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Thermodynamics of higher-dimensional Brans–Dicke black holes in the presence of a conformal-invariant field inspired by power-Maxwell electrodynamics

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By use of the conformal transformations, in addition to translating the Brans–Dicke (BD) action to the Einstein frame (EF), we introduce an electromagnetic Lagrangian which preserves conformal invariance. We solve the EF field equations, which mathematically are confronted with the problem of indeterminacy, by use of an exponential ansatz function. When the self-interacting potential is absent or is taken constant in the BD action, the exact solution of this theory is just that of Einstein-conformal-invariant theory with a trivial scalar field. This is a higher-dimensional (HD) analogue of the same considered in Ref. [R.-G. Cai, Y. S. Myung, Phys. Rev. D **56**, 3466 (1997)]. The EF general solution admits two classes of black holes (BHs) with non-flat and non-AdS asymptotic behavior which can produce extreme and multi-horizon ones. We obtain the exact solutions of BD-conformal-invariant theory, by applying inverse conformal transformations, which show two classes of extreme and multi-horizon BHs too. Based on the fact that thermodynamic quantities remain unchanged under conformal transformations, we show that the first law of BH thermodynamics is valid in the Jordan frame. We analyze the thermal stability of the HD BD-conformal-invariant BHs by use of the canonical ensemble method.

Subject Index E00, E03, E31

1. Introduction

The scalar–tensor theory is a modified gravity theory in which the scalar field is non-minimally coupled to gravity. It can be considered as the simplest extension of Einstein's gravity theory which coincides with the low energy limit of string theory [1]. This theory is often used for resolving the problems of inflation and dark energy in the modern cosmology [2,3]. Brans and Dicke have proposed a version of scalar-tensor theory which is consistent with Mach's principle and Dirac's large number hypothesis [4,5]. It includes a coupling parameter denoted by ω, which recovers Einstein's initial theory as the Brans–Dicke (BD) parameter grows to infinity [6]. The BD theory, which is considered as an alternative theory of gravity, has had some outstanding cosmological achievements. For example: although the advanced precession of Mercury's perihelion was initially explained by Einstein's theory, when the sun's oblateness is taken into account a complete explanation is possible by BD theory [7].

It has been shown that the exact solutions of four-dimensional (4D) BD-Maxwell theory, in the absence of self-interacting scalar potential, is just the Reissner–Nordström (RN) black hole (BH) with a trivial constant scalar field [8]. This is due to the fact that Maxwell's electrodynamics remains conformal-invariant in 4D spacetimes, and the differential equation governed by the scalar field becomes source-free. In the higher-dimensional (HD) spacetimes, however, since Maxwell's electrodynamics is no longer conformal-invariant and plays the role of a non-zero source, the solutions are affected by non-trivial scalar fields [9,10]. Here, we show that similar result can be achieved when the BD theory is considered in the presence of HD conformal-invariant electrodynamics. Maxwell's theory of classical electrodynamics is conformal-invariant only in the 4D spacetimes. It also leads to infinite electric field and self-energy in the position of point-like charged particles. Different models of non-linear electrodynamics were initially proposed with the aim of resolving the related failures [11,12]. Some of the non-linear electromagnetic models, such as Born–Infeld and Born–Infeld-like theories [13–19], have successfully removed the singularities in the classical level. The problem of conformal symmetry breaking in the higher- and lower-dimensional spacetimes may be resolved by suitable use of the power-law non-linear electrodynamics [20,21].

Conformal invariance was initially introduced by showing that Maxwell's equations remain invariant under conformal transformations [22,23]. A detailed discussion on the physical importance of conformal symmetry in physics, and some other conformal-invariant equations, can be seen in Ref. [24]. In flat spacetime, massless fields propagate on the light-cone, and their field equations are conformal-invariant. An important point is that although the conformal invariance guarantees the light-cone propagation, the inverse is not always true [25]. It is well-known that the massless spin-2 particles, the so-called gravitons which propagate with the light speed, are described by a symmetric rank-2 tensor field in Einstein's theory of relativity, and this theory violates conformal symmetry. It has been shown that a theory of relativity which presents symmetries of de Sitter and conformal groups simultaneously can be constructed by use of a mixed symmetry rank-3 tensor field [25–27]. Moreover, the anti-de Sitter/conformal-invariant field theory (AdS/CFT) correspondence states that there is a relation between d-dimensional conformal-invariant field theory (CFT) and a theory of gravity in a $(d + 1)$-dimensional anti-de Sitter (AdS) space, both with the same symmetry group $SO(2, d)$. Nowadays, the AdS/CFT dual and its applications have been extended to almost all diverse areas of physics [28–33].

Since the BHs are systems in high energy levels, investigation of their properties in the presence of quantum-corrected classical perspectives is expected to give more realistic advantages [34,35]. Thus, noting the fundamental role of conformal invariance in constructing AdS/CFT correspondence, and also its importance in establishing conformal quantum gravity theory [36,37], we study the impacts of conformal symmetry, via consideration of conformal-invariant electrodynamics, on the thermodynamic behavior of HD BD BHs. Briefly, the main purpose of the present work is to obtain the exact charged BD BH solutions in the presence of HD conformal-invariant electrodynamics. Also, through study of thermodynamic properties, we examine the impacts of conformal-invariant electrodynamics on the thermodynamic quantities, the first law of BH thermodynamics and the stability properties of the BD BHs.

The paper will be organized as follows: In Sect. 2, by applying a mathematical tool named as the conformal transformation, the action of HD BD theory is translated from the Jordan frame (JF) to that of Einstein-dilaton (Ed) theory in the Einstein frame (EF). Through this procedure, the action of conformal-invariant electrodynamics has been introduced for the HD spacetimes. In Sect. 3, two novel classes of exact HD solutions are introduced in the presence of conformal-invariant electrodynamics. In Sect. 4, the exact solutions of the BD-conformal-invariant gravity theory are obtained, from their Ed analogues, by use of the inverse conformal transformations. Regarding the fact that under conformal transformations the thermodynamic quantities remain invariant, we show that the first law of BH thermodynamics is valid for our novel BD-conformal-invariant BHs. We discuss thermal stability of the BD BHs by use of the canonical ensemble method. The results will be summarized and discussed in Sect. 5.

2. The general formalism

The action of BD theory can be written either in the JF or in its conformally related spacetime, named as the EF. The field equations obtained by the variational principle in the JF are highly non-linear such that they cannot be solved directly [38,39]. Fortunately, there is a mathematical tool known as the conformal transformations by use of which one can translate the JF action to that of EF, where the equations of gravitational and scalar fields are decoupled [40,41]. The EF action is just that of Ed theory, and the related differential equations can be easily solved for the spherically symmetric BHs. Then, by using inverse conformal transformations, one can obtain the exact BD BH solutions from the corresponding ones in the EF [42,43]. The action of $(n+1)$-dimensional BD gravity theory, in the presence of an electromagnetic Lagrangian, takes the following general form [44,45]:

$$I^{BD} = -\frac{1}{16\pi}\int\sqrt{-\bar{g}}\left[\psi\bar{\mathcal{R}} - \frac{\omega}{\psi}\,\bar{g}^{\mu\nu}\partial_\mu\psi\,\partial_\nu\psi - U(\psi) + L(\bar{X})\right]d^{n+1}x, \tag{1}$$

where $\bar{\mathcal{R}} = \bar{g}^{\mu\nu}\bar{\mathcal{R}}_{\mu\nu}$ is the Ricci scalar, ω is the BD parameter, and ψ is a scalar field. The term $L(\bar{X})$ states the Lagrangian density of non-linear electrodynamics as a function of $\bar{X} = \bar{F}^{\mu\nu}\bar{F}_{\mu\nu}$. $\bar{F}_{\mu\nu} = \partial_\mu\bar{A}_\nu - \partial_\nu\bar{A}_\mu$ is the Farady's tensor and $\bar{A}_\mu$ is the electromagnetic four-potential. Here, we use the power-Maxwell non-linear electrodynamics with the Lagrangian density as [46,47]

$$L(\bar{X}) = \left(-\bar{X}\right)^p, \tag{2}$$

and p is the parameter of non-linearity. Note that, in the case of $p = 1$, Maxwell's electrodynamics is recovered.

Making use of the variational principle, the various field equations can be achieved. One can show that the gravitational field equation takes the following form [48]:

$$\psi\left(\bar{\mathcal{R}}_{\alpha\beta} - \frac{1}{2}\bar{\mathcal{R}}\bar{g}_{\alpha\beta}\right) - \left(\bar{\nabla}_\alpha\bar{\nabla}_\beta - \bar{g}_{\alpha\beta}\bar{\Box}\right)\psi = T^{(s)}_{\alpha\beta} + T^{(em)}_{\alpha\beta}, \tag{3}$$

where $\bar{\Box} = \bar{\nabla}_\mu\bar{\nabla}^\mu$ is the dAlembertian operator, and $T^{(em)}_{\alpha\beta}$ and $T^{(s)}_{\alpha\beta}$ are stress-energy tensors of the scalar and electromagnetic fields

$$T^{(s)}_{\alpha\beta} = \frac{\omega}{\psi}\bar{\nabla}_\alpha\psi\bar{\nabla}_\beta\psi - \frac{1}{2}\left[U(\psi) + \frac{\omega}{\psi}(\bar{\nabla}\psi)^2\right]\bar{g}_{\alpha\beta}, \tag{4}$$

$$T^{(em)}_{\alpha\beta} = \frac{1}{2}\left(-\bar{X}\right)^p\bar{g}_{\alpha\beta} + 2p\left(-\bar{X}\right)^{p-1}\bar{F}_{\alpha\nu}\bar{F}_\beta{}^\nu. \tag{5}$$

Also, for the scalar field equation, we have

$$2[\omega(n-1)+n]\bar{\Box}\psi = (n-1)\psi U'(\psi) - (n+1)U(\psi) + \frac{1}{2}(n+1-4p)\left(-\bar{X}\right)^p. \tag{6}$$

and for the electromagnetic field equation, we obtain

$$\partial_\alpha\left[\sqrt{-\bar{g}}\left(-\bar{X}\right)^{p-1}\bar{F}^{\alpha\beta}\right] = 0. \tag{7}$$

Because of the strong coupling between the scalar and gravitational field equations it is very difficult to obtain the analytical exact solutions directly. Therefore we tend to translate the action (1) from the JF to the EF, where the exact analytical solutions may be obtained simply. An interesting transformation is the so-called conformal transformation which is defined through the following relation:

$$\bar{g}_{\mu\nu} \longrightarrow \bar{g}_{\mu\nu} = \Omega^2(\psi) g_{\mu\nu}, \tag{8}$$

where $g_{\mu\nu}$ is the metric tensor in the EF and $\Omega(\psi)$ is, in general, a well-defined function of the spacetime coordinates. Note that the transformation (8) is not a coordinate transformation, but it acts on the geometry and shrinks or stretches manifold. In addition to Eq. (8) we assume that the Farady's tensor transforms as $\bar{F}_{\mu\nu} \longrightarrow F_{\mu\nu}$. Under the conformal transformations presented in Eq. (8), the Ricci scalar can be calculated easily. That is [49],

$$\bar{\mathcal{R}} \longrightarrow \bar{\mathcal{R}} = \Omega^{-2}\mathcal{R} - \Omega^{-4} n(n-3) g^{\mu\nu} \partial_\mu \Omega \, \partial_\nu \Omega - 2n\Omega^{-3} \Box \Omega. \tag{9}$$

Also, for the metric determinant and Maxwell invariant, we have

$$\bar{g} \longrightarrow \bar{g} = \Omega^{2(n+1)} g, \tag{10}$$

$$\bar{X} \longrightarrow \bar{X} = \Omega^{-4} X. \tag{11}$$

Consequently, one can write

$$L(\bar{X}) \longrightarrow L(\bar{X}) = \Omega^{n+1} L(\Omega^{-4} X), \tag{12}$$

and noting Eq. (2), one can conclude that if we set $p = (n+1)/4$, then $L(\bar{X}) = L(X)$ and, the trace of electromagnetic energy-momentum tensor (5) vanishes (i.e. $\bar{g}^{\alpha\beta} T^{(em)}_{\alpha\beta} = 0$) [50]. Thus the Lagrangian density is invariant, and we achieve the HD conformal-invariant electrodynamics.

In addition, for the translation from JF to the EF, we need a scalar field in the EF which we call ϕ and assume that $\psi = \psi(\phi)$. Therefore, we can write

$$\partial_\mu \psi \, \partial_\nu \psi = \left(\frac{d\psi}{d\phi}\right)^2 \partial_\mu \phi \, \partial_\nu \phi, \tag{13}$$

$$\partial_\mu \Omega \, \partial_\nu \Omega = \left(\frac{d\Omega}{d\phi}\right)^2 \partial_\mu \phi \, \partial_\nu \phi. \tag{14}$$

Now, combining the above-mentioned results shows that the BD action (1) in the EF may be written as [51–53]

$$I^{Ed} = -\frac{1}{16\pi} \int \sqrt{-g} \left[\mathcal{R} - \frac{4}{n-1} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - V(\phi) + L(X) \right] d^{n+1}x, \tag{15}$$

if the following relations are fulfilled:

$$\Omega(\psi) = \psi^{-1/(n-1)}, \tag{16}$$

$$U(\psi)\Omega^{n+1}(\psi) = V(\phi), \tag{17}$$

$$n(n-1)\left(\frac{d\ln\Omega}{d\psi}\right)^2 \left(\frac{d\psi}{d\phi}\right)^2 + \frac{\omega}{\psi^2}\left(\frac{d\psi}{d\phi}\right)^2 = \frac{4}{n-1}. \tag{18}$$

By combining Eqs. (16) and (18), and integrating the obtained equation we have

$$\ln\psi = \frac{\pm 2}{\sqrt{n + (n-1)\omega}} \phi. \tag{19}$$

Therefore, we have

$$\psi = e^{2\beta\phi}, \tag{20}$$

where $\beta = \frac{1}{\sqrt{n+(n-1)\omega}}$, with $\omega > -\frac{n}{n-1}$, and for a suitable function $\phi(r)$, the physical scalar field $\psi(r)$ has to vanish at infinity.

3. The Ed-conformal-invariant BHs

The action (15) which has been written in the EF is known as the action of HD Ed gravity theory coupled to conformal-invariant electrodynamics as the matter field. Making use of the variational principle, it leads to the following field equations [54,55]:

$$\mathcal{R}_{\mu\nu} = \frac{1}{n-1}\left[g_{\mu\nu}V(\phi) + 4\partial_\mu\phi(r)\partial_\nu\phi(r)\right] + \frac{g_{\mu\nu}}{2}\left(\frac{n-3}{n-1}\right)L(X) - 2L'(X)F_{\mu\alpha}F_\nu^{\ \alpha}, \quad (21)$$

$$\nabla_\mu\left[L'(X)F^{\mu\nu}\right] = 0, \quad (22)$$

$$8\Box\phi(r) = (n-1)\frac{dV(\phi)}{d\phi}, \quad (23)$$

which may be named as the equations of gravitational, electromagnetic and scalar fields, in order. Here, we are interested in obtaining the exact solutions in a static and spherically symmetric $(n+1)$-dimensional spacetime. Thus we start with the following ansatz [14,19,56,57]:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2a^2(r)h_{ij}dx^i dx^j. \quad (24)$$

The unknowns $f(r)$ and $a(r)$ will be determined later. Also, $h_{ij}dx^i dx^j$ is the line element of an $(n-1)$-dimensional subspace [58].

It has been shown that, in obtaining the exact solutions, this theory is confronted with the problem of indeterminacy [59,60]. This means that the number of unknowns is one more than the unique equations. This problem can be resolved by use of an exponential ansatz [16,61,62]. That is,

$$a(r) = e^{(2\alpha\phi)/(n-1)}, \quad (25)$$

where α is a constant coefficient. Evidently, in the absence of a dilaton field it reduces to unity and the line element of Einstein gravity is recovered. Then the scalar field $\phi(r)$ is obtained as [59]

$$\phi(r) = \gamma\ln\left(\frac{b}{r}\right), \qquad \text{with} \qquad \gamma = \frac{(n-1)\alpha}{2(1+\alpha^2)}. \quad (26)$$

Note that b is a positive constant. The combination of Eqs. (20) and (6) shows that, for positive values of α and β, ψ vanishes at infinity. Also, for the non-vanishing component of Farady's tensor, in terms of an integration constant q, we have

$$F_{tr} = q\,r^{-2/(1+\alpha^2)}, \quad (27)$$

and noting the relation $F_{tr} = -\partial_r A_t(r)$ one obtains

$$A_t = \begin{cases} -q\ln\left(\frac{r}{\ell}\right), & \text{for } \alpha^2 = 1, \\ q\left(\frac{1+\alpha^2}{1-\alpha^2}\right)r^{(\alpha^2-1)/(\alpha^2+1)}, & \text{for } \alpha^2 < 1. \end{cases} \quad (28)$$

The remaining unknowns $f(r)$ and $V(\phi)$ are governed by the following differential equations:

$$\frac{n-1-2\alpha\gamma}{(n-1)r}\left[f'(r) + (n-2-2\alpha\gamma)\frac{f(r)}{r}\right] = \frac{n-2}{r^2a^2(r)} - \frac{V(\phi)}{n-1} - \frac{L(X)}{2}, \quad (29)$$

$$-\frac{\gamma}{r}\left[f'(r) + (n-2-2\alpha\gamma)\frac{f(r)}{r}\right] = \frac{n-1}{8}\frac{dV(\phi)}{d\phi}. \quad (30)$$

Before obtaining the general solution of these equations, we consider an special case in which $V(\phi) = 0$ or $V(\phi) = constant$, for example $V(\phi) = 2\Lambda$. Thus we have $dV(\phi)/d\phi = 0$, and Eqs. (29) and (30) are incompatible unless $\gamma = 0$. Then, noting Eq. (26), we have $\phi = 0$ and, Eq. (20) results in $\psi = 1$. Under these conditions, the set of Eqs. (29) and (30) lead to

$$f(r) = 1 - \frac{m}{r^{n-2}} - \frac{2\Lambda r^2}{n(n-1)} + \frac{\left(2q^2\right)^{(n+1)/4}}{2r^{n-1}}, \tag{31}$$

which is nothing but the metric function of HD Einstein's gravity with a conformal-invariant electrodynamics. Therefore, the BD-conformal-invariant theory, when the action does not include the self-interacting potential $U(\psi)$, is just the HD RN-conformal-invariant theory coupled to a trivial scalar field $\psi = 1$. This issue is the HD correspondence of the 4D BD-Maxwell theory, where the Maxwell's electrodynamics is conformal-invariant, considered in Ref. [8].

Now, the general solutions of Eqs. (29) and (30) can be written in the following forms [59]:

$$V(\phi) = \begin{cases} 2\Lambda_{eff} e^{4\phi/(n-1)} + 2\Lambda_1 \phi e^{4\phi/(n-1)} + 2\Lambda_2 e^{[2(n+1)\phi]/(n-1)}, & \text{for } \alpha^2 = 1. \\ 2\Lambda e^{4\alpha\phi/(n-1)} + 2\Lambda_3 e^{4\phi/[\alpha(n-1)]} + 2\Lambda_4 e^{[2(n+1)\phi]/[\alpha(n-1)]}, & \text{for } \alpha^2 < 1, \end{cases} \tag{32}$$

where, Λ_{eff} is the cosmological constant $\Lambda = -[n(n-1)]/2\ell^2$ with a constant absorbed in it, and

$$\Lambda_1 = \frac{-2(n-2)}{b^2}, \quad \Lambda_2 = \frac{\left(2q^2\right)^{(n+1)/4}}{2b^{(n+1)/2}}, \quad \Lambda_3 = \frac{\alpha^2(n-1)(n-2)}{2b^2(\alpha^2-1)}, \quad \Lambda_4 = \frac{\alpha^2(n-1)\left(2q^2\right)^{(n+1)/4}}{2(n+1-2\alpha^2)\, b^{(n+1)/(1+\alpha^2)}}, \tag{33}$$

and, in terms of the integration constant m, Eq. (29) gives the following exact solution for the metric function:

$$f(r) = \begin{cases} -mr^{(3-n)/2} - \frac{8\Lambda_{eff} br}{(n-1)^2} + \frac{4r(n-2)}{b(n-1)}\left[\frac{n+1}{n-1} + \ln\left(\frac{b}{r}\right)\right] - \left(\frac{n+1}{n-1}\right)\left(2q^2\right)^{(n+1)/4} r^{(3-n)/2} \ln\left(\frac{r}{\ell}\right), \\ \qquad \text{for } \alpha^2 = 1, \\ -m\, r^{1-(2\gamma/\alpha)} - 2\Lambda b^2 \Upsilon \left(\frac{r}{b}\right)^{2/(1+\alpha^2)} + \Upsilon_1 \left(\frac{r}{b}\right)^{(2\alpha^2)/(1+\alpha^2)} + \Upsilon_2 \left(2q^2\right)^{(n+1)/4} r^{(1-n+2\alpha^2)/(1+\alpha^2)}, \\ \qquad \text{for } \alpha^2 < 1. \end{cases} \tag{34}$$

We have used the definitions $\Upsilon = [(1+\alpha^2)^2]/[(n-1)(n-\alpha^2)]$, $\Upsilon_1 = [(1+\alpha^2)^2(n-2)]/[(\alpha^2+n-2)(1-\alpha^2)]$ and $\Upsilon_2 = [(1+\alpha^2)^2(n+1)]/[2(n+1-2\alpha^2)(1-\alpha^2)]$, for simplicity. Note that the statements presented in Eqs. (32), (33) and (34) reduce to the corresponding quantities in the four-dimensional case of Brans–Dicke–Maxwell theory, where the applied electromagnetic theory is conformal-invariant too [63]. Also, in the absence of dilaton field (i.e. $\alpha = 0$), the asymptotically AdS metric function (31) is recovered. This means that our solutions, which are asymptotically unusual, are affected by the scalar field. Indeed, the inclusion of dilatonic scalar field causes the metric function to be asymptomatically non-flat and non-AdS.

The plots of $f(r) - r$, for $n = 4$ and $n = 5$ cases, have been depicted in Figs. 1 and 2. They illustrate that our solutions reveal that extreme, naked singularity, one-horizon, two-horizon and three-horizon BHs can occur for the properly chosen parameters. The appearance of the multi-horizon BHs, which shows a quantum effect known as the anti-evaporation [64], arises from the consideration of conformal-invariant electrodynamics.

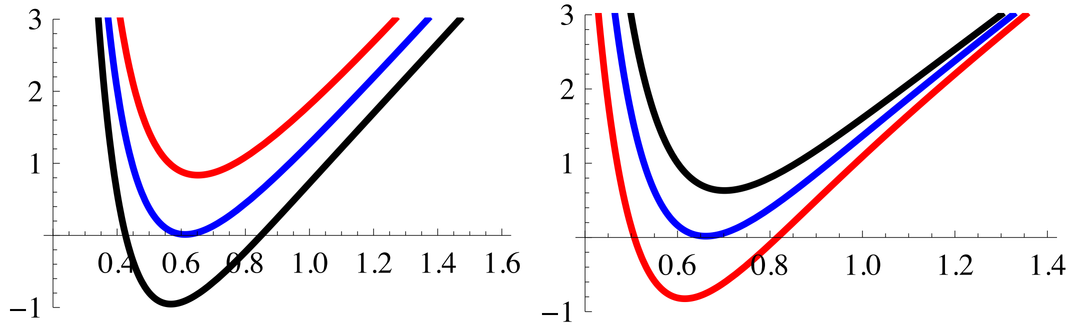

Fig. 1. $f(r)$ versus r, for $\ell = 1$, $b = 1.5$, $q = 1$: Left: $n = 4$, $m = 4$, $\alpha = 0.38$(black), 0.43(blue), 0.47(red), Right: $n = 5$, $\alpha = 0.4$, $m = 3.7$ (black), 3.94 (blue), 4.22 (red).

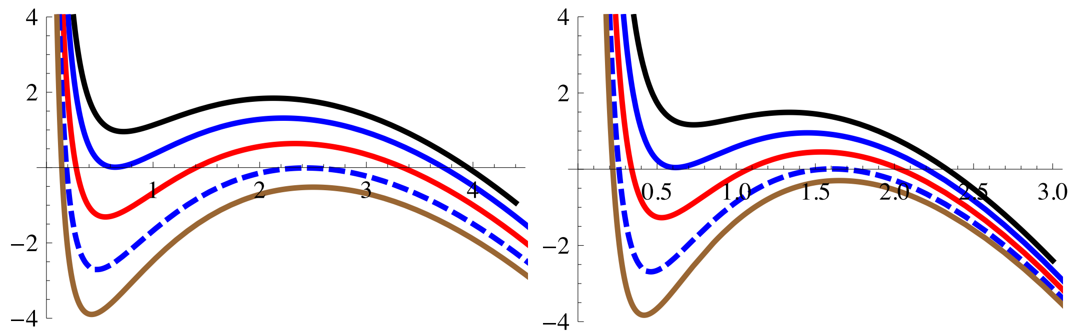

Fig. 2. $f(r)$ versus r, for $\alpha = 1$, $\ell = 2$, $q = 1$: Left: $n = 4$, $b = 0.8$, $m = 7.5$ (black), 8.28 (blue), 9.3 (red), 10.3 (blue dashed), 11.1 (brown), Right: $n = 5$, $b = 0.6$, $m = 7.3$ (black), 8.05 (blue), 8.8 (red), 9.5 (blue dashed), 10 (brown).

4. The BD-conformal-invariant BHs

Up to now, the exact solutions have been obtained in the presence of an HD conformal-invariant electrodynamics inspired by nonlinear power-Maxwell field, in the EF where the static and spherically symmetric spacetime takes the form of Eq. (24). Now, we explore BD BH solutions in the JF by use of their EF counterparts. Then, we study thermodynamics and thermal stability of our novel BD BHs. To this end we proceed with the following line element [65,66]:

$$ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = -\mathcal{A}(r)dt^2 + \frac{dr^2}{\mathcal{B}(r)} + r^2\mathcal{C}^2(r)h_{ij}dx^i dx^j. \tag{35}$$

Note that the functions $\mathcal{A}(r)$, $\mathcal{B}(r)$, and $\mathcal{C}(r)$ are unknown metric coefficients, and will be fixed by use of the conformal transformations (8). Noting Eqs. (16) and (20), we have $\Omega(\phi) = \exp\{-[(2\beta)/(n-1)]\phi\}$ and $\bar{g}_{\mu\nu} = (b/r)^{-(4\beta\gamma)/(n-1)} g_{\mu\nu}$. Thus, for the metric coefficients, we have

$$\mathcal{A}(r) = \left(\frac{b}{r}\right)^{-(4\beta\gamma)/(n-1)} f(r),$$

$$\mathcal{B}(r) = \left(\frac{b}{r}\right)^{(4\beta\gamma)/(n-1)} f(r),$$

$$\mathcal{C}(r) = \left(\frac{b}{r}\right)^{-(2\beta\gamma)/(n-1)} a(r). \tag{36}$$

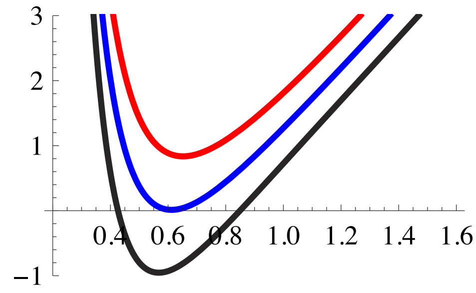

Fig. 3. $\mathcal{B}(r)$ versus r, for $\ell = 1$, $b = 1.5$, $q = 1$, $\beta = 2$: Left: $n = 4$, $m = 4$, $\alpha = 0.38$ (black), 0.43 (blue), 0.47 (red), Right: $n = 5$, $\alpha = 0.4$, $m = 3.7$ (black), 3.94 (blue), 4.22 (red).

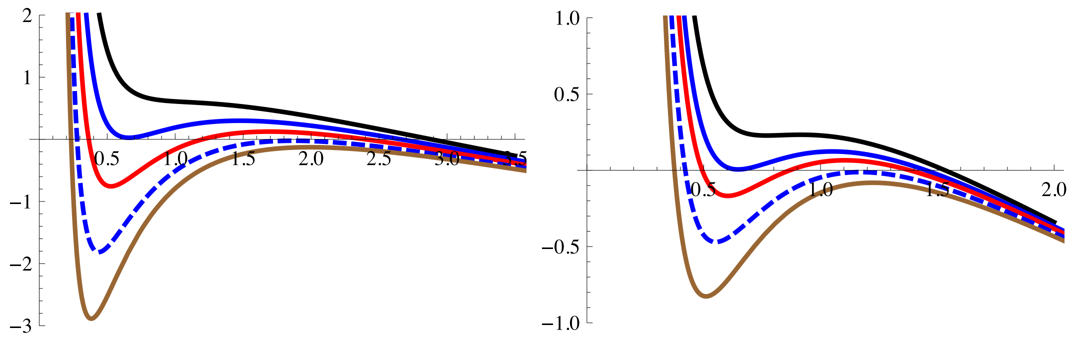

Fig. 4. $\mathcal{B}(r)$ versus r, for $\alpha = 1$, $\ell = 2$, $q = 1$, $\beta = 1$: Left: $n = 4$, $b = 0.7$, $m = 7.8$ (black), 8.44 (blue), 8.9 (red), 9.4 (blue dashed), 9.8 (brown), Right: $n = 5$, $b = 0.5$, $m = 7.95$ (black), 8.165 (blue), 8.3 (red), 8.5 (blue dashed), 8.7 (brown).

Note that $a(r)$ and $f(r)$ have been presented in Eqs. (25) and (34), respectively. The graphs of $\mathcal{B}(r)$ versus r, for five- and six-dimensional cases (i.e. $n = 4$ and $n = 5$) have been plotted in Figs. 3 and 4. The graphs show that three-horizon, two-horizon, one-horizon, extreme and horizon-less BHs appear for the BD exact solutions obtained here. The appearance of the extreme BHs with two extreme horizons (blue curves of Figs. 3 and 4), and the existence of the multi-horizon BHs reveals the impacts of the conformal-invariant electrodynamics under consideration.

Indeed the horizon radii are the real roots of equation $\bar{g}^{rr} = \mathcal{B}(r) = 0$. Although it is not easy to determine its real roots analytically, based on the method of Ref. [67] we can give some analysis by use of plots. The relation $\mathcal{B}(r_{_h}) = 0$, where the $r_{_h}$ is the horizon radius, gives

$$m(r_{_h}) = \begin{cases} 4r_{_h}^{(n-1)/2}\left\{\frac{n-2}{b(n-1)}\left[\frac{n+1}{n-1} + \ln\left(\frac{b}{r_{_h}}\right)\right] - \frac{2\Lambda_{eff}b}{(n-1)^2}\right\} - \left(\frac{n+1}{n-1}\right)\left(2q^2\right)^{(n+1)/4}\ln\left(\frac{r_{_h}}{\ell}\right), & \text{for } \alpha^2 = 1, \\ r_{_h}^{(2\gamma/\alpha)-1}\left[\Upsilon_1\left(\frac{r_{_h}}{b}\right)^{(2\alpha^2)/(1+\alpha^2)} - 2\Lambda b^2\Upsilon\left(\frac{r_{_h}}{b}\right)^{2/(1+\alpha^2)}\right] + \Upsilon_2\left(2q^2\right)^{(n+1)/4} r_{_h}^{(\alpha^2-1)/(\alpha^2+1)}, & \text{for } \alpha^2 < 1, \end{cases} \tag{37}$$

which is useful for exploring the existence of BH and cosmological horizons. To do this, we consider the cases with and without cosmological horizon, separately.

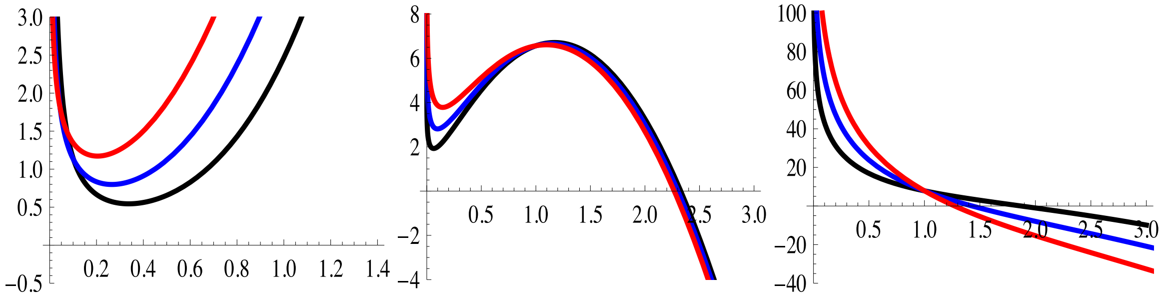

Fig. 5. $m(r_h)$ versus r_h, for $n = 4,\ \ell = 1$: Left: $q = 0.4,\ b = 1.5,\ \alpha = 0.25$ (black), 0.5 (blue), 0.65 (red), Middle: $\alpha = 1,\ b = 0.35,\ q = 0.45$ (black), 0.55 (blue), 0.65 (red), Right: $\alpha = 1,\ b = 0.5,\ q = 1.85$ (black), 2.2 (blue), 2.5 (red).

When there is no cosmological horizon (r_c), by varying r_h from zero to infinity, the function $m(r_h)$ starts from infinity and goes to infinity again. In other words $m(r_h)$ has a minimum value at $m = m_{ext}$. The intersections of lines $m = constant$ with curve $m(r_h)$ determine the location of BH horizons. Therefore, noting the left-hand panel of Fig. 5, our solutions show BHs with inner (r_-) and outer (r_+) horizons provided that $m > m_{ext}$ is chosen. For $m = m_{ext}$ and $m < m_{ext}$ they show extreme BHs and naked singularities, respectively. This issue corresponds to the case presented in Fig. 3.

In the presence of cosmological horizons, $m(r_h)$ starts from infinity and goes to minus infinity as r_h changes from zero to infinity. Thus, noting the number of real roots of $\partial m/\partial r_h = 0$, the following cases are possible: (1) There are two real roots and, thus $m(r_h)$ has a minimum and a maximum at $m = m_{ext}$ and $m = m_{crit}$, respectively. As shown in the middle panel of Fig. 5, the BHs with three inner (r_-), outer (r_+) and cosmological (r_c) horizons exist if m is in the interval $m_{ext} < m < m_{crit}$. Extreme BHs occur for $m = m_{ext}$, and naked singularities with a cosmological horizon can exist for the cases $m < m_{ext}$ and $m \geq m_{crit}$. (2) There is no real root and no minimum or maximum occurs (see the right-hand panel of Fig. 5). Therefore the exact solutions present a naked singularity with cosmological horizon, which does not show a BH. Note that these cases are in agreement with the plots of Fig. 4.

Now, the JF/physical scalar potential $U(\psi)$ can be determined by use of Eqs. (16), (17), (20) and (32). That is,

$$U(\psi) = \begin{cases} 2\psi^{(n+1)/(n-1)}\left[\Lambda_{eff}\psi^{2/((n-1)\beta)} + \Lambda_1 \frac{\ln\psi}{2\beta}\psi^{2/((n-1)\beta)} + \Lambda_2\psi^{(n+1)/[(n-1)\beta]}\right], & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \alpha^2 = 1, & \\ 2\psi^{(n+1)/(n-1)}\left[\Lambda\psi^{(2\alpha)/[(n-1)\beta]} + \Lambda_3\psi^{2/[(n-1)\alpha\beta]} + \Lambda_4\psi^{(n+1)/[(n-1)\alpha\beta]}\right], & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \alpha^2 < 1, & \end{cases} \tag{38}$$

which can be considered as a typical power-law function of ψ. Application of similar functions has produced acceptable results in the context of scalar-tensor cosmology [68,69]. Note that $U(\psi)$, in addition to a scalar term, includes two additional terms which reflect the impacts of charge q and cosmological constant Λ. Also, it recovers the same quantity in the four-dimensional Brans–Dicke–Maxwell theory [63].

Making use of the BH's surface gravity, one can calculate the horizon temperature of BD-conformal-invariant BHs. It can be shown that $\bar{T} = (1/4\pi)\left\{\sqrt{[\mathcal{B}(r)/\mathcal{A}(r)]}d\mathcal{A}(r)/dr\right\}_{r=r_+}$ [70–73]. For our case, noting Eq. (36), one can show that

$$\bar{T} = \frac{1}{4\pi}\left(\frac{b}{r_+}\right)^{(4\beta\gamma)/(n-1)} \frac{d}{dr}\left[\left(\frac{b}{r}\right)^{-(4\beta\gamma)/(n-1)} f(r)\right]_{r=r_+} = \frac{1}{4\pi}\frac{df(r)}{dr}|_{r=r_+} = T. \tag{39}$$

Note that we have used the fact that $f(r_+) = 0$ here. These calculations show that the horizon temperature is the same for both Ed and BD-conformal-invariant BHs. In other words, the BH temperature remains invariant under conformal transformations. Thus the Hawking temperature associated with the outer horizon (r_+) is explicitly written as [59]

$$\bar{T} = \begin{cases} \frac{r_+}{2\pi}\left\{\frac{n-2}{r_+ b}\left[1+\ln\left(\frac{b}{r_+}\right)\right] - \frac{2\Lambda b}{(n-1)r_+} + \frac{(2q^2)^{(n+1)/4}}{n-1}\left[\left(\frac{r_+}{b}\right)^{(n-1)/2} - \frac{n+1}{2}\right] r_+^{-(n+1)/2}\right\}, & \text{for } \alpha^2 = 1, \\ \frac{1+\alpha^2}{4\pi r_+}\left[\frac{n-2}{1-\alpha^2}\left(\frac{b}{r_+}\right)^{-(2\alpha^2)/(1+\alpha^2)} - \frac{2\Lambda b^2}{n-1}\left(\frac{r_+}{b}\right)^{2/(1+\alpha^2)} - \frac{(n+1)(2q^2)^{(n+1)/4}}{2(n+1-2\alpha^2)} r_+^{2-\varepsilon}\right], \text{ with } \varepsilon = \frac{n+1}{1+\alpha^2}, & \text{for } \alpha^2 < 1. \end{cases} \tag{40}$$

The electric charge of BD-conformal-invariant BHs can be calculated by the use of Gauss' electric law [67,74,75]. It can be shown that

$$\bar{Q} = \frac{\omega_{n-1}}{4\pi} q^{(n-1)/2}, \quad \text{for} \quad \alpha^2 \leq 1. \tag{41}$$

Thus, in comparison with the corresponding quantity in Ed theory, the electric charge remains invariant under conformal transformations [59].

As it has been proved, by use of the Euclidean action method, mass, entropy and electric potential of BHs preserve conformal-invariant symmetry [8,67,76]. Thus, for the mass $\bar{M}$, entropy $\bar{S}$ and electric potential $\bar{\Phi}$, we have [59]

$$\bar{M} = \frac{(n-1)\omega_{n-1}}{16\pi(1+\alpha^2)} b^{2\alpha\gamma} m,$$

$$\bar{S} = \frac{\omega_{n-1}}{4} r_+^{n-1} e^{2\alpha\phi}, \tag{42}$$

$$\bar{\Phi}(r_+) = \begin{cases} -cq\ln\left(\frac{r_+}{\ell}\right), & \text{for} \quad \alpha^2 = 1, \\ cq\left(\frac{1+\alpha^2}{1-\alpha^2}\right) r_+^{(\alpha^2-1)/(\alpha^2+1)}, & \text{for} \quad \alpha^2 < 1, \end{cases} \tag{43}$$

where $c = [2/(2-\alpha^2)]b^{(2\alpha^2)/(1+\alpha^2)}$, and $\omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is equal to area of a unit S^{n-1}-sphere [77].

Now, by use of the condition $\mathcal{B}(r_+) = 0$, we calculate the mass parameter $m(r_+)$. The relation between $\mathcal{B}(r)$ and $f(r)$ in Eq. (36) shows that $m(r_+)$ is identical for the EF and JF BHs. Therefore the Smarr mass formula is the same for both Ed -and BD-conformal-invariant BHs [59]. That is,

$$\bar{M}(\bar{S}, \bar{Q}) = M(S, Q). \tag{44}$$

Through a direct calculation, one can show that

$$\bar{\Phi} = \frac{\partial \bar{M}(\bar{S}, \bar{Q})}{\partial \bar{Q}} = \Phi \quad \text{and} \quad \bar{T} = \frac{\partial \bar{M}(\bar{S}, \bar{Q})}{\partial \bar{S}} = T. \tag{45}$$

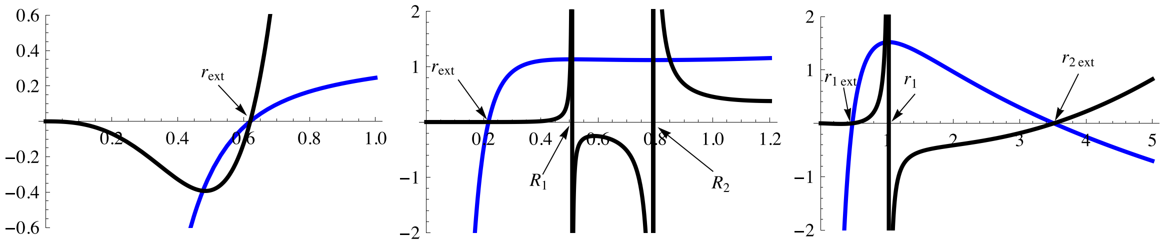

Fig. 6. $\bar{T}$ (blue) and $\bar{C}_{\bar{Q}}$ (black) versus r_+: Left: $n = 4$, $\alpha = 0.4$, $\ell = 1$, $q = 1$, $b = 1.5$, $0.5\bar{T}$, $\bar{C}_{\bar{Q}}$. Middle: $n = 4$, $\alpha = 0.4$, $\ell = 2$, $q = 0.3$, $b = 0.5$, $2\bar{T}$, $0.004\bar{C}_{\bar{Q}}$. Right: $n = 4$, $\alpha = 1$, $\ell = 2$, $q = 1$, $b = 0.7$, $5\bar{T}$, $0.05\bar{C}_{\bar{Q}}$.

Thus, the first law of BH thermodynamics is also valid for the BD-conformal-invariant BHs. That is,

$$d\bar{M} = \bar{T}d\bar{S} + \bar{\Phi}d\bar{Q}. \tag{46}$$

Thermal stability of the BHs can be investigated by use of the canonical ensemble method and noting the signature of the heat capacity. A physically reasonable BH, with positive temperature, is in the stable phase if its heat capacity is positive. Unstable BHs undergo phase transition to be stabilized. The vanishing and diverging points of heat capacity are named as the first- and second-type phase transition points, respectively [78–81].

The BH heat capacity is given via the $\bar{C}_{\bar{Q}} = (\bar{T}\partial\bar{S})/\partial\bar{T}$. As a matter of calculation, one is able to show that

$$\frac{\partial\bar{T}}{\partial\bar{S}} = \begin{cases} \frac{4b^{(1-n)/2}r_+^{(3-n)/2}}{\pi(n-1)\omega_{n-1}}\left[\frac{n+1}{4}(2q^2)^{(n+1)/4}r_+^{-(n+1)/2} - \frac{n-2}{r_+b}\right], & \text{for } \alpha^2 = 1, \\ \frac{-(1+\alpha^2)(r_+/b)^{2\alpha\gamma}}{\pi r_+^n(n-1)\omega_{n-1}}\left[(n-2)\left(\frac{r_+}{b}\right)^{(2\alpha^2)/(1+\alpha^2)} + \frac{2\Lambda b^2(1-\alpha^2)}{n-1}\left(\frac{r_+}{b}\right)^{2/(1+\alpha^2)} + \Gamma\, r_+^{2-\varepsilon}\right], & \text{for } \alpha^2 < 1, \end{cases} \tag{47}$$

where the parameter Γ is defined as $\Gamma = \left[(n+1)(\alpha^2 - n)(2q^2)^{(n+1)/4}\right] / \left[2(n+1-2\alpha^2)\right]$. The plots of $\bar{T}$ and $\bar{C}_{\bar{Q}}$ are shown in Fig. 6 simultaneously, for analyzing thermal stability of physically reasonable BD-conformal-invariant BHs.

The plots show that two different situations are possible for the BHs with $\alpha^2 < 1$: (1) There is only one point of first-type phase transition located at $r_+ = r_{ext}$, and BHs with horizon radii greater than r_{ext} are locally stable (left-hand panel). (2) There is one point of first-type phase transition located at $r_+ = r_{ext}$ and two points of second-type phase transition, which we label as R_1 and R_2, such that $r_{ext} < R_1 < R_2$ (middle panel). The BHs with horizon radii in the ranges $r_{ext} < r_+ < R_1$ and $r_+ > R_2$ are stable. For the BHs with $\alpha^2 = 1$ there are two points of first-type phase transition which we call r_{1ext} and r_{2ext}. Also, there is a point of second-type phase transition which we label as r_1 such that $r_{1ext} < r_1 < r_{2ext}$. The BHs with horizon radii in the interval $r_{1ext} < r_+ < r_1$ are locally stable (right-hand panel).

5. Conclusion

We have started with the action of $(n+1)$-dimensional BD gravity in the presence of power-law non-linear electrodynamics. The related field equations are highly non-linear and too difficult to be solved directly. Thus we have converted the BD action to that of Ed theory by utilizing a mathematical tool known as the conformal transformation. We found that an HD conformal-invariant electrodynamics can be achieved for a suitable choice of non-linearity parameter. Since the static and spherically symmetric Ed-conformal-invariant field equations form an indeterminate system, we have used an exponential ansatz for overcoming this problem. In the special case, when the scalar potential $V(\phi)$ is absent (or is treated as a constant), we found that the BD-conformal-invariant theory is just the HD RN-conformal-invariant which is coupled to the trivial constant scalar field $\psi = 1$. A similar issue which is due to the consideration of conformal-invariant electrodynamics has been reported in the 4D BD-Maxwell theory too [8]. The general solution of Ed theory leads to two classes of conformal-invariant BHs. They are asymptotically non-flat and non-AdS, but, due to conformal-invariant symmetry of electrodynamics, in addition to the horizon-less, extreme, one-horizon and two-horizon BHs, this theory reveals multi-horizon ones too. The appearance of the multi-horizon BHs reveals a quantum effect known as the anti-evaporation phenomena [64]. We obtained two classes of exact solutions for BD-conformal-invariant theory by use of the inverse conformal transformations on the corresponding ones in the Ed theory. They also can produce multi-horizon BHs and exhibit a quantum anti-evaporation effect (Figs. 3 and 4).

The conserved and thermodynamic quantities of BD-conformal-invariant BHs have been calculated by use of the appropriate methods. Through direct calculations, we have shown that the electric charge and Hawking temperature remain invariant under conformal transformations. It is well-known that the entropy-area law is no longer valid for the BD BHs. Making use of the Euclidean action method, it has been shown that BH mass, entropy and electric potential are conformal-invariant and remain unchanged too [67,76]. Then, through the Smarr mass relation, we have shown that the first law of BH thermodynamics is valid for our new BD-conformal-invariant BHs.

The thermal stability of BD BHs has been studied by use of the canonical ensemble method and under the influence of conformal-invariant electrodynamics. It has been found that, for the BHs with $\alpha^2 < 1$, there are two possibilities: (1) The BHs with $r_+ = r_{ext}$ undergo first-type phase transition, and those with horizon radii greater than r_{ext} are locally stable. (2) The BHs with $r_+ = r_{ext}$ experience first-type phase transition while those with horizon radii equal to R_1 and R_2 undergo second-type phase transition. The BHs with horizon radii in the ranges $r_{ext} < r_+ < R_1$ and $r_+ > R_2$ are stable. In the case where $\alpha^2 = 1$, there are two points of first-type phase transition, located at $r_+ = r_{1ext}$ and $r_+ = r_{2ext}$, which is due to consideration of the conformal-invariant electrodynamics. Also, there is a second-type phase transition point, which we named $r_+ = r_1$. It means that stability of the BHs has been affected by conformal symmetry of the utilized electrodynamics. The second-type phase transition has not been reported in the previous works. The BHs with horizon radii in the range $r_{1ext} < r_+ < r_1$ are locally stable (see Fig. 6).

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