Journal of Physics: Conference Series **343** (2012) 012057 doi:10.1088/1742-6596/343/1/012057

Vortex-type solutions in ABJM theory

Chanju Kim

Institute for the Early Universe and Department of Physics, Ewha Womans University, Seoul 120-750, Korea

E-mail: cjkim@ewha.ac.kr

Abstract. We review our recent investigations on vortex type-solutions in ABJM theory without and with mass deformation. By imposing suitable supersymmetric conditions, we obtain vortex-type half-BPS equations and also its energy bound. For the undeformed ABJM theory, the resulting half-BPS equation is the same as that in SYM theory. For the mass-deformed ABJM theory, the half-BPS equations for U(2)×U(2) case reduce to the vortex equation in Maxwell-Higgs theory, which supports static regular multi-vortex solutions. In U(N)×U(N) case with $N > 2$ we obtain the nonabelian vortex equation of Yang-Mills-Higgs theory with a suitable ansatz. We also discuss less supersymmetric cases.

1. Introduction

The Aharony-Bergman-Jafferis-Maldacena (ABJM) theory [1] is known to be the low-energy limit of world-volume theory of multiple M2-branes. The ABJM theory is given in the basis of brane constructions and is described by (1+2)-dimensional $\mathcal{N}$=6 supersymmetric Chern-Simons-matter theories with U(N)×U(N) or SU(N)×SU(N) gauge group and a sextic scalar potential. In large N limit, the ABJM theory is dual to M-theory on $\mathrm{AdS}_4 \times \mathrm{S}^7/\mathbb{Z}_k$, where k is related with the discrete level of Chern-Simons term.

Chern-Simons-Higgs theory with a sextic scalar potential has been first introduced in order to saturate the BPS bound for the static multi-BPS vortex solutions [2]. It also arises in the supersymmetric abelian Chern-Simons-Higgs theories [3]. The sextic potential of the theory has both the symmetric and broken vacua, which allows a rich spectrum of solition solutions. In addition to the topologically stable multi-BPS vortices and domain walls, marginally stable nontopological solitons (or Q-balls) and nontopological vortices (or Q-vortices) also exist [4]. Extension to U(1)×U(1) gauge group [5] and nonabelian gauge group [6] has also been made. Therefore, in the scheme of (1+2)-dimensional quantum field theories, the mass-deformed ABJM theories may be understood as complicated Chern-Simons-Higgs theories with extended supersymmetries and the undeformed ABJM theories as their superconformal limit.

In this paper, we review our recent works on vortex-type solutions in ABJM theory [7, 8]. We will first consider the simplest $\mathcal{N} = 2$ U(1) Chern-Simons-Higgs theory by focusing on obtaining BPS bound and the Bogomoln'yi equation. This will serve as a prototype of later analysis. Then we discuss the vortex-type half-BPS objects of the ABJM theory without and with mass deformation. We also comment on less sypersymmetric BPS cases.

2. Brief review of U(1) Chern-Simons-Higgs theory

In this section we will briefly review U(1) Chern-Simons-Higgs theory and its generalization to $\mathcal{N}=2$ supersymmetric theory. The result will be used later when analyzing the ABJM theory. The pure Chern-Simons Higgs theory is described by the Lagrangian

$$\mathcal{L}_{\rm B} = -|D_\mu\phi|^2 + \frac{\kappa}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho - V(\phi)\,, \tag{1}$$

where $D_\mu\phi = (\partial_\mu + iqA_\mu)\phi$. A characteristic feature of the Chern-Simons theory is that any excitations with the electric charge must also carry magnetic flux. This can be seen by considering the Gauss constraint,

$$-\kappa B = -\kappa(\partial_1 A_2 - \partial_2 A_1) = qJ_0, \tag{2}$$

where J_μ is the conserved current $J_\mu = i(\phi^* D_\mu\phi - \phi D_\mu\phi^*)$.

The energy functional is given by

$$E = \int d^2x \left[|D_0\phi|^2 + |D_i\phi|^2 + V(\phi)\right]\,. \tag{3}$$

After some reshuffling, we can rewrite the energy as

$$\begin{aligned} E = \int d^2x \Bigg\{ &(\partial_0|\phi|)^2 + |(D_1 \pm iD_2)\phi|^2 + \left|\frac{\kappa B}{2q\phi} \pm \frac{q^2}{\kappa}\phi^*(v^2 - |\phi|^2)\right|^2 \\ &+ \left[V - \frac{q^4}{\kappa^2}|\phi|^2(|\phi|^2 - v^2)^2\right]\Bigg\} \pm qv^2\Phi\,, \end{aligned} \tag{4}$$

where Φ is the magnetic flux and we have eliminated A_0 by use of the Gauss law. Hence with the choice,

$$V(\phi) = \frac{q^4}{\kappa^2}|\phi|^2(v^2 - |\phi|^2)^2\,, \tag{5}$$

we have a lower bound on the energy,

$$E \geq qv^2|\Phi|. \tag{6}$$

The equality is saturated by the static fields obeying the Bogomoln'yi equations [2],

$$\begin{aligned} &(D_1 \pm iD_2)\phi = 0\,, \\ &qB = \pm\frac{m_H^2}{2}\frac{|\phi|^2}{v^2}\left(1 - \frac{|\phi|^2}{v^2}\right)\,, \end{aligned} \tag{7}$$

where $m_H = 2q^2v^2/|\kappa|$ is the mass of the Higgs field and the upper (lower) sign corresponds to positive (negative) values of Φ. Note that the right hand side of the second equation has additional multiplicative $|\phi|^2$ factor compared with that of Maxwell-Higgs case. A consequence of this difference is that the magnetic field vanishes at vortex points and hence the magnetic field is ring-shaped for $n \neq 0$.

The theory (1) with the potential (5) is actually a bosonic part of the $\mathcal{N}=2$ supersymmetric theory

$$\mathcal{L} = \mathcal{L}_{\rm B} + i\bar{\psi}\gamma^\mu D_\mu\psi + \frac{q^2}{\kappa}(3|\phi|^2 - v^2)\bar{\psi}\psi, \tag{8}$$

which is invariant under the supersymmetric variation

$$\begin{aligned}\delta\phi &= \bar{\eta}\psi,\\ \delta A_\mu &= i\frac{q}{\kappa}(\bar{\psi}\gamma_\mu\eta\phi + \bar{\eta}\gamma_\mu\psi\phi^*),\\ \delta\psi &= \gamma^\mu\eta D_\mu\phi + \frac{q^2}{\kappa}\eta\phi(|\phi|^2 - v^2),\end{aligned} \tag{9}$$

where η is a complex spinor.

It is not difficult to show that requiring $\delta\psi = 0$ with the supersymmetric condition

$$\gamma^0\eta = \mp i\eta, \tag{10}$$

reproduces (7). Note that (10) breaks half of the supersymmetry. Therefore the solutions of (7) are half-BPS objects.

On substituting the first equation of (7) into the second equation, we obtain a single second order nonlinear differential equation,

$$\nabla^2 \ln\frac{|\phi|^2}{v^2} = -m_H^2\frac{|\phi|^2}{v^2}\left(1 - \frac{|\phi|^2}{v^2}\right) + 4\pi\sum_{p=1}^{n}\delta(\mathbf{x} - \mathbf{x}_p)\,, \tag{11}$$

where $\mathbf{x}_p$'s are zeros of ϕ and n, the number of zeros, is the vorticity. The boundary condition at infinity is determined from the finite energy condition, i.e.,

$$|\phi| \longrightarrow 0 \text{ or } v, \quad \text{as } r \longrightarrow \infty. \tag{12}$$

Therefore both the topological ($|\phi(\infty)| = v$) and the nontopological ($|\phi(\infty)| = 0$) soliton solutions are possible. A characteristic feature of the solutions is that, unlike those in the Maxwell-Higgs theory, they carry nonzero angular momenta. Suppose that the scalar field behaves $\phi \sim \frac{1}{r^\alpha}$ as $r \to \infty$ ($\alpha = 0$ for topological solitons). In the rotationally symmetric case, the angular momentum is calculated as $J = \frac{\pi\kappa}{q^2}(\alpha^2 - n^2)$. Thus the vortex solutions behave as anyons obeying fractional statistics.

3. ABJM theory and its vortex-type BPS solutions

The ABJM theory is an $\mathcal{N} = 6$ superconformal $\mathrm{U}(N) \times \mathrm{U}(N)$ Chern-Simons gauge theory with level $(k, -k)$, coupled to four complex scalars and four fermions in the bifundamental representation,

$$\begin{aligned}S_{\mathrm{ABJM}} = \int d^3x \Bigg\{ &\frac{k}{4\pi}\epsilon^{\mu\nu\lambda}\mathrm{tr}\left(A_\mu\partial_\nu A_\lambda + \frac{2i}{3}A_\mu A_\nu A_\lambda - \hat{A}_\mu\partial_\nu\hat{A}_\lambda - \frac{2i}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda\right)\\ &- \mathrm{tr}\big(D_\mu Y_A^\dagger D^\mu Y^A\big) + \mathrm{tr}\big(\psi^{A\dagger}i\gamma^\mu D_\mu\psi_A\big) - V_{\mathrm{ferm}} - V_0\Bigg\},\end{aligned} \tag{13}$$

where $A = 1, \ldots, 4$ and

$$D_\mu Y^A = \partial_\mu Y^A + iA_\mu Y^A - iY^A\hat{A}_\mu. \tag{14}$$

In the ABJM action (13), V_{ferm} is the Yukawa-type quartic-interaction term,

$$\begin{aligned}V_{\mathrm{ferm}} = \frac{2i\pi}{k}\,\mathrm{tr}\big(&Y_A^\dagger Y^A\psi^{B\dagger}\psi_B - Y^A Y_A^\dagger\psi_B\psi^{B\dagger} + 2Y^A Y_B^\dagger\psi_A\psi^{B\dagger} - 2Y_A^\dagger Y^B\psi^{A\dagger}\psi_B\\ &- \epsilon^{ABCD}Y_A^\dagger\psi_B Y_C^\dagger\psi_D + \epsilon_{ABCD}Y^A\psi^{B\dagger}Y^C\psi^{D\dagger}\big),\end{aligned} \tag{15}$$

and V_0 is the sixth-order scalar potential,

$$V_0 = -\frac{4\pi^2}{3k^2}\,\mathrm{tr}\big(Y^A Y_A^\dagger Y^B Y_B^\dagger Y^C Y_C^\dagger + Y_A^\dagger Y^A Y_B^\dagger Y^B Y_C^\dagger Y^C + 4Y^A Y_B^\dagger Y^C Y_A^\dagger Y^B Y_C^\dagger - 6Y^A Y_B^\dagger Y^B Y_A^\dagger Y^C Y_C^\dagger\big). \tag{16}$$

The action (13) is invariant under the $\mathcal{N}=6$ supersymmetry transformation [1, 9, 10, 11],

$$\begin{aligned}
\delta Y^A &= i\omega^{AB}\psi_B,\\
\delta\psi_A &= \gamma^\mu \omega_{AB} D_\mu Y^B + \omega_{BC}\left(\beta_A^{BC} + \delta_A^{[B}\beta_D^{C]D}\right),\\
\delta A_\mu &= -\frac{2\pi}{k}\big(Y^A\psi^{B\dagger}\gamma_\mu\omega_{AB} + \omega^{AB}\gamma_\mu\psi_A Y_B^\dagger\big),\\
\delta\hat{A}_\mu &= \frac{2\pi}{k}\big(\psi^{A\dagger}Y^B\gamma_\mu\omega_{AB} + \omega^{AB}\gamma_\mu Y_A^\dagger\psi_B\big),
\end{aligned} \tag{17}$$

where ω_{AB} are supersymmetry transformation parameters with

$$\omega^{AB} = \omega_{AB}^* = \frac{1}{2}\epsilon^{ABCD}\omega_{CD}, \tag{18}$$

and

$$\beta_C^{AB} = \frac{4\pi}{k} Y^{[A} Y_C^\dagger Y^{B]}. \tag{19}$$

There exists a unique mass deformation of the ABJM theory which respects the full $\mathcal{N}=6$ supersymmetry [9]. For the mass-deformed theory, the transformations (17) remain unchanged except the fermion fields for which there is an additional transformation,

$$\delta_{\mathrm{m}}\psi_A = \mu M_A^{\ B}\omega_{BC}Y^C, \tag{20}$$

where μ is the mass deformation parameter and $M_A^B = \mathrm{diag}(1,1,-1,-1)$. The modified supersymmetry transformation leads to the following additional terms to the Lagrangian,

$$\begin{aligned}
\Delta V_{\mathrm{ferm}} &= \mathrm{tr}\,\mu\psi^{\dagger A} M_A^{\ B}\psi_B,\\
\Delta V_0 &= \mathrm{tr}\left(\frac{4\pi\mu}{k}Y^A Y_A^\dagger Y^B M_B^{\ C} Y_C^\dagger - \frac{4\pi\mu}{k}Y_A^\dagger Y^A Y_B^\dagger M_C^{\ B} Y^C + \mu^2 Y_A^\dagger Y^A\right).
\end{aligned} \tag{21}$$

The form of the potential is not manifestly positive-definite. As in U(1) case of the previous section, however, it can be written in a positive-definite form [12, 13],

$$V_{\mathrm{m}} = V_0 + \Delta V_0 = \frac{2}{3}\left|\beta_A^{BC} + \delta_A^{[B}\beta_D^{C]D} + \mu M_A^{\ [B}Y^{C]}\right|^2, \tag{22}$$

where we have introduced the notation $|\mathcal{O}|^2 \equiv \mathrm{tr}\mathcal{O}^\dagger\mathcal{O}$.

From (22) the vacuum equation of the mass-deformed theory is

$$\beta_A^{BC} + \delta_A^{[B}\beta_D^{C]D} + \mu M_A^{\ [B}Y^{C]} = 0. \tag{23}$$

There is essentially a unique irreducible solution [14]:

$$Y_{mn}^1 = \delta_{mn}\sqrt{\frac{k\mu}{2\pi}}\sqrt{m-1}\,, \qquad Y_{mn}^2 = \delta_{m,n+1}\sqrt{\frac{k\mu}{2\pi}}\sqrt{N-m}, \qquad Y^3 = Y^4 = 0. \tag{24}$$

General vacuum configuration would be direct sums of these irreducible solutions or those with the substitution $Y^1, Y^2 \leftrightarrow Y_3^\dagger, Y_4^\dagger$. See [15] for details.

3.1. Half-BPS equations

In this paper, we are interested in half-BPS vortex-type equations. Considering (10) in U(1) case, it is natural to impose the supersymmetric condition

$$\gamma^0 \omega_{AB} = i s_{AB} \omega_{AB}, \qquad s_{AB} = s_{BA} = \pm 1. \tag{25}$$

After some algebra, we find that $\delta\psi_A = 0$ gives

$$\begin{aligned} (D_1 - i s_{AB} D_2) Y^B &= 0, \\ \delta_A^{[B} D_0 Y^{C]} - i s_{BC} \left(\beta_A^{BC} + \delta_A^{[B} \beta_D^{C]D} + \mu M_A^{\ [B} Y^{C]} \right) &= 0, \qquad \text{(no sum over } B, C). \end{aligned} \tag{26}$$

The same equation can also be obtained from the bosonic part of the energy,

$$\begin{aligned} E = \frac{1}{3} \int d^2x \, \mathrm{tr} \Bigg\{ & 2 \sum_{A,B,C} \left| \delta_A^{[B} D_0 Y^{C]} - i s_{BC} \left(\beta_A^{BC} + \delta_A^{[B} \beta_D^{C]D} + \mu M_A^{\ [B} Y^{C]} \right) \right|^2 \\ & + \sum_{A \neq B} |(D_1 - i s_{AB} D_2) Y^A|^2 \Bigg\} \\ + i s \, \mathrm{tr} \int d^2x \epsilon_{ij} \partial_i & \left(Y_1^\dagger D_j Y^1 - \frac{1}{3} \sum_{A=2}^{4} Y_A^\dagger D_j Y^A \right) - \frac{s}{3} \mu \, \mathrm{tr} \int d^2x (j^0 + 2 J_{12}^0), \end{aligned} \tag{27}$$

where j^0 and J_{12}^0 are respectively the charge density associated with the overall U(1) rotation and that for an SU(4) roration $Y^1 \to e^{-i\alpha} Y^1$, $Y^2 \to e^{i\alpha} Y^2$, i.e.,

$$\begin{aligned} j_\mu &= i(Y^A D_\mu Y_A^\dagger - D_\mu Y^A Y_A^\dagger), \\ J_{12\mu} &= i(Y^1 D_\mu Y_1^\dagger - D_\mu Y^1 Y_1^\dagger) - i(Y^2 D_\mu Y_2^\dagger - D_\mu Y^2 Y_2^\dagger). \end{aligned} \tag{28}$$

For any well-behaved BPS configuration satisfying the Bogomolnyi equations (26), the energy is bounded by both the U(1) charge $Q = \mathrm{tr} \int d^2x \, j^0$ and the R-charge $R_{12} = \mathrm{tr} \int d^2x \, J_{12}^0$,

$$E \geq \frac{1}{3} |\mu (Q + 2R_{12})|. \tag{29}$$

Note that the energy bound is proportional to the mass-deformation parameter μ. Therefore, in the undeformed theory, there would be no finite energy regular solution to the half-BPS equations other than vacuum configurations. One can neverthless consider solutions with infinite enrgy, which may be physically important in the context of string theory.

For the original ABJM theory without mass deformation ($\mu = 0$), it can be shown that the Bogomolnyi equation (26) is equivalent to [7]

$$\begin{aligned} (D_1 - i s D_2) Y^1 &= 0, \\ Y^A &= v^A I, \qquad (A = 2, 3, 4), \\ B = \hat{B} &= -\frac{s}{2} \left(\frac{2\pi v}{k} \right)^2 [Y^1, Y_1^\dagger], \end{aligned} \tag{30}$$

where v^A $(A = 2, 3, 4)$ are constants and $v^2 = \sum_{A=2}^4 |v^A|^2$. The equation (30) is the same as the half-BPS equation of super Yang-Mills theory with the identification $g_{\mathrm{YM}} = \frac{2\pi v}{k}$ which has appeared in the contexed of the compactification of ABJM theory (from M2 to D2) [16, 17].

Under a suitable ansatz [7], (30) is reduced to (affine-) Toda-type equation,

$$\partial\bar{\partial}\ln|y_a|^2 = 4v\left(\frac{2\pi}{k}\right)^2\sum_{b=1}^{N-1}K_{ab}\left(|y_b|^2 - \frac{|G(z)|^2}{|c_b|^2\prod_{c=1}^{N-1}|y_c|^2}\right),$$
$$y_M = \frac{G(z)}{\prod_{a=1}^{N-1}y_a}, \tag{31}$$

where $G(z)$ is an arbitrary holomorphic function. For SU(2), this becomes Liouville-type equation (with $G = 0$) or Sinh-Gordon-type equation (with G =const.).

In the mass-deformed theory ($\mu \neq 0$), it is very difficult to anlyze the equations in the most general way except $N = 2, 3$. Here we content ourselves with discussing $N = 2$ case in detail. Then we will briefly mention higher rank cases. See [7] for details. For $N = 2$, we can solve the equations in (26) which do not contain derivatives and find that the scalar fields can be written in the form

$$Y^1 = \sqrt{\frac{k\mu}{2\pi}}\begin{pmatrix}0 & f\\ 0 & 0\end{pmatrix}, \qquad Y^2 = \sqrt{\frac{k\mu}{2\pi}}\begin{pmatrix}a & 0\\ 0 & \sqrt{a^2+1}\end{pmatrix},$$
$$Y^3 = Y^4 = 0, \tag{32}$$

while the magnetic fields take the diagonal form

$$B = \hat{B} = -2s\mu^2\begin{pmatrix}a^2(1+|f|^2) & 0\\ 0 & (a^2+1)(1-|f|^2)\end{pmatrix}, \tag{33}$$

where a is a nonnegative constant. Combining these two using the first equation in (26) gives

$$\partial\bar{\partial}\ln|f|^2 = \mu^2\left[(2a^2+1)|f|^2 - 1\right] + \pi\sum_{p=1}^{n}\delta(\mathbf{x}-\mathbf{x}_p), \tag{34}$$

This is nothing but the vortex equation appearing in Maxwell-Higgs theory. Comparing with the vortex equation (11) in U(1) theory, we see that $|\phi|^2$ term is missing in the right-hand side of (34). The energy of the solution is a sum of two terms,

$$E = \frac{nk\mu}{2a^2+1} + \left|\frac{k\mu}{2\pi}B_0\,\mathrm{tr}\int d^2x\right|, \tag{35}$$

where $B_0 = -4s\mu^2a^2(a^2+1)/(2a^2+1)$ is the asymptotic value of the magnetic field in (33). Therefore solutions with nonzero a may be interpreted as vortices in a constant magnetic field.

Since we have seen in section 2 that solutions in U(1) theory carry nonzero angular momentum, one may wonder if this is also the case in the ABJM theory. One can however show that the angular momentum here vanishes contrary to the U(1) theory. This is essentially because fields do not carry both charge and vorticity, i.e., either D_0Y^A or D_iY^A vanishes in the ABJM theory.

It would be worthwile to examine the origin of the Maxwell-Higgs vortex equation (34) in the mass-deformed ABJM theory which is a Chern-Simons gauge theory. For this purpose we consider the ansatz

$$Y^1 = \sqrt{\frac{k\mu}{2\pi}}\begin{pmatrix}0 & f\\ 0 & 0\end{pmatrix}, \qquad Y^2 = \sqrt{\frac{k\mu}{2\pi}}\begin{pmatrix}0 & 0\\ 0 & g\end{pmatrix}, \qquad Y^3 = Y^4 = 0, \tag{36}$$

and calculate the form of the potential as a function of f and g. With $g = 1$ this reduces to (32) with $a = 0$ so we would like to see how the potential changes as g changes. Plugging (36) into (22), we have the following potential in the mass-deformed theory,

$$V_{\rm m}(f, g) = \left(\frac{k}{2\pi}\right)^2 \mu^3[|f|^2(|g|^2 - 1)^2 + |g|^2(|f|^2 - 1)^2]. \tag{37}$$

Then $V_{\rm m}$ vanishes at $f = g = 0$ and $|f| = |g| = 1$ as it should be. From this potential we get the quartic potential $(|f|^2 - 1)^2$ with $g = 1$ which is the potential appearing in Maxwell-Higgs theory. Note that the point $f = 0,\ g = 1$ is *not* a local maximum of the potential since $V_{\rm m}(f = 0, g) \sim |g|^2$. One may then wonder why the configuration does not roll down to the origin. This is due to the special nature of the Gauss law in Chern-Simons gauge theory, namely the magnetic field is proportional to the charge density as discussed in section 2. Replacing D_0Y^2 by the magnetic field in the energy expression, we obtain an effective potential term $|B/g|^2$ (see (4)) which acts as a barrier at the origin ($g \to 0$). This can be interpreted as a centrifugal term inversely proportional to $1/g^2$ due to the rotation in Y^2 plane. Along the direction $f = g$, (37) becomes the sextic potential $|f|^2(|f|^2 - 1)^2$ which appears in (5) of U(1) theory. It turns out that this direction corresponds to a less supersymmetric BPS case [8].

Closing the subsection we briefly comment on the case of U(N)×U(N) theories. An interesting nontrivial solution is obtained with the ansatz generalizing (32) with $a = 0$ to a block matrix form,

$$Y^1 = \sqrt{\frac{k\mu}{2\pi}} \begin{pmatrix} 0_{N_1\times N_1} & F_{N_1\times N_2} \\ 0_{N_2\times N_1} & 0_{N_2\times N_2} \end{pmatrix}, \qquad Y^2 = \sqrt{\frac{k\mu}{2\pi}} \begin{pmatrix} 0_{N_1\times N_1} & 0_{N_1\times N_2} \\ 0_{N_2\times N_1} & G_{N_2\times N_2} \end{pmatrix}, \tag{38}$$

where N_1, N_2 are positive integers satisfying $N_1 + N_2 = N$ and the subscript denotes the dimensionality of each block. This form of ansatz results in the nonabelian vortex equation in U(N_2) Yang-Mills theory with N_1 fundamental scalar fields and has been studied extensively [18, 19]. Another interesting solution is obtained with a sort of irreducible nondegenerate ansatz similar to (24),

$$Y^1_{mn} = \delta_{m+1,n}\sqrt{\frac{k\mu}{2\pi}} f_m, \qquad Y^2_{mn} = \delta_{mn}\sqrt{\frac{k\mu}{2\pi}} a_m, \tag{39}$$

where

$$a_m = \sqrt{a_1^2 + m - 1}\,. \tag{40}$$

Here a_1 is a nonnegative constant and $f_1, \ldots, f_{N-1}$ are functions to be determined. Then we obtain $N - 1$ coupled differential equations,

$$\partial\bar{\partial} \ln |f_n|^2 = -\mu^2[a_n^2|f_{n-1}|^2 - (a_n^2 + a_{n+1}^2)|f_n|^2 + a_{n+1}^2 + 1]. \tag{41}$$

(Here we omit the usual delta funtion term in the right-hand side.) This type of coupled equations has appeared in U$(1)^{N-1}$ gauge theories with $N - 1$ Higgs fields which couple to the gauge fields [20].

3.2. Less supersymmetric cases

Imposing supersymmetric conditions in addition to (25) further breaks supersymmetry. In this subsection, we will consider a few representative cases and see how the resulting BPS equation and its energy bound change. Details will be published in [8].

The condition $\omega_{12} = 0$ as well as (25) on the supersymmetry paramter ω_{AB} reduces the number of supersymmetries to four. With these conditions, $\delta\psi_A = 0$ leads to the following set of 1/3-BPS equations:

$$\begin{aligned}
&(D_1 - isD_2)Y^1 = 0, && (D_1 + isD_2)Y^2 = 0,\\
&D_1Y^c = D_2Y^c = 0, \quad (c = 3, 4), && \\
&D_0Y^1 + is(\beta^{21}_2 + \mu Y^1) = 0, && D_0Y^2 - is(\beta^{12}_1 + \mu Y^2) = 0,\\
&D_0Y^c + is(\beta^{2c}_2 - \beta^{1c}_1) = 0, \quad (c = 3, 4), && \\
&\beta^{3a}_3 = \beta^{4a}_4 \quad (a = 1, 2), && \beta^{43}_4 - \mu Y^3 = \beta^{34}_3 - \mu Y^4 = 0,\\
&\beta^{23}_1 = \beta^{24}_1 = \beta^{13}_2 = \beta^{14}_2 = \beta^{14}_3 = \beta^{24}_3 = \beta^{13}_4 = \beta^{23}_4 = 0. &&
\end{aligned} \tag{42}$$

The energy is then given by

$$E = |\mu \operatorname{tr} R_{12}|. \tag{43}$$

Note the difference from the half-BPS case (29).

When two among the three supersymmetry parameters vanish, e.g., $\omega_{13} = \omega_{14} = 0$, the Bogomolnyi equations of all four complex scalar fields become nontrivial,

$$\begin{aligned}
&(D_1 - isD_2)Y^a = 0, \quad (a = 1, 2), && (D_1 + isD_2)Y^c = 0, \quad (c = 3, 4),\\
&D_0Y^1 + is(\beta^{C1}_C - 2\beta^{21}_2 - \mu Y^1) = 0, && D_0Y^2 + is(\beta^{C2}_C - 2\beta^{12}_1 - \mu Y^2) = 0,\\
&D_0Y^3 - is(\beta^{C3}_C - 2\beta^{43}_4 + \mu Y^3) = 0, && D_0Y^4 - is(\beta^{C4}_C - 2\beta^{34}_3 + \mu Y^4) = 0,\\
&\beta^{34}_1 = \beta^{34}_2 = \beta^{12}_3 = \beta^{12}_4 = 0. &&
\end{aligned} \tag{44}$$

In this case, only $\mathcal{N} = 1$ supersymmetry is unbroken and the solutions are 1/6-BPS objects. The energy turns out to be bounded by the U(1) charge,

$$E = |\mu Q|. \tag{45}$$

As in the half-BPS case, one can assume various form of ansatz to find nontrivial vortex-type solutions. In particular, when $N = 2$ the vortex equations are obtained which have appeared in Maxwell-Higgs model with an independent Chern-Simons term [5]. The result will be reported in the forthcoming publication [8].

4. Conclusion

In this paper we reviewed our recent investigations on vortex-type half-BPS equations in the ABJM theory with or without mass deformation. We obtained the energy bound (29) which is proportional to the mass-deformation parameter.

For the undeformed ABJM theory, we found that the BPS equation reduces to that in half-BPS equation of supersymmetric Yang-Mills theory. It has no finite energy regular solution. In the mass-deformed theory, we showed that the BPS equations for U(2)×U(2) case reduce to the Maxwell-Higgs vortex equation which is known to have multi-vortex solutions. We explored the origin of Maxwell-Higgs vortex in the Chern-Simons gauge theory. For U(N)×U(N) case with $N > 2$, we obtained the nonabelian vortex equation of Yang-Mills-Higgs theory and also more general equations.

By imposing further supersymmetric conditions, we found 1/3-BPS and 1/6-BPS equations. The detailed analysis as well as its physical implication will be reported in the forthcoming publication.

Acknowledgments
This work was supported by the World Class University grant R32-2009-000-10130-0 through the National Research Foundation, Ministry of Education, Science and Technology of Korea.

References
[1] Aharony O, Bergman O, Jafferis D L and Maldacena J 2008 *J. High Energy Phys.* JHEP10(2008)091 (*Preprint* arXiv:0806.1218 [hep-th])
[2] Hong J, Kim Y, and Pac P Y 1990 *Phys. Rev. Lett.* **64** 2230
Jackiw R and Weinberg E J 1990 *Phys. Rev. Lett.* **64** 2234
[3] Lee C, Lee K M and Weinberg E J 1990 *Phys. Lett.* B **243** 105
[4] Jackiw R, Lee K M and Weinberg E J 1990 *Phys. Rev.* D **42** 3488
[5] Kim C 1993 *Phys. Rev.* **D47**, 673-684 (*Preprint* hep-th/9209110)
[6] Cugliandolo L F, Lozano G, Manias M V and Schaposnik F A 1991 *Mod. Phys. Lett.* A **6** 479
Lee K M 1991 *Phys. Rev. Lett.* **66** 553
Lee K M 1991 *Phys. Lett.* B **255** 381
[7] Kim C, Kim Y, Kwon O K and Nakajima H 2009 *Phys. Rev.* D **80** 045013 (*Preprint* arXiv:0905.1759 [hep-th])
[8] Go G, Kim C, Kim Y, Kwon O K and Nakajima H 2011 *to appear.*
[9] Hosomichi K, Lee K M and Lee S 2008 *Phys. Rev.* D **78** 066015 (*Preprint* arXiv:0804.2519 [hep-th])
[10] Gaiotto D, Giombi S and Yin X 2009 *J. High Energy Phys.* JHEP04(2009)066 (*Preprint* arXiv:0806.4589 [hep-th])
[11] Terashima S 2008 *J. High Energy Phys.* JHEP08(2008)080 (*Preprint* arXiv:0807.0197 [hep-th])
[12] Bandres M A, Lipstein A E and Schwarz J M 2008 *J. High Energy Phys.* JHEP09(2008)027 (*Preprint* arXiv:0807.0880 [hep-th])
[13] Hosomichi K, Lee K M, Lee S, Lee S, Park J and Yi P 2008 *J. High Energy Phys.* JHEP11(2008)058 (*Preprint* arXiv:0809.1771 [hep-th])
[14] Gomis J, Rodriguez-Gomez D, Van Raamsdonk M and Verlinde H 2008 *J. High Energy Phys.* JHEP09(2008)113 (*Preprint* arXiv:0807.1074 [hep-th])
[15] Kim H C and Kim S 2010 *Nucl. Phys.* B **839** 96 (*Preprint* arXiv:1001.3153 [hep-th])
[16] Mukhi S and Papageorgakis C 2008 *J. High Energy Phys.* JHEP05(2008)085 (*Preprint* arXiv:0803.3218 [hep-th])
[17] Pang Y and Wang T 2008 *Phys. Rev.* D **78** 125007 (*Preprint* arXiv:0807.1444 [hep-th])
[18] Hanany A and Tong D 2003 *J. High Energy Phys.* JHEP07(2003)037 (*Preprint* hep-th/0306150)
[19] Eto M, Isozumi Y, Nitta M, Ohashi K and Sakai N 2006 *J. Phys.* A **39** R315 (*Preprint* hep-th/0602170)
[20] Kim C, Lee C, Ko P, Lee B H and Min H 1993 *Phys. Rev.* D **48** 1821 (*Preprint* hep-th/9303131)