EFFECT OF FREE OSCILLATIONS ON THE PARTICLE REVOLUTION PERIOD IN A RELATIVISTIC CYCLOTRON

V. P. Dmitrievskiĭ, B. I. Zamolodchikov, V. V. Kol'ga, T. M. Prilipko

Joint Institute of Nuclear Research (OIYaI), Dubna

(Presented by B. I. Zamolodchikov)

The present paper studies the effect of the departure from isochronous motion of particles having various amplitudes of free oscillations relative to the closed orbit*. Isochronous closed orbits within an arbitrary magnetic field may be achieved, as is well known, if the magnetic field strength varies according to the equation

$$H(L) = H_0 \frac{E}{E_0}, \tag{1}$$

where H(L) is the average value of the field over the closed orbit; E is the total energy of the ion.

For a magnetic field whose strength within the median plane is given by the function

$$H_z = H(r)\left[1 + \varepsilon(r)\sin\{\alpha(r) - N\varphi\}\right], \tag{2}$$

the condition for the existence of the isochronous closed orbit can be achieved with an arbitrary value of the momentum $p = e/cH(R)R$, if the configuration of the magnetic field is described by the expression

$$H(R) = \frac{H_0}{\sqrt{1-\frac{R^2}{r_\infty^2}}}\left[1 + \frac{\Delta R}{R} + \frac{1}{4}\cdot\frac{\varepsilon^2 N^2}{(N^2-n-1)^2}\right], \tag{3}$$

*The possibility of such an effect was first pointed out by G. I. Budker.

where the first correction term* $\Delta R/R = -\frac{\varepsilon^2}{2N^2(1+n)}\left[\frac{3}{2}+n+\frac{R}{\varepsilon}\cdot\frac{d\varepsilon}{dr}\Big|_{r=R}\right]$ characterizes the decrease in the average radius of the orbit with the variation of the magnetic field, while the second such term represents $\frac{1}{4}\cdot\frac{\varepsilon^2N^2}{(N^2-n-1)^2}$, the lengthening of the orbit due to its wave shape:

$$n=\frac{R}{H(R)}\cdot\frac{dH}{dr}\Big|_{r=R}.$$

The period of rotation for an arbitrary ion is found from the equation

$$T=\frac{E}{pc^2}\int_0^{2\pi}\sqrt{r'^2+r^2+z'^2}\,d\varphi. \tag{4}$$

Since the amplitude and frequency of the axial oscillations are small, compared to the amplitude and frequencies of the radial oscillations, one can neglect the influence of the axial oscillations on the isochronous behavior. If we denote by $r_1(\varphi)$ the function which corresponds to the periodic solution (closed orbit) of the equation

$$r''-\frac{2r'^2}{r}-r=-\frac{e}{pc}\,\frac{(r'^2+r^2)^{3/2}}{r}\,H_z(r,\varphi), \tag{5}$$

and write down the free oscillations in the form a cos $Q_2\varphi+\bar{\varrho}$, then the general solution may be written as

$$r=r_1+a\cos Q_r\varphi+\bar{\varrho}, \tag{6}$$

where $\bar{\varrho}$ is the change of the average radius of the ion due to the free oscillations; Q_r is the frequency of radial oscillations relative to one orbit of the ion.

By substituting equation (6) into the expression (4) we can find the change in the period of rotation of the ion which takes into account the free oscillations:

$$\frac{\Delta T}{T}=\frac{\bar{\varrho}}{r_0}+\frac{Q_r^2}{4}\cdot\frac{a^2}{r_0^2}, \tag{7}$$

where r_0 is the average radius of the closed orbit ($r_0 = R + \Delta R$).

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*The equation was verified on an electronic computer for the case $d\varepsilon/dr = 0$, the accuracy was 10 percent.

$\bar{\varrho}$ can be determined from equations (5) and (6). Neglecting the quadratic terms, we obtain

$$\bar{\varrho} = -\frac{1}{2r_0}\left[1+2n+d-\frac{1}{2}Q_r^2\right]\frac{a^2}{Q_r^2}, \quad (8)$$

where

$$d = \frac{1}{2}\cdot\frac{R^2}{H(R)}\cdot\frac{d^2H(r)}{dr^2}\bigg|_{r=R}.$$

Since for a relativistic cyclotron $d = \frac{1}{2}n + \frac{3}{2}n^2$, $Q_r^2 \approx 1 + n$, the expression (7) can be represented in the form:

$$\frac{\Delta T}{T} = -\frac{n}{2}\frac{a^2}{r_0^2}. \quad (9)$$

A similar result was obtained independently in [1].

Below, we list the solutions of equations (4) and (5) on the electronic computer (ESM) of the OIYaI.

№	n	ε	a, см	R, см	$\Delta T/T$	
					Calculated on ESM	Using formula (9)
1	1,94	0	4	322	$-1,50\cdot10^{-4}$	$-1,495\cdot10^{-4}$
2	1,94	0,268	3,82	322	$-1,85\cdot10^{-4}$	$-1,365\cdot10^{-4}$
3	1,94	0,268	5,23	322	$-3,35\cdot10^{-4}$	$-2,57\cdot10^{-4}$
4	0.102	0,202	4,70	122	$-1,10\cdot10^{-4}$	$-0,77\cdot10^{-4}$

We also calculated on the electronic computer the length of the orbit averaged over 500 turns:

$$L = \frac{1}{\nu}\int_0^{2\pi\nu}(r'^2+r^2)^{1/2}\,d\varphi. \quad (10)$$

Values of r and r' were taken from the solution of equation (5).

From expression (9) and the table it follows that with radial oscillation amplitudes of the order of 3--4 cm, the relative changes in the period may exceed 10^{-4} and the effect due to the free oscillations may be decisive in the choice of the accelerating voltage for relativistic cyclotrons having energies of 600--800 MeV.

The table shows that for $\varepsilon \neq 0$ there exist significant deviations in the calculated quantities. These deviations are related to the inaccuracy of the method outlined which neglected the quadratic terms. It follows from expression (9) that the departure from isochronous behavior is essentially dependent on energy (for a relativistic cyclotron $n = E^2/E_0^2 - 1$) and, consequently, the full phase shift during the accelerating process is calculated by integrating the equation

$$\frac{d\varphi}{d\nu} = -2\pi\frac{\Delta T}{T}. \tag{11}$$

If we denote the initial and final phases of the acceleration by φ_i, φ_f we obtain the following expression for the minimum energy increment per turn:

$$eV = \frac{\pi E_0}{3(\sin\varphi_f - \sin\varphi_i)}\left(\frac{E^3}{E_0^3} - 1\right)\frac{a^2}{r_\infty^2}, \tag{12}$$

where

$$r_\infty = \frac{E_0}{eH_0}.$$

If we limit the azimuthal beam widening during the acceleration process within $\pi/3$, then for a relativistic cyclotron of 700 MeV [2] and r_∞ = 396.4 cm the expression (12) can be written in the form

$$eV \approx 31.3\ a^2\ \text{keV}, \tag{13}$$

and, consequently, for a = 3 cm the necessary energy increment per turn must be approximately equal to 280 keV.

The damping of free oscillation according to the law

$$a \approx a_0\frac{E_0}{E} \tag{14}$$

contributes to a decrease in the above-mentioned effect. With this in mind, the integration of equations (11), (9) yields the following result:

$$eV = \frac{\pi(E - E_0)}{\sin\varphi_f - \sin\varphi_i}\cdot\frac{a_0^2}{r_\infty^2}. \tag{15}$$

For the above quoted parameters

$$eV = 16.3\ a_0^2\ \text{keV}. \tag{16}$$

One should, however, notice that the presence of several resonant zones such as Q_r = 1.5, Q_r = 1.6 within which may occur certain increases in the amplitude of radial oscillations generates difficulties when utilizing equation (16). Within real accelerators one must expect that the perturbation of the isochronous behavior caused by free oscillations may occupy a certain intermediate position between 31.3 a^2 and 16.3 a_0^2 keV.

It is essential that the departures from isochronous behavior studied above are only very slightly connected with the errors in the variation of the mean magnetic field and with the errors of stabilization.

The given effect results in an increase of the azimuthal extension of the beam during the acceleration process. In connection with this, the choice of the amplitude of the accelerating hf voltage is influenced by this effect only in the case when the necessary energy increment per turn of 31.3 a^2 keV exceeds the energy increment determined by the phase motion of the "center of mass" of the beam.

Numerical calculations on the ESM computer supplied the estimate of the influence the axial free oscillations have on the isochronous motion [term z'^2 in expression (4)]. The additional lengthening of the trajectory as a result of this new term reduces the given effect slightly. Nevertheless, numerical results for the relative increase of the orbital length does not exceed 10^{-15} for a 2 cm amplitude of axial oscillation.

BIBLIOGRAPHY

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