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Seiberg-Witten equations in all dimensions

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To create a little flower is the labour of ages.
-William Blake, *Proverbs of Hell*

Abstract

Starting with an n-dimensional oriented Riemannian manifold with a Spin^c-structure, we describe an elliptic system of equations which recover the Seiberg–Witten equations when $n = 3, 4$. The equations are for a $U(1)$-connection A and spinor ϕ, as usual, and also an odd degree form β (generally of inhomogeneous degree). From A and β we define a Dirac operator $D_{A,\beta}$ using the Clifford action of β and $*\beta$ on spinors (with carefully chosen coefficients) to modify D_A. The first equation in our system is $D_{A,\beta}(\phi) = 0$. The left-hand side of the second equation is the principal part of the Weitzenböck remainder for $D^*_{A,\beta} D_{A,\beta}$. The equation sets this equal to $q(\phi)$, the trace-free part of projection against ϕ, as is familiar from the cases $n = 3, 4$. In dimensions $n = 4m$ and $n = 2m + 1$, this gives an elliptic system modulo gauge. To obtain a system which is elliptic modulo gauge in dimensions $n = 4m + 2$, we use two spinors and two connections, and so have two Dirac and two curvature equations, which are coupled via the form β. We then prove a collection of a priori estimates for solutions to these equations. Unfortunately they are not sufficient to prove compactness modulo gauge, instead leaving the possibility that bubbling may occur. We also construct several examples of solutions of these equations in dimensions $5, 6$ and 8. And finally we describe a modified version of these Seiberg–Witten equations on manifolds with a $\mathrm{Spin}(7)$-structure and construct a solution when the $\mathrm{Spin}(7)$-structure is torsion-free.

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Part I

Introduction

In the fall of 1994 Edward Witten announced a "new gauge theory of 4-manifolds" [25], capable of giving results analogous to the earlier theory of Donaldson [7], but where the computations involved are "at least a thousand times easier" (Taubes). The equations involved in this new gauge theory are well known as the Seiberg–Witten equations. The equations introduced in [33], led quickly to a revolution in 3- and 4-dimensional differential geometry and they remain at the forefront of research today. Shortly after their appearance, Witten showed how one could count solutions to the equations, defining an invariant of the underlying smooth 4-manifold [33]. The equations in dimension 3 and 4 and the resulting moduli spaces have had a profound impact on low dimensional geometry and topology, some examples being dramatic discoveries of homeomorphic but non-diffeomorphic 4-manifolds, distinguished by their Seiberg–Witten invariants, Kronheimer and Mrowka's proof of the Thom conjecture [18], Taubes' proof of the Weinstein conjecture in dimension 3 [31], Taubes' work showing that symplectic 4-manifolds have non-vanishing SW invariant [32] and the close link between Seiberg–Witten equations and J-holomoprhic curves [28],[30],[29] and many more.

Despite a lot of effort, higher-dimensional generalisations of the Seiberg–Witten equations were unknown (at least without an additional structure being present). In this thesis we introduce an elliptic system of equations over a Spin^c-manifold of any dimension which generalise the Seiberg–Witten equations in the cases $n = 3, 4$.

Let (M, g) be a Riemannian manifold of dimension n, which admits a Spin^c-structure. We begin by fixing notation. Write $S \to M^n$ for the spin bundle of the Spin^c-structure. When n is even, $S = S_+ \oplus S_-$ splits into subbundles of positive and negative spinors. We write $c \colon \Lambda^* \to \mathrm{End}(S)$ for the Clifford action of differential forms on spinors. We follow the conventions of [20]. In particular, (real) 1-forms act as skew-Hermitian endomorphisms. Meanwhile, in dimension $n = 2m$, the volume form satisfies $i^m c(\mathrm{dvol})) = \pm 1$ on $S_\pm$ whilst in dimension $n = 2m - 1$, $i^m c(\mathrm{dvol}) = 1$ on all of S.

Let L denote the determinant line bundle of S when n is odd, and of S_+ (or equivalently S_-) when n is even. Let $\mathcal{A}$ denote the set of unitary connections in L. Given $A \in \mathcal{A}$, we write D_A for the associated Dirac operator.

When n is even, a spinor $\phi \in S_\pm$ defines a trace-free Hermitian endomorphism $E_\phi \colon S_\pm \to S_\pm$ via

$$E_\phi(\psi) = \langle \psi, \phi \rangle \, \phi - \frac{1}{r} |\phi|^2 \psi. \tag{1}$$

where r is the rank of $S_\pm$. When n is odd and $\phi \in S$ we use E_ϕ to denote the analogous trace-free endomorphism of S, where now r is the rank of S.

Before giving the n-dimensional version of the Seiberg–Witten equations, we first recall the three and four dimensional cases, highlighting the features we will generalise. In dimension 4, Clifford multiplication gives an isomorphism

$$c \colon i\Lambda^2_+ \to i\mathfrak{su}(S_+) \tag{2}$$

between imaginary self-dual 2-forms and trace-free Hermitian endomorphisms of S_+. Given $\phi \in \Gamma(S_+)$, we write $q(\phi) \in i\Omega^2_+$ for the imaginary self-dual 2-form corresponding to E_ϕ under (2). The Seiberg–Witten equations for $A \in \mathcal{A}$ and $\phi \in \Gamma(S_+)$ are:

$$D_A \phi = 0, \tag{3}$$

$$F_A^+ = q(\phi). \tag{4}$$

The gauge group $\mathcal{G} = \mathrm{Map}(M, S^1)$ acts on (A, ϕ) by pull-back and this action preserves the space of solutions to the equations.

The whole 4-dimensional story is now based on two crucial facts. Firstly, these equations are *elliptic modulo gauge*. This is ultimately because the linearisation of the map $A \mapsto F_A^+$ combined with Coulomb gauge, to work modulo the gauge action, gives the truncated de Rahm complex, in the form

$$\Omega^1 \stackrel{d^*+d^+}{\longrightarrow} \Omega^0 \oplus \Omega^2_+.$$

Secondly, the space of solutions to the equations are *compact modulo gauge*. This follows ultimately from the fact that the left-hand side of the curvature equation (4) is directly related to the Weitzenböck formula[1]:

$$D_A^2 - \nabla_A^* \nabla_A = \frac{s}{4} + \frac{1}{2} c(F_A^+), \tag{5}$$

where s is the scalar curvature of (M, g).

To summarise: *on a 4-manifold, using Dirac operators D_A parametrised by $A \in \mathcal{A}$ ensures that prescribing the Weitzenböck remainder gives an elliptic system modulo gauge.*

In dimension 3, meanwhile, Clifford multiplication gives an isomorphism

$$c \colon i\Lambda^2 \to i\mathfrak{su}(S) \tag{6}$$

Given $\phi \in \Gamma(S)$, we now write $q(\phi) \in i\Omega^2$ for the imaginary 2-form corresponding to E_ϕ under (6). The 3-dimensional Seiberg–Witten equations for $A \in \mathcal{A}$, $\phi \in \Gamma(S)$ and $\beta \in \Omega^3$ are:

$$(D_A - c(i\beta))\,\phi = 0, \tag{7}$$

$$F_A - 2id^*\beta = q(\phi). \tag{8}$$

One often sees these equations written with $\beta = 0$. This is because when ϕ is not identically zero and M is compact, one can check that the equations actually force $\beta = 0$. It is also perhaps more common to see the equations with the 3-form β replaced by the function $*\beta$. (Note $c(\beta)$ is multiplication by $- * \beta$.) We choose the above version of the equations because they fit more cleanly with our generalisation. The equations are elliptic modulo gauge because the linearisation of $(A, \beta) \mapsto F_A - 2id^*\beta$, in combination with Coulomb gauge, produces the de Rahm complex, in the form

$$\Omega^1 \oplus \Omega^3 \stackrel{d+d^*}{\longrightarrow} \Omega^0 \oplus \Omega^2.$$

Just as in the 4-dimensional case, the curvature equation (8) is related to a Weitzenböck remainder. This time

$$D_{A,\beta}^* D_{A,\beta} - \nabla_A^* \nabla_A = \frac{s}{4} + \frac{1}{2} c(F_A - 2id^*\beta) + |\beta|^2. \tag{9}$$

Here $D_{A,\beta} = D_A - c(i\beta)$ is the Dirac operator appearing in (7).

To summarise: *on a 3-manifold, using Dirac operators $D_{A,\beta}$ parametrised by $A \in \mathcal{A}$ and $\beta \in \Omega^3$ ensures that prescribing the principal part of the Weitzenböck remainder gives an elliptic system modulo gauge.*

[1] The formula relating the Dirac Laplacian to the rough Laplacian is due to Schrödinger [24] in 1932, and was subsequently rediscovered by Lichnerowicz [19] in 1962. The analogous formula relating a generalised Laplacian to a rough Laplacian is seemingly due to Weitzenböck. Historically a better name for (5) would probably be the "Schrödinger–Lichnerowicz formula" but we follow a relatively common practice by calling this and similar equations "Weitzenböck formulae".

These features are what we generalise to give Seiberg–Witten equations in arbitrary dimensions: we consider a family of Dirac operators $D_{A,\beta}$ parametrised by $A \in \mathcal{A}$ and a certain choice of odd degree form β (of inhomogeneous degree). The odd degree forms ensure that prescribing the principal part of the Weitzenböck remainder of $D_{A,\beta}$ is an elliptic system, modulo gauge. When combined with the Dirac equation, the equations have the potential for good analytic properties. We will give some analytic results in this direction. We stress from the outset however that when $n > 4$ our results are not sufficient to prove that the solution space is compact. Instead they leave open the possibility that "bubbling" can occur.

An obvious question is what purpose might these higher-dimensional Seiberg–Witten equations serve? In higher dimensions, there is no need for a gauge theoretic approach to study smooth structures since, for example, the h-cobordism theorem holds [26]. Instead, one might speculate that higher dimensional Seiberg–Witten equations could prove useful when studying manifolds with geometric structures. This fantasy is inspired by Taubes' work on symplectic 4-manifolds. Taubes proved that for a compact symplectic 4-manifold with $b_2^+ > 1$ the Seiberg–Witten invariant for the canonical Spin^c structure is always 1 [32]. This gives an obstruction to the existence of symplectic structures. There is no known obstruction in higher dimensions, beyond the most obvious that there must be a degree 2 cohomology class with non-zero top power. Even deeper is Taubes' Theorem that the Seiberg–Witten invariants are equal to the Gromov–Witten invariants [28, 30, 29]. In particular, for a symplectic manifold with $b_2^+ > 1$, the canonical class is always represented by a J-holomorphic curve. In higher dimensions there are no known general existence results of this kind for J-holomorphic curves. It is, of course, very speculative to hope that these higher dimensional Seiberg–Witten equations could tell us something about higher dimensional symplectic manifolds (especially in light of the fact that the analysis appears much more complicated; see §V!), but at least it does not seem completely impossible.

Dirac operators of the form $D + c(\beta)$ where $\beta \in \Omega^{\mathrm{odd}}$ have appeared in several contexts, going back at least as far as Bismut's pioneering work on Dirac operators associated to metric connections with torsion [4]. This same paper also gives a Weitzenböck formula which is very similar to the general Weitzenböck formula we deduce. Since Bismut's work, there has been a huge amount of work on these particular Dirac operators; so much so that it is futile to give a survey here. To the best of our knowledge, however, there is only one paper which considers this kind of Dirac opertaor in the context of the Seiberg–Witten equations, namely the work [27] of Tanaka. Tanaka formulates a version of the Seiberg–Witten equations on a *symplectic* 6-manifold, which have some similarity to the equations described here. To write down Tanaka's equations one must first pick an almost complex structure compatible with the symplectic form. This is in contrast to our equations which need nothing more than a Riemannian metric and a Spin^c-structure. There is an interesting point in common however: Tanaka perturbs the Dirac operator by adding a $(0,3)$-form to it; this is similar to the point of view taken here, where in 6-dimensions we also perturb the Dirac operator, this time by an arbitrary 3-form.

The main results of the article are structured as follows. In the next section §II, we describe the n-dimensional Seiberg–Witten equations. In §III we show that the equations are elliptic, modulo gauge, and compute the index. In §IV we show that the curvature equation is precisely the principal part of the Weitzenböck remainder term. In §V we exploit this to prove some preliminary a priori estimates on solutions to the equations. §VI is dedicated to the construction of several examples of solutions to these equations and in the last section §VII we describe special modified versions of these Seiberg–Witten equations on manifolds with a Spin(7)-structure and construct a solution when the Spin(7)-structure is torsion-free.

Part II

Seiberg–Witten equations on all Spin^c-manifolds

0.1 The Seiberg–Witten equations in odd dimensions

In odd dimension $n = 2m+1$, it is relatively straightforward to generalise the 3-dimensional story. Only the notation becomes more complicated. To ease things a little, we define a function $s\colon \mathbb{N} \to \{1, i\}$ by

$$s_k = \begin{cases} 1 & \text{if } k \equiv 0 \text{ or } 3 \bmod 4 \\ i & \text{if } k \equiv 1 \text{ or } 2 \bmod 4 \end{cases}$$

Notice that $s_{2k+1} = s_{2k+2}$ and if $\beta_k \in \Omega^k$ then $s_k c(\beta_k)$ is a self-adjoint endomorphism of the spin bundle.

In dimension $2m+1$, Clifford multiplication gives an isomorphism

$$c\colon i\Lambda^2 \oplus \Lambda^4 \oplus \cdots s_{2m}\Lambda^{2m} \to i\mathfrak{su}(S) \tag{10}$$

Given $\phi \in S$, we write $q(\phi)$ for the differential form corresponding to E_ϕ under (10). We consider equations for (A, β, ϕ) where $A \in \mathcal{A}$, $\beta = \beta_3 + \beta_5 + \cdots + \beta_{2m+1}$ with $\beta_k \in \Omega^k$, and $\phi \in \Gamma(S)$. We set

$$D_{A,\beta} = D_A + \sum_{k=1}^{m-1} \Big(s_{2k+1} c(\beta_{2k+1}) + i s_{2m-2k} c(*\beta_{2k+1}) \Big) + ic(*\beta_{2m+1}) \tag{11}$$

Write

$$F_\beta = 2 \sum_{k=1}^{m-1} s_{2k+2} d\beta_{2k+1}, \tag{12}$$

$$C_\beta = 2 \sum_{k=1}^{m} (-1)^{\lfloor \frac{m+1}{2} \rfloor + (m+1)(k+1)} s_{2k} d^*\beta_{2k+1}. \tag{13}$$

The notation F_β and C_β is explained below in Remark 2. The key thing to keep in mind for now is that, up to some factors of $2i$, F_β is essentially $d\beta$ whilst, again up to some factors of $2i$ and also some cumbursome signs (an artefact of how Clifford multiplication works), C_β is essentially $d^*\beta$.

Definition 1. Let M^{2m+1} be an oriented $2m+1$-dimensional manifold with Spin^c-structure. The $(2m+1)$-dimensional Seiberg–Witten equations for (A, β, ϕ) are:

$$D_{A,\beta}(\phi) = 0, \tag{14}$$

$$F_A + F_\beta + C_\beta = q(\phi), \tag{15}$$

where $D_{A,\beta}$, F_β and C_β are given by (11), (12) and (13) respectively.

When $m = 1$, $F_\beta = 0$, $C_\beta = -2id^*\beta_3$ and we recover the ordinary 3-dimensional Seiberg–Witten equations.

The point of these equations is that, as is shown in §IV, the Dirac operator $D_{A,\beta}$ has a Weitzenböck formula of the form

$$D_{A,\beta}^* D_{A,\beta} - \nabla_{A,\beta}^* \nabla_{A,\beta} = \frac{s}{4} + \frac{1}{2} c(F_A + F_\beta + C_\beta) + Q(\beta)$$

where $\nabla_{A,\beta}$ is a unitary connection on S determined by A and β and $Q(\beta)$ is a zeroth order term which is purely algebraic in β. (For example, when $m = 1$ this is equation (9) above where

$Q(\beta) = |\beta|^2$.) So (15) prescribes the principal part of the Weitzenböck remainder. Moreover, the system is elliptic modulo gauge, essentially because the de Rahm complex is elliptic. This is the reason behind the various factors for $c(\beta_{2k+1})$ and $c(*\beta_{2k+1})$ in (11): they are chosen precisely to make $d\beta$ and $d^*\beta$ appear in the Weitzenböck remainder. (See §III for the details.)

Remark 2. If we think of β as a collection of connection k-forms, in the sense of $U(1)$-gerbes (or "k-form gauge fields" as they are called in the physics literature) then, up to the various factors of i and 2, F_β is the sum of the curvatures of the β_k. Meanwhile $C_\beta = 0$ is the Coulomb gauge condition. With this in mind it is tempting to think of $D_{A,\beta}$ as a Dirac operator coupled to various connections on appropriate $U(1)$-gerbes.

One reason to want to do this is to put F_β on a similar footing to F_A. In 4-dimensional Seiberg–Witten theory it is important to be able to vary the cohomology class $[F_A]$. In particular, for some classes there are no solutions. In the above description, however, $[F_\beta] = 0$ is fixed. To get non-zero classes, one would need to interpret β_{2k+1} as a connection in a 2k-gerbe with non-zero characteristic class. However we have been unable to make sense of "spinors with values in a gerbe" or of the action of gerbe gauge-transformations in this setting (or "$(k-1)$-form gauge transformations" as they are sometimes called in the physics literature).

0.2 The Seiberg–Witten equations in dimension $4m$

We next give the direct generalisation of the 4-dimensional Seiberg–Witten equations to dimension $n = 4m$. Here, Clifford multiplication gives the following isomorphism:

$$c\colon i\Lambda^2 \oplus \Lambda^4 \oplus \cdots \oplus s_{2m}\Lambda^{2m}_+ \to i\mathfrak{su}(S_+) \tag{16}$$

where Λ^{2m}_+ is the $+1$ eigenspace of $*$ acting on Λ^{2m}. Given $\phi \in S_+$, we write $q(\phi)$ for the form which corresponds under (16) to $E_\phi \in i\mathfrak{su}(S_+)$. We consider equations for (A, β, ϕ) where $A \in \mathcal{A}$, $\beta = \beta_3 + \beta_5 + \cdots + \beta_{2m-1}$ with $\beta_k \in \Omega^k$ and $\phi \in \Gamma(S_+)$. We set

$$D_{A,\beta} = D_A + \sum_{k=1}^{m-1} \Big(s_{2k+1} c(\beta_{2k+1}) + s_{4m-2k-1} c(*\beta_{2k+1}) \Big). \tag{17}$$

This is a self-adjoint operator $D_{A,\beta}\colon \Gamma(S_+) \to \Gamma(S_-)$. Write

$$F^+_\beta = 2s_{2m} d^+ \beta_{2m-1} + 2 \sum_{k=1}^{m-2} s_{2k+2} d\beta_{2k+1}; \tag{18}$$

$$C_\beta = 2 \sum_{k=1}^{m-1} (-1)^{m+k+1} s_{2k} d^* \beta_{2k+1}. \tag{19}$$

Here, $d^+\beta_{2m-1}$ is the Λ^{2m}_+-component of $d\beta_{2m-1}$.

Definition 3. Let M^{4m} be an oriented Riemannian $4m$-manifold with Spin^c-structure. The $4m$-dimensional Seiberg–Witten equations on M for (A, β, ϕ) are:

$$D_{A,\beta}(\phi) = 0, \tag{20}$$

$$F_A + F^+_\beta + C_\beta = q(\phi), \tag{21}$$

where $D_{A,\beta}$, F^+_β and C_β are given by (17), (18) and (19) respectively.

Again, the point of our equations is that there is a Weitzenböck formula for $D_{A,\beta}$ (see §IV) and the principal part of the remainder is exactly $\frac{1}{2}(F_A + F_\beta^+ + C_\beta)$. Moreover, the equations are elliptic modulo gauge, essentially because the truncated de Rahm complex is elliptic:

$$\Omega^0 \to \Omega^1 \to \cdots \to \Omega^{2m}_+$$

(See §III for the details.)

Remark 4. In the case $M^{4m} = X^{4m-1} \times \mathbb{R}$ is a Riemannian product, one can consider solutions to the equations which are $\mathbb{R}$-invariant, leading to a system of equations for fields defined purely on X.. It turns out that these so-called "reduced" equations are equivalent to the $(4m-1)$-dimensional Seiberg–Witten equations on X from Definition 1. To see this, note first that a 1-form α on X can be made to act on S_+ via $c'(\alpha) = c(\alpha) \circ c(dt)$ where c is the Clifford action on M and t is the coordinate on $\mathbb{R}$ (with dt unit length and positively oriented). In this way we identify S_+ as the spin bundle $S \to X$, with c' being the Clifford product on X; we also identify $\mathbb{R}$-invariant sections of S_+ with $\Gamma(X, S)$. Next note that an $\mathbb{R}$-invariant connection A on M is of the form $A' + if dt$ where A' is a connection on X and $f \in \Omega^0(X)$. To recover the Seiberg–Witten equations on X, one should take the top degree odd form to be $*f$. Similarly, an $\mathbb{R}$-invariant form $\beta_{2k+1} \in \Omega^{2k+1}(M)$ has the shape $\beta_{2k+1} = \beta'_{2k+1} + \beta'_{2k} \wedge dt$ for forms β' on X. It is the odd-degree forms β'_{2k+1}, $*\beta'_{2k}$, $*f$ (with appropriate signs) the connection A and spinor ϕ, which solve the Seiberg–Witten equations on X.

0.3 The Seiberg–Witten equations in dimension $4m-2$

This leaves dimension $n = 4m-2$ and here things are more complicated. Clifford multiplication gives isomorphisms

$$c\colon i\Lambda^2 \oplus \Lambda^4 \oplus \cdots \oplus s_{2m-2}\Lambda^{2m-2} \to i\mathfrak{su}(S_\pm) \tag{22}$$

$$c\colon s_{2m+2}\Lambda^{2m+2} \oplus \cdots \oplus \Lambda^{4m-4} \to i\mathfrak{su}(S_\pm) \tag{23}$$

This time there is no corresponding bundle of odd degree forms with the correct rank to set up an elliptic system. For example, in dimension 6, $i\mathfrak{su}(S_+)$ has rank 15, so the curvature equation will give 15 constraints whilst gauge fixing provides one more. Meanwhile, the connection A accounts for 6 degrees of freedom and so we are left looking for 10 more degrees of freedom, but there is no bundle of forms with this rank which we can use to parametrise Dirac operators. The way out is to use *two* spinors and connections, leading to two Dirac equations and two curvature equations, with everything coupled via the odd degree forms.

We do this as follows. Given $\phi \in S_+$ and $\psi \in S_-$ we write $q(\phi)$ and $q(\psi)$ for the differential forms corresponding to E_ϕ and E_ψ respectively under (23). We consider equations for $(A, B, \beta, \phi, \psi)$ where $A, B \in \mathcal{A}$, $\beta = \beta_3 + \beta_5 + \cdots + \beta_{4m-5}$ where $\beta_k \in \Omega^k$, $\phi \in \Gamma(S_+)$ and $\psi \in \Gamma(S_-)$. We write

$$D_{A,\beta,+} = D_A + \sum_{k=1}^{m-2} \Big(s_{2k+1} c(\beta_{2k+1}) + s_{4m-2k-3} c(*\beta_{2k+1}) \Big) + s_{2m-1} c(\beta_{2m-1}), \tag{24}$$

$$D_{B,\beta,-} = D_B + s_{2m-1} c(*\beta_{2m-1}) + \sum_{k=m}^{2m-3} \Big(s_{2k+1} c(\beta_{2k+1}) + s_{4m-2k-3} c(*\beta_{2k+1}) \Big). \tag{25}$$

We regard $D_{A,\beta,+}$ as acting on sections of S_+ and $D_{B,\beta,-}$ as acting on sections of S_-. Notice that β_{2m-1} is the only part of β which appears in both Dirac operators. The terms β_k for $k < 2m-1$ appear in $D_{A,\beta,+}$ and the terms β_k for $k > 2m-1$ appear in $D_{B,\beta,-}$. We now write

$$F_{\beta,+} = 2\sum_{k=1}^{m-1} s_{2k+2} d\beta_{2k+1} \tag{26}$$

$$C_{\beta,+} = 2\sum_{k=1}^{m-2} (-1)^m s_{2k} d^*\beta_{2k+1} \tag{27}$$

$$F_{\beta,-} = 2\sum_{k=m}^{2m-3} s_{2k+2} d\beta_{2k+1} \tag{28}$$

$$C_{\beta,-} = 2\sum_{k=m-1}^{2m-3} (-1)^{m+1} s_{2k} d^*\beta_{2k+1} \tag{29}$$

The $\pm$ in the suffices $F_{\beta,\pm}, C_{\beta,\pm}$ are related to whether the β_k involved appear in the Dirac operator acting on positive or negative spinors.

Definition 5. Let M^{4m-2} be an oriented Riemannian $(4m-2)$-manifold with Spin^c -structure. The $(4m-2)$-dimensional Seiberg–Witten equations on M for $(A, B, \beta, \phi, \psi)$ are

$$D_{A,\beta,+}\phi = 0, \tag{30}$$

$$F_A + F_{\beta,+} + C_{\beta,+} = q(\phi), \tag{31}$$

$$D_{B,\beta,-}\psi = 0, \tag{32}$$

$$F_A + F_{\beta,-} + C_{\beta,-} = q(\psi), \tag{33}$$

where $D_{A,\beta,+}$, $D_{B,\beta,-}$, $F_{\beta,\pm}$ and $C_{\beta,\pm}$ are defined in equations (24)–(29)

Once again, the point is that the Dirac operators $D_{A,\beta,+}$ and $D_{B,\beta,-}$ have Weitzenböck formulae (see §IV) in which the principal parts of the remainders are precisely $\frac{1}{2}(F_A+F_{\beta,+}+C_{\beta,+})$ and $\frac{1}{2}(F_B + F_{\beta,-} + C_{\beta,-})$ respectively.

Since there are two spinors and two connections in play, the appropriate gauge group is now $\mathcal{G} = \mathrm{Map}(M, S^1 \times S^1)$ where the first factor acts by pull-back on (A, ϕ), the second by pull-back on (B, ψ) and both factors act trivially on β. This action preserves the above equations. With this in mind, the equations (30)–(33) are elliptic modulo gauge, something which essentially comes down to ellipticity of the de Rahm complex again. The details are found in §III but we explain here how the numerology works out in dimension 6. The important variables here are the connections A, B and the 3-form β. (The Dirac equations are already elliptic and so ϕ, ψ do not concern us for this discussion.) Each curvature equation is 15 constraints, whilst fixing for both factors in the gauge action brings another 2 constraints, making $15 + 15 + 2 = 32$ in total; meanwhile, each connection gives 6 degrees of freedom (the rank of Λ^1) and the rank of Λ^3 is 20, so (A, B, β_3) corresponds to $6+6+20 = 32$ degrees of freedom, which is equal to the number of constraints.

Remarks 6. We make three remarks concerning the $4m-2$-dimensional equations.

1. We could equally have taken both spinors to be sections of S_+. One reason to take sections of both S_+ and S_- is that in dimension $4m-2$, geometric structures do not single out a

"preferred orientation". For example, in dimension 4, a symplectic manifold (M, ω) has a preferred orientation, given by ω^2 and typically M will not have a symplectic structure inducing the opposite orientation. However, if $\dim M = 6$, then ω and $-\omega$ induce opposite orientations. Since geometric structures do not single out an orientation in dimension $4m-2$, we choose the same behaviour for the Seiberg–Witten equations. This choice affects the index of the equations, as computed in §III, but not the analytic estimates in §V.

2. Another interesting choice is to begin with *two* Spin^c-structures $S_\pm$ and $W_\pm$. We may then take $A \in \mathcal{A}(\det S_+)$, $\phi \in \Gamma(S_+)$ whilst $B \in \mathcal{A}(\det W_-)$, $\psi \in \Gamma(W_-)$. Again, this choice will affect the index of the equations, but not the analytic estimates proved later. This choice is convenient in our construction of examples of solutions of the equations over Kähler threefolds.

3. When $M^{4m-2} = X \times \mathbb{R}$ is a Riemannian product, one can certainly consider solutions to the equations over M which are invariant in the $\mathbb{R}$ direction. This gives a system of equations on the odd-dimensional manifold X. Unlike dimensional reduction from $4m$ to $4m-1$ dimensions, however, this time the resulting equations on X are more complicated than the odd-dimensional Seiberg–Witten equations as in Definition 1.

   One could also dimensionally reduce the $(4m-1)$-dimensional Seiberg–Witten equations on $M^{4m-2} \times \mathbb{R}$ to obtain a system on M^{4m-2}. This also gives a system of equations which is more complicated that the $(4m-2)$-dimensional Seiberg–Witten equations of Definition 5.

Part III

Ellipticity and the index

Proposition 7. *The odd-dimesional Seiberg–Witten equations* (14) *and* (15) *over* M^{2m+1} *are elliptic modulo gauge, with index zero.*

Proof. The equations (14), (15) define a map

$$\begin{aligned} \mathrm{SW}\colon \mathcal{A} \times \Omega^{\mathrm{odd},\geq 3} \times \Gamma(S) &\to \left(i\Omega^2 \oplus \Omega^4 \oplus \cdots \oplus s_{2m}\Omega^{2m}\right) \times \Gamma(S) \\ \mathrm{SW}(A,\beta,\phi) &= (F_A + F_\beta + C_\beta - q(\phi), D_{A,\beta}\phi) \end{aligned}$$

Suppose $\delta(A,\beta,\phi) = (2ia, b, \sigma)$ is an infinitesimal perturbation of (A,β,ϕ), where $a \in \Omega^1$, $b \in \Omega^{\mathrm{odd},\geq 3}$ and $\sigma \in \Gamma(S)$. As a, b vary, $a+b$ fills out the space Ω^{odd} of *all* odd degree forms. The linearisation of SW at (A,β,ϕ) is

$$\begin{aligned} d_{(A,\beta,\phi)}\,\mathrm{SW}\colon \Omega^{\mathrm{odd}} \oplus \Gamma(S) &\to \left(i\Omega^2 \oplus \Omega^4 \oplus \cdots \oplus s_{2m}\Omega^{2m}\right) \oplus \Gamma(S) \\ d_{(A,\beta,\phi)}\,\mathrm{SW}(a+b,\sigma) &= (2ida + F_b + C_b - d_\phi q(\sigma), D_{A,\beta}\sigma + c(ia+b)\phi) \end{aligned}$$

(Note F_β and C_β are linear in β and so equal to their own derivative.) We supplement this with the Coulomb gauge condition $2d^*\colon \Omega^1 \to \Omega^0$ and discard the zeroth order terms $d_\phi q(\psi)$ and $c(ia+b)\phi$ which do not affect ellipticity or the index. This leaves the map

$$\begin{aligned} L\colon \Omega^{\mathrm{odd}} \oplus \Gamma(S) &\to s\Omega^{\mathrm{even}} \oplus \Gamma(S) \\ L(a,b,\sigma) &= (2d^*a + 2ida + F_b + C_b, D_A\sigma) \end{aligned}$$

where $s\Omega^{\mathrm{even}} = \bigoplus_{k=0}^{m} s_{2k}\Omega^{2k}$. The point is that the first component, $2(d^*a + ida) + F_b + C_b$, is essentially the operator $2(d+d^*)$ acting on $a+b$, just with some extraneous signs and factors of i. This doesn't affect invertibility of the symbol, nor does it change the kernel and cokernel. Since both $d+d^*$ and D_A are elliptic with index zero [1], the same is true for L. $\square$

Proposition 8. *The* $4m$*-dimensional Seiberg–Witten equations* (20) *and* (21) *are elliptic modulo gauge, with index:*

$$-(1 - b_1 + b_2 - \cdots - b_{2m-1} + b_{2m}^+) + 2\int_M c_1(L) \wedge \mathrm{Td}(M)$$

Proof. The equations (20) and (21) define a map

$$\begin{aligned} \mathrm{SW}\colon \mathcal{A} \times \left(\Omega^3 \oplus \Omega^5 \oplus \cdots \oplus \Omega^{2m-1}\right) \times \Gamma(S_+) &\to \left(i\Omega^2 \oplus \Omega^4 \oplus \cdots \oplus s_{2m}\Omega^{2m}_+\right) \times \Gamma(S_+) \\ \mathrm{SW}(A,\beta,\phi) &= \left(F_A + F_\beta^+ + C_\beta - q(\phi), D_{A,\beta}\phi\right) \end{aligned}$$

Following the same arguments as in the proof of Proposition 7, we conclude that the ellipticity of the equations modulo gauge is equivalent to that of the operator

$$\begin{aligned} L\colon \Omega^1 \oplus \Omega^3 \oplus \cdots \oplus \Omega^{2m-1} \oplus \Gamma(S_+) &\to \Omega^0 \oplus i\Omega^2 \oplus \cdots \oplus s_{2m}\Omega^{2m}_+ \oplus \Gamma(S_-) \\ L(a+b,\sigma) &= \left(2d^*a + 2ida + F_b^+ + C_b, D_A\sigma\right) \end{aligned}$$

where F_b^+ and C_b are defined by (18) and (19) respectively. The point now is that the first component, $(2d^*a + ida) + F_b^+ + C_b$ is essentially the elliptic operator corresponding to the truncated de Rahm complex

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots \to \Omega^{2m-1} \xrightarrow{d^+} \Omega^{2m}_+$$

The fact that L has some additional signs and factors of i affects neither the invertibility of the symbol nor the kernel and cokernel. It follows that the equations are elliptic and have the same index as $\operatorname{ind}(D_A) - \big(\sum_{k=0}^{2m-1}(-1)^k b_k + b_{2m}^+\big)$. The result now follows from the Atiyah–Singer index theorem [1]. $\square$

Proposition 9. *The* $4m-2$*-dimensional Seiberg–Witten equations* (30), (31), (32) *and* (33) *are elliptic modulo gauge, with index* $-\chi(M)$*,* $\chi(M)$ *being the Euler characteristic of* M*.*

Proof. The equations (30), (31), (32), (33) define a map

$$\begin{aligned}
&SW\colon \mathcal{A}\times\mathcal{A}\times\bigoplus_{k=1}^{2m-3}\Omega^{2k+1}\times\Gamma(S_+)\times\Gamma(S_-)\to\bigoplus_{k=1}^{2m-2}s_{2k}\Omega^{2k}\times\Gamma(S_-)\times\Gamma(S_+)\\
&SW(A,B,\beta,\phi,\psi)\\
&=\big((F_A+F_{\beta,+}+C_{\beta,+}-q(\phi))+(F_B+F_{\beta,-}+C_{\beta,-}-q(\psi)),D_{A,\beta,+}\phi,D_{B,\beta,-}\psi\big)
\end{aligned}$$

$\mathcal{A}$ is an affine space modelled on $i\Omega^1$, or equivalently on $i\Omega^{4m-3}$ (using the isomorphism $*:\Omega^1\to\Omega^{4m-3}$). Suppose $\delta(A,B,\beta,\phi,\psi)=(2ia,2i\tilde{a},b,\sigma,\xi)$ is an infinitesimal perturbation of (A,B,β,ϕ,ψ), where $a\in\Omega^1,\tilde{a}\in\Omega^{4m-3},b\in\oplus_{k=1}^{2m-3}\Omega^{2k+1},\sigma\in\Gamma(S_+)$ and $\xi\in\Gamma(S_-)$. As $a,\tilde{a}$ and b vary, $a+b+\tilde{a}$ fills out the space Ω^{odd} of *all* odd degree forms. The linearisation of SW at (A,B,β,ϕ,ψ) is

$$\begin{aligned}
&d_{(A,B,\beta,\phi,\psi)}SW\colon\Omega^{\mathrm{odd}}\oplus\Gamma(S_+)\oplus\Gamma(S_-)\to\big(i\Omega^2\oplus\Omega^4\oplus\cdots\oplus\Omega^{4m-4}\big)\oplus\Gamma(S_-)\oplus\Gamma(S_+)\\
&d_{(A,B,\beta,\phi,\psi)}SW(a+b+\tilde{a},\sigma,\xi)\\
&=\Big(2ida+F_{b,+}+C_{b,+}-d_\phi q(\sigma)+F_{b,-}+C_{b,-}+2id^*\tilde{a}-d_\psi q(\xi),\\
&\quad D_{A,\beta,+}\sigma+c(ia)\phi+\sum_{k=1}^{m-2}\big(s_{2k+1}c(b_{2k+1})+s_{4m-2k-3}c(*b_{2k+1})\big)\phi+s_{2m-1}c(b_{2m-1})\phi,\\
&\quad D_{B,\beta,-}\xi+c(i\tilde{a})\psi+s_{2m-1}c(*b_{2m-1})\psi+\sum_{k=m}^{2m-3}\big(s_{2k+1}c(b_{2k+1})+s_{4m-2k-3}c(*b_{2k+1})\big)\psi\Big)
\end{aligned}$$

We supplement this with the Coulomb gauge condition $2d^*\colon\Omega^1\to\Omega^0$ or equivalently, $-2d:\Omega^{4n-3}\to\Omega^{4m-2}$ (using Hodge-star) and discard the zeroth order terms which do not affect ellipticity or the index. This leaves the map

$$\begin{aligned}
L\colon\Omega^{\mathrm{odd}}\oplus\Gamma(S_+)\oplus\Gamma(S_-)\to s\Omega^{\mathrm{even}}\oplus\Gamma(S_-)\oplus\Gamma(S_+)\\
L(a,b,\tilde{a},\sigma,\xi)=(2d^*a+2ida+F_{b,+}+C_{b,+}+F_{b,-}+C_{b,-}+2d^*\tilde{a}-2id\tilde{a},D_A\sigma,D_B\xi)
\end{aligned}$$

where $s\Omega^{\mathrm{even}}=\bigoplus_{k=0}^{2m-1}s_{2k}\Omega^{2k}$. The point is that the first component, $2(d^*a+ida)+F_{b,+}+C_{b,+}+F_{b,-}+C_{b,-}+2(d^*\tilde{a}-id\tilde{a})$, is essentially the operator $2(d+d^*)$ acting on $a+b+\tilde{a}$, just with some extra signs and factors of i. This doesn't affect invertibility of the symbol, nor does it change the kernel and cokernel. Since all three operators $d+d^*$, D_A and D_B are elliptic, the same is true for L. The index of $2(d+d^*)$ is $-\big(\sum_{k=0}^{4m-2}(-1)^k b_k\big)=-\chi(M)$. The index of the two Dirac operators cancel each other since, index of $D_A=-$ index of $D_A^*=-$ index of D_B. So index of D_A+ index of $D_B=0$. Hence the total index is $-\chi(M)$. $\square$

Part IV

The Weitzenböck formulae

Let (M, g) be a Riemannian manifold of dimension n and (E, h) be a hermitian vector bundle on M. We also pick a unitary connection A on E. In the following Lemma and throughout this section we have picked a coframe e_j which is stationary at a point $p \in M$ with respect to the Levi-Civita connection. We write ∇_j for the corresponding directional derivative in the direction dual to e_j with respect to ∇_A.

Lemma 10. *Let $\widetilde{B} \in \Omega^1(\mathfrak{u}(E))$, be locally given by $\widetilde{B} = \sum_j e_j \otimes B_j$ for $B_j \in \Omega^0(\mathfrak{u}(E))$. Then*

$$(\nabla_A + \widetilde{B})^*(\nabla_A + \widetilde{B}) = \nabla_A^* \nabla_A - 2\sum_j B_j \circ \nabla_j - \sum_j (\nabla_j(B_j) + B_j^2).$$

Proof. For $\phi \in \Gamma(E)$,

$$\begin{aligned}(\nabla_A + \widetilde{B})^*(\nabla_A + \widetilde{B})(\phi) &= \nabla_A^* \nabla_A \phi + \nabla_A^*(\widetilde{B}\phi) + \widetilde{B}^*(\nabla_A \phi) + \widetilde{B}^* \widetilde{B}\phi \\ &= \nabla_A^* \nabla_A \phi - \sum_j \nabla_j(B_j \phi) - \sum_j B_j(\nabla_j \phi) - \sum_j B_j^2 \phi \\ &= \nabla_A^* \nabla_A \phi - \sum_j \nabla_j(B_j)\phi - 2\sum_j B_j(\nabla_j \phi) - \sum_j B_j^2 \phi \qquad \square\end{aligned}$$

This lemma will be used in the calculation of all of the Weitzenböck formulae below. We follow the notation used in §I. For a Spin^c-manifold M^n with $S \to M$ being the spin bundle of a Spin^c-structure, a unitary connection A on the determinant bundle L determines a connection on S. We change the associated Dirac operator D_A by a form $\beta \in \Omega^*(M; \mathbb{C})$ (possibly with inhomogeneous degree) and define:

$$D_{A,\beta} := D_A + c(\beta)$$

Proposition 11. *Let $\widetilde{B} = -\frac{1}{2}\sum_j e_j \otimes (c(e_j) \circ c(\beta) + c(\beta)^* \circ c(e_j))$, then the connection $\nabla_{A,\beta} := \nabla_A + \widetilde{B}$ and the Dirac operator $D_{A,\beta}$ satisfies the Weitzenböck formula:*

$$\begin{aligned}D_{A,\beta}^* D_{A,\beta} = \nabla_{A,\beta}^* \nabla_{A,\beta} + \frac{s}{4} + \frac{1}{2}c(F_A) + \frac{1}{2}\sum_j (c(e_j) \circ c(\nabla_j \beta) - c(\nabla_j \beta)^* \circ c(e_j)) \\ + \frac{1}{4}\sum_j (c(e_j) \circ c(\beta) + c(\beta)^* \circ c(e_j))^2 + c(\beta)^* \circ c(\beta) \qquad (34)\end{aligned}$$

where s is the scalar curvature of M.

Proof.

$$
\begin{aligned}
&D^*_{A,\beta}D_{A,\beta}\\
&= (D_A + c(\beta)^*)(D_A + c(\beta))\\
&= D_A^2 + D_A \circ c(\beta) + c(\beta)^* \circ D_A + c(\beta)^* \circ c(\beta)\\
&= \nabla_A^*\nabla_A + \frac{s}{4} + \frac{1}{2}c(F_A) + \sum_j c(e_j)\circ c(\nabla_j\beta) + \sum_j c(e_j)\circ c(\beta)\circ\nabla_j + \sum_j c(\beta)^*\circ c(e_j)\circ\nabla_j\\
&\quad + c(\beta)^*\circ c(\beta)\\
&= \big(\nabla_A^*\nabla_A + \sum_j (c(e_j)\circ c(\beta) + c(\beta)^*\circ c(e_j))\circ\nabla_j + \frac{1}{2}\sum_j (c(e_j)\circ c(\nabla_j\beta) + c(\nabla_j\beta)^*\circ c(e_j))\\
&\quad - \frac{1}{4}\sum_j (c(e_j)\circ c(\beta) + c(\beta)^*\circ c(e_j))^2\big) + \frac{s}{4} + \frac{1}{2}c(F_A) + c(\beta)^*\circ c(\beta)\\
&\quad + \frac{1}{2}\sum_j \big(c(e_j)\circ c(\nabla_j\beta) - c(\nabla_j\beta)^*\circ c(e_j)\big) + \frac{1}{4}\sum_j (c(e_j)\circ c(\beta) + c(\beta)^*\circ c(e_j))^2\\
&= \nabla^*_{A,\beta}\nabla_{A,\beta} + \frac{s}{4} + \frac{1}{2}c(F_A) + \frac{1}{2}\sum_j (c(e_j)\circ c(\nabla_j\beta) - c(\nabla_j\beta)^*\circ c(e_j))\\
&\quad + \frac{1}{4}\sum_j (c(e_j)\circ c(\beta) + c(\beta)^*\circ c(e_j))^2 + c(\beta)^*\circ c(\beta)
\end{aligned}
$$

□

Notice that the last two terms in the Weitzenböck formula (34) are quadratic in β and do not involve derivatives of β. The term $\frac{1}{2}c(F_A) + \frac{1}{2}\sum_j (c(e_j)\circ c(\nabla_j\beta) - c(\nabla_j\beta)^*\circ c(e_j))$ is the principal part of the Weitzenböck remainder. In the remaining part of this section, we will explicitly calculate the principal parts for different choices of β in different dimensions. Before we proceed, we would like to fix some conventions regarding Clifford multiplications [10],[21]: For $\eta \in \Omega^1, \gamma \in \Omega^k$,

$$
\begin{aligned}
c(\eta)\circ c(\gamma) + (-1)^{k+1}c(\gamma)\circ c(\eta) &= -2c(\eta\lrcorner\gamma)\\
c(\eta\wedge\gamma) &= c(\eta)\circ c(\gamma) + c(\eta\lrcorner\gamma)
\end{aligned}
$$

$\lrcorner$ is the contraction operator defined in the following way. At a point $p \in M$,

$$
\eta\lrcorner(e_1\wedge\cdots\wedge e_k) := \sum_{i=1}^{k} (-1)^{i-1}\langle e_i, \eta\rangle e_1\wedge\cdots\wedge\cdots\wedge\hat{e_i}\wedge\cdots\wedge e_k
$$

Hence, for $\gamma \in \Omega^{odd}$,

$$
c(d\gamma) = c\big(\sum_j e_j\wedge\nabla_j\gamma\big) = \frac{1}{2}\sum_j \big(c(e_j)\circ c(\nabla_j\gamma) - c(\nabla_j\gamma)\circ c(e_j)\big) \tag{35}
$$

$$
c(d^*\gamma) = c\big(-\sum_j e_j\lrcorner\nabla_j\gamma\big) = \frac{1}{2}\sum_j \big(c(e_j)\circ c(\nabla_j\gamma) + c(\nabla_j\gamma)\circ c(e_j)\big) \tag{36}
$$

and for $\gamma \in \Omega^{\text{even}}$,

$$c(d\gamma) = c\big(\sum_j e_j \wedge \nabla_j\gamma\big) = \frac{1}{2}\sum_j \big(c(e_j)\circ c(\nabla_j\gamma) + c(\nabla_j\gamma)\circ c(e_j)\big) \tag{37}$$

$$c(d^*\gamma) = c\big(-\sum_j e_j \lrcorner \nabla_j\gamma\big) = \frac{1}{2}\sum_j \big(c(e_j)\circ c(\nabla_j\gamma) - c(\nabla_j\gamma)\circ c(e_j)\big). \tag{38}$$

0.3.1 Dimension 3

$\beta = i*\beta_3, \beta_3 \in \Omega^3$, $\nabla_{A,\beta} = \nabla_A - \frac{i}{2}\sum_j e_j \otimes (c(e_j)\circ c(*\beta_3) - c(*\beta_3)\circ c(e_j)) = \nabla_A$. We calculate the principal part of the Weitzenböck formula:

$$\begin{aligned}
&\frac{1}{2}c(F_A) + \frac{i}{2}\sum_j \big(c(e_j)\circ c(\nabla_j * \beta_3) + c(\nabla_j * \beta_3)\circ c(e_j)\big)\\
&= \frac{1}{2}c(F_A) + c(id * \beta_3)\\
&= \frac{1}{2}c(F_A - 2id^*\beta_3)
\end{aligned}$$

The Weitzenböck formula reads:

$$D^*_{A,\beta}D_{A,\beta} = \nabla^*_A\nabla_A + \frac{s}{4} + \frac{1}{2}c(F_A - 2id^*\beta_3) + c(\beta_3)^2 \tag{39}$$

$$= \nabla^*_A\nabla_A + \frac{s}{4} + \frac{1}{2}c(F_A - 2id^*\beta_3) + |\beta_3|^2 \tag{40}$$

0.3.2 Dimension 4

$\beta = 0, \nabla_{A,\beta} = \nabla_A$. The Weitzenböck formula (on a positive spinor) reads:

$$D^2_A = \nabla^*_A\nabla_A + \frac{s}{4} + \frac{1}{2}c(F^+_A) \tag{41}$$

0.3.3 Dimension 5

$\beta = \beta_3 - *\beta_3 + i*\beta_5, \beta_l \in \Omega^l$. Notice $c\big(\beta_3 - *\beta_3 + i*\beta_5\big) = c\big((1-i)\beta_3 + i*\beta_5\big)$.

Say in our chosen coordinate neighbourhood, $\beta_3 = \sum\limits_{i_1<i_2<i_3} \beta_{i_1i_2i_3} e_{i_1}\wedge e_{i_2}\wedge e_{i_3}$, we get

$$\begin{aligned}
&\sum_{j=1}^6 c(e_j)\circ c\big(\sum_{i_1<i_2<i_3}\beta_{i_1i_2i_3}e_{i_1}\wedge e_{i_2}\wedge e_{i_3}\big)\circ c(e_j)\\
&= \sum_{i_1<i_2<i_3}\beta_{i_1i_2i_3}\big(\sum_{j=1}^6 c(e_j)\circ c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})\circ c(e_j)\big)
\end{aligned}$$

We notice that for $j \in \{i_1, i_2, i_3\}, c(e_j)\circ c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})\circ c(e_j) = -c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})$ and for $j \notin \{i_1, i_2, i_3\}, c(e_j)\circ c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})\circ c(e_j) = c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})$. So,

$$\sum_j (c(e_j)\circ c(\beta_3)\circ c(e_j)) = -c(\beta_3)$$

$c(\beta_3)^2$ will potentially give us Clifford actions of real $0, 2$ and 4-forms. Notice that $c(\beta_3)^2$ is a Hermitian endomorphism of the positive spinors, hence we only get to see the Clifford actions of a 0-form and a 4-form, since Clifford actions of real 2-forms are skew-Hermitian. For the 0-forms, we have $\big(c(e_{i_1}) \circ c(e_{i_2}) \circ c(e_{i_3})\big)^2 = 1$. Hence, we get $c(\beta_3)^2 = |\beta_3|^2 + c(\theta_4)$, where θ_4 is a four-form. We notice

$$\begin{aligned}
&\text{for } j \neq i_1 \neq i_2 \neq i_3 \neq i_4,\\
&c(e_j) \circ c(e_{i_1}) \circ c(e_{i_2}) \circ c(e_{i_3}) \circ c(e_{i_4}) \circ c(e_j) = -c(e_{i_1}) \circ c(e_{i_2}) \circ c(e_{i_3}) \circ c(e_{i_4}),\\
&\text{and for } i_1 \neq i_2 \neq i_3 \neq i_4, j \in \{i_1, i_2, i_3, i_4\},\\
&c(e_j) \circ c(e_{i_1}) \circ c(e_{i_2}) \circ c(e_{i_3}) \circ c(e_{i_4}) \circ c(e_j) = c(e_{i_1}) \circ c(e_{i_2}) \circ c(e_{i_3}) \circ c(e_{i_4})
\end{aligned}$$

Hence, $\sum_{j=1}^{5} c(e_j) \circ c(\theta_4) \circ c(e_j) = 3c(\theta_4)$ and $\sum_{j=1}^{5} c(e_j)\big(\sum_{i_1<i_2<i_3} \beta^2_{i_1 i_2 i_3}\big) \circ c(e_j) = -5|\beta_3|^2$. We get,

$$\sum_j c(e_j) \circ c(\beta_3)^2 \circ c(e_j) = -5|\beta_3|^2 + 3c(\theta_4) = 3c(\beta_3)^2 - 8|\beta_3|^2$$

$$\begin{aligned}
c(\beta) \circ c(\beta)^* &= c\big((1-i)\beta_3 + i * \beta_5\big) \circ c\big((1+i)\beta_3 - i * \beta_5\big) = 2c(\beta_3)^2 - 2(*\beta_5)c(\beta_3) + |\beta_5|^2\\
c(\beta)^* \circ c(\beta) &= c\big((1+i)\beta_3 - i * \beta_5\big) \circ c\big((1-i)\beta_3 + i * \beta_5\big) = 2c(\beta_3)^2 - 2(*\beta_5)c(\beta_3) + |\beta_5|^2
\end{aligned}$$

Assembling all the pieces we have,

$$\begin{aligned}
&\sum_j \big(c(e_j) \circ c(\beta) + c(\beta)^* \circ c(e_j)\big)^2\\
&= \sum_j \big(c(e_j) \circ c(\beta) \circ c(\beta)^* \circ c(e_j) - c(\beta)^* \circ c(\beta)\big)\\
&\quad + \big(\sum_j c(e_j) \circ c(\beta) \circ c(e_j)\big) \circ c(\beta) + c(\beta)^* \circ \big(\sum_j c(e_j) \circ c(\beta)^* \circ c(e_j)\big)\\
&= 2(3c(\beta_3)^2 - 8|\beta_3|^2) + 2(*\beta_5)c(\beta_3) - 5|\beta_5|^2 - 5(2c(\beta_3)^2 - 2(*\beta_5)c(\beta_3) + |\beta_5|^2)\\
&\quad + (2ic(\beta_3)^2 + 5|\beta_5|^2 - 6i(1-i)(*\beta_5)c(\beta_3)) + (-2ic(\beta_3)^2 + 5|\beta_5|^2 + 6i(1+i)(*\beta_5)c(\beta_3))\\
&= -4c(\beta_3)^2 - 16|\beta_3|^2
\end{aligned}$$

The principal part of the curvature equations is:

$$\begin{aligned}
&\frac{1}{2}c(F_A) + \frac{1}{2}\sum_j \big(c(e_j) \circ (1-i)\nabla_j\beta_3 - (1+i)\nabla_j\beta_3 \circ c(e_j)\big)\\
&+ \frac{i}{2}\sum_j \big(c(e_j) \circ c(\nabla_j(*\beta_5)) + c(\nabla_j(*\beta_5)) \circ c(e_j)\big)\\
&= \frac{1}{2}c(F_A + 2d\beta_3 - 2id^*\beta_3 + 2id(*\beta_5))\\
&= \frac{1}{2}c(F_A + 2d\beta_3 - 2id^*\beta_3 + 2d^*\beta_5)
\end{aligned}$$

Finally, we get the Weitzenböck formula

$$\begin{aligned}D^*_{A,\beta}D_{A,\beta} =&\nabla^*_{A,\beta}\nabla_{A,\beta} + \frac{s}{4} + \frac{1}{2}c(F_A + 2d\beta_3 - 2id^*\beta_3 + 2d^*\beta_5)\\ &+ c(\beta_3)^2 - 4|\beta_3|^2 - 2(*\beta_5)c(\beta_3) + |\beta_5|^2\\ =&\nabla^*_{A,\beta}\nabla_{A,\beta} + \frac{s}{4} + \frac{1}{2}c(F_A + 2d\beta_3 - 2id^*\beta_3 + 2d^*\beta_5)\\ &+ c(\beta_3 - *\beta_5)^2 - 4|\beta_3|^2\end{aligned}$$

0.3.4 Dimension 6

$\beta = \beta_3 \in \Omega^3$. Notice $c(\beta) = c(\beta)^*$. First, we calculate the quadratic term:

$$\begin{aligned}&\sum_j \big(c(e_j)\circ c(\beta) + c(\beta)\circ c(e_j)\big)^2\\ =&\sum_j \big(c(e_j)\circ c(\beta)^2\circ c(e_j) + c(\beta)\circ c(e_j)^2\circ c(\beta) + c(e_j)\circ c(\beta)\circ c(e_j)\circ c(\beta)\\ &+ c(\beta)\circ c(e_j)\circ c(\beta)\circ c(e_j)\big)\end{aligned}$$

Say in our chosen coordinate neighbourhood, $\beta = \sum\limits_{i_1<i_2<i_3} \beta_{i_1i_2i_3} e_{i_1}\wedge e_{i_2}\wedge e_{i_3}$, we get

$$\begin{aligned}&\sum_{j=1}^{6} c(e_j)\circ c\big(\sum_{i_1<i_2<i_3}\beta_{i_1i_2i_3}e_{i_1}\wedge e_{i_2}\wedge e_{i_3}\big)\circ c(e_j)\\ &= \sum_{i_1<i_2<i_3}\beta_{i_1i_2i_3}\big(\sum_{j=1}^{6} c(e_j)\circ c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})\circ c(e_j)\big)\end{aligned}$$

We notice that for $j\in\{i_1,i_2,i_3\}, c(e_j)\circ c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})\circ c(e_j) = -c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})$ and for $j\notin\{i_1,i_2,i_3\}, c(e_j)\circ c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})\circ c(e_j) = c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})$.
So, $\sum\limits_{j=1}^{6} c(e_j)\circ c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})\circ c(e_j) = 0$.

$c(\beta)^2$ will potentially give us Clifford actions of $0, 2, 4$ and 6-forms. Notice that $c(\beta_3)^2$ is a Hermitian endomorphism of the positive spinors, hence we only get to see the Clifford actions of a 0-form and a 4-form, since Clifford actions of real 2-forms and 6-forms are skew-Hermitian. For the 0-forms, we have $\big(c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})\big)^2 = 1$. Hence, we get $c(\beta)^2 = |\beta|^2 + c(\theta_4)$, where θ_4 is a four-form. We notice

$$\begin{aligned}&\text{for } j\neq i_1\neq i_2\neq i_3\neq i_4,\\ &c(e_j)\circ c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})\circ c(e_{i_4})\circ c(e_j) = -c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})\circ c(e_{i_4}),\\ &\text{and for } i_1\neq i_2\neq i_3\neq i_4, j\in\{i_1,i_2,i_3,i_4\},\\ &c(e_j)\circ c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})\circ c(e_{i_4})\circ c(e_j) = c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})\circ c(e_{i_4})\end{aligned}$$

Hence, $\sum\limits_{j=1}^{6} c(e_j)\circ c(\theta_4)\circ c(e_j) = 2c(\theta_4)$ and $\sum\limits_{j=1}^{6} c(e_j)\big(\sum\limits_{i_1<i_2<i_3}\beta^2_{i_1i_2i_3}\big)\circ c(e_j) = -6|\beta|^2$. We get,

$$\sum_j c(e_j)\circ c(\beta)^2\circ c(e_j) = -6|\beta|^2 + 2c(\theta_4) = 2c(\beta)^2 - 8|\beta|^2$$

and

$$\sum_j \big(c(e_j)\circ c(\beta)+c(\beta)\circ c(e_j)\big)^2 = -6c(\beta)^2 + \sum_j c(e_j)\circ c(\beta)^2\circ c(e_j) = -4c(\beta)^2 - 8|\beta|^2$$

The principal part of the remainder is:

$$\frac{1}{2}c(F_A) + \frac{1}{2}\sum_j \big(c(e_j)\circ c(\nabla_j\beta_3) - c(\nabla_j\beta_3)\circ c(e_j)\big) = \frac{1}{2}c\big(F_A + 2d\beta_3\big)$$

Finally, assembling all the pieces we get the Weitzenböck formula on positive spinors:

$$D^2_{A,\beta} = \nabla^*_{A,\beta}\nabla_{A,\beta} + \frac{s}{4} + \frac{1}{2}c(F_A + 2d\beta_3) - 2|\beta_3|^2 \tag{42}$$

For $\beta = *\beta_3$, the principal part of the remainder is:

$$\begin{aligned}&\frac{1}{2}c(F_B) + \frac{1}{2}\sum_j \big(c(e_j)\circ c(\nabla_j\beta_3) - c(\nabla_j\beta_3)\circ c(e_j)\big)\\ &= \frac{1}{2}c\big(F_B + 2d(*\beta_3)\big)\\ &= \frac{1}{2}c\big(F_B - 2id^*\beta_3\big) \quad [\text{ on } \Gamma(S_-)]\end{aligned}$$

The Weitzenböck formula on negative spinors reads:

$$D^2_{B,\beta} = \nabla^*_{B,\beta}\nabla_{B,\beta} + \frac{s}{4} + \frac{1}{2}c(F_B - 2id^*\beta_3) - 2|\beta_3|^2 \tag{43}$$

0.3.5 Dimension 7

$\beta = (\beta_3 + i*\beta_3) + (i\beta_5 - *\beta_5) + i*\beta_7, \beta_l \in \Omega^l$.
Notice $c\big((\beta_3 + i*\beta_3) + (i\beta_5 - *\beta_5) + i*\beta_7\big) = c\big((1+i)\beta_3 + (1+i)\beta_5 + i*\beta_7\big)$

$$\begin{aligned}&\sum_j \big(c(e_j)\circ c(\beta) + c(\beta)^*\circ c(e_j)\big)^2\\ &= \sum_j \big((1+i)c(e_j)\circ c(\beta_3+\beta_5) + i(*\beta_7)c(e_j) + (1-i)c(\beta_3-\beta_5)\circ c(e_j) - i(*\beta_7)c(e_j)\big)^2\\ &= \sum_j \big((1+i)c(e_j)\circ c(\beta_3+\beta_5) + (1-i)c(\beta_3-\beta_5)\circ c(e_j)\big)^2\\ &= 2i\Big(\sum_j c(e_j)\circ c(\beta_3+\beta_5)\circ c(e_j)\Big)\circ c(\beta_3+\beta_5)\\ &\quad - 2ic(\beta_3-\beta_5)\circ\Big(\sum_j c(e_j)\circ c(\beta_3-\beta_5)\circ c(e_j)\Big)\\ &\quad + 2\sum_j c(e_j)\circ c(\beta_3+\beta_5)\circ c(\beta_3-\beta_5)\circ c(e_j) - 14c(\beta_3-\beta_5)\circ c(\beta_3+\beta_5)\end{aligned}$$

Say in our chosen coordinate neighbourhood, $\beta_3 = \sum_{i_1<i_2<i_3} \beta_{i_1i_2i_3} e_{i_1} \wedge e_{i_2} \wedge e_{i_3}$, we get

$$\begin{aligned}&\sum_{j=1}^{7} c(e_j)\circ c\Big(\sum_{i_1<i_2<i_3} \beta_{i_1i_2i_3} e_{i_1} \wedge e_{i_2} \wedge e_{i_3}\Big)\circ c(e_j)\\ &= \sum_{i_1<i_2<i_3} \beta_{i_1i_2i_3}\Big(\sum_{j=1}^{7} c(e_j)\circ c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})\circ c(e_j)\Big)\end{aligned}$$

Notice that for $j \in \{i_1, i_2, i_3\}$, $c(e_j)\circ c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})\circ c(e_j) = -c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})$ and for $j \notin \{i_1, i_2, i_3\}$, $c(e_j)\circ c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})\circ c(e_j) = c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})$.

Hence $\sum_{j=1}^{7} c(e_j)\circ c(\beta_3)\circ c(e_j) = c(\beta_3)$.

And if $\beta_5 = \sum_{i_1<i_2<i_3<i_4<i_5} \gamma_{i_1i_2i_3i_4i_5} e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge e_{i_4} \wedge e_{i_5}$,

$$\begin{aligned}&\sum_{j=1}^{7} c(e_j)\circ c(\beta_5)\circ c(e_j)\\ &= \sum_{i_1<i_2<i_3<i_4<i_5} \gamma_{i_1i_2i_3i_4i_5}\Big(\sum_{j=1}^{7} c(e_j)\circ c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})\circ c(e_{i_4})\circ c(e_{i_5})\circ c(e_j)\Big)\end{aligned}$$

Notice that for $j \in \{i_1, i_2, i_3, i_4, i_5\}$,
$c(e_j)\circ c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})\circ c(e_{i_4})\circ c(e_{i_5})\circ c(e_j) = -c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})\circ c(e_{i_4})\circ c(e_{i_5})$
and for $j \notin \{i_1, i_2, i_3, i_4, i_5\}$,
$c(e_j)\circ c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})\circ c(e_{i_4})\circ c(e_{i_5})\circ c(e_j) = c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})\circ c(e_{i_4})\circ c(e_{i_5})$

Hence, $\sum_{j=1}^{7} c(e_j)\circ c(\beta_5)\circ c(e_j) = -3c(\beta_5)$

$c(\beta_3)^2$ will potentially give us Clifford actions of $0, 2, 4$ and 6-forms. Notice that $c(\beta_3)^2$ is a Hermitian endomorphism of the positive spinors, hence we only get to see the Clifford actions of a 0-form and a 4-form, since Clifford actions of real 2-forms and 6-forms are skew-Hermitian. For the 0-forms, we have $\big(c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})\big)^2 = 1$. Hence, we get $c(\beta_3)^2 = |\beta_3|^2 + c(\theta_4)$, where θ_4 is a four-form. We notice

for $j \neq i_1 \neq i_2 \neq i_3 \neq i_4$,
$c(e_j)\circ c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})\circ c(e_{i_4})\circ c(e_j) = -c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})\circ c(e_{i_4})$,
and for $i_1 \neq i_2 \neq i_3 \neq i_4, j \in \{i_1, i_2, i_3, i_4\}$,
$c(e_j)\circ c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})\circ c(e_{i_4})\circ c(e_j) = c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})\circ c(e_{i_4})$

Hence, $\sum_{j=1}^{7} c(e_j)\circ c(\theta_4)\circ c(e_j) = c(\theta_4)$ and $\sum_{j=1}^{7} c(e_j)\Big(\sum_{i_1<i_2<i_3} \beta^2_{i_1i_2i_3}\Big)c(e_j) = -7|\beta_3|^2$.

We get

$$\sum_{i=1}^{7} c(e_j)\circ c(\beta_3)^2\circ c(e_j) = -7|\beta_3|^2 + c(\theta_4) = c(\beta_3)^2 - 8|\beta_3|^2$$

Similarly, $c(\beta_5)^2$ will potentially give us Clifford actions of real $0, 2, 4$ and 6-forms. Notice that $c(\beta_5)^2$ is a Hermitian endomorphism of the positive spinors, hence we only get to see the Clifford actions of a 0-form and a 4-form, since Clifford actions of real 2-forms and 6-forms are skew-Hermitian.

For the 0-form, we observe that for $i_1 \neq i_2 \neq i_3 \neq i_4 \neq i_5$,

$$\big(c(e_{i_1}) \circ c(e_{i_2}) \circ c(e_{i_3}) \circ c(e_{i_4}) \circ c(e_{i_5})\big) \circ \big(c(e_{i_1}) \circ c(e_{i_2}) \circ c(e_{i_3}) \circ c(e_{i_4}) \circ c(e_{i_5})\big) = -\operatorname{Id}$$

Hence, $c(\beta_5)^2 = -|\beta_5|^2 + c(\gamma_4), \gamma_4 \in \Omega^4$.

$$\sum_j c(e_j) \circ c(\beta_5)^2 \circ c(e_j) = 7|\beta_5|^2 + c(\gamma_4) = c(\beta_5)^2 + 8|\beta_5|^2$$

$\big(c(\beta_3) \circ c(\beta_5) - c(\beta_5) \circ c(\beta_3)\big)$ will potentially give us Clifford actions of $2, 4$ and 6-forms. Notice that $\big(c(\beta_3) \circ c(\beta_5) - c(\beta_5) \circ c(\beta_3)\big)$ is a Hermitian endomorphism of spinors. Since Clifford actions of real 2 and 6-forms are skew-Hermitian endomorphisms, we only get to see Clifford action of a 4-form. So, we have

$$\sum_j c(e_j) \circ \big(c(\beta_3) \circ c(\beta_5) - c(\beta_5) \circ c(\beta_3)\big) \circ c(e_j) = \big(c(\beta_3) \circ c(\beta_5) - c(\beta_5) \circ c(\beta_3)\big)$$

Hence

$$\begin{aligned}
&\sum_j \big(c(e_j) \circ c(\beta) + c(\beta)^* \circ c(e_j)\big)^2 \\
&= -12\big(c(\beta_3)^2 - c(\beta_5)^2\big) - 16(|\beta_3|^2 + |\beta_5|^2) - 16\big(c(\beta_3) \circ c(\beta_5) - c(\beta_5) \circ c(\beta_3)\big) \\
&\quad - 4i\big(c(\beta_3) \circ c(\beta_5) + c(\beta_5) \circ c(\beta_3)\big)
\end{aligned}$$

$$\begin{aligned}
&c(\beta)^* c(\beta) \\
&= \big((1-i)c(\beta_3 - \beta_5) - i(*\beta_7)\big) \circ \big((1+i)c(\beta_3 + \beta_5) + i(*\beta_7)\big) \\
&= 2\big(c(\beta_3)^2 - c(\beta_5)^2 + c(\beta_3) \circ c(\beta_5) - c(\beta_5) \circ c(\beta_3)\big) + |\beta_7|^2 \\
&\quad + 2 * (\beta_7) c(\beta_3 - i\beta_5)
\end{aligned}$$

The principal part of the remainder is

$$\begin{aligned}
&\frac{1}{2} c\big(F_A + (1+i)c(e_j) \circ c(\nabla_j \beta_3) - (1-i)c(\nabla_j \beta_3) \circ c(e_j)\big) \\
&+ \frac{1}{2} c\big((1+i)c(e_j) \circ c(\nabla_j \beta_5) + (1-i)c(\nabla_j \beta_5) \circ c(e_j)\big) \\
&+ \frac{i}{2}\big(c(e_j) \circ c(\nabla_j(*\beta_7)) + c(\nabla_j(*\beta_7)) \circ c(e_j)\big) \\
&= \frac{1}{2} c\big(F_A + 2d\beta_3 + 2id^*\beta_3 + 2id\beta_5 + 2d^*\beta_5 + 2id(*\beta_7)\big) \\
&= \frac{1}{2} c(F_A + 2d\beta_3 + 2id^*\beta_3 + 2id\beta_5 + 2d^*\beta_5 + 2id^*\beta_7)
\end{aligned}$$

Finally we get the Weitzenböck formula

$$
\begin{aligned}
&D^*_{A,\beta}D_{A,\beta}\\
&= \nabla^*_{A,\beta}\nabla_{A,\beta} + \frac{s}{4} + \frac{1}{2}c(F_A + 2d\beta_3 + 2id^*\beta_3 + 2id\beta_5 + 2d^*\beta_5 + 2id^*\beta_7)\\
&\quad - \big(c(\beta_3)^2 - c(\beta_5)^2\big) - 4(|\beta_3|^2 + |\beta_5|^2) - 2\big(c(\beta_3)\circ c(\beta_5) - c(\beta)_5\circ c(\beta_3)\big)\\
&\quad - i\big(c(\beta_3)\circ c(\beta_5) + c(\beta)_5\circ c(\beta_3)\big) + 2(*\beta_7)\big(c(\beta_3) - ic(\beta_5)\big) + |\beta_7|^2
\end{aligned}
$$

Notice $\big(c(\beta_3)\circ c(\beta_5) - c(\beta)_5\circ c(\beta_3)\big)$ and $i\big(c(\beta_3)\circ c(\beta_5) + c(\beta)_5\circ c(\beta_3)\big)$ give us trace-free Hermitian endomorphisms of the spinors. $\big(c(\beta_3)\circ c(\beta_5) - c(\beta)_5\circ c(\beta_3)\big)$ comes from Clifford action of a real 4-form and $i\big(c(\beta_3)\circ c(\beta_5) + c(\beta)_5\circ c(\beta_3)\big)$ comes from Clifford action of an imaginary 2-form and an imaginary 6-form.

0.3.6 Dimension 8

$\beta = \beta_3 + i*\beta_3, \beta_l \in \Omega^l$. Notice $c(\beta_3 + i*\beta_3) = (1+i)c(\beta_3)$.

$$
\begin{aligned}
&\sum_j \big(c(e_j)\circ c(\beta) + c(\beta)^*\circ c(e_j)\big)^2\\
&\sum_j \big((1+i)c(e_j)\circ c(\beta_3) + (1-i)c(\beta_3)\circ c(e_j)\big)^2\\
&= 2i\Big(\sum_j c(e_j)\circ c(\beta_3)\circ c(e_j)\Big)\circ c(\beta_3) - 2ic(\beta_3)\circ\Big(\sum_j c(e_j)\circ c(\beta_3)\circ c(e_j)\Big)\\
&\quad + 2\sum_j c(e_j)\circ c(\beta_3)^2\circ c(e_j) + 2\sum_j c(\beta_3)\circ c(e_j)^2\circ c(\beta_3)
\end{aligned}
$$

Say in our chosen coordinate neighbourhood, $\beta_3 = \sum_{i_1<i_2<i_3} \beta_{i_1i_2i_3} e_{i_1}\wedge e_{i_2}\wedge e_{i_3}$, we get

$$
\begin{aligned}
&\sum_{j=1}^{8} c(e_j)\circ c\Big(\sum_{i_1<i_2<i_3} \beta_{i_1i_2i_3} e_{i_1}\wedge e_{i_2}\wedge e_{i_3}\Big)\circ c(e_j)\\
&= \sum_{i_1<i_2<i_3} \beta_{i_1i_2i_3}\Big(\sum_{j=1}^{8} c(e_j)\circ c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})\circ c(e_j)\Big)
\end{aligned}
$$

Notice that for $j\in\{i_1,i_2,i_3\}, c(e_j)\circ c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})\circ c(e_j) = -c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})$ and for $j\notin\{i_1,i_2,i_3\}, c(e_j)\circ c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})\circ c(e_j) = c(e_{i_1})\circ c(e_{i_2})\circ c(e_{i_3})$.

$$
\sum_{j=1}^{8} c(e_j)\circ c(\beta_3)\circ c(e_j) = 2c(\beta_3) \tag{44}
$$

$c(\beta_3)^2$ will potentially give us Clifford actions of $0, 2, 4$ and 6-forms. Notice that $c(\beta_3)^2$ is a Hermitian endomorphism of the positive spinors, hence we only get to see the Clifford actions of a 0-form and a 4-form, since Clifford actions of real 2-forms and 6-forms are skew-Hermitian.

For the 0-forms, we have $\left(c(e_{i_1}) \circ c(e_{i_2}) \circ c(e_{i_3})\right)^2 = 1$. Hence, we get $c(\beta_3)^2 = |\beta_3|^2 + c(\theta_4)$, where θ_4 is a four-form. We notice

$$\begin{aligned}
&\text{for } j \neq i_1 \neq i_2 \neq i_3 \neq i_4, \\
&c(e_j) \circ c(e_{i_1}) \circ c(e_{i_2}) \circ c(e_{i_3}) \circ c(e_{i_4}) \circ c(e_j) = -c(e_{i_1}) \circ c(e_{i_2}) \circ c(e_{i_3}) \circ c(e_{i_4}), \\
&\text{and for } i_1 \neq i_2 \neq i_3 \neq i_4, j \in \{i_1, i_2, i_3, i_4\}, \\
&c(e_j) \circ c(e_{i_1}) \circ c(e_{i_2}) \circ c(e_{i_3}) \circ c(e_{i_4}) \circ c(e_j) = c(e_{i_1}) \circ c(e_{i_2}) \circ c(e_{i_3}) \circ c(e_{i_4})
\end{aligned}$$

Hence, $\sum_{j=1}^{8} c(e_j) \circ c(\theta_4) \circ c(e_j) = 0$ and $\sum_{j=1}^{8} c(e_j)\Big(\sum_{i_1<i_2<i_3} \beta^2_{i_1 i_2 i_3}\Big) \circ c(e_j) = -8|\beta_3|^2$. So, we get

$$\sum_{i=1}^{8} c(e_j) \circ c(\beta_3)^2 \circ c(e_j) = -8|\beta_3|^2 \tag{45}$$

The principal part of the remainder is

$$\begin{aligned}
&\frac{1}{2}c(F_A) + \frac{1}{2}\sum_j \big((1+i)c(e_j) \circ c(\nabla_j \beta_3) - (1-i)c(\nabla_j \beta_3) \circ c(e_j)\big) \\
&= \frac{1}{2}c(F_A + 2d\beta_3 + 2d^*\beta_3) \\
&= \frac{1}{2}c(F_A + 2d\beta_3^+ + 2id^*\beta_3) \quad \text{[on positive spinors]}
\end{aligned}$$

Assembling all the pieces we get,

$$\begin{aligned}
&(D_A + c(\beta))^2 \\
&= \nabla^*_{A,\beta}\nabla_{A,\beta} + \frac{s}{4} + \frac{1}{2}c(F_A + d\beta_3^+ + 2id^*\beta_3) - 2c(\beta_3)^2 - 4|\beta_3|^2 \quad [\text{on } \Gamma(S_+)]
\end{aligned}$$

0.3.7 Dimension $2m+1$

Following §0.1, $\beta = \sum_{k=1}^{m-1}\left(s_{2k+1}\beta_{2k+1} + is_{2m-2k} * \beta_{2k+1}\right) + i * \beta_{2m+1}, \beta_l \in \Omega^l$.

$$\begin{aligned}
&\frac{1}{2}\sum_j (c(e_j) \circ c(\nabla_j \beta) - c(\nabla_j \beta)^* \circ c(e_j)) \\
&= \frac{1}{2}\sum_{k=1}^{m-1} s_{2k+1}\sum_j \big(c(e_j) \circ c(\nabla_j \beta_{2k+1}) - c(\nabla_j \beta_{2k+1}) \circ c(e_j)\big) \\
&\quad + \frac{i}{2}\sum_{k=1}^{m} s_{2m-2k}\sum_j \big(c(e_j) \circ c(\nabla_j(*\beta_{2k+1})) + c(\nabla_j(*\beta_{2k+1})) \circ c(e_j)\big) \\
&= \sum_{k=1}^{m-1} s_{2k+2}c(d\beta_{2k+1}) + i\sum_{k=1}^{m} s_{2m-2k}c(d^*\beta_{2k+1}) \\
&= \sum_{k=1}^{m-1} s_{2k+2}c(d\beta_{2k+1}) + \sum_{k=1}^{m} (-1)^{\lfloor \frac{m+1}{2} \rfloor + (m+1)(k+1)} s_{2k}c(d^*\beta_{2k+1})
\end{aligned}$$

The Weitzenböck formula reads

$$
\begin{aligned}
&D_{A,\beta}^{*}D_{A,\beta} \\
&= \nabla_{A,\beta}^{*}\nabla_{A,\beta} + \frac{s}{4} + \frac{1}{2}c\Big(F_A + 2\sum_{k=1}^{m-1} s_{2k+2}d\beta_{2k+1} + 2\sum_{k=1}^{m}(-1)^{\lfloor\frac{m+1}{2}\rfloor+(m+1)(k+1)}s_{2k}d^{*}\beta_{2k+1}\Big) \\
&\quad + \text{quadratic terms in } \beta
\end{aligned}
\tag{46}
$$

0.3.8 Dimension $4m$

Following §0.2, $\beta = \sum_{k=1}^{m-1}(s_{2k+1}\beta_{2k+1} + s_{4m-2k-1} * \beta_{2k+1}), \beta_l \in \Omega^l$.

$$
\begin{aligned}
&\frac{1}{2}\sum_j (c(e_j)\circ c(\nabla_j\beta) - c(\nabla_j\beta)^{*}\circ c(e_j)) \\
&= \frac{1}{2}\sum_{k=1}^{m-1} s_{2k+1}\sum_j \big(c(e_j)\circ c(\nabla_j\beta_{2k+1}) - c(\nabla_j\beta_{2k+1})\circ c(e_j)\big) \\
&\quad + \frac{1}{2}\sum_{k=1}^{m-1} s_{4m-2k-1}\sum_j \big(c(e_j)\circ c(\nabla_j(*\beta_{2k+1})) - c(\nabla_j(*\beta_{2k+1}))\circ c(e_j)\big) \\
&= \sum_{k=1}^{m-1}\big(s_{2k+2}c(d\beta_{2k+1}) + s_{4m-2k-1}c(d(*\beta_{2k+1}))\big) \\
&= \sum_{k=1}^{m-1}\big(s_{2k+2}c(d\beta_{2k+1}) + (-1)^{m+k+1}s_{2k}c(d^{*}\beta_{2k+1})\big) \quad [\text{ on } \Gamma(S_+)]
\end{aligned}
$$

We get the Weitzenböck formula (acting on positive spinors):

$$
\begin{aligned}
&D_{A,\beta}^{2} \\
&= \nabla_{A,\beta}^{*}\nabla_{A,\beta} + \frac{s}{4} + \frac{1}{2}c\Big(F_A + 2\sum_{k=1}^{m-2} s_{2k+2}d\beta_{2k+1} + 2s_{2m}d^{+}\beta_{2m-1} + 2\sum_{k=1}^{m-1}(-1)^{m+k+1}s_{2k}d^{*}\beta_{2k+1}\Big) \\
&\quad + \text{quadratic terms in } \beta
\end{aligned}
\tag{47}
$$

0.3.9 Dimension $4m-2$

Following §0.3, $\beta = \sum_{k=1}^{m-2}(s_{2k+1}\beta_{2k+1} + s_{4m-2k-3} * \beta_{2k+1}) + s_{2m-1}\beta_{2m-1}, \beta_l \in \Omega^l$.

$$\begin{aligned}
&\frac{1}{2}\sum_j (c(e_j)\circ c(\nabla_j\beta) - c(\nabla_j\beta)^*\circ c(e_j))\\
&= \frac{1}{2}\sum_{k=1}^{m-1} s_{2k+1}\Big(\sum_j (c(e_j)\circ c(\nabla_j\beta_{2k+1}) - c(\nabla_j\beta_{2k+1})\circ c(e_j))\Big)\\
&\quad + \frac{1}{2}\sum_{k=1}^{m-2} s_{4m-2k-3}\Big(\sum_j (c(e_j)\circ c(\nabla_j(*\beta_{2k+1})) - c(\nabla_j(*\beta_{2k+1}))\circ c(e_j))\Big)\\
&= \sum_{k=1}^{m-1} s_{2k+2}c(d\beta_{2k+1}) + \sum_{k=1}^{m-2} s_{4m-2k-3}c(d(*\beta_{2k+1}))\\
&= \sum_{k=1}^{m-1} s_{2k+2}c(d\beta_{2k+1}) + \sum_{k=1}^{m-2} (-1)^m s_{2k}c(d^*\beta_{2k+1}) \quad [\text{ on } \Gamma(S_+)]
\end{aligned}$$

The Weitzenböck formula for the positive spinors reads:

$$\begin{aligned}
D^2_{A,\beta} &= \nabla^*_{A,\beta}\nabla_{A,\beta} + \frac{s}{4} + \frac{1}{2}c\Big(F_A + 2\sum_{k=1}^{m-1} s_{2k+2}d\beta_{2k+1} + 2\sum_{k=1}^{m-2}(-1)^m s_{2k}d^*\beta_{2k+1}\Big)\\
&\quad + \text{quadratic terms in } \beta
\end{aligned} \tag{48}$$

If we take $\beta = s_{2m-1} * \beta_{2m-1} + \sum_{k=m}^{2m-3}(s_{2k+1}\beta_{2k+1} + s_{4m-2k-3} * \beta_{2k+1}), \beta_l \in \Omega^l$.

$$\begin{aligned}
&\frac{1}{2}\sum_j (c(e_j)\circ c(\nabla_j\beta) - c(\nabla_j\beta)^*\circ c(e_j))\\
&= s_{2m-1}\frac{1}{2}\sum_j \big(c(e_j)\circ c(\nabla_j * (\beta_{2m-1})) - c(\nabla_j(*\beta_{2m-1}))\circ c(e_j)\big)\\
&\quad + \sum_{k=m}^{2m-3} s_{2k+1}\big(c(e_j)\circ c(\nabla_j\beta_{2k+1}) - c(\nabla_j\beta_{2k+1})\circ c(e_j)\big)\\
&\quad + \sum_{k=m}^{2m-3} s_{4m-2k-3}\big(c(e_j)\circ c(\nabla_j(*\beta_{2k+1})) - c(\nabla_j(*\beta_{2k+1}))\circ c(e_j)\big)\\
&= s_{2m-1}c(d(*\beta_{2m-1})) + \sum_{k=m}^{2m-3} s_{2k+2}c(d\beta_{2k+1}) + \sum_{k=m}^{2m-3} s_{4m-2k-3}c(d(*\beta_{2k+1}))\\
&= \sum_{k=m}^{2m-3} s_{2k+2}c(d\beta_{2k+1}) + \sum_{k=m-1}^{2m-3}(-1)^{m+1}s_{2k}c(d^*\beta_{2k+1}) \quad [\text{ on } \Gamma(S_-)]
\end{aligned}$$

The Weitzenböck formula for the negative spinors reads:

$$\begin{aligned}
D^2_{B,\beta} &= \nabla^*_{B,\beta}\nabla_{B,\beta} + \frac{s}{4} + \frac{1}{2}c\Big(F_B + 2\sum_{k=m}^{2m-3} s_{2k+2}d\beta_{2k+1} + 2\sum_{k=m-1}^{2m-3}(-1)^{m+1}s_{2k}d^*\beta_{2k+1}\Big)\\
&\quad + \text{quadratic terms in } \beta
\end{aligned} \tag{49}$$

Part V

A priori estimates

We prove some a priori estimates for the 8d Seiberg–Witten equations. Similar techniques can be applied to get bounds in other dimensions as well. The Laplacian used here is $\Delta = dd^* + d^*d$. Before we get to the 8d equations, we sketch a proof of compactness of the moduli space of the 4d SW equations. The a priori estimates we prove for the 8d equations are in parallel to the analysis in the 4d case. For more details of the compactness result of 4d SW equations one can consult [20]. The 4d SW equations for $A \in \mathcal{A}$ and $\phi \in \Gamma(S_+)$ on an orientable Riemannian four manifold (M, g) are:

$$D_A\phi = 0 \tag{50}$$

$$F_A^+ = q(\phi) \tag{51}$$

- The first step is to get a uniform C^0 bound of ϕ. We use the Weitzenböck formula:

  $$D_A^2 = \nabla_A^*\nabla_A + \frac{s}{4} + \frac{1}{2}c(F_A^+) \qquad \text{[s is the scalar curvature]}$$

  Evaluating it on (A, ϕ), a solution of the equations (50), (51) and taking pointwise inner product with ϕ yields

  $$\langle \nabla_A^*\nabla_A\phi, \phi\rangle + \frac{s}{4}|\phi|^2 + \frac{1}{4}|\phi|^4 = 0 \tag{52}$$

  We get

  $$\begin{aligned}\frac{1}{2}\Delta|\phi|^2 &= \langle \nabla_A^*\nabla_A\phi, \phi\rangle - |\nabla_A\phi|^2 \\ &= -\frac{s}{4}|\phi|^2 - \frac{1}{4}|\phi|^4 - |\nabla_A\phi|^2\end{aligned}$$

  Thus at a point x_0 where $|\phi|^2$ achieves its maximum, we have

  $$\frac{1}{4}|\phi(x_0)|^4 \le -\frac{s(x_0)}{4}|\phi(x_0)|^2$$

  Hence $\exists C > 0$ such that $||\phi||_{C^0}^2 \le C$.

- Using the C^0 bound of ϕ, we get some more estimates:

  $$\begin{aligned}||F_A^+||_{C^0} = ||q(\phi)||_{C^0} &\le \frac{1}{2}||\phi||_{C^0}^2 \le \frac{C}{2} \\ \text{and } ||F_A^+||_{L^2}^2 &\le \frac{C^2}{4}\text{vol}(M)\end{aligned}$$

  Now if we assume that the virtual dimension of the moduli space is compact i.e.,

  $$c_1(L)^2 - 2\big(\chi(M) + 3\ \text{sign}(M)\big) \ge 0$$

  [L is the determinant bundle of the positive Spin^c bundle]

  and also use the identity

  $$c_1(L)^2 = \frac{1}{4\pi^2}\left(||F_A^+||_{L^2}^2 - ||F_A^-||_{L^2}^2\right)$$

we get

$$||F_A^-||_{L^2}^2 \leq \frac{C^2}{4}\text{vol}(M) - 8\pi^2\chi(M) - 12\pi^2\ \text{sign}(M)$$

Eq. (52) gives us:

$$||\nabla_A\phi||_{L^2} \leq \frac{C}{4}\ \text{vol}(M)$$

- Notice since

$$F_A^+ = \phi \otimes \phi^* - \frac{|\phi|^2}{2}\ \text{Id}$$

we have

$$\nabla_{L.C.}F_A^+ = \nabla_A\phi \otimes \phi^* + \phi \otimes \nabla_A\phi^* - \ \text{Re}\langle\nabla_A\phi, \phi\rangle\ \text{Id}$$

Using the a priori bound on $||\nabla_A\phi||_{L^2}^2$ and the L^∞ bound on ϕ, we get an a priori bound on $||\nabla_{L.C.}F_A^+||_{L^2}^2$. Once we get an a priori L_1^2 bound on F_A^+, using Uhlenbeck compactness (or elementary Hodge theory in our case since the gauge group is abelian) we get an L_2^2 bound on the connection upto gauge:

Let A_0 be a fixed C^∞ connection on L. There's a constant C_1 depending only on M and A_0 such that for any solution (A, ϕ) to the SW equations we have a connection $A^{'} = A_0 + \alpha$ gauge equivalent to A with

$$d^*\alpha = 0 \text{ and } ||\alpha||_{L^2}^2 \leq C_1$$

- Now we parley the L_2^2 bound on A and the L^∞ bound on ϕ into C^∞ bounds on both A and ϕ using standard bootstrap technique for elliptic equations. After some careful use of bootstrap and Sobolev embedding we end up with the following compactness result:

  Let (A_n, ϕ_n) be any sequence to the SW equations. Then after passing to a subsequence, and applying L_3^2 changes of gauge we can arrange that the (A_n, ϕ_n) are C^∞ objects and they converge in the C^∞ topology to a limit (A, ϕ) which is also a solution to the SW equations. In particular, the moduli space of solutions to the SW equations is compact.

Now we get back to the SW equations in dimension 8. The equations in dimension 8 are the following for $\phi \in \Gamma(S_+), A \in \mathcal{A}, \beta \in \Omega^3$:

$$(D_A + (1+i)c(\beta))\phi = 0 \tag{53}$$

$$F_A + 2id^*\beta + 2d^+\beta = q(\phi) \tag{54}$$

Lemma 12. *There is a constant* $C > 0$, *such that if* ϕ, A, β *solve* (53) *and* (54), *then*

$$||\phi||_{C^0}^2 \leq C(||\beta||_{C^0}^2 + 1).$$

In particular, if A_j, ϕ_j, β_j *is a sequence of solutions in which* ϕ_j *is unbounded, then* β_j *is also unbounded.*

Proof. If ϕ, A, β solve (53) and (54), then using the Weitzenböck formula proved in §0.3.6 for any $x \in M$, we get

$$\nabla^*_{A,\beta}\nabla_{A,\beta}(\phi(x)) + \frac{s(x)}{4}\phi(x) + \frac{7}{16}|\phi(x)|^2\phi(x) - 2(c(\beta)^2\phi)(x) - 4|\beta(x)|^2\phi(x) = 0$$

Taking point-wise inner product with ϕ yields

$$\langle\nabla^*_{A,\beta}\nabla_{A,\beta}(\phi(x)), \phi(x)\rangle + \frac{s(x)}{4}|\phi(x)|^2 + \frac{7}{16}|\phi(x)|^4 - 2|c(\beta(x))\phi(x)|^2 - 4|\beta(x)|^2|\phi(x)|^2 = 0$$

We have

$$\begin{aligned}\frac{1}{2}\Delta|\phi|^2 = \langle\nabla^*_{A,\beta}\nabla_{A,\beta}\phi, \phi\rangle - |\nabla_{A,\beta}\phi|^2 &\leq 4|\beta|^2|\phi|^2 + 2|c(\beta)\phi|^2 - \frac{s}{4}|\phi|^2 - \frac{7}{16}|\phi|^4 \\ &\leq (6|\beta|^2 - \frac{s}{4})|\phi|^2 - \frac{7}{16}|\phi|^4 \qquad (55)\end{aligned}$$

It follows that at a point $x_0 \in M$, where $|\phi|^2$ achieves its maximum and hence $\Delta(|\phi(x_0)|^2) \geq 0$, we have

$$\left(6|\beta(x_0)|^2 - \frac{s(x_0)}{4}\right)|\phi(x_0)|^2 - \frac{7}{16}|\phi(x_0|^4 \leq 0$$

As ϕ is not identically zero, $|\phi(x_0)|^2 > 0$, and hence we get

$$\begin{aligned}\frac{7}{16}|\phi(x_0)|^2 &\leq \left(6|\beta(x_0)|^2 - \frac{s(x_0)}{4}\right) \\ &\leq C_1\left(|\beta(x_0)|^2 + 1\right) \quad [C_1 = 6 \times \max\left(1, \frac{|s(x_0)|}{24}\right)]\end{aligned}$$

This implies

$$||\phi||^2_{C^0} \leq C(||\beta||^2_{C^0} + 1) \quad [C = \frac{16}{7}C_1] \qquad \square$$

Lemma 13. *Given any solution to the equations* (53) *and* (54), β *is determined in the following way:*

$$\beta = -\frac{i}{2}dGq(\phi)_2 - *dGq(\phi)_4 + \beta_h$$

Here G *denotes the Green's operator for Laplacians on* Ω^2 *and* Ω^4 *respectively (with abuse of notation we call it* G *in both cases) and* β_h *denotes the harmonic part of* β.

Proof. We write $q(\phi)_2$ for the part of $q(\phi)$ which lies in $i\Omega^2$ and $q(\phi)_4$ for the part of $q(\phi)$ which lies in Ω^4_+. Applying d to the curvature equation we get,

$$dd^*\beta = -\frac{i}{2}dq(\phi)_2 \text{ and } d^*d\beta = - * (dq(\phi)_4)$$

Together these two equations give

$$\Delta\beta = (dd^* + d^*d)\beta = -\frac{i}{2}dq(\phi)_2 - *(dq(\phi)_4)$$

The result now follows from applying Green's operator and using that G commutes with both d and $*$. $\square$

Lemma 14. *There is a constant* C*, such that for any solution to the equations* (53) *and* (54)*,*

$$||\beta||_{L_1^p} \leq C\left(\int_M |\phi|^{2p}\right)^{1/p} + C||\beta_h||$$

(where, at the expense of changing C*, we can use any norm on harmonic* 3*-forms since they are all equivalent). It follows (by Sobolev embedding) that*

$$||\beta||_{C^0} \leq C\left(\int_M |\phi|^{18}\right)^{1/9} + C||\beta_h||$$

Proof. We estimate

$$\begin{aligned} ||dGq(\phi)_2||_{L_1^p} &\leq C||Gq(\phi)||_{L_2^p} \leq C||q(\phi)||_{L^p} \leq C\big(|\phi|^{2p}\big)^{1/p} \\ ||*dGq(\phi)_4||_{L_1^p} &\leq C||Gq(\phi)||_{L_2^p} \leq C||q(\phi)||_{L^p} \leq C\big(|\phi|^{2p}\big)^{1/p} \end{aligned}$$

The lemma now follows from these two inequalities. □

Lemma 15. *There is a constant* $C > 0$*, such that for any solution to the equations* (53) *and* (54)*,*

$$\Delta(|\beta|^2 + |\phi|^2) \leq C(|\beta|^4 + 1)$$

Proof. We use a Bochner–Weitzenböck formula for the three-form β [22]:

$$\frac{1}{2}\Delta|\beta|^2 = \langle\Delta\beta, \beta\rangle - |\nabla\beta|^2 + F(\beta)$$

Here F is a quadratic term in β, related to the curvature tensor. Using the lemmas above we get,

$$\begin{aligned} \Delta|\beta|^2 + \Delta|\phi|^2 &= 2\big(\langle\Delta\beta, \beta\rangle - |\nabla\beta|^2 + F(\beta) + \big(6|\beta|^2 - \frac{s}{4}\big)|\phi|^2 - \frac{7}{16}|\phi|^4 - |\nabla_{A,\beta}\phi|^2\big) \\ &\leq C|\nabla q(\phi)||\beta| + C|\beta|^2 + C(|\beta|^2+1)|\phi|^2 - \frac{7}{8}|\phi|^4 - 2|\nabla_{A,\beta}\phi|^2 \\ &\leq C|\nabla_A\phi||\phi||\beta| + C|\beta|^2 + C(|\beta|^2+1)|\phi|^2 - \frac{7}{8}|\phi|^4 - 2|\nabla_{A,\beta}\phi|^2 \\ &\leq C(|\beta|^2|\phi|^2 + |\beta|^2 + |\phi|^2) - \frac{7}{8}|\phi|^4 \\ &\leq C(|\beta|^4 + 1) \end{aligned}$$

□

Remark 16. Notice that from the proof of the last lemma, we also get an ϵ–regularity type inequality

$$\begin{aligned} \Delta(|\beta|^2 + |\phi|^2) &\leq C(|\beta|^2|\phi|^2 + |\beta|^2| + \phi|^2) \\ &\leq C\big((|\beta|^2| + \phi|^2)^2 + 1\big) \end{aligned}$$

for a constant $C > 0$.

Lemma 12 says that given a uniform C^0 bound of the three-form β, we get a uniform C^0 bound of the spinor ϕ. We do a Nash–Moser type iteration to improve this result:

Lemma 17. *There exists* $q > 0$ *such that if* β *is uniformly bounded in* L^q, *then* ϕ *is uniformly bounded in* L^∞.

Proof. Equation (55) gives us

$$\frac{1}{2}\Delta|\phi|^2 \leq \left(6|\beta|^2 - \frac{s}{4}\right)|\phi|^2 - \frac{7}{16}|\phi|^4$$

Now let's call $|\phi|^2 = f$ and $\left(12|\beta|^2 - \frac{s}{2}\right) = g$. So we get

$$\begin{aligned}
&\Delta f \leq gf - \frac{7}{8}f^2 \\
&\Rightarrow \int f^{p+1}\Delta f \leq \int (gf^{p+2} - \frac{7}{8}f^{p+3}) \\
&\Rightarrow \frac{4(p+1)}{(p+2)^2}\int |\nabla(f^{\frac{p+2}{2}})|^2 \leq \int |g|f^{p+2} - \frac{7}{8}\int f^{p+3} \\
&\Rightarrow \frac{4(p+1)}{(p+2)^2}\int |\nabla(f^{\frac{p+2}{2}})|^2 \leq \int \frac{(c|g|)^{p+3}}{p+3} + \int \frac{f^{p+3}}{c^{\frac{p+3}{p+2}}\frac{p+3}{p+2}} - \frac{7}{8}\int f^{p+3} \quad \text{[c any positive constant]}
\end{aligned}$$

For any p, choose $c = \frac{8}{7}$. We get

$$\begin{aligned}
&c^{\frac{p+3}{p+2}} \geq \frac{8}{7}, \text{ and for that c we get,} \\
&c^{\frac{p+3}{p+2}} > \frac{8}{7} \times \frac{p+2}{p+3} \Rightarrow \left(\frac{1}{c^{\frac{p+3}{p+2}}\frac{p+3}{p+2}} - \frac{7}{8}\right) < 0
\end{aligned}$$

Hence we can write,

$$\int f^{p+3} \leq C(p)\int |g|^{p+3}$$

where the constant $C(p)$ depends on p but not on f or g. Hence, we get an L^{p+3} bound of f from an L^{p+3} bound of g.

We can actually do better! Start with the inequality:

$$\begin{aligned}
&\frac{4(p+1)}{(p+2)^2}\int |\nabla(f^{\frac{p+2}{2}})|^2 \leq \int |g|f^{p+2} - \frac{7}{8}\int f^{p+3} \\
&\quad \Rightarrow \frac{4(p+1)}{(p+2)^2}\left(\int |\nabla(f^{\frac{p+2}{2}})|^2 + \int (f^{\frac{p+2}{2}})^2\right) \leq \frac{4(p+1)}{(p+2)^2}\int |\nabla(f^{\frac{p+2}{2}})|^2 + 4\int f^{p+2} \\
&\qquad \leq \int (|g|+4)f^{p+2} - \frac{7}{8}\int f^{p+3} \qquad (56)
\end{aligned}$$

We play the same trick on the right hand side as before but we do it with $(|g|+4)$ and f instead of $|g|$ and f (the trick is nothing but Young's inequality) and we end up with

$$\int |\nabla(f^{\frac{p+2}{2}})|^2 + \int \left(f^{\frac{p+2}{2}}\right)^2 \leq \widetilde{C}(p)\int (|g|+4)^{p+3}$$

Where the constant $\widetilde{C}(p)$ depends on p but not on f or g. Now use Sobolev embedding $L^2_1 \hookrightarrow L^{\frac{8}{3}}$ in dimension 8 and this gives us:

$$\Big(\int f^{\frac{4(p+2)}{3}}\Big)^{\frac{3}{8}} \leq C_S\Big(\int |\nabla(f^{\frac{p+2}{2}})|^2 + \int (f^{\frac{p+2}{2}})^2\Big)^{\frac{1}{2}}$$

So, we can control the $L^{\frac{4(p+2)}{3}}$ norm of f using the L^{p+3} norm of g (notice $\frac{4(p+2)}{3} > p+3$ for $p > 1$). Next we start with the inequality (56).

$$\frac{4(p+1)}{(p+2)^2}\Big(\int |\nabla(f^{\frac{p+2}{2}})|^2 + \int (f^{\frac{p+2}{2}})^2\Big) \leq \frac{4(p+1)}{(p+2)^2}\int |\nabla(f^{\frac{p+2}{2}})|^2 + 4\int f^{p+2} \leq \int (|g|+4)f^{p+2} - \frac{7}{8}\int f^{p+3}$$
$$\Rightarrow \Big(\int |\nabla(f^{\frac{p+2}{2}})|^2 + \int (f^{\frac{p+2}{2}})^2\Big) \leq C(p+2)\int (|g|+4)f^{p+2}$$

We use Sobolev embedding $L^2_1 \hookrightarrow L^{\frac{8}{3}}$ for the function $f^{\frac{p+2}{2}}$ and get

$$\Big(\int f^{\frac{4}{3}(p+2)}\Big)^{\frac{3}{8}} \leq C_S\Big(\int |\nabla(f^{\frac{p+2}{2}})|^2 + \int (f^{\frac{p+2}{2}})^2\Big)^{\frac{1}{2}}$$
$$\Rightarrow \Big(\int f^{\frac{4}{3}(p+2)}\Big)^{\frac{3}{4}} \leq C_S^2\Big(\int |\nabla(f^{\frac{p+2}{2}})|^2 + \int (f^{\frac{p+2}{2}})^2\Big)$$
$$\Rightarrow \Big(\int f^{\frac{4}{3}(p+2)}\Big)^{\frac{3}{4}} \leq C\cdot C_S^2(p+2)\int (|g|+4)f^{p+2}$$
$$\Rightarrow \|f\|_{L^{\frac{4}{3}(p+2)}} \leq \big(C\cdot C_S^2(p+2)\big)^{\frac{1}{p+2}}\Big(\int (|g|+4)f^{p+2}\Big)^{\frac{1}{p+2}}$$

Using Hölder inequality with $m > 4$, and n such that $\frac{1}{m} + \frac{1}{n} = 1$, we get

$$\int (|g|+4)f^{p+2} \leq \|(|g|+4)\|_{L^m}\|f^{p+2}\|_{L^n} = \|(|g|+4)\|_{L^m}\|f\|^{p+2}_{L^{n(p+2)}}$$
$$\Rightarrow \|f\|_{L^{\frac{4}{3}(p+2)}} \leq \big(C\cdot C_S^2(p+2)\big)^{\frac{1}{p+2}}\|(|g|+4)\|^{\frac{1}{p+2}}_{L^m}\|f\|_{L^{n(p+2)}}$$

Since $m > 4$, we get $n < \frac{4}{3}$. Hence, if $\exists N > 0$, such that $(|g|+4)\|_{L^m} \leq N$, then we can start the iteration to get an L^∞ norm of f in terms of some L^p norm of f and some L^q norm of g. Thereafter if we control high enough L^q norm of g ($q > 4$), we control the L^∞ norm of f. □

Part VI

Construction of solutions

In this section we give several examples of solutions of the SW equations in dimension 5, 6 and 8. We explain the procedure we follow in constructing solutions of SW equations on $\Sigma \times \mathbb{R}^3$, on $\Sigma \times \mathbb{C}^2$, on closed Kähler 3- and 4-folds. We choose the three-form β to be of the form $(\bar\partial f + \partial \bar f) \wedge \omega$ for a complex-valued smooth function f, ω being the either the Kähler form on Σ or on the manifold depending on the case. Next we choose the spinor (or two spinors) as a function of f which solve the Dirac equation (or equations) for the Chern connection on the determinant line bundle and finally using identities in Kähler geometry, we change the curvature equation (or equations) into Kadzan–Warner type pdes [16] in f and solve for f. The constructions have striking similarities with Bradlow's work on vortices in holomorphic line bundles over closed Kähler manifolds [5].

0.4 Solution of 5-dimensional SW equations

The SW equations in dimension 5 are the following (for a spinor ϕ, a unitary connection A, $\beta_j \in \Omega^j$) :

$$(D_A + (1-i)c(\beta_3) + i * \beta_5)\phi = 0 \tag{57}$$
$$F_A - 2id^*\beta_3 + 2d\beta_3 + 2d^*\beta_5 = q(\phi) \tag{58}$$

0.4.1 Solution on a circle bundle over a del Pezzo surface

We produce a solution of the SW equations on an S^1-bundle over a Kähler surface (X, ω), ω denoting the Kähler form on X. We assume a condition: $F_{A_K} = ic^2\omega, c$ is a constant. This implies (X, ω) is Fano and Kähler-Einstein. In particular X is diffeomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$ or $\mathbb{CP}^2$ blown up in at most 8 points [12]. These are called del Pezzo surfaces. A_K is the connection on the canonical bundle K induced by the Kähler metric on X.

There is a tautological Spin^c-bundle on $X : \Lambda^0 \oplus \Lambda^{0,1} \oplus \Lambda^{0,2}$ [20]. We take a new Spin^c-bundle by twisting the tautological Spin^c-bundle by the canonical line bundle K :

$$S(X) = K \oplus (\Lambda^{0,1} \otimes K) \oplus \Lambda^0$$

We take the 5-manifold M by taking the S^1-bundle inside K :

$$M := \{(x, v_x) : x \in X, v_x \in K_x, \|v_x\| = 1\}$$

The $\| \ \|$ on K is induced by the Kähler metric again. The induced metric and the connection on K from the Kähler metric also defines a metric and a connection on M and we have

$$T_{(x,v_x)}M = \pi^* T_x X \oplus \{w_x \in K_x : \langle w_x, v_x \rangle = 0\}$$

$S^1 \to M \xrightarrow{\pi} X$ is an S^1-bundle over X. One can define a Spin^c-bundle over M by pulling back the Spin^c-bundle on X :

$$S(M) = \pi^* K \oplus \pi^*(\Lambda^{0,1} \otimes K) \oplus \pi^* \Lambda^0$$
$$\mathrm{Det}\big(S(M)\big) = \pi^*(K^2)$$

The Clifford action of the volume form on a fiber (let's call it vol_f) is the following:

$$c(vol_f) = -i\,\mathrm{Id} \text{ on } \left(\pi^*K \oplus \pi^*\Lambda^0\right)$$
$$c(vol_f) = i\,\mathrm{Id} \text{ on } \left(\pi^*(\Lambda^{0,1} \otimes K)\right)$$

There exists a "tautological" spinor (say φ) on M defined by

$$\varphi(x, v_x) = \pi^*(v_x)$$

Since $\varphi \in \Omega^0(M; \pi^*K)$, $q(\varphi)$ can be written in diagonal matrix form for its action on spinors in $\pi^*(K) \oplus \pi^*(\Lambda^{0,1} \otimes K) \oplus \pi^*\Lambda^0$.

$$c(q(\varphi)) = \begin{bmatrix} \frac{3}{4}\mathrm{Id} & 0 & 0 \\ 0 & -\frac{1}{4}\mathrm{Id} & 0 \\ 0 & 0 & -\frac{1}{4}\mathrm{Id} \end{bmatrix}$$

Clifford actions of $i(\pi^*\omega)$ and $\pi^*(\omega^2)$ on the spinors are described as follows.

	π^*K	$\pi^*(\Lambda^{0,1} \otimes K)$	$\pi^*\Lambda^0$
$i(\pi^*\omega)$	$2\mathrm{Id}$	0	$-2\mathrm{Id}$
$\pi^*(\omega^2)$	$-2\mathrm{Id}$	$2\mathrm{Id}$	$-2\mathrm{Id}$

Hence, we deduce

$$q(\varphi) = \frac{1}{8}\left(2i\pi^*\omega - \pi^*(\omega^2)\right)$$

Notice for an orthonormal basis $e_0, \ldots, e_5$ of forms at a point (x, v_x) such that e_0 is a form on the fibre, $\nabla_0 \varphi = e_0$ and for $j \in \{1, \ldots, 4\}, \nabla_j \varphi = 0$. We get $D_{\pi^*(2A_K)}\varphi = -\varphi$.

Choose $\phi = 2\sqrt{2}c\varphi, A = 2\pi^*A_K, \beta_3 = -\frac{1}{2}vol_f \wedge \pi^*\omega$ and $*\beta_5 = 1$, then ϕ, A, β_3 and β_5 solve the Dirac equation (57). The curvature equation (58) reads:

$$2\pi^*F_{A_K} + i\pi^*F_{A_K} \wedge \pi^*\omega = c^2\left(2i\pi^*\omega - \pi^*(\omega^2)\right)$$

The above equation has a solution iff $F_{A_K} = ic^2\omega$. Since this was in the assumption in our choice of X, both equations (57) and (58) are solved.

Remark 18. For an explicit ansatz, one can take $X = \mathbb{CP}^2$, since $K_{\mathbb{CP}^2} \cong \mathcal{O}(-3)$. Then the circle bundle of $\mathcal{O}(-3)$ can be identified with $S^5/\mathbb{Z}_3$ (a Lens space), where $S^5 = \{z \in \mathbb{C}^3 : \|z\| = 1\}$, which can be identified with the circle bundle of $\mathcal{O}(-1) \to \mathbb{CP}^2$ and the action of $\mathbb{Z}_3$ is given by $z \sim \zeta_k z$, where ζ_k is a 3-rd root of unity for $k = 1, 2, 3$. One way to see this is the following.

$\mathbb{CP}^2$ is the space of lines in $\mathbb{C}^3$. The total space of circle bundle of $\mathcal{O}(-3) \to \mathbb{CP}^2$ is $\{\ell \in \mathbb{CP}^2, v \in (\mathbb{C}^3)^{\otimes k} : v \in \ell^{\otimes k}, \|v\| = 1\}$. Now there is a isomorphism

$$S^3/\mathbb{Z}_3 = \{\ell \in \mathbb{CP}^2, z \in \ell : \|z\| = 1\}/(z \sim \zeta_k z) \simeq \{\ell \in \mathbb{CP}^2, v \in \ell^{\otimes k} : \|v\| = 1\}$$
$$(\ell, z) \mapsto (\ell, z \otimes \cdots \otimes z).$$

0.4.2 Solution of perturbed 5d SW equations on $\Sigma \times \mathbb{R}^3$

We take the base manifold to be $M^5 = \Sigma \times \mathbb{R}^3$, where Σ is a closed compact Riemann surface. We call the Kähler form on Σ by ω, we also choose a Riemannian metric on Σ compatible with the almost complex structure such that ω becomes the volume form. Let x_1, x_2, x_3 denote coordinates in the $\mathbb{R}^3$ direction. $\mathbb{R}^3$ has the standard Euclidean metric. The spinor bundle on $\mathbb{R}^3$ can be chosen as a trivial $\mathbb{C}^2$ bundle, we follow the following convention for the Clifford action of one forms:

$$c(dx_1) = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, c(dx_2) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, c(dx_3) = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

The spinor bundles on $\Sigma \times \mathbb{R}^3$ can be taken as follows [17]:

$$S(\Sigma \times \mathbb{R}^3) = \left(\Lambda^0(\Sigma, L) \otimes \mathbb{C}^2\right) \oplus \left(\Lambda^{0,1}(\Sigma, L) \otimes \mathbb{C}^2\right)$$

The explicit description should involve pullback of the spinor-bundles on Σ and $\mathbb{R}^3$. We avoid writing pull backs with abuse of notations. $\mathbb{C}^2$ denotes the trivial bundle on $\mathbb{R}^3$ (with abuse of notation again) and L is a holomorphic line bundle on Σ. So, $S(\Sigma \times \mathbb{R}^3)$ is a direct sum of four line bundles.

$$\det(S) = L^4 \otimes K_\Sigma^{-2}$$

To solve the SW equations, take $\beta_3 = (\bar{\partial} f + \partial \bar{f}) \wedge dx_1 \wedge dx_2, f \in C^\infty(\Sigma, \mathbb{C}), \beta_5 = 0$.

We choose a hermitian metric h on L and take the corresponding Chern connection A_h on L, the Kähler metric induces a connection A_{K_Σ} on K_Σ, and we take the usual flat connection on $\mathbb{C}^2$. So we get a connection on $L^4 \otimes K_\Sigma^{-2}$. With abuse of notation we call it $A := 4A_h - 2A_{K_\Sigma}$.

For the spinor, we start with φ, a non-zero holomorphic section of L, i.e., $\bar{\partial}_{A_h}\varphi = 0$. Say e_1, e_2 denote the standard basis elements of $\mathbb{C}^2$, here we will think of them as spinors on $\mathbb{R}^3$. With the standard metric and the flat connection on $\mathbb{C}^2$, e_1 and e_2 form standard basis of $\mathbb{C}^2$ giving unit length nowhere vanishing parallel spinors on $\mathbb{R}^3$. Define a spinor on M, $\phi := e^{i(1-i)f}\varphi \otimes e_1$. The following table describes Clifford actions of some forms on the spinors.

	$\Lambda^0 \otimes \mathbb{C}\{e_1\}$	$\Lambda^0 \otimes \mathbb{C}\{e_2\}$	$\Lambda^{0,1} \otimes \mathbb{C}\{e_1\}$	$\Lambda^{0,1} \otimes \mathbb{C}\{e_2\}$
$i\omega$	Id	Id	-Id	-Id
$idx_1 \wedge dx_2$	Id	-Id	Id	-Id
$\omega \wedge dx_1 \wedge dx_2$	-Id	Id	Id	-Id

$$c(\bar{\partial} f + \partial \bar{f})\varphi = c(\bar{\partial} f)\varphi = \sqrt{2}\bar{\partial} f \wedge \varphi$$
$$c\big((\bar{\partial} f + \partial \bar{f}) \wedge dx_1 \wedge dx_2\big)(\varphi \otimes e_1) = -\sqrt{2}i(\bar{\partial} f \wedge \varphi) \otimes e_1$$
$$D_A(e^{i(1-i)f}\varphi \otimes e_1) = \sqrt{2}\bar{\partial}_{A_h}(e^{i(1-i)f}\varphi) \otimes e_1 = \sqrt{2}i(1-i)e^{i(1-i)f}(\bar{\partial} f \wedge \varphi) \otimes e_1$$

Observe that ϕ, A, β solves the Dirac equation (57).

$$\begin{aligned} & d\big((\bar{\partial} f + \partial \bar{f}) \wedge dx_1 \wedge dx_2\big) \\ & = 2i\partial\bar{\partial}(\mathrm{Im} f) \wedge dx_1 \wedge dx_2 \\ & = -\Delta \mathrm{Im}(f)\omega \wedge dx_1 \wedge dx_2 \end{aligned}$$

$$
\begin{aligned}
&d^*\big((\bar\partial f + \partial \bar f) \wedge dx_1 \wedge dx_2\big)\\
&= - * d\big(i(\bar\partial f - \partial \bar f) \wedge dx_3\big)\\
&= -2i * \big(\partial\bar\partial(\mathrm{Re} f) \wedge dx_3\big)\\
&= \Delta \mathrm{Re}(f) dx_1 \wedge dx_2
\end{aligned}
$$

$$
\begin{aligned}
q(\phi) &= \frac{|\phi|^2}{4}(i\omega + i dx_1 \wedge dx_2 - \omega \wedge dx_1 \wedge dx_2)\\
&= \frac{1}{4} e^{2(\mathrm{Re} f - \mathrm{Im} f)} |\varphi|^2 (i\omega + i dx_1 \wedge dx_2 - \omega \wedge dx_1 \wedge dx_2)
\end{aligned}
$$

The curvature equation (58) reads:

$$
\begin{aligned}
&F_A - 2\Delta(\mathrm{Im} f)\omega \wedge dx_1 \wedge dx_2 - 2i\Delta(\mathrm{Re}(f)) dx_1 \wedge dx_2\\
&= \frac{1}{4} e^{2(\mathrm{Re} f - \mathrm{Im} f)} |\varphi|^2 (i\omega + i dx_1 \wedge dx_2 - \omega \wedge dx_1 \wedge dx_2)
\end{aligned}
$$

If we add this extra term: $c(i dx_1 \wedge dx_2 - \omega \wedge dx_1 \wedge dx_2)$ on the left hand side of the curvature equation we get:

$$
\begin{aligned}
&F_A - 2\Delta(\mathrm{Im} f)\omega \wedge dx_1 \wedge dx_2 - 2i\Delta(\mathrm{Re}(f)) dx_1 \wedge dx_2\\
&+ c(i dx_1 \wedge dx_2 - \omega \wedge dx_1 \wedge dx_2)\\
&= \frac{1}{4} e^{2(\mathrm{Re} f - \mathrm{Im} f)} |\varphi|^2 (i\omega + i dx_1 \wedge dx_2 - \omega \wedge dx_1 \wedge dx_2)
\end{aligned}
$$

This breaks into three equations:

$$
\begin{aligned}
&F_A = \frac{i}{4} e^{2(\mathrm{Re} f - \mathrm{Im} f)} |\varphi|^2 \omega,\\
&- 2\Delta(\mathrm{Im} f) + \frac{1}{4} e^{2(\mathrm{Re} f - \mathrm{Im} f)} |\varphi|^2 = c\\
&\text{and } 2\Delta(\mathrm{Re} f) + \frac{1}{4} e^{2(\mathrm{Re} f - \mathrm{Im} f)} |\varphi|^2 = c
\end{aligned}
$$

The last two equations give us:

$$
\begin{aligned}
&\Delta(\mathrm{Re} f + \mathrm{Im} f) = 0\\
&\Rightarrow \mathrm{Re} f + \mathrm{Im} f = a_1, \text{ for a constant } a_1
\end{aligned}
$$

So, the last two equations become one single equation:

$$
2\Delta(\mathrm{Re} f) + \frac{1}{4} e^{4\mathrm{Re} f} e^{-2a_1} |\varphi|^2 = c
$$

For a fixed a_1, there exists a unique Ref which solves the pde for a constant $c > 0$ [16]. Once we know f, we solve the remaining equation

$$
4F_{A_h} - F_{K_\Sigma} = \frac{i}{4} e^{4\mathrm{Re} f} e^{-2a_1} |\varphi|^2 \omega
$$

We perturb the initial metric h on L by e^λ, $\lambda \in C^\infty(\Sigma, \mathbb{R})$. The new metric being $h' := e^\lambda h$. Notice this conformal change in the metric doesn't change the holomorphic structure of L. We write the equation now with respect to the Chern connection $A_{h'}$ and the metric h'.

$$\begin{aligned}
&4F_{A_{h'}} - 2F_{K_\Sigma} = \frac{i}{4} e^{4\mathrm{Re}f} e^{-2a_1} |\varphi|^2_{h'} \omega \\
&\Leftrightarrow 4F_{A_h} - 4\partial\bar\partial\lambda - 2F_{K_\Sigma} = \frac{i}{4} e^{4\mathrm{Re}f} e^{-2a_1} e^{2\lambda} |\varphi|^2 \omega \\
&\Leftrightarrow \langle (4F_{A_h} - 2F_{K_\Sigma}), \omega\rangle \omega - 2i\Delta\lambda\omega = \frac{i}{4} e^{4\mathrm{Re}f} e^{-2a_1} e^{2\lambda} |\varphi|^2 \omega \\
&\Leftrightarrow 2\Delta\lambda + \frac{1}{4} e^{4\mathrm{Re}f} e^{-2a_1} e^{2\lambda} |\varphi|^2 = 2i\langle F_{K_\Sigma}, \omega\rangle - 4i\langle F_{A_h}, \omega\rangle
\end{aligned}$$

As we already know f and φ is not everywhere zero, there exists a unique solution for λ [16] if

$$\int_\Sigma \big(2i\langle F_{K_\Sigma}, \omega\rangle - 4i\langle F_{A_h}, \omega\rangle\big)\omega > 0 \Leftrightarrow \int_\Sigma c_1(K_\Sigma) > 2\int_\Sigma c_1(L)$$

Notice $\int_\Sigma c_1(L) \geq 0$ since, $0 \neq \varphi \in H^0(\Sigma, L)$. So, we found a solution of the following perturbed version of 5-dimensional Seiberg–Witten equations:

$$\big(D_A + c((1-i)\beta_3 + i * \beta_5)\big)\phi = 0 \tag{59}$$

$$F_A - 2id^*\beta_3 + 2d\beta_3 + 2d^*\beta_5 + \eta = q(\phi) \tag{60}$$

$$\eta = c(idx_1 \wedge dx_2 - \omega \wedge dx_1 \wedge dx_2)$$

Remark 19. Notice that the perturbation η is *harmonic* and we can think of it as playing the role of the cohomology class of $2d\beta$, $\beta = \beta_3 + \beta_5$. If we actually can make sense of "spinors with values in a gerbe", the perturbation should be absorbed in the differential terms in the left hand side of the curvature equation.

Now if we take the spinor to be $\phi := e^{i(1-i)f_1}\varphi \otimes e_2$, and $\beta_3 = -(\bar\partial f_1 + \partial \bar{f_1}) \wedge dx_1 \wedge dx_2$, $\beta_5 = 0$; notice that $\phi, \beta, A = 4A_h - 2A_{K_\Sigma}$ solves the Dirac equation (59).

$$q(\phi) = \frac{1}{4}|\phi|^2(i\omega - idx_1 \wedge dx_2 + \omega \wedge dx_1 \wedge dx_2)$$

The perturbed curvature equation (60) reads:

$$\begin{aligned}
&F_A + 2\Delta(\mathrm{Im}f_1)\omega \wedge dx_1 \wedge dx_2 + 2i\Delta(\mathrm{Re}f_1)dx_1 \wedge dx_2 + c(idx_1 \wedge dx_2 - \omega \wedge dx_1 \wedge dx_2) \\
&= \frac{1}{4} e^{2(\mathrm{Re}f_1 - \mathrm{Im}f_1)}|\varphi|^2(i\omega - idx_1 \wedge dx_2 + \omega \wedge dx_1 \wedge dx_2)
\end{aligned}$$

This breaks into the following three equations:

$$\begin{aligned}
F_A &= \frac{i}{4} e^{2(\mathrm{Re}f_1 - \mathrm{Im}f_1)}|\varphi|^2\omega \\
-2\Delta(\mathrm{Im}f_1) + \frac{1}{4} e^{2(\mathrm{Re}f_1 - \mathrm{Im}f_1)}|\varphi|^2 &= -c \\
2\Delta(\mathrm{Re}f_1) + \frac{1}{4} e^{2(\mathrm{Re}f_1 - \mathrm{Im}f_1)}|\varphi|^2 &= -c
\end{aligned}$$

The last two equations give:

$$\begin{aligned}&\Delta(\mathrm{Re} f_1 + \mathrm{Im} f_1) = 0\\ &\Rightarrow \mathrm{Re} f_1 + \mathrm{Im} f_1 = a_2, \text{ for a constant } a_2.\end{aligned}$$

So the last two equations become one single equation:

$$2\Delta(\mathrm{Re} f_1) + \frac{1}{4} e^{4\mathrm{Re} f_1} e^{-2a_2} |\varphi|^2 = -c$$

For a fixed c_1, there exists a unique $\mathrm{Re} f_1$ which solves the pde for a constant $c < 0$ [16]. Once we know f_1, we solve the remaining equation (written below) in the same way we did before.

$$4F_{A_h} - 2F_{K_\Sigma} = \frac{i}{4} e^{4\mathrm{Re} f_1} e^{-2a_2} |\varphi|^2 \omega$$

The required condition for the existence of solution is same as before:

$$\int_\Sigma c_1(K_\Sigma) > 2\int_\Sigma c_1(L)$$

Proposition 20. *There exists a solution of the system of equations* (59) *and* (60) *on* $\Sigma \times \mathbb{R}^3$ *under the following conditions:*

$$\begin{aligned}&\textit{for } c \neq 0: \ \textit{dim } H^0(\Sigma, L) > 0 \textit{ and deg}(K_\Sigma - 2L) > 0\\ &\textit{for } c = 0: \ \textit{deg}(K_\Sigma - 2L) = 0\end{aligned}$$

Moreover for all c, *there exists a solution which is translation invariant in any direction in* $\mathbb{R}^3$.

Proof. Both $c > 0$ and $c < 0$ cases are discussed above. The remaining case is $c = 0$ which gives us back the 5d Seiberg–Witten equations (57), (58) without any perturbation term. We take $\phi = 0, \beta_3 = 0, \beta_5 = 0$. This solves the Dirac equation (57) for any unitary connection on L. So, the curvature equation (58) reads: $F_A = 0$. This has a solution iff $\deg(L^4 \otimes K_\Sigma^{-2}) = 0$, i.e., $\deg(K_\Sigma - 2L) = 0$. The solutions constructed are all translation invariant in any direction in $\mathbb{R}^3$. □

Remark 21. For an explicit ansatz, one can simply take any compact Riemann surface Σ with genus $g > 1$. If L is the trivial line bundle, it satisfies both the conditions when $c \neq 0$ since, dim $H^0(\Sigma, L) = 1$ and $\deg(K_\Sigma - 2L) = \deg(K_\Sigma) = 2g - 2 > 0$.

One can add a different perturbation term in the curvature equation and get the following system of equations:

$$\big(D_A + c((1-i)\beta_3 + i * \beta_5)\big)\phi = 0 \tag{61}$$

$$F_A - 2id^*\beta_3 + 2d\beta_3 + 2d^*\beta_5 + \eta = q(\phi) \tag{62}$$

$$\eta = c(idx_1 \wedge dx_2 + \omega \wedge dx_1 \wedge dx_2)$$

To construct a solution of (61) and (62), we take the three-form to be $\beta_3 = -(\bar\partial g_1 + \partial \bar g_1) \wedge dx_1 \wedge dx_2, g_1 \in C^\infty(\Sigma, \mathbb{C})$ and $\beta_5 = 0$. We start with the same set up with the hermitian metric

h and the corresponding Chern connection A_h on L. Finally for the spinor, first we take a non-zero section $\psi \in \Gamma(K_\Sigma^{-1} \otimes L)$ such that $\bar{\partial}^*_{A_h}\psi = 0$. In another words ψ is an anti-holomorphic section of $K_\Sigma^{-1} \otimes L = L - K_\Sigma$, hence $\bar{\psi}$ is a holomorphic section of $K_\Sigma - L$. We define a spinor $\phi := e^{-(1+i)\bar{g}_1}\psi \otimes e_1$.

For a smooth section $\xi \in \Omega^0(\Sigma, L)$,

$$\begin{aligned}
&\langle \bar{\partial}^*_{A_h}(e^{-(1+i)\bar{g}_1}\psi), \xi\rangle_{L^2} \\
&= \langle \psi, e^{-(1-i)g_1}\bar{\partial}_{A_h}\xi\rangle_{L^2} \\
&= \langle \psi, \bar{\partial}_{A_h}(e^{-(1-i)g_1}\xi)\rangle_{L^2} - \langle \psi, -(1-i)e^{-(1-i)g_1}\xi\bar{\partial}g\rangle_{L^2} \\
&= \int_\Sigma (1+i)e^{-(1+i)\bar{g}_1}\psi \wedge *\big(\overline{\xi\bar{\partial}g}\big) \\
&= -\int_\Sigma i(1+i)e^{-(1+i)\bar{g}_1}\bar{\xi}(\psi \wedge \partial\bar{g}_1) \\
&= \overline{\int_\Sigma i(1-i)e^{-(1-i)g_1}\xi(\bar{\psi} \wedge \bar{\partial}g_1)} \\
&= \overline{\langle \xi, -i(1+i)e^{-(1+i)\bar{g}_1} * (\psi \wedge \partial\bar{g}_1)\rangle_{L^2}} \\
&= \langle -i(1+i)e^{-(1+i)\bar{g}_1} * (\psi \wedge \partial\bar{g}_1), \xi\rangle_{L^2}
\end{aligned}$$

Hence

$$\begin{aligned}
\bar{\partial}^*_{A_h}(e^{-(1+i)\bar{g}_1}\psi) &= i(1+i)e^{-(1+i)\bar{g}_1} * (\partial\bar{g}_1 \wedge \psi) \\
&= -(1-i)e^{-(1+i)\bar{g}_1} * (\partial\bar{g}_1 \wedge \psi)
\end{aligned}$$

$$\begin{aligned}
c(\bar{\partial}g_1 + \partial\bar{g}_1)\psi &= -\sqrt{2}(\bar{\partial}g_1 \lrcorner \psi) = -\sqrt{2}i * (\partial\bar{g}_1 \wedge \psi) \\
c\big(-(\bar{\partial}g_1 + \partial\bar{g}_1) \wedge dx_1 \wedge dx_2\big)(\psi \otimes e_1) &= \sqrt{2} * (\partial\bar{g}_1 \wedge \psi) \otimes e_1 \\
D_A\big(e^{-(1+i)\bar{g}}\psi \otimes e_1\big) &= \sqrt{2}\bar{\partial}^*_{A_h}(e^{-(1+i)\bar{g}_1}\psi) \otimes e_1
\end{aligned}$$

So, ϕ, A_h, β solves the Dirac equation (61).

$$\begin{aligned}
q(\phi) &= \frac{|\phi|^2}{4}(-i\omega + i dx_1 \wedge dx_2 + \omega \wedge dx_1 \wedge dx_2) \\
&= \frac{1}{4}e^{-2(\mathrm{Re}g_1 + \mathrm{Im}g_1)}|\psi|^2(-i\omega + i dx_1 \wedge dx_2 + \omega \wedge dx_1 \wedge dx_2)
\end{aligned}$$

The curvature equation (62) reads:

$$\begin{aligned}
&F_A + 2\Delta(\mathrm{Im}g_1)\omega \wedge dx_1 \wedge dx_2 + 2i\Delta(\mathrm{Re}g_1)dx_1 \wedge dx_2 + c(i dx_1 \wedge dx_2 + \omega \wedge dx_1 \wedge dx_2) \\
&= \frac{1}{4}e^{-2(\mathrm{Re}g_1 + \mathrm{Im}g_1)}|\psi|^2(-i\omega + i dx_1 \wedge dx_2 + \omega \wedge dx_1 \wedge dx_2)
\end{aligned}$$

This breaks into three equations:

$$\begin{aligned}
F_A &= -\frac{i}{4}e^{-2(\mathrm{Re}g_1 + \mathrm{Im}g_1)}|\psi|^2\omega \\
-2\Delta(\mathrm{Im}g_1) + \frac{1}{4}e^{-2(\mathrm{Re}g_1 + \mathrm{Im}g_1)}|\psi|^2 &= c \\
-2\Delta(\mathrm{Re}g_1) + \frac{1}{4}e^{-2(\mathrm{Re}g_1 + \mathrm{Im}g_1)}|\psi|^2 &= c
\end{aligned}$$

We get

$$\begin{aligned}
&\Delta\big(\mathrm{Re}g_1 - \mathrm{Im}g_1\big) = 0 \\
&\Rightarrow \mathrm{Re}g_1 = \mathrm{Im}g_1 + a_3, \text{ for a constant } a_3
\end{aligned}$$

So, the last two equations become one single equation:

$$-2\Delta(\mathrm{Re}g_1) + \frac{1}{4}e^{-4(\mathrm{Re}g_1)}e^{-2a_3}|\psi|^2 = c$$

There exists a unique $\mathrm{Re}g_1$ which solves the pde for a constant $c > 0$ [16]. Once we know g_1, we solve the remaining equation:

$$F_A = -\frac{i}{4}e^{-4\mathrm{Re}g_1}e^{-2a_3}|\psi|^2\omega$$

We perturb the initial metric h on L by $e^{\widetilde{\lambda}}$, $\widetilde{\lambda} \in C^\infty(\Sigma, \mathbb{R})$. The new metric being $\widetilde{h} := e^{\widetilde{\lambda}}h$. Notice this conformal change in the metric doesn't change the holomorphic structure of L. We write the equation now with respect to the Chern connection $A_{\widetilde{h}}$ and the metric $\widetilde{h}$ on L. This conformal change in metric of L, induces a conformal change by $e^{-\widetilde{\lambda}}$ on the corresponding hermitian metric on L^{-1}. So, $\bar{\psi}$ remains a holomorphic section of $K_\Sigma - L$, and now when we go back to the corresponding anti-holomorphic section on $L - K_\Sigma$, the norm changes by $e^{-\widetilde{\lambda}}$. The curvature equation becomes

$$\begin{aligned}
&4F_{A_{\widetilde{h}}} - 2F_{K_\Sigma} = -\frac{i}{4}e^{-4\mathrm{Re}g_1}e^{-2c_3}e^{-2\widetilde{\lambda}}|\psi|^2\omega \\
&\Leftrightarrow 4F_A - 4\partial\bar{\partial}\widetilde{\lambda} - 2F_{K_\Sigma} = -\frac{i}{4}e^{-4\mathrm{Re}g_1}e^{-2c_3}e^{-2\widetilde{\lambda}}|\psi|^2\omega \\
&\Leftrightarrow \langle(4F_{A_h} - 2F_{K_\Sigma}), \omega\rangle\omega - 2i\Delta\widetilde{\lambda}\omega = -\frac{i}{4}e^{-4\mathrm{Re}g_1}e^{-2c_3}e^{-2\widetilde{\lambda}}|\psi|^2\omega \\
&\Leftrightarrow -2\Delta\widetilde{\lambda} + \frac{1}{4}e^{-4\mathrm{Re}g_1}e^{-2c_3}e^{-2\widetilde{\lambda}}|\psi|^2 = i\langle(4F_{A_h} - 2F_{K_\Sigma}), \omega\rangle
\end{aligned}$$

Since g_1 is already known and ψ is not everywhere zero, there exists a unique solution for $\widetilde{\lambda}$ from the pde [16] iff

$$\int_\Sigma i(4F_{A_h} - 2F_{K_\Sigma}) > 0 \Leftrightarrow \int_\Sigma c_1(2L - K_\Sigma) > 0$$

We also give a solution for $c < 0$. Take $\phi := e^{-(1+i)\bar{g}_2}\psi \otimes e_2$, and $\beta_3 = (\bar{\partial}g_2 + \partial\bar{g_2}) \wedge dx_1 \wedge dx_2, \beta_5 = 0$. Notice that $\phi, \beta, A = 4A_h - 2A_{K_\Sigma}$ solves the Dirac equation (61).

$$q(\psi) = -\frac{1}{4}|\phi|^2(i\omega + i dx_1 \wedge dx_2 + \omega \wedge dx_1 \wedge dx_2)$$

The curvature equation (62) reads:

$$\begin{aligned}
&F_A - 2\Delta(\mathrm{Im}g_2)\omega \wedge dx_1 \wedge dx_2 - 2i\Delta(\mathrm{Re}g_2)dx_1 \wedge dx_2 + c(i dx_1 \wedge dx_2 + \omega \wedge dx_1 \wedge dx_2) \\
&= -\frac{1}{4}e^{-2(\mathrm{Re}g_2 + \mathrm{Im}g_2)}|\psi|^2(i\omega + i dx_1 \wedge dx_2 + \omega \wedge dx_1 \wedge dx_2)
\end{aligned}$$

This breaks into three parts:

$$F_A = -\frac{i}{4} e^{-2(\mathrm{Re} g_2 + \mathrm{Im} g_2)} |\psi|^2 \omega$$
$$-2\Delta(\mathrm{Im} g_2) + \frac{1}{4} e^{-2(\mathrm{Re} g_2 + \mathrm{Im} g_2)} |\psi|^2 = -c$$
$$-2\Delta(\mathrm{Re} g_2) + \frac{1}{4} e^{-2(\mathrm{Re} g_2 + \mathrm{Im} g_2)} |\psi|^2 = -c$$

This gives:

$$\Delta(\mathrm{Re} g_2 - \mathrm{Im} g_2) = 0$$
$$\Rightarrow \mathrm{Re} g_2 = \mathrm{Im} g_2 + a_4, \text{ for a constant } a_4$$

There exists a unique solution for $\mathrm{Re} g_2$ from the following pde for $c < 0$ [16]:

$$-2\Delta(\mathrm{Re} g_2) + \frac{1}{4} e^{-4\mathrm{Re} g_2} e^{-2a_4} |\psi|^2 = -c$$

Once we know g_2, we solve the remaining equation in the same way we did for the $c > 0$ case.

$$F_A = -\frac{i}{4} e^{-4\mathrm{Re} g_2} e^{-2a_4} |\psi|^2 \omega$$

The required condition to solve it being

$$\int_\Sigma i(4F_{A_h} - 2F_{K_\Sigma}) > 0 \Leftrightarrow \int_\Sigma c_1(2L - K_\Sigma) > 0$$

Proposition 22. *There exists a solution of the system of equations* (61) *and* (62) *on* $\Sigma \times \mathbb{R}^3$ *under the following conditions:*

$$\text{for } c \neq 0: \ \dim H^0(\Sigma, K_\Sigma - L) > 0 \text{ and } \deg(K_\Sigma - 2L) < 0$$
$$\text{for } c = 0: \ \deg(K_\Sigma - 2L) = 0$$

Moreover for all c, *there exists a solution which is translation invariant in any direction in* $\mathbb{R}^3$.

Proof. Both $c > 0$ and $c < 0$ cases are discussed above. The remaining case $c = 0$ is explained in proposition 20. □

Remark 23. For an explicit ansatz, we take a compact Riemann surface Σ with genus $g > 1$. If $L = K_\Sigma$, it satisfies both the conditions when $c \neq 0$ since, $\dim H^0(\Sigma, K_\Sigma - L) = 1$ and $\deg(K_\Sigma - 2L) = -\deg(K_\Sigma) = 2 - 2g < 0$.

0.5 Solution of 6-dimensional SW equations

Let (M, g) be a 6-dimensional manifold Spin^c manifold. Moreover say $S = S_+ \oplus S_-$ and $\widetilde{S} = \widetilde{S}_+ \oplus \widetilde{S}_-$ be two (potentially different) Spin^c-bundles on M. Let $\mathcal{A}$ and $\mathcal{B}$ denote the set of unitary connections on $\det(S_+)$ and $\det(\widetilde{S}_-)$ respectively. Then the SW equations for $\phi \in \Gamma(S_+), \psi \in \Gamma(\widetilde{S}_-), \beta \in \Omega^3, A \in \mathcal{A}$ and $B \in \mathcal{B}$ are:

$$(D_A + c(\beta))\phi = 0 \tag{63}$$

$$F_A - 2i * d\beta = q(\phi) \tag{64}$$

$$(D_B + c(*\beta))\psi = 0 \tag{65}$$

$$F_B + 2i * d * \beta = q(\psi) \tag{66}$$

Remarks 24. We make a few remarks about the moduli space of the SW equations.

1. Notice that analogous to the 8d case, we can prove similar a priori estimates for the 6d SW equations using the same techniques used in §V. I.e., given a C^0 bound of β, we get a C^0 bound of the spinors ϕ and ψ and for a large enough p, L^p bounds of $|\phi|, |\psi|$ and some bound of the harmonic part of β (with respect to any norm), we get a C^0 bound of β.

2. We explain a possible way the moduli space can be non-compact and produce explicit examples in §§0.5.2 on Kähler 3-folds where this phenomenon occurs. The idea is the following. Say $(\phi, \psi, \beta, A, B)$ is a solution of the equations (63),(64), (65),(66). Now say α is a non-trivial harmonic three-form such that $c(\alpha)\phi = 0$ and $c(*\alpha)\psi = 0$, then for any $r \in \mathbb{R}$, $(\phi, \psi, \beta + r\alpha, A, B)$ is also a solution of the SW equations.

   This brings a question what if we fix the harmonic part of the three-form, can we prove that the moduli space is compact in that scenario? The answer is still unknown to the author.

0.5.1 Spin geometry in dimension 6

Some representation theory of Spin(6) **and** Spin^c(6) :

Spin(6) $\cong$ SU(4)**:** We start with a description of the fundamental group of SO(6), i.e., Spin(6). This is one of the exceptional isomorphisms of Lie groups. Enough to show that SU(4) is a double cover of SO(6) since $\pi_1\big(SO(6)\big) = \mathbb{Z}/2\mathbb{Z}$. The double cover is constructed as follows:

SU(4) has a natural action on $\mathbb{C}^4$, it induces an action of SU(4) on $\Lambda^2\mathbb{C}^4$. $\Lambda^2\mathbb{C}^4$ is six dimensional (in $\mathbb{C}$) and has a hermitian inner-product induced by the one on $\mathbb{C}^4$. This can be defined by saying if $\{e_1, \dots, e_4\}$ is an orthonormal basis for $\mathbb{C}^4$ then $\{e_i \wedge e_j\}$ is an orthonormal basis for $\Lambda^2\mathbb{C}^4$ or, more invariantly,

$$\langle v_1 \wedge v_2, w_1 \wedge w_2\rangle = \det\langle v_i, w_j\rangle$$

Where $\langle v, w\rangle$ is the Hermitian inner product on $\mathbb{C}^4$. The action of SU(4) on $\Lambda^2\mathbb{C}^4$ preserves this inner product. Moreover we have a hodge star operator $*$ on $\Lambda^2\mathbb{C}^4$ induced by the inner product such that $*^2 = 1$, Hence it splits the space into self-dual and anti self-dual forms:

$$\Lambda^2\mathbb{C}^4 = \Lambda^2_+\mathbb{C}^4 \oplus \Lambda^2_-\mathbb{C}^4$$

Notice that the action of SU(4) is preserved under this splitting and the real dimension of $\Lambda^2_+\mathbb{C}^4$ is 6. Since, SU(4) preserves the inner product and has determinant one, we get a map from $SU(4) \to SO(6)$, one checks that the kernel is $\pm$Id, and therefore, by dimensionality reasons, must be a surjection.

Remark 25. The discussion above says that if our 6-dimensional manifold (say M) is spin, and $S = S_+ \oplus S_-$ is a spin bundle on M, then $\Lambda^2_+(S_+) \cong TM$.

Positive and negative spinors are dual to each other: For both the groups $\mathrm{Spin}(6) \cong SU(4)$ and $\mathrm{Spin}^c(6)$, they have two irreducible representations of dimension 4 and they are dual to each other [11]. Since the representations corresponding to positive and negative spinors have different highest weights: $(1/2, 1/2, 1/2)$ and $(1/2, 1/2, -1/2)$ [11], the positive and negative spinors must come from the two different irreducible representations and hence they are dual to each other.

If we start from $SU(4)$, which has a standard 4-dimensional irreducible representation (say V). The wedge product $\Lambda^3 V$ is also 4-dimensional and irreducible. These two representations are not isomorphic to each other because their highest weights are different, thus V and $\Lambda^3 V$ correspond to two spinor representations of $\mathrm{Spin}(6)$. However, $SU(4)$ admits an outer automorphism that exchanges the highest weights of V and $\Lambda^3 V$. So the positive spinor can be either V or $\Lambda^3 V$, depending on the isomorphism between $SU(4)$ and $\mathrm{Spin}(6)$ we choose. Despite this outer automorphism, we can still observe that two spinors are dual to each other since the wedge product gives us a non-degenerate pairing: $V \times \Lambda^3 V \to \mathbb{C}$.

Clifford multiplication:

Now we would work with Spin^c-manifolds. Let M be a Spin^c-manifold of dimension 6. We choose a Spin^c-structure on M and write $S = S_+ \oplus S_- \to M$, to be the corresponding spin bundle, which splits into the bundle of positive and negative spinors. Endomorphisms of spinors of the same chirality are given by the Clifford action of even-degree forms and morphisms between spinors of opposite chirality are given by the Clifford action of odd-degree forms:

$$c : \oplus_k \Lambda^{2k}(M) \otimes \mathbb{C} \to \mathrm{End}_{\mathbb{C}}(S_+, S_+) \cong \mathrm{End}_{\mathbb{C}}(S_-, S_-)$$
$$c : \oplus_k \Lambda^{2k+1}(M) \otimes \mathbb{C} \to \mathrm{End}_{\mathbb{C}}(S_+, S_-) \cong \mathrm{End}_{\mathbb{C}}(S_-, S_+)$$

Below we give a more detailed description of these maps.

The Clifford action of the volume form is i times identity on positive spinors and $-i$ times identity on negative spinors. This says that for any form $\alpha \in \Lambda^*(M)$, $c(\alpha) = \pm i c(*\alpha)$, depending on the degree of α and the chirality of the spinor. Clifford action of complexified 0 and 2-forms give the isomorphisms:

$$\mathrm{End}_{\mathbb{C}}(S_+, S_+) \cong \big(\Lambda^0(M) \oplus \Lambda^2(M)\big) \otimes \mathbb{C} \cong \mathrm{End}_{\mathbb{C}}(S_-, S_-)$$

For a hermitian endomorphism of spinors of same chirality, we can split it into a trace-free part and the trace-part. Λ^0 is the trace part and the imaginary 2-forms (or equivalently the real 4-forms) (real dimension 15) act as trace-free hermitian endomorphisms via Clifford multiplication:

$$c : i\Lambda^2(M) \cong i\mathfrak{su}(S_+) \cong i\mathfrak{su}(S_-)$$
$$c : \Lambda^4(M) \cong i\mathfrak{su}(S_+) \cong i\mathfrak{su}(S_-)$$

Coming back to $\mathrm{End}_{\mathbb{C}}(S_+, S_-)$, it is a complex-vector bundle of rank 16 over M. As explained before, Clifford action of a form is same as the Clifford action of its Hodge-star upto a constant. So, we get

$$c : \big(\Lambda^1(M) \oplus \Lambda^3(M)\big) \otimes \mathbb{C} \to \mathrm{End}_{\mathbb{C}}(S_+, S_-) \cong \mathrm{End}_{\mathbb{C}}(S_-, S_+)$$

We make an important observation on the space of three forms of the manifold. on $\Lambda^3(M;\mathbb{C})$, $*^2 = (-1)^{3\times(6-3)} = -1$. Hence $\Lambda^3(M;\mathbb{C})$ splits as $\Lambda^3_+(X,\mathbb{C}) \oplus \Lambda^3_-(M;\mathbb{C})$. The subscript $+$ and $-$ respectively denote the eigen-spaces of $+i$ and $-i$. Here $*$ is a complex linear extension of the Hodge-star operator on $\Lambda^3(M;\mathbb{R})$.

Lemma 26. *For a* Spin^c *bundle* $S = S_+ \oplus S_-$ *on* M, *the Clifford action of* $\Lambda^3_+(X,\mathbb{C})$ *on the negative spinors is trivial and the Clifford action of* $\Lambda^3_-(X,\mathbb{C})$ *on the positive spinors is trivial.*

Proof. The proof is given for a specific three-form. The other cases can be checked similarly. Around a point $p \in M$, we choose a normal coordinate system $(x_1, \ldots, x_6)$ and the corresponding co-vectors at p be $e_1, \ldots, e_6$. From now on in this proof, all the equations are supposed to be thought of at the point p with respect to the chosen normal neighborhood. Let's ϕ_+ be a positive spinor, i.e.,

$$c(-ie_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge e_5 \wedge e_6)\phi_+ = \phi_+$$

Let's take the 3-form $\beta = (e_1 \wedge e_2 \wedge e_3) + i(e_4 \wedge e_5 \wedge e_6)$.

$$\begin{aligned} *\beta &= *(e_1 \wedge e_2 \wedge e_3) + i * (e_4 \wedge e_5 \wedge e_6) \\ &= (e_4 \wedge e_5 \wedge e_6) - i(e_1 \wedge e_2 \wedge e_3) \\ &= -i\big((e_1 \wedge e_2 \wedge e_3) + i(e_4 \wedge e_5 \wedge e_6)\big) \end{aligned}$$

So, β is a $-i$ eigen-vector of the Hodge-star operator on three-forms.

$$\begin{aligned} c(\beta)\phi_+ &= c(\beta)c(-ie_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge e_5 \wedge e_6)\phi_+ \\ &= -ic\big((e_1 \wedge e_2 \wedge e_3) + i(e_4 \wedge e_5 \wedge e_6)\big)c(e_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge e_5 \wedge e_6)\phi_+ \\ &= -i\big((c(e_1)c(e_2)c(e_3))^2 c(e_4)c(e_5)c(e_6) \\ &\qquad + ic(e_4)c(e_5)c(e_6)c(e_1)c(e_2)c(e_3)c(e_4)c(e_5)c(e_6)\big)\phi_+ \\ &= -i\big(c(e_4)c(e_5)c(e_6) - ic(e_1)c(e_2)c(e_3)\big)\phi_+ \\ &= -c\big((e_1 \wedge e_2 \wedge e_3) + i(e_4 \wedge e_5 \wedge e_6)\big)\phi_+ \\ &= -c(\beta)\phi_+ \\ \Rightarrow c(\beta)\phi_+ &= 0 \end{aligned}$$

Identical calculation proves that for β, a $+i$ eigen-vector of the Hodge-star operator on three-forms, and a negative spinor ϕ_-

$$c(\beta)\phi_- = 0$$

□

Therefore, we have the following isomorphisms:

$$\begin{aligned} c : \big(\Lambda^1(M) \oplus \Lambda^3_+(M)\big) \otimes \mathbb{C} &\cong \mathrm{End}_{\mathbb{C}}(S_+, S_-) \\ c : \big(\Lambda^1(M) \oplus \Lambda^3_-(M)\big) \otimes \mathbb{C} &\cong \mathrm{End}_{\mathbb{C}}(S_-, S_+) \end{aligned}$$

An immediate question arises: *how does one differentiate a one-form from a three form as an element of* $End_{\mathbb{C}}(S_+, S_-)$ *or* $End_{\mathbb{C}}(S_-, S_+)$? *In other words, what is the induced splitting* *on the right hand side from the splitting on the left hand side?* Let's see the case for $\mathrm{End}_{\mathbb{C}}(S_-, S_+)$, the other one is identical. Notice, $\mathrm{End}_{\mathbb{C}}(S_-, S_+) \cong S_-^* \otimes S_+ \cong S_+ \otimes S_+$. This splits into symmetric and anti-symmetric tensors:

$$S_+ \otimes S_+ \cong \mathrm{Sym}(S_+) \oplus \Lambda^2(S_+)$$

$\Lambda^1 \otimes \mathbb{C}$ identifies with $\Lambda^2(S_+)$ (both has rank 6), and $\Lambda^3_- \otimes \mathbb{C}$ identifies with $\mathrm{Sym}(S_+)$ (both has rank 10). Similarly, for $\mathrm{End}_{\mathbb{C}}(S_+, S_-) \cong S_+^* \otimes S_- \cong S_- \otimes S_- \cong \mathrm{Sym}(S_-) \oplus \Lambda^2(S_-)$; $\Lambda^1 \otimes \mathbb{C}$ identifies with $\Lambda^2(S_-)$ (both has rank 6), and $\Lambda^3_+ \otimes \mathbb{C}$ identifies with $\mathrm{Sym}(S_-)$ (both has rank 10).

0.5.2 Solution on a closed Kähler 3-fold

The equations on a Kähler 3-fold

We study the SW equations when the oriented Riemannian 6-manifold is a 3 dimensional complex Kähler manifold say $(X, \omega); \omega$ being the Kähler form. The Kähler form together with a Riemannian metric determines a unique compatible J.

First let's see which (p, q)-forms are in Ω^3_+ and which are in Ω^3_-. We start with the $(1, 2)$ and $(2, 1)$ forms. We take local holomorphic coordinates $\{z_k = x_k + iy_k\}_{k=1,2,3}$ centered at a point $x \in X$ so that the Kähler metric is standard to second order at the point. In local coordinates we see that at x, $\{dz_k \wedge d\bar{z}_j \wedge d\bar{z}_l\}_{\{j \neq l\}}$ span the $(1, 2)$ forms. Observe

$$\begin{aligned} &* (d\bar{z}_2 \wedge dx_1 \wedge dy_1) = i(d\bar{z}_2 \wedge dx_3 \wedge dy_3) \quad \text{and} \\ &* (d\bar{z}_2 \wedge dx_3 \wedge dy_3) = i(d\bar{z}_2 \wedge dx_1 \wedge dy_1) \end{aligned}$$

Hence we get

$$\begin{aligned} &* \big(d\bar{z}_2 \wedge (dx_1 \wedge dy_1 + dx_3 \wedge dy_3)\big) = i\big(d\bar{z}_2 \wedge (dx_3 \wedge dy_3 + dx_1 \wedge dy_1)\big) \\ &\Rightarrow *(d\bar{z}_2 \wedge \omega) = i(d\bar{z}_2 \wedge \omega) \quad \text{and} \\ &* \big(d\bar{z}_2 \wedge (dx_1 \wedge dy_1 - dx_3 \wedge dy_3)\big) = i\big(d\bar{z}_2 \wedge (dx_3 \wedge dy_3 - dx_1 \wedge dy_1)\big) \\ &\Rightarrow *\big(d\bar{z}_2 \wedge (dx_1 \wedge dy_1 - dx_3 \wedge dy_3)\big) = -i\big(d\bar{z}_2 \wedge (dx_1 \wedge dy_1 - dx_3 \wedge dy_3)\big) \end{aligned}$$

Also for $j \neq k \neq l$,

$$*(dz_j \wedge d\bar{z}_k \wedge d\bar{z}_l) = -i(dz_j \wedge d\bar{z}_k \wedge d\bar{z}_l)$$

Notice that for $j \neq k \neq l, (dz_j \wedge d\bar{z}_k \wedge d\bar{z}_l) \wedge \omega = 0$ and similarly we also get

$$\begin{aligned} &* \big(d\bar{z}_j \wedge (d\bar{z}_k - d\bar{z}_l)\big) = -i(d\bar{z}_j \wedge (d\bar{z}_k - d\bar{z}_l) \quad \text{and} \\ &\big(d\bar{z}_j \wedge (d\bar{z}_k - d\bar{z}_l)\big) \wedge \omega = 0. \end{aligned}$$

One can do similar calculations for a $(2, 1)$-form and we end up with the following proposition.

Proposition 27. *On a Kähler 3-fold (X, ω), any real valued 3-form β can be written as*

$$\beta = \beta^{3,0}_- + (\eta \wedge \omega)_+ + \gamma_- + \overline{\beta^{3,0}}_+ + (\bar{\eta} \wedge \omega)_- + \bar{\gamma}_+$$

Where $\eta \in \Omega^{0,1}(X, \mathbb{C}), \gamma \in Ker\big(\wedge \omega : \Omega^{1,2}(X, \mathbb{C}) \to \Omega^{2,3}(X, \mathbb{C})\big)$. The subscript '+' or '−' respectively denote the form as an eigen-vector of the Hodge-star operator with eigenvalue $+i$ or $-i$. The decomposition is same as the Lefschetz decomposition [13].

Proof. We notice that in local holomorphic coordinates $\{z_k = x_k + iy_k\}_{k=1,2,3}$ we have

$$*(d\bar z_1 \wedge d\bar z_2 \wedge d\bar z_3) = i(d\bar z_1 \wedge d\bar z_2 \wedge d\bar z_3) \quad \text{and}$$
$$*(dz_1 \wedge dz_2 \wedge dz_3) = -i(dz_1 \wedge dz_2 \wedge dz_3)$$

The rest follows from the discussion above for $(1,2)$-forms and identical calculations for $(2,1)$-forms. □

We have an induced Spin^c-structure $\widetilde P_J \to X$ on X from its almost complex structure $J : TX \to TX$. The Spin^c-bundles are given by

$$S_+(\widetilde P_J) = \Lambda^0(X,\mathbb C) \oplus \Lambda^{0,2}(X,\mathbb C)$$
$$S_-(\widetilde P_J) = \Lambda^{0,1}(X,\mathbb C) \oplus \Lambda^{0,3}(X,\mathbb C)$$

$\mathrm{Det}\big(S_+(\widetilde P_J)\big) \cong \mathrm{Det}\big(S_-(\widetilde P_J)\big) = K_X^{-2}$. Furthermore, if the almost complex structure is in fact a complex structure for which the Riemannian metric is a Kähler metric, then the Dirac operator on positive spinors associated to this Spin^c-structure and the natural holomorphic, hermitian connection on K_X^{-2} is

$$\sqrt2(\bar\partial + \bar\partial^*) : \Omega^0(X,\mathbb C) \oplus \Omega^{0,2}(X,\mathbb C) \to \Omega^{0,1}(X,\mathbb C) \oplus \Omega^{0,3}(X,\mathbb C)$$

Any other Spin^c-structure $\widetilde P$ differs from $\widetilde P_J$ by tensoring with some $U(1)$ bundle $Q \to X$. let $\mathcal L_0$ be the complex line bundle associated to Q. Then the spin bundles for $\widetilde P$ are given by

$$S_+(\widetilde P) = \Lambda^0(X,\mathcal L_0) \oplus \Lambda^{0,2}(X,\mathcal L_0)$$
$$S_-(\widetilde P) = \Lambda^{0,1}(X,\mathcal L_0) \oplus \Lambda^{0,3}(X,\mathcal L_0)$$

The Clifford multiplication of the forms on the spinors are discussed in detail in appendix A.1. We will use the formulae from the appendix throughout this section. The determinant of $\widetilde P$ is identified with $K_X^{-2} \otimes \mathcal L_0^4$, or to put it another way, $\mathcal L_0^4 = K_X^2 \otimes \mathcal L$, where $\mathcal L = \det(\widetilde P)$. A $U(1)$-connection A on $\mathcal L$ is equivalent to a unitary connection A_0 on $\mathcal L_0$, the equivalence being $A_0^4 = A_{K_X}^2 \otimes A$ (with some possible abuse of notation) where A_{K_X} is the holomorphic connection on K_X. The Dirac operator associated to the connection A on $\mathcal L$ is [20]

$$\sqrt2(\bar\partial_{A_0} + \bar\partial^*_{A_0}) : \Omega^0(X,\mathcal L_0) \oplus \Omega^{0,2}(X,\mathcal L_0) \to \Omega^{0,1}(X,\mathcal L_0) \oplus \Omega^{0,3}(X,\mathcal L_0),$$

the operator obtained by coupling $\sqrt2(\bar\partial + \bar\partial^*)$ with the covariant derivative ∇_{A_0} on $\mathcal L_0$.

We clarify some notations which will be used in the next part of the article. For a form in $\Omega^{p,q}(X,\mathcal L_0)$ we define $* : \bigwedge^{p,q} X \otimes \mathcal L_0 \to \bigwedge^{n-q,n-p} X \otimes \mathcal L_0$

$$*(\phi \otimes s) = *(\phi) \otimes s \ (\phi \in \Omega^{p,q}(X,\mathbb C) \text{ locally, } * \text{ is complex-linear on } \Omega^{p,q}(X,\mathbb C))$$

Notice that this is different from the usual $\bar *_h$ operator, which can be defined if we put a hermitian metric h on $\mathcal L_0$. h can be also interpreted as a $\mathbb C$-antilinear isomorphism $h : \mathcal L_0 \cong \mathcal L_0^*$. We get $\bar *_h : \bigwedge^{p,q} X \otimes \mathcal L_0 \to \bigwedge^{n-p,n-q} X \otimes \mathcal L_0^*$

$$\bar *_h(\phi \otimes s) = \bar *(\phi) \otimes h(s) = \overline{*(\phi)} \otimes h(s) = *(\bar\phi) \otimes h(s)$$

First pair of equations

Let's have a look at the first Dirac equation.

$$(D_A + c(\beta))\phi = 0, \qquad \phi = \phi_1 + \phi_2 \in \Omega^0(X, \mathcal{L}_0) \oplus \Omega^{0,2}(X, \mathcal{L}_0)$$
$$\beta = (\beta^{3,0} + \overline{\beta^{3,0}}) + (\beta^{1,2} + \overline{\beta^{1,2}}) \in \Omega^{3,0}(X, \mathbb{C}) \oplus \Omega^{0,3}(X, \mathbb{C}) \oplus \Omega^{1,2}(X, \mathbb{C}) \oplus \Omega^{2,1}(X, \mathbb{C})$$

In this Kähler set up, the first Dirac equation reads:

$$\sqrt{2}(\bar{\partial}_{A_0}\phi_1 + \bar{\partial}^*_{A_0}\phi_2) + c(\beta^{1,2})\phi_1 + c(\overline{\beta^{1,2}})\phi_2 = 0 \tag{67}$$
$$\sqrt{2}\bar{\partial}_{A_0}\phi_2 + c(\overline{\beta^{3,0}})\phi_1 + c(\beta^{1,2})\phi_2 = 0 \tag{68}$$

We turn to the first curvature equation: $F_A - 2i * d\beta = q(\phi)$. In matrix form the curvature equation reads:

$$\begin{bmatrix} -i\langle (F_A - 2i * d\beta)^{1,1}, \omega\rangle & -2(F_A - 2i * d\beta)^{2,0} \\ 2(F_A - 2i * d\beta)^{0,2} & -2i * \left((F_A - 2i * d\beta)^{1,1} \wedge (\ )\right) + i\langle F_A - 2i * d\beta, \omega\rangle \end{bmatrix}$$
$$= \begin{bmatrix} \frac{3}{4}|\phi_1|^2 - \frac{1}{4}|\phi_2|^2 & \bar{\phi}_2\phi_1 \\ \bar{\phi}_1\phi_2 & \bar{\phi}_2\phi_2 - \frac{1}{4}|\phi_1|^2 - \frac{1}{4}|\phi_2|^2 \end{bmatrix} \tag{69}$$

Second pair of equations

For the second pair of equations we work with a potentially different Spin^c-structure on X. Similar to the last case, any other Spin^c structure $\widetilde{Q}$ differs from $\widetilde{P}_J$ by tensoring with some $U(1)$ bundle $Q_1 \to X$. Let $\mathcal{L}_1$ be the complex line bundle associated to Q_1. Then the spin bundles for $\widetilde{Q}$ are given by

$$S_+(\widetilde{Q}) = \Lambda^0(X, \mathcal{L}_1) \oplus \Lambda^{0,2}(X, \mathcal{L}_1)$$
$$S_-(\widetilde{Q}) = \Lambda^{0,1}(X, \mathcal{L}_1) \oplus \Lambda^{0,3}(X, \mathcal{L}_1)$$

The determinant of $\widetilde{Q}$ is identified with $K_X^{-2} \otimes \mathcal{L}_1^4$. or to put it another way, $\mathcal{L}_1^4 = K_X^2 \otimes \widetilde{\mathcal{L}}$, where $\widetilde{\mathcal{L}} = \det(\widetilde{Q})$. A $U(1)$-connection B on $\widetilde{\mathcal{L}}$ is equivalent to a unitary connection B_0 on $\mathcal{L}_1$, the equivalence being $B_0^4 = A_{K_X}^2 \otimes B$ where A_{K_X} is the holomorphic connection on K_X. The Dirac operator associated to the connection B on $\widetilde{\mathcal{L}}$ is [20]

$$\sqrt{2}(\bar{\partial}_{B_0} + \bar{\partial}^*_{B_0}) : \Omega^0(X, \mathcal{L}_1) \oplus \Omega^{0,2}(X, \mathcal{L}_1) \to \Omega^{0,1}(X, \mathcal{L}_1) \oplus \Omega^{0,3}(X, \mathcal{L}_1),$$

the operator obtained by coupling $\sqrt{2}(\bar{\partial} + \bar{\partial}^*)$ with the covariant derivative ∇_{B_0} on $\mathcal{L}_1$. The second Dirac equation says

$$(D_B + c(*\beta))\psi = 0, \qquad \psi = \psi_1 + \psi_2 \in \Omega^{0,3}(X, \mathcal{L}_1) \oplus \Omega^{0,1}(X, \mathcal{L}_1)$$
$$\beta = (\beta^{3,0} + \overline{\beta^{3,0}}) + (\beta^{1,2} + \overline{\beta^{1,2}}) \in \Omega^{3,0}(X, \mathbb{C}) \oplus \Omega^{0,3}(X, \mathbb{C}) \oplus \Omega^{1,2}(X, \mathbb{C}) \oplus \Omega^{2,1}(X, \mathbb{C})$$
$$\Rightarrow *\beta = (*\beta^{3,0} + *\overline{\beta^{3,0}}) + (*\beta^{1,2} + *\overline{\beta^{1,2}}) \in \Omega^{3,0}(X, \mathbb{C}) \oplus \Omega^{0,3}(X, \mathbb{C}) \oplus \Omega^{1,2}(X, \mathbb{C}) \oplus \Omega^{2,1}(X, \mathbb{C})$$

In the Kähler set up, the equation reads:

$$\sqrt{2}\,\bar{\partial}^*_{B_0}\psi_2 + c(*\beta^{3,0})\psi_1 + c(*\overline{\beta^{1,2}})\psi_2 = 0 \tag{70}$$
$$\sqrt{2}(\bar{\partial}^*_{B_0}\psi_1 + \bar{\partial}_{B_0}\psi_2) + c(*\beta^{1,2})\psi_2 + c(*\overline{\beta^{1,2}})\psi_1 = 0 \tag{71}$$

We write down the second curvature equation in its matrix form. $F_B + 2i * d * \beta = q(\psi)$ reads:

$$\begin{bmatrix} i\langle(F_B + 2i * d * \beta), \omega\rangle & 2(F_B + 2i * d * \beta)^{0,2} \wedge (\ ) \\ 2i * \big((F_B + 2i * d * \beta)^{2,0} \wedge (\ )\big) & 2 * \big(* (F_B + 2i * d * \beta)^{1,1} \wedge (\ )\big) - i\langle F_B + 2i * d * \beta, \omega\rangle \end{bmatrix}$$
$$= \begin{bmatrix} \frac{3}{4}|\psi_1|^2 - \frac{1}{4}|\psi_2|^2 & \bar{\psi}_2\psi_1 \\ \bar{\psi}_1\psi_2 & \bar{\psi}_2\psi_2 - \frac{1}{4}|\psi_1|^2 - \frac{1}{4}|\psi_2|^2 \end{bmatrix} \tag{72}$$

Construction of a solution

We start with a smooth function $f : X \to \mathbb{C}$ and choose the three-form β to be

$$\beta = (\bar{\partial} f + \partial \bar{f}) \wedge \omega$$

Now we choose the two spinors ϕ and ψ carefully (they will depend on f) so that they solve the two Dirac equations for any holomorphic connections on the line bundles $\mathcal{L}_0$ and $\mathcal{L}_1$. Since β is already first-order in f, the curvature equations turn into two second-order pdes in f. Using two lemmas (proved below) from Kähler geometry, we turn the curvature equations into Kazdan-Warner type pdes [16] and solve for f. The first curvature equation gives us the imaginary part of f and the second one gives us the real part of f.

Before proceeding further, we prove the two lemmas, which we will use to simplify the curvature equations.

Lemma 28. *On a compact Kähler three-fold* (X, ω), *for any smooth complex valued function* h, *we have*

$$*(\partial\bar{\partial} h \wedge \omega) = -\partial\bar{\partial} h + \frac{i}{2}\Delta(h)\omega$$

Proof. We take local holomorphic coordinates $\{z_k = x_k + iy_k\}_{k=1,2,3}$ centered at a point $x \in X$ so that the Kähler metric is standard to second order at the point. All the calculations below is done at the point x.

$$\begin{aligned} *(\partial\bar{\partial} h \wedge \omega) &= *\Big(\sum_{j,k=1}^{3} \frac{\partial^2 h}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k \wedge \omega\Big) \\ &= *\Big(\sum_{j\neq k} \frac{\partial^2 h}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k \wedge \omega + \sum_{k=1}^{3} \frac{\partial^2 h}{\partial z_k \partial \bar{z}_k} dz_k \wedge d\bar{z}_k \wedge \omega\Big) \end{aligned}$$

Now we notice what happens to the case $j \neq k$, let's take a specific case.

$$\begin{aligned} *(dz_1 \wedge d\bar{z}_2 \wedge \omega) &= *(dz_1 \wedge d\bar{z}_2 \wedge dx_3 \wedge dx_3) \\ &= *\{(dx_1 + idy_1) \wedge (dx_2 - idy_2) \wedge dx_3 \wedge dy_3\} \\ &= *\{(dx_1 \wedge dx_2 + dy_1 \wedge dy_2) \wedge dx_3 \wedge dy_3 + i(dy_1 \wedge dx_2 - dx_1 \wedge dy_2) \wedge dx_3 \wedge dy_3\} \\ &= -\{(dx_1 \wedge dx_2 + dy_1 \wedge dy_2) + i(dy_1 \wedge dx_2 - dx_1 \wedge dy_2)\} \\ &= -(dz_1 \wedge d\bar{z}_2) \end{aligned}$$

Similar calculations for all other cases, show that for $j \neq k$,

$$*(dz_j \wedge d\bar{z}_k \wedge \omega) = -(dz_j \wedge d\bar{z}_k)$$

Now for $j = k$, we get

$$\begin{aligned}*\big(dz_k \wedge d\bar{z}_k \wedge \omega\big) &= *\big(-2idx_k \wedge dy_k \wedge (\sum_{m=1}^{3} dx_m \wedge dy_m)\big)\\ &= -2i\big(\sum_{m\neq k} dx_m \wedge dy_m\big)\\ &= -2i(\omega - dx_k \wedge dy_k)\\ &= -dz_k \wedge d\bar{z}_k - 2i\omega\end{aligned}$$

So, we get

$$\begin{aligned}&*\big(\sum_{j\neq k} \frac{\partial^2 h}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k \wedge \omega + \sum_{k=1}^{3} \frac{\partial^2 h}{\partial z_k \partial \bar{z}_k} dz_k \wedge d\bar{z}_k \wedge \omega\big)\\ &= -\sum_{j\neq k} \frac{\partial^2 h}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k - \sum_{k=1}^{3} \frac{\partial^2 h}{\partial z_k \partial \bar{z}_k} dz_k \wedge d\bar{z}_k - 2i\big(\sum_{k=1}^{3} \frac{\partial^2 h}{\partial z_k \partial \bar{z}_k}\big)\omega\\ &= -\big(\sum_{j,k=1}^{3} \frac{\partial^2 h}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k\big) - 2i\big(\sum_{k=1}^{3} \frac{1}{2}(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k})\frac{1}{2}(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k})h\big)\omega\\ &= -\partial\bar{\partial}h - \frac{2i}{4}\big(\sum_{k=1}^{3}(\frac{\partial^2 h}{\partial x_k^2} + \frac{\partial^2 h}{\partial y_k^2})\big)\omega\\ &= -\partial\bar{\partial}h + \frac{i}{2}\Delta(h)\omega\end{aligned}$$

□

Lemma 29. *On a compact Kähler three-fold* (X, ω), *for any smooth complex valued function* h, *we have*

$$\langle \partial\bar{\partial}h, \omega\rangle = \frac{i}{2}\Delta(h)$$

Proof. We take local holomorphic coordinates $\{z_k = x_k + iy_k\}_{k=1,2,3}$ centered at a point $x \in X$ so that the Kähler metric is standard to second order at the point. All the calculations below is done at the point x.

$$\begin{aligned}\langle \partial\bar{\partial}h, \omega\rangle &= *\big(\partial\bar{\partial}h \wedge (*\omega)\big)\\ &= *\big(\sum_{j,k=1}^{3} \frac{\partial^2 h}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k \wedge (*\omega)\big)\end{aligned}$$

Notice that

$$\begin{aligned}&*\omega\\ &= *(dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + dx_3 \wedge dy_3)\\ &= (dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 + dx_2 \wedge dy_2 \wedge dx_3 \wedge dy_3 + dx_3 \wedge dy_3 \wedge dx_1 \wedge dy_1)\end{aligned}$$

So, for $j \neq k, dz_j \wedge d\bar{z}_k \wedge (*\omega) = 0$. Hence we get

$$
\begin{aligned}
&*\Big(\sum_{j,k=1}^{3} \frac{\partial^2 h}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k \wedge (*\omega)\Big)\\
&= *\Big(\sum_{k=1}^{3} \frac{\partial^2 h}{\partial z_k \partial \bar{z}_k} dz_k \wedge d\bar{z}_k \wedge (*\omega)\Big)\\
&= -2i *\Big(\sum_{k=1}^{3} \frac{\partial^2 h}{\partial z_k \partial \bar{z}_k} dx_k \wedge dy_k \wedge (*\omega)\Big)\\
&= -2i *\Big(\sum_{k=1}^{3} \frac{1}{2}\big(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\big)\frac{1}{2}\big(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}\big) h \, dV\Big) \quad (dV \text{ is the volume form})\\
&= \frac{-2i}{4}\Big(\sum_{k=1}^{3}\big(\frac{\partial^2 h}{\partial x_k^2} + \frac{\partial^2 h}{\partial y_k^2}\big)\Big)\\
&= \frac{i}{2}\Delta(h)
\end{aligned}
$$

□

For any Kähler manifold (X^n, ω) and a holomorphic line bundle L on it, let's define *the degree of* L *to be* $\int_X c_1(L) \wedge w^{n-1}$. $c_1(L)$ being the first Chern class of L.

Let's see what happens to the first set of Dirac equations (67),(68), if we assume $\phi_2 = 0 \in \Omega^{0,2}(X, \mathcal{L}_0)$. There's nothing to solve for in (68), as $\overline{\beta^{3,0}} = 0$. Equation 2.1 gives

$$
\begin{aligned}
&\sqrt{2}\bar{\partial}_{A_0}\phi_1 + c(\beta^{1,2})\phi_1 = 0\\
&\Rightarrow \sqrt{2}\bar{\partial}_{A_0}\phi_1 + c(\bar{\partial} f \wedge \omega)\phi_1 = 0\\
&\Rightarrow \sqrt{2}\bar{\partial}_{A_0}\phi_1 - 2\sqrt{2} i \bar{\partial} f \wedge \phi_1 = 0\\
&\Rightarrow \bar{\partial}_{A_0}\phi_1 = 2i\bar{\partial} f \wedge \phi_1
\end{aligned}
$$

To solve this we take $\phi_1 = e^{2if}\alpha_1$ where $\alpha_1 \in \Omega^0(X, \mathcal{L}_0)$ and $\bar{\partial}_{A_0}\alpha_1 = 0$. Then

$$
\begin{aligned}
\bar{\partial}_{A_0}\phi_1 &= \bar{\partial}_{A_0}(e^{2if}\alpha_1)\\
&= 2ie^{2if}\bar{\partial} f \wedge \alpha_1\\
&= 2i\bar{\partial} f \wedge \phi_1
\end{aligned}
$$

For $\phi_2 = 0, \eta = \bar{\partial} f, \beta^{3,0} = 0, \gamma = 0$, the curvature equation reads

$$
\begin{aligned}
&-i\langle(F_A - 2i * d\beta, \omega\rangle = \frac{3}{4}|\phi_1|^2\\
&F_A^{0,2} = 0 \text{ and}\\
&2i * \big((F_A - 2i * d\beta)^{1,1} \wedge (\;)\big) + i\langle F_A - 2i * d\beta, \omega\rangle = -\frac{1}{4}|\phi_1|^2
\end{aligned}
$$

The second condition says that $F_A^{0,2} = 4F_{A_0}^{0,2} = 0$. Hence $\mathcal{L}_0$ must have a holomorphic structure. We put a Hermitian metric h on $\mathcal{L}_0$ and choose A_0 to be the Chern connection with respect to

h. Also, the first and third condition combines to give

$$2i * \big((F_A - 2i * d\beta)^{1,1} \wedge (\ )\big) = \frac{1}{2}|\phi_1|^2$$

Following the description of Clifford multiplication of $(1,1)$-forms, $\mu \in \Omega^{1,1}(X, \mathbb{C})$ acts on $\lambda \in \Omega^{0,2}(X, \mathcal{L}_0)$ to give back a form in $(0,2)$ form in $\Omega^{0,2}(X, \mathcal{L}_0)$ by the action $c(\mu)\lambda = *(\mu \wedge \lambda)$ and for $\mu = \omega$, we get $*(\omega \wedge \lambda) = \lambda$. Hence point-wise $(F_A - 2i * d\beta)^{1,1}$ must be a multiple of ω and we observe that

$$(F_A - 2i * d\beta)^{1,1} = \frac{i}{4}|\phi_1|^2\omega$$

solves it for us. Hence, the first curvature equation reads:

$$\begin{aligned}
&(F_A - 2i * d\beta)^{1,1} = \frac{i}{4}|\phi_1|^2\omega \\
&\Rightarrow F_A - 2i * \big((\partial\bar{\partial} f + \bar{\partial}\partial\bar{f}) \wedge \omega\big) = \frac{i}{4} * (\phi_1 \wedge *\overline{\phi_1})\omega \\
&\Rightarrow F_A - 2i * \big(\partial\bar{\partial}(f - \bar{f}) \wedge \omega\big) = \frac{i}{4} * (e^{2if}\alpha_1 \wedge e^{-2i\bar{f}} * \overline{\alpha_1})\omega \\
&\Rightarrow F_A - 2i * \big(\partial\bar{\partial}(2i\mathrm{Im}(f)) \wedge \omega\big) = \frac{i}{4}e^{2i(f-\bar{f})} * (\alpha_1 \wedge *\overline{\alpha_1})\omega \\
&\Rightarrow F_A + 4 * \big(\partial\bar{\partial}(\mathrm{Im}(f)) \wedge \omega\big) = \frac{i}{4}e^{-4\mathrm{Im} f}|\alpha_1|^2\omega \\
&\Rightarrow F_A - 4\partial\bar{\partial}(\mathrm{Im}(f)) + 4 \times \frac{i}{2}\Delta(\mathrm{Im}(f))\omega = \frac{i}{4}e^{-4\mathrm{Im} f}|\alpha_1|^2\omega
\end{aligned}$$

Say, the initial Hermitian metric on $\mathcal{L}_0$ is h, we make a conformal change by e^λ for a smooth function $\lambda : X \to \mathbb{R}$. So with the new metric $h' = e^\lambda h$, and the corresponding connection A' on the determinant bundle $\mathcal{L} = K_X^{-1} \otimes \mathcal{L}_0^4$, the equation becomes

$$\begin{aligned}
&F_{A'} - 4\partial\bar{\partial}(\mathrm{Im}(f)) + 4 \times \frac{i}{2}\Delta(\mathrm{Im}(f))\omega = \frac{i}{4}e^{-4\mathrm{Im} f}|\alpha_1|^2_{h'}\omega \\
&\Rightarrow F_A - 4\partial\bar{\partial}\lambda - 4\partial\bar{\partial}(\mathrm{Im}(f)) + 4 \times \frac{i}{2}\Delta(\mathrm{Im}(f))\omega = \frac{i}{4}e^{-4\mathrm{Im} f + \lambda}|\alpha_1|^2\omega \\
&\Rightarrow F_A = \partial\bar{\partial}(4\lambda + 4\mathrm{Im}(f)) + i\big(\Delta(-2\mathrm{Im}(f)) + \frac{e^\lambda|\alpha_1|^2}{4}e^{-4\mathrm{Im}(f)}\big)\omega
\end{aligned}$$

Now F_A is a d-closed 2 form. So, $F_A = d\theta + F_A^{\mathcal{H}}$ for a suitable 1 form θ and a Harmonic form $F_A^{\mathcal{H}}$. Now $d\theta$ is a closed form and it's d-exact. Hence, it's $\partial\bar{\partial}$ exact by $\partial\bar{\partial}$-lemma [13]. So, we have $g : X \to \mathbb{R}$ (notice that g must be real valued for $\partial\bar{\partial}(g)$ to be an imaginary-valued form) and a Harmonic form $F_A^{\mathcal{H}}$ such that $F_A = 4\partial\bar{\partial}(g) + F_A^{\mathcal{H}}$. Moreover if we assume that the first Chern class of the determinant line bundle is a constant multiple of the Kähler form ω (this is a topological assumption on the line bundle), we can write

$$F_A = 4\partial\bar{\partial}(g) + ic\omega \qquad \text{(for some constant c)}$$

Hence we need to solve for

$$g = \lambda + \mathrm{Im}(f) \quad \text{and} \quad \Delta(-2\mathrm{Im}(f)) + \frac{e^\lambda|\alpha_1|^2}{4}e^{-4\mathrm{Im}(f)} = c$$

Putting $\lambda = g - \mathrm{Im}(f)$ in the second equation we get

$$\Delta(-2\mathrm{Im}(f)) + \frac{e^{g}|\alpha_1|^2}{4} e^{-5\mathrm{Im}(f)} = c$$

This pde is of Kazdan-Warner type and has a unique solution for any g if $c > 0$ and α_1 is not identically 0 [16] (we would we need degree of the determinant line bundle $\mathcal{L}$ to be negative for c to be positive). Once we get $\mathrm{Im}(f)$, we get λ from the first equation.

Next, we turn our attention to the second pair of equations. We start with a holomorphic structure on $\mathcal{L}_1$ and choose a compatible Hermitian metric $\widetilde{h}$ on $\mathcal{L}_1$ and let's say $\widetilde{h}^*$ be the induced metric on $\mathcal{L}_1^*$. We put the Chern connection B_0 on $\mathcal{L}_1$ so that F_{B_0} and hence F_B has only nontrivial $(1,1)$ part. Again we choose $\psi_2 = 0 \in \Omega^{0,1}(X, \mathcal{L}_1)$. Then equations (70) and (71) read:

$$\begin{aligned}
&\sqrt{2}\bar{\partial}^*_{B_0}\psi_1 + c\big(*(\partial\bar{f}\wedge\omega)\big)\psi_1 = 0\\
&\Rightarrow \sqrt{2}\bar{\partial}^*_{B_0}\psi_1 - ic(\partial\bar{f}\wedge\omega)\psi_1 = 0\\
&\Rightarrow \sqrt{2}\bar{\partial}^*_{B_0}\psi_1 - 2i\sqrt{2}*(\partial\bar{f}\wedge\psi_1) = 0
\end{aligned}$$

To solve this we take $\psi_1 = e^{-2\bar{f}}\xi_1$, where $\bar{\partial}^*_{B_0}\xi_1 = 0$. For smooth section $\alpha \in \Omega^{0,2}(X, \mathcal{L}_1)$,

$$\begin{aligned}
&\langle\bar{\partial}^*_{B_0}(e^{-2\bar{f}}\xi_1), \alpha\rangle_{L^2}\\
&= \langle\xi_1, e^{-2f}\bar{\partial}_{B_0}\alpha\rangle_{L^2}\\
&= 2\langle\xi_1, e^{-2f}\bar{\partial}f\wedge\alpha\rangle_{L^2}\\
&= 2\langle\psi_1, \bar{\partial}f\wedge\alpha\rangle_{L^2}\\
&= 2\overline{\langle\bar{\partial}f\wedge\alpha, \psi_1\rangle_{L^2}}\\
&= 2\overline{\int_X (\bar{\partial}f\wedge\alpha\wedge *\bar{\psi}_1)}\\
&= 2\overline{\int_X -i(\bar{\partial}f\wedge\alpha\wedge\bar{\psi}_1)}\\
&= 2\overline{\int_X \alpha\wedge *\overline{\big(*(i\partial\bar{f}\wedge\psi_1)\big)}}\\
&= \overline{\langle\alpha, 2*(i\partial\bar{f}\wedge\psi_1)\rangle_{L^2}}\\
&= \langle 2i*(\partial\bar{f}\wedge\psi_1), \alpha\rangle_{L^2}
\end{aligned}$$

So, $\bar{\partial}^*_{B_0}\psi_1 = 2i*(\partial\bar{f}\wedge\psi_1)$. What remains is the curvature equation, similar to the case of the curvature equation for positive spinors one is able to deduce that in our case, the curvature equation breaks into two equations:

$(F_B + 2i*d*\beta)^{1,1} = -\frac{i}{4}|\psi_1|^2\omega$ (This solves the diagonal parts) and

$F_B^{0,2} = 0$ (This solves the anti-diagonal part)

$$d*\beta = i(\partial\bar{\partial}f - \bar{\partial}\partial\bar{f})\wedge\omega = 2i(\partial\bar{\partial}\mathrm{Re}f)\wedge\omega$$

So we get,

$$
\begin{aligned}
&F_B + 2i * \big(2i(\partial\bar{\partial}\mathrm{Re} f) \wedge \omega\big) = -\frac{i}{4}|\psi_1|^2\omega \\
&\Rightarrow F_B - 4 * \big((\partial\bar{\partial}\mathrm{Re} f) \wedge \omega\big) = -\frac{i}{4}|\psi_1|^2\omega \\
&\Rightarrow F_B - 4\big(-\partial\bar{\partial}\mathrm{Re} f + \frac{i}{2}\Delta(\mathrm{Re} f)\omega\big) = -\frac{i}{4}|\psi_1|^2\omega \\
&\Rightarrow F_B + 4\partial\bar{\partial}\mathrm{Re} f = -i\big(-2\Delta(\mathrm{Re} f) + \frac{|\psi_1|^2}{4}\big)\omega \\
&\Rightarrow F_B + 4\partial\bar{\partial}\mathrm{Re} f = -i\big(-2\Delta(\mathrm{Re} f) + \frac{e^{-4\mathrm{Re} f}|\xi_1|^2}{4}\big)\omega
\end{aligned}
$$

Say the initial Hermitian metric on $\mathcal{L}_1$ was h_1, we make a conformal change on the metric by e^{g_1} for a smooth function $g_1 : X \to \mathbb{R}$. We notice that choosing ξ_1 an anti-holomorphic section of $\mathcal{L}_1$ is same as choosing the corresponding holomorphic section $\bar{\xi}_1$ of $\bar{\mathcal{L}}_1 = \mathcal{L}_1^*$. Now with the natural induced metric from h_1 say, h_1^* on $\mathcal{L}_1^*$ we also have $|\bar{\xi}_1|_{h_1^*} = |\xi_1|_{h_1}$. For a conformal change by e^{g_1} on h_1, we get a scaling by e^{-g_1} on h_1^*. As this is a conformal change in the metric, $\bar{\xi}_1$ is still a holomorphic section of $\bar{\mathcal{L}}_1 = \mathcal{L}_1^*$, and now when we go back to the corresponding anti-holomorphic section of $\mathcal{L}_1$, the norm of this section changes by e^{-g_1}. With the new metric $h_1' = e^{g_1}$, on $\mathcal{L}_1$ and the corresponding connection B' on the determinant bundle $K_X^{-2} \otimes \mathcal{L}_1^4$, the equation becomes

$$
\begin{aligned}
&F_{B'} + 4\partial\bar{\partial}\mathrm{Re} f = -i\big(-2\Delta(\mathrm{Re} f) + \frac{e^{-4\mathrm{Re} f}|\bar{\xi_1}|^2}{4}\big)\omega \\
&\Rightarrow F_B - 4\partial\bar{\partial}g_1 + 4\partial\bar{\partial}\mathrm{Re} f = -i\big(-2\Delta(\mathrm{Re} f) + \frac{e^{-4\mathrm{Re} f}e^{-g_1}|\xi_1|^2}{4}\big)\omega \\
&\Rightarrow F_B = \partial\bar{\partial}(4g_1 - 4\mathrm{Re} f) - i\big(-\Delta(2\mathrm{Re} f) + \frac{e^{-4\mathrm{Re} f - g_1}|\xi_1|^2}{4}\big)\omega
\end{aligned}
$$

If $F_B = 4\partial\bar{\partial}g' - ic'\omega$ for some smooth function $g' : X \to \mathbb{R}$ and a constant c', then we need to solve for g_1 and $\mathrm{Re}(f)$ such that

$$
g_1 - \mathrm{Re} f = g' \quad \text{and} \quad \Delta(-2\mathrm{Re} f) + \frac{e^{-4\mathrm{Re} f - g_1}|\xi_1|^2}{4} = c'
$$

Putting $-g_1 = -g' - \mathrm{Re} f$, we get

$$
\Delta(-2\mathrm{Re} f) + \frac{e^{-g'}|\xi_1|^2}{4}e^{-5\mathrm{Re} f} = c'
$$

Since, ξ_1 is not identically zero, this equation has a unique solution for $c' > 0$ [16], i.e., the determinant line bundle $\widetilde{\mathcal{L}}$ needs to have positive degree.

Hence, we see that for each $\alpha_1 \in H^0(X, \mathcal{L}_0)$ and $\bar{\xi}_1 \in H^0(X, K_X \otimes \mathcal{L}_1^{-1})$, there is a unique way to solve for $f : X \to \mathbb{C}$ and the two unitary connections A_0 and B_0 on $\mathcal{L}_0$ and $\mathcal{L}_1$ such that

$$
\big(\phi = e^{2if}\alpha_1, \psi = e^{-2\bar{f}}\xi_1, A = (-2A_{K_X} + 4A_0), B = (-2A_{K_X} + 4B_0), \beta = (\bar{\partial}f + \partial\bar{f}) \wedge \omega\big)
$$

solves the 6d SW equations.

Let's see how the solution changes if we scale α_1, ξ_1 by two non-zero constants $a, b \in \mathbb{C}^*$. Define

$$f_{a,b} := f + \frac{1}{2}\ln|b| + \frac{i}{2}\ln|a|$$

If we would have started with $a\alpha_1$ and $b\xi_1$ instead of α_1 and ξ_1, the above construction would give us a new solution of the equations:

$$\big(e^{2if_{a,b}}a\alpha_1, e^{-2\bar{f}_{a,b}}b\xi_1, (-2A_{K_X} + 4A_0), (-2A_{K_X} + 4B_0), (\bar{\partial} f_{a,b} + \partial \bar{f}_{a,b}) \wedge \omega\big)$$
$$= \Big(\frac{a}{|a|}e^{i\ln|b|}e^{2if}\alpha_1, \frac{b}{|b|}e^{i\ln|a|}e^{-2\bar{f}}\xi_1, (-2A_{K_X} + 4A_0), (-2A_{K_X} + 4B_0), (\bar{\partial} f + \partial \bar{f}) \wedge \omega\Big)$$

Which is gauge-equivalent to the original solution

$$\big(e^{2if}\alpha_1, e^{-2\bar{f}}\xi_1, (-2A_{K_X} + 4A_0), (-2A_{K_X} + 4B_0), \beta = (\bar{\partial} f + \partial \bar{f}) \wedge \omega\big)$$

And given two different sets of holomorphic sections of of $\mathcal{L}_0$ and $K_X \otimes \mathcal{L}_1^{-1}$ which are not in the same conformal classes lead to two different sets of solutions. Hence modulo gauge the space of solutions we constructed is the product of two projective spaces: $\mathbb{CP}\big(H^0(X, \mathcal{L}_0)\big) \times \mathbb{CP}\big(H^0(X, K_X \otimes \mathcal{L}_1^{-1})\big)$.

Remarks 30. We make three remarks regarding the solutions we constructed above.

- We make one important observation about how the moduli space can be non-compact, using the above construction. Take a three form $\alpha = (\gamma + \bar{\gamma})$, where $\gamma \in \Omega^{1,2}(X, \mathbb{C})$ and $\gamma \wedge \omega = 0$. Notice that for any $\phi \in \Omega^0(X, \mathcal{L}_0), \psi \in \Omega^0(X, K_X \otimes \mathcal{L}_1^{-1}), c(\alpha)\phi = 0$ and $c(*\alpha)\psi = 0$. If γ is harmonic, we can add any constant multiple of α to our three-form and get another solution. If γ is harmonic and $\gamma \wedge \omega = 0$, it is a harmonic primitive $(1, 2)$-form; i.e., $\gamma \in H^{1,2}(X)_p$ [13]. So, modulo gauge the space of solutions we constructed actually is $\mathbb{CP}\big(H^0(X, \mathcal{L}_0)\big) \times \mathbb{CP}\big(H^0(X, K_X \otimes \mathcal{L}_1^{-1})\big) \times H^{1,2}(X)_p$.
- Notice that if $b_3(X) = 0$ (this would imply $H^{1,2}(X)_p = \{0\}$), then for the choice of $\mathcal{L}_0$ to be the trivial line bundle and $\mathcal{L}_1 = K_X$, modulo gauge the space of solutions we constructed is a singleton set. This is reminiscent to the moduli space of 4d SW equations on Kähler surfaces: when we twist the canonical Spin^c-bundle by the trivial or the canonical line bundle, the moduli space is again a singleton set in that case.
- Notice that in our solution, the zero set of the spinors i.e., $\phi^{-1}(0)$ and $\psi^{-1}(0)$ are two divisors. Hence they intersect (at least generically) along a complex curve inside X.

A class of examples

Now we give a class of explicit examples where we have the necessary conditions to have non-trivial solutions of our equations. From the construction explained above, we see that for the construction to work, we need two holomorphic line bundles $\mathcal{L}_0$ and $\mathcal{L}_1$ on X with the following conditions:

1. $\dim H^0(X, \mathcal{L}_0) > 0$
2. $c_1(K_X^{-2} \otimes \mathcal{L}_0^4) = a_0[\omega]$ with $a_0 < 0$ [This implies $\deg(K_X^{-2} \otimes \mathcal{L}_0^4) < 0$]
3. $\dim H^0(X, K_X \otimes \mathcal{L}_1^{-1}) > 0$
4. $c_1(K_X^{-2} \otimes \mathcal{L}_1^4) = a_1[\omega]$ with $a_1 > 0$ [This implies $\deg(K_X^{-2} \otimes \mathcal{L}_1^4) > 0$]

Example 1. Let's take 3 compact Riemann surfaces $(X_i, \omega_i)_{i=1,2,3}$ of the same genus $g > 1$. ω_i denotes the normalized Kähler form on X_i such that $\int_{X_i} \omega_i = 1$. Define $X := X_1 \times X_2 \times X_3$. The Kähler form on X is $\omega := \sum_{i=1}^{3} \pi_i^* \omega_i$ where π_i is the projection of X onto X_i.

Lemma 31. *Let* (A, ω_A) *and* (B, ω_B) *be two compact Riemann surfaces with* ω_A, ω_B *denoting the respective normalized Kähler forms on them such that the integration of the Kähler form on the manifold gives* 1*. Now say* L *is a holomorphic line bundle on* A*, then deg* L*=deg* $\pi_A^*(L)$*. where* π_A *is the projection of* $A \times B$ *onto* A*.*

Proof.

$$
\begin{aligned}
&\int_{A\times B} c_1(\pi_A^* L) \wedge (\pi_A^* \omega_A + \pi_B^* \omega_B) \\
&= \int_{A\times B} \pi_A^* c_1(L) \wedge \pi_B^* \omega_B + \int_{A\times B} \pi_A^* c_1(L) \wedge \pi_A^* \omega_A \\
&= \int_{A\times B} \pi_A^* c_1(L) \wedge \pi_B^* \omega_B + \int_{A\times B} \pi_A^* (c_1(L) \wedge \omega_A) \\
&= \int_{A\times B} \pi_A^* c_1(L) \wedge \pi_B^* \omega_B + 0 \\
&= \int_A c_1(L) \times \int_B \omega_B \\
&= \int_A c_1(L)
\end{aligned}
$$

□

$K_X = \otimes_{i=1}^{3} \pi_i^*(K_{X_i})$. We take $\mathcal{L}_0$ to be the trivial complex line bundle on X and $\mathcal{L}_1$ to be the canonical line bundle K_X. We have

$$H^0(X, \mathcal{L}_0) \cong \mathbb{C} \text{ and } c_1(K_X^{-2} \otimes \mathcal{L}_0^4) = -2(2g-2)\omega$$

and for $\mathcal{L}_1 = K_X$, we have

$$H^0(X, K_X \otimes K_X^{-1}) \cong \mathbb{C} \text{ and } c_1(K_X^{-2} \otimes K_X^4) = 2(2g-2)\omega$$

Example 2. The idea is to take X a hypersurface in $\mathbb{CP}^4$ of very high degree, let's take a holomorphic section of $\mathcal{O}(d) \to \mathbb{CP}^4$, i.e., a homogeneous polynomial of degree d in 5 variables (d to be determined later). If we choose this generically, the zero locus is a smooth algebraic variety X.

The Kahler form ω on X is given by restricting the Fubini-Study form ω_{FS} on X and it lies in the cohomology class obtained by restricting $c_1(\mathcal{O}(1))$ on X. Meanwhile, by the adjunction formula, $K_X = \mathcal{O}(d-5)|_X$.

Now we take $\mathcal{L}_0 = \mathcal{O}(k_0)|_X$ and $\mathcal{L}_1 = \mathcal{O}(k_1)|_X$ (k_0 and k_1 to be determined later). For $m > 0$, dim $H^0(\mathbb{CP}^4; \mathcal{O}(m)) > 0$. Restricting these holomorphic sections to X we can find line bundles on X with non-trivial holomorphic sections. We need k_0 and k_1 to satisfy the following conditions:

1. $k_0 > 0$ so that there are holomorphic sections of $\mathcal{L}_0$.
2. $2(5-d)+4k_0 < 0$. This ensures that $c_1(K_X^{-2} \otimes \mathcal{L}^4)$ is a negative multiple of ω.
3. $d-5-k_1 > 0$ so that there are holomorphic sections of $K_X \otimes \mathcal{L}_1^{-1}$.
4. $2(5-d)+4k_1 > 0$. This ensures that $c_1(K_X^{-2} \otimes \mathcal{L}_1^4)$ is a positive multiple of ω.

Putting this conditions together we get

$$0 < k_0 < \frac{d-5}{2} < k_1 < d-5$$

So, in this way we get many examples by first choosing $d > 7$ and then picking k_0 and k_1 as we like in the above ranges.

0.5.3 Solution of perturbed SW equations on $\Sigma \times \mathbb{C}^2$

In this section we construct solutions of a perturbed version of the equations on $X = \Sigma \times \mathbb{C}^2$, where Σ is a compact Riemann surface. We find solutions which are invariant in the $\mathbb{C}^2$ direction. The discussion is very similar to the theory of vortex equations on a Riemann surface (e.g. see [5]).

We take a holomorphic line bundle $\mathcal{L} \to \Sigma$ over Σ. We define spinor bundles on Σ to be

$$S_+(\Sigma) = \mathcal{L}$$
$$S_-(\Sigma) = K_\Sigma^{-1} \otimes \mathcal{L}$$

and on $\mathbb{C}^2$ to be

$$S_+(\mathbb{C}^2) = \Lambda^0(\mathbb{C}^2) \oplus \Lambda^{0,2}(\mathbb{C}^2)$$
$$S_-(\mathbb{C}^2) = \Lambda^{0,1}(\mathbb{C}^2)$$

One can construct a Spin^c-bundle on $\Sigma \times \mathbb{C}^2$ in the following way [17]:

$$S_+(\Sigma \times \mathbb{C}^2) = \big(S_+(\Sigma) \otimes S_+(\mathbb{C}^2)\big) \oplus \big(S_-(\Sigma) \otimes S_-(\mathbb{C}^2)\big)$$
$$S_-(\Sigma \times \mathbb{C}^2) = \big(S_+(\Sigma) \otimes S_-(\mathbb{C}^2)\big) \oplus \big(S_-(\Sigma) \otimes S_+(\mathbb{C}^2)\big)$$

So, the determinant bundle is:

$$\mathrm{Det}\big(S_+(\Sigma \times \mathbb{C}^2)\big) = \mathcal{L}^4 \otimes K_\Sigma^{-2}$$

The Kähler metric on Σ induces a natural metric and a holomorphic connection on K_Σ and hence on K_Σ^{-2}. We start with a hermitian metric h on $\mathcal{L}$ and the usual flat metric on $\mathbb{C}^2$. So, giving a unitary connection $\widetilde{A}$ on $\mathcal{L}$ is enough to determine a unitary connection A on the determinant bundle. With abuse of notation we say $A = -2A_{K\Sigma} + 4\widetilde{A}$. We denote the natural Kähler form on $\mathbb{C}^2$ by ω, ω_Σ is the Kähler form on Σ and we take $(\omega + \omega_\Sigma)$ to be the Kähler form on $\Sigma \times \mathbb{C}^2$.

Now we write the perturbed equations. The perturbation terms are added in the curvature equations.

$$D_A(\phi) + c(\beta)(\phi) = 0 \tag{73}$$
$$F_A - 2i * d\beta + \eta_1 = q(\phi), \quad \eta_1 = 2F_{K_\Sigma} + ir\omega + i\tau\omega_\Sigma \;\; (r, \tau > 0) \tag{74}$$
$$D_B(\psi) + c(*\beta)(\psi) = 0 \tag{75}$$
$$F_B + 2i * d * \beta + \eta_2 = q(\psi), \quad \eta_2 = 2F_{K_\Sigma} - ir_1\omega - i\tau_1\omega_\Sigma \;\; (r_1, \tau_1 > 0) \tag{76}$$

To be very precise, we should be writing $\pi^*_\Sigma(F_\Sigma), \pi^*_{\mathbb{C}^2}(\omega)$ etc in our discussion, hopefully it won't cause any confusion to the reader.

Take a smooth function $f : \Sigma \to \mathbb{C}$. We take $\beta = (\bar{\partial} f + \partial \bar{f}) \wedge \omega$. To solve the first Dirac equation take the Chern connection $\widetilde{A}$ w.r.t. h and a holomoprhic section $\alpha \in H^0(\Sigma, \mathcal{L})$ and define $\phi := e^{2if}\alpha \otimes 1$. Then,

$$\begin{aligned} D_A(\phi) &= \sqrt{2}\bar{\partial}_{\widetilde{A}}(e^{2if}\alpha) \otimes 1 \\ &= 2\sqrt{2}ie^{2if}\bar{\partial} f \wedge \alpha \otimes 1 \\ &= -(\sqrt{2}e^{2if}\alpha) \otimes (-2i) \\ &= -c(\bar{\partial} f \wedge \omega)(e^{2if}\alpha \otimes 1) \end{aligned}$$

The first curvature equation reads:

$$\begin{aligned} F_A - 2i * d\beta + 2F_{K_\Sigma} + ir\omega + i\tau\omega_\Sigma &= \frac{i}{4}|\phi|^2(\omega + \omega_\Sigma) \\ 4F_{\widetilde{A}} + 4 * \big(\partial\bar{\partial}(\mathrm{Im} f) \wedge \omega\big) + ir\omega + i\tau\omega_\Sigma &= \frac{i}{4}|\phi|^2(\omega + \omega_\Sigma) \end{aligned}$$

Say $z_1 = x_1 + iy_1$ is the coordinate on Σ around a fixed point and $z_2 = x_2 + iy_2, z_3 = x_3 + iy_3$ be the coordinates on $\mathbb{C}^2$. We notice that

$$\begin{aligned} *\big(\partial\bar{\partial}(\mathrm{Im} f) \wedge \omega\big) &= *\Big(\frac{\partial^2}{\partial z_1 \partial \bar{z}_1}(\mathrm{Im} f) dz_1 \wedge d\bar{z}_1 \wedge \omega\Big) \\ &= *\Big(\frac{1}{4}\big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2}\big)(\mathrm{Im} f)(-2i dx_1 \wedge dy_1) \wedge (dx_2 \wedge dy_2 + dx_3 \wedge dy_3)\Big) \\ &= \frac{i}{2}\big(\Delta_\Sigma(\mathrm{Im} f)\omega\big) \end{aligned}$$

So, the first curvature equation becomes

$$4F_{\widetilde{A}} + 2i\Delta_\Sigma(\mathrm{Im} f)\omega + ir\omega + i\tau\omega_\Sigma = \frac{i}{4}e^{-4\mathrm{Im} f}|\alpha|^2(\omega + \omega_\Sigma)$$

and it splits into two equations on the Riemann surface:

$$4F_{\widetilde{A}} + i\tau\omega_\Sigma = \frac{i}{4}e^{-4\mathrm{Im} f}|\alpha|^2\omega_\Sigma \quad \text{and} \quad -2\Delta_\Sigma(\mathrm{Im} f) + \frac{1}{4}e^{-4\mathrm{Im} f}|\alpha|^2 = r$$

First we solve for $\mathrm{Im}(f)$ from the second equation for a fixed α (we can do this since $r > 0$ [16]). Now we turn to the first equation. Notice $F_{\widetilde{A}}$ is a holomorphic connection on the holomorphic line bundle $\mathcal{L}$ and hence only the $(1,1)$ part of the curvature is nontrivial. We make a conformal change in the metric h: $h' = e^{\lambda} h$ for some smooth function $\lambda : \Sigma \to \mathbb{R}$ which induces a new Chern connection on $\mathcal{L}$, say A'. The equation becomes

$$\begin{aligned} &F_{A'} - \partial\bar{\partial}\lambda + i\frac{\tau}{4}\omega_{\Sigma} = \frac{i}{16}e^{-4\mathrm{Im}f}e^{\lambda}|\alpha|^2\omega_{\Sigma} \\ &\Leftrightarrow \langle F_{A'}, \omega_{\Sigma}\rangle_{-}\langle\partial\bar{\partial}\lambda, \omega_{\Sigma}\rangle + i\frac{\tau}{4} = \frac{i}{16}e^{-4\mathrm{Im}f}e^{\lambda}|\alpha|^2 \\ &\Leftrightarrow \langle F_{A'}, \omega_{\Sigma}\rangle + i\frac{\tau}{4} = \frac{i}{2}\Delta_{\Sigma}\lambda + \frac{i}{16}e^{-4\mathrm{Im}f}e^{\lambda}|\alpha|^2 \end{aligned}$$

The real valued function Imf is already known. So we can solve for λ if

$$\begin{aligned} &\int_{\Sigma}\big(-i4\langle F_{A'}, \omega_{\Sigma}\rangle + \tau\big) > 0 \\ &\Leftrightarrow \tau\frac{\mathrm{Vol}(\Sigma)}{2\pi} \geq 4\int_{\Sigma}\langle\frac{iF_{A'}}{2\pi}, \omega_{\Sigma}\rangle \\ &\Leftrightarrow \tau > \frac{8\pi}{\mathrm{Vol}(\Sigma)}\deg(\mathcal{L}) \end{aligned}$$

For the second set of equations, we start with a holomorphic line bundle $\mathcal{L}_1$ (potentially different from $\mathcal{L}$). One can construct a Spin^c-bundle on $\Sigma \times \mathbb{C}^2$ as [17]:

$$\begin{aligned} \widetilde{S}_{+}(\Sigma \times \mathbb{C}^2) &= \big(\mathcal{L}_1 \otimes S_{+}(\mathbb{C}^2)\big) \oplus \big((K_{\Sigma}^{-1} \otimes \mathcal{L}_1) \otimes S_{-}(\mathbb{C}^2)\big) \\ \widetilde{S}_{-}(\Sigma \times \mathbb{C}^2) &= \big(\mathcal{L}_1 \otimes S_{-}(\mathbb{C}^2)\big) \oplus \big((K_{\Sigma}^{-1} \otimes \mathcal{L}_1) \otimes S_{+}(\mathbb{C}^2)\big) \end{aligned}$$

$\mathrm{Det}(\widetilde{S}_{-}) \cong K_{\Sigma}^{-2} \otimes \mathcal{L}_1^4$. If we put a hermitian metric (say h_1 on $\mathcal{L}_1$ and the usual flat metric on $\mathbb{C}^2$, then giving a unitary connection (say $\widetilde{B}$ on $\mathcal{L}$ is enough to determine a unitary connection B on the determinant bundle $K_{\Sigma}^{-2} \otimes \mathcal{L}_1^4$. With abuse of notation we write $B = -2A_{K_{\Sigma}} + 4\widetilde{B}$. $A_{K_{\Sigma}}$ is the holomorphic connection on K_{Σ} determined by the Kähler metric on Σ.

Take a negative spinor $\psi := \psi_1 \otimes 1 \in \Gamma\big((K_{\Sigma}^{-1} \otimes \mathcal{L}_1) \otimes \Lambda^0(\mathbb{C}^2)\big)$. The second Dirac equation: $(D_B + c(*\beta))\psi = 0$ reads

$$\begin{aligned} &\sqrt{2}(\bar{\partial}_{\widetilde{B}}^{*}\psi_1) \otimes 1 + c\big(*(\partial\bar{f} \wedge \omega)\big)\psi_1 \otimes 1 = 0 \\ &\Rightarrow \sqrt{2}(\bar{\partial}_{\widetilde{B}}^{*}\psi_1) \otimes 1 - ic(\partial\bar{f} \wedge \omega)\psi_1 \otimes 1 = 0 \\ &\Rightarrow \sqrt{2}(\bar{\partial}_{\widetilde{B}}^{*}\psi_1) \otimes 1 - 2i\sqrt{2}\big(*_{\Sigma}(\partial\bar{f} \wedge \psi_1)\big) \otimes 1 = 0 \end{aligned}$$

To solve this we take $\psi_1 = e^{-2\bar{f}}\xi_1$, where $\bar{\partial}^*_{\widetilde{B}}\xi_1 = 0$. For smooth section $\alpha \in \Omega^0(\Sigma, \mathcal{L}_1)$,

$$
\begin{aligned}
&\langle \bar{\partial}^*_{\widetilde{B}}(e^{-2\bar{f}}\xi_1), \alpha\rangle_{L^2} \\
&= \langle \xi_1, e^{-2f}\bar{\partial}_{\widetilde{B}}\alpha\rangle_{L^2} \\
&= 2\langle \xi_1, e^{-2f}\bar{\partial} f \wedge \alpha\rangle_{L^2} \\
&= 2\langle \psi_1, \bar{\partial} f \wedge \alpha\rangle_{L^2} \\
&= 2\overline{\langle \bar{\partial} f \wedge \alpha, \psi_1\rangle_{L^2}} \\
&= 2\overline{\int_X (\bar{\partial} f \wedge \alpha \wedge *_\Sigma \bar{\psi}_1)} \\
&= 2\overline{\int_X -i(\bar{\partial} f \wedge \alpha \wedge \bar{\psi}_1)} \\
&= 2\overline{\int_X \alpha \wedge *_\Sigma \overline{\left(*_\Sigma (i\partial \bar{f} \wedge \psi_1)\right)}} \\
&= \overline{\langle \alpha, 2 *_\Sigma (i\partial \bar{f} \wedge \psi_1)\rangle_{L^2}} \\
&= \langle 2i *_\Sigma (\partial \bar{f} \wedge \psi_1), \alpha\rangle_{L^2}
\end{aligned}
$$

So, $\bar{\partial}^*_{\widetilde{B}}\psi_1 = 2i *_\Sigma (\partial\bar{f} \wedge \psi_1)$. What remains to solve is the second curvature equation:

$$
\begin{aligned}
4F_{\widetilde{B}} + 2i * d * \beta - ir_1\omega - i\tau_1\omega_\Sigma &= -\frac{i}{4}|\psi_1|^2(\omega + \omega_\Sigma) \\
4F_{\widetilde{B}} - 2i(\Delta_\Sigma \mathrm{Re} f)\omega - ir_1\omega - i\tau_1\omega_\Sigma &= -\frac{i}{4}e^{-\mathrm{Re} f}|\xi_1|^2(\omega + \omega_\Sigma)
\end{aligned}
$$

This splits into two equations:

$$
4F_{\widetilde{B}} + \frac{i}{4}e^{-\mathrm{Re} f}|\xi_1|^2\omega_\Sigma = i\tau_1\omega_\Sigma \text{ and } 2\Delta_\Sigma(-\mathrm{Re} f) + \frac{1}{4}e^{-\mathrm{Re} f}|\xi_1|^2 = r_1
$$

Since ξ is not identically zero and $r_1 > 0$, there exists a unique solution for Ref [16] in the second equation. Now that we know Ref, we go back to solve the first equation. The initial Hermitian metric on $\mathcal{L}_1$ was h_1, we make a conformal change on the metric by e^{λ_1} for a smooth function $\lambda_1 : X \to \mathbb{R}$. We notice that choosing ξ_1 an anti-holomorphic section of $\mathcal{L}_1$ is same as choosing the corresponding holomorphic section $\bar{\xi}_1$ of $\bar{\mathcal{L}}_1 = \mathcal{L}_1^*$. Now with the natural induced metric from h_1 say, h_1^* on $\mathcal{L}_1^*$ we also have $|\bar{\xi}_1|_{h_1^*} = |\xi_1|_{h_1}$. For a conformal change by e^{λ_1} on h_1, we get a scaling by $e^{-\lambda_1}$ on h_1^*. As this is a conformal change in the metric, $\bar{\xi}_1$ is still a holomorphic section of $\bar{\mathcal{L}}_1 = \mathcal{L}_1^*$, and now when we go back to the corresponding anti-holomorphic section of $\mathcal{L}_1$, the norm of this section changes by $e^{-\lambda_1}$. With the new metric $h_1' = e^{\lambda_1}$, on $\mathcal{L}_1$ and the corresponding connection $\widetilde{B}'$ on $\mathcal{L}_1$, the equation becomes:

$$
\begin{aligned}
&4F_{\widetilde{B}'} + \frac{i}{4}e^{-\mathrm{Re} f}|\xi_1|^2_{h_1'}\omega_\Sigma = i\tau_1\omega_\Sigma \\
&\Leftrightarrow 4F_{\widetilde{B}} - 4\partial\bar{\partial}\lambda_1 + \frac{i}{4}e^{-\mathrm{Re} f}e^{-\lambda_1}|\xi_1|^2\omega_\Sigma = i\tau_1\omega_\Sigma \\
&\Leftrightarrow 4\langle F_{\widetilde{B}}, \omega_\Sigma\rangle - 2i\Delta_\Sigma\lambda_1 + \frac{i}{4}e^{-\mathrm{Re} f}e^{-\lambda_1}|\xi_1|^2 = i\tau_1 \\
&\Leftrightarrow 2\Delta_\Sigma(-\lambda_1) + \frac{1}{4}e^{-\mathrm{Re} f}e^{-\lambda_1}|\xi_1|^2 = \tau_1 + 4\langle iF_{\widetilde{B}}, \omega_\Sigma\rangle
\end{aligned}
$$

Ref is a known function, hence we can uniquely solve for λ_1 [16] if

$$\int_\Sigma (\tau_1 + 4\langle iF_{\widetilde{B}}, \omega_\Sigma\rangle) > 0$$
$$\Leftrightarrow \tau_1 > -\frac{8\pi}{\mathrm{Vol}(\Sigma)}\deg(\mathcal{L}_1)$$

So we proved the existence of non-trivial solutions of the equations (73),(74),(75),(76) under the following conditions on the two holomorphic line bundles $\mathcal{L}$ and $\mathcal{L}_1$ on Σ :

1. $\dim H^0(\Sigma, \mathcal{L}) > 0$
2. $\tau > \frac{8\pi}{\mathrm{Vol}(\Sigma)}\deg(\mathcal{L})$
3. $\dim H^0(\Sigma, K_\Sigma \otimes \mathcal{L}_1^{-1}) > 0$
4. $\tau > -\frac{8\pi}{\mathrm{Vol}(\Sigma)}\deg(\mathcal{L}_1)$

Remark 32. For an explicit example, take any closed Riemann surface Σ of genus g, take $\mathcal{L}$ to be the trivial line bundle, then for any $\tau > 0$, we satisfy the first two conditions. And if we take $\mathcal{L}_1 = K_\Sigma, \tau_1 > \frac{16\pi(g-1)}{\mathrm{Vol}(\Sigma)}$; we also satisfy the second two conditions.

0.5.4 Solution of perturbed SW equations on $\mathbb{R}^6$

In this section we find solution of a perturbed version of the SW equations on $\mathbb{R}^6$, by reducing the equations to certain odes as a function of $r = \sqrt{\sum_{i=1}^6 x_i^2}$ $\big((x_1, \dots, x_6)$ denote the coordinates in $\mathbb{R}^6\big)$. $\mathbb{R}^6 \cong \mathbb{C}^3$ has a natural Kähler form $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + dx_5 \wedge dx_6$. The perturbation involves adding a positive multiple of ω on the left hand side of the first curvature equation.

The spinor bundles on $\mathbb{R}^6 = \mathbb{C}^3$ are trivial. We can in fact think of them as follows:

$$S_+(\mathbb{C}^3) = \Lambda^{0,0} \oplus \Lambda^{0,2} \cong \mathbb{C} \oplus \mathbb{C}^3$$
$$S_-(\mathbb{C}^3) = \Lambda^{0,1} \oplus \Lambda^{0,3} \cong \mathbb{C}^3 \oplus \mathbb{C}$$

With abuse of notation we write the trivial complex line bundle on $\mathbb{C}^3$ as $\mathbb{C}$. Take the three-form $\beta = ir(\partial - \bar\partial)r \wedge \omega$. Choose a connection A on the determinant bundle of S_+ and a spinor ϕ in $\Omega^{0,0}(\mathbb{C}^3)$, such that $D_A\phi = 0$ (we explain in a bit how to find such A and ϕ). Say $g(r)$ is a smooth function in r. We solve for g to solve the first Dirac equation (63).

$$\big(D_A + c(ir(\partial - \bar\partial)r \wedge \omega)\big)g(r)\phi = 0$$
$$\Rightarrow g'(r)c(dr)\phi + g(r)D_A\phi + g(r)c\big(ir(\partial - \bar\partial)r \wedge \omega\big)\phi = 0$$
$$\Rightarrow \sqrt{2}g'(r)\bar\partial r \wedge \phi = irg(r)(-2\sqrt{2}i\bar\partial r \wedge \phi)$$
$$\Rightarrow g'(r)(\bar\partial r \wedge \phi) = 2rg(r)(\bar\partial r \wedge \phi)$$

$g(r) = ce^{r^2}$ solves it for any constant c. Now we explain how to solve $D_A\phi = 0$. We denote the connection on the determinant line bundle by $A = d + ia$. Then $D_A = D + \frac{1}{2}c(ia)$. Choose

$ia = -2(\partial - \bar{\partial})r^2$.

$$\begin{aligned}
&(\partial - \bar{\partial})r^2 \\
&= (\partial - \bar{\partial})(z_1\bar{z}_1 + z_2\bar{z}_2 + z_3\bar{z}_3)\,, \text{where } z_1 = x_1 + ix_2, z_2 = x_3 + ix_4, z_3 = x_5 + ix_6 \\
&= (\bar{z}_1 dz_1 - z_1 d\bar{z}_1) + (\bar{z}_2 dz_2 - z_2 d\bar{z}_2) + (\bar{z}_3 dz_3 - z_3 d\bar{z}_3) \\
&= 2i(-x_2 dx_1 + x_1 dx_2 - x_4 dx_3 + x_3 dx_4 - x_6 dx_5 + x_5 dx_6)
\end{aligned}$$

We choose a point-wise basis $\{e_1, e_2, e_3, e_4\}$ of $S_+ \cong \mathbb{C}^4$ such that e_1 denotes the constant function $1 \in \Lambda^{0,0}_p(\mathbb{C}^3)$ at a point $p \in \mathbb{C}^3$ and the point-wise basis of $S_- \cong \mathbb{C}^4$ is chosen as $\{c(dx_1)e_1, c(dx_1)e_2, c(dx_1)e_3, c(dx_1)e_4\}$. Then the matrix representations of Clifford multiplication are

$$c(dx_1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, c(dx_2) = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}, c(dx_3) = \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{bmatrix},$$

$$c(dx_4) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, c(dx_5) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, c(dx_6) = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}$$

For $\phi \in \Omega^{0,0}(\mathbb{C}^3)$ and $ia = -2(\partial - \bar{\partial})r^2$, $D_A\phi = 0$ breaks down into the following three equations:

$$\begin{aligned}
\frac{\partial\phi}{\partial x_1} + i\frac{\partial\phi}{\partial x_2} - 2i(-x_2\phi + ix_1\phi) = 0 \\
i\frac{\partial\phi}{\partial x_3} - \frac{\partial\phi}{\partial x_4} - 2i(-ix_4\phi - x_3\phi) = 0 \\
-\frac{\partial\phi}{\partial x_5} - i\frac{\partial\phi}{\partial x_6} - 2i(x_6\phi - ix_5\phi) = 0
\end{aligned}$$

For any constant $c_1, \phi = c_1 e^{-\sum_{i=1}^{6} x_i^2}$ solves all three equations simultaneously.

The curvature equation (64) reads:

$$\begin{aligned}
&ida - 2i * d\big(ir(\partial - \bar{\partial})r \wedge \omega\big) = \frac{i}{4}|g(r)|^2|\phi|^2\omega \\
&\Rightarrow ida + 2 * \big((\partial + \bar{\partial})r \wedge (\partial - \bar{\partial})r \wedge \omega + r(\partial + \bar{\partial})(\partial - \bar{\partial})r \wedge \omega\big) = \frac{i}{4}|g(r)|^2|\phi|^2\omega \\
&\Rightarrow ida + 2 * \big(-2\partial r \wedge \bar{\partial} r \wedge \omega - 2r\partial\bar{\partial} r \wedge \omega\big) = \frac{i}{4}|g(r)|^2|\phi|^2\omega \\
&\Rightarrow ida + 2 * (-\partial\bar{\partial}(r^2)) = \frac{i}{4}|g(r)|^2|\phi|^2\omega \\
&\Rightarrow ida + 2\partial\bar{\partial}(r^2) - i\Delta(r^2)\omega = \frac{i}{4}|g(r)|^2|\phi|^2\omega
\end{aligned}$$

$\Delta(r^2) = -\sum_{i=1}^{6} \frac{\partial^2}{\partial x_i^2} r^2 = -12$ and $ida = -2(\partial + \bar{\partial})(\partial - \bar{\partial})r^2 = 4\partial\bar{\partial} r^2$. We also have

$$\begin{aligned}
\Rightarrow ida &= -i(-dx_2 \wedge dx_1 + dx_1 \wedge dx_2 - dx_4 \wedge dx_3 + dx_3 \wedge dx_4 - dx_6 \wedge dx_5 + dx_5 \wedge dx_6) \\
&= -8i\omega
\end{aligned}$$

So the left hand side of the curvature equation is $6\partial\bar{\partial}(r^2) + 12i\omega = -12i\omega + 12i\omega = 0$. So, we can solve the following set of equations:

$$
\begin{aligned}
(D_A + c(\beta))(g\phi) &= 0 \\
F_A - 2i * d\beta + r_1\omega &= q(g\phi), r_1 > 0
\end{aligned}
$$

for $A = d - 4i(-x_2dx_1 + x_1dx_2 - x_4dx_3 + x_3dx_4 - x_6dx_5 + x_5dx_6)$, $g\phi = 2\sqrt{r_1}$ and $\beta = ir(\partial - \bar{\partial})r \wedge \omega$.

Let's solve the second pair of equations (65), (66) as well. We choose a connection B on the determinant bundle of S_- and a negative-spinor $\psi \in \Omega^{0,3}(\mathbb{C}^3)$, such that $D_B\psi = 0$ (again we explain how to get this in a bit). Now take a smooth function $h(r)$ in r and solve for h, with the same β as above. The Dirac equation (65) reads:

$$
\begin{aligned}
&\big(D_B + c(*(ir(\partial - \bar{\partial})r \wedge \omega))\big)(h(r)\psi) = 0 \\
&\Rightarrow h'(r)c(dr)\psi + h(r)D_B\psi + rh(r)c(\partial r \wedge \omega)\psi = 0 \\
&\Rightarrow h'(r)c(dr)\psi = -rh(r)c(\partial r \wedge \omega)\psi \\
&\Rightarrow -\sqrt{2}h'(r) * (\partial r \wedge \mu) = -rh(r)2\sqrt{2} * (\partial r \wedge \psi)
\end{aligned}
$$

Hence $h(r) = c_2e^{r^2}$ solves it for any constant c_2.

We denote the connection on the determinant line bundle by $B = d + ib$. Then $D_B = D + \frac{1}{2}c(ib)$. We now explain how to choose ψ and b such that $D_B\psi = 0$. We would do this again using explicit calculation. We start by giving matrix representation of Clifford multiplication of $dx_i, i \in \{1, 2, 3, 4, 5, 6\}$. We choose a point-wise basis $\{f_1, f_2, f_3, f_4\}$ of $S_- \cong \mathbb{C}^4$ such that f_1 denotes the basis element $d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3|_p \in \Lambda_p^{0,3}(\mathbb{C}^3)$ at a point $p \in \mathbb{C}^3$ and the point-wise basis of $S_- \cong \mathbb{C}^4$ is chosen as $\{c(dx_1)f_1, c(dx_1)f_2, c(dx_1)f_3, c(dx_1)f_4\}$. The matrix representations of Clifford multiplication of these forms are:

$$
c(dx_1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, c(dx_2) = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}, c(dx_3) = \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{bmatrix},
$$

$$
c(dx_4) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, c(dx_5) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, c(dx_6) = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}
$$

We choose $ib = ia = -2(\partial - \bar{\partial})r^2$. The same calculations as in the first Dirac equation says that for $\varphi = c_3e^{-\sum_{i=1}^{6} x_i^2}$ (c_3 being a constant), $\psi := \varphi d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \in \Omega^{0,3}(\mathbb{C}^3)$ solves $D_B\psi = 0$.

Coming back to the curvature equation (66), we have:

$$
\begin{aligned}
&F_B + 2i * d * \beta = q(\psi) \\
&\Rightarrow idb + 2i * d(rdr \wedge \omega) = -\frac{i}{4}|h(r)|^2|\psi|^2\omega \\
&\Rightarrow idb = -\frac{i}{4}|h(r)|^2|\psi|^2\omega \\
&\Rightarrow -8i\omega = -\frac{i}{4}|h(r)|^2|\psi|^2\omega
\end{aligned}
$$

So, $\psi = 4\sqrt{2}d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3$, $B = d - 4i(-x_2dx_1 + x_1dx_2 - x_4dx_3 + x_3dx_4 - x_6dx_5 + x_5dx_6)$, $\beta = ir(\partial - \bar{\partial})r \wedge \omega$ solve the second set of equations (65), (66). We get the following proposition:

Proposition 33. *There exists a non-trivial solution of the following perturbed version of the SW equations on* $\mathbb{R}^6$:

$$(D_A + c(\beta))\phi = 0 \tag{77}$$
$$F_A - 2i * d\beta + r_1\omega = q(\phi) \tag{78}$$
$$(D_B + c(*\beta))\psi = 0 \tag{79}$$
$$F_B + 2i * d * \beta = q(\psi) \tag{80}$$

$r_1 > 0$ *is a constant.*

0.6 Solution of 8-dimensional SW equations

0.6.1 Spin geometry in dimension 8:

Let M be a Spin^c-manifold of dimension 8 and $S = S_+ \oplus S_- \to M$, be a spinor bundle on M. S_+ and S_- are both complex vector bundles of dimension 8. The complexified forms act on the spinors via Clifford multiplication:

$$c : \big(\Lambda^0(M) \oplus \Lambda^2(M) \oplus \Lambda^4(M)\big) \otimes \mathbb{C} \to \mathrm{End}_{\mathbb{C}}(S_+, S_+) \cong \mathrm{End}_{\mathbb{C}}(S_-, S_-)$$
$$c : \big(\Lambda^1(M) \oplus \Lambda^3(M)\big) \otimes \mathbb{C} \to \mathrm{End}_{\mathbb{C}}(S_+, S_-) \cong \mathrm{End}_{\mathbb{C}}(S_-, S_+)$$

Notice that the Hodge-star operator $*$ squares to identity on four-forms: $*^2 = \mathrm{Id}$ on $\Lambda^4(M)$. Hence, $\Lambda^4(M)$ splits as self-dual and anti self-dual four forms:

$$\Lambda^4(M) = \Lambda^4_+(M) \oplus \Lambda^4_-(M)$$

The self dual part $\Lambda^4_+(M)$ *acts trivially on the negative spinors and the anti self-dual part* $\Lambda^4_-(M)$ *acts trivially on the positive spinors.* The proof is identical to the proof of lemma 26. The following maps are isomorphisms:

$$c : \big(\Lambda^0(M) \oplus \Lambda^2(M) \oplus \Lambda^4_+(M)\big) \otimes \mathbb{C} \to \mathrm{End}_{\mathbb{C}}(S_+, S_+)$$
$$c : \big(\Lambda^0(M) \oplus \Lambda^2(M) \oplus \Lambda^4_-(M)\big) \otimes \mathbb{C} \to \mathrm{End}_{\mathbb{C}}(S_-, S_-)$$

For a hermitian endomorphism of positive spinors, we can split it into a trace-free part and the trace-part. Λ^0 (real dim 1) is the trace part and the imaginary 2-forms and the self-dual real 4-forms (real dimension $28 + 35 = 63$ in total) act as trace-free hermitian endomorphisms via Clifford multiplication:

$$c : i\Lambda^2(M) \oplus \Lambda^4_+(M) \to i\mathfrak{su}(S_+)$$

And similarly for negative spinors we have:

$$c : i\Lambda^2(M) \oplus \Lambda^4_+(M) \to i\mathfrak{su}(S_-)$$

These isomorphisms induce a splitting of $i\mathfrak{su}(S_+)$ and $i\mathfrak{su}(S_-)$, which can be explained as follows. Both S_+ and S_- has a real-structure which is Spin(8)-equivariant [10]. *The self-dual (or the anti self-dual) 4-forms give endomorphisms which commute with the real structure, whereas the imaginary 2-forms anti-commute with the real structure.*

0.6.2 Solution of perturbed SW equations on a closed Kähler 4-fold

In this section we describe a solution of a perturbed version of 8d SW equations on a closed Kähler 4-manifold X with a Kähler form ω. The equations for $A \in \mathcal{A}, \phi \in \Gamma(S_+), \beta \in \Omega^3$ are:

$$\big(D_A + (1+i)c(\beta)\big)\phi = 0 \tag{81}$$
$$F_A + 2(d\beta)^+ + 2id^*\beta + c\omega^2 = q(\phi), \quad c < 0 \tag{82}$$

We take a positive Spin^c bundle on X, by twisting the canonical positive Spin^c bundle by a holomorphic line bundle $\mathcal{L}$ (with a hermitian metric say h) :

$$S_+(X) = \Lambda^0(X,\mathcal{L}) \oplus \Lambda^{0,2}(X,\mathcal{L}) \oplus \Lambda^{0,4}(X,\mathcal{L})$$

The determinant bundle of $S_+(X)$ is $K_X^{-1} \otimes \operatorname{Det}(\Lambda^{0,2}(X)) \otimes \mathcal{L}^8 \cong K_X^{-4} \otimes \mathcal{L}^8$. Hence choosing a unitary connection (say $\widetilde{A}$) on $\mathcal{L}$ gives us a unitary connection (say A) on the determinant bundle. For the construction of solution we will have two cases depending on whether the first Chern class of the determinant bundle is a positive or negative multiple of the Kähler form. This is a necessary assumption we make on the topology of the line-bundle $\mathcal{L}$ for this construction.

The reason behind adding the extra term $c\omega^2$ is the following. As we will see in our construction, we will choose our spinor ϕ to be a non-trivial section of either $\mathcal{L}$ or $K_X^{-1} \otimes \mathcal{L}$ (depending on whether $c_1(K_X^{-4} \otimes \mathcal{L}^8)$ is a negative or positive multiple of ω) and for both these cases the Hodge decomposition of $q(\phi)$ will have a non-zero harmonic four-form, i.e., some constant multiple of ω^2. But the left hand side of the original curvature equation: $F_A + (d\beta + *d\beta) + 2id^*\beta = q(\phi)$ doesn't have any harmonic-four form. Hence, we need this extra term to solve the curvature equation for the choice of our spinor. Notice the perturbation is *harmonic*.

$c_1(K_X^{-4} \otimes \mathcal{L}^8)$ is a negative multiple of ω:

The construction solution is very similar to the 6d case. We take the three-form $\beta = (\bar{\partial}f + \partial\bar{f}) \wedge \omega$ for some smooth complex valued function f on X and choose a spinor ϕ (which depends on f) in $\Omega^0(X,\mathcal{L})$ such that the Dirac equation is solved for any holomorphic connection on $\mathcal{L}$.

Start with the Chern connection A_0 on $\mathcal{L}$ compatible with the holomoprhic structure on $\mathcal{L}$ and we choose a holomorphic section ϕ_0, i.e., $\bar{\partial}_{A_0}\phi_0 = 0$. The Clifford action of $\beta = (\bar{\partial}f + \partial\bar{f}) \wedge \omega$ on ϕ_0 is given by:

$$c(\beta)\phi_0 = c(\bar{\partial}f \wedge \omega)\phi_0 = -3\sqrt{2}i\,\bar{\partial}f \wedge \phi_0$$

Define a positive spinor $\phi := e^{3i(1+i)f}\phi_0 \in \Omega^0(X,\mathcal{L})$. We have

$$\begin{aligned}\big(D_A + (1+i)c(\beta)\big)\phi &= \sqrt{2}\bar{\partial}_{A_0}\phi + (1+i)c(\bar{\partial}f \wedge \omega)\phi \\ &= 3\sqrt{2}i(1+i)e^{3i(1+i)f}\bar{\partial}f \wedge \phi_0 - 3\sqrt{2}i(1+i)e^{3i(1+i)f}\bar{\partial}f \wedge \phi_0 \\ &= 0\end{aligned}$$

Now let's focus on the curvature equation.

$$\begin{aligned}d\beta &= (\partial\bar{\partial}f + \bar{\partial}\partial\bar{f}) \wedge \omega \\ &= \big(\partial\bar{\partial}(f - \bar{f})\big) \wedge \omega \\ &= 2i\big(\partial\bar{\partial}(\operatorname{Im} f)\big) \wedge \omega\end{aligned}$$

To calculate $d\beta^+ = \frac{1}{2}\big(d\beta + *(d\beta)\big)$, we will use local coordinates. We take local holomorphic coordinates $\{z_k = x_k + iy_k\}_{k=1,2,3,4}$ centered at a point $x \in X$ so that the Kähler metric is standard to second order at the point. For a real valued smooth function h, we have

$$
\begin{aligned}
&(\partial\bar{\partial}h) \wedge \omega \\
&= \sum_{j,k=1}^{4} \frac{\partial^2 h}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k \wedge \omega \\
&= \sum_{j \neq k} \frac{\partial^2 h}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k \wedge \omega + \sum_{j=1}^{4} \frac{\partial^2 h}{\partial z_j \partial \bar{z}_j} dz_j \wedge d\bar{z}_j \wedge \omega
\end{aligned}
$$

In these coordinates, $\omega = \sum_{k=1}^{4} dx_k \wedge dy_k$. Let's take $j = 1, k = 2$.

$$
\begin{aligned}
&* (dz_1 \wedge d\bar{z}_2 \wedge \omega) \\
&= *\big((dx_1 + idy_1) \wedge (dx_2 - idy_2) \wedge (dx_3 \wedge dy_3 + dx_4 \wedge dy_4)\big) \\
&= *\big((dx_1 \wedge dx_2 + dy_1 \wedge dy_2 - idx_1 \wedge dy_2 - idx_2 \wedge dy_1) \wedge (dx_3 \wedge dy_3 + dx_4 \wedge dy_4)\big) \\
&= -\big((dx_1 \wedge dx_2 + dy_1 \wedge dy_2 - idx_1 \wedge dy_2 - idx_2 \wedge dy_1) \wedge (dx_3 \wedge dy_3 + dx_4 \wedge dy_4)\big) \\
&= -(dz_1 \wedge d\bar{z}_2 \wedge \omega)
\end{aligned}
$$

Let's take $j = 1$

$$
\begin{aligned}
&* (dz_1 \wedge d\bar{z}_1 \wedge \omega) \\
&= *\big((dx_1 + idy_1) \wedge (dx_1 - idy_1) \wedge (dx_2 \wedge dy_2 + dx_3 \wedge dy_3 + dx_4 \wedge dy_4)\big) \\
&= *\big(-2idx_1 \wedge dy_1 \wedge (dx_2 \wedge dy_2 + dx_3 \wedge dy_3 + dx_4 \wedge dy_4)\big) \\
&= -2i\big(\frac{\omega^2}{2} - dx_1 \wedge dy_1 \wedge (dx_2 \wedge dy_2 + dx_3 \wedge dy_3 + dx_4 \wedge dy_4)\big) \\
&= -dz_1 \wedge d\bar{z}_1 \wedge \omega - i\omega^2
\end{aligned}
$$

Hence,

$$
\begin{aligned}
d\beta^+ &= \frac{1}{2}\big(d\beta + *(d\beta)\big) \\
&= \frac{1}{2}\big(2i \times (-i) \sum_{j=1}^{4} \frac{\partial^2 (\mathrm{Im} f)}{\partial z_j \partial \bar{z}_j} \omega^2\big) \\
&= \big(\sum_{j=1}^{4} \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j})\frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j})(\mathrm{Im} f)\big)\omega^2 \\
&= \frac{1}{4}\big(\sum_{j=1}^{4} (\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2})(\mathrm{Im} f)\big)\omega^2 \\
&= -\frac{1}{4}\Delta(\mathrm{Im} f)\omega^2
\end{aligned}
$$

For calculating $d^*\beta$, we will need a small lemma.

Lemma 34. *On a Kähler* 4*-fold* (X,ω)*, for a smooth real-valued function* h*, we have:*

$$*\big(\partial\bar\partial h\wedge\omega^2\big)=-2\partial\bar\partial h+i(\Delta h)\omega$$

Proof. We take local holomorphic coordinates $\{z_k=x_k+iy_k\}_{k=1,2,3,4}$ centered at a point $x\in X$ so that the Kähler metric is standard to second order at the point. Then at x,

$$\begin{aligned}&(\partial\bar\partial h)\wedge\omega^2\\&=\sum_{j,k=1}^{4}\frac{\partial^2 h}{\partial z_j\partial\bar z_k}dz_j\wedge d\bar z_k\wedge\omega^2\\&=\sum_{j\neq k}\frac{\partial^2 h}{\partial z_j\partial\bar z_k}dz_j\wedge d\bar z_k\wedge\omega^2+\sum_{j=1}^{4}\frac{\partial^2 h}{\partial z_j\partial\bar z_j}dz_j\wedge d\bar z_j\wedge\omega^2\end{aligned}$$

Let's take $j=1,k=2$.

$$\begin{aligned}&*\,(dz_1\wedge d\bar z_2\wedge\omega^2)\\&=*\big((dx_1+idy_1)\wedge(dx_2-idy_2)\wedge 2(dx_3\wedge dy_3\wedge dx_4\wedge dy_4)\big)\\&=*\big((dx_1\wedge dx_2+dy_1\wedge dy_2-idx_1\wedge dy_2-idx_2\wedge dy_1)\wedge 2(dx_3\wedge dy_3\wedge dx_4\wedge dy_4)\big)\\&=-2(dx_1\wedge dx_2+dy_1\wedge dy_2-idx_1\wedge dy_2-idx_2\wedge dy_1)\\&=-2dz_1\wedge d\bar z_2\end{aligned}$$

Let's take $j=1$

$$\begin{aligned}&*\,(dz_1\wedge d\bar z_1\wedge\omega^2)\\&=*\big((dx_1+idy_1)\wedge\omega^2\big)\\&=*\big(-2idx_1\wedge dy_1\wedge\big(dx_2\wedge dy_2\wedge(dx_3\wedge dy_3+dx_4\wedge dy_4)+dx_3\wedge dy_3\wedge(dx_2\wedge dy_2+dx_4\wedge dy_4)\\&\quad+dx_4\wedge dy_4\wedge(dx_2\wedge dy_2+dx_3\wedge dy_3)\big)\big)\\&=-2i\times 2(\omega-dx_1\wedge dy_1)\\&=-2dz_1\wedge d\bar z_1-4i\omega\end{aligned}$$

Hence,

$$\begin{aligned}&*\,\big((\partial\bar\partial h)\wedge\omega^2\big)\\&=-2\partial\bar\partial h+i(\Delta h)\omega\end{aligned}$$

□

$$d^*\beta=-*d*\beta$$

In local coordinates at x,

$$\begin{aligned}*\beta&=*\big((\bar\partial f+\partial\bar f)\wedge\omega\big)\\&=*\big(\sum_{j=1}^{4}(\frac{\partial f}{\partial\bar z_j}d\bar z_j+\frac{\partial f}{\partial z_j}dz_j)\wedge\omega\big)\end{aligned}$$

Take $j = 1$. By explicit calculation one can check that

$$*(d\bar{z}_1 \wedge \omega) = \frac{i}{2} d\bar{z}_1 \wedge \omega^2 \text{ and } *(dz_1 \wedge \omega) = -\frac{i}{2} dz_1 \wedge \omega^2$$

Hence,

$$\begin{aligned}
&* \beta = \frac{i}{2}(\bar{\partial} f - \partial \bar{f}) \wedge \omega^2 \\
&\Rightarrow d * \beta = \frac{i}{2}(\partial\bar{\partial} f - \bar{\partial}\partial \bar{f}) \wedge \omega^2 = i\partial\bar{\partial}(\mathrm{Re} f) \wedge \omega^2 \\
&\Rightarrow *d * \beta = i\big(-2\partial\bar{\partial}(\mathrm{Re} f) + i(\Delta \mathrm{Re} f)\omega\big) \\
&\Rightarrow - * d * \beta = 2i\partial\bar{\partial}(\mathrm{Re} f) + (\Delta \mathrm{Re} f)\omega
\end{aligned}$$

For $\phi \in \Omega^0(X, \mathcal{L}), c(q(\phi))$ can be written in diagonal matrix form for its Clifford action on positive spinors in $\Omega^0(X, \mathcal{L}) \oplus \Omega^{0,2}(X, \mathcal{L}) \oplus \Omega^{0,4}(X, \mathcal{L})$.

$$c(q(\phi)) = \begin{bmatrix} \frac{7}{8}|\phi|^2 \mathrm{Id} & 0 & 0 \\ 0 & -\frac{1}{8}|\phi|^2 \mathrm{Id} & 0 \\ 0 & 0 & -\frac{1}{8}|\phi|^2 \mathrm{Id} \end{bmatrix}$$

We also describe the Clifford multiplication of the forms $i\omega$ and ω^2 on the positive spinors.

	Ω^0	$\Omega^{0,2}$	$\Omega^{0,4}$
$i\omega$	4Id	0	-4Id
ω^2	-12Id	4Id	-12Id

Hence,

$$q(\phi) = \frac{|\phi|^2}{32}(4i\omega - \omega^2)$$

$$|\phi|^2 = e^{3(i-1)f} \times e^{-3(i+1)\bar{f}} |\phi_0|^2 = e^{-6(\mathrm{Re} f + \mathrm{Im} f)} |\phi_0|^2$$

The curvature equation (82) reads

$$\begin{aligned}
&F_A - 4\partial\bar{\partial}(\mathrm{Re} f) + 2i\Delta(\mathrm{Re} f)\omega = \frac{i}{8} e^{-6(\mathrm{Re} f + \mathrm{Im} f)} |\phi_0|^2 \omega \\
&\text{and } -\frac{1}{2}\Delta(\mathrm{Im} f) + c = -\frac{1}{32} e^{-6(\mathrm{Re} f + \mathrm{Im} f)} |\phi_0|^2
\end{aligned}$$

So, to find a solution we have to solve these two equations simultaneously. Say, the initial hermitian metric on $\mathcal{L}$ was h, we make a conformal change in the metric by a smooth function $\lambda : X \to \mathbb{C}$, the new metric being $h' := e^{\lambda} h$. Let's call the corresponding connection on the curvature equation A'. With respect to this new connection the curvature equation reads:

$$\begin{aligned}
&F_{A'} - 4\partial\bar{\partial}(\mathrm{Re} f) + 2i\Delta(\mathrm{Re} f)\omega = \frac{i}{8} e^{-6(\mathrm{Re} f + \mathrm{Im} f)} |\phi_0|^2_{h'} \omega \\
&\Leftrightarrow F_A - 8\partial\bar{\partial}\lambda - 4\partial\bar{\partial}(\mathrm{Re} f) + 2i\Delta(\mathrm{Re} f)\omega = \frac{i}{8} e^{-6(\mathrm{Re} f + \mathrm{Im} f) + \lambda} |\phi_0|^2 \omega
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{1}{2}\Delta(\mathrm{Im} f) + c = -\frac{1}{32} e^{-6(\mathrm{Re} f + \mathrm{Im} f)} |\phi_0|^2_{h'} \\
&\Leftrightarrow -\frac{1}{2}\Delta(\mathrm{Im} f) + c = -\frac{1}{32} e^{-6(\mathrm{Re} f + \mathrm{Im} f) + \lambda} |\phi_0|^2
\end{aligned}$$

Now assume $F_A = \partial\bar{\partial}g + ic'\omega$ (this assumption depends on the first Chern class of $\mathcal{L}$). So, we solve for:

$$g = 8\lambda + 4\mathrm{Re}f, \quad 2\Delta(-\mathrm{Re}f) + \frac{1}{8}e^{-6(\mathrm{Re}f+\mathrm{Im}f)+\lambda}|\phi_0|^2 = c'$$
$$\text{and } -\frac{1}{2}\Delta(\mathrm{Im}f) + c = -\frac{1}{32}e^{-6(\mathrm{Re}f+\mathrm{Im}f)+\lambda}|\phi_0|^2$$

We replace λ by g and Ref in the last two equations and solve them for Ref and Imf and then we get back λ from the first equation. So we solve for Ref and Imf simultaneously from these two equations.

$$\Delta(-\mathrm{Re}f) + \frac{1}{16}e^{\frac{g}{8}}e^{(-\frac{13}{2}\mathrm{Re}f-6\mathrm{Im}f)}|\phi_0|^2 = c'$$
$$\Delta(-\mathrm{Im}f) + \frac{1}{16}e^{\frac{g}{8}}e^{(-\frac{13}{2}\mathrm{Re}f-6\mathrm{Im}f)}|\phi_0|^2 = -c$$

Firstly, these two equations give us $c' = -c$. Hence we also get

$$\Delta(\mathrm{Re}f - \mathrm{Im}f) = 0$$

Which implies

$$\mathrm{Re}f = \mathrm{Im}f + a, \text{ for a real constant } a$$

So, the two second order pdes above become a single one:

$$\Delta(-\mathrm{Re}f) + \frac{1}{16}e^{(\frac{g}{8}-6a)}|\phi_0|^2e^{-\frac{25}{2}\mathrm{Re}f} = -c$$

Since $|\phi_0|^2$ is not everywhere zero, this equation has a unique solution for Ref since $c < 0$ [16]. Notice that $c' = -c > 0$ is the same condition as $-4c_1(K_X) + 8c_1(\mathcal{L})$ being a negative multiple of ω.

Hence, we see that for each $\phi_0 \in H^0(X, \mathcal{L}_0)$, there is a unique way to solve for $f : X \to \mathbb{C}$ and a unitary connection A_0 on $\mathcal{L}_0$ such that

$$\left(\phi = e^{3i(1+i)f}\phi_0, A = (-4A_{K_X} + 8A_0), \beta = (\bar{\partial}f + \partial\bar{f}) \wedge \omega\right)$$

solves the SW equations.

Let's see how the solution changes if we scale α_0 by a non-zero constant θ. Define

$$f_\theta := f + \frac{(1+i)}{6}\ln(\theta)$$

If we start with $\theta\phi_0$ instead of ϕ_0, the above construction would give us a new solution of the equations:

$$\left(e^{3i(1+i)(f_\theta)}\theta\phi_0, (-4A_{K_X} + 8A_0), (\bar{\partial}f_\theta + \partial\bar{f}_\theta) \wedge \omega\right)$$
$$= \left(\phi = e^{3i(1+i)f}\phi_0, A = (-4A_{K_X} + 8A_0), \beta = (\bar{\partial}f + \partial\bar{f}) \wedge \omega\right)$$

And given two different holomorphic sections of $\mathcal{L}$ which are not in the same conformal class lead to two different sets of solutions. So, modulo gauge the space of solutions we found is $\mathbb{CP}(H^0(X, \mathcal{L}))$.

$c_1(K_X^{-4} \otimes \mathcal{L}^8)$ **is a positive multiple of** ω**:**

The strategy for this case is very much similar to the one before, one major difference is that here the choice of our spinor will be a section of $\Lambda^{0,4}(X, \mathcal{L})$ instead of being a section of $\Lambda^0(X, \mathcal{L})$.

We start with a three-form $\beta = (\bar{\partial} f_1 + \partial \bar{f}_1) \wedge \omega$, f_1 being a complex-valued smooth function on X. We also take the Chern connection A_0 on $\mathcal{L}$. For $\phi \in \Omega^{0,4}(X, \mathcal{L}), A = -4A_{K_X} + 8A_0, \beta = (\bar{\partial} f + \partial \bar{f}) \wedge \omega$, the Dirac equation (81) reads:

$$\begin{aligned} &\sqrt{2}\bar{\partial}^*_{A_0}\phi + (1+i)c(\beta)\phi = 0 \\ &\Leftrightarrow \sqrt{2}\bar{\partial}^*_{A_0}\phi - 3\sqrt{2}i(1+i) * (\partial\bar{f}_1 \wedge \phi) = 0 \end{aligned}$$

Choose a section $\xi \in \Gamma\big(\Lambda^{0,4}(X, \mathcal{L})\big)$ such that $\bar{\partial}^*_{A_0}\xi = 0$ and define $\phi = e^{-3i(1+i)\bar{f}_1}\xi$. For For smooth section $\alpha \in \Omega^{0,3}(X, \mathcal{L})$,

$$\begin{aligned} &\langle \bar{\partial}^*_{A_0}(e^{-3i(1+i)\bar{f}_1}\xi), \alpha\rangle_{L^2} \\ &= \langle \xi, e^{3i(1-i)f_1}\bar{\partial}_{A_0}\alpha\rangle_{L^2} \\ &= \langle \xi, -3i(1-i)e^{3i(1-i)f_1}\bar{\partial} f_1 \wedge \alpha\rangle_{L^2} \\ &= 3i(1+i)\langle \phi, \bar{\partial} f_1 \wedge \alpha\rangle_{L^2} \\ &= 3i(1+i)\overline{\langle \bar{\partial} f_1 \wedge \alpha, \phi\rangle_{L^2}} \\ &= 3i(1+i)\overline{\int_X (\bar{\partial} f_1 \wedge \alpha \wedge *\bar{\phi})} \\ &= 3i(1+i)\overline{\int_X (\bar{\partial} f_1 \wedge \alpha \wedge \bar{\phi})} \\ &= 3i(1+i)\overline{\int_X \alpha \wedge *\overline{\big(*(\partial\bar{f}_1 \wedge \phi)\big)}} \\ &= 3i(1+i)\overline{\langle \alpha, *(\partial\bar{f}_1 \wedge \phi)\rangle_{L^2}} \\ &= \langle 3i(1+i) * (\partial\bar{f}_1 \wedge \phi), \alpha\rangle_{L^2} \end{aligned}$$

Hence, ϕ, A, β solves the Dirac equation (81). Notice since $\phi \in \Omega^{0,4}(X, \mathcal{L})$, we have

$$\begin{aligned} q(\phi) &= -\frac{|\phi|^2}{32}(4i\omega + \omega^2) \\ &= -\frac{e^{6(\mathrm{Re} f_1 - \mathrm{Im} f_1)}|\xi|^2}{32}(4i\omega + \omega^2) \end{aligned}$$

Going back to the curvature equation (82) we get

$$\begin{aligned} &F_A - 4\partial\bar{\partial}(\mathrm{Re} f_1) + 2i\Delta(\mathrm{Re} f_1)\omega = -\frac{i}{8}e^{6(\mathrm{Re} f_1 - \mathrm{Im} f_1)}|\xi|^2\omega \\ &\text{and } -\frac{1}{2}\Delta(\mathrm{Im} f_1) + c = -\frac{1}{32}e^{6(\mathrm{Re} f_1 - \mathrm{Im} f_1)}|\xi|^2 \end{aligned}$$

We want to solve these two equations simultaneously. The initial hermitian metric on $\mathcal{L}$ was h, we make a conformal change in the metric by a smooth function $\lambda_1 : X \to \mathbb{C}$, the new metric being $h' := e^{\lambda_1}h$. We notice that choosing ξ an anti-holomorphic section of $\Lambda^{0,4} \otimes \mathcal{L} = K_X^{-1} \otimes \mathcal{L}$ is same as choosing the corresponding holomorphic section $\bar{\xi}$ of $\overline{K_X^{-1} \otimes \mathcal{L}} = (K_X^{-1} \otimes \mathcal{L})^*$. Now with the natural induced metric from h say, h^* on $(K_X^{-1} \otimes \mathcal{L})^*$ we also have $|\bar{\xi}|_{h^*} = |\xi_1|_h$. For a conformal change by e^{λ_1} on h, we get a scaling by $e^{-\lambda_1}$ on h^*. As this is a conformal change in the metric, $\bar{\xi}$ is still a holomorphic section of $\overline{K_X^{-1} \otimes \mathcal{L}} = (K_X^{-1} \otimes \mathcal{L})^*$ and now when we go back to the corresponding anti-holomorphic section of $K_X^{-1} \otimes \mathcal{L}$, the norm of this section changes by $e^{-\lambda_1}$. Let's call the corresponding connection on the curvature equation A'. With respect to this new connection the curvature equation reads:

$$F_{A'} - 4\partial\bar{\partial}(\mathrm{Re} f_1) + 2i\Delta(\mathrm{Re} f_1)\omega = -\frac{i}{8} e^{6(\mathrm{Re} f_1 - \mathrm{Im} f_1)} |\xi|^2_{h'} \omega$$

$$\Leftrightarrow F_A - 8\partial\bar{\partial}\lambda_1 - 4\partial\bar{\partial}(\mathrm{Re} f_1) + 2i\Delta(\mathrm{Re} f_1)\omega = -\frac{i}{8} e^{6(\mathrm{Re} f_1 - \mathrm{Im} f_1) - \lambda_1} |\xi|^2 \omega$$

and

$$-\frac{1}{2}\Delta(\mathrm{Im} f_1) + c = -\frac{1}{32} e^{6(\mathrm{Re} f_1 - \mathrm{Im} f_1)} |\xi|^2_{h'}$$

$$\Leftrightarrow -\frac{1}{2}\Delta(\mathrm{Im} f_1) + c = -\frac{1}{32} e^{6(\mathrm{Re} f_1 - \mathrm{Im} f_1) - \lambda_1} |\xi|^2$$

Assume $F_A = \partial\bar{\partial} g_1 + i\tilde{c}\omega$ (this assumption depends on the first Chern class of $\mathcal{L}$). So, we solve for:

$$g_1 = 8\lambda_1 + 4\mathrm{Re} f_1, \quad -2\Delta(\mathrm{Re} f_1) - \frac{1}{8} e^{6(\mathrm{Re} f_1 - \mathrm{Im} f_1) - \lambda_1} |\xi|^2 = \tilde{c}$$

$$\text{and } -\frac{1}{2}\Delta(\mathrm{Im} f_1) + c = -\frac{1}{32} e^{6(\mathrm{Re} f_1 - \mathrm{Im} f) - \lambda_1} |\xi|^2$$

We replace λ_1 by g_1 and $\mathrm{Re} f_1$ in the last two equations and solve them for $\mathrm{Re} f_1$ and $\mathrm{Im} f_1$ and then we get back λ_1 from the first equation. So we solve for $\mathrm{Re} f_1$ and $\mathrm{Im} f_1$ simultaneously from these two equations.

$$\Delta(\mathrm{Re} f_1) + \frac{1}{16} e^{-\frac{g_1}{8}} e^{(\frac{13}{2}\mathrm{Re} f_1 - 6\mathrm{Im} f_1)} |\xi|^2 = -\tilde{c}$$

$$\Delta(-\mathrm{Im} f_1) + \frac{1}{16} e^{-\frac{g_1}{8}} e^{(\frac{13}{2}\mathrm{Re} f_1 - 6\mathrm{Im} f_1)} |\xi|^2 = -c$$

Firstly, these two equations give us $c = \tilde{c}$. Hence we also get

$$\Delta(\mathrm{Re} f_1 + \mathrm{Im} f_1) = 0$$

Which implies

$$\mathrm{Re} f_1 + \mathrm{Im} f_1 = a_1, \text{ for a real constant } a_1$$

and the two second order pdes above become a single one:

$$\Delta(\mathrm{Re} f_1) + \frac{1}{16} e^{(\frac{g_1}{8} - 6a_1)} |\xi|^2 e^{\frac{25}{2}\mathrm{Re} f_1} = -c$$

Since $|\xi|^2$ is not everywhere zero, this equation has a unique solution for $\mathrm{Re} f_1$ as $c < 0$ [16]. Notice that $\tilde{c} = c < 0$ is the same condition as $-4c_1(K_X) + 8c_1(\mathcal{L})$ being a positive multiple of ω.

Using similar arguments as in the other case, it's straight-forward to see that modulo gauge the space of solutions we constructed is $\mathbb{CP}(H^0(K_X \otimes \mathcal{L}^{-1}))$.

Remark 35. Notice if $\mathcal{L}$ is the trivial bundle, $\mathbb{CP}(H^0(X, \mathcal{L}))$ is a singleton point and for $\mathcal{L} = K_X$, $\mathbb{CP}(H^0(K_X \otimes \mathcal{L}^{-1}))$ is again a singleton set, which is reminiscent of the moduli space of 4d SW equations on Kähler 2-folds.

A class of solutions

We give a class of explicit examples where we have the necessary conditions to have non-trivial solutions of our equations. From the construction explained above, we see that for the construction to work, we need a holomorphic line bundle $\mathcal{L}$ on X with the following conditions:

1. $\dim H^0(X, \mathcal{L}) > 0$
2. $c_1(K_X^{-4} \otimes \mathcal{L}^8) = c_0[\omega]$ with $c_0 < 0$ [This implies $\deg(K_X^{-4} \otimes \mathcal{L}^8) < 0$]

or

1. $\dim H^0(X, K_X \otimes \mathcal{L}^{-1}) > 0$
2. $c_1(K_X^{-4} \otimes \mathcal{L}^8) = \tilde{c_0}[\omega]$ with $\tilde{c_0} > 0$ [This implies $\deg(K_X^{-4} \otimes \mathcal{L}_1^8) > 0$]

Example 1. Take 4 compact Riemann surfaces $(X_i, \omega_i)_{i=1,2,3,4}$ of the same genus $g > 1$. ω_i denotes the normalized Kähler form on X_i such that $\int_{X_i} \omega_i = 1$. Define $X := X_1 \times X_2 \times X_3 \times X_4$. The Kähler form on X is $\omega := \sum_{i=1}^4 \pi_i^* \omega_i$ where π_i is the projection of X onto X_i.

$K_X = \otimes_{i=1}^4 \pi_i^*(K_{X_i})$ and $c_1(K_X) = (2g-2)\omega$. If we choose $\mathcal{L}$ to be the trivial line bundle it satisfies the first two conditions and if we choose $\mathcal{L} = K_X$, it satisfies the alternative two conditions.

Example 2. We take X to be a hypersurface in $\mathbb{CP}^4$ of very high degree, let's take a holomorphic section of $\mathcal{O}(d) \to \mathbb{CP}^5$, i.e., a homogeneous polynomial of degree d in 6 variables (d to be determined later). If we choose this generically, the zero locus is a smooth algebraic variety X.

The Kahler form ω on X is given by restricting the Fubini-Study form ω_{FS} on X and it lies in the cohomology class obtained by restricting $c_1(\mathcal{O}(1))$ on X. Meanwhile, by the adjunction formula, $K_X = \mathcal{O}(d-6)|_X$.

Take $\mathcal{L} = \mathcal{O}(k)|_X$. For $m > 0$, $\dim H^0(\mathbb{CP}^5; \mathcal{O}(m)) > 0$. Restricting these holomorphic sections to X we can find line bundles on X with non-trivial holomorphic sections. We need k to satisfy the following two conditions for $\mathcal{O}(k)|_X$ to satisfy the first two conditions:

1. $k > 0$ so that there are non-trivial holomorphic sections of$\mathcal{L}$
2. $4(6-d) + 8k < 0$. This ensures that $c_1(K_X^{-4} \otimes \mathcal{L}^8)$ is a negative multiple of ω.

and for the alternative two conditions we would need

1. $d - 6 - k > 0$ so that there are non-trivial holomorphic sections of $K_X \otimes \mathcal{L}^{-1}$
2. $4(6-d) + 8k > 0$. This ensures that $c_1(K_X^{-4} \otimes \mathcal{L}^8)$ is a positive multiple of ω.

Putting these all together we would need k to be in the following ranges for existence of non-trivial solution:

$$0 < k < \frac{d-6}{2}$$

or

$$\frac{d-6}{2} < k < d-6$$

So we get many examples by choosing $d > 8$ and choosing k as we like in the above ranges.

Part VII

Seiberg–Witten equations on Spin(7)-manifolds

0.7 Spin(7)-manifolds

0.7.1 Spin(7)-structure on a manifold

In the list of the possible holonomy groups for a non-symmetric, irreducible Riemannian manifold, there are two exceptional cases: G_2 and Spin(7) [2]. There are several ways of defining the group Spin(7), often involving octonians. We will use a more general and certainly one of the more useful definitions, is the one given below.

Choose $x_0, x_1, \ldots, x_7$ to be the standard Euclidean coordinates of $\mathbb{R}^8$. We also equip $\mathbb{R}^8$ with the standard orientation coming from the volume form $dx_0 \wedge \cdots \wedge dx_7$ and the Euclidean metric $g = \sum_{j=0}^{7} dx_j \otimes dx_j$. Define a four form Φ_0 on $\mathbb{R}^8$ by

$$\begin{aligned}\Phi_0 :=& dx_{0123} - dx_{0167} - dx_{0527} - dx_{0563} + dx_{0415} + dx_{0426} + dx_{0437} \\ &+ dx_{4567} - dx_{4523} - dx_{4163} - dx_{4127} + dx_{2637} + dx_{1537} + dx_{1526}\end{aligned}$$

where $dx_{klmn} = dx_k \wedge dx_l \wedge dx_m \wedge dx_n$. Notice that Φ_0 is self-dual, i.e., $*\Phi_0 = \Phi_0$.

The subgroup of $GL(8, \mathbb{R})$ preserving Φ_0 is isomorphic to Spin(7), the double cover of $SO(7)$, which is a compact semi-simple, 21-dimensional Lie group. It is a subgroup of $SO(8)$, so the metric g can be reconstructed from Φ_0.

Let M be an 8-manifold. Consider AM, defined as a sub-bundle of $\Lambda^4 T^*M$ by

$$AM_m = \{\lambda \in (\Lambda^4 T^*M)_m : \exists \text{ an isomorphism } \phi : T_m M \to \mathbb{R}^8, \text{ taking } \lambda \text{ to } \Phi_0\}$$

A smooth section of AM gives rise to a Spin(7)-structure. Let Φ be a smooth section of AM, then Φ is a smooth four form on M and defines a Spin(7)-structure on M. By an abuse of notation we shall often identify a Spin(7)- structure with its associated 4-form Φ. A Spin(7)-structure Φ on M induces a natural metric g on M by the inclusion $\text{Spin}(7) \subset SO(8)$.

The fiber of AM is $GL(8, \mathbb{R})/\text{Spin}(7)$, so AM is not a vector sub-bundle of $\Lambda^4 T^*M$. Now a smooth section of AM gives rise to a Spin(7)-structure on M.

The action of Spin(7) on $\mathbb{R}^8$ gives an action of Spin(7) on $\Lambda^k(\mathbb{R}^8)^*$, which splits $\Lambda^k(\mathbb{R}^8)^*$ into an orthogonal direct sum of irreducible representations of $Spin(7)$. Suppose that M is an oriented 8-manifold with a Spin(7)-structure, so that M has a 4-form Φ and a metric g. Then in the same way, $\Lambda^k T^*M$ splits into an orthogonal direct sum of sub-bundles with irreducible representations of Spin(7) as fibres [15]:

$$\begin{aligned}\Lambda^2 &= \Lambda^2_7 \oplus \Lambda^2_{21} \qquad \Lambda^6 = \Lambda^6_7 \oplus \Lambda^6_{21} \\ \Lambda^3 &= \Lambda^3_8 \oplus \Lambda^3_{48} \qquad \Lambda^5 = \Lambda^5_7 \oplus \Lambda^5_{48} \\ &\Lambda^4 = \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27} \oplus \Lambda^4_{35}\end{aligned}$$

The notation Λ^k_l refers to an l-dimensional irreducible Spin(7)-representation which is a subspace of Λ^k. The decomposition respects the *Hodge star* $*$ operator since $Spin(7) \subset SO(8)$. We define the projection operator on forms $\pi^k_l : \Omega^k \to \Omega^k_l$, defined as the projection of k-forms onto the l-dimensional subspace.

It turns out that Spin(7)-manifolds are spin, i.e., $w_2(M) = 0$. We give an equivalent description of Spin(7)-manifolds from a *spinorial* point of view.

We start with an orientable Riemannian 8-manifold which is spin. Moreover we fix a spin-structure and take the associated real spin-bundle on M, say $S = S_+ \oplus S_-$. This basically comes from the real spinorial representation $\rho : \mathrm{Spin}(8) \to SO(16)$, constructed by restricting the isomorphism $Cl_8 \cong GL(16)$ and equipping $\mathbb{R}^{16}$ with a metric $\langle\cdot,\cdot\rangle$ which makes the Clifford product a skew-symmetric endomorphism.

At each $m \in M$, the action $\mathrm{Spin}(8) \to SO(S_+(M)_m)$ is a double covering, so that the existence of a unitary spinor $\xi \in \Gamma(S_+)$ determines an identification between Spin(7) and the stabilizer of of $\xi_m : \mathrm{stab}(\xi_m)$. We get back the four-form Φ associated to the Spin(7)-structure in the following way [23].

$$\begin{aligned}\Phi(v_1, v_2, v_3, v_4) =& \langle v_1, v_2\rangle\langle v_3, v_4\rangle - \langle v_1, v_3\rangle\langle v_4, v_2\rangle + \langle v_1, v_4\rangle\langle v_2, v_3\rangle \\ &- \langle c(v_2^*)c(v_1^*)\xi, c(v_3^*)c(v_4^*)\xi\rangle\end{aligned}$$

v^* is defined using the canonical identification between the tangent and the cotangent bundle: $v^* := g(v, \cdot)$.

0.7.2 Clifford multiplication

Let's start with a complex spin-bundle $S = S_+ \oplus S_- \to M$ on M, S_+ and S_- are bundles over M of complex dimension 8. There exists a real Spin(8)-equivariant structure on S which anti-commutes with Clifford multiplication of one-forms [10]. So, one can get back the *real* spinors from the *complex* spinors.

The Clifford action of the Cayley form Φ on $\Gamma(S_+)$ has two eigenvalues: 14 and -2. This gives us a splitting of S_+ :

$$S_+ = \langle\eta\rangle \oplus \langle\eta\rangle^\perp$$

η is a unit length positive spinor such that $c(\Phi)\eta = 14\eta$ and for any φ orthogonal to η, $c(\Phi)\varphi = -2\varphi$. Recall that the trace-free hermitian endomorphisms of the positive spinors in dimension 8 are given by the Clifford actions of imaginary two forms and the self-dual four forms:

$$c\colon i\Lambda^2 \oplus \Lambda^4_+ \to i\mathfrak{su}(S_+)$$

Moreover since, M has a Spin(7)-structure, both $i\Lambda^2$ and Λ^4_+ splits:

$$\begin{aligned} i\Lambda^2 &= i\Lambda^2_7 \oplus i\Lambda^2_{21} \\ \Lambda^4_+ &= \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27}\end{aligned}$$

$\Lambda^4_1 = \langle\Phi\rangle$ and as we saw earlier the Clifford action of Φ preserves the splitting of S_+. Together the two 7-dimensional parts correspond to Hermitian maps which send $\langle\eta\rangle$ to $\langle\eta\rangle^\perp$ and $\langle\eta\rangle^\perp$ to $\langle\eta\rangle$:

$$c : i\Lambda^2_7 \oplus \Lambda^4_7 \to \langle\eta\rangle^\perp \otimes \langle\eta\rangle^*$$

Since S_+ has a real structure, it induces a real structure on $\langle\eta\rangle^\perp \otimes \langle\eta\rangle^*$. Hence it can be seen as the direct sum of two real vector spaces and the direct sum decomposition is indeed

$$\langle\eta\rangle^\perp \otimes \langle\eta\rangle^* = c(i\Lambda^2_7) \oplus c(\Lambda^4_7)$$

Notice that both sides have real dimension $7 \times 2 = 14$.

And finally the forms $i\Lambda^2_{21}$ and Λ^4_{27} preserve $\langle\eta\rangle$ and $\langle\eta\rangle^\perp$. In fact their action on η is trivial and the Clifford actions of these forms give us trace-free Hermitian endomorphisms of $\langle\eta\rangle^\perp$, i.e., the following map is an isomorphism.

$$c : i\Lambda^2_{21} \oplus \Lambda^4_{27} \to i\mathfrak{su}(\langle\eta\rangle^\perp)$$

Together with the trace-part, we also get the isomorphism:

$$c : i\Lambda^2_{21} \oplus \Lambda^4_1 \oplus \Lambda^4_{27} \to \langle\eta\rangle^\perp \otimes (\langle\eta\rangle^\perp)^*$$

Notice for a Hermitian 7×7 matrix, the real part is a symmetric 7×7 matrix, so 28 real numbers. Meanwhile the imaginary part is 21 imaginary numbers (because it's skew, not symmetric). $i\Lambda^2_{21}$ contributes to the imaginary part and $\Lambda^4_1 \oplus \Lambda^4_{27}$ contributes to the real part.

0.8 Seiberg–Witten equations on a Spin(7)-manifold

Taking inspiration from the original Seiberg–Witten equations in dimension 4, there have been attempts to devise monopole equations on manifolds with a Spin(7)-structure, e.g., see [3]. We introduce a new set of elliptic equations on a 8-dimensional manifold M with a Spin(7)-structure determined by a Cayley form say Φ. The equations for $\phi \in \Gamma(S_+), \alpha \in i\Omega^1, \beta \in \Omega^3_8$ are:

$$D\phi + c(\alpha + \beta)\phi = 0, \tag{83}$$

$$\pi^2_7(d\alpha) + \pi^4_{1\oplus 7}(d\beta) + \Phi = (\pi^2_7 \oplus \pi^4_{1\oplus 7})\big(q(\phi)\big) \tag{84}$$

$q(\phi) \in i\Omega^2 \oplus \Omega^4_+, \pi^2_7 \oplus \pi^4_{1\oplus 7}$ is the projection of $i\Omega^2 \oplus \Omega^4_+$ onto the subspace $i\Omega^2_7 \oplus \Omega^4_1 \oplus \Omega^4_7$. The gauge group $\mathcal{G} = \mathrm{Map}(M, S^1)$ acts on (ϕ, α, β) in the following way:

$$g \in \mathcal{G}, g \cdot (\phi, \alpha, \beta) = (g\phi, \alpha - g^{-1}dg, \beta)$$

This action preserves the space of solutions to the equations and moreover the equations are elliptic modulo gauge. One gets Seiberg–Witten equations on manifolds with a G_2 and $SU(3)$-structure by dimensional reductions of the equations above.

The equations came out of the following fantasy.

- Given a Spin(7)-structure Φ on a closed 8-manifold M, we hope to "count" solutions to the SW equations (83), (84) and get an invariant of the isotopy class of the Spin(7)-structure (allowing an isotopy through the Cayley form Φ).
- If the Spin(7)-structure is torsion-free, i.e., $d\Phi = 0$, we hope to prove the invariant equals 1.
- With the assumptions of the statements above, let's say we find a Φ for which the invariant is not 1, that would imply that Φ is not isotopic to a torsion-free Spin(7)-structure.

  However the fantasy is still far out of reach, especially since the analysis seems to be much more complicated (see §§0.10) compared to the 4d SW theory. This is a work in progress.

Proposition 36. *The Seiberg–Witten equations* (83), (84) *on a Spin*(7)*-manifold* M *are elliptic modulo gauge, with index zero.*

Proof. The equations (83), (84) define a map

$$SW : i\Omega^1 \times \Omega^3_8 \times \Gamma(S_+) \to \left(i\Omega^2_7 \oplus \Omega^4_1 \oplus \Omega^4_7\right) \times \Gamma(S_-)$$
$$SW(\alpha,\beta,\phi) = \left(\pi^2_7(d\alpha) + \pi^4_{1\oplus 7}(d\beta) + \Phi - (\pi^2_7 \oplus \pi^4_{1\oplus 7})q(\phi), (D + c(\alpha+\beta))\phi\right)$$

If $\delta(A,\beta,\phi) = (a,b,\sigma)$ is an infinitesimal pertubation of (α,β,ϕ), where $a \in i\Omega^1, b \in \Omega^3_8$ and $\sigma \in \Gamma(S_+)$. The linearisation of SW at (α,β,ϕ) is

$$d_{(\alpha,\beta,\phi)}SW : \Omega^1 \oplus \Omega^3_8 \oplus \Gamma(S_+) \to \left(i\Omega^2_7 \oplus \Omega^4_1 \oplus \Omega^4_7\right) \times \Gamma(S_-)$$
$$d_{(A,\beta,\phi)}SW(a,b,\sigma) = \left(\pi^2_7 da + \pi^4_{1\oplus 7} d\beta - (\pi^2_7 + \pi^4_{1\oplus 7})d_\phi q(\sigma), (D + c(\alpha+\beta))\sigma + c(a+b)\phi\right)$$

We supplement this with the Coulomb gauge condition $2d^* : \Omega^1 \to \Omega^0$ and discard the zeroth order terms which do not affect ellipticity or the index. This leaves the map

$$L : i\Omega^1 \oplus \Omega^3_8 \times \Gamma(S_+) \to \left(i\Omega^0 \oplus i\Omega^2_7\right) \times \left(\Omega^4_1 \oplus \Omega^4_7\right) \times \Gamma(S_-)$$
$$L(a,b,\sigma) = \left((d^*a + \pi^2_7(da), \pi^4_1(db) + \pi^4_7(db), D\sigma\right)$$

All three maps

$$d^* + \pi^2_7 \circ d : \Omega^1 \to \Omega^0 \oplus \Omega^2_7, \tag{85}$$
$$\pi^4_{1\oplus 7} \circ d : \Omega^3_8 \to \Omega^4_1 \oplus \Omega^4_7, \tag{86}$$
$$D : \Gamma(S_+) \to \Gamma(S_-)$$

are elliptic [34] and hence so is L.

Joyce [15] showed that $S_+ \cong (\Omega^0 \oplus \Omega^2_7) \otimes \mathbb{C} \cong (\Omega^4_1 \oplus \Omega^4_7) \otimes \mathbb{C}$ and $S_- \cong \Omega^1 \otimes \mathbb{C} \cong \Omega^3_8 \otimes \mathbb{C} \cong \Omega^5_8 \otimes \mathbb{C}$ and the operators described in (85),(86) are essentially $D_- : \Gamma(S_-) \to \Gamma(S_+)$ modulo some rearrangements of constants (D_- is the canonical Dirac operator on negative spinors, the corresponding connection being the lift of the Levi-Civita connection). The canonical Dirac operator on positive spinors can be described as:

$$D_+ : (\Omega^4_1 \oplus \Omega^4_7) \otimes \mathbb{C} \to \Omega^5_8 \otimes \mathbb{C}$$
$$D_+(\xi_1 \oplus \xi_7) = 8\pi^5_8(d\xi_1) + 7\pi^5_8(d\xi_7)$$

Since D_- is the dual of D_+, the total index adds up to be zero. □

0.9 Construction of a solution

We call the Spin(7)-structure (say determined by the Cayley form Φ) torsion free if $\nabla\Phi = 0$. Fernández [9] showed that this is equivalent to $d\Phi = 0$ and moreover such a manifold has holonomy as a subgroup of Spin(7). In fact, it's an if and only if condition [6].

Proposition 37. *If* (M,Φ) *is a Spin(7)-manifold with zero torsion (i.e.,* Φ *being a harmonic form), then the Seiberg–Witten equations* (83), (84) *have a non-trivial solution.*

Proof. An important observation is the spinor η (the unit length spinor corresponding to the form Φ) is parallel [14]: $\nabla\eta = 0$ (the connection being the canonical lift of the Levi-Civita connection). Hence $D\eta = 0$. Notice

$$c\big(q(\eta)\big) = \begin{cases} \frac{7}{8}\mathrm{Id} \text{ on } \langle\eta\rangle \\ -\frac{1}{8}\mathrm{Id} \text{ on } \langle\eta\rangle^\perp \end{cases}$$

Hence

$$q(\eta) = \frac{1}{16}\Phi$$

and $\phi = 4\eta, \alpha = 0, \beta = 0$ solves the Seiberg–Witten equations (83), (84). □

0.10 A priori estimates

Proposition 38. *In dimension 8, for* $\alpha \in i\Omega^1$ *and* $\beta \in \Omega^3$*, the self-adjoint Dirac operator* $D_{\alpha,\beta} := D + c(\alpha+\beta)$ *enjoys the following Weitzenböck formula:*

$$D^2_{\alpha,\beta} = \nabla^*_{\alpha,\beta}\nabla_{\alpha,\beta} + \frac{s}{4} + c(d\alpha + d\beta) - 2|\beta|^2 \tag{87}$$

s *denotes the scalar curvature,* $\nabla_{\alpha,\beta}$ *is a unitary connection depending on both* α *and* β*, an explicit description is given in proposition 11.*

Proof. We pick a coframe e_j which is stationary at a point $p \in M$ with respect to the Levi-Civita connection. Proposition 11 gives us

$$D^2_{\alpha,\beta} = \nabla^*_{\alpha,\beta}\nabla_{\alpha,\beta} + \frac{s}{4} + c(d\alpha + d\beta) + \frac{1}{4}\sum_j \big(c(e_j)\circ c(\alpha+\beta) + c(\alpha+\beta)\circ c(e_j)\big)^2 + c(\alpha+\beta)^2$$

$$\begin{aligned}
&\sum_j \big(c(e_j)\circ c(\alpha+\beta) + c(\alpha+\beta)\circ c(e_j)\big)^2 \\
&= \sum_j c(e_j)\circ \big(c(\alpha)\circ c(\beta) + c(\beta)\circ c(\alpha) + c(\alpha)^2 + c(\beta)^2\big)\circ c(e_j) \\
&\quad + \big(\sum_j c(e_j)\circ c(\alpha+\beta)\circ c(e_j)\big)\circ c(\alpha+\beta) \\
&\quad + c(\alpha+\beta)\circ\big(\sum_j c(e_j)\circ c(\alpha+\beta)\circ c(e_j)\big) - 8c(\alpha+\beta)^2
\end{aligned}$$

Notice that for $\alpha \in i\Omega^1, c(\alpha)^2 = |\alpha|^2$. Since $c(\alpha+\beta)^2, c(\alpha)^2, c(\beta)^2$ are all self-adjoint endomorphisms of positive spinors, so must be $\big(c(\alpha)\circ c(\beta) + c(\beta)\circ c(\alpha)\big)$. Hence, $c(\alpha)\circ c(\beta) + c(\beta)\circ c(\alpha) = c(\theta)$, for some $\theta \in i\Omega^2$. Notice that for $k \neq l$,

$$\begin{aligned}
j \in \{k,l\}, c(e_j)\circ c(e_k)\circ c(e_l)\circ c(e_j) &= c(e_k)\circ c(e_l) \\
j \notin \{k,l\}, c(e_j)\circ c(e_k)\circ c(e_l)\circ c(e_j) &= -c(e_k)\circ c(e_l)
\end{aligned}$$

This says $\sum_j c(e_j)\circ c(\theta)\circ c(e_j) = -4c(\theta)$. By similar observation, we also get $\sum_j \big(c(e_j)\circ c(\alpha)\circ c(e_j)\big) = 6c(\alpha)$. We borrow two formulas from chapter IV, namely equation 44:

$$\sum_j c(e_j)\circ c(\beta)\circ c(e_j) = 2c(\beta)$$

and equation 45

$$\sum_j c(e_j)\circ c(\beta)^2\circ c(e_j) = -8|\beta|^2$$

Assembling all the pieces we get

$$\begin{aligned}&\sum_j \big(c(e_j)\circ c(\alpha+\beta)+c(\alpha+\beta)\circ c(e_j)\big)^2\\&=-4(c(\alpha)\circ c(\beta)+c(\beta)\circ c(\alpha))+8|\alpha|^2-8|\beta|^2+(6c(\alpha)+2c(\beta))\circ(c(\alpha)+c(\beta))\\&\quad+(c(\alpha)+c(\beta))\circ(6c(\alpha)+2c(\beta))-8(c(\alpha)\circ c(\beta)+c(\beta)\circ c(\alpha)+|\alpha|^2+c(\beta)^2)\\&=-4|\alpha|^2-4(c(\alpha)\circ c(\beta)+c(\beta)\circ c(\alpha))-4c(\beta)^2-8|\beta|^2\end{aligned}$$

So,

$$\begin{aligned}&\frac{1}{4}\sum_j \big(c(e_j)\circ c(\alpha+\beta)+c(\alpha+\beta)\circ c(e_j)\big)^2+c(\alpha+\beta)^2\\&=-2|\beta|^2\end{aligned}$$

Finally the Weitzenböck formula reads:

$$D^2_{\alpha,\beta}=\nabla^*_{\alpha,\beta}\nabla_{\alpha,\beta}+\frac{s}{4}+c(d\alpha+d\beta)-2|\beta|^2$$

□

Before proceeding further, let's understand the term $(\pi^2_7\oplus\pi^4_{1\oplus7})\big(q(\phi)\big)$ explicitly. Say $\phi=f\eta+\xi, f\in C^\infty(M,\mathbb{C}),\xi\in\Gamma(\langle\eta\rangle^\perp)$. In terms of its Clifford action on $S_+=\langle\eta\rangle\oplus\langle\eta\rangle^\perp$, the matrix representation of $c(q(\phi))$ is:

$$\begin{bmatrix}\frac{7|f|^2-|\xi|^2}{8} & f\bar{\xi}\\ \bar{f}\xi & (\langle\ ,\xi\rangle-\frac{|\xi|^2}{7})-\frac{7|f|^2-|\xi|^2}{56}\end{bmatrix}$$

Notice the term $\big(\langle\ ,\xi\rangle-\frac{|\xi|^2}{7}\big)\in\Gamma\big(i\mathfrak{su}(\langle\eta\rangle^\perp)\big)\cong i\Omega^2_{21}\oplus\Omega^4_{27}$, hence the matrix representation of $c(\pi^2_7\oplus\pi^4_{1\oplus7})\big(q(\phi)\big)$ is

$$\begin{bmatrix}\frac{7|f|^2-|\xi|^2}{8} & f\bar{\xi}\\ \bar{f}\xi & -\frac{7|f|^2-|\xi|^2}{56}\end{bmatrix}$$

The diagonal entries are given by $\frac{(7|f|^2-|\xi|^2)}{8\times14}\Phi\in\Omega^4_1$ and the off-diagonal parts are given by an element of $i\Omega^2_7\oplus\Omega^4_7$.

$$\begin{aligned}c(\pi^2_7\oplus\pi^4_{1\oplus7})\big(q(\phi)\big)(\phi)&=\frac{7}{8}(|f|^2+|\xi|^2)f\eta+\frac{(49|f|^2+\xi|^2)}{56}\xi\\ \Rightarrow\langle c(\pi^2_7\oplus\pi^4_{1\oplus7})\big(q(\phi)\big)\phi,\phi\rangle&=\frac{7}{8}\big(|f|^4+2|f|^2|\xi|^2+\frac{1}{7^2}|\xi|^4\big)\end{aligned}$$

Lemma 39. *Let* M *be a Spin*(7)*-manifold with a Spin*(7) *structure defined by the Cayley four form* Φ *such that* $d\Phi=0$. *Then if* $\phi=f\eta+\xi,\alpha,\beta$ *solves the SW equations* (83), (84), *then* f *and* ξ *satisfy:*

$$\int(7|f|^2-|\xi|^2)=8\times14$$

Proof. Since Φ is closed we get,

$$\int (d\beta \wedge \Phi) = 0$$
$$\Rightarrow \int \Big(\frac{(7|f|^2 - |\xi|^2)}{8 \times 14} - 1\Big)\Phi \wedge \Phi = 0$$

□

Lemma 40. *Let* M *be a Spin*(7)*-manifold with a Spin*(7) *structure defined by the Cayley four form* Φ *such that* $d\Phi = 0$*. Then for* $\alpha \in i\Omega^2_7$

$$3\int |\pi^2_7(d\alpha)|^2 = \int |\pi^2_{21}(d\alpha)|^2$$

Proof. Since Φ is closed we have

$$\int d\alpha \wedge d\alpha \wedge \Phi = 0$$
$$\Rightarrow \int \langle d\alpha, *(d\alpha \wedge \Phi)\rangle = 0$$
$$\Rightarrow \int \langle \big(\pi^2_7(d\alpha) + \pi^2_{21}(d\alpha)\big), *\big(\pi^2_7(d\alpha) + \pi^2_{21}(d\alpha) \wedge \Phi\big)\rangle = 0$$
$$\Rightarrow \int \langle \pi^2_7(d\alpha) + \pi^2_{21}(d\alpha), 3\pi^2_7(d\alpha) - \pi^2_{21}(d\alpha)\rangle = 0$$
$$\Rightarrow 3\int |\pi^2_7(d\alpha)|^2 = \int |\pi^2_{21}(d\alpha)|^2$$

□

Lemma 41. *Let* M *be a Spin*(7)*-manifold with a torsion-free Spin*(7)*-structure defined by the Cayley four form* Φ*. Then if* $\phi = f\eta + \xi, \alpha, \beta$ *solves the SW equations* (83)*,* (84)*, then for any* $\epsilon > 0, f, \xi$ *and* β *satisfy*

$$\int \frac{7}{8}\big(|f|^4 + 2|f|^2|\xi|^2 + \frac{1}{7^2}|\xi|^4\big) \le 2\int |\beta|^2(|f|^2 + |\xi|^2) + \frac{3}{2\epsilon}\int |f|^2|\xi|^2 + \frac{\epsilon}{2}\int |\xi|^4 + 16 \times 14$$

Proof. Using the Weitzenböck formula (87) we get

$$0 = \nabla^*_{\alpha,\beta}\nabla_{\alpha,\beta}\phi + c(d\alpha + d\beta)\phi - 2|\beta|^2\phi$$

Taking point-wise inner product with ϕ yields

$$0 = \langle \nabla^*_{\alpha,\beta}\nabla_{\alpha,\beta}\phi, \phi\rangle + \langle c(d\alpha + d\beta)\phi, \phi\rangle - 2|\beta|^2|\phi|^2$$

Since we are working with a torsion-free Spin(7)-structure and $\beta \in \Omega^3_8, d\beta^+$ has no component in Ω^4_{27} [8]. Hence we get

$$0 = \langle \nabla^*_{\alpha,\beta}\nabla_{\alpha,\beta}\phi, \phi\rangle + \langle c(\pi^2_7(d\alpha) + d\beta)\phi, \phi\rangle - 2|\beta|^2|\phi|^2 + \langle c(\pi^2_{21}(d\beta))\phi, \phi\rangle$$
$$\Rightarrow 0 = \langle \nabla^*_{\alpha,\beta}\nabla_{\alpha,\beta}\phi, \phi\rangle + \frac{7}{8}\big(|f|^4 + 2|f|^2|\xi|^2 + \frac{1}{7^2}|\xi|^4\big) - 14|f|^2 + 2|\xi|^2 - 2|\beta|^2|\phi|^2 + \langle c(\pi^2_{21}(d\beta))\xi, \xi\rangle$$

We have

$$
\begin{aligned}
\frac{1}{2}\Delta|\phi|^2 &= \langle \nabla^*_{\alpha,\beta}\nabla_{\alpha,\beta}\phi, \phi\rangle - |\nabla_{\alpha,\beta}\phi|^2 \\
&= -\frac{7}{8}\left(|f|^4 + 2|f|^2|\xi|^2 + \frac{1}{7^2}|\xi|^4\right) + 2(7|f|^2 - |\xi|^2) + 2|\beta|^2|\phi|^2 - \langle c(\pi^2_{21}(d\beta))\xi, \xi\rangle - |\nabla_{\alpha,\beta}\phi|^2
\end{aligned}
$$

Integrating both sides we get

$$
\begin{aligned}
\int \frac{7}{8}\left(|f|^4 + 2|f|^2|\xi|^2 + \frac{1}{7^2}|\xi|^4\right) &\leq 2\int |\beta|^2(|f|^2 + |\xi|^2) + \int |\pi^2_{21}(d\beta)||\xi|^2 + 2\int (7|f|^2 - |\xi|^2) \\
&\leq 2\int |\beta|^2(|f|^2 + |\xi|^2) + \frac{1}{2\epsilon}\int |\pi^2_{21}(d\beta)|^2 + \frac{\epsilon}{2}\int |\xi|^4 + 16 \times 14 \\
&= 2\int |\beta|^2(|f|^2 + |\xi|^2) + \frac{3}{2\epsilon}\int |\pi^2_{7}(d\beta)|^2 + \frac{\epsilon}{2}\int |\xi|^4 + 16 \times 14 \\
&\leq 2\int |\beta|^2(|f|^2 + |\xi|^2) + \frac{3}{2\epsilon}\int |f|^2|\xi|^2 + \frac{\epsilon}{2}\int |\xi|^4 + 16 \times 14
\end{aligned}
$$

We are using the Peter-Paul inequality and the two lemmas proved above and also the fact that the following isomorphism is also an isometry with respect to the usual norms on both sides.

$$
c : i\Lambda^2_7 \oplus \Lambda^4_7 \to \langle \eta \rangle^{\perp} \otimes \langle \eta \rangle^* \subset S_+ \otimes S^*_+
$$

□

Appendix A

Clifford multiplication of (p, q)-forms

A.1 Clifford multiplication on Kähler 3-folds

The appendix involves calculating explicit formulae of Clifford multiplication of (p, q)-forms on spinors. From now on, for a form α and a spinor ν we will write $\alpha \cdot \nu$ instead of $c(\alpha)\nu$ to denote Clifford action of α on ν (this is purely to make my typing job easier and hopefully it won't confuse the reader). Clifford multiplication by a real one-form $\alpha \in \Omega^1(X, \mathbb{R})$ on a spinor ν is given by the formula [20]:

$$\alpha \cdot \nu = \sqrt{2}\big(\pi^{0,1}(\alpha) \wedge \nu - \pi^{0,1}(\alpha) \lrcorner \nu\big)$$

Now we need to calculate its complex-linear extension to the complexified forms. For $a, b \in \Omega^1(X, \mathbb{R})$,

$$\begin{aligned}(a + ib) \cdot \nu &= a \cdot \nu + ib \cdot \nu \\ &= \sqrt{2}\big(\pi^{0,1}(a) \wedge \nu - \pi^{0,1}(a) \lrcorner \nu\big) + i\sqrt{2}\big(\pi^{0,1}(b) \wedge \nu - \pi^{0,1}(b) \lrcorner \nu\big)\end{aligned}$$

As contraction is complex anti-linear in the first variable, $i(\pi^{0,1}(b) \lrcorner \nu) = (-i\pi^{0,1}(b)) \lrcorner \nu$. Hence,

$$\begin{aligned}&\sqrt{2}\big(\pi^{0,1}(a) \wedge \nu - \pi^{0,1}(a) \lrcorner \nu\big) + i\sqrt{2}\big(\pi^{0,1}(b) \wedge \nu - \pi^{0,1}(b) \lrcorner \nu\big) \\ &= \sqrt{2}\big((\pi^{0,1}(a) + i\pi^{0,1}(b)) \wedge \nu - (\pi^{0,1}(a) - i\pi^{0,1}(b)) \lrcorner \nu\big) \\ &= \sqrt{2}\big(\pi^{0,1}(a + ib) \wedge \nu - \overline{\pi^{1,0}(a + ib)} \lrcorner \nu\big)\end{aligned}$$

A.1.1 Clifford action on positive spinors

We take local holomorphic coordinates $\{z_k = x_k + iy_k\}_{k=1,2,3}$ centered at a point $x \in X$ so that the Kähler metric is standard to second order at the point.

$\underline{\Omega^{3,0}(X, \mathbb{C}) \text{ acting on } \Omega^0(X, \mathcal{L}_0) \oplus \Omega^{0,2}(X, \mathcal{L}_0):}$ Trivial action as we are contracting three times.

$\underline{\Omega^{0,3}(X, \mathbb{C}) \text{ acting on } \Omega^0(X, \mathcal{L}_0):}$ For $\mu \in \Omega^{0,3}(X, \mathbb{C}), \lambda \in \Omega^0(X, \mathcal{L}_0)$, $\mu \cdot \lambda = 2\sqrt{2}\mu \wedge \lambda \in \Omega^3(X, \mathcal{L}_0)$.

$\underline{\Omega^{0,3}(X,\mathbb{C}) \text{ acting on } \Omega^{0,2}(X,\mathcal{L}_0):}$ Trivial action.

$\underline{\Omega^{1,2}(X,\mathbb{C}) \text{ acting on } \Omega^{0}(X,\mathcal{L}_0):}$ Say $\lambda \in \Omega^0(X,\mathcal{L}_0)$. There are basically two cases we need to consider here.

Case 1: $j \neq l \neq k$, then $(dz_j \wedge d\bar{z}_l \wedge d\bar{z}_k)\cdot\lambda = d\bar{z}_l \cdot d\bar{z}_k \cdot dz_j \cdot \lambda = 0$.

Case 2: $j \neq k$, then $(dz_j \wedge d\bar{z}_j) \wedge d\bar{z}_k = -2i(dx_j \wedge dy_j) \wedge d\bar{z}_k = -2i d\bar{z}_k \wedge (dx_j \wedge dy_j)$. $d\bar{z}_k \wedge (dx_j \wedge dy_j)\cdot\lambda = d\bar{z}_k \cdot dx_j \cdot dy_j \cdot \lambda$ and $\pi^{0,1}(dy_j) = \frac{i}{2}d\bar{z}_j, \pi^{0,1}(dx_j) = \frac{1}{2}d\bar{z}_j$. So,

$$dy_j \cdot \lambda = \sqrt{2}\frac{i}{2}(d\bar{z}_j \wedge \lambda)$$
$$dx_j \cdot dy_j \cdot \lambda = -\frac{i}{2} d\bar{z}_j \lrcorner (d\bar{z}_j \wedge \lambda)$$

$$\begin{aligned} d\bar{z}_k \cdot dx_j \cdot dy_j \cdot \lambda &= -\frac{i}{\sqrt{2}} d\bar{z}_k \wedge \big(d\bar{z}_j \lrcorner (d\bar{z}_j \wedge \lambda)\big) \\ &= -\frac{i}{\sqrt{2}} d\bar{z}_k \wedge \big(|d\bar{z}_j|^2 \lambda\big) \\ &= -\sqrt{2} i \, d\bar{z}_k \wedge \lambda \quad \in \Omega^{0,1}(X,\mathcal{L}_0) \end{aligned}$$

We also notice that for $j \neq l \neq k$, $(dz_j \wedge d\bar{z}_l \wedge d\bar{z}_k) \wedge \omega = 0$, where $\omega = \sum_{m=1}^{3} dx_m \wedge dy_m$ is the Kähler form. To understand the second case let's look at the element $d\bar{z}_2 \wedge dx_1 \wedge dy_2$.

$$(d\bar{z}_2 \wedge dx_1 \wedge dy_1) \wedge \omega = dx_1 \wedge dy_1 \wedge d\bar{z}_2 \wedge dx_3 \wedge dy_3$$

and

$$\begin{aligned} *(d\bar{z}_2 \wedge dx_1 \wedge dy_2 \wedge \omega) &= *(dx_1 \wedge dy_1 \wedge d\bar{z}_2 \wedge dx_3 \wedge dy_3) \\ &= dy_2 + i dx_2 \\ &= i d\bar{z}_2 \end{aligned}$$

Here and on-wards $*$ on complexified forms will mean the complex-linear extension of the Hodge-$*$ operator on real forms. Similarly checking all other cases, we observe that for $k \neq j$,

$$*(d\bar{z}_k \wedge dx_j \wedge dy_j \wedge \omega) = i d\bar{z}_k$$

Hence we see that for any $\mu \in \Omega^{1,2}(X,\mathbb{C}), \lambda \in \Omega^0(X,\mathcal{L}_0)$,

$$\mu \cdot \lambda = -\sqrt{2}\big(*(\mu \wedge \omega) \wedge \lambda\big)$$

Now for $\mu = \eta \wedge \omega$, we get

$$\begin{aligned} (\eta \wedge \omega)\cdot\lambda &= -\sqrt{2}\big(*(\eta \wedge \omega^2) \wedge \lambda\big) \\ &= -2\sqrt{2} i \, \eta \wedge \lambda \end{aligned}$$

$\underline{\Omega^{1,2}(X,\mathbb{C}) \text{ acting on } \Omega^{0,2}(X,\mathcal{L}_0):}$ Say $\lambda \in \Omega^{0,2}(X,\mathcal{L}_0), \mu \in \Omega^{1,2}(X,\mathbb{C})$.

Locally let's say we write $\lambda = \lambda_1 d\bar{z}_1 \wedge d\bar{z}_2 + \lambda_2 d\bar{z}_2 \wedge d\bar{z}_3 + \lambda_3 d\bar{z}_3 \wedge d\bar{z}_1$. Now similar to the last case, for $j \neq k \neq l$,

$$(dz_j \wedge d\bar{z}_k \wedge d\bar{z}_l)\cdot\lambda = dz_j \cdot d\bar{z}_k \cdot d\bar{z}_l \cdot \lambda = 0$$

and for $j \neq k$,

$$(d\bar{z}_k \wedge dx_j \wedge dy_j) \cdot \lambda = d\bar{z}_k \cdot dx_j \cdot dy_j \cdot \lambda$$

Let's look at the case $j = 1, k = 2$.

$$\begin{aligned}
(dx_1 \wedge dy_1) \cdot \lambda &= dx_1 \cdot (dy_1 \cdot \lambda) \\
&= dx_1 \cdot \big(\sqrt{2} \times \frac{i}{2}(d\bar{z}_1 \wedge \lambda + d\bar{z}_1 \lrcorner \lambda)\big) \\
&= (\sqrt{2})^2 \times \frac{i}{2} \times \frac{1}{2}\big(d\bar{z}_1 \wedge (d\bar{z}_1 \lrcorner \lambda) - d\bar{z}_1 \lrcorner (d\bar{z}_1 \wedge \lambda)\big) \\
&= \frac{i}{2}\big(d\bar{z}_1 \wedge (d\bar{z}_1 \lrcorner \lambda) - d\bar{z}_1 \lrcorner (d\bar{z}_1 \wedge \lambda)\big) \\
&= \frac{i}{2}\big(d\bar{z}_1 \wedge (\lambda_1 |d\bar{z}_1|^2 d\bar{z}_2 - \lambda_3 |d\bar{z}_1|^2 d\bar{z}_3) - \lambda_2 d\bar{z}_1 \lrcorner (d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3)\big) \\
&= \frac{i}{2}\big(2\lambda_1 d\bar{z}_1 \wedge d\bar{z}_2 - 2\lambda_2 d\bar{z}_2 \wedge d\bar{z}_3 + 2\lambda_3 d\bar{z}_3 \wedge d\bar{z}_1\big) \\
&= i(\lambda_1 d\bar{z}_1 \wedge d\bar{z}_2 - \lambda_2 d\bar{z}_2 \wedge d\bar{z}_3 + \lambda_3 d\bar{z}_3 \wedge d\bar{z}_1)
\end{aligned}$$

So we get

$$\begin{aligned}
d\bar{z}_2 \cdot dx_1 \cdot dy_1 \cdot \lambda &= i d\bar{z}_2 \cdot \big(\lambda_1 d\bar{z}_1 \wedge d\bar{z}_2 - \lambda_2 d\bar{z}_2 \wedge d\bar{z}_3 + \lambda_3 d\bar{z}_3 \wedge d\bar{z}_1\big) \\
&= \sqrt{2} i \lambda_3 d\bar{z}_2 \wedge d\bar{z}_3 \wedge d\bar{z}_1
\end{aligned}$$

Similar to the last case we notice that for $j \neq l \neq k, (dz_j \wedge d\bar{z}_l \wedge d\bar{z}_k) \wedge \omega = 0$, where $\omega = \sum_{m=1}^{3} dx_m \wedge dy_m$ is the Kähler form and $*(d\bar{z}_2 \wedge dx_1 \wedge dy_1 \wedge \omega) = i d\bar{z}_2$. Hence

$$(d\bar{z}_2 \wedge dx_1 \wedge dy_1) \cdot \lambda = \sqrt{2}\big(* (d\bar{z}_2 \wedge dx_1 \wedge dy_1 \wedge \omega) \wedge \lambda\big)$$

Similarly the same can be proved for any $j \neq k$. Hence we get

$$\mu \cdot \lambda = \sqrt{2}\big(* (\mu \wedge \omega) \wedge \lambda\big)$$

For $\mu = \eta \wedge \omega$, we get

$$\begin{aligned}
(\eta \wedge \omega) \cdot \lambda &= \sqrt{2}\big(* (\eta \wedge \omega^2) \wedge \lambda\big) \\
&= 2\sqrt{2} i \, \eta \wedge \lambda
\end{aligned}$$

$\underline{\Omega^{2,1}(X, \mathbb{C}) \text{ acting on } \Omega^0(X, \mathcal{L}_0)}$: Trivial action.

$\underline{\Omega^{2,1}(X, \mathbb{C}) \text{ acting on } \Omega^{0,2}(X, \mathcal{L}_0)}$: Say $\lambda \in \Omega^{0,2}(X, \mathcal{L}_0), \mu \in \Omega^{2,1}(X, \mathbb{C})$.

Locally let's say we write $\lambda = \lambda_1 d\bar{z}_1 \wedge d\bar{z}_2 + \lambda_2 d\bar{z}_2 \wedge d\bar{z}_3 + \lambda_3 d\bar{z}_3 \wedge d\bar{z}_1$. We observe that $\Omega^{2,1}(X, \mathbb{C})$ consists of two types of forms, namely $dz_k \wedge dz_l \wedge d\bar{z}_j$ for $k \neq l \neq j$ and $dz_k \wedge dz_l \wedge d\bar{z}_l$ for $k \neq l$. We'll prove a formula for the Clifford action of μ for an element of each type, the rest can be checked doing similar calculation.

$$\begin{aligned}
(dz_1 \wedge dz_2 \wedge d\bar{z}_3) \cdot \lambda &= dz_1 \cdot dz_2 \cdot d\bar{z}_3 \cdot \lambda \\
&= dz_1 \cdot dz_2 \cdot (\sqrt{2} \lambda_1 d\bar{z}_3 \wedge d\bar{z}_1 \wedge d\bar{z}_2) \\
&= dz_1 \cdot (-2\lambda_1 |d\bar{z}_2|^2 d\bar{z}_3 \wedge d\bar{z}_1) \\
&= -2\sqrt{2}\lambda_1 |d\bar{z}_1|^2 |d\bar{z}_2|^2 d\bar{z}_3 \\
&= -8\sqrt{2}\lambda_1 d\bar{z}_3
\end{aligned}$$

We also notice that

$$\begin{aligned}
*\big((dz_1 \wedge dz_2 \wedge d\bar{z}_3) \wedge \lambda\big) &= *\big((dz_1 \wedge dz_2 \wedge d\bar{z}_3) \wedge (\lambda_1 d\bar{z}_1 \wedge d\bar{z}_2)\big) \\
&= \lambda_1 * \big(dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_3 \wedge d\bar{z}_2\big) \\
&= -\lambda_1 * \big(dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \wedge d\bar{z}_3\big) \\
&= -\lambda_1 \times (-2i)^2 * \big(dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge (dx_3 - idy_3)\big) \\
&= 4\lambda_1 \times (id\bar{z}_3) \\
&= 4i\lambda_1 d\bar{z}_3
\end{aligned}$$

So,

$$(dz_1 \wedge dz_2 \wedge d\bar{z}_3) \cdot \lambda = 2\sqrt{2}i * \big((dz_1 \wedge dz_2 \wedge d\bar{z}_3) \wedge \lambda\big)$$

Now let's look at another type of form.

$$\begin{aligned}
(dx_1 \wedge dy_1 \wedge dz_2) \cdot \lambda &= dz_2 \cdot dx_1 \cdot dy_1 \cdot \lambda \\
&= idz_2 \cdot (\lambda_1 d\bar{z}_1 \wedge d\bar{z}_2 - \lambda_2 d\bar{z}_2 \wedge d\bar{z}_3 + \lambda_3 d\bar{z}_3 \wedge d\bar{z}_1) \\
&= -\sqrt{2}i(-\lambda_1 |d\bar{z}_2|^2 d\bar{z}_1 - \lambda_2 |d\bar{z}_2|^2 d\bar{z}_3) \\
&= 2\sqrt{2}i(\lambda_1 d\bar{z}_1 + \lambda_2 d\bar{z}_3)
\end{aligned}$$

We also have

$$\begin{aligned}
*\big((dx_1 \wedge dy_1 \wedge dz_2) \wedge \lambda\big) &= *\big(dx_1 \wedge dy_1 \wedge dz_2 \wedge (\lambda_2 d\bar{z}_2 \wedge d\bar{z}_3)\big) \\
&= -2i\lambda_2 * \big(dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge d\bar{z}_3\big) \\
&= -2i\lambda_2 \times id\bar{z}_3 \\
&= 2\lambda_2 d\bar{z}_3
\end{aligned}$$

and

$$\begin{aligned}
\overline{*(dx_1 \wedge dy_1 \wedge dz_2 \wedge \omega)} \lrcorner \lambda &= \overline{*(dx_1 \wedge dy_1 \wedge dz_2 \wedge dx_3 \wedge dy_3)} \lrcorner \lambda \\
&= \overline{(-idz_2)} \lrcorner \lambda \\
&= (id\bar{z}_2) \lrcorner \lambda \\
&= -i(d\bar{z}_2 \lrcorner \lambda) \\
&= -i(-2\lambda_1 d\bar{z}_1 + 2\lambda_2 d\bar{z}_3) \\
&= 2i(\lambda_1 d\bar{z}_1 - \lambda_2 d\bar{z}_3)
\end{aligned}$$

Hence

$$(dx_1 \wedge dy_1 \wedge dz_2) \cdot \lambda = 2\sqrt{2}i * \big((dx_1 \wedge dy_1 \wedge dz_2) \wedge \lambda\big) + \sqrt{2}\big(\overline{*(dx_1 \wedge dy_1 \wedge dz_2 \wedge \omega)} \lrcorner \lambda\big)$$

One can check that all other type of $(2,1)$ forms satisfy the same formula. So, we get for any $\mu \in \Omega^{2,1}(X, \mathbb{C})$ and $\lambda \in \Omega^{0,2}(X, \mathcal{L}_0)$, locally we have

$$\mu \cdot \lambda = 2\sqrt{2}i * \big(\mu \wedge \lambda\big) + \sqrt{2}\big(\overline{*(\mu \wedge \omega)} \lrcorner \lambda\big)$$

Hence for $\mu = \bar{\gamma}$ with $\gamma \wedge \omega = 0$, we get

$$\bar{\gamma} \cdot \lambda = 2\sqrt{2}i * (\bar{\gamma} \wedge \lambda)$$

$\underline{\Omega^{2,0}(X,\mathbb{C}) \text{ acting on } \Omega^{0}(X,\mathcal{L}_0)}$: Trivial action.

$\underline{\Omega^{2,0}(X,\mathbb{C}) \text{ acting on } \Omega^{0,2}(X,\mathcal{L}_0)}$: Say $\lambda \in \Omega^{0,2}(X,\mathcal{L}_0), \mu \in \Omega^{2,0}(X,\mathbb{C}), j \neq k$, $(dz_j \wedge dz_k) \cdot \lambda = 2\big(d\bar{z}_j \lrcorner (d\bar{z}_k \lrcorner \lambda)\big)$. Let's say in local coordinates

$$\lambda = \lambda_1 d\bar{z}_1 \wedge d\bar{z}_2 + \lambda_2 d\bar{z}_2 \wedge d\bar{z}_3 + \lambda_3 d\bar{z}_3 \wedge d\bar{z}_1, \mu = \alpha_1 dz_1 \wedge dz_2 + \alpha_2 dz_2 \wedge dz_3 + \alpha_3 dz_3 \wedge dz_1.$$

$$\begin{aligned}
(dz_1 \wedge dz_2) \cdot \lambda &= dz_1 \cdot dz_2 \cdot \lambda \\
&= dz_1 \cdot (-\sqrt{2})\big(d\bar{z}_2 \lrcorner (\lambda_1 d\bar{z}_1 \wedge d\bar{z}_2 + \lambda_2 d\bar{z}_2 \wedge d\bar{z}_3)\big) \\
&= 2d\bar{z}_1 \cdot (-\lambda_1 |d\bar{z}_2|^2 d\bar{z}_1) \\
&= -2\lambda_1 |d\bar{z}_1|^2 |d\bar{z}_2|^2 \\
&= -8\lambda_1 \\
&= -2\lambda_1 |d\bar{z}_1 \wedge d\bar{z}_2|^2 \\
&= -2\langle \lambda, \overline{dz_1 \wedge dz_2} \rangle
\end{aligned}$$

So, locally we get $\mu \cdot \lambda = -2\langle \lambda, \bar{\mu} \rangle \in \Omega^0(X,\mathcal{L}_0)$. Also observe

$$\begin{aligned}
*\lambda &= \lambda_1 dx_3 \wedge dy_3 \wedge d\bar{z}_1 \wedge d\bar{z}_2 + \lambda_2 dx_1 \wedge dy_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 + \lambda_3 dx_2 \wedge dy_2 \wedge d\bar{z}_3 \wedge d\bar{z}_1 \\
&= \omega \wedge \lambda
\end{aligned}$$

and

$$\begin{aligned}
*\big((dz_1 \wedge dz_2) \wedge *\lambda\big) &= \lambda_1 * (dz_1 \wedge dz_2 \wedge dx_3 \wedge dy_3 \wedge d\bar{z}_1 \wedge d\bar{z}_2) \\
&= \lambda_1 * \big(-(dz_1 \wedge d\bar{z}_1) \wedge (dz_2 \wedge d\bar{z}_2) \wedge dx_3 \wedge dy_3\big) \\
&= -\lambda_1 \times (-2i)^2 \\
&= 4\lambda_1
\end{aligned}$$

Hence we get

$$\mu \cdot \lambda = -2 * (\mu \wedge *\lambda)$$

$\underline{\Omega^{0,2}(X,\mathbb{C}) \text{ acting on } \Omega^{0}(X,\mathcal{L}_0)}$: Say $\lambda \in \Omega^0(X,\mathcal{L}_0), \mu \in \Omega^{0,2}(X,\mathbb{C})$.

$$\mu \cdot \lambda = 2\mu \wedge \lambda \in \Omega^{0,2}(X,\mathcal{L}_0).$$

$\underline{\Omega^{0,2}(X,\mathbb{C}) \text{ acting on } \Omega^{0,2}(X,\mathcal{L}_0)}$: Trivial action.

$\underline{\Omega^{1,1}(X,\mathbb{C}) \text{ acting on } \Omega^{0}(X,\mathcal{L}_0)}$: Say $\lambda \in \Omega^0(X,\mathcal{L}_0), \mu \in \Omega^{1,1}(X,\mathbb{C})$.

Now for $j \neq k, dz_j \wedge d\bar{z}_k \cdot \lambda = \sqrt{2} dz_j \cdot (d\bar{z}_k \wedge \lambda) = 0$.

$$\begin{aligned}
dx_j \cdot dy_j \cdot \lambda &= -\frac{i}{2} d\bar{z}_j \lrcorner (d\bar{z}_j \wedge \lambda) \\
&= -\frac{i}{2} |d\bar{z}_j|^2 \lambda \\
&= -i\lambda
\end{aligned}$$

So, we get $\mu \cdot \lambda = -i\langle \mu, \omega \rangle \lambda \in \Omega^0(X,\mathcal{L}_0)$, where ω is the Kähler form.
$\omega = \sum_{j=1}^{3} dx_j \wedge dy_j$ in local coordinates.

$\underline{\Omega^{1,1}(X,\mathbb{C}) \text{ acting on } \Omega^{0,2}(X,\mathcal{L}_0):}$ Say $\lambda \in \Omega^{0,2}(X,\mathcal{L}_0)$, $\mu \in \Omega^{1,1}(X,\mathbb{C})$. In local coordinates, $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + dx_3 \wedge dy_3$. Say,

$$\lambda = \lambda_1 d\bar{z}_1 \wedge d\bar{z}_2 + \lambda_2 d\bar{z}_2 \wedge d\bar{z}_3 + \lambda_3 d\bar{z}_3 \wedge d\bar{z}_1$$

$$\begin{aligned}(dx_1 \wedge dy_1)\cdot\lambda &= dx_1 \cdot (dy_1 \cdot \lambda)\\ &= dx_1 \cdot \big(\sqrt{2} \times \frac{i}{2}(d\bar{z}_1 \wedge \lambda + d\bar{z}_1 \lrcorner \lambda)\big)\\ &= (\sqrt{2})^2 \times \frac{i}{2} \times \frac{1}{2}\big(d\bar{z}_1 \wedge (d\bar{z}_1 \lrcorner \lambda) - d\bar{z}_1 \lrcorner (d\bar{z}_1 \wedge \lambda)\big)\\ &= \frac{i}{2}\big(d\bar{z}_1 \wedge (d\bar{z}_1 \lrcorner \lambda) - d\bar{z}_1 \lrcorner (d\bar{z}_1 \wedge \lambda)\big)\\ &= \frac{i}{2}\big(d\bar{z}_1 \wedge (\lambda_1 |d\bar{z}_1|^2 d\bar{z}_2 - \lambda_3 |d\bar{z}_1|^2 d\bar{z}_3) - \lambda_2 d\bar{z}_1 \lrcorner (d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3)\big)\\ &= \frac{i}{2}\big(2\lambda_1 d\bar{z}_1 \wedge d\bar{z}_2 - 2\lambda_2 d\bar{z}_2 \wedge d\bar{z}_3 + 2\lambda_3 d\bar{z}_3 \wedge d\bar{z}_1\big)\\ &= i(\lambda_1 d\bar{z}_1 \wedge d\bar{z}_2 - \lambda_2 d\bar{z}_2 \wedge d\bar{z}_3 + \lambda_3 d\bar{z}_3 \wedge d\bar{z}_1)\end{aligned}$$

Similarly we get,

$$\begin{aligned}(dx_2 \wedge dy_2)\cdot\lambda &= i(\lambda_1 d\bar{z}_1 \wedge d\bar{z}_2 + \lambda_2 d\bar{z}_2 \wedge d\bar{z}_3 - \lambda_3 d\bar{z}_3 \wedge d\bar{z}_1)\\ (dx_3 \wedge dy_3)\cdot\lambda &= i(-\lambda_1 d\bar{z}_1 \wedge d\bar{z}_2 + \lambda_2 d\bar{z}_2 \wedge d\bar{z}_3 + \lambda_3 d\bar{z}_3 \wedge d\bar{z}_1)\end{aligned}$$

So,

$$\omega \cdot \lambda = (dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + dx_3 \wedge dy_3)\cdot\lambda = i\lambda$$

Next, we'll try to find out a formula for elements in $\Omega^{1,1}(X,\mathbb{C})$ perpendicular to ω. Let's look at the Clifford action of an element of the form:

$$\mu = \alpha_{12} dz_1 \wedge d\bar{z}_2 + \alpha_{23} dz_2 \wedge d\bar{z}_3 + \alpha_{31} dz_3 \wedge d\bar{z}_1 + \alpha_{21} dz_2 \wedge d\bar{z}_1 + \alpha_{32} dz_3 \wedge d\bar{z}_2 + \alpha_{13} dz_1 \wedge d\bar{z}_3$$

We get

$$\mu\cdot\lambda = -4\big[(\alpha_{31}\lambda_2 + \alpha_{32}\lambda_3) d\bar{z}_1 \wedge d\bar{z}_2 + (\alpha_{12}\lambda_3 + \alpha_{13}\lambda_1) d\bar{z}_2 \wedge d\bar{z}_3 + (\alpha_{23}\lambda_1 + \alpha_{21}\lambda_2) d\bar{z}_3 \wedge d\bar{z}_1\big]$$

and

$$\begin{aligned}\mu \wedge \lambda = {}& \alpha_{12}\lambda_3 dz_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \wedge d\bar{z}_1 + \alpha_{23}\lambda_1 dz_2 \wedge d\bar{z}_3 \wedge d\bar{z}_1 \wedge d\bar{z}_2 + \alpha_{31}\lambda_2 dz_3 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3\\ &+ \alpha_{21}\lambda_2 dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 + \alpha_{32}\lambda_3 dz_3 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \wedge d\bar{z}_1 + \alpha_{13}\lambda_1 dz_1 \wedge d\bar{z}_3 \wedge d\bar{z}_1 \wedge d\bar{z}_2\\ ={}& (\alpha_{12}\lambda_3 + \alpha_{13}\lambda_1) dz_1 \wedge (d\bar{z}_3 \wedge d\bar{z}_1 \wedge d\bar{z}_2) + (\alpha_{23}\lambda_1 + \alpha_{21}\lambda_2) dz_2 \wedge (d\bar{z}_3 \wedge d\bar{z}_1 \wedge d\bar{z}_2)\\ &+ (\alpha_{31}\lambda_2 + \alpha_{32}\lambda_3) dz_3 \wedge (d\bar{z}_3 \wedge d\bar{z}_1 \wedge d\bar{z}_2)\end{aligned}$$

Notice

$$\begin{aligned}*\big(dz_1 \wedge (d\bar{z}_3 \wedge d\bar{z}_1 \wedge d\bar{z}_2)\big) &= *\big((dz_1 \wedge d\bar{z}_1) \wedge (d\bar{z}_2 \wedge d\bar{z}_3)\big)\\ &= *(-2i dx_1 \wedge dy_1) \wedge (d\bar{z}_2 \wedge d\bar{z}_3)\\ &= -2i d\bar{z}_2 \wedge d\bar{z}_3\end{aligned}$$

and

$$*\big(dz_2 \wedge (d\bar{z}_3 \wedge d\bar{z}_1 \wedge d\bar{z}_2)\big) = -2i d\bar{z}_3 \wedge d\bar{z}_1$$
$$*\big(dz_3 \wedge (d\bar{z}_3 \wedge d\bar{z}_1 \wedge d\bar{z}_2)\big) = -2i d\bar{z}_1 \wedge d\bar{z}_2$$

Hence we get

$$\mu \cdot \lambda = -2i * (\mu \wedge \lambda)$$

We claim that for any $\mu \perp \omega, \mu \cdot \lambda = -2i * (\mu \wedge \lambda)$. Enough to check that with this new formula, $(dx_k \wedge dy_k) \cdot \lambda$ gives back the same element we calculated before explicitly for $k = 1, 2, 3$.

$$\begin{aligned} dx_1 \wedge dy_1 &= \big(dx_1 \wedge dy_1 - \langle dx_1 \wedge dy_1, \omega\rangle \frac{\omega}{3}\big) + \frac{\omega}{3} \in \langle\omega\rangle^{\perp} \oplus \langle\omega\rangle^{\parallel} \\ &= \frac{1}{3}\big((2dx_1 \wedge dy_1 - dx_2 \wedge dy_2 - dx_3 \wedge dy_3) + \omega\big) \end{aligned}$$

$$-2i * \big((2dx_1 \wedge dy_1 - dx_2 \wedge dy_2 - dx_3 \wedge dy_3) \wedge \lambda\big) = -2i\big(2\lambda_2 d\bar{z}_2 \wedge d\bar{z}_3 - \lambda_3 d\bar{z}_3 \wedge d\bar{z}_1 - \lambda_1 d\bar{z}_1 \wedge d\bar{z}_2\big)$$

$$-2i\big(2\lambda_2 d\bar{z}_2 \wedge d\bar{z}_3 - \lambda_3 d\bar{z}_3 \wedge d\bar{z}_1 - \lambda_1 d\bar{z}_1 \wedge d\bar{z}_2\big) + i\lambda = i(\lambda_1 d\bar{z}_1 \wedge d\bar{z}_2 - \lambda_2 d\bar{z}_2 \wedge d\bar{z}_3 + \lambda_3 d\bar{z}_3 \wedge d\bar{z}_1)$$

Similarly one can check that the formula works for $k = 2, 3$. So, we get that for any $\mu \in \Omega^{1,1}(X, \mathbb{C}), \lambda \in \Omega^{0,2}(X, \mathcal{L}_0)$,

$$\mu \cdot \lambda = -2i * (\mu^{\perp} \wedge \lambda) + \frac{i}{3}\langle\mu, \omega\rangle\lambda \ \in \Omega^{0,2}(X, \mathcal{L}_0), \quad \text{where} \quad \mu^{\perp} = \mu - \langle\mu, \omega\rangle\frac{\omega}{3}$$

Hence,

$$\begin{aligned} \mu \cdot \lambda &= -2i * \big((\mu - \langle\mu, \omega\rangle\frac{\omega}{3}) \wedge \lambda\big) + \frac{i}{3}\langle\mu, \omega\rangle\lambda \\ &= -2i * (\mu \wedge \lambda) + \frac{2i}{3}\langle\mu, \omega\rangle * (\omega \wedge \lambda) + \frac{i}{3}\langle\mu, \omega\rangle\lambda \\ &= -2i * (\mu \wedge \lambda) + \frac{2i}{3}\langle\mu, \omega\rangle\lambda + \frac{i}{3}\langle\mu, \omega\rangle\lambda \\ &= -2i * (\mu \wedge \lambda) + i\langle\mu, \omega\rangle\lambda \end{aligned}$$

A.1.2 Clifford action on negative spinors

$\Omega^{3,0}(X, \mathbb{C})$ acting on $\Omega^{0,1}(X, \mathcal{L}_1)$: Trivial action.
$\Omega^{3,0}(X, \mathbb{C})$ acting on $\Omega^{0,3}(X, \mathcal{L}_1)$: Let's have $\mu \in \Omega^{3,0}(X, \mathbb{C}), \lambda \in \Omega^{0,3}(X, \mathcal{L}_1)$.

$$\begin{aligned} (dz_1 \wedge dz_2 \wedge dz_3) \cdot (d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3) &= dz_1 \cdot dz_2 \cdot dz_3 \cdot (d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3) \\ &= dz_1 \cdot dz_2 \cdot (-\sqrt{2}|d\bar{z}_3|^2 d\bar{z}_1 \wedge d\bar{z}_2) \\ &= dz_1 \cdot (-2|d\bar{z}_2|^2|d\bar{z}_3|^2 d\bar{z}_1) \\ &= 2\sqrt{2}|d\bar{z}_1|^2|d\bar{z}_2|^2|d\bar{z}_3|^2 \\ &= 2\sqrt{2} \times 8 \end{aligned}$$

We also see

$$\begin{aligned}
dz_1 \wedge dz_2 \wedge dz_3 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 &= dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge dz_3 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \\
&= -dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \wedge dz_3 \wedge d\bar{z}_3 \\
&= -(-2i)^3 dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge dx_3 \wedge dy_3 \\
&= -8i\, dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge dx_3 \wedge dy_3
\end{aligned}$$

Hence

$$*\big((dz_1 \wedge dz_2 \wedge dz_3) \wedge (d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3)\big) = -8i$$

So, we get

$$\mu \cdot \lambda = 2\sqrt{2}i * (\mu \wedge \lambda) \in \Omega^0(X, \mathcal{L}_1)$$

$\underline{\Omega^{0,3}(X,\mathbb{C}) \text{ acting on } \Omega^{0,1}(X,\mathcal{L}_1) \oplus \Omega^{0,3}(X,\mathcal{L}_1)}$: Trivial action.

$\underline{\Omega^{1,2}(X,\mathbb{C}) \text{ acting on } \Omega^{0,1}(X,\mathcal{L}_1)}$: Let's take $\mu \in \Omega^{1,2}(X,\mathbb{C})$ and $\lambda \in \Omega^{0,1}(X,\mathcal{L}_1)$. Say, locally $\lambda = \lambda_1 d\bar{z}_1 + \lambda_2 d\bar{z}_2 + \lambda_3 d\bar{z}_3$

$$\begin{aligned}
(dz_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3) \cdot (\lambda_1 d\bar{z}_1 + \lambda_2 d\bar{z}_2 + \lambda_3 d\bar{z}_3) &= dz_1 \cdot d\bar{z}_2 \cdot (\sqrt{2}\lambda_1 d\bar{z}_3 \wedge d\bar{z}_1 - \sqrt{2} d\bar{z}_2 \wedge d\bar{z}_3) \\
&= \sqrt{2} dz_1 \cdot (\sqrt{2}\lambda_1 d\bar{z}_2 \wedge d\bar{z}_3 \wedge d\bar{z}_1) \\
&= -2\sqrt{2}\lambda_1 |d\bar{z}_1|^2 d\bar{z}_2 \wedge d\bar{z}_3 \\
&= -4\sqrt{2}\lambda_1 d\bar{z}_2 \wedge d\bar{z}_3
\end{aligned}$$

We also observe that

$$\begin{aligned}
*\big((dz_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3) \wedge (\lambda_1 d\bar{z}_1 + \lambda_2 d\bar{z}_2 + \lambda_3 d\bar{z}_3)\big) &= *\big(\lambda_1 dz_1 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3\big) \\
&= *\big(-2i\lambda_1 dx_1 \wedge dy_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3\big) \\
&= -2i\lambda_1 d\bar{z}_2 \wedge d\bar{z}_3
\end{aligned}$$

Hence

$$\begin{aligned}
&(dz_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3) \cdot (\lambda_1 d\bar{z}_1 + \lambda_2 d\bar{z}_2 + \lambda_3 d\bar{z}_3) \\
&= -2\sqrt{2}i * \big((dz_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3) \wedge (\lambda_1 d\bar{z}_1 + \lambda_2 d\bar{z}_2 + \lambda_3 d\bar{z}_3)\big)
\end{aligned}$$

and

$$\begin{aligned}
&(d\bar{z}_2 \wedge dx_1 \wedge dy_1) \cdot (\lambda_1 d\bar{z}_1 + \lambda_2 d\bar{z}_2 + \lambda_3 d\bar{z}_3) \\
&= d\bar{z}_2 \cdot dx_1 \big(\sqrt{2} \times \frac{i}{2}(\lambda_2 d\bar{z}_1 \wedge d\bar{z}_2 + \lambda_3 d\bar{z}_1 \wedge d\bar{z}_3) + \sqrt{2} \times \frac{i}{2}\lambda_1 |d\bar{z}_1|^2\big) \\
&= d\bar{z}_2 \cdot \big(2 \times \frac{i}{2} \times \frac{1}{2}(2\lambda_1 d\bar{z}_1 - 2\lambda_2 d\bar{z}_2 - 2\lambda_3 d\bar{z}_3)\big) \\
&= -i\sqrt{2}(\lambda_1 d\bar{z}_1 \wedge d\bar{z}_2 + \lambda_3 d\bar{z}_2 \wedge d\bar{z}_3)
\end{aligned}$$

Also notice that

$$\begin{aligned}
*\big((d\bar{z}_2 \wedge dx_1 \wedge dy_1) \wedge (\lambda_1 d\bar{z}_1 + \lambda_2 d\bar{z}_2 + \lambda_3 d\bar{z}_3)\big) &= *\big(\lambda_3 dx_1 \wedge dy_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3\big) \\
&= \lambda_3 d\bar{z}_2 \wedge d\bar{z}_3
\end{aligned}$$

and

$$\begin{aligned}
&* \big((d\bar{z}_2 \wedge dx_1 \wedge dy_1) \wedge \omega\big) \wedge (\lambda_1 d\bar{z}_1 + \lambda_2 d\bar{z}_2 + \lambda_3 d\bar{z}_3) \\
&= *\big(dx_1 \wedge dy_1 \wedge d\bar{z}_2 \wedge dx_3 \wedge dy_3\big) \wedge (\lambda_1 d\bar{z}_1 + \lambda_2 d\bar{z}_2 + \lambda_3 d\bar{z}_3) \\
&= i d\bar{z}_2 \wedge (\lambda_1 d\bar{z}_1 + \lambda_2 d\bar{z}_2 + \lambda_3 d\bar{z}_3) \\
&= i(-\lambda_1 d\bar{z}_1 \wedge d\bar{z}_2 + \lambda_3 d\bar{z}_2 \wedge d\bar{z}_3)
\end{aligned}$$

So we get

$$\begin{aligned}
-2\sqrt{2}i * \big((d\bar{z}_2 \wedge dx_1 \wedge dy_1) \wedge \lambda\big) &+ \sqrt{2} * (d\bar{z}_2 \wedge dx_1 \wedge dy_1 \wedge \omega) \wedge \lambda \\
&= -2\sqrt{2}i\lambda_3 d\bar{z}_2 \wedge d\bar{z}_3 + \sqrt{2}i(-\lambda_1 d\bar{z}_1 \wedge d\bar{z}_2 + \lambda_3 d\bar{z}_2 \wedge d\bar{z}_3) \\
&= -\sqrt{2}i(\lambda_1 d\bar{z}_1 \wedge d\bar{z}_2 + \lambda_3 d\bar{z}_2 \wedge d\bar{z}_3)
\end{aligned}$$

One can check that the other forms satisfy the same formula, hence we get

$$\mu \cdot \lambda = -2\sqrt{2}i * (\mu \wedge \lambda) + \sqrt{2} * (\mu \wedge \omega) \wedge \lambda \in \Omega^{0,2}(X, \mathcal{L}_1)$$

So, if $\mu = \gamma$ with $\gamma \wedge \omega = 0$, then we get

$$\gamma \cdot \lambda = -2\sqrt{2}i * (\gamma \wedge \lambda)$$

$\underline{\Omega^{1,2}(X, \mathbb{C}) \text{ acting on } \Omega^{0,3}(X, \mathcal{L}_1)}$: Trivial action.
$\underline{\Omega^{2,1}(X, \mathbb{C}) \text{ acting on } \Omega^{0,1}(X, \mathcal{L}_1)}$: Let's say $\mu \in \Omega^{2,1}(X, \mathbb{C})$ and $\lambda \in \Omega^{0,1}(X, \mathcal{L}_1)$. Say, locally

$$\lambda = \lambda_1 d\bar{z}_1 + \lambda_2 d\bar{z}_2 + \lambda_3 d\bar{z}_3$$

We notice that for $j \neq k \neq l$,

$$(dz_j \wedge dz_k \wedge d\bar{z}_l) \cdot \lambda = d\bar{z}_l \cdot dz_j \cdot dz_k \cdot \lambda = 0$$

and

$$\begin{aligned}
(dz_2 \wedge dx_1 \wedge dy_1) \cdot \lambda &= dz_2 \cdot (dx_1 \wedge dy_1) \cdot \lambda \\
&= dz_2 \cdot \big(i(\lambda_1 d\bar{z}_1 - \lambda_2 d\bar{z}_2 - \lambda_3 d\bar{z}_3)\big) \\
&= 2\sqrt{2}i\lambda_2
\end{aligned}$$

We also see that

$$\begin{aligned}
(dz_2 \wedge dx_1 \wedge dy_1 \wedge \omega) \wedge \lambda &= (dx_1 \wedge dy_1 \wedge dz_2 \wedge dx_3 \wedge dy_3) \wedge \lambda \\
&= dx_1 \wedge dy_1 \wedge dz_2 \wedge dx_3 \wedge dy_3 \wedge \lambda_2 d\bar{z}_2 \\
&= -2i\lambda_2(dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge dx_3 \wedge dy_3)
\end{aligned}$$

and

$$*\big((dz_2 \wedge dx_1 \wedge dy_1) \wedge \omega \wedge \lambda\big) = -2i\lambda_2$$

One can check that all other forms enjoy the same formula. Hence we get

$$\mu \cdot \lambda = -\sqrt{2} * (\mu \wedge \omega \wedge \lambda)$$

We also notice that for any $k \in \{1,2,3\}$

$$\begin{aligned} *dz_k &= *(dx_k + idy_k) \\ &= dy_k \wedge \frac{\omega^2}{2} - idx_k \wedge \frac{\omega^2}{2} \\ &= -idz_k \wedge \frac{\omega^2}{2} \end{aligned}$$

Hence for $\eta \in \Omega^{0,1}(X,\mathbb{C})$ and $\mu = \bar{\eta} \wedge \omega$,

$$\begin{aligned} (\bar{\eta} \wedge \omega) \cdot \lambda &= -\sqrt{2} * (\bar{\eta} \wedge \omega^2 \wedge \lambda) \\ &= -2\sqrt{2}i * (\bar{\eta} \wedge *\lambda) \end{aligned}$$

<u>$\Omega^{2,1}(X,\mathbb{C})$ acting on $\Omega^{0,3}(X,\mathcal{L}_1)$:</u> Let's say $\mu \in \Omega^{2,1}(X,\mathbb{C})$ and $\lambda \in \Omega^{0,3}(X,\mathcal{L}_1)$. We notice that for $j \neq k \neq l$,

$$(dz_j \wedge dz_k \wedge d\bar{z}_l) \cdot \lambda = 0$$

and

$$\begin{aligned} (dz_2 \wedge dx_1 \wedge dy_1) \cdot (d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3) &= dx_1 \cdot dy_1 \cdot (2\sqrt{2}d\bar{z}_1 \wedge d\bar{z}_3) \\ &= dx_1(4 \times \frac{i}{2} \times 2d\bar{z}_3) \\ &= 4i \times \sqrt{2} \times \frac{1}{2}d\bar{z}_1 \wedge d\bar{z}_3 \\ &= 2\sqrt{2}i\, d\bar{z}_1 \wedge d\bar{z}_3 \end{aligned}$$

We also see that

$$\begin{aligned} *\big((dz_2 \wedge dx_1 \wedge dy_1) \wedge \omega\big) &= *\big((dx_1 \wedge dy_1 \wedge dz_2 \wedge dx_3 \wedge dy_3)\big) \\ &= -idz_2 \end{aligned}$$

and

$$\begin{aligned} *\big(* (dz_2 \wedge dx_1 \wedge dy_1 \wedge \omega) \wedge \lambda\big) &= *\big(- idz_2 \wedge \lambda\big) \\ &= -i * (dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3) \\ &= i * (dz_2 \wedge d\bar{z}_2 \wedge d\bar{z}_1 \wedge d\bar{z}_3) \\ &= i \times (-2i) * (dx_2 \wedge dy_2 \wedge d\bar{z}_1 \wedge d\bar{z}_3) \\ &= 2 * (dx_2 \wedge dy_2 \wedge d\bar{z}_1 \wedge d\bar{z}_3) \\ &= 2d\bar{z}_1 \wedge d\bar{z}_3 \end{aligned}$$

Hence we get

$$\mu \cdot \lambda = \sqrt{2}i * \big(* (\mu \wedge \omega) \wedge \lambda\big) \in \Omega^{0,2}(X,\mathcal{L}_1)$$

Hence for $\eta \in \Omega^{0,1}(X,\mathbb{C})$ and $\mu = \bar{\eta} \wedge \omega$,

$$\begin{aligned} (\bar{\eta} \wedge \omega) \cdot \lambda &= -\sqrt{2}i * \big(* (\bar{\eta} \wedge \omega^2) \wedge \lambda\big) \\ &= 2\sqrt{2} * (\bar{\eta} \wedge \lambda) \end{aligned}$$

$\underline{\Omega^{0,2}(X,\mathbb{C}) \text{ acting on } \Omega^{0,1}(X,\mathcal{L}_1):}$ Let's say $\mu \in \Omega^{0,2}(X,\mathbb{C})$ and $\lambda \in \Omega^{0,1}(X,\mathcal{L}_1)$.

$$\mu \cdot \lambda = 2\mu \wedge \lambda \in \Omega^{0,3}(X,\mathcal{L}_1)$$

$\underline{\Omega^{0,2}(X,\mathbb{C}) \text{ acting on } \Omega^{0,3}(X,\mathcal{L}_1):}$ Trivial action.
$\underline{\Omega^{1,1}(X,\mathbb{C}) \text{ acting on } \Omega^{0,1}(X,\mathcal{L}_1):}$ Let's say $\mu \in \Omega^{1,1}(X,\mathbb{C})$ and $\lambda \in \Omega^{0,1}(X,\mathcal{L}_1)$. Say locally

$$\lambda = \lambda_1 d\bar{z}_1 + \lambda_2 d\bar{z}_2 + \lambda_3 d\bar{z}_3$$

$$\begin{aligned}
(dz_1 \wedge d\bar{z}_2) \cdot \lambda &= \sqrt{2} dz_1 \cdot (\lambda_1 d\bar{z}_2 \wedge d\bar{z}_1 + \lambda_3 d\bar{z}_2 \wedge d\bar{z}_3) \\
&= 2\lambda_1 \left(|d\bar{z}_1|^2 d\bar{z}_2 \right) \\
&= 4\lambda_1 d\bar{z}_2
\end{aligned}$$

$$\begin{aligned}
(dx_1 \wedge dy_1) \cdot \lambda &= i(\lambda_1 d\bar{z}_1 - \lambda_2 d\bar{z}_2 - \lambda_3 d\bar{z}_3) \\
(dx_2 \wedge dy_2) \cdot \lambda &= i(-\lambda_1 d\bar{z}_1 + \lambda_2 d\bar{z}_2 - \lambda_3 d\bar{z}_3) \\
(dx_3 \wedge dy_3) \cdot \lambda &= i(-\lambda_1 d\bar{z}_1 - \lambda_2 d\bar{z}_2 + \lambda_3 d\bar{z}_3)
\end{aligned}$$

So we observe

$$\begin{aligned}
\omega \cdot \lambda &= (dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + dx_3 \wedge dy_3) \cdot \lambda \\
&= -i\lambda
\end{aligned}$$

We notice that

$$\begin{aligned}
*\big(*(dz_1 \wedge d\bar{z}_2) \wedge \lambda \big) &= *\big(-(dz_1 \wedge d\bar{z}_2) \wedge (dx_3 \wedge dy_3) \wedge \lambda \big) \\
&= *\big(-(dz_1 \wedge d\bar{z}_2) \wedge (dx_3 \wedge dy_3) \wedge (\lambda_1 d\bar{z}_1) \big) \\
&= *\big(\lambda_1 (dz_1 \wedge d\bar{z}_1) \wedge d\bar{z}_2 \wedge (dx_3 \wedge dy_3) \big) \\
&= -2i\lambda_1 * \big((dx_1 \wedge dy_1) \wedge d\bar{z}_2 \wedge (dx_3 \wedge dy_3) \big) \\
&= -2i\lambda_1 \times (i d\bar{z}_2) \\
&= 2\lambda_1 d\bar{z}_2
\end{aligned}$$

and

$$\begin{aligned}
&*\big(*(2dx_1 \wedge dy_1 - dx_2 \wedge dy_2 - dx_3 \wedge dy_3) \wedge \lambda \big) \\
&= *\big((2dx_2 \wedge dy_2 \wedge dx_3 \wedge dy_3 - dx_1 \wedge dy_1 \wedge dx_3 \wedge dy_3 - dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2) \wedge \lambda \big) \\
&= *\big(2\lambda_1 d\bar{z}_1 \wedge dx_2 \wedge dy_2 \wedge dx_3 \wedge dy_3 - \lambda_2 dx_1 \wedge dy_1 \wedge d\bar{z}_2 \wedge dx_3 \wedge dy_3 - \lambda_3 dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge d\bar{z}_3 \big) \\
&= i(2\lambda_1 d\bar{z}_1 - \lambda_2 d\bar{z}_2 - \lambda_3 d\bar{z}_3)
\end{aligned}$$

Similar calculations for other forms would ultimately prove that for $\mu \perp \omega$,

$$\mu \cdot \lambda = 2 * (*\mu \wedge \lambda)$$

Hence for any μ, we get

$$\begin{aligned}\mu \cdot \lambda &= (\mu - \langle \mu, \omega \rangle \frac{\omega}{3}) \cdot \lambda + \langle \mu, \omega \rangle \frac{\omega}{3} \cdot \lambda \\ &= 2 * \big(* (\mu - \langle \mu, \omega \rangle \frac{\omega}{3}) \wedge \lambda \big) - \frac{i}{3} \langle \mu, \omega \rangle \lambda \\ &= 2 * (*\mu \wedge \lambda) - \frac{2}{3} \langle \mu, \omega \rangle * (*\omega \wedge \lambda) - \frac{i}{3} \langle \mu, \omega \rangle \lambda \\ &= 2 * (*\mu \wedge \lambda) - \frac{2}{3} \langle \mu, \omega \rangle \times (i\lambda) - \frac{i}{3} \langle \mu, \omega \rangle \lambda \\ &= 2 * (*\mu \wedge \lambda) - i \langle \mu, \omega \rangle \lambda \in \Omega^{0,1}(X, \mathcal{L}_1)\end{aligned}$$

$\Omega^{1,1}(X, \mathbb{C})$ acting on $\Omega^{0,3}(X, \mathcal{L}_1)$: Let's say $\mu \in \Omega^{1,1}(X, \mathbb{C})$ and $\lambda \in \Omega^{0,3}(X, \mathcal{L}_1)$. For $k \neq l$,

$$(dz_k \wedge d\bar{z}_l) \cdot (d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3) = 0$$

and for any $k \in \{1, 2, 3\}$

$$(dx_k \wedge dy_k) \cdot (d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3) = i d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3$$

Hence we get

$$\mu \cdot \lambda = i \langle \mu, \omega \rangle \lambda \in \Omega^{0,3}(X, \mathcal{L}_1)$$

$\Omega^{2,0}(X, \mathbb{C})$ acting on $\Omega^{0,1}(X, \mathcal{L}_1)$: Trivial action.
$\Omega^{2,0}(X, \mathbb{C})$ acting on $\Omega^{0,3}(X, \mathcal{L}_1)$: Let's say $\mu \in \Omega^{1,1}(X, \mathbb{C})$ and $\lambda \in \Omega^{0,3}(X, \mathcal{L}_1)$.

$$\begin{aligned}(dz_1 \wedge dz_2) \cdot (d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3) &= dz_1 \cdot (2\sqrt{2} d\bar{z}_1 \wedge d\bar{z}_3) \\ &= -8 d\bar{z}_3\end{aligned}$$

We also notice that

$$\begin{aligned}*\big((dz_1 \wedge dz_2) \wedge (d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3)\big) &= -(-2i)^2 * (dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge d\bar{z}_3) \\ &= 4i d\bar{z}_3\end{aligned}$$

Using similar calculations for other forms, we see that

$$\mu \cdot \lambda = 2i * (\mu \wedge \lambda) \in \Omega^{0,1}(X, \mathcal{L}_1)$$

A.2 Clifford multiplication on Kähler 4-folds

We take local holomorphic coordinates $\{z_k = x_k + i y_k\}_{k=1,2,4}$ centered at a point $x \in X$ so that the Kähler metric is standard to second order at the point.
$(\eta^{0,1} \wedge \omega)$ acting on $\Omega^0(X, \mathcal{L})$: Enough to calculate when $\eta = d\bar{z}_1$. In local coordinates

$$\omega = \sum_{j=1}^{4} dx_j \wedge dy_j$$

Hence, $\eta \wedge \omega = d\bar{z}_1 \wedge \sum_{j=2}^{4} dx_j \wedge dy_j$ and for $\phi \in \Omega^0(X, \mathcal{L})$,

$$\begin{aligned}
(\eta \wedge \omega) \cdot \phi &= d\bar{z}_1 \cdot (\sum_{j=2}^{4} dx_j \wedge dy_j) \cdot \phi \\
&= d\bar{z}_1 \cdot (-3i\phi) \\
&= -3\sqrt{2} i d\bar{z}_1 \wedge \phi
\end{aligned}$$

In general

$$(\eta^{0,1} \wedge \omega) \cdot \phi = -3\sqrt{2} i \eta^{0,1} \wedge \phi$$

$(\eta^{1,0} \wedge \omega)$ acting on $\Omega^{0,4}(X, \mathcal{L})$: Enough to consider the case when $\eta^{1,0} = dz_1$. Notice $\eta \wedge \omega = dz_1 \wedge (\sum_{j=2}^{4} dx_j \wedge dy_j)$. For $\phi \in \Omega^{0,4}(X, \mathcal{L})$,

$$\begin{aligned}
(\eta \wedge \omega) \cdot \phi &= dz_1 \cdot (\sum_{j=2}^{4} dx_j \wedge dy_j) \cdot \phi \\
&= dz_1 \cdot (3i\phi) \\
&= -3\sqrt{2} i (d\bar{z}_1 \lrcorner \phi) \\
&= -3\sqrt{2} i * (dz_1 \wedge \phi)
\end{aligned}$$

In general we get the following formula:

$$(\eta^{1,0} \wedge \omega) \cdot \phi = -3\sqrt{2} i * (\eta^{1,0} \wedge \phi)$$

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