



# Metric of a rotating charged magnetized sphere

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## ABSTRACT

Stationary axisymmetric metric describing the exterior field of a rotating, charged sphere endowed with magnetic dipole moment is presented and discussed. It has a remarkably simple multipole structure defined by only four nonzero Hoenselaers-Perjés relativistic moments.

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## 1. Introduction

In 1984, Simon [1] proposed a definition of the relativistic multipole moments for the stationary asymptotically flat electrovacuum spacetimes generalizing the known Geroch-Hansen definition [2,3] developed for the stationary vacuum case. In the general stationary case, the moments are certain symmetric traceless tensors, and in the axisymmetric case each of these tensors gives rise to a scalar multipole moment defined first by Hansen (see also [4]). The practical computation of Simon's multipoles in the latter case – which is only of interest to us in this paper – is facilitated by the Hoenselaers-Perjés (HP) procedure [5], rectified by Sotiriou and Apostolatos [6], according to which the multipoles are expressible in terms of the coefficients in the series expansions of the modified Ernst potentials [7]  $\xi$  and  $\eta$  evaluated on the symmetry axis and related to the usual Ernst potentials  $\mathcal{E}$  and  $\Phi$  by the formulas

$$\xi = \frac{1 - \mathcal{E}}{1 + \mathcal{E}}, \quad \eta = \frac{2\Phi}{1 + \mathcal{E}}. \quad (1)$$

In the papers on exact electrovacuum solutions it is customary to calculate the first four Simon's complex moments  $P_n$  and  $Q_n$ ,  $n = 0, 1, 2, 3$ , which coincide with the corresponding HP coefficients  $m_n$  and  $q_n$  in the expansions ( $z \rightarrow \infty$ )

$$\xi(\rho = 0, z) = \sum_{n=0}^{\infty} m_n z^{-n-1}, \quad \eta(\rho = 0, z) = \sum_{n=0}^{\infty} q_n z^{-n-1}, \quad (2)$$

because they provide one with important information about the physical characteristics of the sources such as the total mass, angular momentum, electric charge or magnetic dipole moment. For all practical applications, the knowledge of Simon's moments higher than  $n = 3$  is not actually needed, and in this respect it would be worthy to note that even the expressions of  $P_n$  and  $Q_n$  for  $n = 4$  and  $5$  were given in [5] with errors detected only fourteen years later [6].

In our recent work [8] we have introduced the notion of the Fodor-Hoenselaers-Perjés (FHP) multipole moments [9] for vacuum spacetimes as an alternative to the Geroch-Hansen (GH) multipoles [2,3]. The objective of the present paper is to extend our previous results to the electrovacuum case and in particular give arguments in favor of introducing the *HP multipole moments* instead of the Simon's

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ones. As a nontrivial example of a spacetime determined by only four nonzero HP moments we shall consider an electrovac metric for a spinning sphere endowed with electric charge and magnetic dipole moment.

## 2. The 5-parameter solution and its 4-parameter subfamily

As has been observed in the paper [8], it is the GH multipole moments  $P_n$  that should be considered approximations to the FHP quantities  $m_n$ , and not the contrary. This is because the knowledge of the axis value of the Ernst potential, which is uniquely determined by the FHP multipoles  $m_n$ , is sufficient for its holomorphic continuation to the whole space and for calculating the corresponding metric functions, and the GH moments  $P_n$  are completely dropped from such a modern solution generating procedure. At the same time, it is also clear that the quantities  $P_n$  and  $m_n$  are rather closely related, the first four of them being identical,  $P_n = m_n$ ,  $n = 0, 1, 2, 3$ , while in the particular case of the Kerr metric [10] the latter equality holds for all  $n$ . The particular 3-parameter solution for a spinning deformed mass considered in [8] is defined by the axis data

$$\mathcal{E}(\rho = 0, z) = \frac{z^2 - Mz - M^2q - iM^2j}{z^2 + Mz - M^2q + iM^2j}, \quad (3)$$

where  $M$ ,  $q$  and  $j$  are, respectively, the mass, dimensionless mass quadrupole moment and dimensionless angular momentum. In the limit  $q = 0$ , the axis data (3) defines the solution describing the exterior field of a rotating sphere because in this case all the corresponding FHP mass multipole moments, except for the monopole one, become equal to zero.

We find it likely, in view of the potential interest for physical and astrophysical applications, to generalize the results of the paper [8] to the electrovacuum case by introducing the additional parameters of electric charge and magnetic dipole moment. Then we must consider the axis data of the form

$$\begin{aligned} \mathcal{E}(\rho = 0, z) &= \frac{z^2 - Mz - M^2q - iM^2j}{z^2 + Mz - M^2q + iM^2j}, \\ \Phi(\rho = 0, z) &= \frac{Mez + iM^2\mu}{z^2 + Mz - M^2q + iM^2j}, \end{aligned} \quad (4)$$

where the electromagnetic field is described by the Ernst potential  $\Phi$  in which the parameters  $e$  and  $\mu$  are the dimensionless charge and the dimensionless magnetic dipole moment, respectively. The latter interpretation can be confirmed by calculating the first four complex multipole moments with the aid of formulas (2), thus yielding for (4)

$$\begin{aligned} m_0 &= M, \quad m_1 = iM^2j, \quad m_2 = M^3q, \quad m_3 = iM^4qj, \\ q_0 &= Me, \quad q_1 = iM^2\mu, \quad q_2 = M^3qe, \quad q_3 = iM^4q\mu, \end{aligned} \quad (5)$$

and the above HP multipoles coincide with the respective Simon's multipole moments. As will be seen below, the explicit formulas for the coefficients  $m_n$  and  $q_n$  determined by the data (4) can be readily found for any  $n$ , whereas the general expressions of the Simon multipoles  $P_n$  and  $Q_n$  for  $n > 5$  have not been computed to date because of their complicated form and scarce significance. Therefore, taking into account that the knowledge of the axis data (4) is sufficient for the construction of the corresponding Ernst potentials in the entire space [11], and also that in general the axis values of the Ernst potentials are defined uniquely by the quantities  $m_n$  and  $q_n$ , it is natural to come to the conclusion that in the stationary axisymmetric case Simon's multipoles  $P_n$  and  $Q_n$  must make room for what we can rightly call the Hoenselaers-Perjés multipole moments  $m_n$  and  $q_n$  which are better adjusted to the intrinsic structure of the stationary electrovacuum solutions and to the modern solution generating techniques. It would be perhaps worthwhile noting that, in our opinion, the subsidiary role of the FHP and HP moments in previous years could be mainly attributed to the common belief of many researchers in the restricted mathematical potential of the Ernst axis data, clearly formulated for example by one of the classics of the generating methods [12]: "It turns out that, although insufficient to determine the metric itself, the knowledge of the Ernst potential on the symmetry axis... is sufficient to determine the multipole moment behavior of the solution". To a large extent, the FHP and HP multipoles may be viewed as a practical realization and amalgamation of the two important projects on exact solutions both started by the pioneering papers of Geroch [2,13,14].

The general concise expressions for the HP multipole moments of the 5-parameter solution defined by the data (4) can be shown to have the form

$$\begin{aligned} m_{2k} &= M^{2k+1}q^k, \quad m_{2k+1} = iM^{2k+2}q^kj, \\ q_{2k} &= M^{2k+1}q^ke, \quad q_{2k+1} = iM^{2k+2}q^k\mu, \quad k = 0, 1, 2, \dots \end{aligned} \quad (6)$$

and the Ernst potentials possessing the above multipole structure can be constructed from (4) by means of Sibgatullin's integral method [11,15] as a particular specialization of the extended  $N$ -soliton electrovac solution [16]. The resulting expressions for  $\mathcal{E}$  and  $\Phi$  are

$$\begin{aligned} \mathcal{E} &= \frac{A - B}{A + B}, \quad \Phi = \frac{C}{A + B}, \\ A &= \sigma_+\sigma_-[(\sigma_+^2 + \sigma_-^2)(1 - e^2) + 2(j^2 - \mu^2)](R_+ + R_-)(r_+ + r_-) \\ &\quad - [2\sigma_+^2\sigma_-^2(1 - e^2) + (\sigma_+^2 + \sigma_-^2)(j^2 - \mu^2)](R_+ - R_-)(r_+ - r_-) \\ &\quad + 4\sigma_+\sigma_-(j^2 + q - qe^2 - \mu^2)(R_+R_- + r_+r_-) \\ &\quad + id(j - e\mu)[\sigma_+(R_+ + R_-)(r_+ - r_-) - \sigma_-(R_+ - R_-)(r_+ + r_-)], \end{aligned}$$

$$\begin{aligned}
B &= Md\{\sigma_+\sigma_-[d(R_+ + R_- + r_+ + r_-) - (1 - e^2)(R_+ + R_- - r_+ - r_-)] \\
&\quad - i(j + je^2 - 2e\mu)[\sigma_-(R_+ - R_-) - \sigma_+(r_+ - r_-)] + ijd[\sigma_-(R_+ - R_-) + \sigma_+(r_+ - r_-)]\}, \\
C &= Md\{e\sigma_+\sigma_-[d(R_+ + R_- + r_+ + r_-) - (1 - e^2)(R_+ + R_- - r_+ - r_-)] \\
&\quad - i(2je - \mu - e^2\mu)[\sigma_-(R_+ - R_-) - \sigma_+(r_+ - r_-)] \\
&\quad + i\mu d[\sigma_-(R_+ - R_-) + \sigma_+(r_+ - r_-)]\}, \\
R_{\pm} &= \sqrt{\rho^2 + (z \pm M\sigma_+)^2}, \quad r_{\pm} = \sqrt{\rho^2 + (z \pm M\sigma_-)^2}, \\
\sigma_{\pm} &= \sqrt{(1 + 2q - e^2 \pm d)/2}, \\
d &= \sqrt{(1 + 2q - e^2)^2 + 4(j^2 - q^2 - \mu^2)},
\end{aligned} \tag{7}$$

and these have been worked out with the aid of the general formulas of the paper [17].

The well-known Kerr-Newman (KN) solution [18] for a charged rotating mass is contained in (7) as the particular case  $q = -j^2$ ,  $\mu = je$ , for which we get from (6)

$$m_n = M(imj)^n, \quad q_n = Me(imj)^n, \quad n = 0, 1, 2, \dots \tag{8}$$

and these  $m_n$  and  $q_n$  coincide with the Simon multipoles  $P_n$  and  $Q_n$  calculated for the KN solution by Sotiriou and Apostolatos [6]. Apparently, the KN spacetime is determined by an infinite set of multipole moments. At the same time, as it follows from (5) and (6), the solution (7) has a very interesting 4-parameter subfamily defined by only four nonzero HP multipole moments which corresponds to the choice  $q = 0$  in (7). Since all the mass-multipole moments, except for the monopole one, in this case are equal to zero, the resulting solution should be interpreted as describing the exterior geometry of a rotating charged magnetized sphere. Anticipating a possible wide interest this 4-parameter electrovac solution might represent to the researchers due to its remarkable multipole structure, in what follows we shall consider it in more detail. First of all, we note that in the case of vanishing  $q$  the Ernst potentials (7) take the form

$$\begin{aligned}
\mathcal{E} &= \frac{A - B}{A + B}, \quad \Phi = \frac{C}{A + B}, \\
A &= \sigma_+\sigma_-[(1 - e^2)^2 + 2(j^2 - \mu^2)](R_+ + R_-)(r_+ + r_-) \\
&\quad + (j^2 - \mu^2)[(1 - e^2)(R_+ - R_-)(r_+ - r_-) + 4\sigma_+\sigma_-(R_+R_- + r_+r_-)] \\
&\quad + id(j - e\mu)[\sigma_+(R_+ + R_-)(r_+ - r_-) - \sigma_-(R_+ - R_-)(r_+ + r_-)], \\
B &= Md\{\sigma_+\sigma_-[d(R_+ + R_- + r_+ + r_-) - (1 - e^2)(R_+ + R_- - r_+ - r_-)] \\
&\quad - i\sigma_-(j + je^2 - 2e\mu - jd)(R_+ - R_-) + i\sigma_+(j + je^2 - 2e\mu + jd)(r_+ - r_-)\}, \\
C &= eB - iMd(je - \mu)[\sigma_-(1 - e^2 + d)(R_+ - R_-) \\
&\quad - \sigma_+(1 - e^2 - d)(r_+ - r_-)], \\
R_{\pm} &= \sqrt{\rho^2 + (z \pm M\sigma_+)^2}, \quad r_{\pm} = \sqrt{\rho^2 + (z \pm M\sigma_-)^2}, \\
\sigma_{\pm} &= \sqrt{(1 - e^2 \pm d)/2}, \quad d = \sqrt{(1 - e^2)^2 + 4(j^2 - \mu^2)},
\end{aligned} \tag{9}$$

while the corresponding metric functions  $f$ ,  $\gamma$  and  $\omega$ , which enter the line element

$$ds^2 = f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f(dt - \omega d\varphi)^2, \tag{10}$$

can be worked out from the respective general expressions of the paper [17], yielding

$$\begin{aligned}
f &= \frac{A\bar{A} - B\bar{B} + C\bar{C}}{(A + B)(\bar{A} + \bar{B})}, \quad e^{2\gamma} = \frac{A\bar{A} - B\bar{B} + C\bar{C}}{16d^4|\sigma_+|^2|\sigma_-|^2R_+R_-r_+r_-}, \\
\omega &= -\frac{\text{Im}[G(\bar{A} + \bar{B}) + C\bar{I}]}{A\bar{A} - B\bar{B} + C\bar{C}}, \\
G &= 2(imj - z)B + Md\{\sigma_+\sigma_-(2 - e^2)[\sigma_-(R_+ + R_-)(r_+ - r_-) \\
&\quad - \sigma_+(R_+ - R_-)(r_+ + r_-)] - (2j^2 - \mu^2)[\sigma_-(R_+ - R_-)(r_+ + r_-) \\
&\quad - \sigma_+(R_+ + R_-)(r_+ - r_-)] - id(2j - e\mu)(R_+ - R_-)(r_+ - r_-)\} \\
&\quad - iMe(je - \mu)[(1 - e^2)(R_+ - R_-)(r_+ - r_-) \\
&\quad - 2\sigma_+\sigma_-(R_+ + R_-)(r_+ + r_-) + 4\sigma_+\sigma_-(R_+R_- + r_+r_-)] \\
&\quad + 2M^2d[2j^2\sigma_+\sigma_-[\sigma_+(R_+ - R_-) - \sigma_-(r_+ - r_-)] + \mu(je - \mu)[\sigma_-(R_+ - R_-) - \sigma_+(r_+ - r_-)] \\
&\quad - ie\sigma_+\sigma_-(je - \mu)(R_+ + R_- - r_+ - r_-)],
\end{aligned}$$

$$\begin{aligned}
I = & -zC + Md[\sigma_+(e\sigma_-^2 + j\mu)(R_+ + R_-)(r_+ - r_-) - \sigma_-(e\sigma_+^2 + j\mu) \\
& \times (R_+ - R_-)(r_+ + r_-) - 2i\sigma_+\sigma_-(je - \mu)(R_+R_- - r_+r_- + 2M^2d)] \\
& - iM\{\sigma_+\sigma_-(1 + e^2)(je - \mu)[2(R_+R_- + r_+r_-) - (R_+ + R_-)(r_+ + r_-)] \\
& + [2(je + \mu)(j^2 - \mu^2) + e(1 - e^2)(j - e\mu)](R_+ - R_-)(r_+ - r_-)\} \\
& + M^2d[2(j\mu + e\mu^2 - 2j^2e)[\sigma_+(r_+ - r_-) - \sigma_-(R_+ - R_-)] \\
& + i\sigma_+\sigma_-[\mu d(R_+ + R_- + r_+ + r_-) - (4je - 3\mu - e^2\mu)(R_+ + R_- - r_+ - r_-)]\}.
\end{aligned} \tag{11}$$

Moreover, the nonzero electric and magnetic components of the electromagnetic 4-potential are given by the formulas

$$A_t = -\text{Re}\left(\frac{C}{A+B}\right), \quad A_\varphi = \text{Im}\left(\frac{I}{A+B}\right), \tag{12}$$

so that we have fully described the gravitational and electromagnetic fields of the rotating sphere carrying both the electric charge and magnetic dipole moment.

Evidently, due to its finite multipole structure, the 4-parameter solution (9)-(12) does not contain the Kerr and KN spacetimes as particular cases, and therefore it differs from the 4-parameter solution describing the magnetized KN source [19,20]. However, in the limit of zero angular momentum ( $j = 0$ ), both solutions coincide, representing a magnetized Reissner-Nordström mass. It is, therefore, the nonzero angular momentum sector of the solution (7) that makes it very special and attractive from the physical point of view.

### 3. Discussion

It is easy to see that formulas (9) and (11) considerably simplify if the parameters  $j$ ,  $e$  and  $\mu$  are subject to the constraint  $\mu = ej$  which represents the same gyromagnetic ratio of the electron as in the KN metric (see e.g. [21] for a discussion of this ratio in the context of exact electrovac solutions). In that subfamily, the potential  $\mathcal{E}$  becomes an analytic function of  $\Phi$  and hence such a 3-parameter subfamily can be treated within the framework of the well-known Ernst-Harrison (EH) charging transformation [7,22] involving the nonzero parameter  $e$ . Since in the known practical applications of the EH transformation the values of  $e$  are usually restricted to the “undercharged” case  $e < 1$ , it would be instructive to see how this transformation works in the “overcharged” case  $e > 1$  too. With this idea in mind, we first set  $\mu = ej$  in the formulas (9) and (11), and then rewrite the resulting solution by rescaling the quantities  $\sigma_\pm$  and introducing

$$m = M\varepsilon, \quad j = j/\varepsilon, \quad \varepsilon \equiv \sqrt{1 - e^2}, \tag{13}$$

thus finally yielding for the Ernst potentials  $\mathcal{E}$  and  $\Phi$  the expressions

$$\begin{aligned}
\mathcal{E} &= \frac{\varepsilon A - B}{\varepsilon A + B}, \quad \Phi = \frac{eB}{\varepsilon A + B}, \\
A &= \sigma_+\sigma_-(1 + 2j^2)(R_+ + R_-)(r_+ + r_-) + j^2[(R_+ - R_-)(r_+ - r_-) \\
&+ 4\sigma_+\sigma_-(R_+R_- + r_+r_-)] + ij\delta[\sigma_+(R_+ + R_-)(r_+ - r_-) - \sigma_-(R_+ - R_-)(r_+ + r_-)], \\
B &= m\delta\{\sigma_+\sigma_-[(\delta - 1)(R_+ + R_-) + (\delta + 1)(r_+ + r_-)] \\
&+ ij[\sigma_-(\delta - 1)(R_+ - R_-) + \sigma_+(\delta + 1)(r_+ - r_-)]\}, \\
R_\pm &= \sqrt{\rho^2 + (z \pm m\sigma_+)^2}, \quad r_\pm = \sqrt{\rho^2 + (z \pm m\sigma_-)^2}, \\
\sigma_\pm &= \sqrt{(1 \pm \delta)/2}, \quad \delta = \sqrt{1 + 4j^2},
\end{aligned} \tag{14}$$

and for the metric functions  $f$ ,  $\gamma$ ,  $\omega$  the expressions

$$\begin{aligned}
f &= \frac{\varepsilon\bar{\varepsilon}A\bar{A} - \varepsilon^2B\bar{B}}{(\varepsilon A + B)(\bar{\varepsilon}\bar{A} + \bar{B})}, \quad e^{2\gamma} = \frac{\varepsilon\bar{\varepsilon}A\bar{A} - \varepsilon^2B\bar{B}}{16\delta^4|\varepsilon|^2|\sigma_+|^2|\sigma_-|^2R_+R_-r_+r_-}, \\
\omega &= -\frac{\text{Im}[G(\bar{\varepsilon}\bar{A} + \bar{B}) + eB\bar{I}]}{\varepsilon\bar{\varepsilon}A\bar{A} - \varepsilon^2B\bar{B}}, \\
G &= 2(imj - z)B + \frac{1 + \varepsilon^2}{\varepsilon}m\delta\{\sigma_+\sigma_-[\sigma_-(R_+ + R_-)(r_+ - r_-) \\
&- \sigma_+(R_+ - R_-)(r_+ + r_-)] - j^2[\sigma_-(R_+ - R_-)(r_+ + r_-) \\
&- \sigma_+(R_+ + R_-)(r_+ - r_-)] - ij\delta(R_+ - R_-)(r_+ - r_-)\} \\
&+ 4m^2j^2\delta\sigma_+\sigma_-[\sigma_+(R_+ - R_-) - \sigma_-(r_+ - r_-)], \\
I &= -ezB + \frac{e}{\varepsilon}m\delta[\sigma_+(\sigma_-^2 + j^2)(R_+ + R_-)(r_+ - r_-) \\
&- \sigma_-(\sigma_+^2 + j^2)(R_+ - R_-)(r_+ + r_-) - ij\delta(R_+ - R_-)(r_+ - r_-)] \\
&+ m^2je\delta\{2j[\sigma_-(R_+ - R_-) - \sigma_+(r_+ - r_-)] \\
&+ i\sigma_+\sigma_-[(\delta - 1)(R_+ + R_-) + (\delta + 1)(r_+ + r_-)]\}.
\end{aligned} \tag{15}$$

Formulas (13) and (14) determine the essence of the EH transformation: this symmetry transformation generates an electrovac solution from a given vacuum one by introducing the charge parameter  $e$  by means of the relations (14) for  $\mathcal{E}$  and  $\Phi$ . At the same time, as it follows from (13), while in the undercharged case ( $e < 1$ ) the rescaled parameters  $m$  and  $j$  remain real-valued, the latter parameters become pure imaginary in the overcharged case  $e > 1$ , which, however, does not mean that the branch  $e > 1$  of the EH transformation is unphysical – one simply has to take into account the relation of the parameters  $m$  and  $j$  to the physical mass  $M$  and physical angular momentum  $J$  given in (13). Mention also that in the overcharged case the expression for the metric function  $\gamma$ , as can be seen in (15), is not of the same form as in the vacuum solution because  $\varepsilon$  is pure imaginary for  $e > 1$ . All these subtleties may explain why the application of the EH transformation is restricted in the literature to the simpler undercharged case  $e < 1$  only, but we emphasize that our formulas (14)–(15) are valid for arbitrary  $e$ .

Another interesting subclass of the solution (9)–(11) is defined by the condition of vanishing electric charge  $e = 0$ . The resulting 3-parameter metric could be interpreted as a rotating magnetized sphere, thus being appropriate for the description of the exterior field of a neutron star with negligible deformation. The expressions of the potentials  $\mathcal{E}$  and  $\Phi$  in this case take the form

$$\begin{aligned}\mathcal{E} &= \frac{A-B}{A+B}, \quad \Phi = \frac{i\mu C}{A+B}, \\ A &= \sigma_+ \sigma_- (1 + 2j^2 - 2\mu^2)(R_+ + R_-)(r_+ + r_-) \\ &\quad + (j^2 - \mu^2)[(R_+ - R_-)(r_+ - r_-) + 4\sigma_+ \sigma_- (R_+ R_- + r_+ r_-)] \\ &\quad + ijd[\sigma_+(R_+ + R_-)(r_+ - r_-) - \sigma_-(R_+ - R_-)(r_+ + r_-)], \\ B &= Md\{\sigma_+ \sigma_- [(d-1)(R_+ + R_-) + (d+1)(r_+ + r_-)] \\ &\quad + ij[\sigma_-(d-1)(R_+ - R_-) + \sigma_+(d+1)(r_+ - r_-)]\}, \\ C &= Md[\sigma_-(d+1)(R_+ - R_-) + \sigma_+(d-1)(r_+ - r_-)], \\ R_{\pm} &= \sqrt{\rho^2 + (z \pm M\sigma_{\pm})^2}, \quad r_{\pm} = \sqrt{\rho^2 + (z \pm M\sigma_{\pm})^2}, \\ \sigma_{\pm} &= \sqrt{(1 \pm d)/2}, \quad d = \sqrt{1 + 4(j^2 - \mu^2)},\end{aligned}\tag{16}$$

whereas for the corresponding metric coefficients we readily get

$$\begin{aligned}f &= \frac{A\bar{A} - B\bar{B} + \mu^2 C\bar{C}}{(A+B)(\bar{A}+\bar{B})}, \quad e^{2\gamma} = \frac{A\bar{A} - B\bar{B} + \mu^2 C\bar{C}}{16d^4 |\sigma_+|^2 |\sigma_-|^2 R_+ R_- r_+ r_-}, \\ \omega &= -\frac{\text{Im}[G(\bar{A} + \bar{B}) + \mu^2 C\bar{I}]}{A\bar{A} - B\bar{B} + \mu^2 C\bar{C}}, \\ G &= 2(iMj - z)B + Md\{2\sigma_+ \sigma_- [\sigma_-(R_+ + R_-)(r_+ - r_-) \\ &\quad - \sigma_+(R_+ - R_-)(r_+ + r_-)] - (2j^2 - \mu^2)[\sigma_-(R_+ - R_-)(r_+ + r_-) \\ &\quad - \sigma_+(R_+ + R_-)(r_+ - r_-)] - 2ijd(R_+ - R_-)(r_+ - r_-)\} \\ &\quad + 2M^2 d\{2j^2 \sigma_+ \sigma_- [\sigma_+(R_+ - R_-) - \sigma_-(r_+ - r_-)] \\ &\quad - \mu^2 [\sigma_-(R_+ - R_-) - \sigma_+(r_+ - r_-)]\}, \\ I &= -zC + M\{\sigma_+ \sigma_- [2(R_+ R_- + r_+ r_-) - (R_+ + R_-)(r_+ + r_-)] \\ &\quad - 2(j^2 - \mu^2)(R_+ - R_-)(r_+ - r_-)\} \\ &\quad + Md\{2\sigma_+ \sigma_- (R_+ R_- - r_+ r_- + 2M^2 d) \\ &\quad + ij[\sigma_-(R_+ - R_-)(r_+ + r_-) - \sigma_+(R_+ + R_-)(r_+ - r_-)]\} \\ &\quad + M^2 d\{\sigma_+ \sigma_- [(d+3)(R_+ + R_-) + (d-3)(r_+ + r_-)] \\ &\quad + 2ij[\sigma_-(R_+ - R_-) - \sigma_+(r_+ - r_-)]\}.\end{aligned}\tag{17}$$

Note that in the above formulas (16) and (17) the functions  $C$  and  $I$  have been slightly redefined compared with  $C$  and  $I$  in (9) and (11), in order to make more visual the appearance of the factor  $i\mu$  in the zero charge case. With such redefinitions, the expressions (12) for  $A_t$  and  $A_\varphi$  also slightly change, namely,

$$A_t = -\text{Re}\left(\frac{i\mu C}{A+B}\right), \quad A_\varphi = \text{Im}\left(\frac{i\mu I}{A+B}\right).\tag{18}$$

Apparently, although the solution for a rotating magnetized sphere, like the solution (14)–(15) before, is a 3-parameter specialization of the general metric (9)–(11), it has only three nonzero HP multipole moments, while in the previous example the three parameters define four nonzero moments. Since both electrovac solutions considered in this section have the same pure vacuum limit determined by two gravitational multipoles, representing mass and angular momentum, we would like to briefly comment in conclusion on an old misleading statement, originally made in [23] and readily adopted by various authors as a true one, according to which *any stationary, asymptotically flat solution to Einstein's equation approaches asymptotically the Kerr solution*. Intuitively, the idea of this statement might look plausible at first glance, as for instance any stationary axisymmetric asymptotically flat solution with nonzero mass and nonzero total angular momentum

would indeed have the same leading mass and rotational moments as in the Kerr solution. However, a simple counterexample to the above statement is a *stationary* solution for two counter-rotating Kerr sources [24] with nonzero mass and zero total angular momentum because the Kerr solution in absence of the angular momentum reduces to the *static* Schwarzschild spacetime. Moreover, the case of zero total mass which was remarked to be also suitably treated in [23] would have nothing to do with the Kerr solution since the limit  $M = 0$  in the latter is just the Minkowski space, whereas there is an infinite number of stationary asymptotically flat spacetimes with zero total mass.

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