$, a is summed from one to N, this field does not have any index; it is a singlet under the $O(N)$ group of transformations.) To this end, we add to the Lagrangian density (7) a new term, thus defining a new Lagrangian density:

$$\mathcal{L}' = \mathcal{L} + \frac{N}{2\lambda_0}\left(\sigma - \frac{\lambda_0}{2N}\,\phi^a\phi^a - \mu_0^2\right)^2. \tag{10}$$

The σ field does not have kinetic energy. The terms in the added expression are chosen such that the quartic term in ϕ as well as the mass term of ϕ disappear from the new Lagrangian density. The equation of motion (Euler-Lagrange equation) of σ yields its

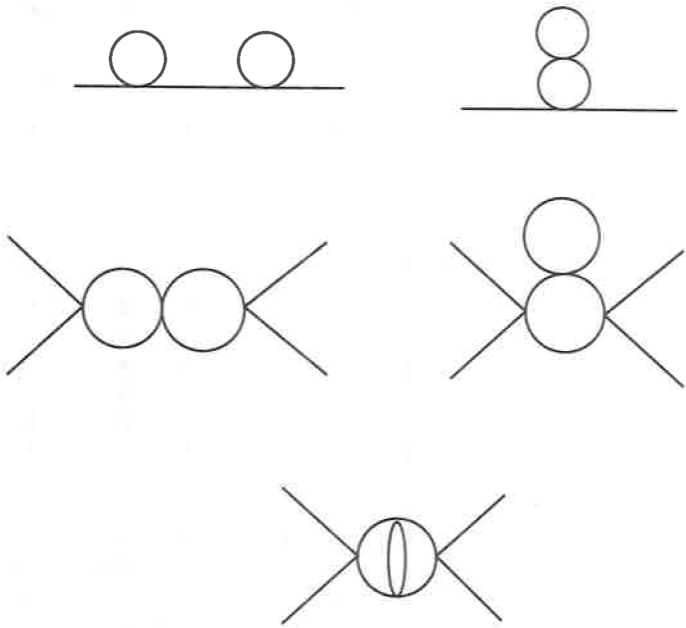

Figure 5: Examples of two-loop diagrams.

definition:

$$\sigma = \frac{\lambda_0}{2N}\,\phi^a\phi^a + \mu_0^2. \tag{11}$$

The latter also shows that the added term in the lagrangian is zero if the equation of motion of σ is used. One may conclude that the new lagrangian is equivalent to the former one:

$$\mathcal{L}' \approx \mathcal{L}. \tag{12}$$

$\mathcal{L}'$ takes the form:

$$\mathcal{L}' = \frac{1}{2}\,\partial_\mu\phi^a\partial^\mu\phi^a + \frac{N}{2\lambda_0}\sigma^2 - \frac{1}{2}\sigma\phi^a\phi^a - \frac{N\mu_0^2}{\lambda_0}\sigma. \tag{13}$$

We have now two types of field, the ϕs and σ, with an interaction term between them which is no longer quartic.

We reanalyze the properties of the propagators, the vertex and the loops with the new Lagrangian.

The bare propagators are

$$D_{0\phi}(p) = \frac{i}{p^2 + i\varepsilon} \iff \text{——————} = O(N^0), \tag{14}$$

$$D_{0\sigma}(p) = \frac{i\lambda_0}{N} \iff \text{- - - - - - - - - - -} = O(N^{-1}). \tag{15}$$

The bare vertex $\sigma\phi\phi$, with coefficient $-i/2$, is represented in Fig. 6.

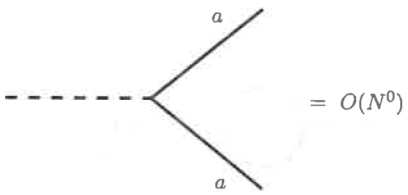

Figure 6: The vertex $\sigma\phi\phi$ and its order in N.

Loops and higher-order diagrams are represented in Figs. 7 and 8.

In the analysis of the above diagrams external lines are not counted. We observe now many new properties with respect to the former situation of Sec. 2.3. First, because of the behavior of the σ progator as $O(N^{-1})$, every internal σ line introduces an additional factor of N^{-1} in the behavior of the corresponding diagram. Therefore, diagrams containing an increasing number of internal σ lines become nondominant. The leading diagrams in their behavior in N will be those that contain the least possible number of internal σ lines. Second, loops of ϕs can occur only if they are joined to σ lines or propagators or within the latter. Multiloop diagrams of ϕs are of the type of the first diagram of Fig. 8 and of its generalizations. However, such diagrams, as well as the second diagram of Fig. 7, are not of the one-particle irreducible type, since by cutting one internal σ propagator,

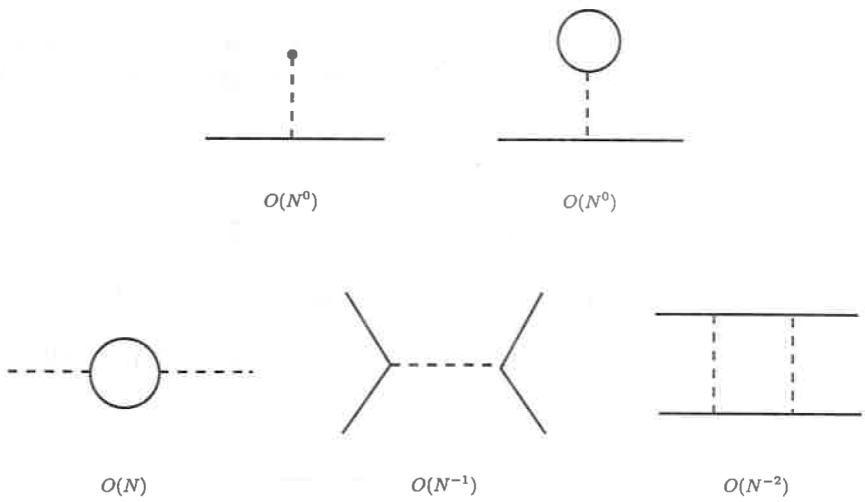

Figure 7: Higher-order diagrams and their order in N.

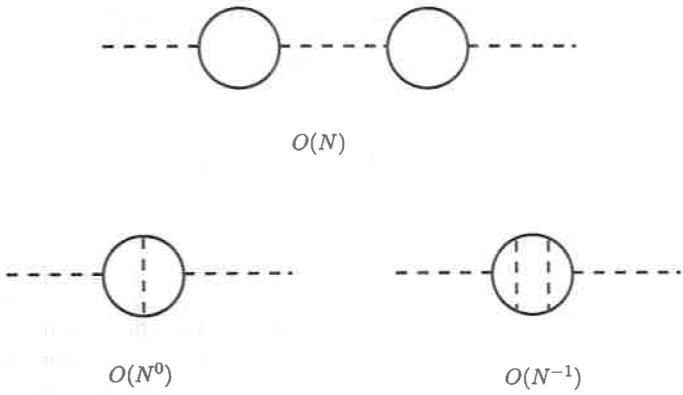

Figure 8: Higher-order diagrams and their order in N.

the diagrams become separated into two disconnected diagrams. Therefore, they cannot enter into the definition of the effective potential. The only loop diagrams of ϕs to be considered are represented by the third diagram of Fig. 7 and the last two diagrams and their generalizations of Fig. 8. The latter, containing internal σ lines, are nondominant with respect to the first one.

In conclusion, in the limit of large N, the leading contributions to the effective potential will come from the third type of diagram of Fig. 7 and of its generalizations, where one can have any number of external σ lines (cf. Fig. 9).

2.5 Renormalization

The effective potential can now be calculated. As expressed in Eq. (6), it is composed of two parts, U_{class} and U_{rad}, corresponding to the classical part and to the part receiving contributions of radiative corrections, respectively:

$$U_{\text{eff}} = U_{\text{class}} + U_{\text{rad}} \ . \tag{16}$$

The classical part is fixed by the content of the Lagrangian density (13):

$$U_{\text{class}} = -\frac{N}{2\lambda_0}\sigma^2 + \frac{1}{2}\sigma\phi^a\phi^a + \frac{N\mu_0^2}{\lambda_0}\sigma. \tag{17}$$

The radiative corrections are given by the sum of ϕ loop diagrams with an increasing number of external σ lines (Fig. 9). Here, according to the definition of the effective potential, the external σ fields should be considered as constants in x-space, or, equivalently, carrying zero momenta in momentum space.

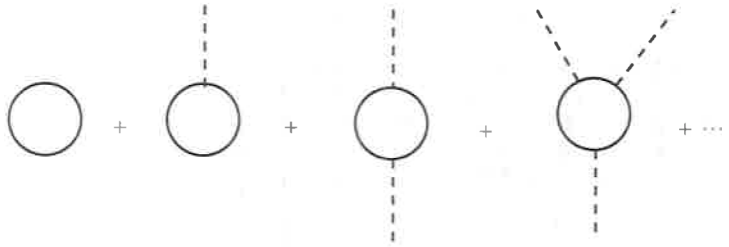

Figure 9: Diagrams contributing to the radiative corrections in the effective potential at leading order in N.

The summation of the above diagrams, with appropriate combinatorial factors, can be done with conventional methods [9, 11]. The loop calculation involves a four-dimensional integration in Minkowski space of ϕ propagators (14). Because of the presence of the $i\varepsilon$ factor in the denominator of the propagator, the latter has a well-defined analyticity property, which allows one to rotate the k_0-integration from the real axis to the imaginary axis and thus calculating the integrals in Euclidean space. This amounts to replacing in

the integrals k_0 with ik_4, with k_4 real. In the following, we shall write the integrals directly in Euclidean space, with the definition $k^2 = \sum_{i=1}^{4} k_i^2 > 0$.

After summation of the diagrams, the effective potential takes the form

$$U_{\text{eff}} = -\frac{N}{2\lambda_0}\sigma^2 + \frac{1}{2}\sigma\phi^a\phi^a + \frac{N\mu_0^2}{\lambda_0}\sigma + \frac{N}{2}\int \frac{d^4k}{(2\pi)^4}\ln(k^2+\sigma). \tag{18}$$

We are interested by the stationary point of U_{eff} (ground state). We therefore calculate its partial derivatives with respect to σ and ϕ^a:

$$\frac{\partial U_{\text{eff}}}{\partial\sigma} = -\frac{N}{\lambda_0}\sigma + \frac{1}{2}\phi^a\phi^a + \frac{N\mu_0^2}{\lambda_0} + \frac{N}{2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+\sigma)}\,, \tag{19}$$

$$\frac{\partial U_{\text{eff}}}{\partial\phi^a} = \phi^a\sigma, \qquad a = 1,\ldots,N. \tag{20}$$

The integral that appears in Eq. (19) is divergent for values of $k \to \infty$. This is a general problem of quantum field theory, which reflects the singular behavior of the theory at high energies or at short distances between the fields in x-space. To cure this difficulty, one generally absorbs the divergent parts into the bare coupling constant and the mass, as well as into redefinitions of the fields, defining finite quantities. If this happens, the theory is classified as being of the renormalizable type. However, not all theories satisfy this requirement. There might arise divergences, with specific structure, which could not be absorbed by existing quantities. As we shall see, the present theory is of the renormalizable type.

To study the possible renormalizability of the theory, we isolate in the integrand of the above integral the dominant parts of the asymptotic behavior:

$$\frac{1}{k^2+\sigma} \underset{k^2\to\infty}{\to} \frac{1}{k^2} - \frac{\sigma}{(k^2)^2} + O(1/(k^2)^3). \tag{21}$$

The first two terms lead to ultraviolet divergences by integration. We cannot, however, manipulate them as they stand, since the second term would lead to a new artificial infrared divergence (when $k \to 0$). We have to incorporate in the second term a mass factor in the denominator to render it softer in the infrared region.

We add and subtract in $\frac{\partial U_{\text{eff}}}{\partial\sigma}$ the following quantities:

$$\frac{N}{2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2} - \frac{N}{2}\sigma\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2(k^2+M^2)}, \tag{22}$$

where M is an arbitrary mass term.

The result is

$$\begin{aligned}\frac{\partial U_{eff}}{\partial\sigma} &= -N\sigma\Big(\frac{1}{\lambda_0} + \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2(k^2+M^2)}\Big) + \frac{1}{2}\phi^a\phi^a \\ &\quad + N\Big(\frac{\mu_0^2}{\lambda_0} + \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\Big) + \frac{N}{32\pi^2}\sigma\ln(\frac{\sigma}{M^2}).\end{aligned} \tag{23}$$

The infinite integrals, together with the bare coupling constant and the bare mass term, may define finite renormalized quantities. This would be possible if we admit that the bare

quantities λ_0 and μ_0^2 are themselves infinite quantities. We thus define a finite coupling constant λ and a finite mass term μ^2 with the equations

$$\frac{1}{\lambda(M)} = \frac{1}{\lambda_0} + \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2(k^2+M^2)}, \tag{24}$$

$$\frac{\mu^2(M)}{\lambda(M)} = \frac{\mu_0^2}{\lambda_0} + \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}. \tag{25}$$

The finite quantities depend on the arbitrary mass parameter M. For each fixed value of M, they take a corresponding value.

One can better study this dependence by comparing for instance the value of the coupling constant $\lambda(M)$ to a reference value $\lambda_1 \equiv \lambda(M_1)$ corresponding to a reference mass parameter M_1. One obtains from the comparison of the corresponding two equations:

$$\frac{1}{\lambda(M)} = \frac{1}{\lambda_1} - \frac{1}{32\pi^2}\ln(\frac{M^2}{M_1^2}). \tag{26}$$

This equation is called the renormalization group equation and plays an important role for the analysis and understanding of the properties of the theory.

We can also express $\lambda(M)$ in terms of λ_1 and the other parameters:

$$\lambda(M) = \frac{\lambda_1}{1 - \frac{\lambda_1}{32\pi^2}\ln(\frac{M^2}{M_1^2})}. \tag{27}$$

λ_1 and generally $\lambda(M)$ are assumed positive for physical reasons (below boundedness of the energy; see the discussion of the classical potential energy density after Eq. (2)). Therefore, we consider domains of solution where this condition is satisfied.

From the last equation we deduce that when M increases starting from M_1, $\lambda(M)$ increases. When M reaches the value $M_{cr} = M_1 \exp(16\pi^2/\lambda_1)$, $\lambda(M)$ diverges and for larger values of M, λ becomes negative. This is a sign of the instability of the theory for large values of λ.

The RGE can also be formulated in the form of a differential equation. Defining

$$M\frac{\partial\lambda(M)}{\partial M} = \beta(\lambda(M)), \tag{28}$$

one finds from the finite form of $\lambda(M)$ [Eq. (27)]

$$\beta(\lambda) = \frac{\lambda^2}{16\pi^2} > 0, \tag{29}$$

which shows that $\lambda(M)$ is an increasing function of M.

What is the interest of the RGE? In perturbation theory, when calculating radiative corrections, one usually finds powers of the quantity $\lambda^2\ln(p^2/M^2)$, where p is a representative of the momenta of the external particles. Even if λ is chosen small, at high energies, i.e., at large values of p, the logarithm may become large enough to invalidate perturbative calculations. However, since the logarithm depends on M for dimensional reasons, one can choose the latter of the order of p to maintain the logarithm small. This

could be useful, if on the other hand λ remains bounded or small at these values of M. The RGE precisely gives us the answer to that question, indicating the way λ behaves at large values of M.

Coming back to our theory, we see from our previous results that this is not the case here: λ, on the contrary, increases with M and diverges at M_{cr}. The theory is ill-defined at high energies. In x-space, high energies correspond to short distances. The interaction becomes stronger when the distance between two sources or two particles decreases. A similar behavior is also found in Quantum Electrodynamics (QED). These theories are better defined at low energies or large distances.

Theories for which the coupling constant decreases at high energies or at short distances are called asymptotically free.

Summarizing the results of this section, we recall that we could get rid of the divergent parts of the integrals by absorbing them in the bare coupling constant and the bare mass term, redefining at the end a finite coupling constant and a finite mass term. The theory we are considering is therefore renormalizable. The price that is paid is the introduction of an arbitrary mass parameter that essentially fixes the mass scale of the physical quantities of the theory.

2.6 The ground state

Using Eqs. (24) and (25), one can express the effective potential and its derivatives [Eqs. (18), (19) and (20)] in terms of the finite coupling constant and mass term. To simplify the notation, we denote $\lambda(M) = \lambda$ and $\mu^2(M) = \mu^2$.

The effective potential is

$$U_{\text{eff}} = \frac{1}{2}\phi^a\phi^a\sigma - \frac{N\sigma^2}{2\lambda}(1 + \frac{\lambda}{64\pi^2}) + \frac{N\mu^2\sigma}{\lambda} + \frac{N\sigma^2}{64\pi^2}\ln(\frac{\sigma}{M^2}). \tag{30}$$

The equations defining the minimum of the effective potential are

$$\frac{\partial U_{\text{eff}}}{\partial\sigma} = -\frac{N\sigma_0}{\lambda} + \frac{1}{2}\phi_0^a\phi_0^a + \frac{N\mu^2}{\lambda} + \frac{N}{32\pi^2}\sigma_0\ln(\frac{\sigma_0}{M^2}) = 0, \tag{31}$$

$$\frac{\partial U_{cff}}{\partial\phi^a} = \phi_0^a\sigma_0 = 0, \qquad a = 1, \ldots, N. \tag{32}$$

From Eqs. (32), one deduces two possibilities: 1) $\phi_0^a = 0$; 2) $\sigma_0 = 0$.
1) $\phi_0^a = 0$ $(a = 1, \ldots, N)$, $\sigma_0 \neq 0$. From Eq. (31) one obtains

$$\sigma_0\Big(1 - \frac{\lambda}{32\pi^2}\ln(\frac{\sigma_0}{M^2})\Big) = \mu^2. \tag{33}$$

The presence of the logarithm imposes $\sigma_0 > 0$. For small values of λ, one has $\mu^2 > 0$. By continuity to larger values, one should search for positive solutions of σ_0. Let σ_0 be the solution of the equation for a certain domain of λ and μ^2. One should redefine the field σ from the value of σ_0:

$$\sigma(x) = \sigma'(x) + \sigma_0. \tag{34}$$

The corresponding shift of σ in the Lagrangian gives back a common mass to the fields ϕ^a. We are in a situation where the $O(N)$ symmetry is realized in the Wigner mode.

2) $\sigma_0 = 0$. From Eq. (31) one obtains

$$\frac{1}{2}\phi_0^a\phi_0^a + \frac{N\mu^2}{\lambda} = 0. \quad \Longrightarrow \quad \mu^2 < 0. \tag{35}$$

$$\phi_0^a\phi_0^a = -\frac{2N\mu^2}{\lambda}. \tag{36}$$

There is a degeneracy of solutions for the ϕ_0^as. One can choose for example:

$$\phi_0^a = 0, \quad a = 1, \ldots, N-1, \qquad \phi_0^N = \sqrt{-\frac{2N\mu^2}{\lambda}} \equiv < \phi > . \tag{37}$$

One then develops ϕ^N around $< \phi >$:

$$\phi^N(x) = \chi(x) + < \phi > . \tag{38}$$

The symmetry is realized in the Goldstone mode, with the presence of $(N-1)$ massless fields ϕ^a and one massive field χ.

We summarize the results obtained so far. The $O(N)$ model defines a renormalizable theory. The radiative corrections introduce limitations into the validity domain of the model. The coupling constant increases at high energies and diverges at some critical mass scale. The ground state equations also introduce new constraints on the parameters of the model. The two modes of realization of the symmetry, found at the classical level, remain valid within the restricted domain of the parameters. The above results are obtained at the leading-order of the $1/N$ expansion, which allows us to simplify in a consistent way the equations of the theory, to solve them and to have an insight on the dynamics, going beyond ordinary perturbation theory. Unfortunately, the $O(N)$ model is not a stable theory as a whole, and persuing the investigations at nonleading orders of the expansion in $1/N$ reveals new restrictions and instabilities.

3 The Gross-Neveu model

3.1 General properties

We consider now an analog of the $O(N)$ model with the boson fields replaced by fermion fields of N different species with the same mass. Since Dirac fermion fields are not generally hermitian, the symmetry group of transformations that leaves the Lagrangian density invariant is $U(N)$. Theories with four-fermion interactions are not renormalizable in four space-time dimensions. This is related to the fact the fermion fields have mass dimension $3/2$, instead of 1 for the boson fields. The interaction term has thus dimension 6, greater than 4, the dimension of the Lagrangian density (dimensions of multiplicative coefficients are not considered here). Such terms do not lead to renormalization. Historically, a model of this kind was considered by Nambu and Jona-Lasinio [13] to implement dynamical chiral symmetry breaking. This model, because of its simplicity and of its ability to

describe nontrivial dynamical phenomena, is used until now as a guiding tool by many authors; on the other hand, because of its nonrenormalizability, it cannot represent a consistent theory, unless it is embedded in a wider theory from which it would emerge as an approximation.

The renormalization difficulty can, however, be circumvented by considering the model in two space-time dimensions. Here, the fermion fields have dimension 1/2 and the interaction term has dimension 2, equal to the dimension of the Lagrangian density. This ensures the renormalizability of the theory. On the other hand, many of the nontrivial dynamical properties of the Nambu–Jona-Lasinio model remain valid in two dimensions and thus allow its study in a consistent way in a simpler framework. This model was considered by Gross and Neveu [14] and is called after them.

In addition to the global $U(N)$ symmetry, the system satisfies also a discrete chiral symmetry, which prevents the fermions from having a mass. In more general versions of the model, a continuous chiral symmetry is imposed rather than a discrete one, but the latter is already sufficient to reproduce the nontrivial effects of the dynamics. We stick here to the discrete chiral symmetry.

The Lagrangian density of the system is

$$\mathcal{L} = \overline{\psi}^a i\gamma^\mu \partial_\mu \psi^a + \frac{g_0}{2N}(\overline{\psi}^a \psi^a)^2, \qquad g_0 > 0. \tag{39}$$

Summation on repeated indices is understood; a runs from 1 to N. $\overline{\psi}^a = \psi^{\dagger a}\gamma^0$. In two space-time dimensions, the Dirac fields have two components and the Dirac matrices γ reduce to the Pauli 2×2 matrices σ:

$$\gamma^0 = \sigma_z, \qquad \gamma^1 = i\sigma_y, \qquad \gamma_5 = \gamma^0\gamma^1 = \sigma_x. \tag{40}$$

The fermion fields are two-component spinors, with indices α or β ($\alpha, \beta = 1, 2$). Spinor indices will often be omitted from our notations; in some cases, for instance in the Lagrangian density or in mass terms, there is an implicit summation on them together with those of the γ matrices.

Discrete chiral transformations are defined in the following way:

$$\psi^a \to \gamma_5 \psi^a, \qquad \overline{\psi}^a = \psi^{\dagger a}\gamma^0 \to -\overline{\psi}^a \gamma_5. \tag{41}$$

A mass term is not invariant under these transformations:

$$m\overline{\psi}^a \psi^a \to -m\overline{\psi}^a \psi^a. \tag{42}$$

Hence, the fermions should be massless (no bare mass term in the Lagrangian density) if discrete chiral invariance is imposed on the theory.

The fermion field has mass dimension 1/2. As a consequence, the bare coupling constant g_0 is dimensionless.

3.2 Asymptotic freedom

The anlalysis of the theory is done in much the same way as for the $O(N)$ model. We first introduce the auxiliary field σ and then retain the leading terms in the $1/N$ expansion.

The auxiliary field is introduced with the addition of a new term to the Lagrangian density:

$$\mathcal{L}' = \mathcal{L} - \frac{N}{2g_0}\Big(\sigma + \frac{g_0}{N}\,\overline{\psi}^a\psi^a\Big)^2. \tag{43}$$

The field σ is defined through its equation of motion:

$$\sigma + \frac{g_0}{N}\,\overline{\psi}^a\psi^a = 0. \tag{44}$$

$\mathcal{L}'$ takes the form

$$\mathcal{L}' = \overline{\psi}^a i\gamma^\mu\partial_\mu\psi^a - \frac{N}{2g_0}\sigma^2 - \sigma\,\overline{\psi}^a\psi^a. \tag{45}$$

The bare fermion propagator is defined as

$$S^{ab}_{0,\alpha\beta}(p) = \delta_{ab}S_{0,\alpha\beta}(p) = \int d^4x e^{ip.x} < 0|T(\psi^a_\alpha(x)\overline{\psi}^b_\beta(0))|0 >, \tag{46}$$

where α and β are the fermion field spinor indices and S_0 is given by the expression

$$S_0(p) = \frac{i\gamma.p}{p^2 + i\varepsilon} \iff \text{——————} = O(N^0). \tag{47}$$

The bare σ propagator is

$$D_{0\sigma}(p) = -\frac{ig_0}{N} \iff \text{- - - - - - - - - - -} = O(N^{-1}). \tag{48}$$

The bare vertex $\sigma\overline{\psi}^a\psi^a$, with coefficient $-i/2$, has a graphical representation similar to that of Fig. 6.

The orders in N, when N is large, of the propagators, the vertex and the loops are the same as those found in the $O(N)$ model, the ψs replacing now the ϕs. The leading part of the effective potential will come from single fermion loops associated with constant external sigma fields.

In the calculation of the effective potential one considers constant classical external fields. Such fields may be interpreted as the vacuum expectation value of the quantized field operator: $\phi_{\text{class}} =< 0|\phi(x)|0 >$. Because of the translation invariance of the vacuum state, the vacuum expectation value of the field operator is x independent and hence ϕ_{class} is constant. This property can be applied to scalar fields. When the field is a fermion, it transforms under Lorentz transformations as a spinor, while the vaccum state remains invariant. This immediately implies that the vacuum expectation value of a fermion field is zero. Therefore, for the calculation of the effective potential, one has to consider only external scalar fields (in our case, the σ field). Fermions contribute only into internal lines or loops.

Another property of fermion fields appears also in loop calculations. Loops of fermion fields involve at the end the trace operation on the spinor indices. For massless fermions, the propagator is proportional to the $\gamma.p$ matrix. The loop value of a single fermion (without any external line) is proportional to the trace of the γ matrix, which is zero. This property generalizes easily to loops associated with an odd number of external constant

scalar fields. Therefore, the quantum part of the effective potential will involve only fermion loops with an even number of external lines.

The effective potential is given as usual by the sun of two contributions [Eq. (16)]:

$$U_{\text{eff}} = U_{\text{class}} + U_{\text{rad}} \ , \tag{49}$$

where $U_{\text{class}} = \frac{N}{2g_0}\sigma^2$. U_{eff} is calculated by the infinite sum of the diagrams of Fig. 10.

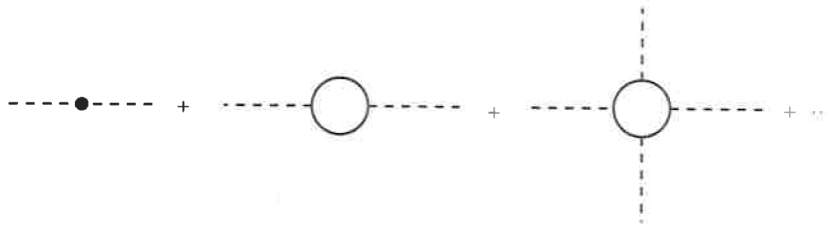

Figure 10: Diagrams, at leading order in the $1/N$ expansion, composing U_{eff}. The first diagram represents U_{class}.

One obtains:

$$U_{\text{eff}} = N\Big(\frac{\sigma^2}{2g_0} - \int \frac{d^2k}{(2\pi)^2}\ln(1+\frac{\sigma^2}{k^2})\Big), \tag{50}$$

$$\frac{\partial U_{\text{eff}}}{\partial\sigma} = 2\sigma N\Big(\frac{1}{2g_0} - \int \frac{d^2k}{(2\pi)^2}\ \frac{1}{(k^2+\sigma^2)}\Big). \tag{51}$$

(The integration momenta are euclidean; see comment before Eq. (18).) Concentrating on Eq. (51), we have to isolate, as for the scalar case, the divergent part of the integral. We introduce a mass parameter μ in the subtracted term to avoid infrared divergence:

$$\frac{\partial U_{\text{eff}}}{\partial\sigma} = \sigma N\Big[\frac{1}{g_0} - 2\int \frac{d^2k}{(2\pi)^2}\ \frac{1}{(k^2+\mu^2)}\Big] + \frac{\sigma N}{2\pi}\ln(\frac{\sigma^2}{\mu^2}). \tag{52}$$

We deduce from the latter expression the renormalization of the coupling constant into a finite value:

$$\frac{1}{g(\mu)} = \frac{1}{g_0} - 2\int \frac{d^2k}{(2\pi)^2}\ \frac{1}{(k^2+\mu^2)}, \qquad g>0. \tag{53}$$

g depends on the mass parameter μ. Its relation to another choice, μ_1, with $g(\mu_1)\equiv g_1$, is:

$$\frac{1}{g(\mu)} = \frac{1}{g_1} + \frac{1}{2\pi}\ln(\frac{\mu^2}{\mu_1^2}), \tag{54}$$

or,

$$g(\mu) = \frac{g_1}{1+\frac{g_1}{2\pi}\ln(\frac{\mu^2}{\mu_1^2})}. \tag{55}$$

$g(\mu)$ decreases when μ increases. We also have:

$$\mu\frac{\partial g}{\partial\mu} = -\frac{g^2}{\pi} = \beta(g) < 0. \tag{56}$$

The theory is therefore asymptotically free. This means that at high energies or at short distances, the interaction becomes weaker and weaker. In these regions perturbation theory can be used with respect to the coupling constant $g(\mu)$. On the opposite side, at low energies or at large distances, the interaction becomes strong. There is even a critical value of μ, $\mu_{\text{crit}} = \mu_1 \exp(-\pi/g_1)$, for which $g(\mu)$ diverges, indicating the occurrence of instabilities in the theory.

3.3 The ground state

The occurrence of instabilities may be rather a sign that the ground state that we are considering and around which calculations are done with free field propagators, is not the correct one. We have to determine from the effective potential the true ground state of the theory.

With respect to the renormalized coupling constant $g(\mu)$, the effective potential takes the form

$$U_{\text{eff}} = \frac{N\sigma^2}{4\pi} \left[\frac{2\pi}{g(\mu)} - 1 + \ln\left(\frac{\sigma^2}{\mu^2}\right) \right]. \tag{57}$$

The minimum of the effective potential is obtained from the equation

$$\frac{\partial U_{\text{eff}}}{\partial \sigma} = \frac{N\sigma}{2\pi} \left[\frac{2\pi}{g(\mu)} + \ln\left(\frac{\sigma^2}{\mu^2}\right) \right] = 0, \tag{58}$$

which possesses two types of solution: 1) $\sigma = 0$; 2) $\sigma = \pm\mu e^{-\pi/g(\mu)}$.

The absolute minimum of U_{eff} can be searched for with a study of the shape of the function U_{eff}. The latter is represented graphically in Fig. 11.

One notes that the solution $\sigma = 0$ is a local maximum, while the solutions $\sigma = \pm\mu e^{-\pi/g(\mu)}$ represent degenerate absolute minima. Because of the symmetry of U_{eff} under the change of sign of σ, any of these can be equivalently chosen. We shall choose for definiteness the solution with a plus sign and shall designate it by σ_0:

$$\sigma_0 = \mu e^{-\pi/g(\mu)}. \tag{59}$$

In order to study the physical properties of the system around the new ground state, we must shift the field σ in the effective potential and in the Lagrangian density by σ_0:

$$\sigma = \sigma' + \sigma_0. \tag{60}$$

In terms of the new field σ', the effective potential becomes

$$U_{\text{eff}} = \frac{N\sigma_0^2}{4\pi} \left(1 + \frac{\sigma'}{\sigma_0}\right)^2 \left[-1 + \ln\left(1 + \frac{\sigma'}{\sigma_0}\right)^2 \right]. \tag{61}$$

We find that $g(\mu)$ and μ have completely disappeared from the new expression of U_{eff} in favor of the single parameter σ_0, which has, by the definition of σ [Eq. (44)], a dimension of mass and thus fixes the mass scale of the theory. No free adjustable parameter has remained.

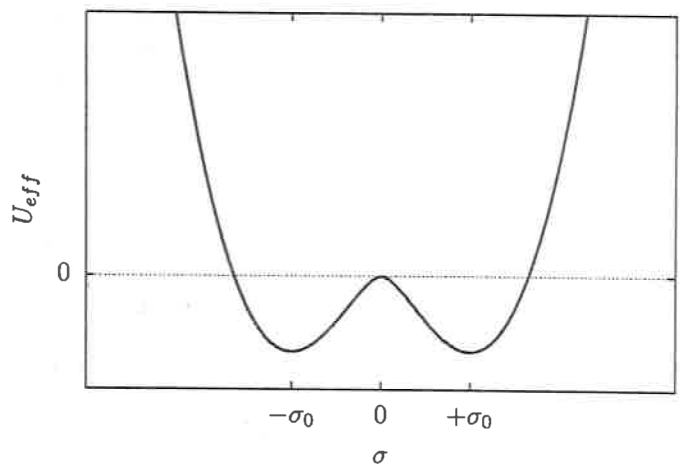

Figure 11: The shape of the function U_{eff} with respect to σ. We have defined $\sigma_0 = \mu e^{-\pi/g(\mu)}$.

The lagrangian density becomes:

$$\mathcal{L}' = \overline{\psi}^a i\gamma^\mu \partial_\mu \psi^a - \frac{N}{2g_0}(\sigma' + \sigma_0)^2 - \sigma_0 \overline{\psi}^a \psi^a - \sigma' \overline{\psi}^a \psi^a. \tag{62}$$

We note the appearance of a mass term of fermions, with value

$$M_F = \sigma_0 = \mu e^{-\pi/g(\mu)}. \tag{63}$$

The Lagrangian density still contains the bare coupling constant g_0. This is necessary, since one still has to calculate and renormalize the σ' propagator. The latter receives contributions from the fermion loops which are divergent and whose divergence should be combined with g_0 to reproduce a finite quantity.

The theory was invariant at the beginning under the discrete chiral transformations, imposing masslessness of the fermions, but now, after renormalization and the shift to the ground state of the energy, the fermions have acquired a mass. This phenomenon is called dynamical mass generation and is due to the spontaneous breaking of the discrete chiral symmetry.

On the other hand, M_F is a physical quantity and should not depend on the particular choices of the arbitrary mass parameter μ. From Eq. (63) one notes that M_F has two types of dependence on μ: an explicit one and an implicit one through the coupling constant $g(\mu)$ [Eq. (56)]. One verifies that M_F is actually independent of μ:

$$\frac{dM_F}{d\mu} = \frac{\partial M_F}{\partial \mu} + \frac{1}{\mu}\beta(g)\frac{\partial M_F}{\partial g} = 0. \tag{64}$$

M_F is said to be renormalization group invariant.

Once M_F is fixed, the two other parameters, g and μ should disappear from the finite quantities of the theory. Any choice of μ is compensated by a corresponding choice of g, to produce M_F. This property was explicitly verified on the expression of U_{eff} [Eq. (61)].

At the beginning, the theory had a dimensionless parameter g and massless fermions. Now it has instead a dimensionful parameter, M_F, the mass of the fermions, which is fixed by the physical conditions and sets the mass scale of the theory. There is no longer a free parameter in the theory. This phenomenon is called mass transmutation.

In summary, the Gross-Neveu model displays many interesting features of quantum field theory and illustrates, in two dimensions, several phenomena – asymptotic freedom, dynamical mass generation, mass transmutation – expected also to occur in four-dimensional theories.

4 QCD

Quantum Chromodynamics (QCD), the theory of the strong interaction, is a gauge theory with the non-Abelian local symmetry group of color $SU(N)_c$, with $N = 3$. We consider henceforth the general case where the paramater N is arbitrary. As "matter" fields, the theory contains quark and antiquark fields, belonging to the defining fundamental representation of the group and to its conjugate, respectively; their number is N. Since this is a gauge theory, there are also gauge fields, the gluons, belonging to the adjoint representation of the group; their number is $(N^2 - 1)$.

There are six "generations" of QCD, distinguished from each other by specific properties of the quarks (charge and mass), called also "flavors". Schematically, there are three "light" quarks, u, d, s and three "heavy" quarks, c, b, t. The free masses of the quarks u and d are of the order of a few MeV, while the mass of the quark s is of the order of 100 MeV. The masses of the heavy quarks c, b, t are approximately 1.3 GeV, 4.2 GeV and 173 GeV, respectively [15]. The gluon, which is the gauge particle, is massless.

The proton, the stable matter particle, is a bound state made mainly of the three quarks u, u, d. Its mass is of the order of 1 GeV. This shows that the mass of the proton is not made of the masses of its quark components, which are nearly massless. Rather, one should expect that it is produced by a dynamical mass generation mechanism, similar to what happened in the Gross-Neveu model. A similar conclusion also holds for the other low-lying hadrons (neutron, ρ meson, etc.). On the other hand, the coupling constant of the QCD Lagrangian is dimensionless. This means that the mass generation phenomenon would be realized by the mechanism of mass transmutation, which also was observed in the Gross-Neveu model.

Since the light quark masses do not seem to play a fundamental role, one can consider the QCD Lagrangian in the ideal situation where the the three light quarks are massless. In this case, the QCD theory also satisfies a global flavor space invariance under the group of continuous chiral transformations $SU(3)_R \times SU(3)_L$ (R for right, L for left). It is expected that this symmetry is realized with the Goldstone mode (no nearly degenerate parity doublets are observed in nature); the corresponding Goldstone bosons are the lowest lying π, K and η mesons. In the limit of vanishing quark masses, the masses of the latter

particles would also vanish. For small values of the quark masses, the Goldstone bosons also acquire a small mass. This explains why these mesons have masses squared much smaller than the other hadron masses squared.

QCD theory has been widely investigated in ordinary perturbation theory and has been shown to be asymptotically free [16, 17]; a relation similar to Eq. (55) has been obtained. This implies that at high energies or at short distances, the QCD interaction becomes weak and ordinary perturbation theory can be applied there. This property has been experimentally verified by many high-energy experiments. The other implication of asymptotic freedom is that at low energies or at large distances the interaction becomes strong enough to forbid perturbative treatments. From asymptotic freedom one also deduces the existence of a renormalization group invariant mass, called $\Lambda_{\rm QCD}$, which realizes mass transmutation in the theory (cf. Eqs. (59), (63), (64)). Contrary to the Gross-Neveu model, however, the resolution of the nonperturbative domain of QCD has not been achieved up to now with analytic calculations. One of the main reasons of this failure is probably related to the fact that quarks and gluons are confined. These particles have not been observed as free asymptotic states like the other known particles. Their presence or existence have been detected mainly in an indirect way. Quarks and gluons are bound by the QCD force to form bound states called hadrons (proton, neutron, π and ρ mesons, etc.) It is at this level that QCD differs from the previous models that were considered or mentioned (Gross-Neveu, Nambu–Jona-Lasinio). Numerical resolution of the strong coupling regime of QCD is successfully realized with Lattice calculations.

The application of the $1/N$ expansion method to QCD leads to some simplifications and well verified qualitative predictions, but fails to solve the theory as a whole. The reason of this last negative result is due to the large number of gluon fields ($\sim N^2$) as compared to that of the quarks (N). While quark loop contributions become negiligible and do not enter in leading expressions in the large-N limit, gluon loop diagrams, and more precisely the class of "planar diagrams", become dominant at large N and an infinite number of them (one-particle irreducible) survive [3, 5, 6, 7, 8]. Their summation in compact form is not an easy task. Examples of planar and non-planar diagrams are presented in Figs. 12 and 13, respectively.

Figure 12: Examples of planar diagrams. Curly lines represent gluons and oriented full lines quarks or antiquarks. When a planar diagram is drawn on a plane, a gluon line does not intersect any other gluon line, except at vertices. The latter are represented on the figures by dots.

Figure 13: Examples of non-planer or crossed diagrams. The intersection points of the gluon lines are not vertices.

One therefore is satisfied at the present time with the qualitative predictions and relative simplifications the $1/N$ expansion method provides. The method, when applied to two-dimensional QCD, has been, however, successful enough to solve the main aspects of the theory in an explicit way [4, 18].

5 Conclusion

The $1/N$ expansion method allows us to solve, at leading order of the expansion, theories and models nonperturbatively, probing directly dynamical phenomena, which would not be reached in ordinary perturbation theory based on expansions with respect to the coupling constant.

The method is also applicable to QCD, but because of the presence of the gluon fields, which belong to the adjoint representation of the gauge symmetry group, its effects are less spectacular. Nevertheless, many simplifications occur and several qualitative features can be drawn about the properties of the theory.

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