Algebraic Operations via Solvable Lattice Models

A DISSERTATION
SUBMITTED TO THE FACULTY OF THE GRADUATE SCHOOL
OF THE UNIVERSITY OF MINNESOTA
BY

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IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

Ben Brubaker

July, 2022

Acknowledgements

Over the past six years of graduate school, there have been a huge number of people who have helped me in many ways and to whom I am grateful. It is impossible to do everyone justice, or even to mention everyone, so any such list is necessarily incomplete.

I would like to start by thanking my committee: Professors Ben Brubaker, Gregg Musiker, Vic Reiner, and Craig Westerland. In addition, I would like to thank my oral exam committee, which included Ben and Vic, and also Professors Sasha Voronov and Peter Webb.

Thanks also to those professors who wrote me recommendation letters during my job search: Ben Brubaker, Daniel Bump, Bryan Mosher, Vic Reiner, and Karen Saxe. Like exam committees, writing letters is an often-thankless task we ask of people we admire and are already indebted to, but it is so important to each individual and for our community, and I thank you for doing this for me.

There are tremendous numbers of professors, postdocs, and students at the University of Minnesota and elsewhere with whom I have collaborated, started a collaboration with, or just had fruitful mathematical discussions with. Frequently, these discussions have been very one-sided and featured me asking many repetitive questions while they patiently answer. It is impossible to thank everyone in this category, but here is a partial list: Esther Banaian, Neelima Borade, Ben Brubaker, Valentin Buciumas, Daniel Bump, Sunita Chepuri, CJ Dowd, Claire Frechette, Darij Grinberg, Henrik Gustafsson, Iva Halacheva, Tom Halverson, Daoji Huang, Matt Huynh, Elizabeth Kelley, Richard Kenyon, Allen Knutson, Christian Korff, Thomas Lam, Jared Marx-Kuo, Vaughan McDonald, Greg Michel, Slava Naprienko, John O'Brien, Nick Ovenhouse, Leonid Petrov, Arun Ram, Vic Reiner, Siddhartha Sahi, Travis Scrimshaw, Ben Strasser, Eric Stucky, Emily Tibor, Justin Troyka, Henry Twiss, Alex Vetter, Katy Weber, Ian Whitehead, Craig Westerland, Lauren Williams, Sylvester Zhang, Valerie Zhang, and Paul Zinn-Justin. In addition, my thesis work was partially supported by NSF RTG grants DMS-1745638 and NSF RTG grant DMS-1148634, and by the University of Minnesota's Doctoral Dissertation Fellowship.

There are many people at Carleton College, where I did my undergraduate degree, who played important roles getting me started in mathematics. These include Professors Eric Egge, Deanna Haunsperger, Mark Krusemeyer, Helen Wong, and many more.

Thanks to my Vincent 524 officemates: Montie Avery, Emily Gullerud, Greg Michel, Nora Mosesov, Ben Strasser, Jasper Weinburd, and others. We never did do anything with our office Facebook page . . . .

I would also like to thank my academic siblings: Heidi Goodson, Will Grodzicki, Ben Strasser, Katy Weber, Claire Frechette, Emily Tibor, Meagan Kenney, and Suki Dasher. To Katy, Claire, and Emily, the three of you were my day-to-day of graduate school. Though our "team name" sometimes changed, our fun never did. It has been a great honor to learn alongside such brilliant and kind people.

To Ben Strasser, you were the biggest reason I finally took the plunge and applied to graduate school, and the reason I first met Ben Brubaker. Your enthusiasm for the math you were doing and the culture of the department, not to mention our shared hobby of talking about the old Carleton swim team days, made me excited to go to graduate school and do math. I likely would not be where I am today without this. Not only that, but you helped me time and time again during my first couple years of graduate school when I was first stepping into the wider world of mathematics. You have been, and remain, one of my closest friends.

To Ben Brubaker, you've been the ideal advisor: a positive role model. Your ability to step back from a problem and see the big picture, to connect to other areas, and also to dive in and isolate the key argument, is remarkable. My goal throughout graduate school has not only been to learn math from you, but to learn how to do math like you. At that, I have only partially succeeded, but it's been a highly important journey for me, and a necessary one since these are the skills that make a mathematician. Add to that your extreme generosity with your time. During some stretches, I've marveled at how you have time to do anything other than meet with me in all the various contexts, never mind simultaneously working with four other students and other collaborators, being Department Head, and raising three kids. Throughout, you've never been anything other than completely understanding when I've had family conflicts or just a slow week.

To my family: my parents John and Jeanne, my siblings Madeleine, Will, and Griffin, plus significant others, aunts, uncles, cousins, and grandparents, along with a few tiny nieces and nephews: thank you for always being there for me throughout many years. I have always felt that I could pursue any career, any life I want, without losing even a drop of emotional support. What a privilege to go through life knowing that the most important thing, family, is always there for me.

Finally, to Skylar Zhang, my wife, my love, my person, you've been the story of my time in graduate school, and of my life going forward. Thanks for so many things, including listening to me and giving suggestions when I'm stuck, supporting me when I need it, and overall helping me appreciate the parts of life that are really the most important.

Dedication

To my grandparents, both living and deceased: Joan Wham, Bill Wham, Mary Ann Wham, John Hardt, and Mary Hardt.

Abstract

This thesis explores solvable lattice models in several contexts. Our overarching goal is to understand and exploit the flexibility of lattice models in their ability to express functorial operations in algebra. In particular, we study lattice models whose partition functions are special functions in representation theory and Schubert calculus. These functions tend to have nice properties relating to their algebraic structure, and we try to connect these properties to combinatorial operations on lattice models.

Chapter 2 studies the connection between solvable lattice models and discrete-time Hamiltonian operators. We give general conditions for the existence of a Hamiltonian operator whose discrete time evolution matches the partition function of certain solvable lattice models. In particular, we examine two classes of lattice models: the classical six-vertex model and a generalized family of $(2n+4)$-vertex models. These models depend on a statistic called charge, and are associated to the quantum group $U_q(\widehat{\mathfrak{gl}}(1|n))$ [1]. Our results show a close and unexpected connection between Hamiltonian operators and solvability.

The six-vertex model can be associated with Hamiltonians from classical Fock space, and we show that such a correspondence exists precisely when the Boltzmann weights are free fermionic. This allows us to prove that the free fermionic partition function is always a (skew) supersymmetric Schur function and then use the Berele-Regev formula to correct a result from [2]. Then, we prove a sharp solvability criterion for the six-vertex model with charge that provides the proper analogue of the free fermion condition. Building on results in [3], we show that this criterion exactly dictates when a charged model has a Hamiltonian operator acting on a Drinfeld twist of q-Fock space. The resulting partition function is then always a (skew) supersymmetric LLT polynomial.

Chapter 3 considers the connections between lattice models and formal group laws. In particular, we exhibit a substitution corresponding to any formal group law into any solution to the Yang-Baxter equation. When applied to the R-matrix from the standard evaluation module for $U_q(\widehat{\mathfrak{sl}}_{n+1})$, the resulting lattice models are related to those studied in [4], and their partition functions may have interpretations in higher cohomology of Schubert varieties. Then, Chapter 4 gives an exposition of lattice model proofs of some well-known identities for Schur polynomials. In addition, we introduce what we call a *symmetrized* version of the Yang-Baxter algebra and show how the Schur polynomial identities come from relations in this algebra.

Running through this work is a thematic aspiration: that lattice models are "unreasonably effective" (in Ben Brubaker's words) in expressing various phenomena throughout mathematics.

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Chapter 1

Introduction

The topic of this thesis is solvable lattice models and their ability to express functorial operations in algebra. We will study in particular the *partition function* of the lattice model. This function has several different algebraic interpretations, and these interpretations encode important facts about the representation theory, geometry, and combinatorics associated to the partition function. Roughly, we are playing the following game:

- Start with a polynomial shadow of a mathematical object (for instance, the character of a representation or a cohomology class representative).
- Find a lattice model whose partition function equals that polynomial (this is often very difficult!).
- Understand functorial operations on the polynomial in terms of operations on the lattice model.

By "functorial operations", what we really mean is transport of structure. An operation on the underlying algebraic or geometric object induces an operation on the associated polynomial. Our goal is to use lattice models to induce the same operation on the partition function. When this happens, our understanding is in a sense "categorical". Rather than simply representing the function, the lattice model also encodes its properties and relationships to similar functions. In addition we hope to observe combinatorial features of the underlying object in the combinatorics of the lattice model. There are many examples of these phenomena in the literature. We briefly explain two examples here.

First, consider Schur polynomials. They are (roughly) characters for $GL_n(\mathbb{C})$-representations. As such, they naturally obey branching rules induced by the restriction $GL_n \longrightarrow GL_{n-1}$. These branching rules have a nice combinatorial model via *Gelfand-Tsetlin* patterns, which are triangular arrays with an interleaving property, and which index basis vectors for each GL_n representation. The Schur polynomial is thus a sum over Gelfand-Tsetlin patterns, and each coefficient in the sum is a monomial. To observe the branching rule, we sum over all possible second rows of the Gelfand-Tsetlin pattern. Moreover, this combinatorial rule is particularly nice in it is manifestly multiplicity one, an unusual property in representation theory, but one that holds for GL_n.

Lattice models naturally encode branching rules in a similar way: fill the top row of the lattice in an admissible way, and sum over all of these partial fillings. See the next figure for the correspondence between lattice model states and Gelfand-Tsetlin patterns. There are several different lattice models whose partition functions are Schur polynomials (see Chapter 4), and this process gives the correct branching rule in each case. Furthermore, in the case of a particular lattice model, called Γ_B in Chapter 4, the states of the model correspond precisely to Gelfand-Tsetlin patterns, and so the lattice model simultaneously understands Schur polynomials both as a generating function and in terms of their branching rule. In addition, adding a row to the lattice model is equivalent to induction from $GL_{n-1} \longrightarrow GL_n$. In is well-known that these functors are adjoint, and in the lattice model, this is naturally expressed by the fact that adding and removing a row are inverses of one another, and therefore give the same structure constants. Finally, one proves the symmetry of the Schur polynomial via repeated applications of the Yang-Baxter equation. This proof is combinatorially evocative: to swap variables in the Schur polynomial we literally swap rows of the lattice model.

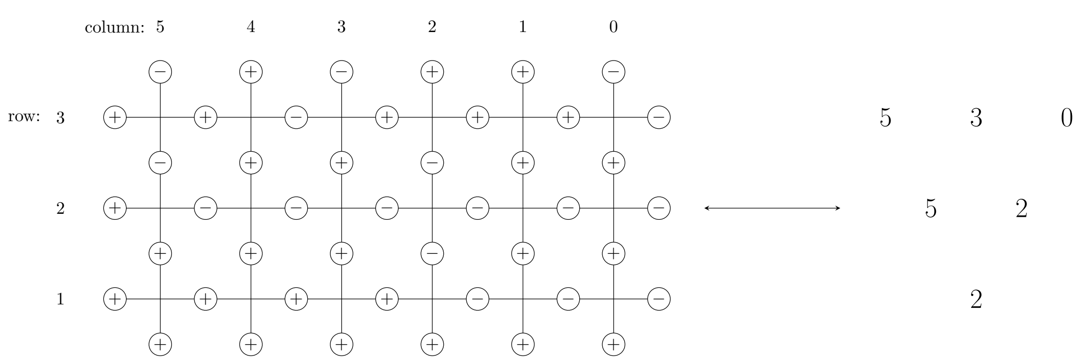


Figure 1.1: Correspondence between states of the six-vertex model and Gelfand-Tsetlin patterns. In both cases, summing over all allowable second rows and removing the top row gives the restriction $GL_n \longrightarrow GL_{n-1}$, while adding an extra row and summing over all possibilities for that row gives the induction $GL_n \longrightarrow GL_{n+1}$.

For the second example, consider (non-Grassmannian) Schubert polynomials. A lattice model for these polynomials is given in [4] ($\beta = q = 0$), after much other work in the area, including by Fomin and Kirillov [5]. These lattice models are generating functions: their states biject with pipe dreams. As pipe dreams relate to the Grobner geometry of the Schubert varieties [6], the lattice models have a geometric connection as a generating function. The Yang-Baxter equation doesn't prove symmetry–after all, the Schubert polynomials aren't symmetric. Rather, we obtain divided-difference operators relating consecutive Schubert polynomials in the Bruhat order. These operators also have a geometric interpretation: they correspond to the action on cohomology of push-pull operators arising from taking a quotient by a parabolic subgroup. The lattice model doesn't "see" this geometry directly; yet it has rich connections to some of the most important geometric operations. The train argument is shown in the Figure below.

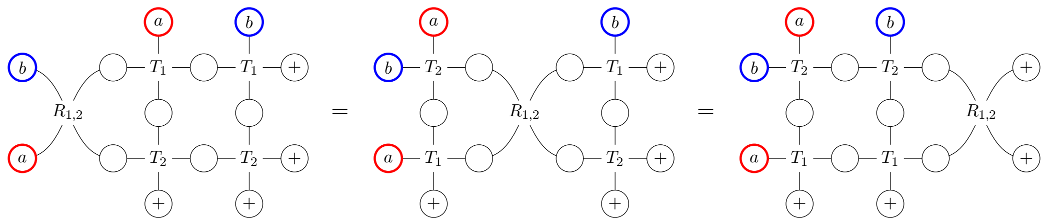


Figure 1.2: An example of the train argument, as it applies to the lattice models in [4]. By evaluating both sides of this equation, one obtains Demazure operators for Schubert polynomials.

There are many other operations in geometry and algebra that have lattice model interpretations, at least in certain cases.

- The use of *color* in lattice models corresponds to refinement of symmetric functions into "atoms". See [7–10].
- Formal group law substitutions in equivariant (higher) cohomology can be done on the level of Boltzmann weights (at least up to connective K-theory), and these substitutions preserve solvability. See [4] and Chapter 3.
- The particle dynamics of Fock space representations match the particle dynamics of free fermionic lattice models. See [11–13], and Chapter 2.
- Both the R-matrix itself and the Demazure operators arising from the Yang-Baxter equation generate a Hecke algebra, and this often reflects a braiding in the module category for some quantum affine algebra or superalgebra In fact, the *Yang-Baxter algebra* of the lattice model, which we discuss in Chapter 4 is sometimes isomorphic to such an algebra. See also [14, 15].
- Cauchy-type identities for symmetric functions are given by commutation relations in the Yang-Baxter algebra, which combinatorially arise from applications of the train argument See [4, 15, 16].

- Product formulas in the form of Knutson-Tao puzzles also arise from the Yang-Baxter equation, and the effect of the Yang-Baxter equation on the path dynamics is precisely the application of the puzzle rule. See [17–20].

This thesis consists of three pieces. The main piece, Chapter 2, whose content also appears in [13], discusses Hamiltonian operators corresponding to lattice models. We study when these two discrete dynamical systems have equivalent particle dynamics. In the case of the six-vertex model, we show that the dynamics of the six-vertex model match those of classical Fock space precisely at the free fermion point, and we use this fact to evaluate the free fermionic partition function for both the "empty" and "domain wall" boundary conditions. Then we solve a similar problem for lattice models with an extra statistic called *charge*. We prove a solvability criterion for these lattice models and show that precisely this criterion ensures a matching Hamiltonian from the *metaplectic Fock spaces* studied in [3].

The other two pieces are more expositive and more speculative. In Chapter 3, we discuss equivariant cohomology, formal group laws, and lattice models. We show that every Yang-Baxter equation yields a parametrized family indexed by formal group laws. For Schubert classes up to connective K-theory, this substitution has the correct cohomological interpretation [4], and we discuss possible extensions and implications.

Finally, in Chapter 4, we explore combinatorial symmetries of the six-vertex model, and use this to give an augmented version of the Yang-Baxter algebra. We pair this with an exposition of lattice models for Schur polynomials, and use relations in the algebra to prove standard identities for Schur polynomials.

Chapter 2

Hamiltonian Operators

2.1 Introduction

This chapter discusses the connections between two types of integrability phenomena arising from quantum groups: solvable lattice models and Hamiltonian operators coming from Heisenberg algebras. Solvable lattice models are parametrized by modules of quantum groups, and are sources of many special functions from geometry, representation theory, and symmetric function theory. The word "solvable" refers to the existence of a solution to the Yang-Baxter equation, and such a solution guarantees that the partition function of the model has symmetries or functional equations. Examples of special functions that have been studied using lattice models include Macdonald polynomials [21, 22], Grothendieck polynomials [4, 18–20, 23], LLT polynomials [24–26], and metaplectic Whittaker functions [1, 27, 28].

Hamiltonian operators have long been studied in physics, including in soliton theory and hierarchies of differential equations [29]. Discrete time evolution Hamiltonians have been used to study special functions like Schur polynomials [12] and Hall-Littlewood polynomials [30]. A general framework for such constructions from a combinatorial perspective was given by Lam [31]. He considered an arbitrary Heisenberg algebra representation and proved a generalized boson-fermion correspondence, obtaining two dual families of polynomials associated to each Hamiltonian operator. Furthermore, Lam showed these functions have Pieri and Cauchy identities determined by the structure constants of the Heisenberg algebra. Both Macdonald polynomials and LLT polynomials fit into this framework, as do many related polynomials and specializations.

We consider Heisenberg algebra representations with bases indexed by partitions, along with dual representations whose dual basis is also indexed by partitions. One obtains families of "skew" functions from these representations by choosing an element of the Heisenberg algebra, acting on a basis vector in the representation and pairing it with a dual basis vector. We use bra-ket notation to express this. If h is an element of the Heisenberg algebra, then

$$\langle\mu|h|\lambda\rangle = \langle\mu|h \otimes 1|\lambda\rangle = \langle\mu|1 \otimes h|\lambda\rangle$$

refers to the pairing of $\langle\mu|$ with the vector obtained from the action of h on $|\lambda\rangle$, or equivalently, the pairing of $\lambda\rangle$ with the vector obtained from the action of h on $\langle\mu|$.

Our Hamiltonians are exponential operators involving infinite sums in the Heisenberg algebra of the form

$$H = \sum_{k\geq 1} s_k J_k, \qquad e^H = \sum_{m=1}^{\infty} \frac{H^m}{m!},$$

where J_k is the kth *current operator* which in the case of *Fock space* acts on particles by displacing them by k units. We thus obtain a family of functions

$$\sigma_{\lambda/\mu} := \langle\mu|e^H|\lambda\rangle,$$

which are sometimes called *τ-functions* in the literature [12, 32].

Since up to isomorphism there is only one Heisenberg algebra and only one highest-weight representation of this algebra [33, Proposition 2.1], our functions really depend on the *realization* of the representation i.e. our choice of basis. There are two ways to modify our family of polynomials:

- Change the Heisenberg algebra generators via a substitution;
- Change the (realization of the) representation of the Heisenberg algebra;

We will see that the first type of modification corresponds to changing the Boltzmann weights of our lattice model, while the second corresponds (at least in the cases we consider) to choosing a lattice model with a different structure. In this chapter, we look at two well known representations of Heisenberg algebras. The first, (classical) Fock space, is a representation of the Lie algebra $\mathfrak{gl}_\infty$ that has been studied in many places e.g. [32,33]. The second, q-Fock space, is a quotient of the tensor algebra of the standard $U_q(\widehat{\mathfrak{sl}}_n)$ evaluation module. At $n = 1$ or $q = 1$, q-Fock space degenerates into classical Fock space.

Our goal is to understand when the τ-function associated to a Hamiltonian operator equals the partition function of a rectangular lattice model. These lattice models are grids of vertices; each edge in the grid is assigned a *spin*, and the spins around a vertex must have one of several *admissible* configurations, in which case the vertex is assigned a nonzero *Boltzmann weight*. If every vertex is admissible, we call this global configuration an *admissible* *state*. Then the partition function is defined as

$$Z := \sum_{\text{state } \mathfrak{s}} \prod_{\text{vertex } v} \text{wt}(v).$$

Our lattice models have fixed side boundary conditions, and we allow the top and bottom boundaries to each depend on a partition. As shorthand, we'll often write $\mathfrak{S}$ to denote the set $\{\mathfrak{S}_{\lambda/\mu}|\lambda, \mu\}$ of lattice models with fixed Boltzmann weights and fixed side boundary conditions, but where the top and bottom boundaries can vary.

We say that a lattice model $\mathfrak{S}$ and a Hamiltonian operator e^H *match* if

$$Z(\mathfrak{S}_{\lambda/\mu}) = * \cdot \langle \mu | e^H | \lambda \rangle \qquad \text{for all strict partitions } \lambda, \mu,$$

where $*$ represents any function of the Boltzmann weights of $\mathfrak{S}$ independent of λ and μ. In practice, these are easily computable as simple products involving the Boltzmann weights. Our condition for a Hamiltonian to match a lattice model is equivalent to requiring that the time-evolution of the Hamiltonian equals the row transfer matrix of the lattice model as an operator on the set of partitions.

Zinn-Justin [12] gives a nice exposition of this connection in the case of the five-vertex model, and the resulting τ-functions are Schur functions. Subsequently, Brubaker and Schultz [11] find Hamiltonians for certain six-vertex lattice models associated to Whittaker functions, and prove that for these models, the partition function is a supersymmetric Schur function.

Our results take steps towards turning these examples into a theory. The following is our main result for six-vertex models.

Theorem 2.1.1 (Theorems 2.4.1 and 2.4.6, and Corollary 2.5.2)**.**

(a) Given a six-vertex model $\mathfrak{S}$ as above, it matches a Hamiltonian from classical Fock space precisely when the Boltzmann weights of the lattice model are free fermionic.

(b) In this case, the partition function is (up to a simple factor) a supersymmetric Schur function.

The free fermion point of the six-vertex model has long been known to be associated with special phenomena. In the five-vertex non-intersecting path model, free fermionic models are those with no attraction or repulsion between lattice paths, and their evolution is governed by entropy. Their partition functions can be expressed as determinants via the Lindström-Gessel-Viennot Lemma [34, 35]. The free fermion point is also central to the solvability of the six-vertex model [2, 36, 37].

Our proof of the first part of Theorem 2.1.1 uses Wick's Theorem [38], along with generating function manipulations. The second part follows directly from the first part using a result by Brubaker and Schultz [11], and corrects a result from [2].

The free fermionic six-vertex model is known to always be solvable [2, Theorem 1]. Having a Hamiltonian operator for these models gives an alternative method to explore the partition functions and prove identities.

We work with two types of six-vertex models, which we call $\mathfrak{S}$ and $\mathfrak{S}^*$, and which are in a sense dual models. Both are four-parameter families which parametrize the free fermionic Boltzmann weights, and we show that both models match with Hamiltonians. The models are generalizations of the Δ and Γ models explored in [2]. In addition, they each generalize the five-vertex vicious and osculating models defined in [39, 40]. One specialization sends $\mathfrak{S}$ to the vicious model and $\mathfrak{S}^*$ to the osculating model, while another specialization does the reverse.

This pair of five-vertex specializations is a special case of a more general phenomenon. We define an involution on our lattice models consisting of simple geometric manipulations that send $\mathfrak{S}$ and $\mathfrak{S}^*$ to each other, while doing the same to the vicious and osculating models. On the Hamiltonian side, this involution exactly matches a generalization of the symmetric function involution explored by Zinn-Justin [12] that arises from particle-hole duality.

In addition, the existence of a correspondence between lattice models and Hamiltonians means that we can use the structure of one to prove identities for the other. In particular, our use of Wick's theorem for classical Fock space provides a Jacobi-Trudi identity for the free fermionic partition function. This can be seen as a generalization of the lattice version of the Lindström-Gessel-Viennot Lemma and reduces to that result in the case of vicious walkers. Somewhat serendipitously, we are also able to use the Edrei-Thoma theorem to prove a positivity result for the free fermionic partition function.

The models $\mathfrak{S}$ require a particular choice of side boundary conditions. In Section 2.6, we use creation and deletion operators to give operator definitions for the free fermionic partition function with arbitrary boundary conditions. In general, this operator is not nicely behaved, but in the case of domain-wall boundary conditions, we get the following result:

Theorem 2.1.2 (Theorem 2.6.6, Corollary 2.6.8)**.** *The partition function of the free fermionic lattice model with domain wall boundary conditions can be expressed as:*

- *A supersymmetric Schur polynomial times an extra factor, and as*
- *A Schur polynomial times a (different) extra factor.*

In both cases, the extra factors are (easily-computed) functions of the Boltzmann weights independent of λ and μ.

The second half of the chapter relates Hamiltonians associated to Drinfeld twists of q-Fock space to six-vertex lattice models with *charge*, an extra parameter giving a "mod n" behavior to the Boltzmann weights. This connection was first explored by Brubaker, Bump, Buciumas, and Gustafsson [3] in their study of metaplectic Whittaker functions. The resulting model has $2n+4$ vertices, a substantial increase in complexity from the standard six-vertex model. For a certain set of Boltzmann weights, Brubaker, Bump, Buciumas, and Gustafsson showed that the model is solvable via a module of $U_q(\widehat{\mathfrak{gl}}(1|n))$.

Reshetikhin [41] defined a large class of Drinfeld twists of quantum groups. When applied to the q-Fock space studied by Kashiwara, Miwa, and Stern [42], a certain subset of these twists are *shift invariant* in the sense that their defining *wedge relations* only depend on the difference between two indices and not the indices themselves. Brubaker, Bump, Buciumas, and Gustafsson show that the associated Hamiltonians match the metaplectic lattice models from [27], and so we will call these spaces *metaplectic Fock spaces.* The rank of the quantum group is the same as the modulus on the charged models, and in the case of the models in [3], the Drinfeld twist gives the Gauss sums for the metaplectic Whittaker functions.

First, we prove a criterion for solvability of charged lattice models. Solvability turns out to be closely related to a condition we call the *generalized free fermion condition*, which consists of the free fermion condition at "zero charge" and an additional *charge condition.* In the case $n = 1$, these conditions reduce to the classical free fermion condition.

Theorem 2.1.3 (Theorem 2.9.1)**.** *The six-vertex model with charge is solvable in precisely two cases:*

1. *The Boltzmann weights satisfy the generalized free fermion condition associated to a metaplectic Fock space.*

2. *The Boltzmann weights satisfy the conditions (2.26) and (2.27).*

The conditions in the second case are quite restrictive and the solution is not very interesting as many of the R-vertex weights are 0. Therefore, for practical purposes, solvability is equivalent to the generalized free fermion condition.

Remarkably, the generalized free fermion condition is also precisely the condition required for a six-vertex model with charge to match a Hamiltonian from a metaplectic Fock space. It was quite unexpected to see the same condition arise naturally from these two very different computations, and we do not currently have an explanation for this phenomenon.

We define charged models $\mathfrak{S}^q$ and $\mathfrak{S}^{*,q}$ which parametrize the generalized free fermionic models. These models generalize $\mathfrak{S}$ and $\mathfrak{S}^*$, as well as the charged lattice models in [3]. We show that $\mathfrak{S}^q$ and $\mathfrak{S}^{*,q}$ match Hamiltonian operators. More precisely,

Theorem 2.1.4 (Theorems 2.8.5, 2.10.7, and 2.11.1)**.**

(a) The six-vertex model with charge matches a Hamiltonian operator on q-Fock space precisely when its Boltzmann weights satisfy the generalized free fermion condition.

(b) In this case, the partition function is (up to a simple factor) a supersymmetric LLT polynomial.

Brubaker, Buciumas, Bump, and Gustafsson showed that their model gives a supersymmetric LLT polynomial; our contribution here is to show that this is true of *all* generalized free fermionic models, and that these are precisely the models associated to Hamiltonians on metaplectic Fock spaces. The generalized free fermionic models give us most, although not all, values of supersymmetric LLT polynomials as their partition functions. Finally, we use Hamiltonians to prove a Cauchy identity for (skew) supersymmetric LLT polynomials. This generalizes results of Lam [43] and Brubaker, Bump, Buciumas, and Gustafsson [3].

In general, the relationship between Hamiltonian operators and solvability is unclear. Both are phenomena involving representations of quantum groups; however, they are different quantum groups! The Hamiltonian operator is associated to a quantum affine algebra, while the R*-matrix* involved in the solvability of our models arises from a quantum affine *super*algebra. Furthermore, the ways that these representations interact with the lattice model are quite different. A vector in Fock space in a sense represents all the vertical edges in a single row of the lattice model at once, while the R-matrix method gives an intertwiner at each vertex of a pair of quantum group modules (and a module interpretation for the vertical edges in [1] is not known).

One nice potential consequence of the relationship of solvability with Hamiltonians is that Hamiltonians could provide a better method for generating algebraic conditions for solvability. It is often difficult to determine whether a complicated lattice model is solvable, and in the case of charged models our proof of solvability is substantially more computationally intense than our proof of a matching Hamiltonian. It is unclear whether a connection between these phenomena exists more generally, and this will be the subject of future work.

Sections 2.2-2.6 are on the topic of classical Fock space and the six-vertex model without charge. Section 2.2 gives preliminaries on Fock space, Hamiltonian operators, and symmetric functions corresponding to Hamiltonians. We give several symmetric function identities, many of which are also proved in [31] or [12]. Then in Section 2.3, we define our lattice models and describe two different relationships between their partition function. In Section 2.4, we prove Theorem 2.1.1(a). Section 2.5 covers several topics relating to the free fermionic partition function, including a proof of Theorem 2.1.1(b), involutions, identities, and positivity. In Section 2.6, we find a Fock space operator that matches with any boundary conditions, and prove Theorem 2.1.2.

Sections 2.7-2.11 concern q-Fock spaces and six-vertex models with charge. Section 2.7 introduces q-Fock space and the action of Hamiltonian operators, while Section 2.8 defines our charged lattice models and the generalized free fermion condition. We present the solvability criterion for charged models in Section 2.9, with some computational details in Section 2.12. The proof of Theorem 2.1.4(a) is in Section 2.10, and the proof of Theorem 2.1.4(b), along with the Cauchy identity for supersymmetric LLT polynomials, is in Section 2.11.

2.2 Fock Space and Hamiltonian Operators

In this section, we define (classical) Fock space and its Hamiltonian operators. We explore the ways in which Hamiltonians generalize symmetric and supersymmetric function theory in terms of an involution as well as Jacobi-Trudi, Cauchy, and Pieri rules.

2.2.1 Partitions

A partition λ of length ℓ is a weakly decreasing sequence of numbers

$$\lambda_1 \geq \lambda_2 \geq \ldots \lambda_\ell \geq 0.$$

It is called *strict* if all the inequalities above except the last one are strict.

Notice that λ can be padded by trailing zeroes. In fact, we will often need pairs of partitions of the same length, so we will often do this.

Let $\rho := \rho_\ell = (\ell - 1, \ell - 2, \ldots, 1, 0)$. We will often *shift* a partition by ρ:

$$\lambda \pm \rho := \lambda \pm \rho_{\ell(\lambda)} = (\lambda_1 \pm (\ell(\lambda) - 1), \lambda_2 \pm (\ell(\lambda) - 2), \ldots, \lambda_{\ell(\lambda)}).$$

ρ-shifting has the nice property that

$$\lambda \text{ is a partition} \quad \text{ if and only if } \quad \lambda + \rho \text{ is a strict partition.}$$

Let λ' be the conjugate partition, $\lambda'_i = |\{k|\lambda_k \geq i\}|$. Notice that the length $\ell(\lambda')$ is not well-defined; this shouldn't be a problem since we may choose λ' to have any number of trailing zeroes.

If λ is a strict partition, choose an integer $M \geq \lambda_1$. Set $\overline{\lambda}$ to be the partition obtained by reversing λ in the range $[0, M]$ and swapping its parts and nonparts:

$$\overline{\lambda} = \rho_{M+1} \setminus (M - \lambda_1, M - \lambda_2, \ldots, M - \lambda_{\ell(\lambda)}),$$

where the \ symbol refers to set subtraction on parts of the partition.

Lemma 2.2.1. *For all strict partitions* λ *and all* $M \geq \lambda_1$, $\overline{\lambda} - \rho = (\lambda - \rho)'$.

Proof. Let $\ell = \ell(\lambda)$, and let $\mu_1 > \mu_2 > \cdots > \mu_{M+1-\ell}$ be the strict partition made up of all the integers in $[0, M]$ that are not parts of λ. Then we have

$$\begin{aligned}(\lambda - \rho)'_i &= |\{k|\lambda_k - (\ell - k) \geq i\}| \\ &= |\{k| \text{ there exist } \geq i \text{ parts } \mu_j \text{ of } \mu \text{ with } \lambda_k > \mu_j\}| \\ &= |\{k|\lambda_k > \mu_{M-\ell+2-i}\}| \\ &= M - \mu_{M-\ell+2-i} - (M - \ell + 1 - i) \\ &= (\overline{\lambda} - \rho)_i,\end{aligned}$$

where the last equality holds since $\ell(\mu) = M - \ell + 1$, $(\overline{\lambda})_i = M - \mu_{M-\ell+2-i}$, and $(\rho_{\ell(\mu)})_i = M - \ell + 1 - i$. □

Let λ, μ be partitions with the same length such that $\lambda_i \geq \mu_i$ for all i. Then we call the pair (λ, μ) a *skew* partition and denote it λ/μ.

2.2.2 Fock Space

Let us define the Clifford algebra

$$A = \langle \psi_i^*, \psi_i | i \in \mathbb{Z} - \frac{1}{2} \rangle,$$

with relations

$$\psi_i\psi_j + \psi_j\psi_i = 0, \qquad \psi_i^*\psi_j^* + \psi_j^*\psi_i^* = 0, \qquad \psi_i\psi_j^* + \psi_j^*\psi_i = \delta_{i,j}.$$

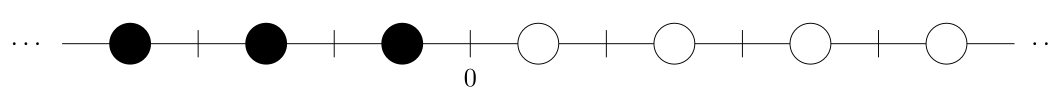
Let $\psi^*(t) = \sum_{k \in \mathbb{Z}-1/2} \psi_k^* t^{k+1/2}$.

The Fock space $\mathcal{F}$ and its dual $\mathcal{F}^*$ are both A-modules. Let $\mathcal{W} = \left(\oplus_{i \in \mathbb{Z}-1/2} \mathbb{C}\psi_i\right) \oplus \left(\oplus_{i \in \mathbb{Z}-1/2} \mathbb{C}\psi_i^*\right)$. We call elements of $\mathcal{W}$ *free fermions.*

Define subspaces $\mathcal{W}_{ann} = (\oplus_{i<0}\mathbb{C}\psi_i) \oplus (\oplus_{i>0}\mathbb{C}\psi_i^*)$ and $\mathcal{W}_{cr} = (\oplus_{i>0}\mathbb{C}\psi_i) \oplus (\oplus_{i<0}\mathbb{C}\psi_i^*)$. Then $\mathcal{F} := A/A\mathcal{W}_{ann}$ is a left A-module, while $\mathcal{F}^* := \mathcal{W}_{cr}A\backslash A$ is a right A-module.

$\mathcal{F}$ is a cyclic module generated by the vector $|0\rangle := 1 \mod A\mathcal{W}_{ann}$ and $\mathcal{F}^*$ is a cyclic module generated by the vector $\langle 0| := 1 \mod \mathcal{W}_{cr}A$.

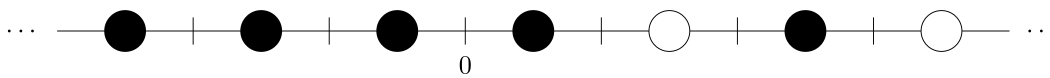
One can represent each of these vectors by the particle diagram



For a strict partition λ, we define

$$|\lambda\rangle := \psi^*_{\lambda_1-\frac{1}{2}}\psi^*_{\lambda_2-\frac{1}{2}}\dots\psi^*_{\lambda_n-\frac{1}{2}}|0\rangle \qquad \text{and} \qquad \langle\lambda| := \langle 0|\psi_{\lambda_n-\frac{1}{2}}\dots\psi_{\lambda_2-\frac{1}{2}}\psi_{\lambda_1-\frac{1}{2}}.$$

We represent the basis vectors of $\mathcal{F}$ and $\mathcal{F}^*$ as particle diagrams as well. For example, the partition $\lambda = (3,1)$ is represented by the following diagram.



$\{|\lambda\rangle\}$ and $\{\langle\lambda|\}$ form a set of dual bases of $\mathcal{F}$ and $\mathcal{F}^*$ with respect to the bilinear form

$$\mathcal{F}^* \otimes_A \mathcal{F} \longrightarrow \mathbb{C}$$

defined by

$$\langle\lambda| \otimes_A |\mu\rangle \mapsto \langle\lambda|\mu\rangle := \delta_{\lambda,\mu},$$

extended linearly.

We can use a similar "bra-ket" notation to write more complicated pairings: we write $\langle\mu|h|\lambda\rangle$ to represent the image under the bilinear form of the quantity $\langle\mu|h \otimes_A 1|\lambda\rangle = \langle\mu|1 \otimes_A h|\lambda\rangle$.

This symmetric bilinear form gives rise to a linear form $\langle\cdot\rangle$ on A, called the *vacuum expectation value.* (In many sources, these definitions are done in reverse).

We define:

$$\langle a\rangle := \langle 0|a|0\rangle.$$

In particular, we have

$$\langle 1\rangle = 1, \quad \langle\psi_i\psi_j\rangle = \langle\psi_i^*\psi_j^*\rangle = 0,$$

$$\langle\psi_i\psi_j^*\rangle = \begin{cases} 1, & \text{if } i = j < 0, \\ 0, & \text{otherwise.} \end{cases} \qquad \langle\psi_i^*\psi_j\rangle = \begin{cases} 1, & \text{if } i = j > 0, \\ 0, & \text{otherwise.} \end{cases}$$

We can use the vacuum expectation value to define the *normal ordering.*

$$:\psi_i\psi_j^*: \; := \psi_i\psi_j^* - \langle\psi_i\psi_j^*\rangle = \begin{cases} \psi_i\psi_j^*, & \text{if } i > 0, \\ -\psi_j^*\psi_i, & \text{if } i < 0. \end{cases}$$

$$:\psi_j^*\psi_i: \; := \psi_j^*\psi_i - \langle\psi_j^*\psi_i\rangle = \begin{cases} \psi_j^*\psi_i, & \text{if } i < 0, \\ -\psi_i\psi_j^*, & \text{if } i > 0. \end{cases}$$

Note that unless $i = j$, $\psi_i\psi_j^* = -\psi_j^*\psi_i$. On the other hand, $\psi_i\psi_i^*|\lambda\rangle \neq 0$ whenever λ does not have a part of size $i + 1/2$, whereas $\psi_i^*\psi_i|\lambda\rangle \neq 0$ whenever $i < 0$ or $i > 0$ and λ does have a part of size $i+1/2$. This would lead to trivial infinite quantities in some of our upcoming definitions. However, using $:\psi_i\psi_j^*: := -\psi_j^*\psi_i$ removes this complication since $:\psi_i\psi_i^*:|\lambda\rangle \neq 0$ only for finitely many i, precisely those $i>0$ where λ has a part of size $i+1/2$.

One can therefore define the Lie algebra,

$$\mathfrak{gl}(\infty) := \left\{ \sum_{ij} a_{ij} : \psi_i\psi_j^* : | \exists N \text{ such that } a_{ij}=0 \text{ if } |i-j|>N \right\} \oplus \mathbb{C}\cdot 1.$$

With this definition, $\mathcal{F}$ and $\mathcal{F}^*$ are both $\mathfrak{gl}(\infty)$-modules. They are reducible, but decompose into irreducible representations

$$\mathcal{F} = \bigoplus_{\ell\in\mathbb{Z}} \mathcal{F}_\ell, \qquad \mathcal{F}^* = \bigoplus_{\ell\in\mathbb{Z}} \mathcal{F}_\ell^*,$$

$$\mathcal{F}_\ell = \mathfrak{gl}(\infty)|\ell\rangle, \qquad \mathcal{F}_\ell^* = \langle\ell|\mathfrak{gl}(\infty)$$

Next, we will define important elements in $\mathfrak{gl}(\infty)$ called *Hamiltonian operators*.

For $n\in\mathbb{Z}$, let

$$J_n = \sum_{i\in\mathbb{Z}-\frac{1}{2}} :\psi_{i-n}^*\psi_i: .$$

These are called *current operators*, and they generate a Heisenberg algebra $\mathcal{H} := \langle J_m | m\in\mathbb{Z}, m\neq 0\rangle$ since $[J_m,J_n]=m\delta_{m,-n}$. Now, let $\{s_k | k\in\mathbb{Z}\setminus\{0\}\}$ be a doubly infinite family of parameters. Let

$$H_+ = \sum_{k\geq 1} s_k J_k, \qquad e^{H_+} = \sum_{m\geq 0}\frac{H_+^m}{m!}.$$

e^{H_+} is the *Hamiltonian operator* of [11, 31, 32]. Similarly, we define

$$H_- := \sum_{k\geq 1} s_{-k}J_{-k}.$$

We will also sometimes write $s_k=\sum_{j=1}^N s_k^{(j)}$, where the $s_k^{(j)}$ are indeterminates. In this case, we have

$$H_\pm = \sum_{j=1}^N \phi_{\pm j}, \qquad \text{where} \qquad \phi_{\pm j} = \sum_{k\geq 1} s_{\pm k}^{(j)} J_{\pm k}.$$

Note that all the ϕ_j commute. If we want to make the parameters $s_{\pm k}^{(j)}$ clear for a Hamiltonian operator, we write

$$H_\pm = H_\pm(s_{\pm 1}^{(1)},\ldots,s_{\pm 1}^{(N)},s_{\pm 2}^{(1)},\ldots,s_{\pm 2}^{(N)},\ldots).$$

The action of a current operator J_k on a vector is to move a particle k spots to the left. The action of e^{H_+} is to move any number of particles any number of spaces to the left. The parameters s_k keep track of which moves we have done. Similarly, the action of e^{H_-} allows us to move any number of particles any number of spaces to the right.

The following result is both useful and classical. It appears in many forms [38].

Proposition 2.2.2 (Wick's Theorem)**.**

$$\langle \psi_{i_1} \dots \psi_{i_r} e^{H_\pm} \psi^*_{j_1} \dots \psi^*_{j_s} \rangle = \begin{cases} \det_{1 \le p,q \le r} \langle \psi_{i_p} e^{H_\pm} \psi^*_{j_q} \rangle, & \textit{if } r = s \\ 0, & \textit{otherwise.} \end{cases}$$

One of the best motivations to study Hamiltonian operators is the boson-fermion correspondence. For all $\ell \in \mathbb{Z}$, let $V_\ell \cong \mathbb{C}[z]$. Let $\mathcal{H}$ act on V_ℓ by the *bosonic action*:

$$J_k \cdot P := \begin{cases} k \cdot \frac{\partial P}{\partial k s_k}, & k > 0 \\ s_k \cdot P, & k < 0, \end{cases}$$

Proposition 2.2.3 (Boson-Fermion Correspondence)**.** *[32, Theorem 1.1] The following map*

$$\mathcal{F}_\ell \longrightarrow V_\ell, \qquad a|0\rangle \mapsto \langle \ell | e^{H_+} a|0\rangle$$

is an isomorphism of $\mathcal{H}$-modules. In other words,

$$h \langle \ell | e^{H_+} a|0\rangle = \langle \ell | e^{H_+} h a|0\rangle, \qquad a \in A, a|0\rangle \in \mathcal{F}_\ell, h \in \mathcal{H}$$

where the action by h on the left side is bosonic, while on the right side it is fermionic.

There is another way to write $\mathcal{F}_\ell$. Let $W = \bigoplus_{i \in \mathbb{Z}} \langle v_i \rangle$ and

$$\bigwedge^{\infty} W = v_{i_1} \wedge v_{i_2} \wedge \dots,$$

with the usual wedge relation $v_i \wedge v_j = -v_j \wedge v_i$. Then

$$\mathcal{F}_\ell = \left\{ v_{i_1} \wedge v_{i_2} \wedge \dots \in \bigwedge^{\infty} W | i_m = \ell - m \text{ for all } m >> 0 \right\},$$

and the action of current operators can be expressed as

$$J_k \cdot (v_{m_1} \wedge v_{m_2} \wedge \dots) = \sum_{i \ge 1} (v_{m_1} \wedge \dots \wedge v_{m_{i-1}} \wedge v_{m_i - k} \wedge v_{m_{i+1}} \wedge \dots).$$

We'll see a more general version of this action in Section 2.7.

2.2.3 Hamiltonians and symmetric functions

We will work with a set of functions that naturally arise from Hamiltonians and generalize some common symmetric and supersymmetric functions such as power sum, homogeneous, elementary, and Schur polynomials. This approach was first taken by Lam [31] and Zinn-Justin [12], and most of the results in this section were proved by one or both of them. An important reference for symmetric functions computations is [44, Chapter I].

A similar idea is due to Korff [45] and Gorbounov-Korff [15], who used operator analogues of symmetric functions to study quantum cohomology via vicious and osculating walkers.

We will use similar notation for our generalizations as for the classical symmetric functions. To avoid confusion, we will always use parentheses for the generalized functions and square brackets for symmetric and supersymmetric functions.

Fix a set of parameters $\boldsymbol{s}_+ := \{s_k^{(j)}, 1 \le j \le N, k \ge 1\}$. We want the negative-index parameters to have a particular relationship with the positive index parameters:

$$\boldsymbol{s}_- := \{s_k^{(j)}, 1 \le j \le N, k \le -1\}, \qquad \text{where} \qquad s_{-k}^{(j)} = (-1)^{k-1} s_k^{(j)}.$$

We will explore symmetric function analogues in the ring $\mathbb{C}[s_k^{(j)} | k \in \mathbb{Z} \setminus \{0\}, 1 \le j \le N]$. In the constructions to follow, specializing $s_k^{(j)} = \frac{1}{k} x_j^k, k > 0$ produces the classical symmetric functions, while specializing $s_k^{(j)} = \frac{1}{k}(x_j^k + (-1)^k y_j^k), k > 0$ produces the supersymmetric functions.

For a partition λ, let $z_\lambda = \prod_{i \ge 1} i^{m_i} m_i!$, where m_i is the number of parts of λ of size i.

Let

$$s_{\pm k} := s_{\pm k}(\boldsymbol{s}_\pm) = \sum_{j=1}^N s_{\pm k}^{(j)}, \qquad p_{\pm k} := p_{\pm k}(\boldsymbol{s}_\pm) = k s_{\pm k}, \qquad k \neq 0,$$

$$s_{\pm\lambda} := s_{\pm\lambda_1} \cdots s_{\pm\lambda_{\ell(\lambda)}}, \qquad p_{\pm\lambda} := p_{\pm\lambda_1} \cdots p_{\pm\lambda_{\ell(\lambda)}}, \qquad \lambda\text{: partition},$$

$$h_{\pm k} := \sum_{\lambda \vdash k} z_\lambda^{-1} p_{\pm\lambda}, \qquad e_{\pm k} := \sum_{\lambda \vdash k} (-1)^{|\lambda| - \ell(\lambda)} z_\lambda^{-1} p_{\pm\lambda}, \qquad k \ge 1,$$

$$s_0 = p_0 = 0, \qquad h_0 = e_0 = 1,$$

and let ω be the involution $\omega(s_k^{(j)}) = (-1)^{k-1} s_k^{(j)}$, extended algebraically. In particular, $\omega(h_k) = e_k$.

As above, let $H_\pm = H_\pm(\boldsymbol{s}_\pm) = \sum_{k \ge 1} \sum_{j=1}^N s_{\pm k}^{(j)} J_{\pm k}$.

Lemma 2.2.4 (Duality)**.**

$$\omega\left(\langle \mu | e^{H_+} | \lambda \rangle\right) = \langle \lambda | e^{H_-} | \mu \rangle. \tag{2.1}$$

Proof. First, if $k \neq 0$,

$$\begin{aligned}
\langle \mu | J_k | \lambda \rangle &= \sum_{i \in \mathbb{Z}-1/2} \langle \mu | : \psi^*_{i-k} \psi_i : | \lambda \rangle \\
&= \sum_{i \in \mathbb{Z}-1/2} \begin{cases} 1, & \text{there exists a partition } \nu \text{ such that } \nu \cup (i) = \lambda, \nu \cup (i-k) = \mu \\ 0, & \text{otherwise.} \end{cases} \\
&= \sum_{i \in \mathbb{Z}-1/2} \begin{cases} 1, & \text{there exists a partition } \nu \text{ such that } \nu \cup (i+k) = \lambda, \nu \cup (i) = \mu \\ 0, & \text{otherwise.} \end{cases} \\
&= \sum_{i \in \mathbb{Z}-1/2} \langle \lambda | : \psi^*_{i+k} \psi_i : | \mu \rangle \\
&= \langle \lambda | J_{-k} | \mu \rangle,
\end{aligned}$$

and so

$$\omega\left(\langle \mu | s_k J_k | \lambda \rangle\right) = \omega(s_k) \omega\left(\langle \mu | J_k | \lambda \rangle\right) = s_{-k} \langle \lambda | J_{-k} | \mu \rangle = \langle \lambda | s_{-k} J_{-k} | \mu \rangle.$$

Then, (2.1) follows by linearity. □

Let

$$S(t) := \sum_{k \geq 1} s_k t^k, \qquad P(t) := S'(t) = \sum_{k \geq 1} p_k t^{k-1},$$

and also

$$H(t) := \sum_{k \geq 0} h_k t^k, \qquad E(t) := \sum_{k \geq 0} e_k t^k.$$

Lemma 2.2.5.

$$S(t) = \log H(t), \tag{2.2}$$

$$h_k = \langle (0) | e^{H_+} | (k) \rangle. \tag{2.3}$$

and similarly

$$-S(-t) = \log E(t), \tag{2.4}$$

$$e_k = \langle (k) | e^{H_-} | (0) \rangle. \tag{2.5}$$

Proof. We start by proving (2.2). Note that z_λ is the product of the parts of λ times the number of permutations on the parts of λ that fix λ. In other words, $\ell(\lambda)! z_\lambda^{-1}$ is the number of compositions of $|\lambda|$ that rearrange to λ, divided by the product of the parts of λ. Using this,

$$h_k = \sum_{\lambda \vdash k} z_\lambda^{-1} p_\lambda = \sum_{r\geq 0} \frac{1}{r!} \sum_{q_1+\ldots+q_r=k} s_{q_1}\ldots s_{q_r},$$

so

$$H(t) = \sum_{k\geq 0} h_k t^k = \sum_{r\geq 0} \frac{1}{r!} \left(\sum_{q_1,\ldots,q_r\geq 1} s_{q_1}\ldots s_{q_r} \right) t^{q_1+\ldots+q_r} = \exp(S(t)),$$

by the definition of the formal exponential.

(2.3) follows from (2.2) since

$$\langle (k)|e^{H_-}|(0)\rangle = \sum_{k\geq 0}\sum_{r\geq 0} \frac{1}{r!} \left(\sum_{q_1+\ldots+q_r=k} s_{q_1}\ldots s_{q_r} \right).$$

Similarly,

$$e_k = \sum_{\lambda \vdash k} z_\lambda^{-1} (-1)^{k-\ell(\lambda)} p_\lambda = \sum_{r\geq 0} \frac{1}{r!} (-1)^{k-r} \sum_{q_1+\ldots+q_r=k} s_{q_1}\ldots s_{q_r},$$

so

$$E(t) = \sum_{k\geq 0} e_k t^k = \sum_{r\geq 0} \frac{1}{r!} (-1)^r \left(\sum_{q_1,\ldots,q_r\geq 1} s_{q_1}\ldots s_{q_r} \right) (-t)^{q_1+\ldots+q_r} = \exp(-S(-t)).$$

(2.5) follows from (2.4) since

$$\begin{aligned}\langle (0)|e^{H_+}|(k)\rangle &= \sum_{r\geq 0} \frac{1}{r!} \left(\sum_{q_1+\ldots+q_r=k} (-1)^{q_1-1} s_{q_1}\ldots (-1)^{q_r-1} s_{q_r} \right) \\ &= \sum_{r\geq 0} \frac{1}{r!} (-1)^{k-r} \left(\sum_{q_1+\ldots+q_r=k} s_{q_1}\ldots s_{q_r} \right).\end{aligned}$$

□

Let

$$H := (h_{i-j})_{0\leq i,j\leq n}, \qquad E := ((-1)^{i-j} e_{i-j})_{0\leq i,j\leq n}.$$

Corollary 2.2.6.

$$H(t)E(-t) = 1,$$

and

$$\sum_{r=0}^{n} (-1)^r e_r h_{n-r} = 0 \qquad \textit{for all } n \geq 1.$$

Furthermore, $E = H^{-1}$*.*

Proof. By the previous lemma,

$$H(t)E(-t) = \exp(S(t))\exp(-S(t)) = 1,$$

and the second equation follows from taking coefficients. The final statement is obtained from a matrix multiplication:

$$(H \cdot E)_{i,j} = \sum_{k=0}^{n} (-1)^{k-j} e_{k-j} h_{i-k} = \sum_{k=j}^{i} (-1)^{k-j} e_{k-j} h_{i-k} = \sum_{r=0}^{i-j} (-1)^r e_r h_{i-j-r} = \delta_{ij}.$$

□

Now let λ and μ be partitions with $\ell(\lambda) = \ell(\mu)$, $\ell(\lambda') = \ell(\mu')$, and $\ell(\lambda) + \ell(\lambda') = n$. (Note that n can be made arbitrarily large, and the partitions can be buffered with trailing zeroes, so this is really no restriction.)

Lemma 2.2.7. *We have*

$$\det\left(h_{\lambda_i - \mu_j - i + j}\right) = \det\left(e_{\lambda'_i - \mu'_j - i + j}\right)$$

Proof. The proof is exactly the same as in Macdonald [44, pp. 22-23] □

Now let $\sigma_{\lambda/\mu} := \langle \mu + \rho | e^{H_+} | \lambda + \rho \rangle$. This is the generalized Schur function, which we denote by σ so as to avoid confusion with the s_k. Note the ρ shift, since Hamiltonians deal with strict partitions. $\sigma_{\lambda/\mu}$ is 0 unless λ and μ have the same length ℓ. Let $\sigma_\lambda := \sigma_{\lambda/(0,\ldots,0)} = \langle \rho | e^{H_+} | \lambda + \rho \rangle$. Note in particular that $h_k(\boldsymbol{s}_+) = s_{(k)/(0)}$. We call $\sigma_{\lambda/\mu}$ the τ *function* corresponding to $\lambda + \rho$ and $\mu + \rho$.

Proposition 2.2.8 (Jacobi-Trudi, Von Nägelsbach–Kostka identities)**.**

$$\sigma_{\lambda/\mu} = \det_{1 \le i,j \le \ell} h_{\lambda_i - \mu_j - i + j} = \det_{1 \le i,j \le \ell} e_{\lambda'_i - \mu'_j - i + j}.$$

Proof. The second equality is the previous proposition. For the first equality,

$$\sigma_{\lambda/\mu} = \langle \mu + \rho | e^{H_+} | \lambda + \rho \rangle = \langle 0 | \psi_{\mu_\ell - 1/2} \cdots \psi_{\mu_1 + \ell - 3/2}\, e^{H_+}\, \psi^*_{\lambda_1 + \ell - 3/2} \cdots \psi^*_{\lambda_\ell - 1/2} | 0 \rangle,$$

and by Wick's Theorem this equals

$$\det_{1 \le i,j \le \ell} \langle 0 | \psi_{\mu_j + \ell - j - 1/2}\, e^{H_+}\, \psi^*_{\lambda_i + \ell - i - 1/2} | 0 \rangle = \det_{1 \le i,j \le \ell} h_{\lambda_i - \mu_j - i + j},$$

by (2.3). □

Paired with results about transformations of lattice models, this can be seen as an analogue of the Lindström-Gessel-Viennot lemma. See Section 2.5.2.

Now we come to our main result of this section, showing that the $\sigma_{\lambda/\mu}$ obey an involutive identity. The proof is now easy.

Proposition 2.2.9.

$$\omega(\sigma_{\lambda/\mu}) = \sigma_{\lambda'/\mu'},$$

and thus

$$\sigma_{\lambda'/\mu'} = \langle \lambda + \rho | e^{H_-} | \mu + \rho \rangle.$$

Proof. Apply the involution ω to the previous identities (2.2.8):

$$\omega(\sigma_{\lambda/\mu}) = \omega \left(\det_{1 \le i,j \le \ell} e_{\lambda'_i - \mu'_j - i + j} \right) = \det_{1 \le i,j \le \ell} h_{\lambda'_i - \mu'_j - i + j} = \sigma_{\lambda'/\mu'}.$$

The second equation follows from the first equation and (2.1). □

This can alternatively be shown by a particle-hole duality (see [12]).

There are two specializations of the parameters $s_i^{(j)}$ that we care about in particular. If we set

$$s_{\pm k}^{(j)} = (\pm 1)^{k-1} \frac{1}{k} x_j^k, \qquad k > 0,$$

we obtain the classical symmetric functions, and $\sigma_{\lambda/\mu}$ is a skew Schur function. If instead we set

$$s_{\pm k}^{(j)} = (\pm 1)^{k-1} \frac{1}{k} (x_j^k + (-1)^{k-1} y_j^k), \qquad k > 0,$$

we obtain supersymmetric functions, and $\sigma_{\lambda/\mu}$ is a skew supersymmetric Schur function.

2.2.4 Cauchy, Pieri, and branching rules for Hamiltonians

For this section, we will use arbitrary sets of parameters. Let $\boldsymbol{s}_+ := \{s_k^{(j)}, 1 \le j \le N, k \ge 1\}$ and $\boldsymbol{t}_- := \{t_{-k}^{(j)}, 1 \le j \le N, k \ge 1\}$ be two half-infinite sets of parameters, and let $H_+ = H_+(\boldsymbol{s}_+)$, $H_- = H_-(\boldsymbol{t}_-)$. Let $\sigma'_{\lambda/\mu} := \langle \lambda + \rho | e^{H_-} | \mu + \rho \rangle$.

Proposition 2.2.10 (Cauchy identity)**.** *For any strict partitions λ and μ,*

$$\sum_{\nu} \sigma_{\lambda/\nu} \sigma'_{\mu/\nu} = \prod_{i,j} \exp \left(\sum_{k \ge 1} k \cdot s_k^{(i)} t_{-k}^{(j)} \right) \cdot \sum_{\nu} \sigma_{\nu/\mu} \sigma'_{\nu/\lambda},$$

where the sums are over all strict partitions ν.

Proof. We evaluate the Hamiltonian $\langle \mu+\rho | e^{H_-} e^{H_+} | \lambda+\rho\rangle$ in two ways. First,

$$\langle \mu+\rho | e^{H_-} e^{H_+} | \lambda+\rho\rangle = \sum_\nu \langle \mu+\rho | e^{H_-} | \nu+\rho\rangle\langle \nu+\rho | e^{H_+} | \lambda+\rho\rangle = \sum_\nu \sigma_{\lambda/\nu}\sigma'_{\mu/\nu}.$$

Next, we apply the commutation relations between H_+ and H'_-.

$$\begin{aligned}\langle \mu+\rho | e^{H'_-} e^{H_+} | \lambda+\rho\rangle &= \exp\left(\sum_{k\geq 1} k\cdot s_k t_{-k}\right)\cdot \langle \mu+\rho | e^{H_+} e^{H_-} | \lambda+\rho\rangle \\ &= \prod_{i,j}\exp\left(\sum_{k\geq 1} k\cdot s_k^{(i)} t_{-k}^{(j)}\right)\cdot \sum_\nu \langle \mu+\rho | e^{H_+} | \nu+\rho\rangle\langle \nu+\rho | e^{H_-} | \lambda+\rho\rangle \\ &= \prod_{i,j}\exp\left(\sum_{k\geq 1} k\cdot s_k^{(i)} s_{-k}^{(j)}\right)\cdot \sum_\nu \sigma_{\nu/\mu}\sigma'_{\nu/\lambda}.\end{aligned}$$

□

Given a variable set $\boldsymbol{s} = \{s_1^{(1)}, \ldots, s_1^{(N)}, s_2^{(1)}, \ldots s_2^{(N)}, \ldots\}$, and some subset I of $[N] := \{1, \ldots, N\}$ let $\boldsymbol{s}|_I$ denote the subset $\bigcup_{i\in I}\{s_1^{(i)}, s_2^{(i)}, \ldots\}$. For example, $\boldsymbol{s}|_{[2,N]} = \boldsymbol{s}\setminus\{s_k^{(1)}\}_{k\geq 1}$.

Proposition 2.2.11 (Branching rule)**.** *For all partitions λ, μ,*

$$\sigma_{\lambda/\mu}(\boldsymbol{s}) = \sum_\nu \sigma_{\lambda/\nu}(\boldsymbol{s}|_{\{1\}})\sigma_{\nu/\mu}(\boldsymbol{s}|_{[2,n]}).$$

Proof. We have

$$\begin{aligned}\sigma_{\lambda/\mu} &= \langle \mu+\rho | e^{H_+} | \lambda+\rho\rangle \\ &= \sum_\nu \langle \mu+\rho | e^{\phi_n}\cdots e^{\phi_2} | \nu+\rho\rangle\langle \nu+\rho | e^{\phi_1} | \lambda+\rho\rangle \\ &= \sum_\nu \sigma_{\lambda/\nu}(\boldsymbol{s}|_{\{1\}})\sigma_{\nu/\mu}(\boldsymbol{s}|_{[2,n]}).\end{aligned}$$

□

Let $J_\mu := J_{\mu_1}\ldots J_{\mu_{\ell(\mu)}}$, and $J_{-\mu} := J_{-\mu_1}\ldots J_{-\mu_{\ell(\mu)}}$. Let

$$D_k = \sum_{\mu\vdash k} z_\mu^{-1} J_\mu, \qquad U_k = \sum_{\mu\vdash k} z_\mu^{-1} J_{-\mu}.$$

Proposition 2.2.12 (Pieri rule)**.**

$$h_k\cdot\sigma_\lambda = \sum_\nu \langle \nu+\rho\, | U_k |\, \lambda+\rho\rangle\, \sigma_\nu.$$

Proof. First note that

$$U_k \cdot 1 = \sum_{\mu \vdash k} z_\mu^{-1} J_{-\mu} \cdot 1 = \sum_{\mu \vdash k} z_\mu^{-1} p_\mu = h_k.$$

Apply the boson-fermion correspondence (Proposition 2.2.3) to obtain

$$\begin{aligned} h_k \cdot \sigma_\lambda &= h_k \cdot \langle \rho | e^{H_+} | \lambda + \rho \rangle \\ &= \langle \rho | e^{H_+} U_k | \lambda + \rho \rangle \\ &= \sum_\nu \langle \rho | e^{H_+} | \nu + \rho \rangle \langle \nu + \rho | U_k | \lambda + \rho \rangle \\ &= \sum_\nu \langle \nu + \rho | U_k | \lambda + \rho \rangle \sigma_\nu. \end{aligned}$$

□

2.3 The six-vertex model

In this section, we will define two related six-vertex models, called $\mathfrak{S}$ and $\mathfrak{S}^*$. Both of them parametrize the space of free fermionic six-vertex models i.e. their Boltzmann weights satisfy the condition

$$\mathtt{a}_1^{(\mathtt{i})}\mathtt{a}_2^{(\mathtt{i})} + \mathtt{b}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{i})} = \mathtt{c}_1^{(\mathtt{i})}\mathtt{c}_2^{(\mathtt{i})}, \qquad \text{for all } i.$$

The Boltzmann weights of these models are a simultaneous generalization of the weights of two pairs of six-vertex models. The first pair are the Γ and Δ models in [2]. The second are the *vicious* and *osculating* models that appear in [45]. There is a duality between the vicious and osculating models that we will generalize. Furthermore, this duality is equivalent to the duality for Hamiltonian operators proven in Lemma 2.2.4.

Our lattice models are finite rectangular grids of intersecting lines, with vertices at the intersection points of each pair of lines. Each edge is assigned a *spin* from a fixed set. An *admissible vertex* is a vertex around which the spins satisfy one of several admissible configurations. If every vertex in the grid is admissible, we call the resulting configuration an *admissible state*.

Each admissible vertex is assigned a *Boltzmann weight*. These weights often depend a row parameter. In this chapter, we will be slightly more general: weights will depend on several row parameters that parametrize certain sets of weights. Our weights will not depend on any column parameters since Hamiltonians do not behave well with respect to column parameters.

By convention, a non-admissible vertex has weight 0. The Boltzmann weight of a state is defined to be the product of all the Boltzmann weights of its constituent vertices, so non-admissible states always have weight 0. The *partition function* Z is the sum of the weights of all admissible states:

$$Z = \sum_{\text{state } \mathfrak{s}} \text{wt}(\mathfrak{s}) = \sum_{\text{state } \mathfrak{s}} \prod_{\text{vertex } v} \text{wt}(v).$$

The partition function Z is a function of its row parameters. A surprising amount of special functions occur as partition functions of a lattice model.

For the six-vertex model, our spin set is $\{+,-\}$. We imagine a $-$ spin to indicate the presence of a particle, and a $+$ spin to the indicate lack of a particle. The admissible vertices for $\mathfrak{S}$ and $\mathfrak{S}^*$ are slightly different, but are chosen in a way so that we can draw paths through the $-$ spins that start and end at the boundary. For $\mathfrak{S}$, these paths move up and left, and for $\mathfrak{S}^*$ they move down and left.

Let $x_i, y_i, z_i, w_i, A_i, B_i$ be arbitrary parameters, depending on a row i. Then the $\mathfrak{S}$ vertices and Boltzmann weights are given in Figure 2.1, and the $\mathfrak{S}^*$ vertices and weights are given in Figure 2.2. We will sometimes refer to the vertex weights using the symbols for the vertices themselves–for instance, writing $\mathtt{a}_1^{(\mathtt{i})} = A_i$ for the $\mathfrak{S}$ weights below. When there is potential for confusion, we will make clear whether we are talking about the vertex itself or its weight, and which set of weights we are using.

$\mathtt{a}_1^{(\mathtt{i})}$	$\mathtt{a}_2^{(\mathtt{i})}$	$\mathtt{b}_1^{(\mathtt{i})}$	$\mathtt{b}_2^{(\mathtt{i})}$	$\mathtt{c}_1^{(\mathtt{i})}$	$\mathtt{c}_2^{(\mathtt{i})}$
A_i	$y_iA_iB_i$	A_iB_i	x_iA_i	$(x_i+y_i)A_iB_i$	A_i

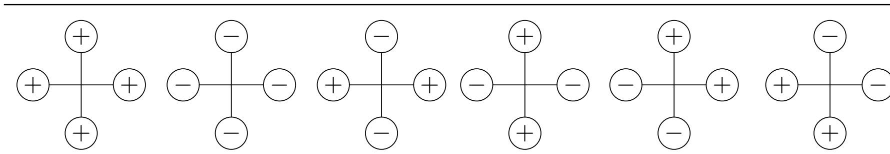


Figure 2.1: The Boltzmann weights for $\mathfrak{S}$. Here, x_i, y_i, A_i, and B_i are parameters associated to each row.

$\mathtt{a}_1^{(\mathtt{i})}$	$\mathtt{a}_2^{(\mathtt{i})}$	$\mathtt{b}_1^{(\mathtt{i})}$	$\mathtt{b}_2^{(\mathtt{i})}$	$\mathtt{c}_1^{(\mathtt{i})}$	$\mathtt{c}_2^{(\mathtt{i})}$
A_i^{-1}	$x_iA_i^{-1}B_i^{-1}$	$A_i^{-1}B_i^{-1}$	$y_iA_i^{-1}$	$(x_i+y_i)A_i^{-1}$	$A_i^{-1}B_i^{-1}$

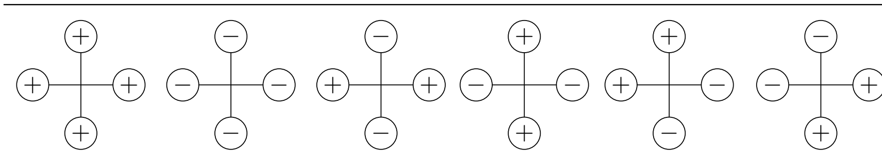


Figure 2.2: The Boltzmann weights for $\mathfrak{S}^*$. Here, x_i, y_i, A_i, and B_i are parameters associated to each row.

The $\mathfrak{S}$ weights are given in Figure 2.1, and the $\mathfrak{S}^*$ weights are given in Figure 2.2. Both sets of Boltzmann weights parametrize the free fermionic weights (with a small caveat; see Remark 2.3.1)).

First, notice that for any values of x_i, y_i, A_i, B_i, we have

$$\mathtt{a}_1^{(\mathtt{i})}\mathtt{a}_2^{(\mathtt{i})} + \mathtt{b}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{i})} - \mathtt{c}_1^{(\mathtt{i})}\mathtt{c}_2^{(\mathtt{i})} = y_i A_i^2 B_i + x_i A_i^2 B_i - (x_i + y_i) A_i^2 B_i = 0,$$

so the $\mathfrak{S}$ weights satisfy the free fermionic condition. Conversely, given a set of free fermionic weights, we can set

$$A_i := \mathtt{a}_1^{(\mathtt{i})}, \quad B_i = \frac{\mathtt{b}_1^{(\mathtt{i})}}{\mathtt{a}_1^{(\mathtt{i})}}, \quad x_i := \frac{\mathtt{b}_2^{(\mathtt{i})}}{\mathtt{a}_1^{(\mathtt{i})}}, \quad y_i := \frac{\mathtt{a}_2^{(\mathtt{i})}}{\mathtt{b}_1^{(\mathtt{i})}}.$$

Then the free fermionic condition ensures that $\mathtt{c}_1^{(\mathtt{i})}\mathtt{c}_2^{(\mathtt{i})} = (x_i + y_i) A_i^2 B_i$, as in Figure 2.1. The $\mathfrak{S}^*$ weights are similar.

Next, we define the boundary conditions for the two models. Let the lattice model

$$\mathfrak{S}_{\lambda/\mu} := \mathfrak{S}_{\lambda/\mu}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{A}, \boldsymbol{B}) = \mathfrak{S}_{\lambda/\mu}(x_1, \dots x_N; y_1, \dots y_N; A_1, \dots A_N; B_1, \dots B_N)$$

be defined as follows.

- N rows, labelled $1, \dots, N$ from bottom to top;
- $M + 1$ columns, where $M \geq \max(\lambda_1, \mu_1)$, labelled $0, \dots, M$ from left to right;
- Left and right boundary edges all $+$;
- Bottom boundary edges $-$ on parts of λ; $+$ otherwise;
- Top boundary edges $-$ on parts of μ; $+$ otherwise.
- Boltzmann weights from Figure 2.1 (in row i, we assign the weights $\mathtt{a}_1^{(\mathtt{i})}, \mathtt{a}_2^{(\mathtt{i})}, \mathtt{b}_1^{(\mathtt{i})}, \mathtt{b}_2^{(\mathtt{i})}, \mathtt{c}_1^{(\mathtt{i})}, \mathtt{c}_2^{(\mathtt{i})}$ from that figure).

See Figure 2.3 for an example of these boundary conditions. Despite the model's dependence on M and N, we suppress them from our notation. Note that the top and bottom boundaries are arbitrary, while the side boundary conditions are all $+$. These "empty" side boundary conditions turn out to be most directly related to Hamiltonian operators. We will consider other side boundary conditions, including domain-wall, in Section 2.6.

Additionally, we will define $\overline{\mathfrak{S}}_{\lambda/\mu}$ to be the same model as $\mathfrak{S}_{\lambda/\mu}$, but with arbitrary Boltzmann weights. We will use this more general model when we don't want to assume free fermionic weights.

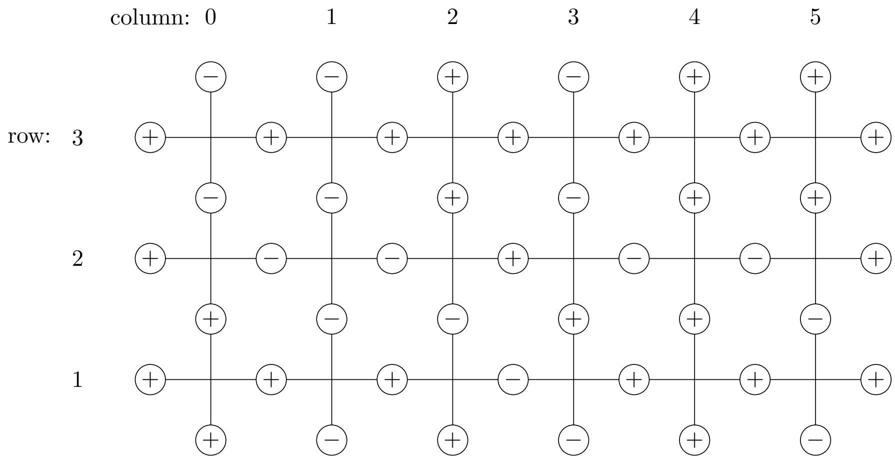


Figure 2.3: A state of the lattice model $\mathfrak{S}_{\lambda/\mu}$, where $\lambda = (5, 3, 1)$ and $\mu = (3, 1, 0)$.

It is often useful to consider the ensemble of lattice models $\{\mathfrak{S}_{\lambda/\mu}\}$ for all (valid) λ, μ, M, N at once. When we want to think of the model in this way, we will simply denote the model $\mathfrak{S}$.

The dual model $\mathfrak{S}^*$ is defined similarly:

$$\mathfrak{S}^*_{\lambda/\mu} := \mathfrak{S}^*_{\lambda/\mu}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{A}, \boldsymbol{B}) = \mathfrak{S}^*_{\lambda/\mu}(x_1, \dots x_N; y_1, \dots y_N; A_1, \dots A_N; B_1, \dots B_N)$$

is the following lattice model.

- N rows, labelled $1, \dots, N$ from top to bottom;
- $M + 1$ columns, where $M \geq \max(\lambda_1, \mu_1)$, labelled $0, \dots, M$ from left to right;
- Left and right boundary edges all $+$;
- Bottom boundary edges $-$ on parts of μ; $+$ otherwise;
- Top boundary edges $-$ on parts of λ; $+$ otherwise.
- Boltzmann weights from Figure 2.2 (in row i, we assign the weights $\mathtt{a}_1^{(\mathtt{i})}, \mathtt{a}_2^{(\mathtt{i})}, \mathtt{b}_1^{(\mathtt{i})}, \mathtt{b}_2^{(\mathtt{i})}$, $\mathtt{c}_1^{(\mathtt{i})}, \mathtt{c}_2^{(\mathtt{i})}$ from that figure).

Remark 2.3.1. To fully parametrize sets of free fermionic weights, we would need a fifth parameter representing $\mathtt{c}_2/\mathtt{c}_1$. However, this turns out not to be necessary. With our boundary conditions, the number of $\mathtt{c}_1^{(\mathtt{i})}$ vertices and the number of $\mathtt{c}_2^{(\mathtt{i})}$ vertices in any state are equal. Thus, changing the relative weights of $\mathtt{c}_1^{(\mathtt{i})}$ and $\mathtt{c}_2^{(\mathtt{i})}$ without changing their product does not change the partition function.

For general side boundary conditions, the number of $\mathtt{c}_1^{(\mathtt{i})}$ vertices and the number of $\mathtt{c}_2^{(\mathtt{i})}$ vertices always differ by a constant that depends only on the boundary conditions. In this case, changing the relative weights of $\mathtt{c}_1^{(\mathtt{i})}$ and $\mathtt{c}_2^{(\mathtt{i})}$ multiplies the partition function by a factor which is easily computed.

2.3.1 Relating $\mathfrak{S}$ and $\mathfrak{S}^*$

We will see in Corollary 2.5.2 that the partition function of both models is a supersymmetric Schur function. For now, note that there are two close relationships between the models. First, let us do the following transformation.

- Rotate the model $\mathfrak{S}_{\lambda/\mu}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{A}, \boldsymbol{B})$ 180°.
- Flip the vertical spins.
- Reverse the ordering on the columns.
- Divide the Boltzmann weights by $A_i^2 B_i$.

This new model has rows labelled from top to bottom, and columns labelled from left to right. If λ is the bottom boundary of the original model, then the top boundary of the new model is the partition $\overline{\lambda}$ obtained by reversing λ and swapping its parts and non-parts. By Lemma 2.2.1, $\overline{\lambda} - \rho = (\lambda - \rho)'$.

The 180° rotation along with the flip of vertical spins sends each of the six $\mathfrak{S}$ vertices to its counterpart $\mathfrak{S}^*$ vertex, swapping types $\mathtt{a}$ and $\mathtt{b}$ vertices. Dividing by $A_i^2 B_i$ sends the $\mathfrak{S}$ Boltzmann weights to the $\mathfrak{S}^*$ Boltzmann weights, again, with vertex types $\mathtt{a}$ and $\mathtt{b}$ swapped. Therefore, we have obtained the model $\mathfrak{S}^*_{\overline{\lambda}/\overline{\mu}}$.

Since each of the first three steps is a weight-preserving bijection of states, this allows us to relate the $\mathfrak{S}$ and $\mathfrak{S}^*$ partition functions.

Proposition 2.3.2.

$$Z(\mathfrak{S}^*_{\overline{\lambda}/\overline{\mu}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{A}, \boldsymbol{B})) = \prod_{i=1}^{N} (A_i^{-2M-2} B_i^{-M-1}) \cdot Z(\mathfrak{S}_{\lambda/\mu}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{A}, \boldsymbol{B})).$$

For our second relationship, we do the following transformation to $\mathfrak{S}$:

- Flip the model vertically (over a horizontal axis).
- Swap the $\mathtt{c}_1$ and $\mathtt{c}_2$ vertices.
- Replace A_i with A_i^{-1} and B_i with B_i^{-1}.

- Swap x_i and y_i.
- Rebalance the $\mathtt{c}_1$ and $\mathtt{c}_2$ vertices by multiplying the former and dividing the latter by $x_i + y_i$.

We have again obtained a $\mathfrak{S}^*$ lattice model, this time simply $\mathfrak{S}^*_{\lambda/\mu}$. Note that only the third and fourth steps change the partition function.

Proposition 2.3.3.

$$Z(\mathfrak{S}^*_{\lambda/\mu}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{A}, \boldsymbol{B})) = Z(\mathfrak{S}_{\lambda/\mu}(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{A}^{-1}, \boldsymbol{B}^{-1})). \tag{2.6}$$

We obtain similar identities by doing similar transformations starting from $\mathfrak{S}^*$. We can now combine these identities to relate the partition functions of $\mathfrak{S}$ and $\mathfrak{S}^*$ to themselves.

Proposition 2.3.4.

(a)

$$Z(\mathfrak{S}^*_{\overline{\lambda}/\overline{\mu}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{A}, \boldsymbol{B})) = \prod_{i=1}^{N}(A_i^{-2M-2} B_i^{-M-1}) \cdot Z(\mathfrak{S}^*_{\lambda/\mu}(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{A}^{-1}, \boldsymbol{B}^{-1})).$$

(b)

$$Z(\mathfrak{S}_{\overline{\lambda}/\overline{\mu}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{A}, \boldsymbol{B})) = \prod_{i=1}^{N}(A_i^{2M+2} B_i^{M+1}) \cdot Z(\mathfrak{S}_{\lambda/\mu}(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{A}^{-1}, \boldsymbol{B}^{-1})).$$

Note that similar identities hold for the general models $\mathfrak{S}$ and $\mathfrak{S}^*$.

The above propositions can be seen as giving us an involution on the partition functions. Let us make that more explicit. Define

$$\tilde{\omega}\left(\mathfrak{S}^*_{\lambda/\mu}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{A}, \boldsymbol{B})\right) := \mathfrak{S}_{\lambda/\mu}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{A}, \boldsymbol{B}), \qquad \tilde{\omega}\left(\mathfrak{S}_{\lambda/\mu}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{A}, \boldsymbol{B})\right) := \mathfrak{S}^*_{\lambda/\mu}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{A}, \boldsymbol{B}),$$

and further define

$$\tilde{\omega}\left(Z(\mathfrak{S}^*_{\lambda/\mu})\right) := Z\left(\tilde{\omega}(\mathfrak{S}^*_{\lambda/\mu})\right), \qquad \tilde{\omega}\left(Z(\mathfrak{S}_{\lambda/\mu})\right) := Z\left(\tilde{\omega}(\mathfrak{S}_{\lambda/\mu})\right).$$

Therefore, by Propositions 2.3.2 and 2.3.3, we have the following description of the action of ω on our partition functions:

Corollary 2.3.5. *For all strict partitions λ, μ,*

$$\tilde{\omega}\left(Z(\mathfrak{S}^*_{\lambda/\mu}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{A}, \boldsymbol{B})))\right) = \textstyle\prod_{i=1}^{N} A_i^{2M+2} B_i^{M+1} Z(\mathfrak{S}^*_{\overline{\lambda}/\overline{\mu}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{A}, \boldsymbol{B})) = Z(\mathfrak{S}^*_{\lambda/\mu}(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{A}^{-1}, \boldsymbol{B}^{-1}))$$

and

$$\tilde{\omega}\left(Z(\mathfrak{S}_{\lambda/\mu}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{A},\boldsymbol{B}))\right) = \prod_{i=1}^{N} A_i^{-2M-2} B_i^{-M-1} Z(\mathfrak{S}_{\overline{\lambda}/\overline{\mu}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{A},\boldsymbol{B})) = Z(\mathfrak{S}_{\overline{\lambda}/\overline{\mu}}(\boldsymbol{y},\boldsymbol{x},\boldsymbol{A}^{-1},\boldsymbol{B}^{-1})).$$

We will see in Section 2.5.2 that when we have a Hamiltonian matching a lattice model, the action of ω is the same as the action of $\tilde{\omega}$.

We conclude this section with a discussion of two specializations. *Vicious walkers* are sets of non-intersecting lattice paths that were introduced by Fisher [39]. These walkers can be interpreted as states of a solvable lattice model, and there is also a dual model made up of *osculating walkers* [40]. Through tableaux combinatorics or the Yang-Baxter equation, one can show that the partition functions of these models are Schur polynomials. The Lindström-Gessel-Viennot (LGV) Lemma [34, 35] expresses the number of sets of non-intersecting paths on a graph as a determinantal formula, and when applied to vicious walkers, the LGV Lemma gives the Jacobi-Trudi formula for Schur polynomials. In addition, the Von Näagelsbach–Kostka formula (dual Jacobi-Trudi) follows from a similar argument by particle-hole duality (see [12]).

The specialization of the model $\mathfrak{S}_{\lambda/\mu}(\boldsymbol{x},\boldsymbol{0},\boldsymbol{1},\boldsymbol{1}))$ gives the vicious model from [45] after reversing the row indices, while $\mathfrak{S}^*_{\lambda/\mu}(\boldsymbol{x},\boldsymbol{0},\boldsymbol{1},\boldsymbol{1}))$ gives the osculating model, after a vertical flip and a rebalancing of the $\mathtt{c}_1$ and $\mathtt{c}_2$ weights. On the other hand, $\mathfrak{S}_{\lambda/\mu}(\boldsymbol{0},\boldsymbol{x},\boldsymbol{1},\boldsymbol{1}))$ gives the osculating model, while $\mathfrak{S}^*_{\lambda/\mu}(\boldsymbol{0},\boldsymbol{x},\boldsymbol{1},\boldsymbol{1}))$ gives the vicious model. By Corollary 2.3.5, $\tilde{\omega}$ interchanges these models.

A second set of specializations obtains the Δ and Γ models from [11]. In Figure 2 of that paper, interpret left and up arrows as $-$ and interpret right and down arrows as $+$. Then $\mathfrak{S}_{\lambda/\mu}(\boldsymbol{x},\boldsymbol{xt},\boldsymbol{1},\boldsymbol{1}))$ is the Δ model from that paper. On the other hand, if we flip $\mathfrak{S}^*$ vertically and rebalance the $\mathtt{c}_1$ and $\mathtt{c}_2$ vertices, then the Γ model in [11] matches $\mathfrak{S}^*_{\lambda/\mu}(\boldsymbol{x},\boldsymbol{x}/\boldsymbol{t},\boldsymbol{1},\boldsymbol{t}))$.

2.4 Six-vertex models and Hamiltonians

The purpose of this section is to prove formulas for the partition functions of $\mathfrak{S}$ and $\mathfrak{S}^*$ in terms of a Hamiltonian operator.

Let T (resp. T^*) be the *row transfer matrix* for $\mathfrak{S}$ (resp. $\mathfrak{S}^*$):

$$\langle\mu|T|\lambda\rangle := Z(\mathfrak{S}_{\lambda/\mu}), \qquad \langle\lambda|T^*|\mu\rangle := Z(\mathfrak{S}^*_{\lambda/\mu}),$$

where for both lattice models, $N = 1$ and $M \geq \max(\lambda_1, \mu_1)$.

Another way to interpret the main result of this section (Theorem 2.4.1) is that the action of the operator e^{ϕ} equals (up the a simple factor) the action of T.

2.4.1 The $\mathfrak{S}$ lattice model

We will say that the lattice model $\mathfrak{S}$ and the Hamiltonian operator e^{H_+} *match* if the following condition holds:

$$Z(\mathfrak{S}_{\lambda/\mu}) = \prod_{i=1}^{N} A_i^{M+1} B_i^{\ell(\lambda)} \cdot \langle \mu | e^{H_+} | \lambda \rangle \qquad \text{for all strict partitions } \lambda, \mu \text{ and all } M, N. \quad (2.7)$$

More generally, we will say that a lattice model $\mathfrak{S}$ and a Hamiltonian operator e^H match if

$$Z(\mathfrak{S}_{\lambda/\mu}) = * \cdot \langle \mu | e^{H} | \lambda \rangle \qquad \text{for all strict partitions } \lambda, \mu \text{ and all } M, N, \quad (2.8)$$

where $*$ represents any easily computable function of the Boltzmann weights of $\mathfrak{S}$.

In a sense, this condition tells us that the Hamiltonian operator e^{H_+} has the *same* structure as the lattice model $\mathfrak{S}$. Although the procedures for calculating the partition function and τ function are different, the result is always the same.

Theorem 2.4.1.

(a) (2.7) holds precisely when

$$s_k^{(j)} = \frac{1}{k}\left(x_i^k + (-1)^{k-1} y_i^k\right) \qquad \textit{for all } k \geq 1, j \in [1, N]. \quad (2.9)$$

(b) If the Boltzmann weights are not free fermionic, (2.7) does not hold for any choice of the $s_k^{(j)}$.

There are a few technical reasons that we look for a relationship of the form (2.7). First, the τ function is independent of M, and increasing M without changing λ or μ adds more $\mathtt{a}_1$ vertices to each state. In other words, the Hamiltonian doesn't "see" $\mathtt{a}_1$ vertices. This means that the τ function is independent of A_i.

Similarly, the Hamiltonian doesn't "see" $\mathtt{b}_1$ vertices. For instance, let $M > \max(\lambda_1, \mu_1)$, and let $\tilde{\lambda} = (M, \lambda_1, \lambda_2, \ldots)$, $\tilde{\mu} = (M, \mu_1, \mu_2 \ldots)$. Then $Z(\mathfrak{S}_{\tilde{\lambda}/\tilde{\mu}}) = \prod_{i=1}^N B_i \cdot Z(\mathfrak{S}_{\lambda/\mu})$, but the τ function is unchanged. Thus, the τ function is independent of B_i as well.

This observation allows us to simplify our analysis. Since A_i appears in every vertex in row i of the lattice, the weight of every state must have the factor $\prod_i A_i^{M+1}$, and therefore so must the partition function. Similarly, note that B_i appears in the weight of precisely the vertices with a $-$ spin on their bottom edge. By particle conservation, there must be the same number of vertical $-$ spins in each row or else both the partition function and τ function are 0. This means that in each admissible state of the model B_i appears precisely $\ell(\lambda)$ times, and we must have $\ell(\lambda) = \ell(\mu)$.

Therefore, we have reduced Part a of Theorem 2.4.1 to the following proposition.

Proposition 2.4.2. *Set $A_i = B_i = 1$ for all i. Then for all strict partitions λ and μ with $\ell(\lambda) = \ell(\mu) = \ell$,*

$$Z(\mathfrak{S}_{\lambda/\mu}) = \langle \mu | e^{H_+} | \lambda \rangle \qquad \text{for all strict partitions } \lambda, \mu \text{ and all } M, N. \tag{2.10}$$

precisely when the Hamiltonian parameters are defined by (2.9).

Our proof will involve cases of increasing generality. We will first prove the proposition in the case that $N = 1$ and $\ell = 1$, and then move on to the case where $N = 1$ and ℓ is arbitrary. This second step involves Wick's theorem in a way that will make Part b of Theorem 2.4.1 easy to prove. When $N = 1$, we only have one set of parameters x_1, y_1, and $s_k = s_k^{(1)}$; we will often leave off the index in this case.

Finally, we will use a simple branching argument to prove Proposition 2.4.2 for arbitrary N.

Lemma 2.4.3. *Proposition 2.4.2 is true in the case where $N = 1$, $\lambda = (r+p)$, and $\mu = (r)$.*

Proof. If $p < 0$, both sides of (2.10) are zero. Otherwise, $\mathfrak{S}_{\lambda/\mu}$ has exactly one admissible state. If $p = 0$, then column r has a vertex of type $\mathtt{b}_1$, and all other vertices are type $\mathtt{a}_1$. If $p > 0$, then column r has a vertex of type $\mathtt{c}_1$, columns $r+1, \dots, r+p-1$ have vertices of type $\mathtt{b}_2$, column $r+p$ has a vertex of type $\mathtt{c}_2$, and all other vertices are type $\mathtt{a}_1$.

This gives

$$Z(\mathfrak{S}_{\lambda/\mu}) = \begin{cases} (x+y)x^{p-1}, & \text{if } p \geq 1 \\ 1, & \text{if } p = 0. \end{cases}$$

Now, we show that the Hamiltonian matches the partition function. Let

$$H(t) = 1 + \sum_{p \geq 1} (x+y)x^{p-1} t^p.$$

On the other hand,

$$\langle \mu | e^{H_+} | \lambda \rangle = \langle 0 | \psi_r e^{H_+} \psi^*_{p+r} | 0 \rangle = \langle 0 | \psi_0 e^{H_+} \psi^*_p | 0 \rangle = \langle 0 | \psi_0 e^{H_+} \psi^*(t) | 0 \rangle |_{t^p},$$

so the result will follow once we can show that

$$H(t) = \tilde{H}(t) := \langle 0 | \psi_0 e^{H_+} \psi^*(t) | 0 \rangle.$$

By Lemma 2.3,

$$h_p = \langle 0|\psi_0 e^{H_+}\psi^*(t)|0\rangle|_{t^p} = \left\langle 0 \middle| \psi_0 \sum_{k\geq 0}\frac{1}{k!}\sum_{q_1+\ldots+q_k=p} s_{q_1}\ldots s_{q_k} J_{q_1}\ldots J_{q_k}\psi_p^* \middle| 0 \right\rangle$$
$$= \sum_{k\geq 0}\frac{1}{k!}\sum_{q_1+\ldots+q_k=p} s_{q_1}\ldots s_{q_k},$$

so

$$\tilde{H}(t) = \sum_{p\geq 0}\left(\sum_{k\geq 0}\frac{1}{k!}\sum_{q_1+\ldots+q_k=p} s_{q_1}\ldots s_{q_k}\right)t^p = \exp\left(\sum_{m\geq 1} s_m t^m\right).$$

Now, we can sum $H(t)$ as a geometric series:

$$H(t) = \frac{1+((x+y)-x)t}{1-xt} = \frac{1+yt}{1-xt},$$

so

$$\log H(t) = \log(1+yt) - \log(1-xt),$$

and

$$\left.\left(\frac{d^n}{dt^n}\log H(t)\right)\right|_{t=0} = \left.\left(-\frac{(n-1)!(-y)^n}{(1+yt)^n} + \frac{(n-1)!x^n}{(1-xt)^n}\right)\right|_{t=0}$$
$$= (n-1)!(x^n + (-1)^{n-1}y^n)$$
$$= n!s_n.$$

Therefore, $H(t) = \tilde{H}(t)$. □

Lemma 2.4.4. *Proposition 2.4.2 is true when $N = 1$, for arbitrary λ, μ.*

Proof. First assume that λ and μ *interleave* i.e. $\lambda_i \geq \mu_i \geq \mu_{i+1}$ for all i. In this case, $\mathfrak{S}_{\lambda/\mu}$ has exactly one admissible state, with vertices determined by the following table.

Vertex in column k	Condition (i arbitrary)
$\mathtt{a}_2$	$\mu_i = \lambda_{i+1} = k$
$\mathtt{b}_1$	$\lambda_i = \mu_i = k$
$\mathtt{b}_2$	$\lambda_i > k > \mu_i$
$\mathtt{c}_1$	$\mu_{i-1} > \lambda_i = k > \mu_i$
$\mathtt{c}_2$	$\lambda_i > k = \mu_i > \lambda_{i+1}$
$\mathtt{a}_1$	else

By the Jacobi-Trudi formula (Proposition 2.2.8; note the ρ shift),

$$\langle \mu | e^{H_+} | \lambda \rangle = \det_{1 \le i,j \le \ell} h_{\lambda_i - \mu_j}.$$

Let $\eta_{i,j} = h_{\lambda_i - \mu_j}$. Since $\lambda_i \geq \mu_i \geq \lambda_{i+1}$ for all i, and since λ, μ are strict, $\eta_{i,j} = 0$ whenever $j \geq i+1$, and

$$\eta_{i,i+1} = \begin{cases} 1, & \mu_i = \lambda_{i+1} \\ 0, & \mu_i > \lambda_{i+1}. \end{cases}$$

On the other hand, by Lemma 2.4.3, for all $p > 0$,

$$h_p = Z(\mathfrak{S}_{(p)/(0)}) = (x+y) \cdot x^{p-1}.$$

Note that $\eta_{i,j}$ with $i \leq j$ doesn't appear in the determinant $\det(\eta_{i,j})$ unless $\mu_i = \lambda_{i+1}, \mu_{i+1} = \lambda_{i+1}, \ldots \mu_{j-1} = \lambda_j$. Therefore, $\det(\eta_{i,j})$ is a product of blocks of the form

$$\begin{bmatrix} \eta_{i,i} & 1 & & & \\ & & 1 & & \\ \cdots & & & \cdots & \\ & & & & 1 \\ \eta_{j,i} & & \cdots & & \eta_{j,j} \end{bmatrix}, \qquad \text{where } i \leq j, \mu_i = \lambda_{i+1}, \mu_{i+1} = \lambda_{i+1}, \ldots \mu_{j-1} = \lambda_j.$$

For all $i \leq b \leq a \leq j$, $\lambda_b > \mu_a$, so $\eta_{x,y} = (x+y)x^{\lambda_b - \mu_a - 1}$, and so the determinant of the block is

$$\det \begin{bmatrix} (x+y)x^{\lambda_i - \mu_i - 1} & 1 & & & \\ & & 1 & & \\ \cdots & & & \cdots & \\ & & & & 1 \\ (x+y)x^{\lambda_i - \mu_j - 1} & & \cdots & & (x+y)x^{\lambda_j - \mu_j - 1} \end{bmatrix} = \frac{x+y}{x} \cdot \left(\frac{x+y}{x} - 1 \right)^{j-i-1} x^{\lambda_i - \mu_j}.$$

Taking the product over all such blocks, we obtain

$$\begin{aligned} \det(\eta_{i,j}) &= x^{|\lambda| - |\mu|} \left(\frac{x+y}{x} - 1 \right)^{\ell} \left(\frac{x+y}{x+y-x} \right)^{\#\mu_i > \lambda_{i+1}} \cdot \left(\frac{x+y}{x} \right)^{-(\#\lambda_i = \mu_i)} \\ &= x^{|\lambda| - |\mu|} \left(\frac{y}{x} \right)^{\ell} \left(\frac{x+y}{y} \right)^{\#\mu_i > \lambda_{i+1}} \cdot \left(\frac{x+y}{x} \right)^{-(\#\lambda_i = \mu_i)} \\ &= x^{|\lambda| - |\mu| - \ell + (\#\lambda_i = \mu_i)} (x+y)^{(\#\mu_i > \lambda_{i+1}) - (\#\lambda_i = \mu_i)} y^{\ell - (\#\mu_i > \lambda_{i+1})}, \end{aligned}$$

where $|\lambda|$ is the sum of the parts of λ, likewise for μ, and expressions like $\#\mu_i > \lambda_{i+1}$

represent the number of indices i such that the statement is true.

Now, from the lattice model side,

$$Z(\mathfrak{S}_{\lambda/\mu}) = x^{|\lambda|-|\mu|-(\#\lambda_i>\mu_i)} \cdot (x+y)^{\ell-(\#\lambda_i=\mu_i)-(\#\mu_i=\lambda_{i+1})} \cdot y^{\#\mu_i=\lambda_{i+1}} = \det(\eta_{i,j}).$$

If λ and μ do not interleave, then it is easy to see that $Z(\mathfrak{S}_{\lambda/\mu}) = 0$. For some a either $\lambda_a < \mu_a$, in which case $\langle\mu|e^{H_+}|\lambda\rangle = 0$, or $\mu_a < \lambda_{a+1}$. Notice that h_{p+r}/h_r is independent of r as long as $r \geq 1$. We can use this fact to do column operations on the matrix $(\eta_{i,j})$, and obtain a matrix (M_{ij}) where $M_{ij} = 0$ whenever $\lambda_{i+1} > \mu_j$. In particular, every nonzero entry in the first a columns will be in the first $a-1$ rows. But this means that $\det(\eta_{ij}) = \det(M_{ij}) = 0$, so the result still holds. □

The proof of Proposition 2.4.2 is now formal. Both lattice models and Hamiltonians "branch" in the same way, so we simply sum over the intermediate partitions. In similar contexts, this is sometimes called a *Miwa transform.* For clarity, we will write $\mathfrak{S}^N_{\lambda/\mu}$ for the usual $\mathfrak{S}$ lattice model with N rows.

Proof of Proposition 2.4.2.

$$\begin{aligned}
\langle\mu|e^{H_+}|\lambda\rangle &= \langle\mu|e^{\phi_N}\dots e^{\phi_1}|\lambda\rangle \\
&= \sum_{\nu_1,\dots,\nu_{N-1}} \langle\mu|e^{\phi_N}|\nu_{N-1}\rangle\langle\nu_{N-1}|e^{\phi_{N-1}}|\nu_{N-2}\rangle\dots\langle\nu_1|e^{\phi_1}|\lambda\rangle \\
&= \sum_{\nu_1,\dots,\nu_{N-1}} Z(\mathfrak{S}^1_{\nu_{N-1}/\mu})\dots Z(\mathfrak{S}^1_{\lambda/\nu_1}) \\
&= Z(\mathfrak{S}^N_{\lambda/\mu}).
\end{aligned}$$

□

This completes the proof of Theorem 2.4.1(a).

Proof of Theorem 2.4.1(b). Let both the Boltzmann weights and the Hamiltonian parameters be arbitrary. In other words, we are using the general model $\overline{\mathfrak{S}}_{\lambda/\mu}$. By Remark 2.3.1 and the discussion after the statement of Theorem 2.4.1, we may set $\mathtt{a}_1^{(\mathtt{j})} = \mathtt{b}_1^{(\mathtt{j})} = \mathtt{c}_2^{(\mathtt{j})} = 1$, and check when (2.10) holds.

By performing the calculations in Lemma 2.4.3, and using the same notation, we get

$$H(t) = \frac{1+(\mathtt{c}_1-\mathtt{b}_2)t}{1+\mathtt{b}_2 t},$$

so

$$\left(\frac{d^n}{dz^n}\log H(z)\right)\Bigg|_{z=0} = (n-1)!({\mathtt{b}_2}^n + (-1)^{n-1}(\mathtt{c}_1-\mathtt{b}_2)^n).$$

This means we must have $s_n = \frac{1}{n}(\mathtt{b_2}^n + (-1)^{n-1}(\mathtt{c_1} - \mathtt{b_2})^n)$.

Now, performing the calculations in Lemma 2.4.4, if λ and μ interleave,

$$\begin{aligned}\langle\mu|e^{H_+}|\lambda\rangle &= \mathtt{b_2}^{|\lambda|-|\mu|} \cdot \left(\frac{\mathtt{c_1}}{\mathtt{b_2}} - 1\right)^{\ell} \cdot \left(\frac{\mathtt{c_1}}{\mathtt{c_1} - \mathtt{b_2}}\right)^{\#\mu_i > \lambda_{i+1}} \cdot \left(\frac{\mathtt{c_1}}{\mathtt{b_2}}\right)^{-(\#\lambda_i = \mu_i)} \\ &= \mathtt{b_2}^{|\lambda|-|\mu|-(\#\lambda_i > \mu_i)} \cdot \mathtt{c_1}^{\ell-(\#\lambda_i=\mu_i)-(\#\mu_i=\lambda_{i+1})} \cdot (\mathtt{c_1} - \mathtt{b_2})^{\#\mu_i = \lambda_{i+1}}.\end{aligned}$$

On the other hand,

$$Z(\mathfrak{S}_{\lambda/\mu}) = \mathtt{b_2}^{|\lambda|-|\mu|-(\#\lambda_i > \mu_i)} \cdot \mathtt{c_1}^{\ell-(\#\lambda_i=\mu_i)-(\#\mu_i=\lambda_{i+1})} \cdot \mathtt{a_2}^{\#\mu_i=\lambda_{i+1}},$$

so equality holds if and only if $\mathtt{a_2} = \mathtt{c_1} - \mathtt{b_2}$, which is the free fermion condition.

□

This completes the proof of Theorem 2.4.1. The following corollary is just restatement of the Theorem in terms of the Boltzmann weights, for convenience.

Corollary 2.4.5. *The general $\mathfrak{S}$ lattice model $\overline{\mathfrak{S}}_{\lambda/\mu}$ matches the Hamiltonian e^{H_+} (see (2.8)) if and only if:*

(a) The Boltzmann weights of $\overline{\mathfrak{S}}_{\lambda/\mu}$ are free fermionic.

(b)

$$s_k^{(j)} = \frac{1}{k}\left(\left(\frac{\mathtt{b}_2^{(\mathtt{j})}}{\mathtt{a}_1^{(\mathtt{j})}}\right)^k + (-1)^{k-1}\left(\frac{\mathtt{a}_2^{(\mathtt{j})}}{\mathtt{b}_1^{(\mathtt{j})}}\right)^k\right) \qquad \textit{for all } k \geq 1, j \in [1, N].$$

(c) The extra factor is $* = \prod_{i=1}^N (\mathtt{a}_1^{(\mathtt{i})})^{M-\ell}(\mathtt{b}_1^{(\mathtt{i})})^{\ell}$.

2.4.2 The $\mathfrak{S}^*$ lattice model

Now we prove a similar identity for the $\mathfrak{S}^*$ lattice model. We will say that the lattice model $\mathfrak{S}$ and the Hamiltonian operator e^{H_-} *match* if the following condition holds:

$$Z(\mathfrak{S}^*_{\lambda/\mu}) = \prod_{i=1}^N A_i^{-(M+1)} B_i^{-\ell(\lambda)} \cdot \langle\lambda|e^{H_-}|\mu\rangle \qquad \text{for all strict partitions } \lambda, \mu \text{ and all } M, N. \tag{2.11}$$

Theorem 2.4.6.

(a) (2.11) holds precisely when

$$s_{-k}^{(j)} = \frac{1}{k}\left(y_i^k + (-1)^{k-1}x_i^k\right) \qquad \text{for all } k \geq 1, j \in [1, N]. \tag{2.12}$$

(b) If the Boltzmann weights are not free fermionic, (2.11) does not hold for any choice of the $s_{-k}^{(j)}$.

Proof. Note that taking (2.9) and (2.12), $s_{-k}^{(j)} = s_k^{(j)}|_{x_i \leftrightarrow y_i}$, so $\langle\lambda|e^{H_-}|\mu\rangle = \langle\mu|e^{H_+}|\lambda\rangle|_{x_i \leftrightarrow y_i}$. Then,

$$\begin{aligned} Z(\mathfrak{S}^*_{\lambda/\mu}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{A}, \boldsymbol{B})) &= Z(\mathfrak{S}_{\lambda/\mu}(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{A}^{-1}, \boldsymbol{B}^{-1})) && \text{by (2.6)} \\ &= \prod_{i=1}^{N} A_i^{M+1} B_i^{\ell(\lambda)} \cdot \langle\mu|e^{H_+}|\lambda\rangle \Big|_{x_i \leftrightarrow y_i, A_i \mapsto A_i^{-1}, B_i \mapsto B_i^{-1}} && \text{by Thm 2.4.1} \\ &= \prod_{i=1}^{N} A_i^{-(M+1)} B_i^{-\ell(\lambda)} \cdot \langle\lambda|e^{H_-}|\mu\rangle \end{aligned}$$

□

Corollary 2.4.7. *The general* $\mathfrak{S}^*$ *lattice model* $\overline{\mathfrak{S}}^*_{\lambda/\mu}$ *matches the Hamiltonian* e^{H_-} *(see (2.8)) if and only if:*

(a) The Boltzmann weights of $\overline{\mathfrak{S}}^*_{\lambda/\mu}$ *are free fermionic.*

(b)

$$s_{-k}^{(j)} = \frac{1}{k}\left(\left(\frac{\mathtt{b}_2^{(\mathtt{j})}}{\mathtt{a}_1^{(\mathtt{j})}}\right)^k + (-1)^{k-1}\left(\frac{\mathtt{a}_2^{(\mathtt{j})}}{\mathtt{b}_1^{(\mathtt{j})}}\right)^k\right) \qquad \text{for all } k \geq 1, j \in [1, N].$$

(c) The extra factor is $* = \prod_{i=1}^{N}(\mathtt{a}_1^{(\mathtt{i})})^{M-\ell}(\mathtt{b}_1^{(\mathtt{i})})^{\ell}$.

2.5 The free fermionic partition function

The main result of this section is that the free fermionic partition function is a (skew) supersymmetric Schur function up to a simple factor. This fact will give us new proofs of Cauchy, Pieri, Jacobi-Trudi, and branching identities for these functions, as well as a six-vertex free fermionic analogue of the Lindström-Gessel-Viennot (LGV) Lemma and a mysterious positivity result.

2.5.1 Supersymmetric Schur functions

Supersymmetric functions are a generalization of symmetric functions with two sets of parameters $\boldsymbol{x} = x_1, \ldots, x_n$ and $\boldsymbol{y} = y_1, \ldots, y_n$. This exposition is taken from Macdonald [46, §6].

Let $[x|\boldsymbol{y}]^r = (x + y_1)(x + y_2) \cdots (x + y_r)$. If λ is a partition, let

$$A_\lambda := \det \left((x_i|\boldsymbol{y})^{\lambda_j} \right).$$

Then

$$s_\lambda[\boldsymbol{x}|\boldsymbol{y}] := \frac{A_{\lambda+\rho}}{A_\rho}$$

is a symmetric polynomial in the x_i, and $s_\lambda[\boldsymbol{x}|\boldsymbol{0}] = s_\lambda[\boldsymbol{x}]$, the Schur polynomial associated to λ.

Then for an integer $r \geq 0$ we define

$$h_r[\boldsymbol{x}|\boldsymbol{y}] := s_{(r)}[\boldsymbol{x}|\boldsymbol{y}], \qquad e_r[\boldsymbol{x}|\boldsymbol{y}] := s_{(1^r)}[\boldsymbol{x}|\boldsymbol{y}].$$

Macdonald then defines skew supersymmetric Schur functions via Jacobi-Trudi formulas: if $\ell(\lambda) = \ell(\mu) = \ell$,

$$s_{\lambda/\mu}[\boldsymbol{x}|\boldsymbol{y}] := \det_{1\leq i,j\leq \ell} h_{\lambda_i - \mu_j - i + j}[\boldsymbol{x}|\boldsymbol{y}] = \det_{1\leq i,j\leq \ell} e_{\lambda'_i - \mu'_j - i + j}[\boldsymbol{x}|\boldsymbol{y}]$$

2.5.2 Evaluating the partition function

Brubaker and Schultz showed in [11] that the partition functions of a large class of free fermionic lattice models are supersymmetric Schur functions.

Proposition 2.5.1. *[11, §A.3.2]*

(a) If $s_k^{(j)} = \frac{1}{k}(x_j^k + (-1)^{k-1} y_j^k)$*, then*

$$\langle \mu + \rho | e^{H_+} | \lambda + \rho \rangle = s_{\lambda/\mu}[x_1, \ldots, x_n | y_1, \ldots, y_n].$$

(b) If $s_{-k}^{(j)} = \frac{1}{k}(y_j^k + (-1)^{k-1} x_j^k)$*, then*

$$\langle \lambda + \rho | e^{H_-} | \mu + \rho \rangle = s_{\lambda/\mu}[y_1, \ldots, y_n | x_1, \ldots, x_n].$$

By scaling these Hamiltonians, or by scaling the lattice models in [11], we can compute the free fermionic partition function.

Corollary 2.5.2.

$$Z(\mathfrak{S}_{\lambda+\rho/\mu+\rho}) = \prod_{i=1}^{N} A_i^{M+1} B_i^{\ell(\lambda)} \cdot s_{\lambda/\mu}[x_1, \dots, x_n | y_1, \dots, y_n],$$

and similarly

$$Z(\mathfrak{S}^*_{\lambda+\rho/\mu+\rho}) = \prod_{i=1}^{N} A_i^{-(M+1)} B_i^{-\ell(\lambda)} \cdot s_{\lambda/\mu}[y_1, \dots, y_n | x_1, \dots, x_n].$$

Proof. This follows from Theorems 2.4.1 and 2.4.6, and Proposition 2.5.1. □

For completeness, we will rewrite these formulas in terms of the vertex weights directly.

$$Z(\mathfrak{S}_{\lambda+\rho/\mu+\rho}) = \prod_{i=1}^{N} \mathtt{a}_1^{(\mathtt{i})\,M+1-\ell(\lambda)} \mathtt{b}_1^{(\mathtt{i})\,\ell(\lambda)} \cdot s_{\lambda/\mu} \left[\frac{\mathtt{b}_2^{(1)}}{\mathtt{a}_1^{(1)}}, \dots, \frac{\mathtt{b}_2^{(\mathtt{N})}}{\mathtt{a}_1^{(\mathtt{N})}} \middle| \frac{\mathtt{a}_2^{(1)}}{\mathtt{b}_1^{(1)}}, \dots, \frac{\mathtt{a}_2^{(\mathtt{N})}}{\mathtt{b}_1^{(\mathtt{N})}} \right], \tag{2.13}$$

$$Z(\mathfrak{S}^*_{\lambda+\rho/\mu+\rho}) = \prod_{i=1}^{N} \mathtt{a}_1^{(\mathtt{i})\,M+1-\ell(\lambda)} \mathtt{b}_1^{(\mathtt{i})\,\ell(\lambda)} \cdot s_{\lambda/\mu} \left[\frac{\mathtt{b}_2^{(1)}}{\mathtt{a}_1^{(1)}}, \dots, \frac{\mathtt{b}_2^{(\mathtt{N})}}{\mathtt{a}_1^{(\mathtt{N})}} \middle| \frac{\mathtt{a}_2^{(1)}}{\mathtt{b}_1^{(1)}}, \dots, \frac{\mathtt{a}_2^{(\mathtt{N})}}{\mathtt{b}_1^{(\mathtt{N})}} \right], \tag{2.14}$$

The weights in (2.13) are the $\mathfrak{S}$ weights (Figure 2.1), while the weights in (2.14) are the $\mathfrak{S}^*$ weights (Figure 2.2).

This result, along with Theorem 2.6.6 can be seen as a replacement of [2, Theorem 9]. That result holds in the case of the 5-vertex free fermionic model, but there is an error in that proof when applied to the full 6-vertex free fermionic model.

We now show that our involution on lattice models defined in Section 2.3 is equivalent to the involution on generalized symmetric functions defined in Section 2.2. In other words, the supersymmetric involution can be written *diagrammatically* in terms of lattice model manipulations.

Extend the involution ω from Section 2.2 so that it sends $A_i \mapsto A_i^{-1}$, $B_i \mapsto B_i^{-1}$. Then, the involutions ω and $\tilde{\omega}$ are the same.

Corollary 2.5.3.

$$\omega(Z(\mathfrak{S}_{\lambda/\mu})) = \tilde{\omega}(Z(\mathfrak{S}_{\lambda/\mu})),$$

and

$$\omega(Z(\mathfrak{S}^*_{\lambda/\mu})) = \tilde{\omega}(Z(\mathfrak{S}^*_{\lambda/\mu})).$$

Proof. By Theorems 2.4.1 and 2.4.6,

$$\begin{aligned}\omega(Z(\mathfrak{S}_{\lambda/\mu})) &= \omega\left(\prod_i A_i^{M+1} B_i^{\ell(\lambda)} \langle \mu | e^{H_+} | \lambda \rangle\right)\\ &= \prod_i A_i^{-(M+1)} B_i^{-\ell(\lambda)} \langle \lambda | e^{H_-} | \mu \rangle\\ &= Z(\mathfrak{S}^*_{\lambda/\mu})\\ &= \tilde{\omega}(Z(\mathfrak{S}_{\lambda/\mu})).\end{aligned}$$

The proof of the second statement is similar □

In particular, using the model $\mathfrak{S}_{\lambda+\rho/\mu+\rho}(\boldsymbol{x}, \boldsymbol{y}, \mathbf{1}, \mathbf{1})$, we have the expected involution on supersymmetric Schur functions:

$$\omega(s_{\lambda/\mu}[\boldsymbol{x}|\boldsymbol{y}]) = \tilde{\omega}(s_{\lambda/\mu}[\boldsymbol{x}|\boldsymbol{y}]) = s_{\lambda/\mu}[\boldsymbol{y}|\boldsymbol{x}].$$

In addition, the Jacobi-Trudi formula (Proposition 2.2.8), applied to the lattice model, is a six-vertex free fermionic analogue of the Lindström-Gessel-Viennot (LGV) Lemma.

For any r, $h_k = Z(\mathfrak{S}_{(k+r)/(r)})|_{A_i=B_i=1}$ and $e_k = Z(\mathfrak{S}^*_{(k+r)/(r)})|_{A_i=B_i=1}$, so we have the following.

Proposition 2.5.4 (Six-vertex LGV Lemma)**.**

$$Z(\mathfrak{S}_{\lambda/\mu}) = \det_{1\le i,j\le \ell(\lambda)} Z(\mathfrak{S}_{(\lambda_i)/(\mu_j)})$$

and

$$Z(\mathfrak{S}^*_{\lambda/\mu}) = \det_{1\le i,j\le \ell(\lambda)} Z(\mathfrak{S}^*_{(\lambda_i)/(\mu_j)}),$$

where if the models on the left side have M columns, and the models on the right side have M_{ij} columns, we have $M = \sum_i M_{i,\sigma(i)}$ for all permutations σ.

This reduces to the usual case of the LGV lemma on a five-vertex model. If we set $y_i = 0, A_i = B_i = 1$, the resulting model gives the well-known equivalence between the tableaux and Jacobi-Trudi definitions of Schur functions, and if we additionally set $x_i = 1$, it gives a determinantal count of sets of non-intersecting lattice paths. We could do the same thing for sets of "osculating paths" by instead setting $x_i = 0, y_i = A_i = B_i = 1$. However, we cannot get an unweighted count of states Boltzmann for the six-vertex model from this formula since specializing all weights to 1 would violate the free fermion condition.

2.5.3 Identities: Pieri rule, Cauchy identity, and branching rule

Combining Corollary 2.5.2 with the results in Section 2.2.4, we get new proofs of the Pieri rule, Cauchy identity, and branching rule for supersymmetric Schur polynomials. Equivalently, we get those same identities for the free fermionic partition function.

The branching rule is straightforward to prove in both the context of the Hamiltonian and the lattice model. The Cauchy identity can also be proved via both methods. Since our Boltzmann weights are free fermionic, the lattice model is solvable by [2, Theorem 1]. Then one can prove the dual Cauchy identity by repeatedly applying the Yang-Baxter equation to a "combined lattice model". See [16, Theorem 7] for an example of this process. On the other hand, the Pieri rule appears to be more easily proved using the Hamiltonian.

Proposition 2.5.5 (Cauchy identity)**.** *For any strict partitions λ and μ,*

$$\sum_{\nu} s_{\lambda/\nu}[\boldsymbol{x}|\boldsymbol{y}]s_{\mu/\nu}[\boldsymbol{z}|\boldsymbol{w}] = \prod_{i,j} \frac{(1-x_i z_j)(1-y_i w_j)}{(1+x_i w_j)(1+y_i z_j)} \cdot \sum_{\nu} s_{\nu/\mu}[\boldsymbol{x}|\boldsymbol{y}]s_{\nu/\lambda}[\boldsymbol{z}|\boldsymbol{w}],$$

where the sums are over all strict partitions ν.

Proof. From Proposition 2.2.10, and using the notation from that proposition, we have the formula

$$\sum_{\nu} \sigma_{\lambda/\nu}\sigma'_{\mu/\nu} = \prod_{i,j} \exp\left(\sum_{k\geq 1} k \cdot s_k^{(i)} s_{-k}^{(j)}\right) \cdot \sum_{\nu} \sigma_{\nu/\mu}\sigma'_{\nu/\lambda},$$

so we just need to prove that when $s_k^{(i)} = \frac{1}{k}\left(x_i^k + (-1)^{k-1}y_i^k\right)$ and $s_{-k}^{(i)} = \frac{1}{k}\left(z_i^k + (-1)^{k-1}w_i^k\right)$,

$$\exp\left(\sum_{k\geq 1} k \cdot s_k^{(i)} s_{-k}^{(j)}\right) = \frac{(1-x_i z_j)(1-y_i w_j)}{(1+x_i w_j)(1+y_i z_j)}.$$

This follows by exponentiating the next string of equalities.

$$\begin{aligned}\sum_{k\geq 1} k \cdot s_k^{(i)} s_{-k}^{(j)} &= \sum_{k\geq 1} k \cdot \frac{1}{k}\left(x_i^k + (-1)^{k-1}y_i^k\right) \cdot \frac{1}{k}\left(z_j^k + (-1)^{k-1}w_j^k\right) \\ &= \sum_{k\geq 1} \frac{1}{k}\left(x_i^k z_j^k + (-1)^{k-1}x_i^k w_j^k + (-1)^{k-1}y_i^k z_j^k + y_i^k w_j^k\right) \\ &= \log(1+x_i w_j) + \log(1+y_i z_j) - \log(1-x_i z_j) - \log(1-y_i w_j) \\ &= \log\left(\frac{(1+x_i w_j)(1+y_i z_j)}{(1-x_i z_j)(1-y_i w_j)}\right).\end{aligned}$$

□

The branching rule and Pieri rule all follow directly from their Hamiltonian analogues in Section 2.2.3: Propositions 2.2.11 and 2.2.12, respectively.

Proposition 2.5.6 (Branching rule)**.** *For all partitions* λ, μ*,*

$$s_{\lambda/\mu}[x_1, \ldots, x_n | y_1, \ldots, y_n] = \sum_{\nu} s_{\mu/\nu}[x_1|y_1] s_{\lambda/\mu}[x_2, \ldots, x_n | y_2, \ldots, y_n].$$

Proposition 2.5.7 (Pieri rule)**.**

$$h_k[\boldsymbol{x}|\boldsymbol{y}] \cdot s_\lambda[\boldsymbol{x}|\boldsymbol{y}] = \sum_{\nu} \langle \nu + \rho | U_k | \lambda + \rho \rangle \, s_\nu[\boldsymbol{x}|\boldsymbol{y}].$$

2.5.4 Positivity

Suppose for this subsection that our lattice model parameters $A_i, B_i, x_i, y_i \in \mathbb{R}$. It turns out that the Hamiltonian interpretation provides a positivity condition for the partition function $Z(\mathfrak{S}_{\lambda/\mu})$.

We will ask when $\mathfrak{S}$ satisfies the following condition:

$$\text{The partition function } Z(\mathfrak{S}_{\lambda/\mu}(\boldsymbol{x}, \boldsymbol{y}, \mathbf{1}, \mathbf{1})) \geq 0 \text{ for all strict partitions } \lambda, \mu. \tag{2.15}$$

Proposition 2.5.8. $\mathfrak{S}$ *satisfies (2.15) if and only if*

$$x_i, y_i \geq 0 \qquad \text{for all } 1 \leq i \leq n.$$

Proof. By Theorem 2.4.1, the map $\phi : \Lambda \longrightarrow \mathbb{R}$ defined by

$$s_k \mapsto \frac{1}{k} \sum_j \left(x_j^k + (-1)^{k-1} y_j^k \right)$$

has the property that $\phi(\sigma_{\lambda/\mu}) = Z(\mathfrak{S}_{\lambda/\mu}(\boldsymbol{x}, \boldsymbol{y}, \mathbf{1}, \mathbf{1}))$ for all strict partitions λ, μ.

By the Edrei-Thoma Theorem (see [47, Proposition 1.3]), $\phi(\sigma_{\lambda/\mu})$ is positive for all λ, μ precisely when $x_j, y_j \geq 0$ for all j. □

In other words, if a free fermionic six-vertex model satisfies

$$\frac{\mathtt{b}_2^{(\mathtt{j})}}{\mathtt{a}_1^{(\mathtt{j})}}, \frac{\mathtt{a}_2^{(\mathtt{j})}}{\mathtt{b}_1^{(\mathtt{j})}} \geq 0 \qquad \text{for all } 1 \leq j \leq n,$$

then the partition function has a predictable sign that only depends on M and $\ell(\lambda)$. It is unclear whether any similar result holds for non-free-fermionic weights, or whether there is a probabilistic interpretation of this result.

2.6 Boundary Conditions

We will use the results of the previous sections to compute the free fermionic partition function for a model with modified left and right boundary conditions. For any number of rows and any left and right boundary conditions, we give an operator on Fock space that matches the partition function; however, this operator is ugly in general. In the case where the left and right boundaries are both uniform (e.g. domain-wall), we can use a different method to give a precise formula for the partition function.

2.6.1 A Fock space operator for any boundary conditions

We'll work with a version of $\mathfrak{S}$ with generalized boundary conditions. Similar results hold for $\mathfrak{S}^*$. If $\alpha, \beta, \lambda, \mu$ are strict partitions where all parts of α and β are positive, let $\mathfrak{S}_{\lambda/\mu}^{\alpha/\beta} := (\mathfrak{S})_{\lambda/\mu}^{\alpha/\beta}$ be defined as follows.

- N rows, where $N \geq \max(\sigma_1, \tau_1)$, labelled $1, \ldots, N$ from bottom to top;
- $M+1$ columns, where $M \geq \max(\lambda_1, \mu_1)$, labelled $0, \ldots, M$ from left to right;
- Right boundary edges $-$ on parts of α; $+$ otherwise;
- Left boundary edges $-$ on parts of β; $+$ otherwise;
- Bottom boundary edges $-$ on parts of λ; $+$ otherwise;
- Top boundary edges $-$ on parts of μ; $+$ otherwise;
- Boltzmann weights from Figure 2.1.

Given $1 \leq i \leq n$, consider 4 cases:

A) Neither α nor β has a part of size i.

B) α has a part of size i, but β does not.

C) β has a part of size i, but α does not.

D) Both α and β have parts of size i.

Define

$$e^{\Phi_i} := \begin{cases} e^{\phi_i}, & \text{if case A,} \\ (x_i + y_i)^{-1} \cdot e^{\phi_i} \psi^*_{M+1/2}, & \text{if case B,} \\ \psi^*_{-3/2} \psi_{-3/2} e^{\phi_i} \psi_{-3/2}, & \text{if case C,} \\ (x_i + y_i)^{-1} \psi^*_{-3/2} \psi_{-3/2} e^{\phi_i} \psi^*_{M+1/2} \psi_{-3/2}, & \text{if case D.} \end{cases}$$

Here is the general idea behind the operators e^{Φ_i}. In cases B and D, we introduce a particle in column $M+1$, while in cases C and D, we remove a particle from column -1. This adjustment corresponds on the lattice model side to creating a "ghost vertex" of type $\mathtt{c}_1^{(\mathtt{i})}$ on the right of the row (resp. type $\mathtt{c}_2^{(\mathtt{i})}$ on the left of the row. We then divide by the weight of the ghost vertex since it doesn't appear in the lattice model.

The particle removal has another wrinkle. We first remove a particle from column -1 by applying $\psi_{-3/2}$, which introduces a factor of $(-1)^{\ell(\lambda)}$. Then the operator e^{ϕ_i} fills that empty spot up. Then the operator $\psi^*_{-3/2}\psi_{-3/2}$ serves as a "check", killing the state unless there is a particle in column -1.

Let $S : \mathcal{F}_\ell \longrightarrow \mathcal{F}_{\ell+1}$ be the shift operator defined by $S|\rho_\ell\rangle = |\rho_{\ell+1}\rangle$. Note that S commutes with current operators, and therefore with e^{ϕ_i}. Let $\emptyset$ denote the empty partition (no parts).

Proposition 2.6.1.

$$Z((\mathfrak{S})^{\alpha/\beta}_{\lambda/\mu}) = \prod_{i=0}^{N}(-1)^{\ell(\lambda)+\delta_i} A_i^{M+1} B_i^{\ell(\lambda)+\delta_i} \langle\mu|e^{\Phi_N} e^{\Phi_{N-1}} \dots e^{\Phi_1} e^{\Phi_0}|\lambda\rangle, \tag{2.16}$$

where

$$\delta_i = |\{j|\alpha_j < i\}| - |\{j|\beta_j < i\}|$$

Proof. We prove this first for a single row, and drop the subscripts $i = 0$. If $N = 1$, we want to prove

$$Z((\mathfrak{S})^{\alpha/\beta}_{\lambda/\mu}) = A_1^M B_1^{\ell(\lambda)} \langle\mu|e^{\Phi_1}|\lambda\rangle$$

for all possible $\alpha, \beta \in \{(1), \emptyset\}$. We have four cases, corresponding to the cases above: A) $\alpha = \beta = \emptyset$; B) $\alpha = (1), \beta = \emptyset$; C) $\alpha = \emptyset, \beta = (1)$; D) $\alpha = \beta = (1)$. We will prove case D, and the others are similar.

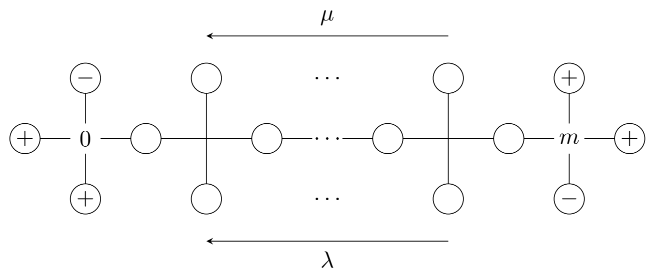
Let $\tilde{\lambda} = (M+2, \lambda_1 + 1, \dots, \lambda_{\ell(\lambda)} + 1), \tilde{\mu} = (\mu_1 + 1, \dots, \mu_{\ell(\mu)} + 1, 0)$. Note that

$$|\tilde{\lambda}\rangle = (-1)^{\ell(\lambda)} S\psi^*_{M+1/2}\psi_{-3/2}|\lambda\rangle \qquad \text{and} \qquad \langle\tilde{\mu}| = \langle\mu|\psi^*_{-3/2}\psi_{-3/2}S^{-1},$$

so

$$\begin{aligned}
\langle\mu|e^{\Phi}|\lambda\rangle &= (x+y)^{-1}\langle\mu|\psi^*_{-3/2}\psi_{-3/2}e^{\phi}\psi^*_{M+1/2}\psi_{-3/2}|\lambda\rangle \\
&= (-1)^{\ell(\lambda)}(x+y)^{-1}\langle\tilde{\mu}|Se^{\phi}S^{-1}|\tilde{\lambda}\rangle \\
&= (-1)^{\ell(\lambda)}(x+y)^{-1}\langle\tilde{\mu}|e^{\phi}|\tilde{\lambda}\rangle \\
&= (-1)^{\ell(\lambda)}(x+y)^{-1}A^{-(M+3)}B^{-\ell(\lambda)-1}Z(\mathfrak{S}_{\tilde{\lambda}/\tilde{\mu}}) \\
&= (-1)^{\ell(\lambda)}A^{-(M+1)}B^{-\ell(\lambda)}Z(\mathfrak{S}^{(1)/(1)}_{\tilde{\lambda}/\tilde{\mu}}).
\end{aligned}$$

Here, the last two equalities are because $\mathfrak{S}_{\tilde{\lambda}/\tilde{\mu}}$ is the following lattice model,



and the rightmost vertex of $\mathfrak{S}_{\tilde{\lambda}/\tilde{\mu}}$ must be type $\mathtt{c}_1$, and the leftmost vertex must be type $\mathtt{c}_2$, so

$$Z(\mathfrak{S}^{(1)/(1)}_{\lambda/\mu}) = \frac{Z(\mathfrak{S}_{\tilde{\lambda}/\tilde{\mu}})}{(x+y)A^2B}.$$

Thus, (2.16) holds for case D, and the other cases are similar.

Now, for multiple rows, the result follows by induction from the single row case, using the branching rules for both lattice models and Hamiltonians. δ_i is the number of parts of λ_i where $|\lambda_i\rangle$ appears in $|e^{\Phi_{i-1}} \cdots e^{\Phi_0}|\lambda\rangle$. $\square$

2.6.2 Computation of the partition function with uniform side boundary conditions

Recall that $\rho_\ell = (\ell-1, \dots, 1, 0)$. Let $\rho^+_\ell = (\ell, \dots, 2, 1)$.

We make the definitions

$$L(\beta) := Z(\mathfrak{S}^{\beta/\emptyset}_{\emptyset/\rho_{\ell(\beta)}}), \qquad R(\alpha) := Z(\mathfrak{S}^{\emptyset/\alpha}_{\rho_{\ell(\alpha)}/\emptyset}).$$

Given integers s, t, let λ, μ be strict partitions with $\ell(\lambda) = \ell, \ell(\mu) = k$, $\ell + s = k + t$. Let $\ell(\alpha) = s, \ell(\beta) = t$, and as usual let $M \geq \max(\lambda_1, \mu_1), N \geq \max(\alpha_1, \beta_1)$. We want to evaluate the partition function $Z(\mathfrak{S}^{\alpha/\beta}_{\lambda/\mu})$.

Let

$$\tilde{\lambda} = (M+s+t, M+s+t-1, \dots, M+t+1, \lambda_1+t, \lambda_2+t, \dots, \lambda_\ell+t),$$

$$\tilde{\mu} = (\mu_1+t, \mu_2+t, \dots, \mu_k+t, t-1, t-2, \dots, 0).$$

Note that if $t = 0$, $\tilde{\mu} = \mu$ and if $s = t = 0$, then $\tilde{\lambda} = \lambda$.

Proposition 2.6.2.

$$\prod_{i=1}^{N} A_i^{M+1+s+t} B_i^{\ell+s} \langle \tilde{\mu} | e^{H_+} | \tilde{\lambda} \rangle = \sum_{\alpha,\beta} L(\beta) R(\alpha) Z(\mathfrak{S}_{\lambda/\mu}^{\alpha/\beta}),$$

where the sum is over all α, β *where* $\alpha_1, \beta_1 \leq N$ *and* $\ell(\alpha) = s, \ell(\beta) = t$.

Proof. By Theorem 2.4.1,

$$\prod_{i=1}^{N} A_i^{M+1+s+t} B_i^{\ell+s} \langle \tilde{\mu} | e^{H_+} | \tilde{\lambda} \rangle = Z(\mathfrak{S}_{\tilde{\lambda}/\tilde{\mu}}).$$

The result follows from breaking the lattice model into three parts: the left t columns, the middle M columns, and the right s columns. □

We will look more closely at the four special cases where s and t are each either 0 or N. In these cases, there is only one choice for α and β, so the sum from Proposition 2.6.2 disappears. Let $L(N) := L(\rho_{N+1}^+), R(N) := R(\rho_{N+1}^+)$. Then, the previous proposition has the following immediate corollary.

Corollary 2.6.3. *Let* $\ell = \ell(\lambda)$.

A)

$$Z(\mathfrak{S}_{\lambda/\mu}^{\emptyset/\emptyset}) = \prod_{i=1}^{N} A_i^{M+1} B_i^{\ell} \cdot \langle \mu | e^{H_+} | \lambda \rangle,$$

B)

$$Z(\mathfrak{S}_{\lambda/\mu}^{\rho_N^+/\emptyset}) = \frac{1}{R(N)} \cdot \prod_{i=1}^{N} A_i^{M+1+N} B_i^{\ell+N} \cdot \langle \mu | e^{H_+} | \tilde{\lambda} \rangle,$$

C)

$$Z(\mathfrak{S}_{\lambda/\mu}^{\emptyset/\rho_N^+}) = \frac{1}{L(N)} \cdot \prod_{i=1}^{N} A_i^{M+1+N} B_i^{\ell} \cdot \langle \tilde{\mu} | e^{H_+} | \tilde{\lambda} \rangle,$$

D)

$$Z(\mathfrak{S}_{\lambda/\mu}^{\rho_N^+/\rho_N^+}) = \frac{1}{L(N)R(N)} \cdot \prod_{i=1}^{N} A_i^{M+1+2N} B_i^{\ell+N} \cdot \langle \tilde{\mu} | e^{H_+} | \tilde{\lambda} \rangle.$$

Case A is just Theorem 2.4.1 restated. To evaluate the other three partition functions, we need to evaluate $L(N)$ and $R(N)$. For the rest of the section, we will use the vertices and the weights from Figure 2.1 interchangeably.

Lemma 2.6.4. *[2, Lemma 10]*

$$L(N) \cdot \left[\prod_{i<j} (\mathtt{a}_1^{(\mathtt{j})}\mathtt{a}_2^{(\mathtt{i})} + \mathtt{b}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{j})})\right]^{-1}$$

is symmetric with respect to the row parameters, and expressible as a polynomial in the variables $\mathtt{a}_1^{(\mathtt{i})}, \mathtt{a}_2^{(\mathtt{i})}, \mathtt{b}_1^{(\mathtt{i})}, \mathtt{b}_2^{(\mathtt{i})}$ *with integer coefficients.*

Proposition 2.6.5.

$$L(N) = \prod_{k=1}^{N} \mathtt{c}_2^{(\mathtt{k})} \cdot \prod_{i<j} \left(\mathtt{a}_1^{(\mathtt{j})}\mathtt{a}_2^{(\mathtt{i})} + \mathtt{b}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{j})}\right) = \prod_{k=1}^{N} A_k^N B_k^{N-k} \cdot \prod_{i<j} (y_i + x_j),$$

and

$$R(N) = \prod_{k=1}^{N} \mathtt{c}_1^{(\mathtt{k})} \cdot \prod_{i<j} \left(\mathtt{a}_1^{(\mathtt{i})}\mathtt{a}_2^{(\mathtt{j})} + \mathtt{b}_1^{(\mathtt{j})}\mathtt{b}_2^{(\mathtt{i})}\right) = \prod_{k=1}^{N} A_k^N B_k^{k-1}(x_k + y_k) \cdot \prod_{i<j} (x_i + y_j).$$

Proof. For the first equation, by the previous lemma, $L(N)$ is a multiple of

$$\left[\prod_{i<j} (\mathtt{a}_1^{(\mathtt{j})}\mathtt{a}_2^{(\mathtt{i})} + \mathtt{b}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{j})})\right]$$

as a polynomial in the Boltzmann weights. In addition, each state of $\mathfrak{S}_{\emptyset/\rho_N}^{\rho_N^+/\emptyset}$ must have precisely 1 more $\mathtt{c}_2$ vertex than $\mathtt{c}_1$ vertex in each row, so $L(N)$ is also divisible by $\prod_{k=1}^{N} \mathtt{c}_2^{(\mathtt{k})}$. Each state of $\mathfrak{S}_{\emptyset/\rho_N}^{\rho_N^+/\emptyset}$ has N^2 vertices, so $L(N)$ must have degree N^2. The product of the factors we have already determined also has degree N, so

$$L(N) = \prod_{k=1}^{N} \mathtt{c}_2^{(\mathtt{k})} \cdot \prod_{i<j} \left(\mathtt{a}_1^{(\mathtt{j})}\mathtt{a}_2^{(\mathtt{i})} + \mathtt{b}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{j})}\right)$$

as desired. The second equality in the first equation follows from plugging in the weights from Figure 2.1. The second equation follows by reversing the edges in $\mathfrak{S}_{\emptyset/\rho_N}^{\rho_N^+/\emptyset}$ to obtain $\mathfrak{S}_{\rho_N/\emptyset}^{\emptyset/\rho_N^+}$. □

All together, we have the following result, where we have also applied Theorem 2.4.1. Case B is the important case of domain-wall boundary conditions.

Theorem 2.6.6. *Let* $\ell = \ell(\lambda)$.

A)

$$Z(\mathfrak{S}^{\emptyset/\emptyset}_{\lambda+\rho/\mu+\rho}) = \prod_{i=1}^{N} (\mathtt{a}_1^{(\mathtt{i})})^{M+1-\ell} (\mathtt{b}_1^{(\mathtt{i})})^{\ell} \cdot s_{\lambda/\mu}\left[\frac{\mathtt{b}_2^{(\mathtt{1})}}{\mathtt{a}_1^{(\mathtt{1})}}, \dots, \frac{\mathtt{b}_2^{(\mathtt{N})}}{\mathtt{a}_1^{(\mathtt{N})}} \middle| \frac{\mathtt{a}_2^{(\mathtt{1})}}{\mathtt{b}_1^{(\mathtt{1})}}, \dots, \frac{\mathtt{a}_2^{(\mathtt{N})}}{\mathtt{b}_1^{(\mathtt{N})}} \right],$$

B)

$$Z(\mathfrak{S}^{\rho_N^+/\emptyset}_{\lambda+\rho/\mu+\rho}) = \frac{\prod_{i=1}^{N} (\mathtt{a}_1^{(\mathtt{i})})^{M+1-\ell} (\mathtt{b}_1^{(\mathtt{i})})^{\ell+N} \cdot s_{\tilde{\lambda}/\mu}\left[\frac{\mathtt{b}_2^{(\mathtt{1})}}{\mathtt{a}_1^{(\mathtt{1})}}, \dots, \frac{\mathtt{b}_2^{(\mathtt{N})}}{\mathtt{a}_1^{(\mathtt{N})}} \middle| \frac{\mathtt{a}_2^{(\mathtt{1})}}{\mathtt{b}_1^{(\mathtt{1})}}, \dots, \frac{\mathtt{a}_2^{(\mathtt{N})}}{\mathtt{b}_1^{(\mathtt{N})}} \right]}{\prod_{k=1}^{N} \mathtt{c}_1^{(\mathtt{k})} \cdot \prod_{i<j} \left(\mathtt{a}_1^{(\mathtt{i})}\mathtt{a}_2^{(\mathtt{j})} + \mathtt{b}_1^{(\mathtt{j})}\mathtt{b}_2^{(\mathtt{i})} \right)},$$

C)

$$Z(\mathfrak{S}^{\emptyset/\rho_N^+}_{\lambda+\rho/\mu+\rho}) = \frac{\prod_{i=1}^{N} (\mathtt{a}_1^{(\mathtt{i})})^{M+1+N-\ell} (\mathtt{b}_1^{(\mathtt{i})})^{\ell} \cdot s_{\tilde{\lambda}/\tilde{\mu}}\left[\frac{\mathtt{b}_2^{(\mathtt{1})}}{\mathtt{a}_1^{(\mathtt{1})}}, \dots, \frac{\mathtt{b}_2^{(\mathtt{N})}}{\mathtt{a}_1^{(\mathtt{N})}} \middle| \frac{\mathtt{a}_2^{(\mathtt{1})}}{\mathtt{b}_1^{(\mathtt{1})}}, \dots, \frac{\mathtt{a}_2^{(\mathtt{N})}}{\mathtt{b}_1^{(\mathtt{N})}} \right]}{\prod_{k=1}^{N} \mathtt{c}_2^{(\mathtt{k})} \cdot \prod_{i<j} \left(\mathtt{a}_1^{(\mathtt{j})}\mathtt{a}_2^{(\mathtt{i})} + \mathtt{b}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{j})} \right)},$$

D)

$$Z(\mathfrak{S}^{\rho_N^+/\rho_N^+}_{\lambda+\rho/\mu+\rho}) = \frac{\prod_{i=1}^{N} (\mathtt{a}_1^{(\mathtt{i})})^{M+1+N-\ell} (\mathtt{b}_1^{(\mathtt{i})})^{\ell+N} \cdot s_{\tilde{\lambda}/\tilde{\mu}}\left[\frac{\mathtt{b}_2^{(\mathtt{1})}}{\mathtt{a}_1^{(\mathtt{1})}}, \dots, \frac{\mathtt{b}_2^{(\mathtt{N})}}{\mathtt{a}_1^{(\mathtt{N})}} \middle| \frac{\mathtt{a}_2^{(\mathtt{1})}}{\mathtt{b}_1^{(\mathtt{1})}}, \dots, \frac{\mathtt{a}_2^{(\mathtt{N})}}{\mathtt{b}_1^{(\mathtt{N})}} \right]}{\prod_{k=1}^{N} \mathtt{c}_1^{(\mathtt{k})}\mathtt{c}_2^{(\mathtt{k})} \cdot \prod_{i\neq j} \left(\mathtt{a}_1^{(\mathtt{j})}\mathtt{a}_2^{(\mathtt{i})} + \mathtt{b}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{j})} \right)}.$$

2.6.3 Berele-Regev formula and Schur functions

For this subsection, let $Z_\lambda = Z(\mathfrak{S}^{\rho_N^+/\emptyset}_{\emptyset/\lambda+\rho})$. We will use the Berele-Regev formula to show that Z_λ has another expression as a Schur function times a deformed denominator. This corrects the result [2, Theorem 9], whose proof is circular.

We will use the *supertableau* (or *bitableau*) formula for supersymmetric Schur functions [44, § I.5, Exercise 23].

A supertableau of shape λ/μ is a filling of λ/μ with the entries $1, \dots, N, 1', \dots, N'$ such that

(i) The entries weakly increase across rows and columns under the ordering $1 < \dots < N < 1' < \dots, N'$.

(ii) There is at most one j' in every row, and at most one i in every column.

Then,

$$s_{\lambda/\mu}[\boldsymbol{x}|\boldsymbol{y}] = \sum_T [\boldsymbol{x}|\boldsymbol{y}]^T, \qquad [\boldsymbol{x}|\boldsymbol{y}]^T = \prod_{i=1}^{N} x_i^{m_i} \prod_{j=1}^{N} y_j^{n_j}, \tag{2.17}$$

where the sum is over all supertableaux T of shape λ/μ, and m_i (resp. n_j) is the number of entries i (resp. j') in T.

Lemma 2.6.7. *Let λ/μ be a skew shape, and let γ/σ be the skew shape obtained from rotating λ/μ by $180°$. Then $s_{\lambda/\mu}[\boldsymbol{x}|\boldsymbol{y}] = s_{\gamma/\sigma}[\boldsymbol{x}|\boldsymbol{y}]$.*

Proof. By [44, § I.5, Exercise 23(c)], (2.17) holds if the ordering on $\{1, \ldots, N, 1', \ldots, N'\}$ is replaced by any total ordering. It is easy to see that a $180°$ rotation gives a weight-preserving bijection between supertableaux for λ/μ and supertableaux for γ/σ with the ordering $N' < \ldots < 1' < N < \ldots < 1$. □

Corollary 2.6.8. *Let τ be the partition obtained by taking the complement of λ in a $N \times (M+1-N)$ box and rotating $180°$. Then,*

$$Z_\lambda = \prod_{i=1}^{N} (\mathtt{a}_1^{(\mathtt{i})})^{M+1-N} \mathtt{c}_2^{(\mathtt{i})} \cdot \prod_{i<j} \left(\mathtt{a}_1^{(\mathtt{j})} \mathtt{a}_2^{(\mathtt{i})} + \mathtt{b}_1^{(\mathtt{i})} \mathtt{b}_2^{(\mathtt{j})} \right) \cdot s_\tau \left[\frac{\mathtt{b}_2^{(1)}}{\mathtt{a}_1^{(1)}}, \ldots, \frac{\mathtt{b}_2^{(1)}}{\mathtt{a}_1^{(1)}} \right] \tag{2.18}$$

Proof. By Theorem 2.6.6B, plugging in $\emptyset$ for λ, and $\lambda + \rho$ for μ,

$$Z_\lambda = \frac{\prod_{i=1}^{N} (\mathtt{a}_1^{(\mathtt{i})})^{M+1} (\mathtt{b}_1^{(\mathtt{i})})^{N} \cdot s_{\nu/\lambda} \left[\frac{\mathtt{b}_2^{(1)}}{\mathtt{a}_1^{(1)}}, \ldots, \frac{\mathtt{b}_2^{(\mathtt{N})}}{\mathtt{a}_1^{(\mathtt{N})}} \middle| \frac{\mathtt{a}_2^{(1)}}{\mathtt{b}_1^{(1)}}, \ldots, \frac{\mathtt{a}_2^{(\mathtt{N})}}{\mathtt{b}_1^{(\mathtt{N})}} \right]}{\prod_{k=1}^{N} \mathtt{c}_1^{(\mathtt{k})} \cdot \prod_{i<j} \left(\mathtt{a}_1^{(\mathtt{i})} \mathtt{a}_2^{(\mathtt{j})} + \mathtt{b}_1^{(\mathtt{j})} \mathtt{b}_2^{(\mathtt{i})} \right)},$$

where ν is the $N \times (M+1)$ block partition: $(M+1, \ldots, M+1)$.

The $180°$ rotation of ν/λ is the (nonskew) partition $\mu := (M+1-\lambda_N, M+1-\lambda_{N-1}, \ldots M+1-\lambda_1)$. Notice that all parts of μ have size $\geq N$ since we must have $M \geq \lambda_1 + N - 1$ for $\lambda + \rho$ to fit on the top boundary. By Lemma 2.6.7,

$$Z_\lambda = \frac{\prod_{i=1}^{N} (\mathtt{a}_1^{(\mathtt{i})})^{M+1} (\mathtt{b}_1^{(\mathtt{i})})^{N} \cdot s_{\mu} \left[\frac{\mathtt{b}_2^{(1)}}{\mathtt{a}_1^{(1)}}, \ldots, \frac{\mathtt{b}_2^{(\mathtt{N})}}{\mathtt{a}_1^{(\mathtt{N})}} \middle| \frac{\mathtt{a}_2^{(1)}}{\mathtt{b}_1^{(1)}}, \ldots, \frac{\mathtt{a}_2^{(\mathtt{N})}}{\mathtt{b}_1^{(\mathtt{N})}} \right]}{\prod_{k=1}^{N} \mathtt{c}_1^{(\mathtt{k})} \cdot \prod_{i<j} \left(\mathtt{a}_1^{(\mathtt{i})} \mathtt{a}_2^{(\mathtt{j})} + \mathtt{b}_1^{(\mathtt{j})} \mathtt{b}_2^{(\mathtt{i})} \right)}. \tag{2.19}$$

Next we apply the Berele-Regev formula [48, Theorem 6.20]. τ is also the partition obtained by subtracting N from every part of μ. Then the Berele-Regev formula says that

$$s_\mu \left[x_1, \ldots, x_N \middle| y_1, \ldots, y_N \right] = \prod_{i,j} (x_j + y_i) \cdot s_\tau [x_1, \ldots, x_n].$$

Applying this to (2.19) gives

$$\begin{aligned} Z_\lambda &= \frac{\prod_{i=1}^N (\mathtt{a}_1^{(\mathtt{i})})^{M+1} (\mathtt{b}_1^{(\mathtt{i})})^N \cdot \prod_{i,j} \left(\frac{\mathtt{b}_2^{(\mathtt{j})}}{\mathtt{a}_1^{(\mathtt{j})}} + \frac{\mathtt{a}_2^{(\mathtt{i})}}{\mathtt{b}_1^{(\mathtt{i})}} \right) \cdot s_\tau \left[\frac{\mathtt{b}_2^{(1)}}{\mathtt{a}_1^{(1)}}, \ldots, \frac{\mathtt{b}_2^{(1)}}{\mathtt{a}_1^{(1)}} \right]}{\prod_{k=1}^N \mathtt{c}_1^{(\mathtt{k})} \cdot \prod_{i<j} \left(\mathtt{a}_1^{(\mathtt{i})} \mathtt{a}_2^{(\mathtt{j})} + \mathtt{b}_1^{(\mathtt{j})} \mathtt{b}_2^{(\mathtt{i})} \right)} \\ &= \frac{\prod_{i=1}^N (\mathtt{a}_1^{(\mathtt{i})})^{M+1-N} \cdot \prod_{i,j} \left(\mathtt{a}_1^{(\mathtt{j})} \mathtt{a}_2^{(\mathtt{i})} + \mathtt{b}_1^{(\mathtt{i})} \mathtt{b}_2^{(\mathtt{j})} \right) \cdot s_\tau \left[\frac{\mathtt{b}_2^{(1)}}{\mathtt{a}_1^{(1)}}, \ldots, \frac{\mathtt{b}_2^{(1)}}{\mathtt{a}_1^{(1)}} \right]}{\prod_{k=1}^N \mathtt{c}_1^{(\mathtt{k})} \cdot \prod_{i<j} \left(\mathtt{a}_1^{(\mathtt{i})} \mathtt{a}_2^{(\mathtt{j})} + \mathtt{b}_1^{(\mathtt{j})} \mathtt{b}_2^{(\mathtt{i})} \right)} \\ &= \prod_{i=1}^N (\mathtt{a}_1^{(\mathtt{i})})^{M+1-N} \mathtt{c}_2^{(\mathtt{i})} \cdot \prod_{i<j} \left(\mathtt{a}_1^{(\mathtt{j})} \mathtt{a}_2^{(\mathtt{i})} + \mathtt{b}_1^{(\mathtt{i})} \mathtt{b}_2^{(\mathtt{j})} \right) \cdot s_\tau \left[\frac{\mathtt{b}_2^{(1)}}{\mathtt{a}_1^{(1)}}, \ldots, \frac{\mathtt{b}_2^{(1)}}{\mathtt{a}_1^{(1)}} \right], \end{aligned}$$

where the last equality uses the free fermion condition. Note that this is a polynomial in the Boltzmann weights of degree $(M+1) \cdot N$. □

A very similar result was stated by Brubaker, Bump, and Friedberg [2, Theorem 9]; they give exactly the formula (2.18) except that their power of $\mathtt{a}_1^{(\mathtt{i})}$ is different. The proof in [2] is circular: they observe that after normalizing $\mathtt{c}_2^{(\mathtt{i})} = 1$, using the free fermion condition, Z_λ can be expressed as a polynomial in $\mathtt{a}_1^{(\mathtt{i})}, \mathtt{a}_2^{(\mathtt{i})}, \mathtt{b}_1^{(\mathtt{i})}, \mathtt{b}_2^{(\mathtt{i})}$. However, they then consider only the states without vertices of type $\mathtt{c}_1^{(\mathtt{i})}$, rather than all admissible states. Despite the fact that $\mathtt{c}_1^{(\mathtt{i})}$ can be removed algebraically from an expression of the partition function, it is still necessary to consider states involving type $\mathtt{c}_1^{(\mathtt{i})}$ vertices. Corollary 2.6.8 is therefore a correction of that proof. The fact that the formula given in [2] is so close to correct suggests that maybe their proof technique can be salvaged.

In [49], the ArXiv version of [2], Brubaker, Bump, and Friedberg give a correct proof of Tokuyama's Theorem by two different evaluations of two free fermionic six-vertex models they call Γ and Δ ice. One side of Tokuyama's formula is given as a sum over Gelfand-Tsetlin patterns, while the other is a Schur function times a deformed denominator.

It is easy to see that is the case of these weights, Corollary 2.6.8 specializes to the Schur function times deformed denominator. In the case of the Γ weights in [49, Table 1], Corollary 2.6.8 yields

$$Z_\lambda = \prod_{i<j} (t_i z_j + z_i) s_\tau[z_1, \ldots, z_n],$$

which is [49, Theorem 5].

Aggarwal, Borodin, Petrov, and Wheeler consider the Berele-Regev formula from the opposite direction. Using their lattice models and [2, Theorem 9], they prove a generalized Berele-Regev formula [50, Corollary 4.13].

This completes our analysis of the classical six-vertex model and classical Fock space.

2.7 Metaplectic Fock spaces

We now turn our attention to the six-vertex model with charge, which will turn out to match Hamiltonian operators arising from Drinfeld twists of q-Fock space.

q-Fock space [42] is a quantum analogue of the classical Fock space defined in Section 2.2. It is formed as a quotient by a Hecke algebra action of the infinite tensor power of the standard evaluation module of the quantum group $U_q(\widehat{\mathfrak{sl}_n})$.

We will work with a family of related spaces defined by Brubaker, Buciumas, Bump, and Gustafsson [3]. Instead of being a module for $U_q(\widehat{\mathfrak{sl}_n})$, these spaces are modules for Drinfeld twists of $U_q(\widehat{\mathfrak{sl}_n})$. Reshetikhin [41] defined a large class of Drinfeld twists of quantum groups, and Brubaker, Buciumas, Bump, and Gustafsson applied these twists to q-Fock space. Drinfeld twisting doesn't affect the algebra structure of a quantum group, but it does affect the coalgebra structure, so the result is a set of genuinely distinct modules, which we will now describe.

Choose an integer $n \geq 1$. Let

$$F = \exp\left(\sum_{1\leq i<j\leq n} a_{ij}(H_i \otimes H_j - H_j \otimes H_i)\right),$$

where the H_i are certain generators of the topological Hopf algebra associated to $U_q(\widehat{\mathfrak{sl}_n})$ (see [41]). If $1 \leq i, j \leq n$, set

$$\alpha_{ij} = \begin{cases} \exp(2a_{i,j} - 2a_{i-1,j} - 2a_{i,j-1} + 2a_{i-1,j-1}), & i \neq j \\ 1, & i = j, \end{cases}$$

and let α_{ij} be defined modulo n so that $\alpha_{i+kn,j+mn} = \alpha_{ij}$.

These α_{ij} determine the relations in our twists of q-Fock space. There is no Clifford algebra structure associated to q-Fock space, so we will define it using wedges. The twisted q-Fock space $\mathcal{F} := \mathcal{F}(F)$ is the space with basis

$$u_{m_1} \wedge u_{m_2} \wedge u_{m_3} \wedge \ldots, \qquad (m_1 > m_2 > \ldots).$$

Wedges with decreasing index like these are called *normally ordered.*

Additionally, $\mathcal{F}$ has the following *normal ordering relations.*

$$u_l \wedge u_m = \begin{cases} -u_m \wedge u_l, & l \equiv m \mod n \\ -q\alpha_{lm} u_m \wedge u_l + (q^2-1)(u_{m-i} \wedge u_{l+i} + -q\alpha_{lm} u_{m-n} \wedge u_{l+n} + \\ \qquad q^2 u_{m-n-i} \wedge u_{l+n+i} - q^3 \alpha_{lm} u_{m-2n} \wedge u_{l+2n} + \ldots), & \text{otherwise,} \end{cases}$$

where $l \leq m$. In particular, $u_m \wedge u_m = 0$. Here, i satisfies $0 < i < n, m - i \equiv l \mod n$. The sum in the second case continues while the wedges are normally ordered. Notice that these relations depend only on the numbers α_{ij}.

We will say a Drinfeld twist of q-Fock space is *shift invariant* if α_{ij} depends only on $i - j$. In this case, we define a function g on integers modulo n let

$$g(i-j) := \begin{cases} -q\alpha_{ij}, & i \neq j \mod n \\ -q^2, & i = j \mod n. \end{cases}$$

We call the integers modulo n *charge*, and will often take as representatives the integers $0, 1, \ldots, n-1$. For convenience, let $v = q^2$. The relation $\alpha_{ij}\alpha_{ji} = 1$ becomes

$$g(a)g(-a) = -g(0), \qquad \text{for all } a \not\equiv 0 \mod n.$$

Under the assumption of shift invariance, the wedge relations become:

$$u_l \wedge u_m = \begin{cases} -u_m \wedge u_l, & l \equiv m \mod n \\ g(l-m)u_m \wedge u_l + (q^2-1)(u_{m-i} \wedge u_{l+i} + g(l-m)u_{m-n} \wedge u_{l+n} + \\ \qquad q^2 u_{m-i} \wedge u_{l+i} + q^2 g(l-m) u_{m-n} \wedge u_{l+n} + \ldots), & \text{otherwise.} \end{cases}$$

Notice that these relations depend only on the function g. As a result, we will often view g as being synonymous with $\mathcal{F}$.

The shift invariant Drinfeld-Reshetikhin twists of q-Fock space were shown in [3] to be related to lattice models for metaplectic Whittaker functions, so we will call these spaces *metaplectic Fock spaces*. They will also turn out to be closely related to the solvability of charged models. The shift invariance property is a natural one to require; as we will see in the next section, charged models have a sort of shift invariance themselves, so it doesn't make sense to compare non-shift-invariant spaces with charged lattice models.

For a strict partition λ, we define

$$|\lambda\rangle := v_{\lambda_1} \wedge v_{\lambda_2} \wedge \ldots \wedge v_{\lambda_{\ell(\lambda)}} \wedge v_{-1} \wedge v_{-2} \wedge \ldots.$$

Define the dual basis $\{\langle\lambda|\}$ of $\mathcal{F}^*$ by the pairing

$$\langle\mu|\lambda\rangle = \begin{cases} 1, & \text{if } \lambda = \mu \\ 0, & \text{otherwise.} \end{cases}$$

There is a Heisenberg action

$$J_k \cdot (u_{m_1} \wedge u_{m_2} \wedge \ldots) = \sum_{i \geq 0} (u_{m_1} \wedge \ldots \wedge u_{m_{i-1}} \wedge u_{m_i - kn} \wedge u_{m_{i+1}} \wedge \ldots).$$

It has been shown [3, 42] that

$$[J_k, J_l] = k \cdot \frac{1 - v^{n|k|}}{1 - v^{|k|}} \delta_{k,-l}.$$

Once again, we fix parameters $s_k^{(j)}$, $k \neq 0, 1 \leq j \leq N$ and set

$$s_k := \sum_{j=1}^{N} s_k^{(j)}, \qquad H_\pm = \sum_{k \geq 1} s_{\pm k} J_{\pm k}, \qquad e^{H_\pm} = \sum_{m \geq 0} \frac{1}{m!} H^m.$$

2.8 Six-vertex models with charge

The lattice models in this section are similar to the six-vertex model, but use an extra statistic called *charge*, an integer modulo n associated to each horizontal $-$ spin. We call the resulting combination of spin and charge (or simply a $+$ spin) a *decorated spin*. As first seen in [3], the charge statistic turns out to be the right way to represent the q-Fock space for $U_q(\widehat{\mathfrak{sl}}_n)$, along with the metaplectic data from its Drinfeld twists.

Informally, the use of charge forces particles to travel a multiple of n columns in each row, mirroring the action of the current operators in the previous section. Therefore, the $\mathtt{a}_2$ and $\mathtt{b}_2$ vertices in charged models depend on the charges on their horizontal edges, while the other four vertices either have $+$ spins on their horizontal edges or are restricted to specified charges on their horizontal $-$ edges. This results in a total of $2n + 4$ vertices, which makes the analysis both by the Yang-Baxter equation and by Hamiltonians more challenging than for the classical six-vertex model. Note that if $n = 1$, this model reduces to that one.

$\mathtt{a}_1^{(\mathtt{i})}$	$\mathtt{a}_2^{(\mathtt{i})}(a)$	$\mathtt{b}_1^{(\mathtt{i})}$	$\mathtt{b}_2^{(\mathtt{i})}(a)$	$\mathtt{c}_1^{(\mathtt{i})}$	$\mathtt{c}_2^{(\mathtt{i})}$
A_i	$h(a) y_i A_i B_i$	$A_i B_i$	$f(a) x_i A_i$	$(f(0) x_i + h(0) y_i) A_i B_i$	A_i

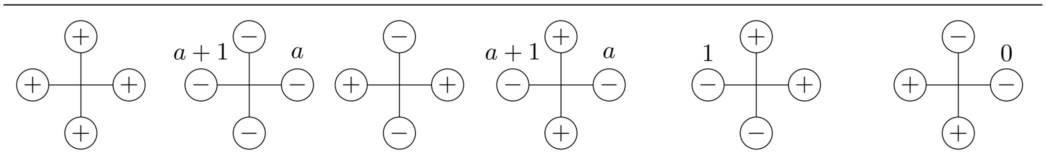
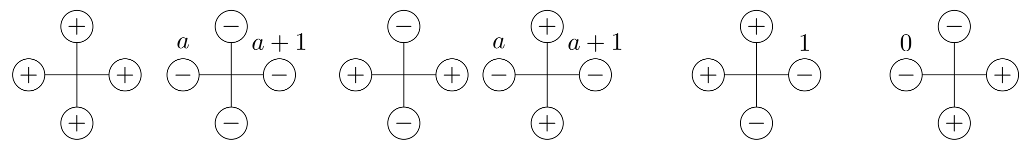


Figure 2.4: The Boltzmann weights for $\mathfrak{S}$ with charge. Here, x_i, y_i, A_i, and B_i are parameters associated to each row, while $f(a)$ and $h(a)$ depend only on the change a.

Define the charged model $\mathfrak{S}_{\lambda/\mu}^q := \mathfrak{S}_{\lambda/\mu}^q(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{A}, \boldsymbol{B}, f, h)$ to be as follows:



$\mathtt{a}_1^{(\mathtt{i})}$	$\mathtt{a}_2^{(\mathtt{i})}(a)$	$\mathtt{b}_1^{(\mathtt{i})}$	$\mathtt{b}_2^{(\mathtt{i})}(a)$	$\mathtt{c}_1^{(\mathtt{i})}$	$\mathtt{c}_2^{(\mathtt{i})}$
A_i^{-1}	$f(a)x_iA_i^{-1}B_i^{-1}$	$A_i^{-1}B_i^{-1}$	$h(a)y_iA_i^{-1}$	$(f(0)x_i+h(0)y_i)A_i^{-1}$	$A_i^{-1}B_i^{-1}$

Figure 2.5: The Boltzmann weights for $\mathfrak{S}^*$ with charge. Here, z_i, w_i, A_i, and B_i are parameters associated to each row, while $f(a)$ and $h(a)$ depend only on the change a.

- N rows, labelled $1, \ldots, N$ from bottom to top;
- $M+1$ columns, where $M \geq \max(\lambda_1, \mu_1)$, labelled $0, \ldots, M$ from left to right;
- Left and right boundary edges all $+$;
- Bottom boundary edges $-$ on parts of λ; $+$ otherwise;
- Top boundary edges $-$ on parts of μ; $+$ otherwise.
- Boltzmann weights from Figure 2.4,

and the charged model $\mathfrak{S}^{*,q}_{\lambda/\mu} := \mathfrak{S}^q_{\lambda/\mu}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{A}, \boldsymbol{B}, f, h)$ is defined similarly:

- N rows, labelled $1, \ldots, N$ from top to bottom;
- $M+1$ columns, where $M \geq \max(\lambda_1, \mu_1)$, labelled $0, \ldots, M$ from left to right;
- Left and right boundary edges all $+$;
- Bottom boundary edges $-$ on parts of μ; $+$ otherwise;
- Top boundary edges $-$ on parts of λ; $+$ otherwise.
- Boltzmann weights from Figure 2.5.

Notice that in the case $n = 1$, these models become $\mathfrak{S}$ and $\mathfrak{S}^*$ from Section 2.3, up to a scaling of the parameters x_i, y_i. As in that case, there are two ways to transform $\mathfrak{S}^q_{\lambda/\mu}$ into $\mathfrak{S}^{*,q}_{\lambda/\mu}$ by manipulating the model. First,

- Rotate the model $\mathfrak{S}^q_{\lambda/\mu}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{A}, \boldsymbol{B}, f, h)$ $180°$.
- Flip the vertical spins.
- Reverse the ordering on the columns.
- Divide the Boltzmann weights by $A_i^2 B_i$.

This results in the model $\mathfrak{S}^{*,q}_{\overline{\lambda}/\overline{\mu}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{A},\boldsymbol{B},f,h)$,so we have the following relationship between partition functions.

Proposition 2.8.1.

$$Z(\mathfrak{S}^{*,q}_{\overline{\lambda}/\overline{\mu}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{A},\boldsymbol{B},f,h)) = \prod_{i=1}^{N}(A_i^{-2M-2}B_i^{-M-1}) \cdot Z(\mathfrak{S}^{q}_{\lambda/\mu}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{A},\boldsymbol{B},f,h)).$$

Second,

- Flip $\mathfrak{S}^{q}_{\lambda/\mu}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{A},\boldsymbol{B},f,h)$ vertically (over a horizontal axis).
- Replace each charge a with the representative modulo n of $n-a+1$ in the range $[0,n-1]$.
- Swap the $\mathtt{c}_1$ and $\mathtt{c}_2$ vertices.
- Replace A_i with A_i^{-1} and B_i with B_i^{-1}.
- Swap x_i and y_i.
- Replace $f(a)$ with $h(-a)$ and $h(a)$ with $f(-a)$;
- Rebalance the $\mathtt{c}_1$ and $\mathtt{c}_2$ vertices by multiplying the former and dividing the latter by $f(0)z_i + h(0)w_i$.

The resulting model is $\mathfrak{S}^{*,q}_{\lambda/\mu}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{A},\boldsymbol{B},f,h)$. Let $\overline{f}(a) = f(a)$, and similarly for $\overline{h}$. Then we have

Proposition 2.8.2.

$$Z(\mathfrak{S}^{*,q}_{\lambda/\mu}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{A},\boldsymbol{B},f,h)) = Z(\mathfrak{S}^{q}_{\lambda/\mu}(\boldsymbol{y},\boldsymbol{x},\boldsymbol{A}^{-1},\boldsymbol{B}^{-1},\overline{h},\overline{f})).$$

We can now combine these identities to relate the partition functions of $\mathfrak{S}^{*,q}$ and $\mathfrak{S}^{q}$ to themselves.

Proposition 2.8.3.

(a)

$$Z(\mathfrak{S}^{*,q}_{\overline{\lambda}/\overline{\mu}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{A},\boldsymbol{B},f,h)) = \prod_{i=1}^{N}(A_i^{-2M-2}B_i^{-M-1}) \cdot Z(\mathfrak{S}^{q}_{\lambda/\mu}(\boldsymbol{y},\boldsymbol{x},\boldsymbol{A}^{-1},\boldsymbol{B}^{-1},\overline{h},\overline{f})).$$

(b)

$$Z(\mathfrak{S}^{q}_{\overline{\lambda}/\overline{\mu}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{A},\boldsymbol{B},f,h)) = \prod_{i=1}^{N}(A_i^{2M+2}B_i^{M+1}) \cdot Z(\mathfrak{S}^{*,q}_{\lambda/\mu}(\boldsymbol{y},\boldsymbol{x},\boldsymbol{A}^{-1},\boldsymbol{B}^{-1},\overline{h},\overline{f})).$$

Similar identities hold when the weights are left arbitrary.

We'll also need the row transfer matrices for both models. Define:

$$\langle\mu|T^q|\lambda\rangle := Z(\mathfrak{S}^q_{\lambda/\mu}), \qquad \langle\lambda|T^{*,q}|\mu\rangle := Z(\mathfrak{S}^{*,q}_{\lambda/\mu}),$$

where for both lattice models, $N = 1$ and $M \geq \max(\lambda_1, \mu_1)$.

Let $\mathcal{F}$ be a metaplectic Fock space as in the previous subsection, and let g be the associated function modulo n. We say the Boltzmann weights for a lattice model with charge (either $\mathfrak{S}^q$ or $\mathfrak{S}^{*,q}$ above) satisfy the *$\mathcal{F}$-free fermion condition* if the following two conditions hold:

- Zero-charge free fermion condition:

$$\mathtt{a}_1\mathtt{a}_2(0) + \mathtt{b}_1\mathtt{b}_2(0) - \mathtt{c}_1\mathtt{c}_2 = 0, \tag{2.20}$$

- $\mathcal{F}$-charge condition: for any $0 \leq a \leq n-1$,

$$\frac{\mathtt{a}_1^{(\mathtt{i})}\mathtt{a}_2^{(\mathtt{i})}(a)}{\mathtt{b}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{i})}(a)} = g(a). \tag{2.21}$$

We will say that a set of Boltzmann weights satisfies the *generalized free fermion condition* if it satisfies the *$\mathcal{F}$-free fermion condition* for any metaplectic Fock space $\mathcal{F}$. The $\mathcal{F}$-charge condition involves values of g, which depend on $\mathcal{F}$, so we need an equivalent expression that is independent of g. To get this, we use the relation $g(a)g(-a) = -g(0)$, and use the $\mathcal{F}$-charge condition to replace each factor with a ratio of weights. Doing this, we arrive with the following conditions.

- Zero charge free fermion condition:

$$\mathtt{a}_1\mathtt{a}_2(0) + \mathtt{b}_1\mathtt{b}_2(0) - \mathtt{c}_1\mathtt{c}_2 = 0,$$

- Charge condition: for any $1 \leq a \leq n-1$,

$$\frac{\mathtt{a}_1^{(\mathtt{i})}\mathtt{a}_2^{(\mathtt{i})}(a)\mathtt{a}_2^{(\mathtt{i})}(-a)}{\mathtt{b}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{i})}(a)\mathtt{b}_2^{(\mathtt{i})}(-a)} = -\frac{\mathtt{a}_2^{(\mathtt{i})}(0)}{\mathtt{b}_2^{(\mathtt{i})}(0)}. \tag{2.22}$$

If our weights satisfy the generalized free fermion condition, we can determine g (and therefore $\mathcal{F}$) from (2.21). If $\mathfrak{S}$ is a lattice model with charge, we will denote the corresponding metaplectic Fock space $\mathcal{F}(\mathfrak{S})$. Note that if $n = 1$, the charge condition goes away, and the generalized free fermion condition reduces to the usual free fermion condition. In this case, $\mathcal{F}(\mathfrak{S})$ is the classical Fock space of Section 2.2.

By construction, both the $\mathfrak{S}$ and $\mathfrak{S}^*$ weights above satisfy the free fermion condition. The left side of the charge condition must be independent of a. Using the parameters in the $\mathfrak{S}$ weights above, it can be expressed more simply:

$$\frac{h(a)h(-a)}{f(a)f(-a)} = -\frac{h(0)}{f(0)}.$$

Notice that the $\mathfrak{S}$ and $\mathfrak{S}^*$ weights above are almost completely general, aside from satisfying the zero-charge free fermion condition. Similarly to the $n = 1$ case, we could change the relative weights of $\mathtt{c}_1$ and $\mathtt{c}_2$, which would create a model that is only trivially different from the original. The last piece is that we could have let the functions $f(a)$ and $h(a)$ depend on their row. However, the ratio $\frac{h(a)}{f(a)}$ for any a needs to be independent of the row in order for the model to match a Hamiltonian of a single metaplectic Fock space, so we simplify notation by having both f and h be independent of i.

Remark 2.8.4. Six-vertex models with charge can be considered as a subset of the colored models studied in [8]. In that context $\mathtt{a}_2$ vertices depend on two parameters, one for each color. The charge in our models corresponds to the difference between the colors. Our models have a shift invariance property: adding an integer uniformly to every column index does not change any of the Boltzmann weights in Figures 2.4 and 2.5, whereas the same operation would change the Boltzmann weights in [8, Figure 7]. In this light, shift invariance is a natural and necessary condition for our Fock spaces. The more general colored models may not need this condition, and will be the subject of future work.

The following is our main result relating Hamiltonian operators to charged lattice models. We will prove it in Section 2.10.

$$Z(\mathfrak{S}^q_{\lambda/\mu}) = \prod_{i=1}^{N} A_i^{M+1} B_i^{\ell(\lambda)} \cdot \langle \mu | e^{H_+} | \lambda \rangle \qquad \text{for all strict partitions } \lambda, \mu \text{ and all } M, N. \tag{2.23}$$

Theorem 2.8.5.

(a) *(2.23) holds precisely when the weights of $\mathfrak{S}^q$ satisfy the generalized free fermion condition and for all $k \geq 1, j \in [1, N]$,*

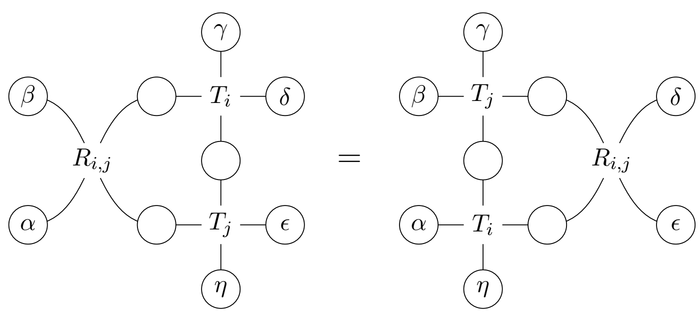
$$s_k^{(j)} = \frac{1}{k}\left(x_i^{nk} \left(\prod_{a=0}^{n-1} f(a) \right)^k + (-1)^{k-1} (g(0))^k y_i^k x_i^{(n-1)k} \left(\prod_{a=0}^{n-1} f(a) \right)^k \right). \tag{2.24}$$

(b) If the Boltzmann weights are not generalized free fermionic, (2.23) does not hold for any choice of the $s_k^{(j)}$.

2.9 Solvability

In this section, we demonstrate that the generalized free fermion condition is important for solvability of the six-vertex model with charge. Consider the vertices in Figure 2.6. We will describe conditions on their Boltzmann weights that are necessary and sufficient for solvability, and in the case where the model is solvable, compute the weights of the R-vertices.

A (rectangular) lattice model is called solvable if there exist a set of *R-vertices* that satisfy the *Yang-Baxter equation*. Let T_i denote a vertex with row index i. The Yang-Baxter equation is satisfied if there exists a set of vertex weights R_{ij} such that for all possible choices of decorated spins $\alpha, \beta, \gamma, \delta, \epsilon, \eta$, we have equality of partition functions:



(2.25)

In particular, the edge labels on the R-vertices must also be decorated spins. We assume a further condition: *conservation of decorated spin.* Namely, the pair of decorated spins entering R_{ij} from the right equals the pair of decorated spins exiting to the left, in some order. The produces the vertices in Figure 2.7. Each R-vertex depends on a pair of row indices (i, j), where i is the index of the top right and bottom left edges, and j is the index of the bottom right and top left edges. We suppress this notation for readability.

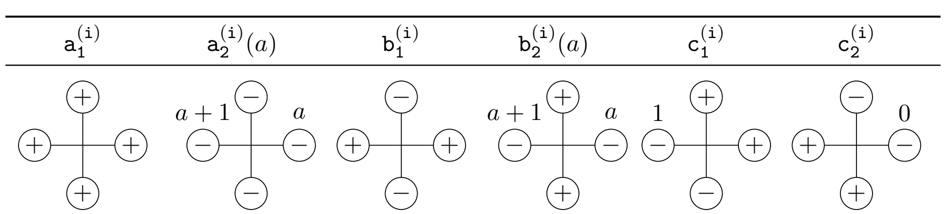


Figure 2.6: A set of admissible vertices for the six-vertex model with charge.

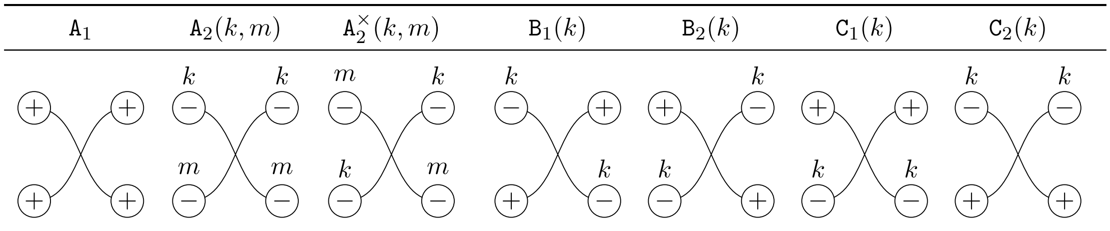


Figure 2.7: A set of R-vertices for the six-vertex model with charge.

The solvability computation (4.3) was done by hand. It consists of 20 cases, all choices of boundary spins such that α, β, and γ have the same number of $-$ spins as δ, ϵ, and η, and many of these cases break up into subcases based on charge. From the resulting equations, we can solve the model and determine conditions on its solvability.

The equations are given in Section 2.12. Here, we present the results. The interested reader can check that the stated conditions are indeed necessary to satisfy the equations in Section 2.12, and that the solutions given do indeed satisfy those conditions.

Let $n > 1$. First, we require that

$$\frac{\mathtt{a}_1^{(\mathtt{i})}\mathtt{a}_2^{(\mathtt{i})}(k)}{\mathtt{b}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{i})}(k)}, \qquad \frac{\mathtt{a}_1^{(\mathtt{i})}\mathtt{a}_2^{(\mathtt{i})}(0)}{\mathtt{c}_1^{(\mathtt{i})}\mathtt{c}_2^{(\mathtt{i})}}, \qquad \text{and} \qquad \frac{\mathtt{b}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{i})}(0)}{\mathtt{c}_1^{(\mathtt{i})}\mathtt{c}_2^{(\mathtt{i})}} \tag{2.26}$$

are independent of i. The first of these conditions along with one of the other two imply the third. Additionally, they imply that

$$\Delta := \frac{\mathtt{a}_1^{(\mathtt{i})}\mathtt{a}_2^{(\mathtt{i})}(0) + \mathtt{b}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{i})}(0) - \mathtt{c}_1^{(\mathtt{i})}\mathtt{c}_2^{(\mathtt{i})}}{2\sqrt{\mathtt{a}_1^{(\mathtt{i})}\mathtt{a}_2^{(\mathtt{i})}(0)\mathtt{b}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{i})}(0)}}$$

is independent of i i.e. constant.

There are now two cases. If $\Delta \neq 0$, then the model is solvable if and only if

$$\prod_{p=0}^{n-1} \mathtt{a}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{j})}(p) = \prod_{p=0}^{n-1} \mathtt{a}_1^{(\mathtt{j})}\mathtt{b}_2^{(\mathtt{i})}(p) \tag{2.27}$$

holds, and the R-vertex weights are given in Table 2.2. Note that while the model is solvable in this case, it is not a very interesting solution. In particular, the weights of $\mathtt{B}_1(k), \mathtt{B}_2(k)$, and $\mathtt{A}_2^{\times}(k, m), k \neq m$ are all 0.

If $\Delta = 0$, the zero charge free fermion condition holds, and the following additional condition is equivalent to solvability

$$\frac{\mathtt{a}_1^{(\mathtt{i})}\mathtt{a}_2^{(\mathtt{i})}(k)\mathtt{a}_1^{(\mathtt{i})}\mathtt{a}_2^{(\mathtt{i})}(-k)}{\mathtt{b}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{i})}(k)\mathtt{b}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{i})}(-k)} = -\frac{\mathtt{a}_1^{(\mathtt{i})}\mathtt{a}_2^{(\mathtt{i})}(0)}{\mathtt{b}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{i})}(0)}.$$

Vertex	Boltzmann weight
$\mathtt{A}_1$	$\mathtt{a}_1^{(\mathtt{i})}\mathtt{a}_2^{(\mathtt{j})}(0)\cdot\prod_{p=1}^{n-1}\mathtt{a}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{j})}(p)+\mathtt{b}_1^{(\mathtt{j})}\mathtt{b}_2^{(\mathtt{i})}(0)\cdot\prod_{p=1}^{n-1}\mathtt{a}_1^{(\mathtt{j})}\mathtt{b}_2^{(\mathtt{i})}(p)$
$\mathtt{A}_2(k,k)$	$\mathtt{a}_1^{(\mathtt{j})}\mathtt{a}_2^{(\mathtt{i})}(0)\prod_{p=1}^{n-1}\mathtt{a}_1^{(\mathtt{j})}\mathtt{b}_2^{(\mathtt{i})}(p)+\mathtt{b}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{j})}(0)\prod_{p=1}^{n-1}\mathtt{a}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{j})}(p)$
$\mathtt{A}_2(k,m), k<m$	$\frac{\mathtt{c}_1^{(\mathtt{j})}\mathtt{c}_2^{(\mathtt{j})}\mathtt{a}_2^{(\mathtt{i})}(0)}{\mathtt{a}_2^{(\mathtt{j})}(0)}\prod_{p=1}^{m-1}\mathtt{a}_1^{(\mathtt{j})}\mathtt{b}_2^{(\mathtt{i})}(p)\prod_{p=m}^{n-1}\mathtt{a}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{j})}(p)\prod_{p=0}^{k-1}\frac{\mathtt{a}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{j})}(p)}{\mathtt{a}_1^{(\mathtt{j})}\mathtt{b}_2^{(\mathtt{i})}(p)}$
$\mathtt{A}_2(k,m), k>m$	$\frac{\mathtt{c}_1^{(\mathtt{j})}\mathtt{c}_2^{(\mathtt{j})}\mathtt{b}_1^{(\mathtt{i})}}{\mathtt{b}_1^{(\mathtt{j})}}\prod_{p=1}^{k-1}\mathtt{a}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{j})}(p)\prod_{p=k}^{n-1}\mathtt{a}_1^{(\mathtt{j})}\mathtt{b}_2^{(\mathtt{i})}(p)\prod_{p=0}^{m-1}\frac{\mathtt{a}_1^{(\mathtt{j})}\mathtt{b}_2^{(\mathtt{i})}(p)}{\mathtt{a}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{j})}(p)}$
$\mathtt{A}_2^{\times}(k,m), k\neq m$	$\frac{\mathtt{a}_2^{(\mathtt{i})}(k-m)}{\mathtt{b}_2^{(\mathtt{i})}(k-m)}\left(\prod_{p=0}^{n-1}\mathtt{a}_1^{(\mathtt{j})}\mathtt{b}_2^{(\mathtt{i})}(p)-\prod_{j=0}^{n-1}\mathtt{a}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{j})}(p)\right)$ $=\frac{\mathtt{b}_2^{(\mathtt{j})}(m-k)}{\mathtt{a}_1^{(\mathtt{j})}(m-k)}\left(\mathtt{a}_2^{(\mathtt{j})}(0)\mathtt{b}_1^{(\mathtt{i})}\prod_{p=1}^{n-1}\mathtt{a}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{j})}(p)-\mathtt{a}_2^{(\mathtt{i})}(0)\mathtt{b}_1^{(\mathtt{j})}\prod_{p=1}^{n-1}\mathtt{a}_1^{(\mathtt{j})}\mathtt{b}_2^{(\mathtt{i})}(p)\right)$
$\mathtt{B}_1(k)$	$\mathtt{a}_2^{(\mathtt{j})}(0)\mathtt{b}_1^{(\mathtt{i})}\cdot\prod_{p=1}^{n-1}\mathtt{a}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{j})}(p)-\mathtt{a}_2^{(\mathtt{i})}(0)\mathtt{b}_1^{(\mathtt{j})}\cdot\prod_{p=1}^{n-1}\mathtt{a}_1^{(\mathtt{j})}\mathtt{b}_2^{(\mathtt{i})}(p)$
$\mathtt{B}_2(k)$	$\prod_{p=0}^{n-1}\mathtt{a}_1^{(\mathtt{j})}\mathtt{b}_2^{(\mathtt{i})}(p)-\prod_{p=0}^{n-1}\mathtt{a}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{j})}(p)$
$\mathtt{C}_1(k)$	$\mathtt{c}_1^{(\mathtt{i})}\mathtt{c}_2^{(\mathtt{j})}\cdot\prod_{p=1}^{k-1}\mathtt{a}_1^{(\mathtt{j})}\mathtt{b}_2^{(\mathtt{i})}(p)\cdot\prod_{p=k}^{n-1}\mathtt{a}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{j})}(p)$
$\mathtt{C}_2(k)$	$\mathtt{c}_1^{(\mathtt{j})}\mathtt{c}_2^{(\mathtt{i})}\cdot\prod_{p=1}^{k-1}\mathtt{a}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{j})}(p)\cdot\prod_{p=k}^{n-1}\mathtt{a}_1^{(\mathtt{j})}\mathtt{b}_2^{(\mathtt{i})}(p)$

Table 2.1: A set of R-vertex weights for the generalized free fermion case (Theorem 2.9.1(a)). For vertices $\mathtt{C}_1(k)$ and $\mathtt{C}_2(k)$, k is taken to be $1\leq k\leq n$, while for vertices $\mathtt{A}_2(k,m)$, k and m are taken to be $0\leq k,m\leq n$. The formulas for vertices $\mathtt{A}_2(k,m)$ and $\mathtt{A}_2^{\times}(k,m)$ hold when $k\neq m$ modulo n. In the case where $k=m$, both vertices equal $\mathtt{A}_2(k,k)$. Note that $\mathtt{B}_1(k),\mathtt{B}_2(k)$, and $\mathtt{A}_2(k,k)$ are independent of k.

Both sides of this equation are independent of i, and when simplified, it reduces to the charge condition (2.22). The R-vertex weights for this case are given in Table 2.1.

In fact, the conditions $\Delta=0$ (for all i), (2.22), and the first condition in (2.26) imply the rest of (2.26). This is equivalent to each row of the lattice model satisfying the $\mathcal{F}$-free-fermion condition for *the same* metaplectic Fock space $\mathcal{F}$.

The following theorem summarizes these results.

Theorem 2.9.1. *The six-vertex model with charge is solvable in precisely two cases:*

1. *The rectangular weights satisfy the $\mathcal{F}$-free-fermion condition for some fixed metaplectic Fock space $\mathcal{F}$. The R-vertex weights are given in Table 2.1.*
2. *(2.26) and (2.27) hold and $\Delta\neq 0$. The R-vertex weights are given in Table 2.2.*

Remark 2.9.2. A comparison of Theorem 2.9.1 and the solvability criterion for the six-vertex model given in [49, Theorem 1] gives evidence of the utility of the Hamiltonian perspective. The equations in Section 2.12 and the process of solving them depend on the condition $n>1$ so that we have at least one non-zero charge, and indeed the criteria given by Theorem 2.9.1(b) do not descend to the six-vertex ($n=1$) case.

However, for generalized free fermion models (Theorem 2.9.1(a)), our criteria do in fact still hold when setting $n=1$. The Hamiltonian perspective explains why this happens.

Vertex	Boltzmann weight
$\mathtt{A}_1$	$\frac{\mathtt{a}_1^{(\mathtt{j})}\mathtt{a}_2^{(\mathtt{j})}(0)+\mathtt{b}_1^{(\mathtt{j})}\mathtt{b}_2^{(\mathtt{j})}(0)}{\mathtt{a}_1^{(\mathtt{j})}\mathtt{b}_2^{(\mathtt{j})}(0)} \cdot \prod_{p=0}^{n-1} \mathtt{a}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{j})}(p)$
$\mathtt{A}_2(k,k)$	$\frac{\mathtt{a}_1^{(\mathtt{i})}\mathtt{a}_2^{(\mathtt{i})}(0)+\mathtt{b}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{i})}(0)}{\mathtt{a}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{i})}(0)} \cdot \prod_{p=0}^{n-1} \mathtt{a}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{j})}(p)$
$\mathtt{A}_2(k,m), k<m$	$\frac{\mathtt{c}_1^{(\mathtt{j})}\mathtt{c}_2^{(\mathtt{j})}\mathtt{a}_2^{(\mathtt{i})}(0)}{\mathtt{a}_2^{(\mathtt{j})}(0)} \prod_{p=1}^{m-1} \mathtt{a}_1^{(\mathtt{j})}\mathtt{b}_2^{(\mathtt{i})}(p) \prod_{p=m}^{n-1} \mathtt{a}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{j})}(p) \prod_{p=0}^{k-1} \frac{\mathtt{a}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{j})}(p)}{\mathtt{a}_1^{(\mathtt{j})}\mathtt{b}_2^{(\mathtt{i})}(p)}$
$\mathtt{A}_2(k,m), k>m$	$\frac{\mathtt{c}_1^{(\mathtt{j})}\mathtt{c}_2^{(\mathtt{j})}\mathtt{b}_1^{(\mathtt{i})}}{\mathtt{b}_1^{(\mathtt{j})}} \prod_{p=1}^{k-1} \mathtt{a}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{j})}(p) \prod_{p=k}^{n-1} \mathtt{a}_1^{(\mathtt{j})}\mathtt{b}_2^{(\mathtt{i})}(p) \prod_{p=0}^{m-1} \frac{\mathtt{a}_1^{(\mathtt{j})}\mathtt{b}_2^{(\mathtt{i})}(p)}{\mathtt{a}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{j})}(p)}$
$\mathtt{A}_2^{\times}(k,m), k \neq m$	0
$\mathtt{B}_1(k)$	0
$\mathtt{B}_2(k)$	0
$\mathtt{C}_1(k)$	$\mathtt{c}_1^{(\mathtt{i})}\mathtt{c}_2^{(\mathtt{j})} \cdot \prod_{p=1}^{k-1} \mathtt{a}_1^{(\mathtt{j})}\mathtt{b}_2^{(\mathtt{i})}(p) \cdot \prod_{p=k}^{n-1} \mathtt{a}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{j})}(p)$
$\mathtt{C}_2(k)$	$\mathtt{c}_1^{(\mathtt{j})}\mathtt{c}_2^{(\mathtt{i})} \cdot \prod_{p=1}^{k-1} \mathtt{a}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{j})}(p) \cdot \prod_{p=k}^{n-1} \mathtt{a}_1^{(\mathtt{j})}\mathtt{b}_2^{(\mathtt{i})}(p)$

Table 2.2: A set of R-vertex weights for the non-free-fermion case (Theorem 2.9.1(b)). The same charge conventions are used as in Table 2.1. These vertex weights are precisely the same as those in Table 2.1 when we impose the additional condition (2.27).

Theorem 2.8.5 and its proof in Section 2.10 do still hold when $n = 1$, and so the connection of solvability to Hamiltonian operators explains why the solution does descend in this case.

2.10 Proof of Theorem 2.8.5

In this section, we prove Theorem 2.8.5. We cannot use the same approach to prove this theorem that we used in the $n = 1$ case, as Wick's theorem is not available to us in this context, so we will use an induction argument due to Brubaker, Buciumas, Bump, and Gustafsson [3, § 4]. Our proof closely mirrors theirs, with generalizations in certain places, and so we will sometimes refer to their proof for steps that are identical.

We will prove the one-row case first, and as a simple scaling gets us the proper powers of A_i and B_i, we set them equal to 1 for convenience.

Proposition 2.10.1. *Theorem 2.8.5(a) holds in the case $N = 1$, $A_1 = B_1 = 1$.*

Let $x = x_1, y = y_1$. Suppose $\mathfrak{S}^q$ satisfies the generalized free fermionic condition. and let $\mathcal{F} = \mathcal{F}(\mathfrak{S}^q)$ and g be the corresponding function $g(a) = h(a)/f(a)$.

If $k - n \leq s \leq k$, define

$$\zeta := x^n \prod_{a=0}^{n-1} f(a), \qquad \tau := -\frac{g(0)\zeta y}{x}, \qquad \zeta_s = x^{s-k+n} \prod_{k-n \leq a \leq s-1} f(a).$$

The following lemma is the base case of Proposition 2.10.1.

Lemma 2.10.2. *If $M < n$, then Proposition 2.10.1 holds.*

Proof. Since there are at most n vertices in the lattice model, there is not enough room for a particle to travel from one column to another. Thus, $Z(\mathfrak{S}^q_{\lambda/\mu}) = 1$ if $\lambda = \mu$, and otherwise is zero. Similarly, any current operator $J_k, k \geq 1$ will send a particle at least n spaces to the left. Therefore, $\langle \mu | e^{H_+} | \lambda \rangle = 1$ if $\lambda = \mu$, and otherwise is zero, so the two sides are equal. □

For the inductive step, we will introduce an operator $\rho^*_j(t)$ that has a similar relationship to both the partition function and the Hamiltonian.

Let ψ^*_j be the creation operator, $\psi^*_j \cdot \underline{u} := u_j \wedge \underline{u}$. A deletion operator is not straightforward to define in the $q \neq -1$ case, so instead we will compute the relationship between the actions of ψ^*_j and e^{H_+}.

Let $\psi^*(t) = \sum_{j \in \mathbb{Z}} \psi^*_j t^j$. Recall from (2.2) that

$$S(t^n) = \sum_{k \geq 1} s_k t^{kn} = \log \left(\sum_{m \geq 0} h_m t^{mn} \right) = \log \left(H(t^n) \right).$$

Lemma 2.10.3. *If (2.24) holds and $N = 1$,*

$$H(t) = 1 + \sum_{k \geq 1} (1 - \tau) \zeta^{k-1} t^k. \tag{2.28}$$

Proof. We start with (2.28), and derive (2.24).

Assuming (2.28), we have

$$H(t) = 1 + \frac{(1-\tau)t}{1 - \zeta t} = \frac{1 - \tau t}{1 - \zeta t},$$

so

$$\log H(t) = \log(1 - \tau t) - \log(1 - \zeta t)$$

and

$$\frac{d^k}{dt^k} (\log H(t)) \bigg|_{t=0} = (k-1)!(\zeta^k - \tau^k),$$

so

$$s_k = \frac{1}{k} (\zeta^k - \tau^k),$$

as in (2.24). All the steps are reversible, so (2.24) implies (2.28). □

Lemma 2.10.4. *(a)*

$$e^{H_+} \psi^*(t) e^{-H_+} = H(t^n) \psi^*(t).$$

(b) Let $\rho_k^*(t) = \psi_k^* - t\psi_{j-n}^*$*. Then,* $e^{H_+}\rho_k^*(\zeta)e^{-H} = \rho_k^*(\tau)$*.*

Proof. (a) Observe that $[J_m, \psi_j^*] = \psi_{j-mn}^*$. Thus,

$$\begin{aligned}
[H, \psi^*(t)] &= \sum_{m\geq 1}\sum_{j\in\mathbb{Z}} s_m t^j \psi_{j-mn}^* \\
&= \sum_{m\geq 1} s_m t^{mn}\psi^*(t) \\
&= S(t^n)\psi^*(t) \\
&= \log\left(H(t^n)\right)\psi^*(t).
\end{aligned}$$

Then, (a) follows from the Baker-Campbell-Hausdoff Theorem.

(b) Recall that $h_0 = 1$, and for $j \geq 1, h_j = s_1\zeta^{j-1}$. By matching coefficients in part (a),

$$e^{H_+}\psi_k^* e^{-H_+} = \psi_k^* + \sum_{j\geq 1}\psi_{k-jn}^* h_j,$$

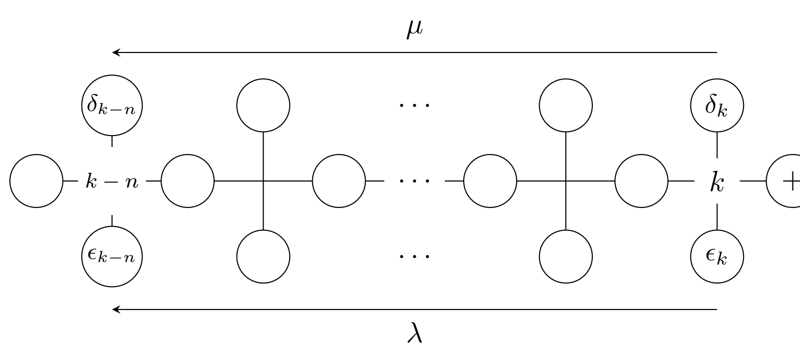
so

$$\begin{aligned}
e^{H_+}\rho_k^*(\zeta)e^{-H_+} &= \psi_k^* + \sum_{j\geq 1}\psi_{k-jn}^* h_j - \zeta\psi_{k-n}^* - \sum_{j\geq 1}\psi_{k-n-jn}^*\zeta h_j \\
&= \psi_k^* + \sum_{j\geq 1}\psi_{k-jn}^* h_j - \zeta\psi_{k-n}^* + \psi_{k-n}^* h_1 - \sum_{j\geq 1}\psi_{k-jn}^* h_j \\
&= \psi_k^* + (h_1 - \zeta)\psi_{k-n}^*.
\end{aligned}$$

Finally, note that $\zeta - h_1 = \tau$.

□

We'll do a similar conjugation on the lattice model side. To do this, we'll need a column-restricted version $\hat{T}_k$ of the row transfer matrix T^q. We let $\hat{T}_k$ be the operator such that $\langle\mu|\hat{T}_k|\lambda\rangle$ is the partition function of the following lattice model



,

where

$$\epsilon_i = \begin{cases} -, & \text{if } \lambda \text{ has a part of size } i, \\ +, & \text{otherwise,} \end{cases} \qquad \delta_i = \begin{cases} -, & \text{if } \mu \text{ has a part of size } i, \\ +, & \text{otherwise.} \end{cases}$$

The leftmost spin is undetermined, and has at most a unique possibility. Colloquially, $\hat{T}_k$ is the partition function of columns $k-n$ through k of the one row lattice model. $\mathfrak{S}^q_{\lambda/\mu}$,

Let us define several quantities that will be useful in the proof of the next lemma, some of which depend on the spins $\epsilon_{k-n}, \dots, \epsilon_k$ associated to λ. Let $\gamma = \frac{f(0)x+h(0)y}{f(0)x} = 1 + g(0)\frac{y}{x}$. Given integers k and s where $k \geq s > k-n$ let

$$G := \prod_{\substack{k-n+1 \leq i \leq k-1 \\ \epsilon_i = -}} \frac{g(k-i)\cdot y}{x}, \qquad G_s := \prod_{\substack{k-n+1 \leq i \leq s-1 \\ \epsilon_i = -}} \frac{g(s-i)\cdot y}{x},$$

$$G'_s := \prod_{\substack{k-n+1 \leq i \leq k-1 \\ \epsilon_i = - \\ i \neq s}} g(k-i).$$

Lemma 2.10.5. *For all strict partitions λ, μ,*

$$\langle \mu | \hat{T}_k \rho^*_k(\zeta) | \lambda \rangle = \langle \mu | \rho^*_k(\tau) \hat{T}_k | \lambda \rangle.$$

Proof. We prove this by cases. The result follows because in every row of the following two tables, the sum of the first two columns is equal to the sum of the last two columns.

First, suppose that for all $k > i > k-n$, $\epsilon_i = \delta_i$. The following table gives the relevant expressions for all choices of the spins in columns k and $k-n$.

$(\epsilon_k, \epsilon_{k-n}, \delta_k, \delta_{k-n})$	$\langle\eta\vert\hat{T}\vert\xi\rangle$	$\langle\eta\vert\hat{T}\psi_k^*\vert\xi\rangle$	$-\zeta\langle\eta\vert\hat{T}\psi_{k-n}^*\vert\xi\rangle$	$\langle\eta\vert\psi_k^*\hat{T}\vert\xi\rangle$	$-\tau\langle\eta\vert\psi_{k-n}^*\hat{T}\vert\xi\rangle$
$(+,+,+,+)$	1	$\gamma\zeta G$	$-\gamma\zeta G$	0	0
$(+,+,+,-)$	0	$\gamma\zeta G$	$-\zeta G$	0	$-\tau G$
$(+,+,-,+)$	0	1	0	1	0
$(+,-,+,+)$	γ	0	0	0	0
$(-,+,+,+)$	$\gamma\zeta G$	0	0	0	0
$(+,+,-,-)$	0	0	0	0	0
$(+,-,+,-)$	1	$-\gamma\tau G$	0	0	$-\gamma\tau G$
$(+,-,-,+)$	0	γ	0	γ	0
$(-,+,+,-)$	$\gamma\zeta G$	0	$-\gamma\tau\zeta G^2$	0	$-\gamma\tau\zeta G^2$
$(-,+,-,+)$	1	0	$\gamma\zeta G$	$\gamma\zeta G$	0
$(-,-,+,+)$	0	0	0	0	0
$(+,-,-,-)$	0	1	0	1	0
$(-,+,-,-)$	0	0	ζG	$\gamma\zeta G$	τG
$(-,-,+,-)$	$-\gamma\tau G$	0	0	0	0
$(-,-,-,+)$	γ	0	0	0	0
$(-,-,-,-)$	1	0	0	$-\gamma\tau G$	$\gamma\tau G$

Now, suppose that for a unique $k > s > k-n$, $\epsilon_s = -, \delta_s = +$. The next table enumerates these cases.

$(\epsilon_k, \epsilon_{k-n}, \delta_k, \delta_{k-n})$	$\langle\eta\vert\hat{T}\vert\xi\rangle$	$\langle\eta\vert\hat{T}\psi_k^*\vert\xi\rangle$	$-\zeta\langle\eta\vert\hat{T}\psi_{k-n}^*\vert\xi\rangle$	$\langle\eta\vert\psi_k^*\hat{T}\vert\xi\rangle$	$-\tau\langle\eta\vert\psi_{k-n}^*\hat{T}\vert\xi\rangle$
$(+,+,+,+)$	$\gamma G_s\zeta_s$	0	0	0	0
$(+,+,+,-)$	0	0	$-\gamma\tau G_s G'_s\zeta_s$	0	$-\gamma\tau G_s G'_s\zeta_s$
$(+,+,-,+)$	0	$\gamma G_s\zeta_s$	0	$\gamma G_s\zeta_s$	0
$(+,-,+,+)$	0	0	0	0	0
$(-,+,+,+)$	0	0	0	0	0
$(+,+,-,-)$	0	0	0	0	0
$(+,-,+,-)$	$\gamma G_s\zeta_s g(s-k)yx^{-1}$	0	0	0	0
$(+,-,-,+)$	0	0	0	0	0
$(-,+,+,-)$	0	0	0	0	0
$(-,+,-,+)$	$\gamma G_s\zeta_s$	0	0	0	0
$(-,-,+,+)$	0	0	0	0	0
$(+,-,-,-)$	0	$\gamma G_s\zeta_s g(s-k)yx^{-1}$	0	$\gamma G_s\zeta_s g(s-k)yx^{-1}$	0
$(-,+,-,-)$	0	0	$\gamma\tau G_s G'_s\zeta_s$	0	$\gamma\tau G_s G'_s\zeta_s$
$(-,-,+,-)$	0	0	0	0	0
$(-,-,-,+)$	0	0	0	0	0
$(-,-,-,-)$	$\gamma G_s\zeta_s g(s-k)yx^{-1}$	0	0	0	0

□

Lemma 2.10.6. $\langle\eta|T\rho_k^*(\zeta)|\xi\rangle = \langle\eta|\rho_k^*(\tau)T|\xi\rangle$.

Proof. This follows from Lemma 2.10.5 by the same argument as [3, Proposition 4.4]. □

Proof of Proposition 2.10.1. This follows from Propositions 2.10.4 and 2.10.6 by the same argument as the proof of Theorem A in [3]. □

The rest of the proof is similar to the proof of Proposition 2.4.2. Let $\mathfrak{S}^{q,N}_{\lambda/\mu}$ denote the usual $\mathfrak{S}$ lattice model with N rows. Recall that $H_+ = \sum_{j=1}^N \phi_j$.

Proof of Theorem 2.8.5. For part a,

$$\begin{aligned}\prod_{i=1}^N A_i^{M+1} B_i^{\ell(\lambda)} \langle \mu | e^{H_+} | \lambda \rangle &= \prod_{i=1}^N A_i^{M+1} B_i^{\ell(\lambda)} \langle \mu | e^{\phi_N} \dots e^{\phi_1} | \lambda \rangle \\ &= \sum_{\nu_1,\dots,\nu_{N-1}} \langle \mu | e^{\phi_N} | \nu_{N-1} \rangle \langle \nu_{N-1} | e^{\phi_{N-1}} | \nu_{N-2} \rangle \dots \langle \nu_1 | e^{\phi_1} | \lambda \rangle \\ &= \sum_{\nu_1,\dots,\nu_{N-1}} Z(\mathfrak{S}^{q,1}_{\nu_{N-1}/\mu}) \dots Z(\mathfrak{S}^{q,1}_{\lambda/\nu_1}) \\ &= Z(\mathfrak{S}^{q,N}_{\lambda/\mu}).\end{aligned}$$

To see that the solution (2.24) for $s_k^{(j)}$ is unique, note that the partition function $Z(\mathfrak{S}^q_{(mn)/(0)})$ depends only on $s_k^{(j)}$ for $1 \le k \le m$.

For part b, we need to show that both the free fermion condition and the charge condition (assuming $n > 1$) are required in order to satisfy (2.23). Consider the model $\overline{\mathfrak{S}}^q$, which is equal to $\mathfrak{S}^q$ but with unspecified Boltzmann weights $\mathtt{a}_1^{(\mathtt{i})}, \mathtt{a}_2^{(\mathtt{i})}(a), \mathtt{b}_1^{(\mathtt{i})}, \mathtt{b}_2^{(\mathtt{i})}(a), \mathtt{c}_1^{(\mathtt{i})}, \mathtt{c}_2^{(\mathtt{i})}$. By rescaling the weights as we do above, we can assume without loss of generality that $\mathtt{a}_1^{(\mathtt{i})} = \mathtt{b}_1^{(\mathtt{i})} = \mathtt{c}_2^{(\mathtt{i})} = 1$ for all i. Then we assume there exist Hamiltonian parameters $s_k^{(j)}, k \ge 1, 1 \le j \le N$ such that

$$Z(\overline{\mathfrak{S}}^q_{\lambda/\mu}) = * \cdot \langle \mu | e^{H_+} | \lambda \rangle \qquad \text{for all strict partitions } \lambda, \mu \text{ and all } M, N. \tag{2.29}$$

For the zero charge free fermion condition, we observe that if M and all the parts of λ and μ are multiples of n, then there are no vertices of type $\mathtt{a}_2^{(\mathtt{i})}(a)$ where $a \neq 0$. This means that both the lattice model and Hamiltonian reduce to the $n = 1$ case. Namely, let $\mathfrak{S}_{red}$ be the lattice model with the following weights and boundary conditions

$$(\mathtt{a}_1^{(\mathtt{i})})_{red} := (\mathtt{a}_1^{(\mathtt{i})})^n, \quad (\mathtt{a}_2^{(\mathtt{i})})_{red} := \mathtt{a}_2^{(\mathtt{i})}(0) \prod_{a=1}^N \mathtt{b}_2^{(\mathtt{i})}(a), \quad (\mathtt{b}_1^{(\mathtt{i})})_{red} = \mathtt{b}_1^{(\mathtt{i})} (\mathtt{a}_1^{(\mathtt{i})})^{n-1},$$

$$(\mathtt{b}_2^{(\mathtt{i})})_{red} = \mathtt{b}_2^{(\mathtt{i})}(0) \prod_{a=1}^N \mathtt{b}_2^{(\mathtt{i})}(a), \quad (\mathtt{c}_1^{(\mathtt{i})})_{red} = \mathtt{c}_1^{(\mathtt{i})} \prod_{a=1}^N \mathtt{b}_2^{(\mathtt{i})}(a), \quad (\mathtt{c}_2^{(\mathtt{i})})_{red} = \mathtt{c}_2^{(\mathtt{i})} (\mathtt{a}_1^{(\mathtt{i})})^{n-1}.$$

$$M_{red} := \frac{M}{n}, \quad \lambda_{red} := \left(\frac{\lambda_1}{n}, \quad \frac{\lambda_2}{n} \dots \right), \quad \mu_{red} := \left(\frac{\mu_1}{n}, \frac{\mu_2}{n} \dots \right).$$

In this case, (2.29) is satisfied precisely when $(\mathfrak{S}_{red})_{\lambda_{red}/\mu_{red}}$ and $H_{red} := \sum_{k\geq 1} s_k J_k$ satisfy (2.8), so $\mathfrak{S}_{red}$ must satisfy the free fermion condition. The reduction hasn't changed the partition function or τ function, so thus $\mathfrak{S}^q$ must also satisfy the free fermion condition.

For the charge condition, if $0 < p < n$, then

$$\begin{aligned}\langle (p,0)|e^{H_+}|(n,p)\rangle &= \langle v_p \wedge v_0 \ldots |e^{H_+}|v_n \wedge v_p \wedge \ldots\rangle \\ &= g(n-p)\langle v_p \wedge v_0 \ldots |e^{H_+}|v_p \wedge v_n \wedge \ldots\rangle \\ &= g(n-p)s_1,\end{aligned}$$

while

$$Z(\overline{\mathfrak{S}}^q_{(n,p)/(p,0)}) = \frac{\mathtt{a}_2^{(1)}(n-p)\mathtt{c}_2^{(1)}}{\mathtt{b}_2^{(1)}(n-p)\mathtt{b}_2^{(1)}(0)}\left(\prod_{i=1}^{n} \mathtt{b}_2^{(1)}(a)\right).$$

Meanwhile,

$$\langle (0)|e^{H_+}|(n)\rangle = s_1 \qquad \text{and} \qquad Z(\overline{\mathfrak{S}}^q_{(n)/(0)}) = \frac{\mathtt{c}_2^{(1)}}{\mathtt{b}_2^{(1)}(0)}\left(\prod_{i=1}^{n} \mathtt{b}_2^{(1)}(a)\right),$$

so we must have $\frac{\mathtt{a}_2^{(1)}(a)}{\mathtt{b}_2^{(1)}(a)} = g(a)$ for all $0 < a < n$. Combining this with the condition $g(a)g(-a) = -g(0)$ gives the charge condition for the first row. For the other weights, use the branching rules

$$Z(\overline{\mathfrak{S}}^q_{\lambda/\mu}) = \sum_{\nu} Z(\overline{\mathfrak{S}}^{q,N-1}_{\lambda/\nu})Z(\overline{\mathfrak{S}}^{q,1}_{\nu/\mu})$$

$$\langle \mu|e^{H_+}|\lambda\rangle = \sum_{\nu}\langle \mu|e^{\phi_N}|\nu\rangle\langle \nu|e^{\phi_{N-1}}\cdots e^{\phi_1}|\lambda\rangle$$

and induction. □

Let $\mathcal{F}$ be a Fock space, and let $\mathcal{F}_*$ equal $\mathcal{F}$ as a vector space, but with wedge relations defined by $g_*(a) := (g(-a))^{-1}$. In particular, we are sending $q \mapsto q^{-1}$. We'll write $|\lambda\rangle_*$ and $\langle\lambda|_*$ to refer to basis vector in $\mathcal{F}_*$ and its dual, and any current operator J_k acting on $\mathcal{F}_*$ or its dual is assumed to be the relevant current operator. Then we have

Theorem 2.10.7.

(a) The equation

$$Z(\mathfrak{S}^{*,q}_{\lambda/\mu}) = \prod_{i=1}^{N} A_i^{-(M+1)}B_i^{-\ell(\lambda)} \cdot \langle\lambda|_* e^{H_-}|\mu\rangle_* \qquad \forall \text{ strict partitions } \lambda,\mu, \forall M,N. \tag{2.30}$$

holds precisely when the weights of $\mathfrak{S}^{*,q}$ satisfy the generalized free fermion condition *and for all $k \geq 1, j \in [1, N]$,*

$$s_{-k}^{(j)} = \frac{1}{k}\left(y_i^{nk}\left(\prod_{a=0}^{n-1} h(a)\right)^k + (-1)^{k-1}(g(0))^{-k}x_i^k y_i^{(n-1)k}\left(\prod_{a=0}^{n-1} h(a)\right)^k\right). \tag{2.31}$$

(b) *If the Boltzmann weights are not generalized free fermionic, (2.30) does not hold for any choice of the $s_k^{(j)}$.*

Proof. This follows from Theorem 2.8.5 and Proposition 2.8.2. □

2.11 Supersymmetric LLT Polynomials

In this final section, we prove two main results. The first result (Theorem 2.11.1) is that our charged partition function is a supersymmetric LLT polynomials, and the second (Theorem 2.11.2) is a Cauchy identity for supersymmetric LLT polynomials. Although our partition functions do not give all specializations of supersymmetric LLT polynomials, the Cauchy identity only uses Hamiltonian operators and is therefore fully general.

2.11.1 The partition function of the charged models

Let

$$L_+(\boldsymbol{x}) = \sum_{k=1}^{\infty}\sum_{j=1}^{N} \frac{1}{k} x_j^k J_k.$$

Brubaker, Buciumas, Bump, and Gustafsson showed [3, Theorem 5.8] that the supersymmetric LLT polynomial can be expressed as

$$\mathcal{G}_{\lambda/\mu}[\boldsymbol{x}|\boldsymbol{y}] := \mathcal{G}_{\lambda/\mu}[\boldsymbol{x}|\boldsymbol{y}; q] = \langle \mu + \rho | e^{L_+(\boldsymbol{x}^n) - L_+(\boldsymbol{y}^n)} | \lambda + \rho \rangle. \tag{2.32}$$

Let $F = \prod_{a=0}^{n-1} f(a)$ and $H = \prod_{a=0}^{n-1} h(a)$. Then,

$$H_+ = \sum_{k=1}^{\infty}\sum_{j=1}^{N} \frac{1}{k}(x_i^n F)^k - \sum_{k=1}^{\infty}\sum_{j=1}^{N} \frac{1}{k}(-g(0)x_i^{n-1}y_i F)^k = L(\boldsymbol{\theta}) - L(\boldsymbol{\pi}),$$

and

$$H_- = \sum_{k=1}^{\infty}\sum_{j=1}^{N} \frac{1}{k}(y_i^n H)^k - \sum_{k=1}^{\infty}\sum_{j=1}^{N} \frac{1}{k}(-(g(0))^{-1}y_i^{n-1}x_i H)^k = L(\boldsymbol{\eta}) - L(\boldsymbol{\tau}),$$

where

$$\theta_i = x_i^n F, \quad \pi_i = -g(0)x_i^{n-1}y_i F, \quad \eta_i = w_i^n H, \quad \tau_i = -g_*(0)w_i^{n-1}z_i H.$$

Therefore, by Theorems 2.8.5 and 2.10.7, we have

Theorem 2.11.1.

$$Z(\mathfrak{S}^q_{\lambda+\rho/\mu+\rho}) = \prod_{i=1}^{N} A_i^{M+1} B_i^{\ell(\lambda)} \mathcal{G}_{\lambda/\mu}(\boldsymbol{\theta}|\boldsymbol{\pi}; q),$$

where $\theta_i = x_i^n F$ *and* $\pi_i = -g(0)x_i^{n-1}y_i F$, *and similarly*

$$Z(\mathfrak{S}^{*,q}_{\lambda+\rho/\mu+\rho}) = \prod_{i=1}^{N} A_i^{-(M+1)} B_i^{-\ell(\lambda)} \mathcal{G}_{\lambda/\mu}(\boldsymbol{\eta}|\boldsymbol{\tau}; q^{-1}),$$

where $\eta_i = w_i^n H$ *and* $\tau_i = -g_*(0)w_i^{n-1}z_i H$.

By varying $\boldsymbol{x}$ and $\boldsymbol{y}$, we obtain almost every value of the supersymmetric LLT polynomial as a partition function of $\mathfrak{S}^q$ and $\mathfrak{S}^{*,q}$. The exceptions are the values $\theta_i = 0$ or $\pi_i = 0$, since setting $x_i = 0, y_i = 0$, $f(a) = 0$, or $h(a) = 0$ causes the model to degenerate, and Theorem 2.8.5 no longer holds in this setting. Unfortunately, the case $\boldsymbol{\pi} = 0$ of the classical LLT polynomials is one of these exceptional cases.

2.11.2 Cauchy identity

Using results from previous sections, one can prove some similar facts involving charged lattice models, q-Fock space and supersymmetric LLT polynomials to those proved in Sections 2.5 and 2.6 about uncharged models, classical Fock space, and supersymmetric Schur polynomials. These include Fock space operators for general side boundary conditions, as well as branching, Pieri, and Cauchy identities. As these proofs take similar forms to those in Sections 2.5 and 2.6, we will only prove the Cauchy identity, and leave the rest to the interested reader.

A similar Cauchy identity for specializations are proved by Lam [43] for LLT polynomials and Brubaker, Buciumas, Bump, and Gustafsson [3] for metaplectic symmetric functions. Our proof technique is similar to the proofs of those results.

See also Curran, Frechette, Yost-Wolff, Zhang, and Zhang [26] for an interesting exploration of this Cauchy identity in the context of lattice models. Our Proposition 2.11.2 is similar to their Corollary 3.3.

Let $L_+(\boldsymbol{x}|\boldsymbol{y}) := L_+(\boldsymbol{x}^n) - L_+(\boldsymbol{y}^n)$. Similarly, let

$$L_-(\boldsymbol{x}|\boldsymbol{y}) := L_-(\boldsymbol{x}^n) - L_-(\boldsymbol{y}^n), \qquad \text{where} \qquad L_-(\boldsymbol{x}) = \sum_{k=1}^{\infty}\sum_{j=1}^{N} \frac{1}{k} x_j^k J_{-k}.$$

Note that by adjointness [3, Proposition 4.9], we also have

$$\mathcal{G}_{\lambda/\mu}[\boldsymbol{x}|\boldsymbol{y}] = \langle \lambda+\rho | e^{L_-(\boldsymbol{x}|\boldsymbol{y})} | \mu+\rho\rangle.$$

Let

$$\Omega(\boldsymbol{x}|\boldsymbol{y};\boldsymbol{z}|\boldsymbol{w}) := \prod_{t=0}^{n-1}\prod_{i,j}\frac{(1-v^t x_i^n w_j^n)(1-v^t y_i^n z_j^n)}{(1-v^t x_i^n z_j^n)(1-v^t y_i^n w_j^n)}.$$

Proposition 2.11.2 (Supersymmetric LLT Cauchy identity)**.** *For any strict partitions* λ *and* μ*,*

$$\sum_{\nu}\mathcal{G}_{\lambda/\nu}[\boldsymbol{x}|\boldsymbol{y}]\mathcal{G}_{\mu/\nu}[\boldsymbol{z}|\boldsymbol{w}] = \Omega(\boldsymbol{x}|\boldsymbol{y};\boldsymbol{z}|\boldsymbol{w})\sum_{\nu}\mathcal{G}_{\nu/\mu}[\boldsymbol{x}|\boldsymbol{y}]\mathcal{G}_{\nu/\lambda}[\boldsymbol{z}|\boldsymbol{w}], \tag{2.33}$$

where the sums are over all strict partitions ν*.*

Note that both $\mathcal{G}_{\lambda/\nu}$ and $\mathcal{G}_{\nu/\lambda}$ are zero unless $|\lambda|-|\nu|$ is a multiple of n.

Proof. We evaluate the Hamiltonian $\langle\mu+\rho|e^{L_-(\boldsymbol{z}|\boldsymbol{w})}e^{L_+(\boldsymbol{x}|\boldsymbol{y})}|\lambda+\rho\rangle$ in two ways. First,

$$\begin{aligned}\langle\mu+\rho|e^{L_-(\boldsymbol{z}|\boldsymbol{w})}e^{L_+(\boldsymbol{x}|\boldsymbol{y})}|\lambda+\rho\rangle &= \sum_{\nu}\langle\mu+\rho|e^{L_-(\boldsymbol{z}|\boldsymbol{w})}|\nu+\rho\rangle\langle\nu+\rho|e^{L_+(\boldsymbol{w}|\boldsymbol{y})}|\lambda+\rho\rangle\\ &= \sum_{\nu}\mathcal{G}_{\lambda/\nu}[\boldsymbol{x}|\boldsymbol{y}]\mathcal{G}_{\mu/\nu}[\boldsymbol{z}|\boldsymbol{w}].\end{aligned}$$

Next, we apply the commutation relations between L_+ and L_-. Let $s_k^{(j)} = x_j^{nk} - y_j^{nk}$, $t_{-k}^{(j)} = z_j^{nk} - w_j^{nk}$, and $s_k = \sum_j s_k^{(j)}$, $t_{-k} = \sum_j t_{-k}^{(j)}$. Recall that

$$[J_k, J_l] = k\cdot\frac{1-v^{n|k|}}{1-v^{|k|}}\delta_{k,-l}.$$

Then,

$$\begin{aligned}&\langle\mu+\rho|e^{L_-(\boldsymbol{z}|\boldsymbol{w})}e^{L_+(\boldsymbol{x}|\boldsymbol{y})}|\lambda+\rho\rangle\\ &\qquad= \exp\left(\sum_{k\geq 1}k\frac{1-v^{nk}}{1-v^k}\cdot s_k t_{-k}\right)\cdot\langle\mu+\rho|e^{L_+(\boldsymbol{x}|\boldsymbol{y})}e^{L_-(\boldsymbol{z}|\boldsymbol{w})}|\lambda+\rho\rangle\\ &\qquad= \prod_{i,j}\exp\left(\sum_{k\geq 1}k\frac{1-v^{nk}}{1-v^k}\cdot s_k^{(i)}t_{-k}^{(j)}\right)\cdot\sum_{\nu}\mathcal{G}_{\nu/\mu}[\boldsymbol{x}|\boldsymbol{y}]\mathcal{G}_{\nu/\lambda}[\boldsymbol{z}|\boldsymbol{w}].\end{aligned}$$

Now,

$$\begin{aligned}\sum_{k\geq 1} k\frac{1-v^{nk}}{1-v^k}\cdot s_k^{(i)}t_{-k}^{(j)} &= \sum_{k\geq 1}\frac{1}{k}\frac{1-v^{nk}}{1-v^k}\cdot(x_i^{nk}-y_i^{nk})(z_j^{nk}-w_j^{nk})\\ &= \sum_{t=0}^{n-1}\sum_{k\geq 1}\frac{1}{k}v^{tk}(x_i^{nk}z_j^{nk}-x_i^{nk}w_j^{nk}-y_i^{nk}z_j^{nk}+y_i^{nk}w_j^{nk})\\ &= \log\prod_{t=0}^{n-1}\frac{(1-v^t x_i^n w_j^n)(1-v^t y_i^n z_j^n)}{(1-v^t x_i^n z_j^n)(1-v^t y_i^n w_j^n)},\end{aligned}$$

and combining this with the first equation gives (2.33). □

If we specialize $x_i^n \mapsto x_i^n F, y_i^n \mapsto -g(0)x_i^{n-1}y_i F, z_i^n \mapsto w_i^n H$, and $w_i^n \mapsto -g_*(0)w_i^{n-1}z_i H$, then we obtain a Cauchy identity for our partition functions $Z(\mathfrak{S}^q)$ and $Z(\mathfrak{S}^{*,q})$.

If we instead let $y_i \mapsto 0, w_i \mapsto 0$, then we obtain the Cauchy identity for classical LLT polynomials [43, Theorem 26], while if we let $y_i^n \mapsto vx_i^n, w_i^n \mapsto vz_i^n$, we obtain the Cauchy identity for metaplectic symmetric functions [3, Theorem 5.10]. Both of those results are given for the special case $\lambda = \mu = \delta$, where δ is an *n-core partition.*

2.12 Charged Model Equations for Solvability

Below, we list the inequivalent equations obtained from (4.3), varying $\alpha, \beta, \gamma, \delta, \epsilon, \eta$ across all decorated spins. Charges k and m are taken modulo n. Solving these equations gives the conditions in Theorem 2.9.1.

$$\mathtt{B}_1(k) = \mathtt{B}_1(k+1), \qquad \mathtt{B}_2(k) = \mathtt{B}_2(k+1), \qquad \forall k;$$

Set $\mathtt{B}_1 := \mathtt{B}_1(k), \mathtt{B}_2 := \mathtt{B}_2(k)$ (independent of k).

$$\frac{\mathtt{C}_1(k+1)}{\mathtt{C}_1(k)} = \frac{\mathtt{a}_1^{(\mathtt{j})}\mathtt{b}_2^{(\mathtt{i})}(k)}{\mathtt{a}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{j})}(k)} = \frac{\mathtt{a}_2^{(\mathtt{i})}(k)\mathtt{b}_1^{(\mathtt{j})}}{\mathtt{a}_2^{(\mathtt{j})}(k)\mathtt{b}_1^{(\mathtt{i})}}, \qquad \forall k \neq 0;$$

$$\frac{\mathtt{C}_2(k+1)}{\mathtt{C}_2(k)} = \frac{\mathtt{a}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{j})}(k)}{\mathtt{a}_1^{(\mathtt{j})}\mathtt{b}_2^{(\mathtt{i})}(k)} = \frac{\mathtt{a}_2^{(\mathtt{j})}(k)\mathtt{b}_1^{(\mathtt{i})}}{\mathtt{a}_2^{(\mathtt{i})}(k)\mathtt{b}_1^{(\mathtt{j})}}, \qquad \forall k \neq 0;$$

$$\mathtt{A}_2^{\times}(k+1, m+1) = \mathtt{A}_2^{\times}(k, m), \qquad \forall k, m;$$

$$\frac{\mathtt{A}_2^{\times}(0,m)}{\mathtt{B}_1} = \frac{\mathtt{b}_2^{(\mathtt{j})}(m)}{\mathtt{a}_2^{(\mathtt{j})}(m)}, \qquad \forall m \neq 0;$$

$$\frac{\mathtt{A}_2^{\times}(k,0)}{\mathtt{B}_2} = \frac{\mathtt{a}_2^{(\mathtt{i})}(k)}{\mathtt{b}_2^{(\mathtt{i})}(k)}, \qquad \forall k \neq 0;$$

$$\frac{\mathtt{A}_2(k+1,m+1)}{\mathtt{A}_2(k,m)} = \frac{\mathtt{b}_2^{(\mathtt{j})}(k)\mathtt{b}_2^{(\mathtt{i})}(m)}{\mathtt{b}_2^{(\mathtt{i})}(k)\mathtt{b}_2^{(\mathtt{j})}(m)} = \frac{\mathtt{a}_2^{(\mathtt{j})}(k)\mathtt{a}_2^{(\mathtt{i})}(m)}{\mathtt{a}_2^{(\mathtt{i})}(k)\mathtt{a}_2^{(\mathtt{j})}(m)}, \qquad \forall k, m;$$

Set $\mathtt{A}_2 := \mathtt{A}_2(k,k) = \mathtt{A}_2^{\times}(k,k)$ (independent of k).

$$\frac{\mathtt{A}_2(k,0)}{\mathtt{C}_2(k+1)} = \frac{\mathtt{a}_2^{(\mathtt{i})}(k)\mathtt{c}_2^{(\mathtt{j})}}{\mathtt{a}_2^{(\mathtt{j})}(k)\mathtt{c}_2^{(\mathtt{i})}}, \qquad \frac{\mathtt{A}_2(k+1,1)}{\mathtt{C}_2(k)} = \frac{\mathtt{b}_2^{(\mathtt{j})}(k)\mathtt{c}_1^{(\mathtt{i})}}{\mathtt{b}_2^{(\mathtt{i})}(k)\mathtt{c}_1^{(\mathtt{j})}}, \qquad \forall k \neq 0;$$

$$\frac{\mathtt{A}_2(0,k)}{\mathtt{C}_1(k+1)} = \frac{\mathtt{b}_2^{(\mathtt{j})}(k)\mathtt{c}_2^{(\mathtt{i})}}{\mathtt{b}_2^{(\mathtt{i})}(k)\mathtt{c}_2^{(\mathtt{j})}}, \qquad \frac{\mathtt{A}_2(1,k+1)}{\mathtt{C}_1(k)} = \frac{\mathtt{a}_2^{(\mathtt{i})}(k)\mathtt{c}_1^{(\mathtt{j})}}{\mathtt{a}_2^{(\mathtt{j})}(k)\mathtt{c}_1^{(\mathtt{i})}}, \qquad \forall k \neq 0;$$

$$\frac{\mathtt{C}_2(1)}{\mathtt{C}_1(0)} = \frac{\mathtt{c}_1^{(\mathtt{j})}\mathtt{c}_2^{(\mathtt{i})}}{\mathtt{c}_1^{(\mathtt{i})}\mathtt{c}_2^{(\mathtt{j})}}, \qquad \frac{\mathtt{C}_1(1)}{\mathtt{C}_2(0)} = \frac{\mathtt{c}_1^{(\mathtt{i})}\mathtt{c}_2^{(\mathtt{j})}}{\mathtt{c}_1^{(\mathtt{j})}\mathtt{c}_2^{(\mathtt{i})}}, \qquad \forall k \neq 0;$$

$$\mathtt{C}_2(1)\mathtt{b}_2^{(\mathtt{i})}(0)\mathtt{a}_1^{(\mathtt{j})} = \mathtt{b}_2^{(\mathtt{j})}(0)\mathtt{a}_1^{(\mathtt{i})}\mathtt{C}_2(0) + \mathtt{c}_1^{(\mathtt{j})}\mathtt{c}_2^{(\mathtt{i})}\mathtt{B}_2;$$

$$\mathtt{C}_1(1)\mathtt{a}_1^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{j})}(0) + \mathtt{B}_2\mathtt{c}_1^{(\mathtt{i})}\mathtt{c}_2^{(\mathtt{j})} = \mathtt{a}_1^{(\mathtt{j})}\mathtt{b}_2^{(\mathtt{i})}(0)\mathtt{C}_1(0);$$

$$\mathtt{C}_1(1)\mathtt{b}_1^{(\mathtt{i})}\mathtt{a}_2^{(\mathtt{j})}(0) = \mathtt{b}_1^{(\mathtt{j})}\mathtt{a}_2^{(\mathtt{i})}(0)\mathtt{C}_1(0) + \mathtt{c}_2^{(\mathtt{j})}\mathtt{c}_1^{(\mathtt{i})}\mathtt{B}_1;$$

$$\mathtt{C}_2(1)\mathtt{a}_2^{(\mathtt{i})}(0)\mathtt{b}_1^{(\mathtt{j})} + \mathtt{B}_1\mathtt{c}_2^{(\mathtt{i})}\mathtt{c}_1^{(\mathtt{j})} = \mathtt{a}_2^{(\mathtt{j})}(0)\mathtt{b}_1^{(\mathtt{i})}(0)\mathtt{C}_2(0);$$

$$\mathtt{A}_1\mathtt{c}_2^{(\mathtt{i})}\mathtt{a}_1^{(\mathtt{j})} = \mathtt{c}_2^{(\mathtt{j})}\mathtt{a}_1^{(\mathtt{i})}\mathtt{C}_2(0) + \mathtt{b}_1^{(\mathtt{j})}\mathtt{c}_2^{(\mathtt{i})}\mathtt{B}_2;$$

$$\mathtt{C}_1(1)\mathtt{a}_1^{(\mathtt{i})}\mathtt{c}_1^{(\mathtt{j})} + \mathtt{B}_2\mathtt{c}_1^{(\mathtt{i})}\mathtt{b}_1^{(\mathtt{j})} = \mathtt{a}_1^{(\mathtt{j})}\mathtt{c}_1^{(\mathtt{i})}(0)\mathtt{A}_1;$$

$$\mathtt{A}_1\mathtt{b}_1^{(\mathtt{i})}\mathtt{c}_2^{(\mathtt{j})} = \mathtt{c}_2^{(\mathtt{j})}\mathtt{a}_1^{(\mathtt{i})}\mathtt{B}_2 + \mathtt{b}_1^{(\mathtt{j})}\mathtt{c}_2^{(\mathtt{i})}\mathtt{C}_1(0);$$

$$\mathtt{B}_1\mathtt{a}_1^{(\mathtt{i})}\mathtt{c}_1^{(\mathtt{j})} + \mathtt{C}_2(1)\mathtt{c}_1^{(\mathtt{i})}\mathtt{b}_1^{(\mathtt{j})} = \mathtt{c}_1^{(\mathtt{j})}\mathtt{b}_1^{(\mathtt{i})}\mathtt{A}_1;$$

$$\mathtt{A}_2\mathtt{c}_1^{(\mathtt{i})}\mathtt{a}_2^{(\mathtt{j})}(0) = \mathtt{c}_1^{(\mathtt{j})}\mathtt{a}_2^{(\mathtt{i})}(0)\mathtt{C}_1(0) + \mathtt{b}_2^{(\mathtt{j})}(0)\mathtt{c}_1^{(\mathtt{i})}\mathtt{B}_1;$$

$$\mathtt{C}_2(1)\mathtt{a}_2^{(\mathtt{i})}(0)\mathtt{c}_2^{(\mathtt{j})} + \mathtt{B}_1\mathtt{c}_2^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{j})}(0) = \mathtt{a}_2^{(\mathtt{j})}(0)\mathtt{c}_2^{(\mathtt{i})}\mathtt{A}_2;$$

$$\mathtt{A}_2\mathtt{b}_2^{(\mathtt{i})}(0)\mathtt{c}_1^{(\mathtt{j})} = \mathtt{b}_2^{(\mathtt{j})}(0)\mathtt{c}_1^{(\mathtt{i})}\mathtt{C}_2(0) + \mathtt{c}_1^{(\mathtt{j})}\mathtt{a}_2^{(\mathtt{i})}(0)\mathtt{B}_2;$$

$$\mathtt{C}_1(1)\mathtt{c}_2^{(\mathtt{i})}\mathtt{b}_2^{(\mathtt{j})}(0) + \mathtt{B}_2\mathtt{a}_2^{(\mathtt{i})}(0)\mathtt{c}_2^{(\mathtt{j})} = \mathtt{c}_2^{(\mathtt{j})}\mathtt{b}_2^{(\mathtt{i})}(0)\mathtt{A}_2;$$

Chapter 3

Formal Group Laws and Solvable Lattice Models

This chapter explores the possibility of using solvable lattice models to study higher cohomology on the flag variety, specifically to express polynomial representatives for higher cohomology classes of Schubert and Bott-Samelson varieties. We adopt Quillen's perspective, and study higher cohomology by the formal group laws associated to them. This perspective allows for a unified framework (at least sometimes), as Bott-Samelson classes can be computed by applying certain *push-pull* operators to a *seed* representing the cohomology of a point. Therefore, key parts of the theory can be accessed via a study of the algebra associated to these operators.

Some useful references for this perspective are [51–54]. Much of my understanding of this topic stems from several very useful discussions with Ben Brubaker, Dan Bump, Arun Ram, and Craig Westerland.

3.1 Cohomiology theories and formal group laws

The definition of an oriented cohomology theory, and the idea to connect these cohomology theories to formal group laws is due to Quillen [55]. He worked in the setting of complex differentiable manifolds. Levine and Morel then extended these ideas to the algebraic setting of smooth quasi-projective schemes. We follow their book [56] in this section.

Definition 3.1.1. Let A be a commutative ring that is torsion free. A formal power series

$$F(X,Y) = \sum_{i,j=0}^{\infty} a_{i,j} X^i Y^j \in A[[X,Y]]$$

is a *formal group law* with coefficients in A precisely when it satisfies the following conditions for all $X, Y, Z \in A$:

1. $F(X, 0) = F(0, X) = X$.
2. $F(X, Y) = F(Y, X)$.
3. $F(X, F(Y, Z)) = F(F(X, Y), Z)$.

More precisely, F is a commutative formal group law of rank 1. F depends on the ring A, so we will sometimes write the ordered pair (A, F) for a formal group law F on the ring A.

There is a universal formal group law $(\mathbb{L}, F_{\mathbb{L}})$, where $\mathbb{L}$ is called the *Lazard ring*. Let $\tilde{\mathbb{L}} := \mathbb{Z}[A_{i,j} | i, j \in \mathbb{N}]$ and $\tilde{F}(X, Y) = \sum_{i,j} A_{i,j} X^i Y^j$. To make this into a formal group law, we impose the formal group law conditions above on the coefficients. Let $\mathbb{L}$ be the quotient of $\tilde{\mathbb{L}}$ given by the relations on the $A_{i,j}$ required by the conditions from Definition 3.1.1, and let $a_{i,j}$ be the image in this ring of $A_{i,j}$. For instance, Condition 1 tells us that $a_{i,0} = a_{0,i} = \delta_{i,1}$, and Condition 2 tells us that $a_{i,j} = a_{j,i}$ for all $i, j \geq 0$. The effect of Condition 3 is more complicated, but can be computed for any given coefficients. For instance, equating the coefficients of X^2YZ on both sides of Condition 3 results in the equation $a_{11}a_{21} = a_{22}$.

Formal group laws correspond to oriented cohomology theories in a precise sense. Let k be a field of characteristic 0, and let $\mathbf{Sm}_k$ be the category of schemes over Spec_k that are smooth, separated, and quasi-projective. Let A^* be an oriented cohomology theory over $\mathbf{Sm}_k$, and let c_1 be the associated first Chern class (see [56] for precise definitions). Given $X \in \mathbf{Sm}_k$ and a line bundle $L \longrightarrow X$, $c_1(L)$ is an element of $A^2(X)$. Then the tensor product of line bundles induces a formal group law on cohomology.

Proposition 3.1.2 (Quillen, Levine-Morel)**.** *Let $X \in \mathbf{Sm}_k$. Then for any oriented cohomology theory A^* on $\mathbf{Sm}_k$, there exists a unique formal group law $(A^*(k), F_A)$, which determines A^*, such that for any line bundles L, M over X,*

$$c_1(L \otimes M) = F_A(c_1(L), c_1(M)). \tag{3.1}$$

In particular, the coeffieicent $a_{i,j}$ is an element of $A^{2-2i-2j}(k)$.

(3.1) can be represented by the commutativity of the following diagram. Let $X \in \mathbf{Sm}_k$, and let $\mathcal{L}$ be the set of line bundles over X.

$$\begin{array}{ccc} \mathcal{L} \times \mathcal{L} & \xrightarrow{\otimes} & \mathcal{L} \\ \downarrow{\scriptstyle c_1} & & \downarrow{\scriptstyle c_1} \\ A^*(X) \otimes_{A^*(k)} A^*(X) & \xrightarrow{F_A} & A^*(X). \end{array}$$

Theorem 3.1.3 (Quillen, Levine-Morel)**.** *There is a universal oriented cohomology theory, Ω^*, called algebraic cobordism. The associated formal group law under Proposition 3.1.2 is the universal formal group law $(\mathbb{L}, F_{\mathbb{L}})$.*

Since we want to view a formal group law F as a binary operator on the cohomology ring, we will often write $X \oplus Y = X \oplus_F Y := F(X, Y)$.

We conclude this section by listing several of the most important cases.

- Ordinary cohomology H^* is associated to the additive formal group law $X \oplus_{F_a} Y = X + Y$.
- K-theory K^* is associated to the multiplicative formal group law $X \oplus_{F_m} Y = X + Y - XY$.
- Connective K-theory βK^* is associated to the multiplicative formal group law $X \oplus_{F_{conn}} Y = X + Y + \beta XY$. Notice that when $\beta = 0$, this becomes the FGL for ordinary cohomology, and when $\beta = -1$ it becomes the FGL for K-theory.
- Elliptic cohomology E^* is associated to the hyperbolic formal group law $X \oplus_{F_{hyp}} Y = \frac{X+Y+\beta XY}{1+\alpha XY}$. When $\alpha = 0$, the becomes the FGL for connective K-theory.
- As mentioned above, algebraic cobordism Ω^* is associated to the universal formal group law $X \oplus_{F_{\mathbb{L}}} Y = F_{\mathbb{L}}(X, Y)$.

3.2 Equivariant cohomology of Schubert varieties

Given a group G, suppose there exists a contractible topological space EG, such that G acts (continuously and) freely on EG. Then for any topological space M with a continuous G-action, the digonal action of G on $EG \times M$ is free. Define the *G-equivariant (higher) cohomology* $A^*_G(M)$ by

$$A^*_G(M) := A^*((EG \times M)/G).$$

Let $BG := EG/G$. Applying the previous definition to the case where M is a point, we see that

$$A^*_G(pt) = A^*(BG).$$

Let $G = GL_r(\mathbb{C})$, and let $B := B^+$ (resp. B^-) be the Borel subgroup of upper (resp. lower) triangular matrices. Then $X := G/B$ is the (type A, complex) *flag variety.* G has a Bruhat decomposition

$$G = \bigsqcup_{w \in S_n} B^- w B.$$

Definition 3.2.1. The Schubert variety X^w corresponding to $w \in S_n$ is the Zariski closure of B^-wB in G/B.

Note that this definition is often called the *opposite Schubert variety.* With this convetion, if w_0 is the longest element in S_n, then X_{w_0} is a point, and X_{id} is the whole flag variety G/B. Given any subvariety $Y \subset X$, there is a fundamental class $[Y] \in H_T^*(X)$ of Y in the T-equivariant cohomology ring of X. It is well-known that the Schubert classes $[X_w]$ form a basis of $H_T^*(X)$.

As Schubert varieties are often singular, in general it is not straightforward to define higher cohomology classes for Schubert varieties. An important approach is to instead consider *Bott-Samelson varieties*, $Z_{\underline{w}}$, which are nonsingular projective varieties with canonical B-equivariant morphisms to the corresponding Schubert varieties. However, Bott-Samelson varieties are associated to each *reduced expression* $\underline{w}$ of an element $w \in S$, rather than to w itself.

For more on Bott-Samelson varieties and related geometry, see [57]. We move instead to the Bott-Samelson classes. The equivariant Bott-Samelson higher cohomology classes can be recursively defined as follows. This definition appears in e.g. [52, Definition 4.6].

Definition 3.2.2. The Bott-Samelson classes $\mathcal{R}_{\underline{w}}^A$ are defined reduced expressions $\underline{w}$ of elements $w \in S_n$.

$$\mathcal{R}_\emptyset = \prod_{i+j \leq n+1} X_i \oplus Y_j. \tag{3.2}$$

If $\underline{w}$ is a reduced expression for w and $ws_i > w$, then if F is the FGL corresponding to the cohomology theory A,

$$\mathcal{R}_{\underline{w}s_i}^A = \partial_i^{(F)} \mathcal{R}_{\underline{w}}^A \qquad \text{where} \qquad \partial_i^{(F)} = (1 + s_i) \frac{1}{X_i \ominus_F X_{i+1}}. \tag{3.3}$$

The operator $\partial^{(F)}$ is called a *push-pull* operator. It arises from the map $G/B \longrightarrow G/P_i$ where P_i is the minimal parabolic subgroup corresponding to the simple root α_i. In the case where these operators braid, the Bott-Samelson classes are independent of reduced word and we have

$$\mathcal{R}_{\underline{w}} = [Z_{w_0\underline{w}}]_A = [S_{w_0 w}]_A. \tag{3.4}$$

3.3 Formal group law Yang-Baxter Equation

Let us assume that we have an *additive* Yang-Baxter equation (YBE):

$$R_{12}(u-v)R_{13}(u-w)R_{23}(v-w) = R_{23}(v-w)R_{13}(u-w)R_{12}(u-v) \tag{3.5}$$

for all $u, v, w \in A$. Here, $R(u)$ is just a matrix-valued function of u. Note that we can easily turn this into a multiplicative YBE. If $R'(e^u) = R(u)$, and $x = e^u, y = e^v, z = e^w$, the above equation becomes

$$R'_{12}(x/y)R'_{13}(x/z)R'_{23}(y/z) = R'_{23}(y/z)R'_{13}(x/z)R'_{12}(x/y). \tag{3.6}$$

See [58, Remark 12.5.B.2].

Now, the operation $\oplus_F$ has an inverse, $\ominus_F$, such that $(X \oplus Y) \ominus Y = X \oplus (Y \ominus Y) = X$ for all $X, Y \in A$.

Proposition 3.3.1.

a. *[59, Proposition IV.4.2] There exists a unique* normailized invariant differential *ω for F, given by $\omega = F_X(0,T)^{-1}dT$, where F_X indicates the derivative in the first component.*

b. *[59, Proposition IV.5.2] If we set $\log_F(T) := \int \omega$, then*

$$\log_F(X \oplus_F Y) = \log_F(X) \oplus_{F_a} \log_F(Y) = \log_F(X) + \log_F(Y). \tag{3.7}$$

Corollary 3.3.2. *Set $R^{(F)}(T) := R(\log_F(T))$ (don't confuse with the notation for a Drinfeld twist). Then,*

$$\begin{aligned} &R^{(F)}_{12}(X_j \ominus_F X_i)R^{(F)}_{13}(X_j \oplus_F Y)R^{(F)}_{23}(X_i \oplus_F Y) \\ &\qquad = R^{(F)}_{23}(X_i \oplus_F Y)R^{(F)}_{13}(X_j \oplus_F Y)R^{(F)}_{12}(X_j \ominus_F X_i). \end{aligned}$$

Proof. Set $u - v = \log_F(X_j \ominus_F X_j), v - w = \log_F(X_i \oplus_F Y)$ in (3.5). Then by (3.7),

$$\begin{aligned} u - w &= (u-v) + (v-w) \\ &= \log_F(X_j \ominus_F X_j) + \log_F(X_i \oplus_F Y) \\ &= \log_F(X_j \ominus_F X_i \oplus_F X_i \oplus_F Y) \\ &= \log_F(X_j \oplus Y). \end{aligned}$$

∎

Example 3.3.3. Let $F = F_{conn}$, $X \oplus_F Y = X + Y + \beta XY$, so $X \ominus_F Y = \frac{X-Y}{1+\beta Y}$. Then $F_X = 1 + \beta Y$, so $\omega = (1+\beta T)^{-1} dT$. Integrating, we have

$$\log_F(T) = \int \frac{1}{1+\beta T} dT = \sum_{n \geq 1} \frac{(-1)^{n-1}\beta^n T^n}{n} = \log(1+\beta T).$$

This means that $R^{(F)}(T) := R(\log(1+\beta T)) = R'(1+\beta T)$. In this context, Corollary 3.3.2 gives:

$$\begin{aligned} &R'_{12}(1+\beta(X_j \ominus_F X_i))R'_{13}(1+\beta(X_j \oplus_F Y)))R'_{23}(1+\beta(X_i \oplus_F Y)) \\ &\qquad = R'_{23}(1+\beta(X_i \oplus_F Y))R'_{13}(1+\beta(X_j \oplus_F Y)))R'_{12}(1+\beta(X_j \ominus_F X_i)), \end{aligned}$$

Since

$$1+\beta(X+Y+\beta XY) = (1+\beta X)(1+\beta Y) \qquad \text{and} \qquad 1+\beta \cdot \frac{X-Y}{1+\beta Y} = \frac{1+\beta X}{1+\beta Y},$$

this is equivalent to:

$$\begin{aligned} &R'_{12}\left(\frac{1+\beta X_j}{1+\beta X_i}\right)R'_{13}((1+\beta X_j)(1+\beta Y))R'_{23}((1+\beta X_i)(1+\beta Y)) \\ &\qquad = R'_{23}((1+\beta X_i)(1+\beta Y))R'_{13}((1+\beta X_j)(1+\beta Y))R'_{12}\left(\frac{1+\beta X_j}{1+\beta X_i}\right). \end{aligned}$$

If $R(u)$ is the standard $U_q(\widehat{\mathfrak{sl}}_{n+1})$ evaluation module, this is the same YBE as in [4, Theorem 4.1].

Example 3.3.4. Now let $F = F_{hyp}$ denote the hyperbolic formal group law:

$$X \oplus_F Y = \frac{X+Y+\beta XY}{1+\alpha XY}, \qquad X \ominus_F Y = \frac{X-Y}{1+\beta Y - \alpha XY}.$$

Then

$$F_X = \frac{(1+\beta Y)(1+\alpha XY) - (X+Y+\beta XY)\alpha Y}{(1+\alpha XY)^2},$$

so

$$\omega = F_X(0,T)^{-1} dT = (1+\beta T - \alpha T^2)^{-1} dT.$$

Let γ and δ be the roots of $T^2 - \beta/\alpha \cdot T - 1/\alpha$. Then

$$\omega = -\frac{1}{\alpha}\frac{1}{(T-\gamma)(T-\delta)} dT = \frac{1}{\alpha(\delta-\gamma)}\left(\frac{1}{(T-\gamma)} - \frac{1}{(T-\delta)}\right) dT,$$

so

$$\log_F(T) = \int \omega = \frac{1}{\alpha(\delta-\gamma)} \left(\log(1-T\gamma^{-1}) - \log(1-T\delta^{-1})\right),$$

and

$$R^{(F)}(T) = R'(\exp\circ\log_F(T)) = R'\left(\frac{(1-T\gamma^{-1})^\eta}{(1-T\delta^{-1})^\eta}\right) = R'\left(\left(\frac{\delta(T-\gamma)}{\gamma(T-\delta)}\right)^\eta\right),$$

where $\eta = \alpha^{-1}(\delta-\gamma)^{-1}$. In this setting, Corollary 3.3.2 becomes

$$R'_{12}\left(\left(\frac{\delta(X_j \ominus_F X_i - \gamma)}{\gamma(X_j \ominus_F X_i - \delta)}\right)^\eta\right) R'_{13}\left(\left(\frac{\delta(X_j \oplus_F Y - \gamma)}{\gamma(X_j \oplus_F Y - \delta)}\right)^\eta\right) R'_{23}\left(\left(\frac{\delta(X_i \oplus_F Y - \gamma)}{\gamma(X_i \oplus_F Y - \delta)}\right)^\eta\right)$$
$$= R'_{23}\left(\left(\frac{\delta(X_i \oplus_F Y - \gamma)}{\gamma(X_i \oplus_F Y - \delta)}\right)^\eta\right) R'_{13}\left(\left(\frac{\delta(X_j \oplus_F Y - \gamma)}{\gamma(X_j \oplus_F Y - \delta)}\right)^\eta\right) R'_{12}\left(\left(\frac{\delta(X_j \ominus_F X_i - \gamma)}{\gamma(X_j \ominus_F X_i - \delta)}\right)^\eta\right).$$

Note that these manipulations amount to clever substitutions. For instance, the second example also follows from (3.6) and the fact that

$$\left(\frac{\delta(X_j \ominus_F X_i - \gamma)}{\gamma(X_j \ominus_F X_i - \delta)}\right)^\eta \cdot \left(\frac{\delta(X_i \oplus_F Y - \gamma)}{\gamma(X_i \oplus_F Y - \delta)}\right)^\eta = \left(\frac{\delta(X_j \oplus_F Y - \gamma)}{\gamma(X_j \oplus_F Y - \delta)}\right)^\eta.$$

However, without the machinery of the formal group law, it is a much longer computation to show this, and would be even harder to notice.

3.4 R-matrices

Let us now apply our results to the R-matrix from the standard evaluation module for $U_q(\widehat{\mathfrak{sl}}_{n+1})$:

$$R(z) = \begin{bmatrix} 1-q^2z & 0 & 0 & 0 \\ 0 & (z-1)q & 1-q^2 & 0 \\ 0 & (1-q^2)z & (z-1)q & 0 \\ 0 & 0 & 0 & 1-q^2z \end{bmatrix}. \tag{3.8}$$

The interpretation of this matrix as the Boltzmann weights for a lattice model is given in [4, Section 3].

For F a formal group law, we have

$$R^{(F)}(T) = \begin{bmatrix} 1-q^2f(T) & 0 & 0 & 0 \\ 0 & (f(T)-1)q & 1-q^2 & 0 \\ 0 & (1-q^2)f(T) & (f(T)-1)q & 0 \\ 0 & 0 & 0 & 1-q^2f(T) \end{bmatrix}, \tag{3.9}$$

where $f(T) := f_F(T) = \exp(\log_F(T))$. Note that by (3.7), $f(X \oplus_F Y) = f(X)f(Y)$ and $f(X \ominus_F Y) = f(X)/f(Y)$.

Fix $X_i, X_j, Y \in A$. Call the vertices arising from the matrices $R^{(F)}(X_j \oplus_F Y)$ and $R^{(F)}(X_i \oplus_F Y)$ *rectangular* vertices, and vertices arising from the matrix $R^{(F)}(X_j \ominus_F X_i)$ *diagonal* vertices. We have:

$$R^{(F)}(X \oplus Y) = \begin{bmatrix} 1 - q^2 f(X \oplus Y) & 0 & 0 & 0 \\ 0 & (f(X \oplus Y) - 1)q & 1 - q^2 & 0 \\ 0 & (1 - q^2)f(X \oplus Y) & (f(X \oplus Y) - 1)q & 0 \\ 0 & 0 & 0 & 1 - q^2 f(X \oplus Y) \end{bmatrix}, \quad (3.10)$$

$$R^{(F)}(X_j \ominus X_i) = \begin{bmatrix} f(X_i) - q^2 f(X_j) & 0 & 0 & 0 \\ 0 & (f(X_j) - f(X_i))q & (1 - q^2)f(X_i) & 0 \\ 0 & (1 - q^2)f(X_j) & (f(X_j) - f(X_i))q & 0 \\ 0 & 0 & 0 & f(X_i) - q^2 f(X_j) \end{bmatrix}, \quad (3.11)$$

where we have globally scaled the weights for the diagonal vertices for convenience, and dropped the subscripts on the formal group operations for compactness.

Next, we perform a Drinfeld-Reshetikhin twist [41] by the matrix

$$\tau := \tau(T) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \left(\frac{T}{q(f(T)-1)}\right)^{-1/2} & 0 & 0 \\ 0 & 0 & \left(\frac{T}{q(f(T)-1)}\right)^{1/2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (3.12)$$

Often, a matrix for a Drinfeld twist is called F, but we have used that letter for the formal group law, so we use τ here. Then define

$$R^{(F,\tau)} := \tau_{21} R^{(F)} \tau^{-1}.$$

We have:

$$R^{(F,\tau)}(X \oplus Y) = \begin{bmatrix} 1 - q^2 f(X \oplus Y) & 0 & 0 & 0 \\ 0 & X \oplus Y & 1 - q^2 & 0 \\ 0 & (1 - q^2)f(X \oplus Y) & \frac{(f(X \oplus Y) - 1)^2 q^2}{X \oplus Y} & 0 \\ 0 & 0 & 0 & 1 - q^2 f(X \oplus Y) \end{bmatrix}, \quad (3.13)$$

$$R^{(F,\tau)}(X_j \ominus X_i) = \begin{bmatrix} f(X_i) - q^2 f(X_j) & 0 & 0 & 0 \\ 0 & (X_j \ominus X_i) f(X_i) & (1-q^2) f(X_i) & 0 \\ 0 & (1-q^2) f(X_j) & \frac{(f(X_j)-f(X_i))^2 q^2}{(X_j \ominus X_i) f(X_i)} & 0 \\ 0 & 0 & 0 & f(X_i) - q^2 f(X_j) \end{bmatrix}, \quad (3.14)$$

Finally, set $q = 0$, and we obtain:

$$\tilde{R}^{(F,\tau)}(X \oplus Y) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & X \oplus Y & 1 & 0 \\ 0 & f(X \oplus Y) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (3.15)$$

$$\tilde{R}^{(F,\tau)}(X_j \ominus X_i) = \begin{bmatrix} f(X_i) & 0 & 0 & 0 \\ 0 & (X_j \ominus X_i) f(X_i) & f(X_i) & 0 \\ 0 & f(X_j) & 0 & 0 \\ 0 & 0 & 0 & f(X_i) \end{bmatrix}, \quad (3.16)$$

We continue our examples from above:

Example 3.4.1 (Continuation of Example 3.3.3)**.** We come back to the FGL $F = F_{conn}$. Here, $f(T) = 1 + \beta T$, $f(X \oplus Y) = 1 + \beta(X \oplus Y) = (1 + \beta X)(1 + \beta Y)$, and $f(X_j \ominus X_i) = 1 + \beta(X_j \ominus X_i) = \frac{1+\beta X_j}{1+\beta X_i}$ so

$$\tilde{R}^{(F,\tau)}(X \oplus Y) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & X \oplus Y & 1 & 0 \\ 0 & (1 + \beta(X \oplus Y)) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$R^{(F)}(X_j \ominus X_i) = \begin{bmatrix} 1 + \beta X_i & 0 & 0 & 0 \\ 0 & X_j - X_i & 1 + \beta X_i & 0 \\ 0 & 1 + \beta X_j & 0 & 0 \\ 0 & 0 & 0 & 1 + \beta X_i \end{bmatrix},$$

which match the ($q = 0$) rectangular and diagonal weights in [4].

Example 3.4.2 (Continuation of Example 3.3.4)**.** Here,

$$f(T) = \left(\frac{1 - \gamma^{-1} T}{1 - \delta^{-1} T} \right)^{\eta},$$

$$f(X \oplus Y) = \left(\frac{1-\gamma^{-1}(X \oplus Y)}{1-\delta^{-1}(X \oplus Y)}\right)^{\eta} = \left(\frac{1+\alpha XY - \gamma^{-1}(X+Y+\beta XY)}{1+\alpha XY - \delta^{-1}(X+Y+\beta XY)}\right)^{\eta},$$

and

$$f(X_j \ominus X_i) = \left(\frac{1-\gamma^{-1}(X_j \ominus X_i)}{1-\delta^{-1}(X_j \ominus X_i)}\right)^{\eta} = \left(\frac{1+\beta X_j - \alpha X_j X_i - \gamma^{-1}(X_j - X_i)}{1+\beta X_j - \alpha X_j X_i - \delta^{-1}(X_j - X_i)}\right)^{\eta}$$

so

$$R^{(F,\tau)}(X \oplus Y) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & X \oplus Y & 1 & 0 \\ 0 & \left(\frac{1-\gamma^{-1}(X \oplus Y)}{1-\delta^{-1}(X \oplus Y)}\right)^{\eta} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$R^{(F)}(X_j \ominus X_i) = \begin{bmatrix} \left(\frac{1-\gamma^{-1}X_i}{1-\delta^{-1}X_i}\right)^{\eta} & 0 & 0 & 0 \\ 0 & (X_j \ominus X_i)\left(\frac{1-\gamma^{-1}X_i}{1-\delta^{-1}X_i}\right)^{\eta} & \left(\frac{1-\gamma^{-1}X_i}{1-\delta^{-1}X_i}\right)^{\eta} & 0 \\ 0 & \left(\frac{1-\gamma^{-1}X_j}{1-\delta^{-1}X_j}\right)^{\eta} & 0 & 0 \\ 0 & 0 & 0 & \left(\frac{1-\gamma^{-1}X_i}{1-\delta^{-1}X_i}\right)^{\eta} \end{bmatrix}.$$

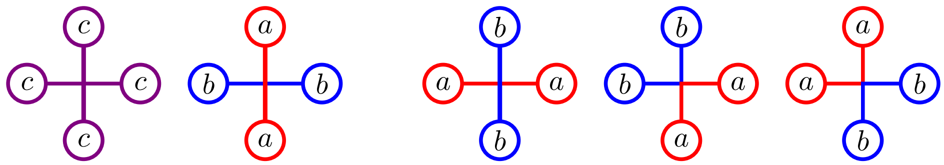
3.5 The partition function

We continue to use the R-matrix $R(z)$ from the standard evaluation module for $U_q(\widehat{\mathfrak{sl}}_{n+1})$. The associated lattice models make use of *color*. The allowed labels on both horizontal and vertical edges are $\{c_1, \ldots, c_n, +\}$, where $c_1, \ldots, c_n$ are a pallette of n "colors". We place the total ordering $c_1 < c_2 < \ldots < c_n$ on the colors, and consider $+$ to be the largest color. The rectangular Boltzmann weights in row i and column k are given by the entries of $\tilde{R}^{(F,\tau)}(X_i \oplus Y_K)$ in (3.13), and can be seen in Figure 3.1. By the constructions in the previous section, this lattice model is solvable. The diagonal weights corresponding to rows i and j are given by the entries of (3.14), and can be seen in Figure 3.2.

Now, choose a positive integer n. Let $w \in S_n$, the symmetric group on n letters, and let $\mathbf{X} = (X_1, \ldots, X_n), \mathbf{Y} = (Y_1, \ldots, Y_n)$. Let $\mathfrak{S}_w(\mathbf{X}, \mathbf{Y}) := \mathfrak{S}_w^{(F)}(\mathbf{X}, \mathbf{Y})$ be a lattice model system with n rows and n columns. X_i is the row parameter for row i and Y_j is the column parameter for column j. The top boundary of $\mathfrak{S}_w$ is comprised of colors increasing 1 to n, ordered from left to right; the left boundary contains colors from $w(1)$ to $w(n)$, ordered from top to bottom; and the other two boundaries are all $+$. Note that these are the same boundary conditions as $\mathfrak{S}_{1,w}$ from [4]. When $w = w_0$, the longest element in S_n, one can check that there is a unique state, and we have

$$Z(\mathfrak{S}_{w_0}) = \prod_{i+j \leq n} X_i \oplus Y_j. \tag{3.17}$$

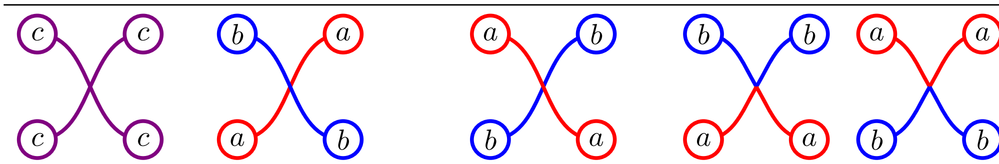
Rectangular vertex weights:



a	b_1	b_2	c_1	c_2
1	$X_i \oplus Y_j$	0	$f(X_i \oplus Y_j)$	1

Figure 3.1: The Boltzmann weights at a vertex in row i and column j, where $\oplus = \oplus_F$ denotes the operation for the formal group law F, $a < b$, and c is any color. We consider the $+$ label to be larger than any color, and the same weights hold when one or more labels are $+$.

Diagonal vertex weights:



a	b_1	b_2	c_1	c_2
$f(X_i)$	$(X_j \ominus X_i)f(X_i)$	0	$f(X_j)$	$f(X_i)$

Figure 3.2: Boltzmann weights for the diagonal vertex with strands labelled i and j, where $a < b$ and c is any color. Again, the same weights hold when one or more labels is $+$.

See Figure 3.3.

By applying the standard "train argument" (see [4, Section 5]), these partition functions have a recursive definition via Demazure-like operators. If s_i is a simple reflection and $ws_i < w$, then

$$Z(\mathfrak{S}_{ws_i}(\mathbf{X}, \mathbf{Y})) = \pi_i^{(F,\tau)} Z(\mathfrak{S}_{ws_i}(\mathbf{X}, \mathbf{Y})),$$

where

$$\begin{aligned}\pi_i^{(F,\tau)} &= \frac{1}{-(X_{i+1} \ominus X_i)f(X_i)}(1 - s_i)f(X_{i+1}) \\ &= \frac{-f(X_{i+1} \ominus X_i)}{X_{i+1} \ominus X_i} + s_i \frac{1}{X_i \ominus X_{i+1}} \\ &= \frac{1}{X_{i+1} \ominus X_i}(s_i - f(X_{i+1} \ominus X_i)).\end{aligned}$$

Example 3.5.1 (Continuation of Example 3.3.3)**.** Substituting $f(T) = 1 + \beta T$ and $X_i \ominus$

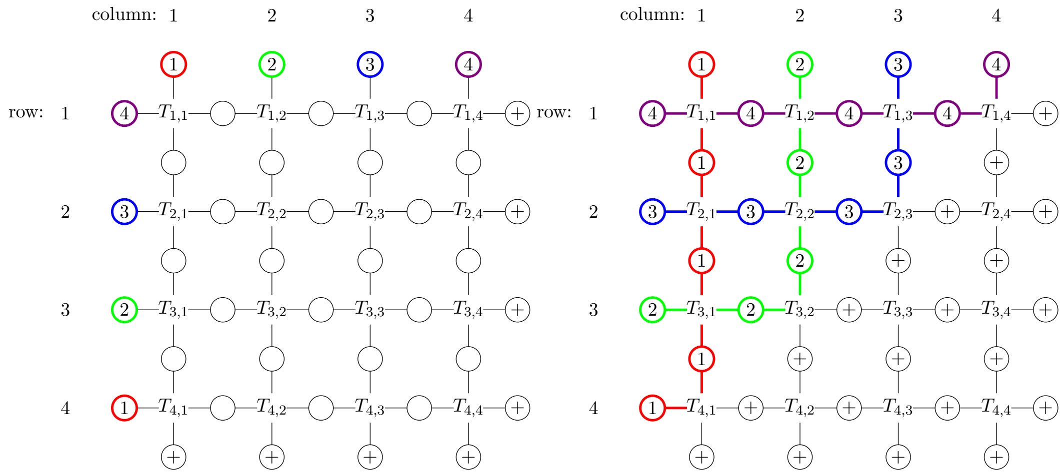


Figure 3.3: Left: boundary conditions for the lattice model system $\mathfrak{S}_{w_0}^{(F)}$ where w_0 is the longest element of S_4. Right: the sole admissible state of this model.

$X_j = \frac{X_i - X_j}{1+\beta X_j}$ into $\pi_i^{(F,\tau)}$ gives

$$\pi_i^{(F,\tau)} = \frac{1}{X_i - X_{i+1}}(1-s_i)(1+\beta X_{i+1}) = (1+s_i)\frac{(1+\beta X_{i+1})}{X_i - X_{i+1}} = (1+s_i)\frac{1}{X_i \ominus X_{i+1}},$$

which equals the push-pull operator for connective K-theory.

Example 3.5.2 (Continuation of Example 3.3.4)**.** Substituting

$$f(T) = \left(\frac{1-\gamma^{-1}T}{1-\delta^{-1}T}\right)^{\eta}, \qquad X_i \ominus X_j = \frac{X_i - X_j}{1+\beta X_j - \alpha X_i X_j}$$

into $\pi_i^{(F,\tau)}$ gives

$$\pi_i^{(F,\tau)} = \frac{1+\beta X_i - \alpha X_i X_{i+1}}{(X_i - X_{i+1})f(X_i)}(1-s_i)f(X_{i+1}).$$

It is unclear whether this expression may be simplified.

3.6 Remarks

We conclude this chapter several remarks about the construction discussed above. The inspiration for the construction was the lattice model in Frozen Pipes [4], written with Ben Brubaker, Claire, Frechette, Emily Tibor, and Katy Weber, where we substitute the connective K-theory FGL into the standard $U_q(\widehat{\mathfrak{sl}}_{n+1})$ R-matrix. Before the substitution, the partition functions are representatives for Schubert classes in equivariant (ordinary) cohomology, and afterwards they are representatives for Schubert classes in equivariant connective K-theory.

One might ask whether the polynomials we get here are representatives for Schubert or Bott-Samelson classes in higher cohomology theories. This is a complicated question. In the case of connective K-theory (and thus ordinary cohomology and K-theory), the Bott-Samelson classes and Schubert classes coincide. In addition, the push-pull operators used to compute these classes braid. For this case, having the lattice model partition function equal the correct polynomials comes down to having the correct "seed" to match the w_0 polynomial, while also obeying a recursive relation given by the push-pull operators.

For higher cohomology theories, the situation is substantially more difficult. For these cohomology theories, the Schubert and Bott-Samelson classes do not usually agree, and in fact the push-pull operators for Bott-Samelson classes do not braid, meaning that they are not independent of their reduced word (and therefore *cannot* equal the Schubert classes). Worse, when a Schubert variety X_w is not smooth, it is unclear how to define its class in higher cohomology at all. See [51] and the references of that paper for more discussion of this problem.

Since the Demazure-like operators arising from a Yang-Baxter equation always braid, we cannot hope that our partition functions equal the Bott-Samelson classes for higher cohomology. But they have the potential to be related to the (yet undefined) Schubert classes. To obtain the Bott-Samelson classes, one needs to break the symmetry that leads to the braid relations. One potential avenue would be instead of performing a Drinfeld-Reshetikhin twist in (3.12), to perform a Drinfeld twist that does not satisfy Reshetikhin's axioms (most of Drinfeld's relevant papers are in Russian; here is one in English [60]). With Reshetikhin's axioms, a Drinfeld twist preserves coassociators, and so if we start with an R-matrix for a quasitriangular Hopf algebra (which is coassociative i.e. trivial coassociator), we obtain another R-matrix for a quasitriangular Hopf algebra. However, without Reshetikhin's axioms, a Drinfeld twist does not preserve coassociators, and in general we obtain an R-matrix only for a quasitriangular *quasi*-Hopf algebra (which lacks coassociativity). These R-matrices satisfy a modified YBE, and so any Demazure-like operators obtained from such a construction might not braid, which seems to be a promising setting for a situation in which we wish not to preserve the braid relation.

It is also worth comparing our construction here to the well-known rational-trigonometric-elliptic hierarachy for solutions to the Yang-Baxter equation. See [61]. Both this construction and that theory give a family of YBE solutions associated to different cohomology theories. In fact, the relationship between rational and trigonometric solutions is precisely the same as our relationship between YBEs for ordinary cohomology and K-theory. However, in the case of elliptic cohomology they are not obviously the same. Jimbo, Konno, Odake, and Shiraishi [62], extending insights of Frønsdal [63], found Drinfeld twists of quantum groups where the resulting quasi-Hopf algebra is an elliptic algebra, and the resulting modified YBE is the "dynamical YBE" [62, 1.7]. The elliptic quantum group has been shown to relate to the equivariant elliptic cohomology of cotangent bundles of Grassmannians [64]. Our construction has the advantages of being much more straightforward (a simple substitution) and applicable to all higher cohomology theories; it would be interesting to see if it has cohomological interpretations.

Finally, although this construction makes sense for any parametrized (additive or multiplicative) Yang-Baxter equation, it is unclear whether in every case it is cohomologically useful, even in connective K-theory. More precisely, given a solvable lattice model whose partition function has an interpretation in equivariant ordinary cohomology, does substituting the connective K-theory FGL always give the analogous construction in equivariant connective K-theory? The answer is likely "no", although this is an interesting question to explore. In work-in-progress with Ben Brubaker and Dan Bump, we have constructed a solvable lattice model whose partition function is an equivariant analogue of a Demazure character. In this case, in contrast to [4], the weights do not depend only on the difference between the row and column parameters (in common parlance, the weights come from an L-matrix, not an R-matrix). There is a way to substitute the FGL for connective K-theory into the weights of this lattice model such that it is still solvable, and we believe the resulting partition function will have relevance in connective K-theory. However, the substitution is not of the form discussed here, and so there may be versions of this construction for the RLL relation.

Chapter 4

Lattice Models for Schur Polynomials

In this final chapter, we consider lattice models for Schur polynomials. Schur polynomials have already entered into this thesis in key ways: they are Schubert polynomials for Grassmannian permutations and they are also supersymmetric Schur polynomials with second parameter set zero.

There is a vast literature of solvable lattice models designed to study symmetric functions, and a very large number of these lattice models produce Schur polynomials as a special case. The same is true for many other constructions in combinatorial representation theory, such as the Heisenberg Hamiltonians from Chapter 2. Therefore, we study Schur polynomials as a "shadow" of more complicated constructions, as well as for their own sake.

The purpose of this chapter is two-fold. The first purpose is to give an exposition of lattice model proofs of several important identities, such as symmetry, branching rules, and the dual Cauchy identity. These results are classical and appear in some form in many places; however, it is our goal to provide a contained exposition. See [2, 12, 45] and their references for (often more general) versions of this story.

The second purpose is to introduce a new algebraic structure, related to the *Yang-Baxter algebra*. That algebra is generated by (coefficients of) the one-row transfer matrix for a given lattice model, with different boundary conditions. Product and coproduct structures are defined in terms of lattice model manipulations, and the resulting algebra is a quasitriangular Hopf algebra that in certain contexts is closely related to Yangians and quantum groups.

Our algebra is similar, but augmented by a group action on the one-row transfer-matrices. This action arises from simple lattice model mainpulations related to symmetries of the eight-vertex model: horizontal and vertical reflections and the interchanging of +

and $-$- spins. Therefore, we call our algbera the *symmetrized Yang-Baxter algebra*. We use this tool to define a plethora of lattice models whose partition functions are (closely related to) Schur functions, and also to prove symmetry and dual Cauchy identity. One purpose of this algebra is to study Cauchy-type identities, since these identities often result from the relationship of a lattice model with transformations of its Boltzmann weights.

A more detailed study of the symmetrized Yang-Baxter algebra could take several forms. First, one could study more closely the effect of the group action on the Yang-Baxter equation, and perhaps express the dual Cauchy identity as a relation in the algebra. Second, one could define similar algebras for other lattice models. Third, one could do a deeper study of our algebra as an object to itself. For instance, it has an obvious coproduct structure in the same way as the Yang-Baxter algebra, so it would be interesting to check whether the symmetrized Yang-Baxter algebra has a bialgebra and potentially a Hopf algebra structure. If so, what can be said about it as a "quantum object"?

We'll start by definiing the (skew) Schur polynomials, the main object of this chapter.

A *semistandard Young tableau* (SSYT) of shape λ/μ is a filling of λ/μ with entries in $\{1, \ldots, N\}$ such that each row is weakly increasing and each column is strictly increasing. For any SSYT T, let $c_i(T)$ be the number of boxes in T containing the number i. Then if $\mathbf{x} = (x_1, x_2, \ldots, x_N)$, we define $\mathbf{x}^T := x_1^{c_1(T)} \cdots x_N^{c_N(T)}$.

Then, the skew Schur polynomial $s_{\lambda/\mu}(\mathbf{x})$ is given by

$$s_{\lambda/\mu} = \sum_T \mathbf{x}^T, \tag{4.1}$$

where the sum is over all SSYT of shape λ/μ.

Setting $x_N = 0$ in (4.1) results in the following branching rule:

$$s_{\lambda/\mu}(x_1, \ldots, x_N) = \sum_\nu x_N^{|\lambda|-|\nu|} s_{\nu/\mu}(x_1, \ldots, x_{N-1}), \tag{4.2}$$

where the sum is over all $\mu \subset \nu \subset \lambda$ such that λ/ν is a horizontal strip. Equivalently, λ and ν must satisfy the *interleaving condition* $\lambda_1 \geq \nu_1 \geq \lambda_2 \geq \nu_2 \geq \ldots$.

4.1 Yang-Baxter algebras

For any partition λ, let ρ_λ be the staircase partition $\rho_\lambda = (\ell(\lambda)-1, \ell(\lambda)-2, \ldots, 1, 0)$, and let $\lambda^+ = \lambda+\rho_\lambda$, where the addition is done componentwise. Notice that λ^+ is always a strict partition, and every strict partition can be represented as λ^+ for a unique λ. Uniqueness in the other direction is not quite as straightforward. Adding trailing zeroes to λ changes λ^+ in a nontrivial way. However, for λ itself, this distinction is often meaningless. Often, we will require two strict partitions λ^+ and μ^+ to have the same (or closely related) number of parts, and we will let ourselves add the required number of trailing zeroes to λ or μ whenever needed. Let $\mathcal{P}$ be the set of partitions and $\mathcal{P}^+$ be the set of strict partitions. Let $\mathcal{P}^+_M$ be the set of strict partitions with all parts $\leq M$.

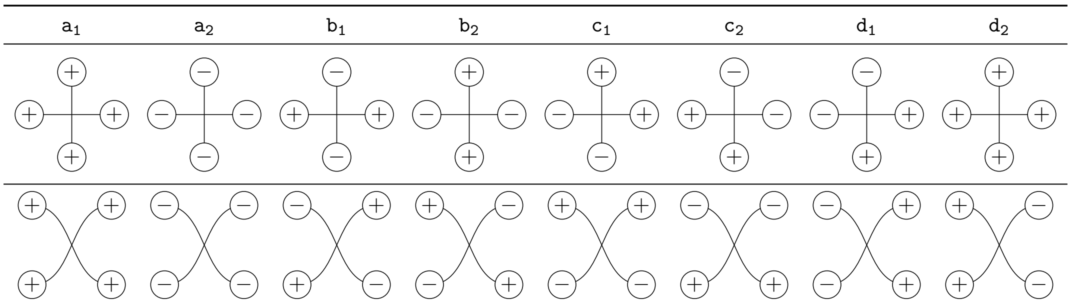


Figure 4.1: The admissible rectangular and diagonal vertices for the eight-vertex model. In every case here, either $\mathtt{c}_1$ and $\mathtt{c}_2$ or $\mathtt{d}_1$ and $\mathtt{d}_2$ will have weight zero, so our weights always live in one of two six-vertex models inside the eight-vertex model.

Let $\mathcal{W} := \mathcal{W}(x)$ denote the set of all sets of Boltzmann weights for the eight vertex model depending on a (row) parameter x (see Figure 4.1). We will eventually look only at five vertex models living inside complementary six-vertex models, but for now the generality is useful. Given $M \geq 0$, $S \in \mathcal{W}$ defines four different operators on $\mathbb{C}(x)[\mathcal{P}^+_M]$, as follows. Let $\lambda^+, \mu^+ \in \mathcal{P}^+_M$. Given $X = A, B, C, D$, let $\langle \lambda^+ | S_X(x) | \mu^+ \rangle$ be the partition function of the following one row lattice model:

- $M + 1$ columns, indexed $M, M-1, \ldots, 1, 0$ from left to right.
- Top boundary $-$ on parts of λ^+; $+$ otherwise.
- Bottom boundary $-$ on parts of μ^+; $+$ otherwise.
- Boltzmann weights given by S.

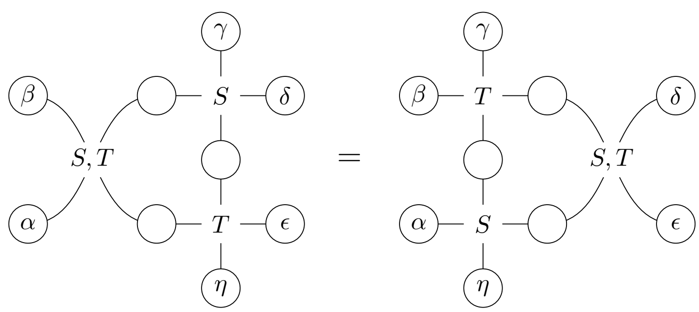
The side boundaries depend on X:

- If $X = A$, both side boundaries are $+$.
- If $X = B$, the left boundary is $+$ and the right is $-$.
- If $X = C$, the left boundary is $-$ and the right is $+$.
- If $X = D$, both side boundaries are $-$.

Applying these operators in sequence gives the partition function of a larger lattice model. Given $\mathbf{X} = (X_1, X_2, \ldots, X_N) \in \{A, B, C, D\}^N$ and parameters $\mathbf{x} = (x_1, \ldots, x_N)$, let $S_{\mathbf{X}}(\mathbf{x}) = S_{X_N}(x_N) \cdots S_{X_1}(x_1)$. Then, $\langle \lambda^+ | S_{\mathbf{X}}(\mathbf{x}) | \mu \rangle$ is the partition function of the following lattice model

- N rows, labelled $1, \ldots, N$ from bottom to top;
- $M+1$ columns, indexed $M, M-1, \ldots, 1, 0$ from left to right.
- Top boundary $-$ on parts of λ; $+$ otherwise.
- Bottom boundary $-$- on parts of μ; $+$ otherwise.
- Left and right boundaries of row i in accordance with X_i
- Boltzmann weights given by S.

Let $T \in \mathcal{W}$ be another set of 8-vertex Boltzmann weights. We say that S and T together satisfy the Yang-Baxter equation (often called the RLL-relation) if there exists a set of weights for the diagonal vertices in (YBE-diagram) such that for all possible choices of spins $\alpha, \beta, \gamma, \delta, \epsilon, \eta$, we have equality of partition functions:



. (4.3)

In this case, we call S and T a *Yang-Baxter pair*. We obtain commutation relations for the operators $S_X := S_X(x)$ and $T_Y := T_Y(y)$. For now, let $\mathtt{a}_1 := \mathtt{a}_1(S,T)$ be the corresponding diagonal vertex weight, and similarly for $\mathtt{a}_2, \mathtt{b}_1$, etc. Then we have the following commutation relations.

Proposition 4.1.1.

$$\mathtt{a}_1 S_A T_A + \mathtt{d}_2 S_C T_C = \mathtt{a}_1 T_A S_A + \mathtt{d}_1 T_B S_B, \tag{4.4}$$

$$\mathtt{a}_1 S_B T_A + \mathtt{d}_2 S_D T_C = \mathtt{b}_2 T_A S_B + \mathtt{c}_2 T_B S_A, \tag{4.5}$$

$$\mathtt{a}_1 S_A T_B + \mathtt{d}_2 S_C T_D = \mathtt{b}_1 T_B S_A + \mathtt{c}_1 T_A S_B, \tag{4.6}$$

$$\mathtt{b}_2 S_C T_A + \mathtt{c}_1 S_A T_C = \mathtt{a}_1 T_A S_C + \mathtt{d}_1 T_B S_D, \tag{4.7}$$

$$\mathtt{b}_1 S_A T_C + \mathtt{c}_2 S_C T_A = \mathtt{a}_1 T_C S_A + \mathtt{d}_1 T_D S_B, \tag{4.8}$$

$$\mathtt{a}_1 S_B T_B + \mathtt{d}_2 S_D T_D = \mathtt{a}_2 T_B S_B + \mathtt{d}_2 T_A S_A, \tag{4.9}$$

$$\mathtt{b}_2 S_D T_A + \mathtt{c}_1 S_B T_C = \mathtt{b}_2 T_A S_D + \mathtt{c}_2 T_B S_C, \tag{4.10}$$

$$\mathtt{b}_1 S_B T_C + \mathtt{c}_2 S_D T_A = \mathtt{b}_2 T_C S_B + \mathtt{c}_1 T_D S_A, \tag{4.11}$$

$$\mathtt{b}_2 S_C T_B + \mathtt{c}_1 S_A T_D = \mathtt{b}_1 T_B S_C + \mathtt{c}_1 T_A S_D, \tag{4.12}$$

$$\mathtt{b}_1 S_A T_D + \mathtt{c}_2 S_C T_B = \mathtt{b}_1 T_D S_A + \mathtt{c}_1 T_C S_B, \tag{4.13}$$

$$\mathtt{a}_2 S_C T_C + \mathtt{d}_1 S_A T_A = \mathtt{a}_1 T_C S_C + \mathtt{d}_1 T_D S_D, \tag{4.14}$$

$$\mathtt{b}_2 S_D T_B + \mathtt{c}_1 S_B T_D = \mathtt{a}_2 T_B S_D + \mathtt{d}_2 T_A S_C, \tag{4.15}$$

$$\mathtt{b}_1 S_B T_D + \mathtt{c}_2 S_D T_B = \mathtt{a}_2 T_D S_B + \mathtt{d}_2 T_C S_A, \tag{4.16}$$

$$\mathtt{a}_2 S_D T_C + \mathtt{d}_1 S_B T_A = \mathtt{b}_2 T_C S_D + \mathtt{c}_2 T_D S_C, \tag{4.17}$$

$$\mathtt{a}_2 S_C T_D + \mathtt{d}_1 S_A T_B = \mathtt{b}_1 T_D S_C + \mathtt{c}_1 T_C S_D, \tag{4.18}$$

$$\mathtt{a}_2 S_D T_D + \mathtt{d}_1 S_B T_B = \mathtt{a}_2 T_D S_D + \mathtt{d}_2 T_C S_C. \tag{4.19}$$

Proof. These relations are proved using the so-called *train argument*. See [2] for an example of this argument. ☐

Next, we introduce a group action on the Boltzmann weights. Let G be the group $G = \{H, V, R, C\}$, where H, V, R, C pairwise commute and have order 2. Therefore, $G \cong (\mathbb{Z}/2\mathbb{Z})^4$. Call an element of G *even* if it is a product of an even number of generators; otherwise call it *odd*. For S a set of Boltzmann weights and $g \in G$, let S^g be the set of Boltzmann weights obtained from the following transformations of S:

- H: reflect over a horizontal axis.,
- V: reflect over a vertical axis,
- R: swap $+$ and $-$ spins on horizontal edges,
- C: swap $+$ and $-$ spins on vertical edges.

These actions commute and have order 2, so we have defined a group action of G on S. This action respects solvability in the following way.

Proposition 4.1.2. *Let $S, T \in \mathcal{W}$ be a Yang-Baxter pair, $g \in G$. Then S^g and T^g are also a Yang-Baxter pair.*

Proof. The action of g on S and T induces an action on the Yang-Baxter diagrams in (4.3), and this action is bijective and weight-perserving on states. $\square$

Acting on S and T individually does not in general preserve solvability. However, it some special cases, it does.

Proposition 4.1.3. *Let $S, T \in \mathcal{W}$. Then S^g and T^h are a Yang-Baxter pair in the following cases:*

1. *S and T are* free fermionic *i.e.*

$$\begin{aligned}&\mathtt{a}_1(S)\mathtt{a}_2(S) + \mathtt{b}_1(S)\mathtt{b}_2(S) - \mathtt{c}_1(S)\mathtt{c}_2(S) - \mathtt{d}_1(S)\mathtt{d}_2(S)\\ &\qquad = \mathtt{a}_1(T)\mathtt{a}_2(T) + \mathtt{b}_1(T)\mathtt{b}_2(T) - \mathtt{c}_1(T)\mathtt{c}_2(T) - \mathtt{d}_1(T)\mathtt{d}_2(T) = 0.\end{aligned}$$

2. *S and T are a Yang-Baxter pair, are compatible six-vertex models (i.e. $\mathtt{d}_1(S) = \mathtt{d}_2(S) = \mathtt{d}_1(T) = \mathtt{d}_2(T) = 0$) and $gh^{-1} \in \{1, HV, HC, VC\}$.*

3. *S and T are Yang-Baxter pair, are compatible symmetric (i.e $\mathtt{a}_1 = \mathtt{a}_2$, etc.) six-vertex models and gh^{-1} is a product of an even number of generators.*

Proof. These statements follow from solvability results for the eight-vertex [65, Section IV] and six-vertex [2, Theorem 1] models. $\square$

In particular, all of the lattice models below are free fermionic five-vertex models, so they are pairwise solvable.

The G-action on the operators $S_X(x)$ is readily expressed in terms of the original operators. For a strict partition $\lambda^+ = (\lambda_1^+, \lambda_2^+, \ldots, \lambda_\ell^+)$, let

- $\overline{\lambda^+} = (M - \lambda_\ell^+, M - \lambda_{\ell-1}^+, \ldots, M - \lambda_1^+$ is the reverse of λ^+.
- $\widehat{\lambda^+} = (M, M-1, \ldots, 2, 1, 0) \setminus \lambda^+$ is the complement of λ^+,
- $\widetilde{\lambda^+} = \overline{\widehat{\lambda^+}}$ is the reverse complement of λ^+.

We abuse notation, and define these operations on $\mathcal{P}$ by $\overline{\lambda}^+ = \overline{\lambda^+}$, and similarly for the other operations. This shouldn't be confusing since whenever we want a partition to be in $\mathcal{P}^+$, we denote it with a superscript $+$. All of these operations have meaning when applied to $\lambda \in \mathcal{P}$. $\widetilde{\lambda} = \lambda'$, the conjugate of λ, $\overline{\lambda}$ is the complement of λ in a $\ell(\lambda) \times M + 1 - \ell(\lambda)$ rectangle, and $\widehat{\lambda}$ is the complement of λ' in a $M + 1 - \ell(\lambda) \times \ell(\lambda)$ rectangle.

The G-action affects the boundary conditions in the following ways:

- H swaps λ^+ and μ^+.
- V sends λ^+ to its reverse, which is the equivalent to sending λ to its complement in a $(\ell(\lambda^+) \times M + 1 - \ell(\lambda^+))$-rectangle, and also swaps boundary conditions B and C.
- R swaps boundary conditions A and D, as well as B and C.
- C sends λ^+ to its complement, which is the equivalent to sending λ to its complement in a $(M + 1 - \ell(\lambda^+) \times \ell(\lambda^+))$-rectangle.

Note that VC sends λ^+ to $\widetilde{\lambda}$, which sends λ to λ'. A full list of all $\langle\lambda^+|S_X^g(x)|\mu^+\rangle$ is given in Table 4.1.

We are now ready to define our main algebraic object, the symmetrized Yang-Baxter algebra. Let $\mathcal{S} \subset \mathcal{W}$. Assuming meromorphicity, define $S_{A_r}, S_{B_r}, S_{C_r}, S_{D_r}$ for all r greater than or equal to some nonpositive integer j by the equation

$$\langle\lambda^+|S_X(x)|\mu^+\rangle = \sum_{r\geq j}\langle\lambda^+|X_r|\mu^+\rangle x^r.$$

Then the *Yang-Baxter* algebra of $\mathcal{S}$ is the algebra $\tilde{\mathcal{A}}(\mathcal{S})$ generated by $\{S_{A_r}, S_{B_r}, S_{C_r}, S_{D_r}|S \in \mathcal{S}\}$, with relations given by Proposition 4.1.1 for each Yang-Baxter pair S and T. We further define, for every $\mathcal{S} \subset \mathcal{W}$, the *symmetrized Yang-Baxter algebra*:

$$\mathcal{A}(\mathcal{S}) := \tilde{\mathcal{A}}(\{S^g|S \in \mathcal{S}, g \in G\}).$$

This algebra is a graded G-module, with action $S_{X_r} \cdot g = S^g_{X_r}$.

4.2 Lattice models for Schur polynomials

Now we use the above machinery to study Schur polynomials. We begin with two five-vertex lattice models whose partition functions we will show are Schur polynomials. Let $\mathcal{S} = \{\Gamma, \Delta\}$, where these weights are given by Figure 4.2. See Figure 4.3 for a state of a lattice model involving the Δ weights. Let $S_A(\mathbf{x}) := S_A(x_N)\cdots S_A(x_1)$, and similarly for B, C, D.

Proposition 4.2.1. *Let $\lambda^+, \mu^+ \in \mathcal{P}_M^+$. Then,*

$$\langle\lambda^+|\Delta_A(x_1,\ldots,x_N)|\mu^+\rangle = \langle\lambda^+|\Gamma_B(x_1,\ldots,x_N)|\mu^+\rangle = s_{\lambda/\mu}.$$

Proof. By inspection of the admissible vertices, the one row partition function $\langle\lambda^+|\Delta_A(x)|\mu^+\rangle$ is nonzero if and only if $\ell(\lambda^+) = \ell(\mu^+)$ and $\lambda_1^+ \geq \mu_1^+ > \lambda_2^+ \geq \mu_2^+ > \ldots$. Similarly,

	A	B	C	D
1	$\langle\lambda^+\vert S_A(x)\vert\mu^+\rangle$	$\langle\lambda^+\vert S_B(x)\vert\mu^+\rangle$	$\langle\lambda^+\vert S_C(x)\vert\mu^+\rangle$	$\langle\lambda^+\vert S_D(x)\vert\mu^+\rangle$
HV	$\langle\overline{\mu^+}\vert S_A(x)\vert\overline{\lambda^+}\rangle$	$\langle\overline{\mu^+}\vert S_C(x)\vert\overline{\lambda^+}\rangle$	$\langle\overline{\mu^+}\vert S_B(x)\vert\overline{\lambda^+}\rangle$	$\langle\overline{\mu^+}\vert S_D(x)\vert\overline{\lambda^+}\rangle$
RC	$\langle\widehat{\lambda^+}\vert S_D(x)\vert\widehat{\mu^+}\rangle$	$\langle\widehat{\lambda^+}\vert S_C(x)\vert\widehat{\mu^+}\rangle$	$\langle\widehat{\lambda^+}\vert S_B(x)\vert\widehat{\mu^+}\rangle$	$\langle\widehat{\lambda^+}\vert S_A(x)\vert\widehat{\mu^+}\rangle$
HR	$\langle\mu^+\vert S_D(x)\vert\lambda^+\rangle$	$\langle\mu^+\vert S_C(x)\vert\lambda^+\rangle$	$\langle\mu^+\vert S_B(x)\vert\lambda^+\rangle$	$\langle\mu^+\vert S_A(x)\vert\lambda^+\rangle$
HC	$\langle\widehat{\mu^+}\vert S_A(x)\vert\widehat{\lambda^+}\rangle$	$\langle\widehat{\mu^+}\vert S_B(x)\vert\widehat{\lambda^+}\rangle$	$\langle\widehat{\mu^+}\vert S_C(x)\vert\widehat{\lambda^+}\rangle$	$\langle\widehat{\mu^+}\vert S_D(x)\vert\widehat{\lambda^+}\rangle$
VR	$\langle\overline{\lambda^+}\vert S_D(x)\vert\overline{\mu^+}\rangle$	$\langle\overline{\lambda^+}\vert S_B(x)\vert\overline{\mu^+}\rangle$	$\langle\overline{\lambda^+}\vert S_C(x)\vert\overline{\mu^+}\rangle$	$\langle\overline{\lambda^+}\vert S_A(x)\vert\overline{\mu^+}\rangle$
VC	$\langle\widetilde{\lambda^+}\vert S_A(x)\vert\widetilde{\mu^+}\rangle$	$\langle\widetilde{\lambda^+}\vert S_C(x)\vert\widetilde{\mu^+}\rangle$	$\langle\widetilde{\lambda^+}\vert S_B(x)\vert\widetilde{\mu^+}\rangle$	$\langle\widetilde{\lambda^+}\vert S_D(x)\vert\widetilde{\mu^+}\rangle$
$HVRC$	$\langle\widetilde{\mu^+}\vert S_D(x)\vert\widetilde{\lambda^+}\rangle$	$\langle\widetilde{\mu^+}\vert S_B(x)\vert\widetilde{\lambda^+}\rangle$	$\langle\widetilde{\mu^+}\vert S_C(x)\vert\widetilde{\lambda^+}\rangle$	$\langle\widetilde{\mu^+}\vert S_A(x)\vert\widetilde{\lambda^+}\rangle$
V	$\langle\overline{\lambda^+}\vert S_A(x)\vert\overline{\mu^+}\rangle$	$\langle\overline{\lambda^+}\vert S_C(x)\vert\overline{\mu^+}\rangle$	$\langle\overline{\lambda^+}\vert S_B(x)\vert\overline{\mu^+}\rangle$	$\langle\overline{\lambda^+}\vert S_D(x)\vert\overline{\mu^+}\rangle$
H	$\langle\mu^+\vert S_A(x)\vert\lambda^+\rangle$	$\langle\mu^+\vert S_B(x)\vert\lambda^+\rangle$	$\langle\mu^+\vert S_C(x)\vert\lambda^+\rangle$	$\langle\mu^+\vert S_D(x)\vert\lambda^+\rangle$
VRC	$\langle\widetilde{\lambda^+}\vert S_D(x)\vert\widetilde{\mu^+}\rangle$	$\langle\widetilde{\lambda^+}\vert S_B(x)\vert\widetilde{\mu^+}\rangle$	$\langle\widetilde{\lambda^+}\vert S_C(x)\vert\widetilde{\mu^+}\rangle$	$\langle\widetilde{\lambda^+}\vert S_A(x)\vert\widetilde{\mu^+}\rangle$
HVR	$\langle\overline{\mu^+}\vert S_D(x)\vert\overline{\lambda^+}\rangle$	$\langle\overline{\mu^+}\vert S_B(x)\vert\overline{\lambda^+}\rangle$	$\langle\overline{\mu^+}\vert S_C(x)\vert\overline{\lambda^+}\rangle$	$\langle\overline{\mu^+}\vert S_A(x)\vert\overline{\lambda^+}\rangle$
HVC	$\langle\widetilde{\mu^+}\vert S_A(x)\vert\widetilde{\lambda^+}\rangle$	$\langle\widetilde{\mu^+}\vert S_C(x)\vert\widetilde{\lambda^+}\rangle$	$\langle\widetilde{\mu^+}\vert S_B(x)\vert\widetilde{\lambda^+}\rangle$	$\langle\widetilde{\mu^+}\vert S_D(x)\vert\widetilde{\lambda^+}\rangle$
R	$\langle\lambda^+\vert S_D(x)\vert\mu^+\rangle$	$\langle\lambda^+\vert S_C(x)\vert\mu^+\rangle$	$\langle\lambda^+\vert S_B(x)\vert\mu^+\rangle$	$\langle\lambda^+\vert S_A(x)\vert\mu^+\rangle$
C	$\langle\widehat{\lambda^+}\vert S_A(x)\vert\widehat{\mu^+}\rangle$	$\langle\widehat{\lambda^+}\vert S_B(x)\vert\widehat{\mu^+}\rangle$	$\langle\widehat{\lambda^+}\vert S_C(x)\vert\widehat{\mu^+}\rangle$	$\langle\widehat{\lambda^+}\vert S_D(x)\vert\widehat{\mu^+}\rangle$
HRC	$\langle\widehat{\mu^+}\vert S_D(x)\vert\widehat{\lambda^+}\rangle$	$\langle\widehat{\mu^+}\vert S_C(x)\vert\widehat{\lambda^+}\rangle$	$\langle\widehat{\mu^+}\vert S_B(x)\vert\widehat{\lambda^+}\rangle$	$\langle\widehat{\mu^+}\vert S_A(x)\vert\widehat{\lambda^+}\rangle$

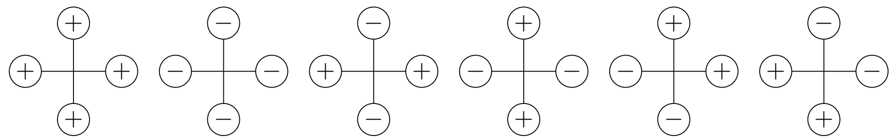
Table 4.1: This table gives the action of S^g in terms of S. The rows of the table are indexed by elements $g \in G$, and the columns are indexed by A, B, C, D. The entry in row g, column X is the one-row partition function $\langle\lambda^+|S^g_X(x)|\mu^+\rangle$.

$\langle\lambda^+|\Gamma_B(x)|\mu^+\rangle$ is nonzero if and only if $\ell(\lambda^+) = \ell(\mu^+) + 1$ and $\lambda^+_1 > \mu^+_1 \geq \lambda^+_2 > \mu^+_2 \geq \dots$. Both of these conditions are equivalent to the statement that λ and μ interleave, and have the correct number of trailing zeros.

If the first condition holds, we claim that

$$\langle\lambda^+|\Delta_A(x)|\mu^+\rangle = x^{|\lambda^+|-|\mu^+|} = x^{|\lambda/\mu|}.$$

To see the first equality, note that in the Δ weights, x_i appears in the weight of a vertex precisely if the vertex has nonzero weight and has a $-$ spin on its left edge. The number of vertices of this sort is precisely the total number of spaces the $-$ spins move left. For the second equation, the interleaving condition tells us that λ/μ is a skew-partition, and the second equation follows from the fact that $\ell(\lambda^+) = \ell(\mu^+)$.



	$\mathtt{a}_1$	$\mathtt{a}_2$	$\mathtt{b}_1$	$\mathtt{b}_2$	$\mathtt{c}_1$	$\mathtt{c}_2$
Γ	1	1	0	x_i	1	1
Δ	1	0	1	x	x	1

Figure 4.2: Two sets of five-vertex Boltzmann weights contained in the same six-vertex model

If the second condition holds, we claim that

$$\langle \lambda^+ | \Gamma_B(x) | \mu^+ \rangle = x^{|\lambda^+|-|\mu^+|} - \ell(\mu) = x^{|\lambda/\mu|}.$$

Here we have a small wrinkle since for the Γ weights, only the $\mathtt{b}_2$ vertex is of weight x. Consider the set Γ' of Boltzmann weights where $\mathtt{a}_2 = \mathtt{c}_1 = x$, and otherwise the weights are the same as Γ. By the same reasoning as above,

$$\langle \lambda^+ | \Gamma'_B(x) | \mu^+ \rangle = x^{|\lambda^+|-|\mu^+|} = x^{|\lambda/\mu|} - \ell(\mu),$$

the second equality following from $\ell(\lambda^+) = \ell(\mu^+) + 1$ Now, the Γ' weights are the same as the Γ weights except that we have multiplied by x the weight of all vertices with a $-$ spin on the bottom edge. But the number of these vertices is precisely $\ell(\mu)$, so we have the claim.

Now the result in both cases follows from the fact that these partition functions satisfy the Schur branching rule. More precisely, we can sum over all ν^+ appearing in the second from top row, and induct on N:

$$
\begin{aligned}
&\langle\lambda^+|\Delta_A(x_1,\dots,x_N)|\mu^+\rangle \\
&\qquad = \langle\lambda^+|A_N(x_N)A_{N-1}(x_{N-1})\cdots A_1(x_1)|\mu^+\rangle \\
&\qquad = \sum_{\nu^+}\langle\lambda^+|A_N(x_N)|\nu^+\rangle\langle\nu^+|A_{N-1}(x_{N-1})\cdots A_1(x_1)|\mu^+\rangle \\
&\qquad = \sum_{\nu^+} x_N^{|\lambda/\nu|}\cdot\begin{cases}1, & \text{if } \lambda \text{ and } \nu \text{ interleave,}\\ 0, & \text{otherwise}\end{cases}\cdot\langle\nu^+|A_{N-1}(x_{N-1})\cdots A_1(x_1)|\mu^+\rangle \\
&\qquad = \sum_{\nu^+} x_N^{|\lambda^+|-|\nu^+|}\cdot\begin{cases}1, & \text{if } \lambda \text{ and } \nu \text{ interleave,}\\ 0, & \text{otherwise}\end{cases}\cdot s_{\nu/\mu}(x_1,\dots,x_{N-1}) \\
&\qquad = s_{\lambda/\mu}(\mathbf{x}),
\end{aligned}
$$

where we have applied the branching rule (4.2). □

We can also mix these operators, and still end up with Schur polynomials.

Corollary 4.2.2. *Choose* $\mathbf{S} = (S_1,\dots,S_N), \mathbf{X} = (X_1,\dots,X_N)$ *such that every* $(S_i)_{X_i} = \Delta_A$ *or* Γ_B. *Then,*

$$\langle\lambda^+|\mathbf{S}_{\mathbf{X}}(\mathbf{x})|\mu^+\rangle = s_{\lambda/\mu}.$$

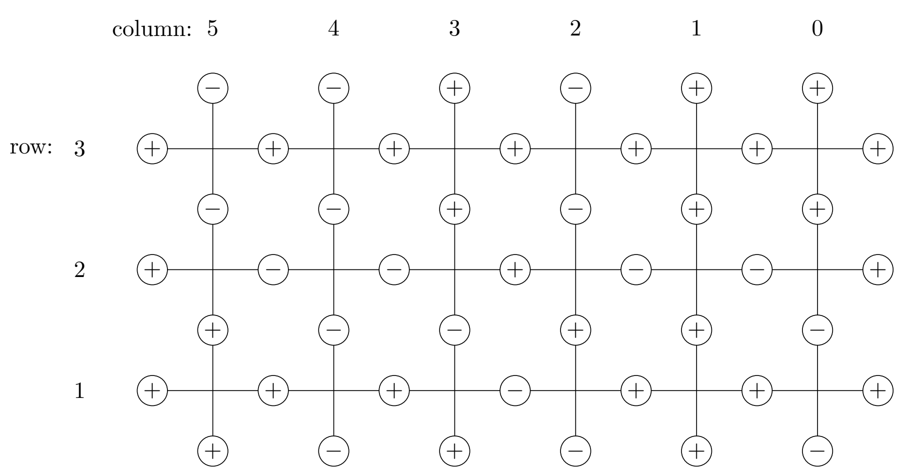


Figure 4.3: A state of the lattice model $\langle|\lambda^+|\Delta_A|\mu^+\rangle$, where $\lambda^+ = (5,4,2)$ and $\mu^+ = (4,2,0)$.

Now we discuss the tranformed Boltzmann weights, Γ^g and Δ^g. These appear in Tables 4.4 (even elements) and 4.5 (odd elements). Both Γ and Δ are six-vertex models, with $\mathtt{d}_1 = \mathtt{d}_2 = 0$. Even elements of G preserve this structure, while odd elements produce a set of weights in a complementary six-vertex model, where $\mathtt{c}_1 = \mathtt{c}_2 = 0$.

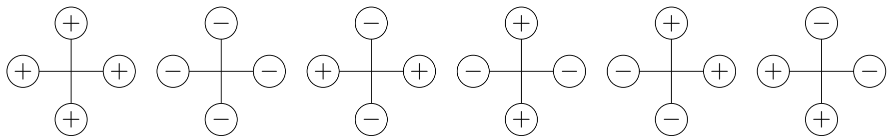
	$\mathtt{a}_1^{(\mathtt{i})}$	$\mathtt{a}_2^{(\mathtt{i})}$	$\mathtt{b}_1^{(\mathtt{i})}$	$\mathtt{b}_2^{(\mathtt{i})}$	$\mathtt{c}_1^{(\mathtt{i})}$	$\mathtt{c}_2^{(\mathtt{i})}$
						
Γ	1	1	0	x_i	1	1
Γ^{HV}	1	1	0	x_i	1	1
Γ^{RC}	1	1	x_i	0	1	1
Γ^{HR}	x_i	0	1	1	1	1
Γ^{HC}	0	x_i	1	1	1	1
Γ^{VR}	x_i	0	1	1	1	1
Γ^{VC}	0	x_i	1	1	1	1
Γ^{HVRC}	1	1	x_i	0	1	1
Δ	1	0	1	x_i	x_i	1
Δ^{HV}	1	0	1	x_i	1	x_i
Δ^{RC}	0	1	x_i	1	1	x_i
Δ^{HR}	x_i	1	0	1	1	x_i
Δ^{HC}	1	x_i	1	0	x_i	1
Δ^{VR}	x_i	1	0	1	x_i	1
Δ^{VC}	1	x_i	1	0	1	x_i
Δ^{HVRC}	0	1	x_i	1	x_i	1

Figure 4.4: Transformed Boltzmann weights Γ^g and Δ^g, for even g

In addition, we mention two symmetries. First, if we substitute $x \mapsto x^{-1}$ and multiply each weight by x_i, we send the Δ^{HR} weights to the Γ' weights mentioned in the proof of Proposition 4.2.1. This transforms the partition function in a predicable way:

$$\langle \lambda^+ | \Gamma_X(x) | \mu^+ \rangle = x^{-\ell(\mu^+)} \langle \lambda^+ | \Gamma'_X(x) | \mu^+ \rangle = x^{|\widehat{\mu^+}|} \langle \lambda^+ | \Delta_X^{HR}(x^{-1}) | \mu^+ \rangle.$$

Second, if we swap the weights for the $\mathtt{c}_1$ and $\mathtt{c}_2$ vertices, this fixes Γ, and sends Δ to Δ^{HV}. In the latter case, by conservation of spin, this multiplies the one-row partition function $\langle \lambda^+ | \Delta_X(x) | \lambda^+ \rangle$ by x if $X = B$, divides it by x if $X = C$, and leaves the one-row partition function unchanged if $X = A$ or D. In other words,

$$x^\epsilon \langle \mu^+ | S_X(x) | \lambda^+ \rangle = \langle \mu^+ | S_X^{HV}(x) | \lambda^+ \rangle,$$

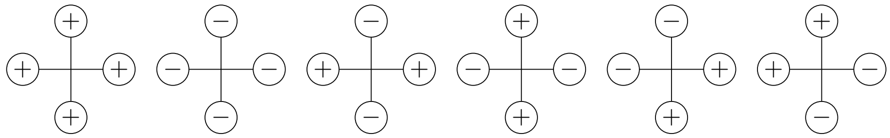
	$\mathtt{a}_1^{(\mathtt{i})}$	$\mathtt{a}_2^{(\mathtt{i})}$	$\mathtt{b}_1^{(\mathtt{i})}$	$\mathtt{b}_2^{(\mathtt{i})}$	$\mathtt{d}_1^{(\mathtt{i})}$	$\mathtt{d}_2^{(\mathtt{i})}$
						
Γ^V	1	1	0	x_i	1	1
Γ^H	1	1	0	x_i	1	1
Γ^{VRC}	1	1	x_i	0	1	1
Γ^{HVR}	x_i	0	1	1	1	1
Γ^{HVC}	0	x_i	1	1	1	1
Γ^R	x_i	0	1	1	1	1
Γ^C	0	x_i	1	1	1	1
Γ^{HRC}	1	1	x_i	0	1	1
Δ^V	1	0	1	x_i	1	x_i
Δ^H	1	0	1	x_i	x_i	1
Δ^{VRC}	0	1	x_i	1	x_i	1
Δ^{HVR}	x_i	1	0	1	x_i	1
Δ^{HVC}	1	x_i	1	0	1	x_i
Δ^R	x_i	1	0	1	1	x_i
Δ^C	1	x_i	1	0	x_i	1
Δ^{HRC}	0	1	x_i	1	1	x_i

Figure 4.5: Transformed Boltzmann weights Γ^g and Δ^g, for odd g

where

$$\epsilon = \begin{cases} 1, & \text{if } S = \Delta \text{ and } X = B, \\ -1, & \text{if } S = \Delta \text{ and } X = C, \\ 0, & \text{if } S = \Gamma, \text{ or if } X = A \text{ or } D. \end{cases}$$

Neither of these relations preserves the power of x, or therefore the grading in the Yang-Baxter algebra.

Using these two symmetries, as well as Table 4.1, one can generalize Corollary 4.2.2. 16 of our one-row partition functions, $\Delta_A, \Gamma_A^{HR}, \Delta_A^{HV}, \Gamma_A^{VR}, \Delta_D^R, \Gamma_D^H, \Delta_D^{HVR}, \Gamma_D^V, \Gamma_B, \Delta_B^{HR}, \Gamma_B^{HV}, \Delta_B^{VR}$,

$\Gamma_C^R, \Delta_C^H, \Gamma_C^{HVR}$, and Δ_C^V, give the branching rule for $s_{\lambda/\mu}$, up to a substitution and/or a factor, so these 16 models could be combined in any way, and the resulting partition function would be expressable by a Schur polynomial. The other cases, for instance $s_{\lambda'/\mu'}$, are similar.

Using Table 4.1 along with Proposition 4.2.1, we obtain many lattice model expressions for Schur polynomials.

Proposition 4.2.3.

$$\begin{aligned}
s_{\lambda/\mu}(\mathbf{x}) &= \langle \mu^+ | \Delta_A(\mathbf{x}) | \lambda^+ \rangle = \langle \overline{\lambda^+} | \Delta_A^{HV}(\mathbf{x}) | \overline{\mu^+} \rangle = \langle \widehat{\mu^+} | \Delta_D^{RC}(\mathbf{x}) | \widehat{\lambda^+} \rangle = \langle \lambda^+ | \Delta_D^{HR}(\mathbf{x}) | \mu^+ \rangle \\
&= \langle \widehat{\lambda^+} | \Delta_A^{HC}(\mathbf{x}) | \widehat{\mu^+} \rangle = \langle \overline{\mu^+} | \Delta_D^{VR}(\mathbf{x}) | \overline{\lambda^+} \rangle = \langle \widetilde{\mu^+} | \Delta_A^{VC}(\mathbf{x}) | \widetilde{\lambda^+} \rangle = \langle \widetilde{\lambda^+} | \Delta_D^{HVRC}(\mathbf{x}) | \widetilde{\mu^+} \rangle \\
&= \langle \overline{\mu^+} | \Delta_A^{V}(\mathbf{x}) | \overline{\lambda^+} \rangle = \langle \lambda^+ | \Delta_A^{H}(\mathbf{x}) | \mu^+ \rangle = \langle \widetilde{\mu^+} | \Delta_D^{VRC}(\mathbf{x}) | \widetilde{\lambda^+} \rangle = \langle \overline{\lambda^+} | \Delta_D^{HVR}(\mathbf{x}) | \overline{\mu^+} \rangle \\
&= \langle \widetilde{\lambda^+} | \Delta_A^{HVC}(\mathbf{x}) | \widetilde{\mu^+} \rangle = \langle \mu^+ | \Delta_D^{R}(\mathbf{x}) | \lambda^+ \rangle = \langle \widehat{\mu^+} | \Delta_A^{C}(\mathbf{x}) | \widehat{\lambda^+} \rangle = \langle \widehat{\lambda^+} | \Delta_D^{HRC}(\mathbf{x}) | \widehat{\mu^+} \rangle \\
&= \langle \mu^+ | \Gamma_B(\mathbf{x}) | \lambda^+ \rangle = \langle \overline{\lambda^+} | \Gamma_C^{HV}(\mathbf{x}) | \overline{\mu^+} \rangle = \langle \widehat{\mu^+} | \Gamma_C^{RC}(\mathbf{x}) | \widehat{\lambda^+} \rangle = \langle \lambda^+ | \Gamma_C^{HR}(\mathbf{x}) | \mu^+ \rangle \\
&= \langle \widehat{\lambda^+} | \Gamma_B^{HC}(\mathbf{x}) | \widehat{\mu^+} \rangle = \langle \overline{\mu^+} | \Gamma_B^{VR}(\mathbf{x}) | \overline{\lambda^+} \rangle = \langle \widetilde{\mu^+} | \Gamma_C^{VC}(\mathbf{x}) | \widetilde{\lambda^+} \rangle = \langle \widetilde{\lambda^+} | \Gamma_B^{HVRC}(\mathbf{x}) | \widetilde{\mu^+} \rangle \\
&= \langle \overline{\mu^+} | \Gamma_C^{V}(\mathbf{x}) | \overline{\lambda^+} \rangle = \langle \lambda^+ | \Gamma_B^{H}(\mathbf{x}) | \mu^+ \rangle = \langle \widetilde{\mu^+} | \Gamma_B^{VRC}(\mathbf{x}) | \widetilde{\lambda^+} \rangle = \langle \overline{\lambda^+} | \Gamma_B^{HVR}(\mathbf{x}) | \overline{\mu^+} \rangle \\
&= \langle \widetilde{\lambda^+} | \Gamma_C^{HVC}(\mathbf{x}) | \widetilde{\mu^+} \rangle = \langle \mu^+ | \Gamma_C^{R}(\mathbf{x}) | \lambda^+ \rangle = \langle \widehat{\mu^+} | \Gamma_B^{C}(\mathbf{x}) | \widehat{\lambda^+} \rangle = \langle \widehat{\lambda^+} | \Gamma_C^{HRC}(\mathbf{x}) | \widehat{\mu^+} \rangle.
\end{aligned}$$

4.3 Solvability

Every set of weights in Figures 4.4 and 4.5 is free fermionic. Therefore, any two of them are a Yang-baxter pair, with the following diagonal vertex weights in the case of 4.4. We can obtain the diagonal weights from the following formulas.

Theorem 4.3.1. *[2, Theorem 1, free fermionic case] Let S and T be sets of free fermionic six-vertex Boltzmann weights, where the weights for both S and T are given by Figure 4.1 with $\mathtt{d}_1 = \mathtt{d}_2 = 0$. Then the following weights R satisfy $R_{12}S_{13}T_{23} = T_{23}S_{13}R_{12}$:*

$$\mathtt{a}_1(R) = \mathtt{b}_2(S)\mathtt{b}_1(T) + \mathtt{a}_1(S)\mathtt{a}_2(T), \qquad \mathtt{a}_2(R) = \mathtt{b}_1(S)\mathtt{b}_2(T) + \mathtt{a}_2(S)\mathtt{a}_1(T),$$

$$\mathtt{b}_1(R) = \mathtt{b}_1(S)\mathtt{a}_2(T) - \mathtt{a}_2(S)\mathtt{b}_1(T), \qquad \mathtt{b}_2(R) = \mathtt{b}_2(S)\mathtt{a}_1(T) - \mathtt{a}_1(S)\mathtt{b}_2(T),$$

$$\mathtt{c}_1(R) = \mathtt{c}_1(S)\mathtt{c}_2(T), \qquad \mathtt{c}_2(R) = \mathtt{c}_2(S)\mathtt{c}_1(T)$$

If S and/or T instead have $\mathtt{c}_1 = \mathtt{c}_2 = 0$, we can get the weights from these by performing the complement operation $+ \leftrightarrow -$ along the corresponding strand.

Select sets of diagonal vertices are given in Figure 4.6.

Among other things, the Yang-Baxter equation tells us that Schur polynomials are symmetric.

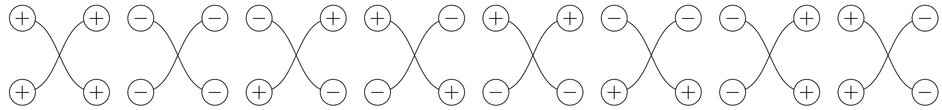
S,T	$\mathtt{A}_1^{(\mathtt{i,j})}$	$\mathtt{A}_2^{(\mathtt{i,j})}$	$\mathtt{B}_1^{(\mathtt{i,j})}$	$\mathtt{B}_2^{(\mathtt{i,j})}$	$\mathtt{C}_1^{(\mathtt{i,j})}$	$\mathtt{C}_2^{(\mathtt{i,j})}$	$\mathtt{D}_1^{(\mathtt{i,j})}$	$\mathtt{D}_2^{(\mathtt{i,j})}$
								
Δ,Δ	x_i	x_j	0	$x_i - x_j$	x_i	x_j	0	0
Γ,Γ	1	1	0	$x_i - x_j$	1	1	0	0
Γ,Δ	x_i	1	-1	$x_i - x_j$	1	1	0	0
Δ,Γ	1	x_j	1	$x_i - x_j$	x_i	1	0	0
Δ^{HVC},Δ	1	-1	$1 + x_i x_j$	1	0	0	x_j	x_i
Δ^{HV},Δ	x_i	x_j	0	$x_i - x_j$	1	$x_i x_j$	0	0
Δ^{HVR},Δ	$x_i - x_i$	0	x_j	x_i	0	0	$x_i x_j$	1
Γ^{R},Γ	$1 - x_i x_j$	1	x_j	x_i	0	0	1	1
Γ^{HVRC},Γ	1	$1 + x_i x_j$	x_i	$-x_j$	1	1	0	0
Γ^{HR},Γ	x_i	x_j	1	$1 - x_i x_j$	1	1	0	0
Γ^{HC},Γ	0	$x_i + x_i$	1	1	1	1	0	0
Γ^{HV},Γ	1	1	0	$x_i - x_j$	1	1	0	0
Γ^{HVR},Γ	$x_i - x_j$	0	1	1	0	0	1	1
Γ^{RC},Γ	1	$1 + x_i x_j$	x_i	$-x_j$	1	1	0	0
Γ^{VR},Γ	x_i	x_j	1	$1 - x_i x_j$	1	1	0	0

Figure 4.6: Boltzmann weights for the diagonal vertices $\mathtt{a_1}(S,T)$, etc. in the Yang-Baxter equation, where a row of weights is labelled by S,T. Here, S has row parameter x_i and T has row parameter x_j.

Proposition 4.3.2. *The Schur polynomial $s_{\lambda/\mu}(x_1,\ldots,x_N)$ is symmetric in the x_i.*

Proof. We will prove this using $S=\Delta, T=\Delta$, although it would be just as straightforward using Γ. By Figure 4.6, $\mathtt{d_1}(S,T)=\mathtt{d_2}(S,T)=0$. By (4.4), $\Gamma_A(x)$ and $\Gamma_A(y)$ commute, so by Proposition 4.2.1, $s_{\lambda/\mu}$ is symmetric in any two adjacent variables, and therefore in all of them. □

4.4 Identities

We will also use the Yang-Baxter equation to prove several identities. All of them rely on the same idea: that the Yang-Baxter equation gives commutation relations in the symmetrized Yang-Baxter algerba. Let $\mathbf{y}=y_1,\ldots,y_{N'}$ be a second set of variables, and let $s_{\lambda/\mu}(\mathbf{x},\mathbf{y})=s_{\lambda/\mu}(x_1,\ldots,x_N,y_1,\ldots,y_{N'})$.

Proposition 4.4.1 (Generalized Branching Rule)**.** *For any* $\lambda, \mu \in \mathcal{P}$*,*

$$s_{\lambda/\mu}(\mathbf{x},\mathbf{y}) = \sum_{\nu} s_{\lambda/\nu}(\mathbf{y}) s_{\nu/\mu}(\mathbf{x}) = \sum_{\nu} s_{\lambda/\nu}(\mathbf{x}) s_{\nu/\mu}(\mathbf{y}).$$

Proof. The first equality is by the branching rule:

$$\begin{aligned} s_{\lambda/\mu}(\mathbf{x},\mathbf{y}) &= \langle \lambda^+ | \Delta_A(\mathbf{y}) \Delta_A(\mathbf{x}) | \mu^+ \rangle \\ &= \sum_{\nu} \langle \lambda^+ | \Delta_A(\mathbf{y}) | \nu^+ \rangle \langle \mu^+ | \Delta_A(\mathbf{x}) | \mu^+ \rangle \\ &= \sum_{\nu} s_{\lambda/\nu}(\mathbf{x}) s_{\nu/\mu}(\mathbf{y}), \end{aligned}$$

while the second is by Proposition 4.3.2. □

Proposition 4.4.2 (Dual Cauchy Identity)**.** *For any* $\lambda, \mu \in \mathcal{P}$*,*

$$\sum_{\nu} s_{\lambda/\nu}(\mathbf{x}) s_{\mu'/\nu'}(\mathbf{y}) = \prod_{i,j} \frac{1}{1+xy} \cdot \sum_{\nu} s_{\nu'/\lambda'}(\mathbf{y}) s_{\nu/\mu}(\mathbf{x}).$$

Proof. By Figure 4.6, we have $\mathtt{a_1}(\Gamma^{HVRC}, \Gamma) = 1$, $\mathtt{a_2}(\Gamma^{HVRC}, \Gamma) = 1+xy$, and $\mathtt{d_1}(\Gamma^{HVRC}, \Gamma) = \mathtt{d_2}(\Gamma^{HVRC}, \Gamma) = 0$. Applying (4.9), we get

$$\Gamma_B^{HVRC}(y)\Gamma_B(x) = (1+xy)\Gamma_B(x)\Gamma_B^{HVRC}(y).$$

This commutation relation, combined with Table 4.1, lead to the identity. The rest is just formal manipulations.

$$\begin{aligned} \sum_{\nu} s_{\lambda/\nu}(\mathbf{x}) s_{\mu'/\nu'}(\mathbf{y}) &= \sum_{\nu} \langle \lambda^+ | \Gamma_B(\mathbf{x}) | \nu^+ \rangle \langle \widetilde{\mu^+} | \Gamma_B(\mathbf{y}) | \widetilde{\nu^+} \rangle \\ &= \sum_{\nu} \langle \lambda^+ | \Gamma_B(\mathbf{x}) | \nu^+ \rangle \langle \nu^+ | \Gamma_B^{HVRC}(\mathbf{y}) | \mu^+ \rangle \\ &= \langle \lambda^+ | \Gamma_B(\mathbf{x}) \Gamma_B^{HVRC}(\mathbf{y}) | \mu^+ \rangle \\ &= \prod_{i,j} \frac{1}{1+xy} \cdot \langle \lambda^+ | \Gamma_B^{HVRC}(\mathbf{y}) \Gamma_B(\mathbf{x}) | \mu^+ \rangle \\ &= \prod_{i,j} \frac{1}{1+xy} \cdot \sum_{\nu} \langle \lambda^+ | \Gamma_B^{HVRC}(\mathbf{y}) | \nu^+ \rangle \langle \nu^+ | \Gamma_B(\mathbf{x}) \mu^+ \rangle \\ &= \prod_{i,j} \frac{1}{1+xy} \cdot \sum_{\nu} \langle \widetilde{\nu^+} | \Gamma_B(\mathbf{y}) | \widetilde{\lambda^+} \rangle \langle \nu^+ | \Gamma_B(\mathbf{x}) | \mu^+ \rangle \\ &= \prod_{i,j} \frac{1}{1+xy} \cdot \sum_{\nu} s_{\nu'/\lambda'}(\mathbf{y}) s_{\nu/\mu}(\mathbf{x}) \end{aligned}$$

□

Note that in the case $\lambda = \mu = \emptyset$, we obtain the more commonly seen form of this identity:

$$\sum_{\nu} s_{\nu}(\mathbf{x}) s_{\nu'}(\mathbf{y}) = \prod_{i,k} (1 + x_i y_j).$$

It is our hope that other identities can be proved using these tools. We'll finish with one such identity.

Proposition 4.4.3. *For any $\lambda, \mu \in \mathcal{P}$,*

$$\sum_{\nu} s_{\lambda/\nu}(\mathbf{x}) s_{\overline{\nu}/\overline{\mu}}(\mathbf{y}) = \prod_{i,j} \frac{x_i}{y_j} \cdot \sum_{\nu} s_{\overline{\lambda'}/\overline{\nu}}(\mathbf{y}) s_{\nu/\mu}(\mathbf{x})$$

where by $\overline{\nu}$ with $\nu \in \mathcal{P}$ we mean the complement of ν in a $\ell(\nu) \times M + 1 - \ell(\nu)$ rectangle.

Proof. By Figure 4.6, we have $\mathtt{a_1}(\Gamma^{VR}, \Gamma) = x_i$, $\mathtt{a_2}(\Gamma^{VR}, \Gamma) = x_j$, and $\mathtt{d_1}(\Gamma^{HVRC}, \Gamma) = \mathtt{d_2}(\Gamma^{HVRC}, \Gamma) = 0$. Applying (4.9), we get

$$x\Gamma_B^{VR}(y)\Gamma_B(x) = y\Gamma_B(x)\Gamma_B^{VR}(y).$$

Therefore,

$$\begin{aligned}
\sum_{\nu} s_{\lambda/\nu}(\mathbf{x}) s_{\overline{\nu}/\overline{\mu}}(\mathbf{y}) &= \sum_{\nu} \langle \lambda^+ | \Gamma_B(\mathbf{x}) | \nu^+ \rangle \langle \overline{\nu^+} | \Gamma_B(\mathbf{y}) | \overline{\mu^+} \rangle \\
&= \sum_{\nu} \langle \lambda^+ | \Gamma_B(\mathbf{x}) | \nu^+ \rangle \langle \nu^+ | \Gamma_B^{VR}(\mathbf{y}) | \mu^+ \rangle \\
&= \langle \lambda^+ | \Gamma_B(\mathbf{x}) \Gamma_B^{VR}(\mathbf{y}) | \mu^+ \rangle \\
&= \prod_{i,j} \frac{x_i}{y_j} \cdot \langle \lambda^+ | \Gamma_B^{VR}(\mathbf{y}) \Gamma_B(\mathbf{x}) | \mu^+ \rangle \\
&= \prod_{i,j} \frac{x_i}{y_j} \cdot \sum_{\nu} \langle \lambda^+ | \Gamma_B^{VR}(\mathbf{y}) | \nu^+ \rangle \langle \nu^+ | \Gamma_B(\mathbf{x}) \mu^+ \rangle \\
&= \prod_{i,j} \frac{x_i}{y_j} \cdot \sum_{\nu} \langle \overline{\lambda^+} | \Gamma_B(\mathbf{y}) | \overline{\nu^+} \rangle \langle \nu^+ | \Gamma_B(\mathbf{x}) | \mu^+ \rangle \\
&= \prod_{i,j} \frac{x_i}{y_j} \cdot \sum_{\nu} s_{\overline{\lambda}/\overline{\nu}}(\mathbf{y}) s_{\nu/\mu}(\mathbf{x}).
\end{aligned}$$

□

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