

**SURVEY ARTICLE**

# A survey on deformations, cohomologies and homotopies of relative Rota–Baxter Lie algebras

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**Abstract**

In this paper, we review deformation, cohomology and homotopy theories of relative Rota–Baxter (RB) Lie algebras, which have attracted quite much interest recently. Using Voronov’s higher derived brackets, one can obtain an  $L_\infty$ -algebra whose Maurer–Cartan elements are relative RB Lie algebras. Then using the twisting method, one can obtain the  $L_\infty$ -algebra that controls deformations of a relative RB Lie algebra. Meanwhile, the cohomologies of relative RB Lie algebras can also be defined with the help of the twisted  $L_\infty$ -algebra. Using the controlling algebra approach, one can also introduce the notion of homotopy relative RB Lie algebras with close connection to pre- $L_\infty$ -algebras. Finally, we briefly review deformation, cohomology and homotopy theories of relative RB Lie algebras of nonzero weights.

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## 1 | INTRODUCTION

The concept of Rota–Baxter (RB) operators on associative algebras was introduced by Baxter [6] in his study of fluctuation theory in probability. Recently it has found many applications, including Connes–Kreimer’s [11] algebraic approach to the renormalization in perturbative quantum field theory. RB operators lead to the splitting of operads [3, 48], and are closely related to quasi-symmetric functions and Hopf algebras [18, 59]. Recently the relationship between RB operators and double Poisson algebras were studied in [25]. In the Lie algebra context, a RB operator was introduced independently in the 1980s as the operator form of the classical Yang–Baxter equation. For further details on RB operators, see [28, 29]. To better understand the classical Yang–Baxter equation and related integrable systems, the more general notion of a relative RB operator (which was called an  $\mathcal{O}$ -operator in the original literature) on a Lie algebra was introduced by Kupersmidt [36]. Relative RB operators provide solutions of the classical Yang–Baxter equation in the semidirect product Lie algebra and give rise to pre-Lie algebras [2].

The concept of a formal deformation of an algebraic structure began with the seminal work of Gerstenhaber [22, 23] for associative algebras. Nijenhuis and Richardson extended this study to Lie algebras [46, 47]. See [26] for more details about the deformation theories of various algebraic structures. More generally, deformation theory for algebras over quadratic operads was developed by Balavoine [4]. For more general operads, we refer the reader to [34, 41, 44], and the references therein. There is a well-known slogan, often attributed to Deligne, Drinfeld and Kontsevich: *every reasonable deformation theory is controlled by a differential graded Lie algebra, determined up to quasi-isomorphism*. This slogan has been made into a rigorous theorem by Lurie and Pridham, cf. [42, 49], and a recent simple treatment in [27]. It is also meaningful to deform *maps* compatible with given algebraic structures. Recently, the deformation theory of morphisms was developed in [7, 20, 21] and the deformation theory of diagrams of algebras was studied in [5, 19] using the minimal model of operads and the method of derived brackets [35, 43, 57]. Sometimes a differential graded Lie algebra up to quasi-isomorphism controlling a deformation theory manifests itself naturally as an  $L_\infty$ -algebra. This often happens when one tries to deform several algebraic structures as well as a compatibility relation between them, such as diagrams of algebras mentioned above.

A classical approach for studying a mathematical structure is associating invariants to it. Prominent among these are cohomological invariants, or simply cohomology, of various types of algebras. Cohomology controls deformations and extension problems of the corresponding algebraic structures. Cohomology theories of various kinds of algebras have been developed and studied in [10, 22, 31, 32]. More recently these classical constructions have been extended to strong homotopy (or infinity) versions of the algebras, cf., for example, [30].

Homotopy invariant algebraic structures play a prominent role in modern mathematical physics. Historically, the first such structure was that of an  $A_\infty$ -algebra introduced by Stasheff

in his study of based loop spaces [52]. Relevant later developments include the work of Lada and Stasheff [37, 53] about  $L_\infty$ -algebras in mathematical physics and the work of Chapoton and Livernet [9] about pre-Lie $_\infty$ -algebras. Strong homotopy (or infinity-) versions of a large class of algebraic structures were studied in the context of operads in [41, 45].

Due to the importance of relative RB Lie and associative algebras, the studies of corresponding deformation, cohomology and homotopy theories attract much interest recently. The first step toward such a study was given in [54], where the deformation and cohomology theories of relative RB operators were established and applications were given to study deformations and cohomologies of skew-symmetric  $r$ -matrices. See also the survey article [55] for more details. Then in [39], applying Voronov's higher derived brackets [57], the controlling algebra of relative RB Lie algebras (a relative RB Lie algebra consists of a Lie algebra  $\mathfrak{g}$ , a representation of  $\mathfrak{g}$  on a vector space  $V$  and a relative RB operator  $T : V \rightarrow \mathfrak{g}$ ) was constructed, which turns out to be an  $L_\infty$ -algebra. Then using the twisting method via Maurer–Cartan elements given in [24], one obtain a twisted  $L_\infty$ -algebra that governs simultaneous deformations of relative RB Lie algebras. Using the  $l_1$  in the twisted  $L_\infty$ -algebra, one can define the cohomology of relative RB Lie algebras. Finally, one can define homotopy relative RB operators via Maurer–Cartan characterization. Voronov's higher derived brackets and the controlling algebras of homotopy relative RB Lie algebras were studied more intrinsically in [40] via the functorial approach. Note that the aforementioned relative RB operators are of weight 0, and deformation, cohomology and homotopy theories of relative RB operators and relative RB Lie algebras of nonzero weights were further studied in [8, 12, 13, 33, 56].

In the associative algebra context, deformations, cohomologies and homotopies of relative RB associative algebras of weight 0 were studied in [14, 15]. Deformations and cohomologies of relative RB operators of nonzero weights were studied in [12]. Independently, deformations, cohomologies and homotopies of RB associative algebras of nonzero weights were studied in [58]. In particular, it was shown in [58] that the operad governing homotopy RB associative algebras is a minimal model of the operad of RB associative algebras. Note that due to the nonhomogeneous relations, the operad of RB algebras are not quadratic, and not covered by the Koszul duality theory. In [17], Dotsenko and Khoroshkin gave a detailed study of the operad of RB associative algebras, and note that it is very difficult to give explicit formulas for differentials in the free resolutions. So it is still curious to give the homotopy theory of RB algebras using the purely operadic approach.

The paper is organized as follows. In Section 2, we recall the main tools which will be used frequently: the Nijenhuis–Richardson bracket and higher derived brackets. In Section 3, we survey the deformation theory of relative RB Lie algebras. Given vector spaces  $\mathfrak{g}$  and  $V$ , first using Voronov's higher derived brackets one obtains an  $L_\infty$ -algebra, whose Maurer–Cartan elements are relative RB Lie algebra structures on  $\mathfrak{g}$  and  $V$ . Then given a relative RB Lie algebra, applying the twisting theory via Maurer–Cartan elements, one obtains a twisted  $L_\infty$ -algebra governs deformations of the relative RB Lie algebra. In Section 4, we survey the cohomology theory of relative RB Lie algebras. Using the  $l_1$  in the above twisted  $L_\infty$ -algebra, one can define the cohomology of relative RB Lie algebras. Moreover, there is a long exact sequence of cohomology groups linking the cohomology of LieRep pairs introduced in [1], the cohomology of  $\mathcal{O}$ -operators introduced in [54] and the cohomology of relative RB Lie algebras. The above general framework has two important special cases: RB Lie algebras and triangular Lie bialgebras. In Section 4.1, one can apply the above general framework to introduce the cohomology of RB Lie algebras. In Section 4.2, one can apply the above general framework to introduce the cohomology of triangular Lie bialgebras. In Section 5, we survey homotopy relative RB Lie algebras that obtained through the Maurer–Cartan approach. In Section 6, we briefly survey deformation, cohomology and homotopy theories of RB Lie and associative algebras of nonzero weights.

## 2 | THE NIJENHUIS–RICHARDSON BRACKET AND HIGHER DERIVED BRACKETS

In this section, we recall the Nijenhuis–Richardson bracket and higher derived brackets which are the main tools in later sections.

### 2.1 | The Nijenhuis–Richardson bracket

Let  $\mathfrak{g}$  be a vector space. For all  $n \geq 0$ , set  $C^n(\mathfrak{g}, \mathfrak{g}) := \text{Hom}(\wedge^{n+1} \mathfrak{g}, \mathfrak{g})$ . Consider the graded vector space  $C^*(\mathfrak{g}, \mathfrak{g}) = \bigoplus_{n=0}^{+\infty} C^n(\mathfrak{g}, \mathfrak{g}) = \bigoplus_{n=0}^{+\infty} \text{Hom}(\wedge^{n+1} \mathfrak{g}, \mathfrak{g})$ . Then  $C^*(\mathfrak{g}, \mathfrak{g})$  equipped with the *Nijenhuis–Richardson bracket* [46, 47]

$$[P, Q]_{\text{NR}} = P \circ Q - (-1)^{pq} Q \circ P, \quad \forall P \in C^p(\mathfrak{g}, \mathfrak{g}), Q \in C^q(\mathfrak{g}, \mathfrak{g}), \tag{1}$$

is a graded Lie algebra, where  $P \circ Q \in C^{p+q}(\mathfrak{g}, \mathfrak{g})$  is defined by

$$(P \circ Q)(x_1, \dots, x_{p+q+1}) = \sum_{\sigma \in \mathbb{S}_{(q+1, p)}} (-1)^\sigma P(Q(x_{\sigma(1)}, \dots, x_{\sigma(q+1)}), x_{\sigma(q+2)}, \dots, x_{\sigma(p+q+1)}). \tag{2}$$

Here  $\mathbb{S}_{(i, n-i)}$  denote the set of  $(i, n - i)$ -shuffles. Recall that a permutation  $\sigma \in \mathbb{S}_n$  is called an  $(i, n - i)$ -*shuffle* if  $\sigma(1) < \dots < \sigma(i)$  and  $\sigma(i + 1) < \dots < \sigma(n)$ . If  $i = 0$  or  $n$ , we assume  $\sigma = \text{Id}$ . The notion of an  $(i_1, \dots, i_k)$ -shuffle and the set  $\mathbb{S}_{(i_1, \dots, i_k)}$  are defined analogously. For  $\mu \in C^1(\mathfrak{g}, \mathfrak{g}) = \text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g})$ , we have

$$[\mu, \mu]_{\text{NR}}(x, y, z) = 2(\mu \circ \mu)(x, y, z) = 2(\mu(\mu(x, y), z) + \mu(\mu(y, z), x) + \mu(\mu(z, x), y)).$$

Thus,  $\mu$  defines a Lie algebra structure on  $\mathfrak{g}$  if and only if  $[\mu, \mu]_{\text{NR}} = 0$ .

Let  $(\mathfrak{g}, \mu)$  be a Lie algebra. Define the set of 0-cochains  $\mathfrak{C}_{\text{Lie}}^0(\mathfrak{g}; \mathfrak{g})$  to be 0, and define the set of  $n$ -cochains  $\mathfrak{C}_{\text{Lie}}^n(\mathfrak{g}; \mathfrak{g})$  to be

$$\mathfrak{C}_{\text{Lie}}^n(\mathfrak{g}; \mathfrak{g}) := \text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{g}) = C^{n-1}(\mathfrak{g}, \mathfrak{g}), \quad n \geq 1.$$

The *Chevalley–Eilenberg coboundary operator*  $d_{\text{CE}}$  of the Lie algebra  $\mathfrak{g}$  with coefficients in the adjoint representation is defined by

$$d_{\text{CE}} f = (-1)^{n-1} [\mu, f]_{\text{NR}}, \quad \forall f \in \mathfrak{C}_{\text{Lie}}^n(\mathfrak{g}; \mathfrak{g}). \tag{3}$$

The resulting cohomology is denoted by  $\mathcal{H}_{\text{Lie}}^*(\mathfrak{g}; \mathfrak{g})$ .

Let  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  be two vector spaces and elements in  $\mathfrak{g}_1$  will be denoted by  $x, y, z, x_i$  and elements in  $\mathfrak{g}_2$  will be denoted by  $u, v, w, v_i$ . For a multilinear map  $f : \wedge^k \mathfrak{g}_1 \otimes \wedge^l \mathfrak{g}_2 \rightarrow \mathfrak{g}_1$ , we define  $\hat{f} \in C^{k+l-1}(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{g}_1 \oplus \mathfrak{g}_2)$  by

$$\hat{f}((x_1, v_1), \dots, (x_{k+l}, v_{k+l})) := \sum_{\tau \in \mathbb{S}_{(k, l)}} (-1)^\tau (f(x_{\tau(1)}, \dots, x_{\tau(k)}, v_{\tau(k+1)}, \dots, v_{\tau(k+l)}), 0).$$

Similarly, for  $f : \wedge^k \mathfrak{g}_1 \otimes \wedge^l \mathfrak{g}_2 \rightarrow \mathfrak{g}_2$ , we define  $\hat{f} \in C^{k+l-1}(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{g}_1 \oplus \mathfrak{g}_2)$  by

$$\hat{f}((x_1, v_1), \dots, (x_{k+l}, v_{k+l})) := \sum_{\tau \in \mathbb{S}(k,l)} (-1)^\tau (0, f(x_{\tau(1)}, \dots, x_{\tau(k)}, v_{\tau(k+1)}, \dots, v_{\tau(k+l)})).$$

The linear map  $\hat{f}$  is called a *lift* of  $f$ . Define  $\mathfrak{g}^{k,l} := \wedge^k \mathfrak{g}_1 \otimes \wedge^l \mathfrak{g}_2$ . The vector space  $\wedge^n(\mathfrak{g}_1 \oplus \mathfrak{g}_2)$  is isomorphic to the direct sum of  $\mathfrak{g}^{k,l}$ ,  $k + l = n$ .

**Definition 2.1.** A linear map  $f \in \text{Hom}(\wedge^{k+l+1}(\mathfrak{g}_1 \oplus \mathfrak{g}_2), \mathfrak{g}_1 \oplus \mathfrak{g}_2)$  has a *bidegree*  $k|l$ , which is denoted by  $\|f\| = k|l$ , if  $f$  satisfies the following two conditions.

- (i) If  $X \in \mathfrak{g}^{k+1,l}$ , then  $f(X) \in \mathfrak{g}_1$  and if  $X \in \mathfrak{g}^{k,l+1}$ , then  $f(X) \in \mathfrak{g}_2$ .
- (ii) In all the other cases,  $f(X) = 0$ .

We denote the set of homogeneous linear maps of bidegree  $k|l$  by  $C^{k|l}(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{g}_1 \oplus \mathfrak{g}_2)$ .

It is clear that this gives a well-defined bigrading on the vector space  $\text{Hom}(\wedge^{k+l+1}(\mathfrak{g}_1 \oplus \mathfrak{g}_2), \mathfrak{g}_1 \oplus \mathfrak{g}_2)$ . We have  $k + l \geq 0$ ,  $k, l \geq -1$  because  $k + l + 1 \geq 1$  and  $k + 1, l + 1 \geq 0$ .

The following lemmas are very important in later studies.

**Lemma 2.2.** *The Nijenhuis–Richardson bracket on  $C^*(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{g}_1 \oplus \mathfrak{g}_2)$  is compatible with the bigrading. More precisely, if  $\|f\| = k_f|l_f$ ,  $\|g\| = k_g|l_g$ , then  $\|[f, g]_{\text{NR}}\| = (k_f + k_g)|(l_f + l_g)$ .*

*Proof.* It follows from direct computation. □

*Remark 2.3.* In later studies, the subspaces  $C^{k|0}(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{g}_1 \oplus \mathfrak{g}_2)$  and  $C^{-1|l}(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{g}_1 \oplus \mathfrak{g}_2)$  will be frequently used. By the above lift map, one has the following isomorphisms:

$$C^{k|0}(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{g}_1 \oplus \mathfrak{g}_2) \cong \text{Hom}(\wedge^{k+1} \mathfrak{g}_1, \mathfrak{g}_1) \oplus \text{Hom}(\wedge^k \mathfrak{g}_1 \otimes \mathfrak{g}_2, \mathfrak{g}_2), \tag{4}$$

$$C^{-1|l}(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{g}_1 \oplus \mathfrak{g}_2) \cong \text{Hom}(\wedge^l \mathfrak{g}_2, \mathfrak{g}_1). \tag{5}$$

**Lemma 2.4.** *If  $\|f\| = (-1)|k$  and  $\|g\| = (-1)|l$ , then  $[f, g]_{\text{NR}} = 0$ . Consequently,  $\bigoplus_{l=1}^{+\infty} C^{-1|l}(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{g}_1 \oplus \mathfrak{g}_2)$  is an abelian subalgebra of the graded Lie algebra  $(C^*(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{g}_1 \oplus \mathfrak{g}_2), [\cdot, \cdot]_{\text{NR}})$ .*

*Proof.* It follows from Lemma 2.2. □

## 2.2 | $L_\infty$ -algebras and higher derived brackets

The notion of an  $L_\infty$ -algebra was introduced by Stasheff in [53]. See [37, 38] for more details.

Let  $V = \bigoplus_{k \in \mathbb{Z}} V^k$  be a  $\mathbb{Z}$ -graded vector space. We will denote by  $S(V)$  the *symmetric algebra* of  $V$ . That is,  $S(V) := T(V)/I$ , where  $T(V)$  is the tensor algebra and  $I$  is the 2-sided ideal of  $T(V)$  generated by all homogeneous elements of the form  $x \otimes y - (-1)^{xy} y \otimes x$ . We will write  $S(V) = \bigoplus_{i=0}^{+\infty} S^i(V)$ . Moreover, we denote the reduced symmetric algebra by  $\tilde{S}(V) := \bigoplus_{i=1}^{+\infty} S^i(V)$ . Denote the product of homogeneous elements  $v_1, \dots, v_n \in V$  in  $S^n(V)$  by  $v_1 \odot \dots \odot v_n$ . The degree of  $v_1 \odot \dots \odot v_n$  is by definition the sum of the degrees of  $v_i$ . For a permutation  $\sigma \in \mathbb{S}_n$  and  $v_1, \dots, v_n \in V$ ,

the Koszul sign  $\varepsilon(\sigma) = \varepsilon(\sigma; v_1, \dots, v_n) \in \{-1, 1\}$  is defined by

$$v_1 \odot \cdots \odot v_n = \varepsilon(\sigma; v_1, \dots, v_n)v_{\sigma(1)} \odot \cdots \odot v_{\sigma(n)}.$$

The *desuspension operator*  $s^{-1}$  changes the grading of  $V$  according to the rule  $(s^{-1}V)^i := V^{i+1}$ . The degree  $-1$  map  $s^{-1} : V \rightarrow s^{-1}V$  is defined by sending  $v \in V$  to its copy  $s^{-1}v \in s^{-1}V$ .

**Definition 2.5.** An  $L_\infty$ -algebra is a  $\mathbb{Z}$ -graded vector space  $\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}^k$  equipped with a collection ( $k \geq 1$ ) of linear maps  $l_k : \bigotimes^k \mathfrak{g} \rightarrow \mathfrak{g}$  of degree 1 with the property that, for any homogeneous elements  $x_1, \dots, x_n \in \mathfrak{g}$ , we have:

(i) (*graded symmetry*) for every  $\sigma \in \mathbb{S}_n$ ,

$$l_n(x_{\sigma(1)}, \dots, x_{\sigma(n-1)}, x_{\sigma(n)}) = \varepsilon(\sigma)l_n(x_1, \dots, x_{n-1}, x_n);$$

(ii) (*generalized Jacobi identity*) for all  $n \geq 1$ ,

$$\sum_{i=1}^n \sum_{\sigma \in \mathbb{S}_{(i, n-i)}} \varepsilon(\sigma)l_{n-i+1}(l_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots, x_{\sigma(n)}) = 0.$$

**Definition 2.6.** An element  $\alpha \in \mathfrak{g}^0$  is called a *Maurer–Cartan element* of an  $L_\infty$ -algebra  $\mathfrak{g}$  if  $\alpha$  satisfies the Maurer–Cartan equation

$$\sum_{k=1}^{+\infty} \frac{1}{k!} l_k(\alpha, \dots, \alpha) = 0. \tag{6}$$

Let  $\alpha$  be a Maurer–Cartan element. Define  $l_k^\alpha : \bigotimes^k \mathfrak{g} \rightarrow \mathfrak{g}$  ( $k \geq 1$ ) by

$$l_k^\alpha(x_1, \dots, x_k) = \sum_{n=0}^{+\infty} \frac{1}{n!} l_{k+n}(\underbrace{\alpha, \dots, \alpha}_n, x_1, \dots, x_k). \tag{7}$$

*Remark 2.7.* To ensure the convergence of the series appearing in the definition of Maurer–Cartan elements and Maurer–Cartan twistings above, one need the  $L_\infty$ -algebra being filtered given by Dolgushev and Rogers in [16], or weakly filtered given in [39]. Since all the  $L_\infty$ -algebras under consideration in the sequel satisfy the weakly filtered condition, so we will not mention this point anymore.

The following result is given by Getzler in [24, Section 4].

**Theorem 2.8.** *With the above notation,  $(\mathfrak{g}, \{l_k^{\alpha}\}_{k=1}^{+\infty})$  is an  $L_\infty$ -algebra, obtained from  $\mathfrak{g}$  by twisting with the Maurer–Cartan element  $\alpha$ . Moreover,  $\alpha + \alpha'$  is a Maurer–Cartan element of  $(\mathfrak{g}, \{l_k^{\alpha}\}_{k=1}^{+\infty})$  if and only if  $\alpha'$  is a Maurer–Cartan element of the twisted  $L_\infty$ -algebra  $(\mathfrak{g}, \{l_k^{\alpha}\}_{k=1}^{+\infty})$ .*

One method for constructing explicit  $L_\infty$ -algebras is given by Voronov’s higher derived brackets [57]. Let us recall this construction.

**Definition 2.9.** A  $V$ -data consists of a quadruple  $(L, H, P, \Delta)$  where:

- $(L, [\cdot, \cdot])$  is a graded Lie algebra;
- $H$  is an abelian graded Lie subalgebra of  $(L, [\cdot, \cdot])$ ;
- $P : L \rightarrow L$  is a projection, that is  $P \circ P = P$ , whose image is  $H$  and kernel is a graded Lie subalgebra of  $(L, [\cdot, \cdot])$ ;
- $\Delta$  is an element in  $\ker(P)^1$  such that  $[\Delta, \Delta] = 0$ .

**Theorem 2.10** [20, 57]. *Let  $(L, H, P, \Delta)$  be a  $V$ -data. Then the graded vector space  $s^{-1}L \oplus H$  is an  $L_\infty$ -algebra, where nontrivial  $l_k$  are given by*

$$\begin{aligned}
 l_1(s^{-1}x, a) &= (-s^{-1}[\Delta, x], P(x + [\Delta, a])), \\
 l_2(s^{-1}x, s^{-1}y) &= (-1)^x s^{-1}[x, y], \\
 l_k(s^{-1}x, a_1, \dots, a_{k-1}) &= P[\dots [[x, a_1], a_2] \dots, a_{k-1}], \quad k \geq 2, \\
 l_k(a_1, \dots, a_{k-1}, a_k) &= P[\dots [[\Delta, a_1], a_2] \dots, a_k], \quad k \geq 2.
 \end{aligned}
 \tag{8}$$

Here  $a, a_1, \dots, a_k$  are homogeneous elements of  $H$  and  $x, y$  are homogeneous elements of  $L$ .

Moreover, if  $L'$  is a graded Lie subalgebra of  $L$  that satisfies  $[\Delta, L'] \subset L'$ , then  $s^{-1}L' \oplus H$  is an  $L_\infty$ -subalgebra of the above  $L_\infty$ -algebra  $(s^{-1}L \oplus H, \{l_k\}_{k=1}^{+\infty})$ .

### 3 | DEFORMATIONS OF RELATIVE ROTA–BAXTER LIE ALGEBRAS

In this section, first we use Voronov’s higher derived brackets to construct the  $L_\infty$ -algebra whose Maurer–Cartan elements are relative RB Lie algebra structures. Then using the twisting method, one obtains the  $L_\infty$ -algebra that controls simultaneous deformations of relative RB Lie algebras.

**Definition 3.1.** A LieRep pair, denoted by  $(\mathfrak{g}, \mu; \rho)$ , consists of a Lie algebra  $(\mathfrak{g}, \mu = [\cdot, \cdot]_\mathfrak{g})$  and a representation  $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  of  $\mathfrak{g}$  on a vector space  $V$ .

Note that  $\mu + \rho \in C^{1|0}(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V)$ . Moreover, the fact that  $\mu$  is a Lie bracket and  $\rho$  is a representation is equivalent to that

$$[\mu + \rho, \mu + \rho]_{NR} = 0.$$

We now recall the notion of a relative RB operator.

**Definition 3.2.**

- (i) A linear operator  $T : \mathfrak{g} \rightarrow \mathfrak{g}$  on a Lie algebra  $\mathfrak{g}$  is called a RB operator if

$$[T(x), T(y)]_\mathfrak{g} = T([T(x), y]_\mathfrak{g} + [x, T(y)]_\mathfrak{g}), \quad \forall x, y \in \mathfrak{g}. \tag{9}$$

Moreover, a Lie algebra  $(\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g})$  with a RB operator  $T$  is called a RB Lie algebra, which is denoted by  $(\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g}, T)$ .

(ii) A *relative RB Lie algebra* is a triple  $((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \rho, T)$ , where  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}; \rho)$  is a LieRep pair and  $T : V \rightarrow \mathfrak{g}$  is a *relative RB operator*, that is,

$$[Tu, Tv]_{\mathfrak{g}} = T(\rho(Tu)(v) - \rho(Tv)(u)), \quad \forall u, v \in V. \tag{10}$$

Note that a RB operator on a Lie algebra is a relative RB operator with respect to the adjoint representation.

Let  $\mathfrak{g}$  and  $V$  be two vector spaces. Then one has a graded Lie algebra  $(\bigoplus_{n=0}^{+\infty} C^n(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V), [\cdot, \cdot]_{\text{NR}})$ . This graded Lie algebra gives rise to a V-data, and an  $L_{\infty}$ -algebra naturally.

**Proposition 3.3.** *One has a V-data  $(L, H, P, \Delta)$  as follows.*

- The graded Lie algebra  $(L, [\cdot, \cdot])$  is given by  $(\bigoplus_{n=0}^{+\infty} C^n(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V), [\cdot, \cdot]_{\text{NR}})$ .
- The abelian graded Lie subalgebra  $H$  is given by

$$H := \bigoplus_{n=0}^{+\infty} C^{-1(n+1)}(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V) = \bigoplus_{n=0}^{+\infty} \text{Hom}(\wedge^{n+1} V, \mathfrak{g}). \tag{11}$$

- $P : L \rightarrow L$  is the projection onto the subspace  $H$ .
- $\Delta = 0$ .

Consequently, one obtains an  $L_{\infty}$ -algebra  $(s^{-1}L \oplus H, \{l_k\}_{k=1}^{+\infty})$ , where  $l_k$  are given by

$$\begin{aligned} l_1(s^{-1}Q, \theta) &= P(Q), \\ l_2(s^{-1}Q, s^{-1}Q') &= (-1)^Q s^{-1}[Q, Q']_{\text{NR}}, \\ l_k(s^{-1}Q, \theta_1, \dots, \theta_{k-1}) &= P[\dots [Q, \theta_1]_{\text{NR}}, \dots, \theta_{k-1}]_{\text{NR}}, \end{aligned}$$

for homogeneous elements  $\theta, \theta_1, \dots, \theta_{k-1} \in H$ , homogeneous elements  $Q, Q' \in L$  and all the other possible combinations vanish.

*Proof.* By Lemma 2.4,  $H$  is an abelian subalgebra of  $(L, [\cdot, \cdot])$ .

Since  $P$  is the projection onto  $H$ , it is obvious that  $P \circ P = P$ . It is also straightforward to see that the kernel of  $P$  is a graded Lie subalgebra of  $(L, [\cdot, \cdot])$ . Thus  $(L, H, P, \Delta = 0)$  is a V-data.

The other conclusions follows immediately from Theorem 2.10. □

By Lemma 2.2, one obtains that

$$\begin{aligned} L' &= \bigoplus_{n=0}^{+\infty} C^{n|0}(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V), \quad \text{where} \\ C^{n|0}(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V) &= \text{Hom}(\wedge^{n+1} \mathfrak{g}, \mathfrak{g}) \oplus \text{Hom}(\wedge^n \mathfrak{g} \otimes V, V) \end{aligned} \tag{12}$$

is a graded Lie subalgebra of  $(\bigoplus_{n=0}^{+\infty} C^n(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V), [\cdot, \cdot]_{\text{NR}})$ .

**Corollary 3.4.** *With the above notation,  $(s^{-1}L' \oplus H, \{l_i\}_{i=1}^{+\infty})$  is an  $L_{\infty}$ -algebra, where  $l_k$  are given by*

$$\begin{aligned} l_2(s^{-1}Q, s^{-1}Q') &= (-1)^Q s^{-1}[Q, Q']_{\text{NR}}, \\ l_k(s^{-1}Q, \theta_1, \dots, \theta_{k-1}) &= P[\dots [Q, \theta_1]_{\text{NR}}, \dots, \theta_{k-1}]_{\text{NR}}, \end{aligned}$$

for homogeneous elements  $\theta_1, \dots, \theta_{k-1} \in H$ , homogeneous elements  $Q, Q' \in L'$ , and all the other possible combinations vanish.

Now we are ready to formulate the main result in this section.

**Theorem 3.5.** *Let  $\mathfrak{g}$  and  $V$  be two vector spaces,  $\mu \in \text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g})$ ,  $\rho \in \text{Hom}(\mathfrak{g} \otimes V, V)$  and  $T \in \text{Hom}(V, \mathfrak{g})$ . Then  $((\mathfrak{g}, \mu), \rho, T)$  is a relative RB Lie algebra if and only if  $(s^{-1}\pi, T)$  is a Maurer–Cartan element of the  $L_\infty$ -algebra  $(s^{-1}L' \oplus H, \{l_i\}_{i=1}^{+\infty})$  given in Corollary 3.4, where  $\pi = \mu + \rho \in C^{1|0}(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V)$ .*

*Proof.* Let  $(s^{-1}\pi, T)$  be a Maurer–Cartan element of  $(s^{-1}L' \oplus H, \{l_i\}_{i=1}^{+\infty})$ . By Lemmas 2.2 and 2.4, we have

$$||[\pi, T]_{\text{NR}}|| = 0|1, \quad ||[[\pi, T]_{\text{NR}}, T]_{\text{NR}}|| = -1|2, \quad ||[[[\pi, T]_{\text{NR}}, T]_{\text{NR}}, T]_{\text{NR}}|| = 0.$$

Then, by Corollary 3.4, we have

$$\begin{aligned} (0, 0) &= \sum_{k=1}^{+\infty} \frac{1}{k!} l_k((s^{-1}\pi, T), \dots, (s^{-1}\pi, T)) \\ &= \frac{1}{2!} l_2((s^{-1}\pi, T), (s^{-1}\pi, T)) + \frac{1}{3!} l_3((s^{-1}\pi, T), (s^{-1}\pi, T), (s^{-1}\pi, T)) \\ &= \left(-s^{-1} \frac{1}{2} [\pi, \pi]_{\text{NR}}, \frac{1}{2} [[\pi, T]_{\text{NR}}, T]_{\text{NR}}\right). \end{aligned}$$

Thus, we obtain  $[\pi, \pi]_{\text{NR}} = 0$  and  $[[\pi, T]_{\text{NR}}, T]_{\text{NR}} = 0$ , which implies that  $(\mathfrak{g}, \mu)$  is a Lie algebra,  $(V; \rho)$  is its representation and  $T$  is a relative RB operator on the Lie algebra  $(\mathfrak{g}, \mu)$  with respect to the representation  $(V; \rho)$ . □

Let  $((\mathfrak{g}, \mu), \rho, T)$  be a relative RB Lie algebra. Denote by  $\pi = \mu + \rho \in C^{1|0}(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V)$ . By Theorem 3.5, we obtain that  $(s^{-1}\pi, T)$  is a Maurer–Cartan element of the  $L_\infty$ -algebra  $(s^{-1}L' \oplus H, \{l_i\}_{i=1}^{+\infty})$  given in Corollary 3.4. Now we are ready to give the  $L_\infty$ -algebra that controls deformations of the relative RB Lie algebra.

**Theorem 3.6.** *With the above notation, one has the twisted  $L_\infty$ -algebra  $(s^{-1}L' \oplus H, \{l_k^{(s^{-1}\pi, T)}\}_{k=1}^{+\infty})$  associated to a relative RB Lie algebra  $((\mathfrak{g}, \mu), \rho, T)$ , where  $\pi = \mu + \rho$ .*

*Moreover, for linear maps  $T' \in \text{Hom}(V, \mathfrak{g})$ ,  $\mu' \in \text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g})$  and  $\rho' \in \text{Hom}(\mathfrak{g}, \mathfrak{gl}(V))$ , the triple  $((\mathfrak{g}, \mu + \mu'), \rho + \rho', T + T')$  is again a relative RB Lie algebra if and only if  $(s^{-1}(\mu' + \rho'), T')$  is a Maurer–Cartan element of the twisted  $L_\infty$ -algebra  $(s^{-1}L' \oplus H, \{l_k^{(s^{-1}\pi, T)}\}_{k=1}^{+\infty})$ .*

*Proof.* If  $((\mathfrak{g}, \mu + \mu'), \rho + \rho', T + T')$  is a relative RB Lie algebra, then by Theorem 3.5,  $(s^{-1}(\mu + \mu' + \rho + \rho'), T + T')$  is a Maurer–Cartan element of the  $L_\infty$ -algebra given in Corollary 3.4. Moreover, by Theorem 2.8,  $(s^{-1}(\mu' + \rho'), T')$  is a Maurer–Cartan element of the  $L_\infty$ -algebra  $(s^{-1}L' \oplus H, \{l_k^{(s^{-1}\pi, T)}\}_{k=1}^{+\infty})$ . □

**Remark 3.7.** The above  $L_\infty$ -algebra controlling deformations of relative RB Lie algebras is an extension of the differential graded Lie algebra controlling deformations of LieRep pairs by the

differential graded Lie algebra controlling deformations of relative RB operators. See [39, Theorem 3.16] for more details.

### 4 | COHOMOLOGIES OF RELATIVE ROTA–BAXTER LIE ALGEBRAS

In this section, we survey the cohomology of relative RB Lie algebras. In particular, one can define the cohomology of RB Lie algebras and the cohomology of triangular Lie bialgebras using this general framework.

One can define the cohomology of a relative RB Lie algebra using the twisted  $L_\infty$ -algebra given in Theorem 3.6.

Let  $((\mathfrak{g}, \mu), \rho, T)$  be a relative RB Lie algebra. Define the set of 0-cochains  $\mathfrak{C}^0(\mathfrak{g}, \rho, T)$  to be 0, and define the set of 1-cochains  $\mathfrak{C}^1(\mathfrak{g}, \rho, T)$  to be  $\mathfrak{gl}(\mathfrak{g}) \oplus \mathfrak{gl}(V)$ . For  $n \geq 2$ , define the space of  $n$ -cochains  $\mathfrak{C}^n(\mathfrak{g}, \rho, T)$  by

$$\begin{aligned} \mathfrak{C}^n(\mathfrak{g}, \rho, T) &:= \mathfrak{C}^n(\mathfrak{g}, \rho) \oplus \mathfrak{C}^n(T) = C^{(n-1)|0}(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V) \oplus C^{-1|(n-1)}(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V) \\ &= (\text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{g}) \oplus \text{Hom}(\wedge^{n-1} \mathfrak{g} \otimes V, V)) \oplus \text{Hom}(\wedge^{n-1} V, \mathfrak{g}). \end{aligned}$$

Define the coboundary operator  $D : \mathfrak{C}^n(\mathfrak{g}, \rho, T) \rightarrow \mathfrak{C}^{n+1}(\mathfrak{g}, \rho, T)$  by

$$D(f, \theta) = (-1)^{n-2} \left( -[\pi, f]_{\text{NR}}, [[\pi, T]_{\text{NR}}, \theta]_{\text{NR}} + \frac{1}{n!} \underbrace{[\dots [ [f, T]_{\text{NR}}, T]_{\text{NR}}, \dots, T]_{\text{NR}}}_n \right), \tag{13}$$

where  $\pi = \mu + \rho$ ,  $f \in \text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{g}) \oplus \text{Hom}(\wedge^{n-1} \mathfrak{g} \otimes V, V)$  and  $\theta \in \text{Hom}(\wedge^{n-1} V, \mathfrak{g})$ .

**Theorem 4.1.** *With the above notation,  $(\bigoplus_{n=0}^{+\infty} \mathfrak{C}^n(\mathfrak{g}, \rho, T), D)$  is a cochain complex, that is,  $D \circ D = 0$ .*

*Proof.* By Theorem 3.6,  $(s^{-1}L' \oplus H, \{l_k^{(s^{-1}\pi, T)}\}_{k=1}^{+\infty})$  is an  $L_\infty$ -algebra, where  $\pi = \mu + \rho$ ,  $H$  and  $L'$  are given by (11) and (12), respectively. For any  $(f, \theta) \in \mathfrak{C}^n(\mathfrak{g}, \rho, T)$ , one has  $(s^{-1}f, \theta) \in (s^{-1}L' \oplus H)^{n-2}$ . By (13), one deduces that

$$D(f, \theta) = (-1)^{n-2} l_1^{(s^{-1}\pi, T)}(s^{-1}f, \theta).$$

Thus,  $(\bigoplus_{n=0}^{+\infty} \mathfrak{C}^n(\mathfrak{g}, \rho, T), D)$  is a cochain complex. □

**Definition 4.2.** The cohomology of the cochain complex  $(\bigoplus_{n=0}^{+\infty} \mathfrak{C}^n(\mathfrak{g}, \rho, T), D)$  is called the *cohomology of the relative RB Lie algebra*  $((\mathfrak{g}, \mu), \rho, T)$ . We denote its  $n$ th cohomology group by  $\mathcal{H}^n(\mathfrak{g}, \rho, T)$ .

Define a linear operator  $h_T : \mathfrak{C}^n(\mathfrak{g}, \rho) \rightarrow \mathfrak{C}^{n+1}(T)$  by

$$h_T f := (-1)^{n-2} \frac{1}{n!} \underbrace{[\dots [ [f, T]_{\text{NR}}, T]_{\text{NR}}, \dots, T]_{\text{NR}}}_n. \tag{14}$$

More precisely,

$$(h_T f)(v_1, \dots, v_n)$$

$$= (-1)^n f_{\mathfrak{g}}(Tv_1, \dots, Tv_n) + \sum_{i=1}^n (-1)^{i+1} T f_V(Tv_1, \dots, Tv_{i-1}, Tv_{i+1}, \dots, Tv_n, v_i), \tag{15}$$

where  $f = (f_{\mathfrak{g}}, f_V)$ , and  $f_{\mathfrak{g}} \in \text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{g})$ ,  $f_V \in \text{Hom}(\wedge^{n-1} \mathfrak{g} \otimes V, V)$  and  $v_1, \dots, v_n \in V$ .

By (13) and (14), the coboundary operator can be written as

$$D(f, \theta) = (\partial f, \delta \theta + h_T f), \tag{16}$$

where  $\partial : \mathfrak{C}^n(\mathfrak{g}, \rho) \rightarrow \mathfrak{C}^{n+1}(\mathfrak{g}, \rho)$  is given by

$$\partial f := (-1)^{n-1} [\mu + \rho, f]_{\text{NR}}. \tag{17}$$

and  $\delta : \mathfrak{C}^n(T) \rightarrow \mathfrak{C}^{n+1}(T)$  is given by

$$\delta \theta = (-1)^n [[\pi, T]_{\text{NR}}, \theta]_{\text{NR}}. \tag{18}$$

The formula of the coboundary operator  $D$  can be well explained by the following diagram:

$$\begin{array}{ccccccc} \dots & \longrightarrow & \mathfrak{C}^n(\mathfrak{g}, \rho) & \xrightarrow{\partial} & \mathfrak{C}^{n+1}(\mathfrak{g}, \rho) & \xrightarrow{\partial} & \mathfrak{C}^{n+2}(\mathfrak{g}, \rho) \longrightarrow \dots \\ & & \searrow h_T & & \searrow h_T & & \\ \dots & \longrightarrow & \mathfrak{C}^n(T) & \xrightarrow{\delta} & \mathfrak{C}^{n+1}(T) & \xrightarrow{\delta} & \mathfrak{C}^{n+2}(T) \longrightarrow \dots \end{array}$$

Since  $D^2 = 0$ , it follows that  $\partial^2 = 0$  and  $\delta^2 = 0$ . Therefore, one has two cochain complexes  $(\oplus_{n=0}^{+\infty} \mathfrak{C}^n(\mathfrak{g}, \rho), \partial)$  and  $(\oplus_{n=0}^{+\infty} \mathfrak{C}^n(T), \delta)$ , whose cohomology are denoted by  $\mathcal{H}^*(\mathfrak{g}, \rho)$  and  $\mathcal{H}^*(T)$ , respectively.

**Theorem 4.3.** *Let  $(\mathfrak{g}, \mu, \rho, T)$  be a relative RB Lie algebra. Then there is a short exact sequence of the cochain complexes:*

$$0 \longrightarrow (\oplus_{n=0}^{+\infty} \mathfrak{C}^n(T), \delta) \xrightarrow{\iota} (\oplus_{n=0}^{+\infty} \mathfrak{C}^n(\mathfrak{g}, \rho, T), D) \xrightarrow{p} (\oplus_{n=0}^{+\infty} \mathfrak{C}^n(\mathfrak{g}, \rho), \partial) \longrightarrow 0,$$

where  $\iota$  and  $p$  are the inclusion map and the projection map.

Consequently, there is a long exact sequence of the cohomology groups:

$$\dots \longrightarrow \mathcal{H}^n(T) \xrightarrow{\mathcal{H}^n(\iota)} \mathcal{H}^n(\mathfrak{g}, \rho, T) \xrightarrow{\mathcal{H}^n(p)} \mathcal{H}^n(\mathfrak{g}, \rho) \xrightarrow{c^n} \mathcal{H}^{n+1}(T) \longrightarrow \dots,$$

where the connecting map  $c^n$  is defined by  $c^n([\alpha]) = [h_T \alpha]$ , for all  $[\alpha] \in \mathcal{H}^n(\mathfrak{g}, \rho)$ .

*Proof.* By (16), one has the short exact sequence of cochain complexes which induces a long exact sequence of cohomology groups. Also by (16),  $c^n$  is given by  $c^n([\alpha]) = [h_T \alpha]$ . □

*Remark 4.4.* The cohomology of the cochain complex  $(\oplus_{n=0}^{+\infty} \mathfrak{C}^n(T), \delta)$  is taken to be the cohomology of the relative RB operator  $T$  [54], and the cohomology of the cochain complex  $(\oplus_{n=0}^{+\infty} \mathfrak{C}^n(\mathfrak{g}, \rho), \partial)$  is taken to be the cohomology of the LieRep pair  $(\mathfrak{g}, \mu; \rho)$  [1]. So the above result establishes the relationship between the cohomology groups of relative RB Lie algebras and the cohomology groups of the underlying relative RB operators and LieRep pairs.

*Remark 4.5.* In the associative algebra context, deformations, cohomologies and homotopies of relative RB operators on associative algebras and relative RB associative algebras were studied in [14] and [15], respectively.

## 4.1 | Cohomology of Rota–Baxter Lie algebras

In this subsection, we survey the cohomology of RB Lie algebras, which is defined with the help of the general framework of the cohomology of relative RB Lie algebras.

Let  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, T)$  be a RB Lie algebra. Define the set of 0-cochains  $\mathfrak{C}_{\text{RB}}^0(\mathfrak{g}, T)$  to be 0, and define the set of 1-cochains  $\mathfrak{C}_{\text{RB}}^1(\mathfrak{g}, T)$  to be  $\mathfrak{C}_{\text{RB}}^1(\mathfrak{g}, T) := \text{Hom}(\mathfrak{g}, \mathfrak{g})$ . For  $n \geq 2$ , define the space of  $n$ -cochains  $\mathfrak{C}_{\text{RB}}^n(\mathfrak{g}, T)$  by

$$\mathfrak{C}_{\text{RB}}^n(\mathfrak{g}, T) := \mathfrak{C}_{\text{Lie}}^n(\mathfrak{g}; \mathfrak{g}) \oplus \mathfrak{C}^n(T) = \text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{g}) \oplus \text{Hom}(\wedge^{n-1} \mathfrak{g}, \mathfrak{g}).$$

Define the embedding  $\mathfrak{i} : \mathfrak{C}_{\text{RB}}^n(\mathfrak{g}, T) \rightarrow \mathfrak{C}^n(\mathfrak{g}, \text{ad}, T)$  by

$$\mathfrak{i}(f, \theta) = (f, f, \theta), \quad \forall f \in \text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{g}), \theta \in \text{Hom}(\wedge^{n-1} \mathfrak{g}, \mathfrak{g}).$$

Denote by  $\text{Im}^n(\mathfrak{i}) = \mathfrak{i}(\mathfrak{C}_{\text{RB}}^n(\mathfrak{g}, T))$ . Then  $(\bigoplus_{n=0}^{+\infty} \text{Im}^n(\mathfrak{i}), \mathcal{D})$  is a subcomplex of the cochain complex  $(\bigoplus_{n=0}^{+\infty} \mathfrak{C}^n(\mathfrak{g}, \text{ad}, T), \mathcal{D})$  associated to the relative RB Lie algebra  $((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}), \text{ad}, T)$ .

Define the projection  $\mathfrak{p} : \text{Im}^n(\mathfrak{i}) \rightarrow \mathfrak{C}_{\text{RB}}^n(\mathfrak{g}, T)$  by

$$\mathfrak{p}(f, f, \theta) = (f, \theta), \quad \forall f \in \text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{g}), \theta \in \text{Hom}(\wedge^{n-1} \mathfrak{g}, \mathfrak{g}).$$

Then for  $n \geq 0$ , define  $D_{\text{RB}} : \mathfrak{C}_{\text{RB}}^n(\mathfrak{g}, T) \rightarrow \mathfrak{C}_{\text{RB}}^{n+1}(\mathfrak{g}, T)$  by  $D_{\text{RB}} = \mathfrak{p} \circ \mathcal{D} \circ \mathfrak{i}$ . More precisely,

$$D_{\text{RB}}(f, \theta) = (d_{\text{CE}}f, \delta\theta + \Omega f), \quad \forall f \in \text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{g}), \theta \in \text{Hom}(\wedge^{n-1} \mathfrak{g}, \mathfrak{g}), \quad (19)$$

where  $\delta$  is given by (18) and  $\Omega : \text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{g}) \rightarrow \text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{g})$  is defined by

$$(\Omega f)(x_1, \dots, x_n) = (-1)^n \left( f(Tx_1, \dots, Tx_n) - \sum_{i=1}^n Tf(Tx_1, \dots, Tx_{i-1}, x_i, Tx_{i+1}, \dots, Tx_n) \right).$$

**Theorem 4.6.** *The map  $D_{\text{RB}}$  is a coboundary operator, that is,  $D_{\text{RB}} \circ D_{\text{RB}} = 0$ .*

*Proof.* One has

$$D_{\text{RB}} \circ D_{\text{RB}} = \mathfrak{p} \circ \mathcal{D} \circ \mathfrak{i} \circ \mathfrak{p} \circ \mathcal{D} \circ \mathfrak{i} = \mathfrak{p} \circ \mathcal{D} \circ \mathcal{D} \circ \mathfrak{i} = 0,$$

which finishes the proof.  $\square$

**Definition 4.7.** Let  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, T)$  be a RB Lie algebra. The cohomology of the cochain complex  $(\bigoplus_{n=0}^{+\infty} \mathfrak{C}_{\text{RB}}^n(\mathfrak{g}, T), D_{\text{RB}})$  is taken to be the *cohomology of the RB Lie algebra*  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, T)$ . Denote the  $n$ th cohomology group by  $\mathcal{H}_{\text{RB}}^n(\mathfrak{g}, T)$ .

### 4.2 | Cohomology of triangular Lie bialgebras

In this subsection, all vector spaces are assumed to be finite dimensional. We survey the cohomology of triangular Lie bialgebras, which is defined with the help of the general cohomological framework for relative RB Lie algebras.

Recall that a Lie bialgebra is a vector space  $\mathfrak{g}$  equipped with a Lie algebra structure  $[\cdot, \cdot]_{\mathfrak{g}} : \wedge^2 \mathfrak{g} \rightarrow \mathfrak{g}$  and a Lie coalgebra structure  $\delta : \mathfrak{g} \rightarrow \wedge^2 \mathfrak{g}$  such that  $\delta$  is a 1-cocycle on  $\mathfrak{g}$  with coefficients in  $\wedge^2 \mathfrak{g}$ . The Lie bracket  $[\cdot, \cdot]_{\mathfrak{g}}$  in a Lie algebra  $\mathfrak{g}$  naturally extends to the *Schouten-Nijenhuis bracket*  $[\cdot, \cdot]_{SN}$  on  $\wedge^* \mathfrak{g} = \bigoplus_{k \geq 0} \wedge^{k+1} \mathfrak{g}$ . More precisely, one has

$$[x_1 \wedge \dots \wedge x_p, y_1 \wedge \dots \wedge y_q]_{SN} = \sum_{\substack{1 \leq i \leq p \\ 1 \leq j \leq q}} (-1)^{i+j} [x_i, y_j]_{\mathfrak{g}} \wedge x_1 \wedge \dots \wedge \hat{x}_i \wedge \dots \wedge x_p \wedge y_1 \wedge \dots \wedge \hat{y}_j \wedge \dots \wedge y_q.$$

An element  $r \in \wedge^2 \mathfrak{g}$  is called a *skew-symmetric r-matrix* [50] if it satisfies the *classical Yang-Baxter equation*  $[r, r]_{SN} = 0$ . It is well known [36] that  $r$  satisfies the classical Yang-Baxter equation if and only if  $r^{\sharp}$  is a relative RB operator on  $\mathfrak{g}$  with respect to the coadjoint representation, where  $r^{\sharp} : \mathfrak{g}^* \rightarrow \mathfrak{g}$  is defined by  $\langle r^{\sharp}(\xi), \eta \rangle = \langle r, \xi \wedge \eta \rangle$  for all  $\xi, \eta \in \mathfrak{g}^*$ .

Let  $r$  be a skew-symmetric  $r$ -matrix. Define  $\delta_r : \mathfrak{g} \rightarrow \wedge^2 \mathfrak{g}$  by  $\delta_r(x) = [x, r]_{SN}$ , for all  $x \in \mathfrak{g}$ . Then  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \delta_r)$  is a Lie bialgebra, which is called a *triangular Lie bialgebra*. From now on, denote a triangular Lie bialgebra by  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, r)$ .

Let  $\mathfrak{g}$  be a Lie algebra and  $r \in \wedge^2 \mathfrak{g}$  a skew-symmetric  $r$ -matrix. Define the set of 0-cochains and 1-cochains to be zero and define the set of  $k$ -cochains to be  $\wedge^k \mathfrak{g}$ . Define  $d_r : \wedge^k \mathfrak{g} \rightarrow \wedge^{k+1} \mathfrak{g}$  by

$$d_r \chi = [r, \chi]_{SN}, \quad \forall \chi \in \wedge^k \mathfrak{g}. \tag{20}$$

Then  $d_r^2 = 0$ . Denote by  $\mathcal{H}^k(r)$  the corresponding  $k$ -th cohomology group, called the *kth cohomology group of the skew-symmetric r-matrix r*.

For any  $k \geq 1$ , define  $\Psi : \wedge^{k+1} \mathfrak{g} \rightarrow \text{Hom}(\wedge^k \mathfrak{g}^*, \mathfrak{g})$

$$\langle \Psi(\chi)(\xi_1, \dots, \xi_k), \xi_{k+1} \rangle = \langle \chi, \xi_1 \wedge \dots \wedge \xi_k \wedge \xi_{k+1} \rangle, \quad \forall \chi \in \wedge^{k+1} \mathfrak{g}, \xi_1, \dots, \xi_{k+1} \in \mathfrak{g}^*. \tag{21}$$

By [54, Theorem 7.7], we have

$$\Psi(d_r \chi) = \delta(\Psi(\chi)), \quad \forall \chi \in \wedge^k \mathfrak{g}. \tag{22}$$

Thus  $(\text{Im}(\Psi), \delta)$  is a subcomplex of the cochain complex  $(\bigoplus_k \mathfrak{C}^k(r^{\sharp}), \delta)$  associated to the relative RB operator  $r^{\sharp}$ , where  $\text{Im}(\Psi) := \bigoplus_k \{\Psi(\chi) | \forall \chi \in \wedge^k \mathfrak{g}\}$  and  $\delta$  is the coboundary operator given by (18) for the relative RB operator  $r^{\sharp}$ .

In the following, we survey the cohomology of a triangular Lie bialgebra  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, r)$ . Define the set of 0-cochains  $\mathfrak{C}_{TLB}^0(\mathfrak{g}, r)$  to be 0, and define the set of 1-cochains to be  $\mathfrak{C}_{TLB}^1(\mathfrak{g}, r) := \text{Hom}(\mathfrak{g}, \mathfrak{g})$ . For  $n \geq 2$ , define the space of  $n$ -cochains  $\mathfrak{C}_{TLB}^n(\mathfrak{g}, r)$  by

$$\mathfrak{C}_{TLB}^n(\mathfrak{g}, r) := \text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{g}) \oplus \wedge^n \mathfrak{g}.$$

Define the embedding  $\mathbf{i} : \mathfrak{C}_{TLB}^n(\mathfrak{g}, r) \rightarrow \mathfrak{C}^n(\mathfrak{g}, \text{ad}^*, r^{\sharp}) = \text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{g}) \oplus \text{Hom}(\wedge^{n-1} \mathfrak{g} \otimes \mathfrak{g}^*, \mathfrak{g}^*) \oplus \text{Hom}(\wedge^{n-1} \mathfrak{g}^*, \mathfrak{g})$  by

$$\mathbf{i}(f, \chi) = (f, f^*, \Psi(\chi)), \quad \forall f \in \text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{g}), \chi \in \wedge^n \mathfrak{g},$$

where  $f^* \in \text{Hom}(\wedge^{n-1} \mathfrak{g} \otimes \mathfrak{g}^*, \mathfrak{g}^*)$  is defined by

$$\langle f^*(x_1, \dots, x_{n-1}, \xi), x_n \rangle = -\langle \xi, f(x_1, \dots, x_{n-1}, x_n) \rangle. \tag{23}$$

Denote by  $\text{Im}^n(\mathfrak{i})$  the image of  $\mathfrak{i}$ , that is,  $\text{Im}^n(\mathfrak{i}) := \{\mathfrak{i}(f, \chi) \mid \forall (f, \chi) \in \mathfrak{C}_{\text{TLB}}^n(\mathfrak{g}, r)\}$ . It was proved in [39] that  $(\oplus_n \text{Im}^n(\mathfrak{i}), D)$  is a subcomplex of the cochain complex  $(\mathfrak{C}^n(\mathfrak{g}, \text{ad}^*, r^\sharp), D)$  associated to the relative RB Lie algebra  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \text{ad}^*, r^\sharp)$ .

Define the projection  $\mathfrak{p} : \text{Im}^n(\mathfrak{i}) \rightarrow \mathfrak{C}_{\text{TLB}}^n(\mathfrak{g}, r)$  by

$$\mathfrak{p}(f, f^*, \theta) = (f, \theta^b), \quad \forall f \in \text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{g}), \theta \in \{\Psi(\chi) \mid \forall \chi \in \wedge^n \mathfrak{g}\},$$

where  $\theta^b \in \wedge^n \mathfrak{g}$  is defined by  $\langle \theta^b, \xi_1 \wedge \dots \wedge \xi_n \rangle = \langle \theta(\xi_1, \dots, \xi_{n-1}), \xi_n \rangle$ . Define the coboundary operator  $D_{\text{TLB}} : \mathfrak{C}_{\text{TLB}}^n(\mathfrak{g}, r) \rightarrow \mathfrak{C}_{\text{TLB}}^{n+1}(\mathfrak{g}, r)$  for a triangular Lie bialgebra by

$$D_{\text{TLB}} = \mathfrak{p} \circ D \circ \mathfrak{i}.$$

**Theorem 4.8.** *The map  $D_{\text{TLB}}$  is a coboundary operator, that is,  $D_{\text{TLB}} \circ D_{\text{TLB}} = 0$ .*

*Proof.* Since  $\mathfrak{i} \circ \mathfrak{p} = \text{Id}$  when restricting on the image of  $\mathfrak{i}$ , one has

$$D_{\text{TLB}} \circ D_{\text{TLB}} = \mathfrak{p} \circ D \circ \mathfrak{i} \circ \mathfrak{p} \circ D \circ \mathfrak{i} = \mathfrak{p} \circ D \circ D \circ \mathfrak{i} = 0,$$

which finishes the proof. □

**Definition 4.9.** Let  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, r)$  be a triangular Lie bialgebra. The cohomology of the cochain complex  $(\oplus_{n=0}^{+\infty} \mathfrak{C}_{\text{TLB}}^n(\mathfrak{g}, r), D_{\text{TLB}})$  is called the *cohomology of the triangular Lie bialgebra*  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, r)$ . Denote the  $n$ th cohomology group by  $\mathcal{H}_{\text{TLB}}^n(\mathfrak{g}, r)$ .

Now we give the precise formula for the coboundary operator  $D_{\text{TLB}}$ . By the definition of  $\mathfrak{i}$ ,  $\mathfrak{p}$ ,  $D$  and (22), one has

$$D_{\text{TLB}}(f, \chi) = (d_{\text{CE}}f, \Theta f + d_r \chi), \quad \forall f \in \text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{g}), \chi \in \wedge^n \mathfrak{g}, \tag{24}$$

where  $d_r$  is given by (20) and  $\Theta : \text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{g}) \rightarrow \wedge^{n+1} \mathfrak{g}$  is defined by  $\Theta f = \Psi^{-1}(h_{r^\sharp}(f, f^*))$ . More precisely,

$$\langle \Theta f, \xi_1 \wedge \dots \wedge \xi_{n+1} \rangle = \sum_{i=1}^{n+1} (-1)^{i+1} \langle \xi_i, f(r^\sharp(\xi_1), \dots, r^\sharp(\xi_{i-1}), r^\sharp(\xi_{i+1}), \dots, r^\sharp(\xi_{n+1})) \rangle, \tag{25}$$

for all  $f \in \text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{g})$  and  $\xi_1, \dots, \xi_{n+1} \in \mathfrak{g}^*$ .

*Remark 4.10.* One can use the cohomology theory developed here to study infinitesimal deformations. More precisely, the cohomology groups  $\mathcal{H}^2(\mathfrak{g}, \rho, T)$ ,  $\mathcal{H}_{\text{RB}}^2(\mathfrak{g}, T)$ ,  $\mathcal{H}_{\text{TLB}}^2(\mathfrak{g}, r)$  classify infinitesimal deformations of the relative RB Lie algebra  $(\mathfrak{g}, \rho, T)$ , the RB Lie algebra  $(\mathfrak{g}, T)$  and the triangular Lie bialgebra  $(\mathfrak{g}, r)$ , respectively.

### 5 | HOMOTOPIES OF RELATIVE ROTA-BAXTER LIE ALGEBRAS

In this section, we survey the notion of a homotopy relative RB Lie algebra, which consists of an  $L_\infty$ -algebra, its representation and a homotopy relative RB operator. Homotopy relative RB operators can be characterized as Maurer–Cartan elements in a certain  $L_\infty$ -algebra.

Denote by  $\text{Hom}^n(\bar{S}(V), V)$  the space of degree  $n$  linear maps from the graded vector space  $\bar{S}(V) = \bigoplus_{i=1}^{+\infty} S^i(V)$  to the  $\mathbb{Z}$ -graded vector space  $V$ . Obviously, an element  $f \in \text{Hom}^n(\bar{S}(V), V)$  is the sum of  $f_i : S^i(V) \rightarrow V$ . We will write  $f = \sum_{i=1}^{+\infty} f_i$ . Set  $C^n(V, V) := \text{Hom}^n(\bar{S}(V), V)$  and  $C^*(V, V) := \bigoplus_{n \in \mathbb{Z}} C^n(V, V)$ . As the graded version of the Nijenhuis–Richardson bracket given in [46, 47], the *graded Nijenhuis–Richardson bracket*  $[\cdot, \cdot]_{\text{NR}}$  on the graded vector space  $C^*(V, V)$  is given by

$$[f, g]_{\text{NR}} := f \bar{o} g - (-1)^{mn} g \bar{o} f, \quad \forall f = \sum_{i=1}^{+\infty} f_i \in C^m(V, V), \quad g = \sum_{j=1}^{+\infty} g_j \in C^n(V, V), \quad (26)$$

where  $f \bar{o} g \in C^{m+n}(V, V)$  is defined by

$$f \bar{o} g = \left( \sum_{i=1}^{+\infty} f_i \right) \bar{o} \left( \sum_{j=1}^{+\infty} g_j \right) := \sum_{k=1}^{+\infty} \left( \sum_{i+j=k+1} f_i \bar{o} g_j \right), \quad (27)$$

while  $f_i \bar{o} g_j \in \text{Hom}(S^{i+j-1}(V), V)$  is defined by

$$(f_i \bar{o} g_j)(v_1, \dots, v_{i+j-1}) := \sum_{\sigma \in \mathbb{S}_{(j, i-1)}} \varepsilon(\sigma) f_i(g_j(v_{\sigma(1)}, \dots, v_{\sigma(j)}, v_{\sigma(j+1)}, \dots, v_{\sigma(i+j-1)})). \quad (28)$$

The following result is well known and, in fact, can be taken as a definition of an  $L_\infty$ -algebra.

**Theorem 5.1.** *With the above notation,  $(C^*(V, V), [\cdot, \cdot]_{\text{NR}})$  is a graded Lie algebra. Its Maurer–Cartan elements  $\sum_{k=1}^{+\infty} l_k$  are the  $L_\infty$ -algebra structures on  $V$ .*

**Definition 5.2** [38]. A *representation* of an  $L_\infty$ -algebra  $(\mathfrak{g}, \{l_k\}_{k=1}^{+\infty})$  on a graded vector space  $V$  consists of linear maps  $\rho_k : S^{k-1}(\mathfrak{g}) \otimes V \rightarrow V, k \geq 1$ , of degree 1 with the property that, for any homogeneous elements  $x_1, \dots, x_{n-1} \in \mathfrak{g}, v \in V$ , we have

$$\sum_{i=1}^{n-1} \sum_{\sigma \in \mathbb{S}_{(i, n-i-1)}} \varepsilon(\sigma) \rho_{n-i+1}(l_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}, x_{\sigma(i+1)}, \dots, x_{\sigma(n-1)}, v)) + \sum_{i=1}^n \sum_{\sigma \in \mathbb{S}_{(n-i, i-1)}} \varepsilon(\sigma) (-1)^{x_{\sigma(1)} + \dots + x_{\sigma(n-i)}} \rho_{n-i+1}(x_{\sigma(1)}, \dots, x_{\sigma(n-i)}, \rho_i(x_{\sigma(n-i+1)}, \dots, x_{\sigma(n-1)}, v)) = 0.$$

Let  $(V, \{\rho_k\}_{k=1}^{+\infty})$  be a representation of an  $L_\infty$ -algebra  $(\mathfrak{g}, \{l_k\}_{k=1}^{+\infty})$ . There is an  $L_\infty$ -algebra structure on the direct sum  $\mathfrak{g} \oplus V$  given by

$$l_k((x_1, v_1), \dots, (x_k, v_k)) := \left( l_k(x_1, \dots, x_k), \sum_{i=1}^k (-1)^{x_i(x_{i+1} + \dots + x_k)} \rho_k(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k, v_i) \right).$$

This  $L_\infty$ -algebra is called the *semidirect product* of the  $L_\infty$ -algebra  $(\mathfrak{g}, \{l_k\}_{k=1}^{+\infty})$  and  $(V, \{\rho_k\}_{k=1}^{+\infty})$ , and denoted by  $\mathfrak{g} \ltimes_\rho V$ .

**Definition 5.3.**

- (i) Let  $(V, \{\rho_k\}_{k=1}^{+\infty})$  be a representation of an  $L_\infty$ -algebra  $(\mathfrak{g}, \{l_k\}_{k=1}^{+\infty})$ . A degree 0 element  $T = \sum_{k=1}^{+\infty} T_k \in \text{Hom}(\tilde{S}(V), \mathfrak{g})$  with  $T_k \in \text{Hom}(S^k(V), \mathfrak{g})$  is called a *homotopy relative RB operator* on an  $L_\infty$ -algebra  $(\mathfrak{g}, \{l_k\}_{k=1}^{+\infty})$  with respect to the representation  $(V, \{\rho_k\}_{k=1}^{+\infty})$  if the following equalities hold for all  $p \geq 1$  and all homogeneous elements  $v_1, \dots, v_p \in V$ ,

$$\sum_{\substack{k_1+\dots+k_m=t \\ 1 \leq i \leq p-1}} \sum_{\sigma \in \mathbb{S}_{(k_1, \dots, k_{m-1}, p-1-t)}} \frac{\varepsilon(\sigma)}{m!} \cdot T_{p-t}(\rho_{m+1}(T_{k_1}(v_{\sigma(1)}, \dots, v_{\sigma(k_1)}), \dots, T_{k_m}(v_{\sigma(k_1+\dots+k_{m-1}+1)}, \dots, v_{\sigma(t)}), v_{\sigma(t+1)}, v_{\sigma(t+2)}, \dots, v_{\sigma(p)})) \\ = \sum_{k_1+\dots+k_n=p} \sum_{\sigma \in \mathbb{S}_{(k_1, \dots, k_n)}} \frac{\varepsilon(\sigma)}{n!} l_n(T_{k_1}(v_{\sigma(1)}, \dots, v_{\sigma(k_1)}), \dots, T_{k_n}(v_{\sigma(k_1+\dots+k_{n-1}+1)}, \dots, v_{\sigma(p)})).$$

- (ii) A *homotopy relative RB Lie algebra* is a triple  $((\mathfrak{g}, \{l_k\}_{k=1}^{+\infty}), \{\rho_k\}_{k=1}^{+\infty}, \{T_k\}_{k=1}^{+\infty})$ , where  $(\mathfrak{g}, \{l_k\}_{k=1}^{+\infty})$  is an  $L_\infty$ -algebra,  $(V, \{\rho_k\}_{k=1}^{+\infty})$  is a representation of  $\mathfrak{g}$  on a graded vector space  $V$  and  $T = \sum_{k=1}^{+\infty} T_k \in \text{Hom}(\tilde{S}(V), \mathfrak{g})$  is a homotopy relative RB operator.

A homotopy relative RB operator on an  $L_\infty$ -algebra is a generalization of an  $\mathcal{O}$ -operator on a Lie 2-algebra introduced in [51].

A representation of an  $L_\infty$ -algebra will give rise to a V-data as well as an  $L_\infty$ -algebra that characterize homotopy relative RB operators as MC elements.

**Proposition 5.4.** *Let  $(\mathfrak{g}, \{l_k\}_{k=1}^{+\infty})$  be an  $L_\infty$ -algebra and  $(V, \{\rho_k\}_{k=1}^{+\infty})$  a representation of  $(\mathfrak{g}, \{l_k\}_{k=1}^{+\infty})$ . Then the following quadruple forms a V-data.*

- The graded Lie algebra  $(L, [\cdot, \cdot])$  is given by  $(C^*(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V), [\cdot, \cdot]_{\text{NR}})$ .
- The abelian graded Lie subalgebra  $H$  is given by  $H := \bigoplus_{n \in \mathbb{Z}} \text{Hom}^n(\tilde{S}(V), \mathfrak{g})$ .
- $P : L \rightarrow L$  is the projection onto the subspace  $H$ .
- $\Delta = \sum_{k=1}^{+\infty} (l_k + \rho_k)$ .

Consequently,  $(H, \{\mathfrak{l}_k\}_{k=1}^{+\infty})$  is an  $L_\infty$ -algebra, where  $\mathfrak{l}_k$  is given by (8).

*Proof.* By Theorem 5.1,  $(C^*(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V), [\cdot, \cdot]_{\text{NR}})$  is a graded Lie algebra. Moreover, by (28),  $\text{Im} P = H$  is an abelian graded Lie subalgebra and  $\ker P$  is a graded Lie subalgebra. Since  $\Delta = \sum_{k=1}^{+\infty} (l_k + \rho_k)$  is the semidirect product  $L_\infty$ -algebra structure on  $\mathfrak{g} \oplus V$ , one has  $[\Delta, \Delta]_{\text{NR}} = 0$  and  $P(\Delta) = 0$ . Thus  $(L, H, P, \Delta)$  is a V-data. Hence by Theorem 2.10, one obtains the higher derived brackets  $\{\mathfrak{l}_k\}_{k=1}^{+\infty}$  on the abelian graded Lie subalgebra  $H$ . □

**Theorem 5.5.** *With the above notation, a degree 0 element  $T = \sum_{k=1}^{+\infty} T_k \in \text{Hom}(\tilde{S}(V), \mathfrak{g})$  is a homotopy relative RB operator on  $(\mathfrak{g}, \{l_k\}_{k=1}^{+\infty})$  with respect to the representation  $(V, \{\rho_k\}_{k=1}^{+\infty})$  if and only if  $T = \sum_{k=1}^{+\infty} T_k$  is a Maurer–Cartan element of the  $L_\infty$ -algebra  $(H, \{\mathfrak{l}_k\}_{k=1}^{+\infty})$ .*

*Proof.* See the proof of [39, Theorem 5.10]. □

A homotopy relative RB operator naturally gives rise to an  $L_\infty$ -algebra structure on  $V$ .

**Proposition 5.6.** *Let  $T = \sum_{k=1}^{+\infty} T_k \in \text{Hom}(\tilde{S}(V), \mathfrak{g})$  be a homotopy relative RB operator on  $(\mathfrak{g}, \{l_k\}_{k=1}^{+\infty})$  with respect to the representation  $(V, \{\rho_k\}_{k=1}^{+\infty})$ .*

- (i)  $e^{[\cdot, T]_{\text{NR}}}(\sum_{k=1}^{+\infty} (l_k + \rho_k))$  is a Maurer–Cartan element of the graded Lie algebra  $(C^*(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V), [\cdot, \cdot]_{\text{NR}})$ .
- (ii) There is an  $L_\infty$ -algebra structure on  $V$  given by

$$l_{t+1}(v_1, \dots, v_{t+1}) = \sum_{k_1 + \dots + k_m = t} \sum_{\sigma \in \mathbb{S}_{(k_1, \dots, k_m, 1)}} \frac{\varepsilon(\sigma)}{m!} \rho_{m+1} \left( T_{k_1} \left( v_{\sigma(1)}, \dots, v_{\sigma(k_1)} \right), \dots, T_{k_m} \left( v_{\sigma(k_1 + \dots + k_{m-1} + 1)}, \dots, v_{\sigma(t)} \right), v_{\sigma(t+1)} \right);$$

- (iii)  $T$  is an  $L_\infty$ -algebra homomorphism from the  $L_\infty$ -algebra  $(V, \{l_k\}_{k=1}^{+\infty})$  to  $(\mathfrak{g}, \{l_k\}_{k=1}^{+\infty})$ .

*Proof.* See the proof of [39, Proposition 5.11]. □

*Remark 5.7.* In the classical case, a relative RB operator induces a pre-Lie algebra [2]. Now a homotopy relative RB operator also induces a pre-Lie $_\infty$ -algebra, which was introduced in [9]. See [39, Section 5.2] for details.

*Remark 5.8.* Dotsenko and Khoroshkin studied the homotopy of RB operators on associative algebras in [17] using the operadic approach, and noted that ‘in general compact formulas are yet to be found’. For RB Lie algebras, one encounters a similarly challenging situation. Nevertheless, we use the controlling algebra and Maurer–Cartan approach to give the concrete formulas of homotopy RB operators, which could provide some guidance for future studies.

## 6 | DEFORMATIONS, COHOMOLOGIES AND HOMOTOPIES OF RB LIE ALGEBRAS OF NONZERO WEIGHTS

Note that there is a more general notion of relative RB Lie algebras of weight  $\lambda$ , and the relative RB Lie algebras studied in previous sections are of weight 0. In this section, we briefly review recent developments of deformations, cohomologies and homotopies of RB Lie algebras of weight  $\lambda$ .

Let  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$  and  $(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}})$  be Lie algebras. Let  $\phi : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{h})$  be a Lie algebra homomorphism, which is called an *action* of  $\mathfrak{g}$  on  $\mathfrak{h}$ .

**Definition 6.1.** Let  $\phi : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{h})$  be an action of a Lie algebra  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$  on a Lie algebra  $(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}})$ . A linear map  $T : \mathfrak{h} \rightarrow \mathfrak{g}$  is called a *relative RB operator* of weight  $\lambda$  on  $\mathfrak{g}$  with respect to  $(\mathfrak{h}; \phi)$  if

$$[T(u), T(v)]_{\mathfrak{g}} = T(\phi(T(u))v - \phi(T(v))u + \lambda[u, v]_{\mathfrak{h}}), \quad \forall u, v \in \mathfrak{h}. \tag{29}$$

In particular, if  $\mathfrak{g} = \mathfrak{h}$  and the action is the adjoint representation of  $\mathfrak{g}$  on itself, then  $T$  is called a RB operator of weight  $\lambda$ . A RB Lie algebra of weight  $\lambda$  is a Lie algebra equipped with a RB operator of weight  $\lambda$ .

In [56], the notion of a homotopy relative RB operator of weight  $\lambda$  on a symmetric Lie algebra was introduced using the controlling algebra approach, which was the first step toward the definition of a homotopy relative RB operator of weight  $\lambda$  on an  $L_\infty$ -algebra. As a byproduct, the controlling algebra of relative RB operators of weight  $\lambda$  was given in [56, Corollary 2.17]. More precisely, let  $\phi : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{h})$  be an action of a Lie algebra  $\mathfrak{g}$  on a Lie algebra  $\mathfrak{h}$ . Then  $(\bigoplus_{n=0}^{+\infty} \text{Hom}(\wedge^n \mathfrak{h}, \mathfrak{g}), \llbracket \cdot, \cdot \rrbracket, d)$  is a differential graded Lie algebra, where the differential  $d : \text{Hom}(\wedge^n \mathfrak{h}, \mathfrak{g}) \rightarrow \text{Hom}(\wedge^{n+1} \mathfrak{h}, \mathfrak{g})$  is given by

$$\begin{aligned} (dg)(v_1, \dots, v_{n+1}) \\ = \sum_{1 \leq i < j \leq n+1} (-1)^{n+i+j-1} g(\lambda[v_i, v_j]_{\mathfrak{h}}, v_1, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_{n+1}), \end{aligned}$$

for all  $g \in \text{Hom}(\wedge^n \mathfrak{h}, \mathfrak{g})$  and  $v_1, \dots, v_{n+1} \in \mathfrak{h}$ , and the graded Lie bracket

$$\llbracket \cdot, \cdot \rrbracket : \text{Hom}(\wedge^n \mathfrak{h}, \mathfrak{g}) \times \text{Hom}(\wedge^m \mathfrak{h}, \mathfrak{g}) \longrightarrow \text{Hom}(\wedge^{m+n} \mathfrak{h}, \mathfrak{g})$$

is given by

$$\begin{aligned} \llbracket g_1, g_2 \rrbracket(v_1, \dots, v_{m+n}) \\ = \sum_{\sigma \in \mathbb{S}_{(m,1,n-1)}} (-1)^{1+\sigma} g_1(\phi(g_2(v_{\sigma(1)}, \dots, v_{\sigma(m)}))v_{\sigma(m+1)}, v_{\sigma(m+2)}, \dots, v_{\sigma(m+n)}) \\ + \sum_{\sigma \in \mathbb{S}_{(n,1,m-1)}} (-1)^{mn+\sigma} g_2(\phi(g_1(v_{\sigma(1)}, \dots, v_{\sigma(n)}))v_{\sigma(n+1)}, v_{\sigma(n+2)}, \dots, v_{\sigma(m+n)}) \\ + \sum_{\sigma \in \mathbb{S}_{(n,m)}} (-1)^{1+mn+\sigma} \llbracket g_1(v_{\sigma(1)}, \dots, v_{\sigma(n)}), g_2(v_{\sigma(n+1)}, \dots, v_{\sigma(m+n)}) \rrbracket_{\mathfrak{g}}, \end{aligned}$$

for all  $g_1 \in \text{Hom}(\wedge^n \mathfrak{h}, \mathfrak{g})$ ,  $g_2 \in \text{Hom}(\wedge^m \mathfrak{h}, \mathfrak{g})$  and  $v_1, \dots, v_{m+n} \in \mathfrak{h}$ . Moreover, a linear map  $T : \mathfrak{h} \rightarrow \mathfrak{g}$  is a relative RB operator of weight  $\lambda$  on  $\mathfrak{g}$  with respect to the action  $\phi$  if and only if  $T$  is a Maurer–Cartan element of the above differential graded Lie algebra.

As soon as one has the above controlling algebra of relative RB operators of weight  $\lambda$ , one can obtain immediately the differential graded Lie algebra that controls deformations of a relative RB operator  $T$  of weight  $\lambda$  using the twisted differential  $d_T := d + \llbracket T, \cdot \rrbracket$ . Meanwhile, one can also define the cohomology of a relative RB operators  $T$  of weight  $\lambda$  using the twisted differential  $d_T$ . See [12] for details. Note that in [12], the controlling algebra of relative RB operators of weight  $\lambda$  on associative algebras were constructed parallelly.

Before [12], the cohomologies of relative RB operators of weight 1 on Lie algebras were given in [33] using a different approach. Namely a relative RB operator  $T : \mathfrak{h} \rightarrow \mathfrak{g}$  of weight 1 induces a new Lie algebra  $(\mathfrak{h}, [\cdot, \cdot]_T)$  and a representation  $\theta : \mathfrak{h} \rightarrow \mathfrak{gl}(\mathfrak{g})$  of  $(\mathfrak{h}, [\cdot, \cdot]_T)$  on the vector space  $\mathfrak{g}$ , where  $[\cdot, \cdot]_T$  and  $\theta$  are given by

$$[u, v]_T = \phi(T(u))v - \phi(T(v))u + [u, v]_{\mathfrak{h}}, \quad (30)$$

$$\theta(u)x = T(\phi(x)u) + [T(u), x]_{\mathfrak{g}}. \quad (31)$$

The Chevalley–Eilenberg cohomology of the Lie algebra  $(\mathfrak{h}, [\cdot, \cdot]_T)$  with coefficients in the representation  $\theta$  is taken to be the cohomology of the relative RB operator  $T$ . In the same paper, the cohomologies of relative RB operators of weight 1 on Lie groups were also introduced and the classical Van Est map was extended to the context of cohomologies of relative RB operators on Lie groups and Lie algebras.

Using Voronov’s higher derived brackets [57], Caseiro and Nunes da Costa succeed in defining homotopy relative RB operators of weight 1 on  $L_\infty$ -algebras with respect to  $L_\infty$ -actions [8], generalized some results in [39] and [56]. In the associative algebra context, Wang and Zhou studied homotopy RB associative algebras of weight  $\lambda$  in [58] and showed that the operad governing homotopy RB associative algebras is a minimal model of the operad of RB associative algebras. The cohomologies of RB associative algebras of weight  $\lambda$  were also given in the same paper. Parallely, the cohomologies of RB Lie algebras of weight  $\lambda$  were given in [13] by which abelian extensions and formal deformations are studied.

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