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Relations between Graviton and Gluon Amplitudes from Twistor Space

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The tree-level amplitudes of gauge theory and gravity in four dimensions have miraculously simple expressions coming from twistor space, graded in complexity by the helicity configuration of the external particles. These are known as the Roiban-Spradlin-Volovich-Witten (RSVW) and Cachazo-Skinner formula respectively. In these proceedings we summarise the results of [1] which finds the form of the double copy relation between these two formulae. The double copy expresses gravity amplitudes as a "square" of gauge theory amplitudes. However, the relation between the RSVW and Cachazo-Skinner formuale has historically been left implicit. We find that the explicit relation is based in twistor space and is naturally helicity-graded. We further show its connection to biadjoint scalar theory.

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1. Introduction

We consider the tree-level scattering of gluons and gravitons. Historically there have been two disconnected organizing principles for aiding the calculation of these amplitudes in gauge theory and gravity: **the double copy** and **the external helicity configuration** of the particles in four dimensions. In these proceedings we summarise the results of [1] which provides a connection between these two principles. This is represented in Figure 1.

The Double Copy Here we will only consider the Kawai-Lewellen-Tye (KLT) double copy [2], with further details on other manifestations of this relation reviewed in [3]. This is a relation between colour-ordered gluon amplitudes and graviton amplitudes at tree level

$$\mathcal{M}^{\mathrm{GR}}(1,\ldots,n) = \sum_{\substack{\alpha\in\Omega_{n-3}\\ \beta\in\tilde{\Omega}_{n-3}}} \mathcal{A}^{\mathrm{YM}}[\alpha]\, S[\alpha|\beta]\, \mathcal{A}^{\mathrm{YM}}[\beta] \tag{1}$$

where $\Omega_{n-3}, \tilde{\Omega}_{n-3}$ are two bases of $(n-3)!$ colour-orderings. The object $S[\alpha|\beta]$ is known as the *KLT (or momentum) kernel*. Choosing the bases

$$\Omega_{n-3} = \{123\sigma|\, \sigma \in S_{n-3}\}, \qquad \tilde{\Omega}_{n-3} = \{213\rho|\, \rho \in S_{n-3}\}, \tag{2}$$

for orderings σ, ρ on $\{3,\ldots,n\}$, the kernel is given in terms of Mandelstam variables $s_{ij} = k_i \cdot k_j$ as

$$S[123\sigma|213\rho] = \prod_{i=4}^{n} \sum_{\substack{j<_{3\sigma} i\\ j<_{3\rho} i}} s_{ij}. \tag{3}$$

The inequalities $(j <_{3\sigma} i, j <_{3\rho} i)$ mean that j precedes i in both the orderings 3σ and 3ρ.

The remarkable aspect of the double copy is that it allows one to express amplitudes of a complicated theory (gravity) in terms of amplitudes of a simpler theory (gauge theory).

External helicity configurations In four dimensions the momenta of massless particles can be described in terms of spinor-helicity variables

$$p_\mu(\sigma^\mu)^{\alpha\dot{\alpha}} := p^{\alpha\dot{\alpha}} = |p\rangle^{\dot{\alpha}}[p|^{\alpha} \tag{4}$$

where $\sigma^\mu = (\mathrm{id}, \sigma^i)$ are the Pauli matrices. These are naturally contracted using the Levi-Civita symbols $\epsilon_{\alpha\beta}, \epsilon_{\dot{\alpha}\dot{\beta}}$. See [4] for more details. The *simplest* amplitudes in Yang-Mills are known as *maximally helicity violating* (MHV) amplitudes. Viewing all particles as ingoing, these are amplitudes with two negative helicity, rest positive helicity gluons[1], and given by the Parke-Taylor formula [5]

$$\mathcal{A}[1^+\ldots i^-\ldots j^-\ldots n^+] = \delta^4\left(\sum_{i=1}^{n} p_i\right) \frac{\langle i\, j\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n\,1\rangle}. \tag{5}$$

[1]The parity conjugate configuration with two positive helicity, rest negative helicity gluons is obtained by complex conjugation.

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where we've restricted to the colour-ordering $12\ldots n$. All other colour-orderings can be obtained by relabelling. The analogous result in gravity has a long history, but the most compact expression was found by Hodges [6]

$$\mathcal{M}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+) = \delta^4\left(\sum_{i=1}^{n} p_i\right)\langle i\,j\rangle^8\,\det{}'(\mathrm{H}_{ij}). \tag{6}$$

where H_{ij} is a matrix whose entries are constructed from the spinor helicity variables of the external gravitons.

The existence of these simple expressions for any number of external particles is remarkable – they are a resummation of factorially growing many Feynman diagrams which fortuitously rearrange into these simple expressions. But a deeper reason for this simplicity is the *integrability* of the self-dual sectors of both Yang-Mills and gravity. The $n-2$ positive helicity particles can be viewed as a self-dual background that the two negative helicity particles are scattering on. This is naturally encoded using twistor theory [7].

One can therefore view generic helicity configuration scattering amplitudes in these theories as *perturbations away from the (anti-)self-dual sector*. Generic N^{d-1}MHV amplitudes, with $d+1$ negative helicity particles and $n-d-1$ positive helicity particles, grow in complexity as they violate either sector. The analogs of the Parke-Taylor and Hodges formulae for these amplitudes can be derived from twistor string theory. They are the Roiban-Spradlin-Volovic-Witten (RSVW) formula [8, 9] for Yang-Mills

$$\mathcal{A}_{n,d}[\rho] = \int \mathrm{d}\mu_{n,d}\,|\tilde{\mathbf{g}}|^4\,\mathrm{PT}_n[\rho] \tag{7}$$

and the Cachazo-Skinner formula [10] for gravity

$$\mathcal{M}_{n,d} = \int \mathrm{d}\mu_{n,d}\,|\tilde{\mathbf{h}}|^8\,\det{}'(\mathbb{H})\,\det{}'(\mathbb{H}^\vee). \tag{8}$$

In these proceedings we will be focusing on the structures of the *integrands*, with more details on the integral to be found in [1].

These formulae provide the simplest closed form expressions for all-multiplicity scattering in these theories. Their simplicity rests on the fact that they harness the helicity configurations of the particles. Indeed, the integral localises onto a set of discrete solutions of size $E(n-3,d-1)$ (where $E(i,j)$ are the Eulerian numbers) known as the *refined (or polarised) scattering equations*. The evaluation is therefore simplified compared to other all-multiplicity formulae coming from the Cachazo-He-Yuan formalism [11].

Despite these two formulae being the simplest expression for all-multiplicity scattering in gauge theory and gravity, the double copy relation between them has always been implicit. In [1] we find the explicit double copy relation at the level of the integrands of the two theories, summarised in Figure 1.

2. Tree-level integrands in gauge theory and gravity

To understand the structures of the formulae (7) and (8), note that the integrals are over the *moduli space of maps of degree d from $\mathbb{P}^1$ to twistor space, with n marked points*. This

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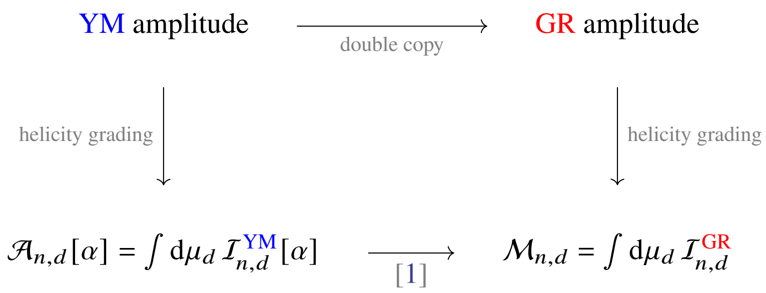

Figure 1: In [1] we find the explicit double copy relation between the helicity-graded amplitudes (7) and (8). This takes the form of a relation between the integrands of the two theories $\mathcal{I}^{\mathrm{YM}}_{n,d}[\alpha]$ and $\mathcal{I}^{\mathrm{GR}}_{n,d}$ in terms of a helicity-graded momentum kernel, shown in (17).

moduli integral, with specified external kinematics, is entirely localised to a set of solutions and a momentum-conserving δ-function.

Let $Z^A = (\mu^{\dot\alpha}, \lambda_\alpha)$ be holomorphic, homogeneous coordinates on $\mathbb{P}^3$ (i.e. defined up to scale $Z^A \sim rZ^A$ for $r \in \mathbb{C}^*$). Twistor space is then defined as

$$\mathbb{PT} = \{Z^A \in \mathbb{P}^3 |\, \lambda_\alpha \neq 0\}. \tag{9}$$

At N^{d-1}MHV, we will consider holomorphic maps of degree d

$$(\mu^{\dot\alpha}, \lambda_\alpha) = Z : \mathbb{P}^1 \to \mathbb{PT}. \tag{10}$$

There is an additional integral over n marked points $\{(\sigma_i^{\mathbf{1}}, \sigma_i^{\mathbf{2}})\}$ on $\mathbb{P}^1$ associated with each particle. The Möbius-invariant inner product between these homogeneous coordinates is $(i\, j) \coloneqq \epsilon_{\mathbf{ba}}\, \sigma_i^{\mathbf{a}} \sigma_j^{\mathbf{b}}$. Other useful definitions will be the *Parke-Taylor factor* and the *Vandermonde determinants*

$$\mathrm{PT}_n[\rho] \coloneqq \prod_{i=1}^{n} \frac{1}{(\rho(i)\, \rho(i+1))}, \qquad |\mathbf{p}| \coloneqq \prod_{\substack{i,j\in \mathbf{p}\\ i<j}} (i\, j). \tag{11}$$

where ρ is an ordering on the n particle labels and $\mathbf{p}$ is some set of points on $\mathbb{P}^1$.

We denote the set of $d+1$ negative helicity particles as $\tilde{\mathbf{g}}$ for gluons and $\tilde{\mathbf{h}}$ for gravitons. Similarly, the $n-d-1$ positive helicity particles are $\mathbf{g}$ and $\mathbf{h}$. The Yang-Mills integrand is

$$\mathcal{I}^{\mathrm{YM}}_{n,d}[\rho] \coloneqq |\tilde{\mathbf{g}}|^4 \mathrm{PT}_n[\rho] \tag{12}$$

with all quantities defined above. The gravity integrand on the other hand is

$$\mathcal{I}^{\mathrm{GR}}_{n,d} \coloneqq |\tilde{\mathbf{h}}|\, \det{}'(\mathbb{H})\, \det{}'(\mathbb{H}^\vee), \tag{13}$$

in terms of the reduced determinants of the Hodges and dual Hodges matrices

$$\mathbb{H}_{ij} = \frac{\left[\frac{\partial}{\partial\mu(\sigma_i)}\, \frac{\partial}{\partial\mu(\sigma_j)}\right]}{(i\, j)}, \qquad \mathbb{H}_{ii} = -\sum_{\substack{j\in\mathbf{h}\\ j\neq i}} \mathbb{H}_{ij} \prod_{l\in\tilde{\mathbf{h}}} \frac{(j\, l)}{(i\, l)}, \qquad i,j \in \mathbf{h}, \tag{14}$$

$$\mathbb{H}^\vee_{ij} = \frac{\langle \lambda(\sigma_i)\, \lambda(\sigma_j)\rangle}{(i\, j)}, \qquad \mathbb{H}^\vee_{ii} = -\sum_{\substack{j\in\tilde{\mathbf{h}}\\ j\neq i}} \mathbb{H}^\vee_{ij} \prod_{l\in\tilde{\mathbf{h}}\setminus\{i,j\}} \frac{(i\, l)}{(j\, l)}, \qquad i,j \in \tilde{\mathbf{h}}. \tag{15}$$

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These are $(d+1) \times (d+1)$ and $(n-d-1) \times (n-d-1)$ matrices respectively, with one null eigenvector each. The components are defined in terms of the map (10). The determinants are zero, but we define the reduced determinants in terms of the minors of these matrices

$$\det{}'(\mathbb{H}) = \frac{|\mathbb{H}^{b}_{b}|}{|\tilde{\mathbf{h}} \cup \{b\}|^2}, \qquad \det{}'(\mathbb{H}^{\vee}) = \frac{|\mathbb{H}^{\vee\, a}_{a}|}{|\tilde{\mathbf{h}} \setminus \{a\}|^2}, \tag{16}$$

independent of the choice of a and b. The Hodges matrices can be viewed as a rescaling of *weighted Laplacian matrices* on the sets $\mathbf{h}$ and $\tilde{\mathbf{h}}$. This gives them a natural interpretation using the matrix tree theorem. Using this and graph theoretic results related to orderings, we find a double copy relation between the two integrands.

3. The helicity-graded double copy

The helicity-graded double copy is expressed in terms of the integrands as

$$\mathcal{I}^{\text{GR}}_{n,d} = \sum_{\substack{\tilde{\rho},\tilde{\sigma} \in S_d \\ \rho,\sigma \in S_{n-d-2}}} \mathcal{I}^{\text{YM}}_{n,d}[a\tilde{\rho}b\rho]\, S_{n,d}[\tilde{\rho},\rho|\tilde{\sigma},\sigma]\, \mathcal{I}^{\text{YM}}_{n,d}[\tilde{\sigma}^T ab\sigma] \tag{17}$$

upon the identification of the particle sets $\mathbf{h} = \mathbf{g}, \tilde{\mathbf{h}} = \tilde{\mathbf{g}}$. The bases of orderings S_d and S_{n-d-2} are over orderings of $\tilde{\mathbf{h}} \setminus \{a\}$ and $\mathbf{h} \setminus \{b\}$ respectively. These bases are fixed by the choice of two arbitrary elements $a \in \tilde{\mathbf{h}}, b \in \mathbf{h}$.

The helicity-graded momentum kernel is

$$S_{n,d}[\tilde{\rho},\rho|\tilde{\sigma},\sigma] = \mathcal{D}[\tilde{\sigma},\sigma] \Bigg(\prod_{i \in \tilde{\mathbf{h}} \setminus \{a\}} \sum_{\substack{j <_{a\tilde{\rho}} i \\ j <_{a\tilde{\sigma}} i}} \tilde{\phi}_{ij} \Bigg) \times \Bigg(\prod_{k \in \mathbf{h} \setminus \{b\}} \sum_{\substack{l <_{b\rho} k \\ l <_{b\sigma} k}} \phi_{kl} \Bigg) \tag{18}$$

where $\mathcal{D}[\tilde{\sigma},\sigma]$ is a pre-factor absorbing projective weights in the integral. The terms $\tilde{\phi}_{ij}$ and ϕ_{ij} should be viewed as analogs of the Mandelstam variables s_{ij} on the sets $\tilde{\mathbf{h}}$ and $\mathbf{h}$ in twistor space. They are defined as

$$\tilde{\phi}_{ij} = \frac{\langle \lambda(\sigma_i)\, \lambda(\sigma_j) \rangle}{(i\, j)} \prod_{k \in (\tilde{\mathbf{h}} \cup \{b,t\}) \setminus \{i,j\}} \frac{1}{(k\, i)(k\, j)}, \qquad \phi_{ij} = \left[\frac{\partial}{\partial \mu(\sigma_i)}\, \frac{\partial}{\partial \mu(\sigma_j)} \right] (i\, j) \prod_{l \in \tilde{\mathbf{h}} \setminus \{a,y\}} (i\, l)(j\, l), \tag{19}$$

with $y \in \tilde{\mathbf{h}}$, $t \in \mathbf{h}$ some arbitrary fixed points, also appearing in $\mathcal{D}[\tilde{\sigma},\sigma]$.

Some comments on this relation are in order. The basis of orderings summed over in (17) has size $(n-d-2)! \times d!$, and is split by chirality of the particles. This is in contrast to the usual KLT double copy (1) which is over a basis of orderings of size $(n-3)!$. The integrands are localised on the set of $E(n-3, d-1)$ polarised scattering equations. This means that in general the number of distinct terms required to evaluate the double copy is greater than the KLT double copy (there is no overall simplification). However when $d = 1$ the helicity-graded double copy reduces directly to the KLT double copy with basis choice (3) (or relabelling thereof).

The matrix inverse of this kernel, integrated against scalar wavefunctions representatives in twistor space, is equal to certain colour-ordered amplitudes $m_n[\alpha|\beta]$ in biadjoint scalar theory in the following sense

$$m_n[a\tilde{\rho}b\rho|\tilde{\sigma}^T ab\sigma] = \int \mathrm{d}\mu_{n,d}\, S^{-1}_{n,d}[\tilde{\rho},\rho|\tilde{\sigma},\sigma]. \tag{20}$$

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This manifests one of the key properties of the field theory KLT kernel: its relation to biadjoint scalary theory [11]. As scalar amplitudes do not have a notion of helicity, the left-hand-side is only dependent on helicity from the partitioning of the colour ordering.

4. Conclusion

In these proceedings we have presented a novel double copy relation at the level of integrands for the helicity-graded representations of gauge theory and gravity scattering amplitudes at tree level, summarised in Figure 1. We hope to further explore this relation for other theories such as Einstein-Yang-Mills and applications towards the double copy on backgrounds (already explored partially in [1]).

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