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Quantization on the Ideal Boundary and the Finite Widths of Resonances

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Abstract: Conformal field theory is quantized on the ideal boundary of a Riemann surface, and the effect on the widths of the resonances of the quantum states is evaluated. The resonances on a surface can be recast in terms of eigenfunctions of a differential operator on the Mandelstam plane. Cusps in this plane, representing Landau singularities, reflect a divergence in the coupling. A cusp on the Riemann surface similarly causes a divergence in the scattering amplitude. The interpretation of the string diagram indicates that the self-interaction of the string in the vicinity of the cusp causes it to implode, which would require an infinite coupling. A consistent physical interpretation of cusps on surfaces requires supersymmetry. The study of unitary minimal models and N = 2 superminimal models indicates that there can exist a set of resonances at the cusps and ends of the surfaces. The uncertainty in the masses of six types of particles at a finite set of cusps is infinitesimal. Tachyon condensation on the ideal boundary would introduce an uncertainty in the mass of a charged particle. The widths of charged particle resonances at the ends of infinite-genus surfaces is not negligible and can be traced to the coupling with tachyons.

Keywords: resonances; Mandelstam plane; cusps; ideal boundary; tachyon condensation; charged particle widths

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1. Introduction

The measure of noncommutativity resulting from quantization on a Riemann surface [1] of genus g can be computed. It is found that there is an extra contribution to the minimal uncertainty in the product $\Delta E \Delta t$ that increases inversely with the genus. Furthermore, coefficients are found to increase by a multiple equal to $\left(1 + \frac{1}{2(g-1)}\right)$ for $g \geq 2$ [2]. The effect on the magnetic moment of the electron may be predicted with this modification of the perturbation theory.

Therefore, in principle, an estimate of the width of a resonance of a particle prepared at the ideal boundary of a Riemann surface may be derived. The boundaries of finite-genus surfaces are border arcs, while those at infinite genus are composed of ends. The coefficient, representing the contribution of the quantum surface to the uncertainty in the string amplitude, tends to 1 in the limit $g \to \infty$. Therefore, the initial estimate is that the states on the ideal boundary of an infinite-genus surface would have the same lifetimes as those in a quantum field theory. The Hausdorff dimension of the limit set Λ_G, of a group of Schottky type, Γ, uniformizing a compact Riemann surface of finite genus [3] or an effectively closed infinite-genus surface [4], is less than or equal to 2. The Hilbert space of square integrable functions is $\mathcal{H} = \mathcal{L}^2(\Lambda_\Gamma, d\mu)$, where $d\mu$ is the Patterson-Sullivan measure [5,6]. The limit set Λ_Γ is contained in a quasi-circle C in the complex projective space $\mathbb{CP}^1$ [7]. The limit set of a Fuchsian Schottky group G belongs to $\mathbb{RP}^1$ [8]. It may be verified that the ideal boundary of a surface uniformized by a group of Schottky type is $(\mathbb{CP}^1 \backslash \Lambda_\Gamma)/\Gamma$, and it would be $(\mathbb{RP}^1 \backslash \Lambda_G)/G$ for a Fuchsian Schottky group. It is known that Fuchsian Schottky groups form a subset of classical Schottky groups, which have limit sets of Hausdorff dimension less than or equal to one when the elements are loxodromic and the number of generators are finite [9]. The ideal boundaries of the Riemann surfaces, consisting of the set of noncompact nonplanar ends, that are uniformized by groups of this type also would have a Hausdorff dimension less than or equal to one. Nonclassical finitely generated Kleinian groups [10] can have limit sets of Hausdorff dimension larger than one [10]. The limit sets of Schottky groups with an infinite number of generators and a sequence of nonoverlapping isometric circles may have fractal dimension less than one [11], greater than one or positive planar area [12,13]. The quantization on the ideal boundaries will be defined on surfaces with ideal boundaries with a Cantor set of ends embedded in the circular boundary of the unit disk covering when the Hausdorff dimension of the limit set does npt exceed one. The procedure will furnish geometrical models of a set of particles, based on three-dimensional handlebodies with solid interiors, below an energy threshold. Above this energy, other geometrical models will be necessary, including manifolds with the ends occupying a two-dimensional region.

When the fundamental domain of the covering group only has a cuspidal end, and the Hausdorff dimension equals zero, and the decay rate of particles travelling along random walks in the surface vanishes. The result, which is entirely semiclassical, confirms the source of indefinite lifetimes of particles as the accumulation of an infinite sequence of handles rather than cuspidal ends. Nevertheless, there are two types of Hilbert spaces on a Riemann surface, one in the interior and the other on the boundary. The uncertainties in the energies of the quantum states in the interior are not affected by the ideal boundary. By contrast, the states in the Hilbert space on the ideal boundary must be quantized separately. It will be established whether the capacity of the surface has an effect on the widths of the resonances.

An analysis of the momentum space propagator of scalar field theory includes a Fourier transform which verifies the standard form when the mass equals zero. If the mass is not zero, the precise form of the configuration space propagator is given by a Bessel function with a leading term that is proportional to the inverse of the squared distance. Substitution of the remaining terms into the integral transform yields a modification of the differential equation for the momentum space propagator. Singularities in the infrared limit are removed in §2 by replacing a recursive system of equations by a higher-order differential equation in momentum space. The differential operator can be transformed to Mandelstam variables, and the resonances may be identified with poles of the resolvent.

The occurrence of cusps in Mandelstam diagrams are known to reflect Landau singularities. The effect of cusps on Riemann surfaces is similar in bosonic string theory. The cusps cause the bosonic string amplitudes to diverge. Therefore, it is necessary to introduce supersymmetry to consistently include cusps on the Euclideanized worldsheets. Unitary conformal field theories and projections of N = 2 superminimal models are examined at cusps. The restriction of cuspidal representations of the Virasoro algebra to two values of the central charge is sufficient to limit the types of massive particles in the vicinity of cusps. A theoretical explanation is given in §3 for the infinitesimal margins of error in the masses of these particles. The masses of a much larger set of particles modelled on the boundary components of solid interiors of infinite-genus surfaces is then enumerated. A maximal value is determined according to the range of Hausdorff dimensions of the ideal boundaries of the surfaces.

Tachyon condensation is a nonperturbative effect in string theory that can be attributed either to dynamics in string field theory or the identification of Hilbert spaces on a Cantor set of ends of an infinite-genus surface. Since the tachyon as negative squared mass, the addition of its imaginary mass to the energy of a resonance would represent a decay of the quantum state. Given that there are tachyons condensing at the ends, the coupling to the electromagnetic field of a charged particle generates a new term in the effective Lagrangian. This new term is identified in §4 a factor contributing to the width of the resonance. It is found to be proportional to the square of the charge. There are several related phenomenlogical findings which place this effect in context. First, the absence of the tachyon in the superstring theory would cause the widths of resonances above a threshold, when supersymmetry is present, to be essentially vanishing. Secondly, the uncertainty in the mass of the electron is infinitesimal, which verifies that the geometry is spherical rather than that of the solid interior of a surface of infinite genus. The tachyon hypothesis, therefore, is compatible with current experimental data.

2. The Description of Resonances in Momentum Space

Resonances on open surfaces have been studied and a relation between the number and dimension of the limit set of the uniformizing group may be given [14]. It is found that when there are cusps, on the surface, the number of resonances, represented by poles of $(\Delta - \frac{1}{4} - \lambda)^{-1}$, increases rapidly. For example, if $N_C(T) = \#\{z \in \mathcal{R}_\Sigma | Im\ z \leq C,\ |Re\ z| \leq T\}$, $N_C(T) = \mathcal{O}(T^{1+\delta})$ for Schottky groups with a limit set of Hausdorff dimension δ [15,16]. This result can be generalized to infiinte-index arithmetic subgroups derived from a quaternion algebra with $\delta > \frac{3}{4}$, including finitely generated Schottky groups [14].

The description of multichannel processes through Riemann sheets may be developed to predict the location of poles in the other sheets, given a resonance on one sheet. The analyticity and unitarity conditions of the scattering matrix together with Le Couter-Newton identities for the Jost determinant are sufficient to give its factorizable form and the locations of the poles [17]. Complex zeros of the scattering matrix elements are compatible with non-zero imaginary parts of complex poles at $\sqrt{s}_i = E_i - i\frac{\Gamma_i}{2}$ representing the finite widths of the resonances.

The scattering matrix elements in the LSZ formula [18] are given by the action of differential operators on the fields. The time-ordered correlation function of two scalar fields is

$$\Delta_F(x-y) = \langle 0|T\phi(x)\phi(y)|0\rangle = \theta(x_0 - y_0)D(x-y) + \theta(y^0 - x^0)D(y-x), \qquad (1)$$

where

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3 2p^0} e^{-ip\cdot(x-y)}. \qquad (2)$$

Then

$$\begin{aligned}(\partial^2 + m^2)\Delta_F(x-y) &= \frac{\partial}{\partial x^0}\langle 0|T(\partial_0\phi(x)\phi(y))|0\rangle + \delta(x^0 - y^0)\langle 0|[\phi(x),\phi(y)]|0\rangle \\ &\quad - \langle 0|T(\nabla_x^2\phi(x)\phi(y))|0\rangle + m^2\langle T\phi(x)\phi(y)|0\rangle \\ &= \langle 0|T(\partial_0^2\phi(x)\phi(y))|0\rangle - i\delta^4(x-y) - \langle 0|T(\nabla_x^2\phi(x)\phi(y))|0\rangle \\ &\quad + m^2\langle T\phi(x)\phi(y)|0\rangle \\ &= -i\delta^4(x-y).\end{aligned} \qquad (3)$$

is the differential for the configuration space-propagator [19]. The Fourier transform converts the differentiation to multiplication by the momentum factors $p^2 - m^2$. The duality in the integral transform allows a formulation in momentum space. The Fourier expansion

$$\Delta_F(p) = \int d^4x \Delta_F(x) e^{ip\cdot x} \tag{4}$$

yields

$$\Box_p \Delta_F(p) = \int \frac{d^4x}{(2\pi)^2}(-x^2)\Delta_F(x)e^{ip\cdot x} = i\delta(p) \tag{5}$$

if

$$\Delta_F(x) = -\frac{i}{(2\pi)^2 x^2}, \tag{6}$$

which is the configuration space propagator for a massless scalar field.

When $m > 0$, Stuckelberg-Feynman propagator for the operator $\Box_x + m^2$ [20,21] is $\frac{im}{4\pi^2 r}K_1(mr)$. As $m \to 0$, $K_1(mr) \to -\frac{1}{mr}$ and the propagator tends to $-\frac{i}{4\pi^2 r^2}$. Since $K_1(mr) = -\frac{1}{mr} + \frac{1}{4}mr + \ldots$, $\frac{im}{4\pi^2 r}K_1(mr) = -\frac{i}{4\pi^2 r^2} + \frac{im^2}{16\pi^2} + \ldots = -\frac{i}{4\pi^2}\left[\frac{1}{r^2} - \frac{m^2}{4} + \ldots\right]$, which will be appromated over a range of infinitesimal m by $-\frac{i}{4\pi^2}\left(\frac{1}{r^2} - \frac{m^2}{4}\right)$. If

$$\Delta_F(x) \simeq -\frac{i}{(2\pi)^2}\left(\frac{1}{x^2} - \frac{m^2}{4}\right), \tag{7}$$

$$\begin{aligned}\Box_p \Delta_F(p)^{(1)} &= \int \frac{d^4x}{(2\pi)^2}\frac{-i}{4\pi^2}\left(\frac{1}{x^2} - \frac{m^2}{4}\right)(-x^2)e^{ip\cdot x} \\ &= i\delta(p) - i\int \frac{d^4x}{(2\pi)^4}\frac{m^2}{4}x^2 e^{ip\cdot x} \\ &= i\delta(p) + i\frac{m^2}{4}\Box_p \int \frac{d^4x}{(2\pi)^4}e^{ip\cdot x} \\ &= i\delta(p) + \frac{m^2}{4}\Box_p^2 \Delta_F(p)^{(0)},\end{aligned} \tag{8}$$

where $\Delta_F(p)^{(0)}$ is the solution to Equation (8). Iteration of this process yields a set recursive equations for $\Delta_F^{(n)}(p)$, with the limit $lim_{n\to\infty}\Delta_F^{(n)}(p)$ being a momentum space propagator for $m > 0$, provided that the mass is less than the characteristic scale of the d'Alembertian in momentum space. The higher-order terms are proportional to the derivatives of the delta function and may be set equal to zero when $p \neq 0$. These singularities in the infrared limit can be removed by replacing the recursive equation by differential operators acting entirely on $\Delta_F(p)$. Consider, for example, the next-order differential equation in momentum space

$$(\Box_p - \frac{m^2}{4}\Box_p^2)\Delta_F(p) = i\delta(p). \tag{9}$$

The resolvent of this differential operator is $\left(\Box_p - \frac{m^2}{4}\Box_p^2 - \lambda\right)^{-1}$. The resonances will be located at the poles of the resolvent.. The series expansion of $\Delta_F(x)$ would generate increasing powers of the d'Alembertian in the equation for $\Delta_F(p)$.

Given the above equation for the propagator, it is necessary to determine the precise transformation to the Mandelstam variables. Writing $s = P^\mu P_\mu$, where P is the total momentum of the incoming states in the s-channel, it follows that

$$\frac{\partial}{\partial P^\mu} = \frac{\partial s}{\partial P^\mu}\frac{\partial}{\partial s} = 2P_\mu \frac{\partial}{\partial s} \tag{10}$$

and

$$\Box_P = 4P^\mu \frac{\partial}{\partial s}\left(P_\mu \frac{\partial}{\partial s}\right). \tag{11}$$

The derivative $\frac{\partial P_\mu}{\partial s}$ may be evaluated through the inverse of a Jacobian matrix of the coordinate transformation between $\{P_\mu\}$ and s and three variables specifying the position on the hyperboloid $P^\mu P_\mu = s$. Then, the differential operator in the Mandelstam plane equals

$$4P^\mu P_\mu \frac{\partial^2}{\partial s^2} + 4P^\mu \frac{\partial P_\mu}{\partial s}\frac{\partial}{\partial s} = 4s\frac{\partial^2}{\partial s^2} + 2\frac{\partial}{\partial s}. \tag{12}$$

By constrast, the Laplacian in the complex s plane is.

$$\frac{\partial}{\partial s}\frac{\partial}{\partial \bar{s}} = \frac{1}{4}\frac{\partial^2}{\partial |s|^2} + \frac{1}{4}\frac{1}{|s|}\frac{\partial}{\partial |s|} + \frac{i}{4}\frac{1}{|s|^2}\frac{\partial}{\partial \theta} + \frac{1}{4}\frac{1}{|s|^2}\frac{\partial^2}{\partial \theta^2} \tag{13}$$

where θ is the angular variable. Given no angular dependence, this differential operator reduces to

$$\frac{1}{4}\frac{\partial^2}{\partial |s|^2} + \frac{1}{4}\frac{1}{|s|}\frac{\partial}{\partial |s|}. \tag{14}$$

It can be verified that no transformation of the form $|s| = w^\lambda$ will alter the ratio of the coefficients.

When s is replaced by its magnitude, the second-order differential operator in Equation (12) may be cast in the form $\frac{\partial^2}{\partial \chi^2}$ for some function $\chi(|s|)$.

$$\frac{\partial^2}{\partial \chi^2} = \left(\frac{d\chi}{d|s|}\right)^{-2}\frac{\partial^2}{\partial |s|^2} - \left(\frac{d\chi}{d|s|}\right)^{-3}\frac{d^2\chi}{d|s|^2}\frac{\partial}{\partial |s|} \tag{15}$$

the matching of the coefficients yields

$$\chi(|s|) = \sqrt{|s|} \tag{16}$$

Cusps in the Mandelstam plane as a result of Landau singularities may originate cusps in a two-dimensional region on a Riemann surface in the Euclidean continuation of a string path integral. This conclusion is valid in bosonic string theory, where the cusp causes the integral representation of an n-point amplitude to diverge. The cusps would generate an increasingly large number of resonances of the Laplacian in regions connected to Landau singularities. For complex poles in the Mandelstam plane at $\sqrt{s}_i = E_i - i\frac{\Gamma_i}{2}$, $|s_i| = E_i{}^2 + \frac{\Gamma_i^2}{4}$ and $\chi(|s_i|) = \sqrt{E_i^2 + \frac{\Gamma_i^2}{4}}$. The transformation from the Mandelstam plane through $\chi(s) = \sqrt{s}$ will preserve the number of resonances in the χ-plane provided that the masses are restricted to be real and non-negative. Then, the number of poles of the resolvent $\left(\frac{\chi^2\partial^2}{\partial\chi^2} - m^2\chi^2\frac{\partial^2}{\partial\chi^2}\left(\chi^2\frac{\partial^2}{\partial\chi^2}\right) - \lambda\right)^{-1}$.

3. The Existence of Particle States at the Cusps and Ends of Riemann Surfaces

The integral over the hyperbolic measure in the vicinity of a cusp on a super-Riemann surface does not diverge. Even when two points coincide on a Riemann surface, the distance between $Z_1 = (z_1, \theta_1)$ and $Z_2 = (z_2, \theta_2)$ in superspace

$$lim_{z_2 \to z_1}|Z_1 - Z_2| = lim_{z_2 \to z_1}|z_1 - z_2 + \theta_1\theta_2| = |\theta_1\theta_2| \tag{17}$$

Expanding a chiral superfield as

$$\Phi(z,\theta) = \phi(z) + \theta\psi(z) \tag{18}$$

the integral of a product two of these superfields yields

$$\begin{aligned}
&\int d^2z_1 d^2z_2 \sqrt{h(z_1)}\sqrt{h(z_2)}\, d\theta_1 d\theta_2 \Phi_1(z_1,\theta_1)\Phi_2(z_2,\theta_2) \\
&= \int d^2z_1 d^2z_2 \sqrt{h(z_1)}\sqrt{h(z_2)}(\phi_1(z_1) + \theta_1\psi(z_1))(\phi_2(z_2) + \theta_2\psi(z_2)) \\
&= \int d^2z_1 d^2z_2 \sqrt{h(z_1)}\sqrt{h(z_2)}\, \theta_1\psi_1(z_1)\theta_2\psi_2(z_2) \\
&= -\int d^2z_1\, d^2z_2 \sqrt{h(z_1)}\sqrt{h(z_2)}\, \psi_1(z_1)\psi_2(z_2)
\end{aligned} \tag{19}$$

generates an interaction between two fermions. The integral of a product of four superfields equals

$$\begin{aligned}
&\int d^2z_1 d^2z_2 \sqrt{h(z_1)}\sqrt{h(z_2)} d\theta_1 d\bar{\theta}_1 d\theta_1 d\bar{\theta}_2\ \overline{\Phi_1(z_1,\theta_1)\Phi_2(z_2,\theta_2)}\Phi_1(z_1\theta_1)\Phi_2(z_2, theta_2) \\
&= \int d^2z_1 d^2z_2 \sqrt{h(z_1)}\sqrt{h(z_2)} d\theta_1 d\bar{\theta}_1 d\theta_2 d\bar{\theta}_2\ \bar{\theta}_1\bar{\psi}_1(\bar{z}_1)\bar{\theta}_2\bar{\psi}_2(\bar{z}_2)\theta_1\psi_1(z_1)\theta_2\psi_2(z_2) \\
&= -\int d^2z_1 d^2z_2 \sqrt{h(z_1)}\sqrt{h(z_2)}\ \bar{\psi}_1(\bar{z}_1)\psi_1(z_1)\bar{\psi}_2(\bar{z}_2)\psi_2(z_2)
\end{aligned} \tag{20}$$

produces a four-fermion interaction.

These interactions can be included in a superconformal field theory on the Riemann surface, even when it has cusps. It may be established that cuspidal representations of the Virasoro algebra are restricted to $m = 1,\ 2$ [22]. Therefore, the number of particles that can be produced in the vicinity of the cusps will be given by the states for these values of m. The unitary minimal conformal field theories in two dimensions have the central charges $c_{m,m+1} = 1 - \frac{6}{m(m+1)}$ [23]. Since the (1,2) and (2,3) minimal models have the central charges -2 and 0 respectively, only the $m = 2$ cuspidal representations of the Virasoro algebra can be interpreted in terms of a pointlike boundaries of surfaces. Divergences in bosonic string theory would prevent such destabilize these cusplike configurations. However, supersymmetry may be introduced to stabilize such particle states. The N = 2 (m,n) superminimal model has the central charge $c = 1 - 2\frac{(m-n)^2}{mn}$ [24]. It is unitary when $|m - n| \equiv 0\ (mod\ 2)$, $\frac{m-n}{2}$ is odd, $gcd(m,n) = 1$ if m and n are odd and $gcd\left(\frac{m}{2}, \frac{n}{2}\right) = 1$ if m and n are even. Then $c_{(1,3)} = -\frac{5}{3}$, $c_{(2,4)} = 0$ and $c_{2,2+2\ell} = 1 - \frac{2\ell^2}{\ell+1}$, which is negative for $\ell > 1$. Reducing the superconformal theory to a bosonic sector, described by a unitary minimal model, after the breaking of supersymmetry, the cuspidal representations may be interpreted in terms of fermionic states. Since the weights of the primary states $|h_{rs}\rangle$ are $h_{rs} = \frac{[(m+1)r-ms]^2-1}{4m(m+1)}$, $1 \le r \le m$, $1 \le s \le m-1$, the values for $m = 2$ are $h_{11} = 0$ and $h_{21} = \frac{5}{8}$. However, the identities $h_{rs} = h_{m-r,m+1-s} = h_{r+m,s+m+1}$ do not exclude $r = s = m$. Instead, $h_{mm} = \frac{m^2-1}{4m(m+1)} > 0$. If r and s are set equal to 2, corresponding weight would be $h_{22} = \frac{1}{8}$. The energy is eigenvalue of the Hamiltonian or the operator $L_0 + \bar{L_0}$ and may be set equal to $h + \bar{h}$. The state with $r = s = 2$ has the least positive energy in an $(m,m+1)$ minimal model because $h_{2,2} = \frac{3}{4m(m+1)}$, and consequently, it is identified with the electron neutrino [25]. The neutrino actually is considered to have a radius of $4.47 \times 10^{-24}m$ [26]. Therefore, it must be verified that with a circular border of this radius would induce a superselection of the representations of the Virasoro algebra. The desingularization of the variety modifies the cuspidal representation [27] only maximally by a logarithmic extension of the minimal model [28]. When the central charge equals zero, there exist logarithmic extensions that preserve this condition. Therefore, the desingularized cuspidal representation of the $m = 2$ minimal model continues to be characterized by a vanishing central charge.

There would be two massive particles for each fermion generation in the vicinity of a cusp. The masses of these six types of particles would be

$$\begin{aligned} m_{\nu_e} &= 0.0076144777238368 \pm 0.000000000002235\ eV/c^2 \\ m_{\chi 1} &= 0.038072388619184 \pm 0.000000000011175 \pm\ eV/c^2 \\ m_{\nu_\mu} &= 1.574432484 \pm 0.000000034\ eV/c^2 \\ m_{\chi 2} &= 7.87216242 \pm 0.00000017\ eV/c^2 \\ m_{\nu_\tau} &= 26.47668 \pm 0.00238\ eV/c^2 \\ m_{\chi 3} &= 132.3834 \pm 0.0119\ eV/c^2 \end{aligned} \tag{21}$$

If the cusp is finite, it will in a point, and any quantum state would be function from the point to $\mathbb{C}$. A univalent function ψ has the value $\psi(z_0) \in \mathbb{C}$ at this point. Since any other wavefunction will have the form $\tilde{\psi}(z_0 = \zeta\psi(z_0)), \zeta \in \mathbb{C}$, it is evident that the Hilbert space at a point will be finite-dimensional and diffeomorphic to $\mathbb{C}$. Normalization of states in $\mathbb{C}$ would reduce the Hilbert space to S^1. Any Hilbert space on a circular boundary of a disk on a Riemann surface cannot be isomorphic to $\mathbb{C}$. A connection between finite widths of resonances and tachyon condensations on ends of surfaces will be discussed in the next section. The absence of an isomorphism from $\mathcal{L}^2(S^1)$ to $\mathbb{C}$ prevents the phenomenon of tachyon condensation on a finite number of cusps, because the Hilbert space on this set of cusps does not allow an amalgamation of an infinite number of tachyon states. For cusplike configurations, with border arcs of non-zero length, the Hilbert space is again defined on a circle. However, a finite number of cusplike geometrical regions will not produce the tachyon condensation necessary to affect the masses of the particles. The widths of the above resonances will be infinitesimal.

An N = 1 minimal, unitary superconformal field theory is invariant under a superalgebra with a central charge $c = \frac{3}{2}\left(1 - \frac{8}{m(m+2)}\right)$ [23]. Identification of the central charge with the scaling dimension of a Cantor set of ends $E(ppp...)$ of an ideal boundary of a surface [25] requires $c = 1 - \frac{1}{p}$. The only central charges with values in [0,1] occur for $m = 2,\ 3,\ 4$. If $m = 3$, $c = \frac{7}{15}$, which does not equal $1 - \frac{1}{p}$ for any positive integer p. For a Cantor set $E(p_1 p_2 p_3 ...)$, with different numbers of subdivisions of intervals at each stage, the average scaling dimension is

$$lim_{\ell\to\infty}\frac{1}{\ell}\sum_{i=1}^{\ell}\left(1 - \frac{1}{p_i}\right) = lim_{\ell\to\infty}\left(1 - \frac{1}{\ell}\sum_{i=1}^{\ell}\frac{1}{p_i}\right). \tag{22}$$

A Cantor set with an average scaling dimension of $\frac{7}{15}$ can be constructed when

$$lim_{\ell\to\infty}\prod_{\ell}\frac{1}{\ell}\sum_{i=1}^{\ell}\frac{1}{p_i} = \frac{8}{15}, \tag{23}$$

The absence of an exact matching of a constant number of subdivisions affects the physical interpretation of these central charges in terms of scaling dimensions.

The central charges of N = 2 (m,n) super minimal models now will be compared with scaling dimensions of Cantor sets.

From Table 1, even though the central charges nearly equal $1 - \frac{1}{p}$, $p \in \mathbb{Z}^+$, for (m,n) = $(2r + 1, 2r + 3)$ with r being a positive integer, it is only an approximation.

Table 1. Central charges larger than $\frac{1}{2}$ of (m,n) N = 2 superconformal field theories with $m < 10$.

(m,n)	c
(4,6)	$\frac{2}{3}$
(5,7)	$\frac{27}{35}$
(6,8)	$\frac{5}{6}$
(7,9)	$\frac{55}{63}$
(7,11)	$\frac{45}{77}$
(8,10)	$\frac{9}{10}$
(9,11)	$\frac{91}{99}$

An exact equality is valid for $p = 4t$ and $q = 4t + 2$, $t \in \mathbb{Z}^+$. Then

$$c = 1 - \frac{8}{4t(4t+2)} = 1 - \frac{1}{t(2t+1)}. \tag{24}$$

The scaling dimension of the Cantor set with subdivisions into p subintervals cam be identified with the central charge of this N = 2 superconformal field theory if $p = t(2t+1)$. The integer p will be odd only when t is odd. It is possible to construct Cantor sets with an even number of subdivisions of the intervals. At lower energies, supersymmetry is broken, and the weights $h_{r,s}$, $\bar{h}_{r,s}$ of primary states in a bosonic sector yield the energies $h_{r,s} + \bar{h}_{r,s} = E_{r,s}$. The ratio of the energy of the highest-weight state $|\phi_{6k-1,1}\rangle$, $E_{6k-1,1} = \frac{(3k-1)(6k-1)}{2}$, in a (6$k$, 6$k$ + 1) minimal, unitary conformal field theory to $E_{2,2} = \frac{1}{8k(6k+1)}$ is $\frac{E_{6k-1,1}}{E_{2,2}} = 4k(3k-1)(36k^2-1)$. A shift in the central charge occurs after coupling to one of the lowest-energy scalar fields in string theory, the massless dilaton and the tachyon with negative squared mass. The coupling of a superminimal model to a dilaton field causes a shift in the background charge β by $\Delta\beta = -2\epsilon$ such that $\Delta c = 12\beta\epsilon$. The central charge also may be reduced in a tachyon background while preserving the structure of a minimal conformal field theory [29]. A shift from $1 - \frac{1}{k(6k+1)}$ to $1 - \frac{1}{k(2k+1)}$ can be achieved by setting

$$\Delta\beta = \frac{1}{3(2k+1)(6k+1)\beta}. \tag{25}$$

Then p may be identified with $k(2k+1)$ in the perturbed conformal field theory, which is compatible with the central charge of the (m,n) superconformal field theory for $k = t$. These marginal deformations can alter the central charge without affect the weights of primary states in the initial (6k, 6k + 1) conformal field theory, Based on the lowest-weight state being the electron neutrino with the mass $m_{h_{22}} = 0.0076144777238368\ eV/c^2$ [30], the masses of two-dimensional fermions may be listed.

The upper limit for the masses of the two-dimensional fermions with masses predicted by this method would be related to the range of validity of the the geometrical model of particles based on Cantor sets of ends in the ideal boundaries of surfaces of infinite genus is valid only in this range [31]. It may be demonstrated that the Hausdorff dimension of the limit point set of the uniformizing group of a surface of infinite genus in the class $O_{AD} \backslash O_{HD}$ belongs to the range $[0, 1-\epsilon]$, where $\epsilon = \frac{1}{1519690.66483}$ [32]. The Hausdorff dimension of the Cantor set $E(ppp...)$, representing the ideal boundary, is $\frac{ln(p-1)}{ln\ p}$. Since it is greater than or equal to the the Hausdorff dimension of the limit point set, the maximum value of p has a lower bound given by $p - max - 1 \geq p_{max}^{1-\epsilon}$ or $p_{max} \geq 129131$. The relation between the number of subdivisions and the integer t, $p = t(2t+1)$, yields $t_{max} \geq 254$. Given that the difference between the Hausdorff dimensions of the limit point set and the ideal boundary is infinitesimal, t_{max} can be set equal to 254. Then, the highest mass of the two-dimensional fermions is 13.8129601372 GeV/c^2. By contrast with a finite number of cusps, there can be condensation on a Cantor set of ends of surfaces of infinite genus, and the margins of error in the above masses will not negligible.

The geometrical model of the pion, based on a quotient of a Euclidean continuation of a three-dimensional anti-de Sitter space-time, consists of the union of two handlebodies, with boundaries that are infinite-genus surfaces, representing a quark and an anti-quark [33]. The quarks, therefore, are three-dimensional, but the Cantor set of ends in the ideal boundary can be embedded in a one-dimensional continuous set. It may be recalled that the mass of the u quark boundary component, necessary for nonperturbative computations such as the computation of $m_{\eta'}$ [25], is $m_{u\infty} = 61.85\ MeV/c^2$. The mass of the quantum state with $t = 66$ in Table 2 approximately 62.1 MeV/c^2, differs by 0.25 MeV/c^2, which is almost equal to the combined rest mass of the particles ψ_7, ψ_{10} and ψ_{16}. The mass at $t = 65$, 58.4196081624 MeV/c^2 nearly coincides with the pion nucleon sigma term of 59.1 MeV/c^2 [34]. It is estimated that additional corrections are required to produce a quark mass parameter from the pion nucleon sigma term. One potential source would be a finer subdivision in the Cantor set of ends in the ideal boundary. The mass of the d quark boundary component may be set equal to 64.85 MeV/c^2, which can be compared to the mass at $t = 67$, 65.9560029775 MeV/c^2. The difference of 1.10600297747 MeV/c^2 can be attributed to an electron-positron pair with a net velocity of 0.3822807179 c or the sum of the rest masses of one of the following groups: $(\psi_3, \psi_4, \psi_9, \psi_{24})$ or $(\psi_6, \psi_{10}, \psi_{15}, \psi_{23})$.

Table 2. Masses of two-dimensional fermions for different (m,n) = (4t,4t + 2) with $t \leq 30$.

(*m*,*n*)	***c***	***p***	m_{ψ_t}
(4,6)	1	3	2.132054 eV/c^2
(8,10)	2	10	43.554808 eV/c^2
(12,14)	3	21	236.109725 eV/c^2
(16,18)	4	36	770.585143 eV/c^2
(20,22)	5	55	1.936351 KeV/c^2
(24,26)	6	78	4.023184 KeV/c^2
(28,30)	7	105	7.517622 KeV/c^2
(32,34)	8	136	12.906601 KeV/c^2
(36,38)	9	171	20.775645 KeV/c^2
(40,42)	10	210	31.789225 KeV/c^2
(44,46)	11	253	46.690578 KeV/c^2
(48,50)	12	300	66.302608 KeV/c^2
(52,54)	13	351	91.526083 KeV/c^2
(56,58)	14	406	123.331543 KeV/c^2
(60,62)	15	465	162.807889 KeV/c^2
(64,66)	16	528	211.063575 KeV/c^2
(68,70)	17	595	269.325598 KeV/c^2
(72,74)	18	666	338.791013 KeV/c^2
(76,78)	19	741	421.080075 KeV/c^2
(80,82)	20	820	517.504880 KeV/c^2
(84,86)	21	903	629.542173 KeV/c^2
(88,90)	22	990	758.855497 KeV/c^2
(92,94)	23	1081	907.135633 KeV/c^2
(96,98)	24	1176	1.076525 MeV/c^2
(100,102)	25	1275	1.267755 MeV/c^2
(104,106)	26	1378	1.483869 MeV/c^2
(108.110)	27	1485	1.726502 MeV/c^2
(112,114)	28	1596	1.997742 MeV/c^2
(116,118)	29	1711	2.299750 MeV/c^2
(120,122)	30	1830	2.634771 MeV/c^2

Purely bosonic states with energies larger than 1.548124523 GeV/c^2 and fermionic states, that are descendants of an $N = 2$ superminimal model with energies larger than

13.8129601372 GeV/c^2 would require a different three-dimensional model such that the fractal set of ends are embedded in a two-dimensional region, and the limit point set must be defined for a three-dimensional Kleinian group. The numerical factors in the bounds for the isoperimetric ratio [35], and the relation between the least eigenvalue of the Laplacian and the Hausdorff dimension [36], differ for hyperbolic three-manifolds and two-dimensional surfaces, generating a different characterization of the ideal boundary and a higher maximal mass for the particles.

4. Tachyon Condensation on the Ends of Surfaces of Infinite Genus

String scattering amplitudes at leading order for tachyons have poles located at resonances with the center-of-momentum squared energy $s = 8(n-1)$, $n = 0, 1, 2, 3, \ldots$ [37]. The tachyon ground state has a negative square mass $s = 4m^2 = -8$ and an imaginary mass $m = \pm\sqrt{2}i$. Tachyon condensation is known to exist at the ends of infinite-genus surfaces [38]. The coupling of tachyons to field describing a particle will produce a wavefunction with an exponential decay, such that the lifetime τ is the inverse of the width Γ of the resonance.

At energies beyond 10^{13} GeV, an E_6 group symmetry is restored. It may be demonstrated that the supersymmetry transformations of a superstring model can be included within the generators of E_6 [39]. The tachyon state is projected out of this supersymmetric theory and widths will have essentially no width. The reduction of the invariance group E_6 of the supersymmetric model to $SO(10) \times U(1)$ will be accompanied by a breaking of supersymmetry [40]. The resulting particle spectrum will now include tachyons of the bosonic string that will condense at the ends of infinite-genus surfaces.

There exist a class of particles that can be modelled by boundary components of surfaces of infinite genus [25], which include the u and d quarks. The relative error in the mass of the u quark is

$$\frac{\delta m_u}{m_u} = \frac{0.122\ MeV/c^2}{305\ MeV/c^2} = 4 \times 10^{-4} \tag{26}$$

The restriction to the ideal boundary of a surface reduces the phase space by a factor of 6, since there are $\binom{4}{2}$ coordinate embeddings of the surface in four dimensions.

Then

$$6\left(\frac{3}{2}\right)^2 \times 4 \times 10^{-4} \simeq 5.4 \times 10^{-4} \tag{27}$$

which approximately equals $\frac{3}{4}$ of the fine structure constant $\alpha \approx \frac{1}{137}$. The relative error in the masses of particles of charge q of this type may be hypothesized to be proportional to $q^2\alpha$.

A theoretical explanation of this effect now will be given. The tachyon has an imaginary mass. The tachyon action will be that of a complex scalar field and has a current of the form $J_T^\mu = q\phi^* \overleftrightarrow{\partial^\mu}\phi$ derived from $U(1)$ invariance. The open string tachyon potential, defined through string field theory action [41]

$$S = \left[\frac{1}{2}\langle\Psi, Q\Psi\rangle + \frac{1}{3}\langle\Psi, \Psi * \Psi\rangle\right] \tag{28}$$

by evaluating it at the tachyon mode $\Psi = Tc_1(0)|0\rangle$ [42,43], without including states of higher mass, is

$$V(T) = \left[\frac{1}{2}T^2 - \frac{27\sqrt{3}}{64}T^3\right]. \tag{29}$$

The dynamics is governed by the BRST kinetic term in the string field theory action away from the tachyon vacuum, which takes the form

$$\frac{1}{2g^2}(\partial_+ T)(\partial_- T) \tag{30}$$

such that the tachyon action is

$$S(T) = \frac{1}{g^2}\int dx_+ dx_- \left[\frac{1}{2}\partial_+ T \partial_- T - \left(\frac{1}{2}T^2 - \frac{27\sqrt{3}}{64}T^3\right)\right]. \tag{31}$$

The computation of the tachyon vacuum and the potential to higher levels yields the tension of the Dirichlet soliton generated by the tachyon condensate [44]. The time evolution of Ψ as $x^+ \to \infty$ to this vacuum [45] also may be established through an analytic representation for Ψ [46,47]. These higher-level states would not contribute significantly to the decay rate of the particle or the width of the resonance. A D-instanton in the rolling string vacuum in the free fermion model describes the addition of a zero-energy fermion to the vacuum state [48].

The dynamics of the tachyon with a homogeneous condensate in closed string field can be described by a timelike Liouville action [49,50]. With an exponential growth in the tachyon field, however, there will be a decouplng from the open string and have a effect on the value of κ_{str} by removing the contribution of the Dirichlet boundaries. The infinite-genus surfaces in closed string theory are noncompact, which replaces this coupling with the open string at finite genus. The exponential time-dependence is sufficient to cover the entire Cantor set of ends of a surface of infinite genus.

Charged particles of charge q and spin-$\frac{1}{2}$ have the current $J^\mu_{elec} = \bar{\psi}\gamma^\mu\psi$. The presence at the ends of the infinite-genus surface modelling the particle introduces a current-current interaction term

$$J_{elec\ \mu} J^\mu_T = q f(\ell)\bar{\psi}\gamma_\mu\psi\phi^* \overleftrightarrow{\partial^\mu}\phi \tag{32}$$

and the entire interaction term in the presence of a gauge field A^μ is

$$q J_{elec\ \mu} A^\mu + J_{elec\ \mu} J^T_\mu = q(A_\mu + f(\ell)\phi^* \overleftrightarrow{\partial^\mu}\phi)\bar{\psi}\gamma_\mu\psi. \tag{33}$$

where $f(\ell)$ is a quadratic function of the length scale to convert a dimension six combination of scalar and spinor fields to a dimension four term. The perturbative expansion of the self-energy now has a radiative correction proportional to $q^2\alpha$ at first order. An additional factor of $\frac{1}{\binom{4}{2}}$ is required by the account for the different coordinate embeddings of the Riemann surface in the four-dimensional embedding space with a metric of positive-definite signature. The coefficient of the interaction term $J^\mu_{elec} J^T_\mu$ will tend to the asymptotic value δ_* of the renormalization group flow. The spinor bilinear may be replaced by $\Box A_\mu$ through the equations of quantum electrodynamics, and the dimensional transmutation of the current-current interaction through the function $f(\ell)$ will yield $\delta_* q \overleftrightarrow{\partial^\mu}\phi A_\mu$. The action describing the interaction of the tachyon with the electromagnetic field of the charged particle would be

$$\begin{aligned} S_{T,elec} = \int d^4x \Big[&\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\bar{\psi}(\gamma^\mu\partial_\mu - m)\psi + q\bar{\psi}\gamma^\mu\psi A_\mu + \frac{1}{2}\partial^\mu\phi^*\partial_\mu\phi \\ &- \frac{1}{2}m_T^2\phi^*\phi + \delta_* q\phi^* \overleftrightarrow{\partial^\mu}\phi A_\mu \Big] \end{aligned} \tag{34}$$

which describes a theory of spinor and scalar electrodynamics. There are two sets of self-energy diagrams, characterized by the presence or absence of scalar loops. Since an arbitrary diagram would be a combination of these two types, the shift in the mass would be given by a product of the two factors given by a series in the coupling. The first factor gives the conventional mas of the charged particle. The second series, resulting from tachyon condensation, the represents the uncertainty.

The first-order diagram describing the interaction of the tachyon with the electromagnetic field without fermion lines only contributes to the self-energy of the charged particle because there is a four-point amplitude with fermions and tachyons, equivalent to a one-loop scalar electrodynamics d, that can be transformed in a self-energy graph by cutting two of the external lines at the vertices.

At first order, the radiative contribution to the mass of the charged particle has the magnitude $2\frac{(2\delta_* q)^2}{e^2}\alpha$. From Equation (26), the leading-order approximation of the relative error in the mass of the charged particle of charge q in this class is $\frac{1}{6}\frac{3}{4}\frac{q^2}{e^2}\alpha = \frac{1}{8}\frac{q^2}{e^2}\alpha$. Therefore, δ_* can be set equal to $\frac{1}{8}$. The coefficient of the amplitude at n^{th} order is $\frac{2^{n-1}}{n!}$, and the number of types of one-particle diagrams is 2^{n-1}. by contrast with quantum electrodynamics, the scalar loops generate ampllitudes of the same sign and

$$\begin{aligned}\delta m &= \left[\frac{1}{8}\frac{q^2}{e^2}\alpha + \sum_{n=2}^{\infty}\frac{2^{2n-2}}{n!}\left(\frac{\left(\frac{1}{4}q\right)^2}{e^2}\alpha\right)^n\right] m \\ &= \left[\frac{1}{16}\frac{q^2}{e^2}\alpha + \frac{1}{4}\left(e^{\frac{1}{4}\frac{q^2}{e^2}\alpha} - 1\right)\right] m.\end{aligned} \tag{35}$$

This result yields a theoretical prediction for the widths of the u and d quark resonances

$$\begin{aligned}\Gamma_{u,th} &= 2\,\delta m_{u,th}c^2 = 2\left[\frac{\alpha}{36} + \frac{1}{4}\left(e^{\frac{\alpha}{9}} - 1\right)\right] m_u c^2 \\ &\simeq 0.24734931089\ MeV/c^2. \\ \Gamma_{d,th} &= 2\,\delta m_{d,th}c^2 = 2\left[\frac{\alpha}{144} + \frac{1}{4}\left(e^{\frac{\alpha}{36}} - 1\right)\right] m_d c^2 \simeq 0.06243606896\ MeV.\end{aligned} \tag{36}$$

A reduced error in the mass of the d quark is predicted by the coupling to tachyons.

5. Conclusions

The finite widths of elementary particle resonances have an origin intrinsically related to geometrical models. It is quite evident that the uncertainty results from the tachyon condensation at the ideal boundaries of surfaces representing these particles, because the squared masses of these modes is negative. Evidence supporting the hypothesis is the set of resonances above the supersymmetry breaking scale with almost no width. The particles in an E_6-invariant superstring model would have ideal boundaries with tachyon condensation since the tachyon is projected out from the supersymmetric spectrum.

The theories of the elementary particle interactions can be embedded within a finite superstring model. Quantum consistency is achieved with the introduction of new worldsheet fields and supersymmetry. One other, more phenomenologically relevant feature of the theory is the vibrational mode of the closed string in the radial direction described by the tachyon. A string field theory has a tachyon potential which determines the nonperturbative vacua. Since the tachyon may be identified with a complex scalar field, the current is proportional to the charge q, It would be coupled to a spin-$\frac{1}{2}$ current in the interaction term $q\phi^*\overleftrightarrow{\partial^\mu}\phi\bar{\psi}\gamma_\mu\psi$. The radiative correction to the self-energy at first order in the perturbation series is proportional to q^2. The relative error in the mass of a charged particle to leading order, with tachyon condensation at the ideal boundary, would be $\frac{1}{8}\frac{q^2}{e^2}\alpha$, where α is the fine structure constant.

This uncertainty has provided theoretical predictions of the margin of error in the mass of the d quark which is less than the current experimental values. The greater precision of future experiments will determine whether these lesser widths can be reached. It remains to be established if these methods can be adapted to the computation of the relative error in the masses of hadrons composed of up and down quarks or leptons. The depiction of quarks through string theoretical methods is likely to have more relevance than a similar model for the electron and muon. The uncertainty in the electron mass is relatively infinitesimal compared to that of the up and down quark. A theoretical explanation for the absence of tachyon condensation in the electron follows from the Lorentz model based on the uniform distribution of charge on the surface of the sphere. There is no Cantor set of ends under these conditions, and the width of the resonance is much less than α.

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