THE STATISTICAL BOOTSTRAP MODEL AND ERICSON FLUCTUATIONS

S. FRAUTSCHI

1. Introduction

Since some of you are rather unfamiliar with statistical models, I shall begin by reminding you what is done with the statistical approach in nuclear physics (where it is well established) and indicating to what extent it has analogues in hadron physics. Then in the second part of my talk I shall focus in more detail on the level density one obtains for hadrons when the statistical approach is supplemented by a bootstrap assumption. Finally in the third portion of the talk I shall return to the nuclear analogy and use it as a guide for suggesting new phenomena in hadron physics - taking as a particular example Ericson fluctuations.

2. The analogy between nuclear and hadron statistical models

In 1936, Bethe [1] proposed a statistical model for the density of excited nuclear levels. He put Z protons and A-Z neutrons in a box with the normal nuclear radius, and considered a free fermion gas. That is, the potential was used only to provide the walls of the box; the residual nucleon-nucleon interactions inside the box were neglected.

As the energy is raised above the Fermi level, the number of nuclear states rises very fast :

i) the <u>first</u> excited single particle level can be filled in a number of ways by raising any of the nucleons near the top of the Fermi sea; each different way leaves a different hole behind and thus a different state;

ii) the <u>first two</u> excited single particle levels can be filled in an even larger number of ways by raising any two of the nucleons near the top of the Fermi sea; again each way leaves a different pair of holes behind and therefore a different state,...

Studying this problem quantitatively, Bethe found that for energies such that most of the fermions are still degenerate, the density of states

$$\rho(E) \equiv \frac{dn}{dE} \tag{1}$$

rises as

$$\rho(E) \propto \exp\left(\sqrt{\frac{AE}{c}}\right) \tag{2}$$

where c is a constant of order 2.5 MeV. Experimentally, excited nuclear levels show up as resonances. At high excitation energies one deduces from nuclear measurements that a rapid rise qualitatively consistent with Bethe's formula occurs.

Of course since this is a statistical model and the potential has been grossly oversimplified, it does not fit many nuclei in detail, especially at low excitations. The model has subsequently been refined by adding effects of the potential which distinguish between even and odd nuclei, put in some shell model effects, etc. These modifications improve the fit to specific nuclei.

To treat <u>reactions</u> statistically, further assumptions are needed. In the popular model of Bohr, for example, reactions proceed via a sum over direct channel resonances which are assumed to add up <u>incoherently</u>. If an average resonance decays into various final states at a rate proportional to phase space, it follows that reaction rates are proportional to the phase space of the final state.

Of course there are many cases in nuclear physics where statistical ideas do not work and one uses a "direct reaction" model. It turns out that what the nuclear physicist calls a "direct reaction" is none other than what a hadron physicist calls an "exchange reaction", particular examples being identifiable as meson exchange, nucleon exchange, Pomeron exchange, and photon exchange. The term "direct" refers here to time : the final state emerges quickly in a direct reaction, whereas the particles in a Bohr reaction spend a long time in the intermediate resonant state. In another language, the difference is that many direct channel resonance contributions to the amplitude add coherently in a "direct reaction", whereas they add incoherently in a Bohr reaction. The direct reaction, when present, provides a larger amplitude than the Bohr model precisely by means of this coherence. Direct reactions are especially prominent when the final state is closely related to the initial state.

Finally, nuclear physics presents cases where a mixed description is most useful. For example at energies above 20 MeV the production of various numbers of neutrons can be treated as a direct reaction which knocks out one or two neutrons, leaving an excited nucleus which boils off further nucleons with a thermodynamic distribution.

It is interesting to note that while Bethe's model of the level density depends on fewer assumptions than the models for reactions, it cannot be tested well without recourse to reaction models. To be sure, at low excitations a direct count of levels with full information on the degeneracy of each is available, analogous to the Rosenfeld Tables. But this is practicable only up to a certain energy, which is too low to test Bethe's asymptotic expression very well. Good information is again available even in heavy nuclei for excitations of about 7 MeV, just above the single neutron threshold, where the average resonance width is still less than the average spacing. At higher energies the resonances overlap and the level density is deduced only with the aid of further hypotheses, such as the "boiling off" picture described above, or Ericson fluctuations.

Table I contains a summary of the models we have just reviewed. To the right of each nuclear model is listed its analogue in hadron physics.

The most familiar hadron analogue is of course the non-statistical case of exchange reactions.

Another well-known case is the pure statistical model for hadron reactions popularized by Fermi [2]. The two incoming particles were assumed to coalesce in an interaction volume where thermodynamic equilibrium at a uniform T was reached, followed by emission proportional to phase space. This picture explains many features of $N\bar{N}$ annihilation. However, at higher energies it fails to produce sufficient forward peaking. [By imposing angular momentum conservation one does find peaking in the model; for example, in a spinless reaction $\ell_z = 0$ and the peaking is due to the fact that Legendre polynomials have a smaller envelope at 90° than at 0°. But this peaking is forward-backward symmetric and is anyway much too small; to fit the data it is absolutely essential to introduce coherence between different partial waves, which takes us outside the statistical picture.]

Hagedorn noted this problem, and also noticed that it was not adequate to count just the phase space for free π, K, and N. One should also include the effects of resonances. But how many ? Hagedorn attacked this problem and the reaction problem simultaneously, applying a bootstrap hypothesis.

As far as the level density of resonances is concerned, we shall show how it is obtained a little later, but let us immediately note Hagedorn's result [3] :

$$\rho(m) \propto e^{bm} \qquad (3)$$

TABLE I : STATISTICAL MODELS

Application	Nuclear	Hadron
1) Level density	Bethe $\rho(E) \propto \exp(\sqrt{AE/c})$	Hagedorn, Frautschi $\rho(m) \propto \exp(bm)$
2) Reactions (additional assumptions required)		
a) pure statistical – rate $\propto$ phase space	Bohr	Fermi, Hamer ($N\bar{N}$ annihilation near threshold)
b) mixed – at higher energies, dynamical reaction followed by evaporation $\propto$ phase space	Serber, Le Couteur, Jackson	Hagedorn
c) purely dynamical –	"direct reaction"	"exchange reaction"
3) Empirical study of level density at high energy	requires use of statistical models of reactions (boiling off of neutrons, Ericson fluctuations)	requires use of statistical models of reactions (boiling off of hadrons, perhaps Ericson fluctuations)

This very rapid growth has an important consequence. Consider the average energy of a set of hadron states in thermodynamic equilibrium at temperature T :

$$\bar{E} = \frac{\int_{m_\pi}^{\infty} dm\,\rho(m) \int d^3p \sqrt{p^2+m^2}\, \exp\left(-\sqrt{m^2+p^2}/kT\right)}{\int_{m_\pi}^{\infty} dm\,\rho(m) \int d^3p \quad \exp\left(-\sqrt{m^2+p^2}/kT\right)} . \tag{4}$$

Employ the level density $\rho(m) = \text{const.}e^{bm}$ (ignoring possible powers of m for simplicity) and approximate E by $m+(p^2/2m)$, a good approximation at high masses where the important behaviour occurs. Then

$$\bar{E} \simeq \frac{\int_{m_\pi}^{\infty} dm\, m \exp\left(bm - \frac{m}{kT}\right) \int d^3p \exp\left(\frac{-p^2}{2mkT}\right)}{\int_{m_\pi}^{\infty} dm \quad \exp\left(bm - \frac{m}{kT}\right) \int d^3p \exp\left(\frac{-p^2}{2mkT}\right)} . \tag{5}$$

Evidently the integrals are undefined unless

$$b < \frac{1}{kT} . \tag{6}$$

If we call

$$b \equiv \frac{1}{kT_0} , \tag{7}$$

the condition becomes

$$T < T_0 . \tag{8}$$

Thus T_0 is a <u>maximum temperature</u>.

Physically, what happens in this model is that if we increase the energy in a fixed volume, it goes into the mass of new particles rather than into raising the kinetic energy of the particles.

Turning now to reactions, we recall that the purely statistical approach of Fermi failed at high energies. I shall describe briefly the approach of Hagedorn [4] (in collaboration with Ranft), to indicate that it is a mixed description consisting of a non-statistical collision followed by thermodynamic "boiling off", somewhat analogous to what is done in nuclear physics at high energies.

In the Hagedorn-Ranft picture of scattering the two colliding bodies never coalesce into a single interaction volume. Each continues on its way, but "heated" internally by the collision, the temperature being higher on the side next to the other projectile where the "friction" has been most intense. By local energy conservation the cool outer side of each projectile retains much of its original longitudinal velocity, while the hot inner side has converted much of its original longitudinal motion into internal energy. This part of the description is evidently non-statistical (it introduces the essential distinction between longitudinal and transverse motion). However, the second step in the description is purely thermodynamic - the internally excited projectiles "boil off" hadrons with distribution controlled by the local temperature.

In this model the momentum and mass dependence of final state particles relative to the decaying projectile are determined essentially by the Boltzmann factor

$$B = \exp\left[-\frac{\sqrt{M^2+p_{\parallel}^2+p_{\perp}^2}}{kT}\right] \tag{9}$$

with T near T_0 for high energy reactions. The longitudinal distribution in the lab. frame also depends strongly on the detailed assumptions made about the velocity distribution of the projectile, which we do not wish to go into here. But the $p_{\perp}$ and mass dependence can be read off directly from the Boltzmann factor, and have the asymptotic behaviour

$$B \xrightarrow[\substack{p_\perp \to \infty \\ M,\, p_\parallel \text{ fixed}}]{} \exp\left(\frac{-p_\perp}{kT}\right) \tag{10}$$

$$B \xrightarrow[\substack{M \to \infty \\ p_\perp,\, p_\parallel \text{ fixed}}]{} \exp\left(\frac{-M}{kT}\right). \tag{11}$$

Comparing with data on the $p_\perp$ distribution, Hagedorn and Ranft [4] have deduced

$$T_0 \simeq 160 \text{ MeV}. \tag{12}$$

In my own work on this subject [5], I made more explicit a point that was already implicit in Hagedorn's work : the hadron level density can be deduced from just two conditions (a statistical condition and a bootstrap condition on the constituents) without direct reference to scattering or to the assumption that local thermodynamic equilibrium is achieved in scattering. I also introduced some technical modifications (discussed below) that make it possible to pin down the level density more precisely.

The situation, as reviewed in Table I, now displays a considerable analogy to nuclear physics :

i) the level density is determined on the basis of a simple statistical assumption, plus one assumption concerning the constituents.

ii) some low energy reactions can be understood statistically [for the latest work on $N\bar{N}$ annihilation see Hamer [6]]. An extra condition must be met in this case - lack of coherence among direct channel resonances.

iii) some high energy reactions can be understood in terms of a more complicated model involving both dynamical and statistical assumptions.

iv) as in nuclear physics, the predicted $\rho(m)$ cannot be conclusively established by direct count of levels (although the existing spectrum is quite compatible with Hagedorn's distribution as far as it goes). In fact, the detailed experimental analysis of resonances in the πN channel has been pushed up near the energy where levels for low J^P are predicted to start overlapping, making further disentangling of individual levels prohibitively difficult. The best evidence for the Hagedorn spectrum is obtained by assuming the Hagedorn-Ranft model for reactions, and comparing the Boltzmann factor (9) which occurs in that model with experimental $p_\perp$ distributions.

3. The hadron level density

Now let us describe in more detail the model for the hadron level density [5]. Just as the nucleus is considered to be a compound with A constituents drawn from two varieties (n and p), we consider the hadron to be a compound with $n \geq 2$ constituents drawn from various varieties (e.g., the three varieties of quark in the quark model, or the many varieties of hadron in the bootstrap model).

The potential is used explicitly only to define the walls of the box - with the radius of order 10^{-13} cm, since we know hadron structure is confined within a distance of this order. Inside the box, constituents will circulate without interacting explicitly. Of course this is the crudest dynamics possible, but it does simplify the problem enough to yield solutions, and it can be improved later if some detailed effects of the interaction are understood, just as Bethe's free fermion gas model was later improved.

Mathematically, the counting proceeds as follows. For <u>one particle</u>, the density of states inside the box is

$$\frac{V\,d^3p_i}{h^3}$$

For <u>n independent particles</u> with total energy m it is

$$\frac{d(\text{number of states})}{dm} \equiv \rho_n(m)$$

$$= \left(\frac{V}{h^3}\right)^{n-1} \prod_{i=1}^{n} \int d^3p_i \, \delta\left(\sum_1^n E_i - m\right) \delta^3\left(\sum_1^n \vec{p}_i\right). \tag{13}$$

Here we have counted only the density of levels with centre-of-mass at rest, because this is the density to be identified with the number of hadron states per unit interval of rest mass.

For example, consider some simple models for the constituents

i) <u>Quark-antiquark model of mesons</u>. Here $n = 2$, the integral is trivial, and one finds

$$\rho(m) \sim m^2. \tag{14}$$

ii) <u>Three-quark model of baryons</u>. Here $n = 3$, and the extra $\int d^3p_i$ increases the density of states to

$$\rho(m) \sim m^5. \tag{15}$$

iii) <u>Single-elementary particle model of mesons</u>. Suppose there were a single elementary boson x, and all mesons were made of xx pairs, xxx triplets, xxxx quartets, etc. ($n = 2,3,\ldots\infty$). In this case the density of states would be

$$\rho(m) = \sum_{n=2}^{\infty} \left(\frac{V}{h^3}\right)^{n-1} \frac{1}{n!} \prod_{i=1}^{n} \int d^3p_i \, \delta\left(\sum_1^n E_i - m\right) \delta^3\left(\sum_1^n \vec{p}_i\right) \tag{16}$$

where the factor $1/n!$ appears because only totally symmetric states of n bosons should be counted. The integrals in (16) can be evaluated approximately and one finds

$$\rho(m) \sim e^{bm^{3/4}}, \tag{17}$$

a much more rapid growth than in previous examples because states of all n are now included.

All of the preceding examples involved elementary constituents. The model we wish to focus on, however, is the <u>bootstrap model of hadrons</u>, in which the constituents are the hadrons themselves. The model can be represented schematically by

$$\begin{pmatrix}\pi\\ K\\ \eta\\ \vdots\end{pmatrix} = \overset{n=2}{\begin{pmatrix}\pi\pi\\ K\pi\\ KK\\ \vdots\end{pmatrix}} + \overset{n=3}{\begin{pmatrix}\pi\pi\pi\\ K\pi\pi\\ \vdots\end{pmatrix}} + \overset{n=4}{\begin{pmatrix}\pi\pi\pi\pi\\ \vdots\end{pmatrix}} + \cdots \tag{18}$$

The equation for the density of states is

$$\rho_{out}(m) = \sum_{n=2}^{\infty}\left(\frac{V}{h^3}\right)^{n-1}\frac{1}{n!}\prod_{i=1}^{n}\int dm_i\,\rho_{in}(m_i)\int d^3p_i\,\delta\Big(\sum_1^n E_i - m\Big)\,\delta^3\Big(\sum_1^n \vec{p}_i\Big) \tag{19}$$

which can be explained as follows :

i) the integral over mass appears on the right-hand side because each particle in the box can take on not only different states of motion with phase space d^3p_i, but also different states of mass with density labelled by $\rho_{in}(m)$. Included in the single particle density $\rho_{in}(m_i)$ are all different states of spin, charge, strangeness, baryon number, etc. For example, π is counted as three states (π^+, π^0, π^-), ρ as nine states $(\rho^+, \rho^0, \rho^-$, each with $2S+1$ spin states) and so on.

ii) note that although we are not taking explicit account of interactions, they are included implicitly to a great extent by counting as constituents all the resonant states which result from interactions. For example, we count both $\pi\pi\pi$ and $\pi\rho$. To understand the connection between counting resonances and including interactions, consider two particles which attract each other moving around in a box. As a result of the attraction, the wave function involving the relative co-ordinate of the two particles oscillates more rapidly than usual when the two particles are close together. The more rapid oscillation means that more states fit into the box; specifically, when there is one extra oscillation (phase shift of 180^o), one extra state can be fit into the box. In this case, although an exact calculation would count states of motion of the original two particles with their mutual potential, it is approximately valid to omit the potential and count states of motion of the original two particles (treated as non-interacting) <u>plus</u> states of motion of the resonance. This is what we have been doing by counting all resonances as independent particles.

iii) the factor $1/n!$, which was required for states consisting of n identical particles, is also needed for states consisting of non-identical particles to avoid double counting.

iv) we are making one error : the Pauli exclusion principle has been ignored. The resulting overestimate of phase space should be slight because states containing pairs of the <u>same</u> fermion are expected to be statistically unimportant in the hadron spectrum [this is confirmed in detail in work by Nahm [7]].

v) on the left side of (19) we introduce the nomenclature $\rho_{out}(m)$ for the total density of states in the box. In a complete bootstrap theory, $\rho_{out}(m)$ would be the same as $\rho_{in}(m)$, but in an approximate model such as ours it is not possible to make them consistent over the entire spectrum, and we must keep the separate labels. However, we can at least require

$$\rho_{out}(m) \xrightarrow[m\to\infty]{} \rho_{in}(m). \tag{20}$$

At low m our statistical approach cannot hope to give the exact self-consistent $\rho(m)$.

The density of states (19) and the bootstrap condition (20) define our version of the bootstrap model. From the previous example (17) of the level density obtained from a single variety of input particle, it is clear that $\rho(m)$ rises at least as fast as $\exp(m^{3/4})$. This motivates us to study the self-consistency of the form

$$\rho_{in}(m) = c m^a \exp(b m^p) . \tag{21}$$

Proceeding in this way, we shall not be able to find a unique solution, but we shall be able to establish the main features that any solution must have at large m.

It can be shown [5] that :

i) for $p < 1$, ρ_{out} grows exponentially faster than ρ_{in}, so this case is not self-consistent.

ii) for $p > 1$, ρ_{out} grows exponentially slower than ρ_{in}, so again this case is not self-consistent.

iii) the remaining case $p = 1$ does allow solutions, self-consistent not only in the exponent but also in the power and numerical coefficient, provided

$$a < -5/2 . \tag{22}$$

The level density discussed thus far is the total one, summing over all quantum numbers. The level densities ρ_{QSB} for particular quantum numbers have also been studied [5]. It is found that a self-consistent solution for all of them is

$$\rho_{QSB}(m) \xrightarrow[m \to \infty]{} c_{QSB}\, m^{a'} e^{bm} \tag{23}$$

$$\rho(m) = \sum_{QSB} \rho_{QSB}(m) \xrightarrow[m \to \infty]{} c\, m^a e^{bm} . \tag{24}$$

The parameters a' and b are common to all Q, S, B. Whether $a' = a$ depends on details of the model [8].

Historically, a number of quite different models have yielded this sort of exponential growth (Table II). Hagedorn [3] was the first to give the reasoning I have followed. He demonstrated the existence of a consistent exponential solution $(\ln \rho_{in} \to \ln \rho_{out})$

TABLE II : MODELS YIELDING $\rho(m) \to cm^a e^{bm}$

Model	a	b $(b_{exp} \approx \frac{1}{160 MeV})$
1) Hagedorn [3] $\ln \rho_{in} \to \ln \rho_{out}$	$a_{in} \leq -5/2$	------
2) Frautschi [5] $\rho_{in} \to \rho_{out}$	$a < -5/2$	------
3) Hamer and Frautschi [8] ($\rho_{in} = \rho_{out}$ above some fixed mass) Nahm [7]	$a = -3$	$\approx 1/m_\pi$
4) Duality [9]	------	------
5) Veneziano [10] $b = 2\pi \left[-\frac{\alpha'}{6}(2a+1)\right]^{\frac{1}{2}}$	$a = -5/2$	1/190 MeV
	-3	1/175 MeV
	$-7/2$	1/160 MeV
	⋮	⋮

but was not able to obtain consistency between a_{in} and a_{out}. The reason was that he effectively included the centre-of-mass phase space d^3p_{CM} on one side of the equation but not on the other, with the result that ρ_{out} grew faster than ρ_{in} by $m^{\frac{3}{2}}$ or more (normally in statistical mechanics the extra degrees of freedom of the centre-of-mass do not matter, but in the present few-body case they do affect the wer behaviour).

In my own work [5] I studied the same model, but calculating directly in phase space (microcanonical ensemble) rather than thermodynamically (canonical ensemble) as Hagedorn had done. Putting the centre-of-mass at rest consistently on both sides of the equation, I was able to make the powers balance and $\rho_{in} \to \rho_{out}$. This stronger condition did not allow the solution $a = -5/2$, which had been allowed by Hagedorn and was in fact the case that had usually been discussed. This may seem like a very small change, but it has important consequences which I shall describe later.

Subsequently Hamer and I [8] (with numerical methods) and Nahm [7] (with analytical methods) tried stronger conditions. For example, Hamer and I required $\rho_{in} = \rho_{out}$ above some finite mass M. These stronger conditions lead to $a = -3$ independent of details. By taking specific models for ρ_{in} below the mass where consistency is required, Hamer and I also derived specific values for b. We tried :

i) input $\pi^+ \pi^0 \pi^-$ $(M = 2m_\pi)$

ii) input lowest 35 mesons $(M \approx 1$ GeV$)$

iii) input lowest 35 mesons and 56 baryons.

Inserting these low mass inputs into the hadron box, we built up the higher mass states on a computer [using (19) and $\rho_{in}(m) = \rho_{out}(m)$ at $m > M$]. In all cases we obtained

$$b \approx 1/m_\pi , \tag{25}$$

near the experimental value. Nahm has obtained similar values.

Apart from the numerical coefficient of $1/m_\pi$, the value of b can be understood on dimensional grounds : m_π is essentially the only mass in the input (the lowest input mass is m_π, and the radius of the box is of the order $\hbar/m_\pi c$).

Recently it has been shown that dual models and Veneziano models, which involve assumptions quite different from Hagedorn's, lead to the same sort of spectrum [9),10)]. In the Veneziano model, even the possible values of the power "a" are similar (Table II)!

Why do dual and statistical models give such similar spectra? An answer has been suggested by Krzywicki and Brout [9),5)]. According to duality the scattering in each channel can be described in terms of direct channel resonances. We make the usual assumption that each resonance has a limited coupling strength in the sense that $<\Gamma_{tot}>$ changes only slowly with energy. Then to describe the scattering in each channel in terms of direct channel resonances, the number of resonances must be comparable to the number of channels. By this type of argument, one obtains a mathematical relation with the structure

$$\rho(m) \propto \frac{1}{2!}\iint \rho(m_1)\rho(m_2)\,dm_1\,dm_2 + \frac{1}{3!}\iiint \rho(m_1)\rho(m_2)\rho(m_3)\,dm_1\,dm_2\,dm_3 \qquad (26)$$
$$+ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

where the terms on the right side give the number of two-particle channels, three-particle channels, etc., and the integrations are over $\sum m_i \leq m$. This equation for the level density in a dual model has the same sort of structure as the statistical model relations (19), (20), and therefore has the same sort of exponentially growing solution.

This result shows that one does not need the full apparatus of the Veneziano model to get an exponentially rising spectrum. Quite possibly the exponentially rising spectrum is a very general feature of any bootstrap model which treats all non-exotic channels on the same footing.

Another question suggested by Table II is, what difference does it make whether $a=-5/2$ or -3 ? Actually, although the exponential factor is clearly the most important part of the predicted level density, the power also makes a significant difference in several respects. Consider the statistically dominant decays of a massive resonance with mass m. This is easy because the model has already provided us with the relative phase space as a function of the kinetic energy and number of particles. Statistically an average resonance decay will go to final states in the same proportion as their phase space. In practice, there will be a decay chain; we consider only the first generation. We find :

i) for any value of a, the phase space for each emitted particle depends on kinetic energy Q_i as $\exp(-bQ_i)=\exp(-Q_i/kT_0)$. Thus phase space is populated most heavily in a strip of width several kT_0 near $\sum m_i \simeq m$.

ii) for $a=-5/2$ there is an appreciable probability for particles to be anywhere in this strip, but for $a < -5/2$ only the corners of the strip (one particle with mass near m, the others with mass near m_π) are heavily populated.

iii) furthermore the average number of particles is $\langle n\rangle \sim \ln m$ for $a=-5/2$, but $\langle n\rangle = 2.4$ <u>independent of m</u> for $a=-3$. For $a=-3$, two-body decays are dominant (69% probability).

The implication of results ii) and iii) for $a=-3$ is that a heavy resonance decays in a long chain, emitting typically one light particle with $E=$ several m_π at each step. This is hard to test directly. But Hagedorn and Ranft [11] point out that their model scales at high energies if combined with this picture of resonance decay (essentially because it predicts the spectrum of low mass particles emitted in resonance decay is independent of the total energy).

In another application, Carlitz [12] has pointed out that at extremely high energy densities such as may occur in the big bang, the phase space for a <u>macroscopic</u> volume will exhibit the same feature as we noted above for a single hadron volume – the statistically favoured state will consist of one very heavy resonance surrounded by a crowd of low mass particles. This very uneven distribution of energy is the source of the large fluctuations one finds in this model, which may have cosmological significance [13].

4. Ericson fluctuations in nuclear physics

Let us now return to the nuclear analogy in order to borrow the idea of Ericson fluctuations, which may be of considerable significance for testing and applying statistical models of hadrons. We shall begin by describing what they look like and how they are analyzed in nuclear physics [14].

The general idea is that at energies high enough for resonances to overlap, one still sees peaks and dips. These are attributed not to individual resonances, but to fluctuations in the number and coupling strength of the overlapping resonances.

The idea can be applied to nuclei even in the presence of "direct reactions" where the simple Bohr model does not hold. Consider for example an elastic reaction such as

$$p + Z \longrightarrow p + Z \, . \tag{27}$$

The imaginary part of the non-flip amplitude is expected to dominate Write it as a sum over direct channel resonance contributions :

$$\begin{aligned} \mathrm{Im}\, A_J^{el} &\simeq \mathrm{Im} \sum_n \frac{\gamma_n \gamma_n}{-E + E_n - i\Gamma_n/2} \\ &\simeq \frac{2}{\langle \Gamma \rangle} \sum_n^{N_J} \gamma_n^2 \end{aligned} \tag{28}$$

where N_J is the number of "overlapping resonances", i.e., the resonances in an interval $\Delta E \sim \Gamma$. Since we have chosen an elasti amplitude, the contributions of individual resonances tend to add up with the same sign *). Nevertheless from one energy interval

*) Although each resonance contribution has in general an extra phase factor when the resonances overlap, which we have ignored in writing (28).

$\Delta E \simeq \Gamma$ to the next there will be statistical fluctuations in the strength of $\sum \gamma^2$, of relative order $1/\sqrt{N_J}$. Thus A_J^{el} can be expressed as the sum of a dominant (mainly imaginary) coherent term A_J^C and a smaller fluctuation term A_J^F :

$$A_J^{el} = A_J^C + A_J^F \tag{29}$$

Similarly, adding all partial waves, we have

$$A^{el}(\theta) = A^C(\theta) + A^F(\theta) . \tag{30}$$

At $\theta = 0^{\circ}$ where all partial waves add coherently, A^F/A^C is expected to be of order $1/\sqrt{N}$ where

$$N = \sum_J N_J \simeq \Gamma\rho . \tag{31}$$

The dominant term $A^C(\theta)$ will exhibit the usual sharp diffraction peak, possibly with diffraction minima at angles determined by the nuclear radius. This structure varies only slowly as a function of energy. By contrast A^F varies rapidly, on a scale $\Delta E \simeq \Gamma$. One looks for these fluctuations in <u>interference</u> with the dominant coherent term. The search is aided by the angular dependence of A^F, which is on the average weak and symmetric about 90° because partial waves do not add coherently in A^F (recall our discussion of the Fermi model in Section 2). What angular dependence $\langle A^F \rangle$ does have comes essentially from the envelope of the Legendre polynomials which is less at 90°. Thus A^F is relatively easier to see at large angles where A^C has fallen far below its peak value.

A classic example is

$$p + {}^{56}Fe \rightarrow p + {}^{56}Fe \tag{32}$$

which has been studied [15] over the range $E_p = 9.3$ to 9.6 MeV at intervals of 2 to 5 keV, at each of several angles between 63° and 171°. The very close spacing of points in energy is necessary to resolve the Ericson fluctuations, which have a width comparable to the average resonance width $\Gamma \approx 3$ keV. The data on $d\sigma/d\Omega$ are shown in Fig. 1. At fixed energy, the dependence on angle is characteristic of a diffraction peak with minima. But at fixed angle, the cross-section exhibits rapid Ericson fluctuations as the energy is varied.

This example illustrates several points :

i) not every peak or dip is an Ericson fluctuation. In the example, the two large dips in the angular distribution are identified as diffraction minima rather than Ericson fluctuations because they persist at all energies.

ii) it is in fact hard to distinguish Ericson fluctuations in the angular dependence at a single energy. The average width $\Delta\Theta$ of a fluctuation peak or dip cannot be much less than $\Delta\Theta \simeq 180^{\circ}/\ell_{max}$ where $\ell_{max} \simeq pR$ is the largest strongly scattered partial wave. In many cases this is not much smaller than the spacing between diffraction dips in A^C.

iii) at fixed Θ or t one sees the Ericson fluctuations more easily. Their typical width ΔE is of order (is in fact a measure of) Γ : in the example above an interval of $\simeq 100\,\Gamma$ was surveyed, allowing many fluctuations to be seen.

It is the relative height of the fluctuations which is sensitive to A^F/A^C and thus to $N = \Gamma\rho$. Nuclear physicists have devised the following quantitative measure [14] to be applied at <u>fixed</u> Θ or t. One measures the differential cross-section, denoted by $\sigma(E)$, at evenly spaced intervals over a range $E_1 \leq E \leq E_2$. The average $\langle\sigma\rangle$ is formed over this range. Then one forms the normalized correlation function

$$C_{exp} = \left\langle \frac{(\sigma(E) - \langle\sigma\rangle)^2}{\langle\sigma\rangle^2} \right\rangle \tag{33}$$

as a measure of fluctuations. Theoretically, if $A^C(\Theta)$ is essentially constant over the range of energies considered, $\langle\sigma\rangle$ can be expressed as

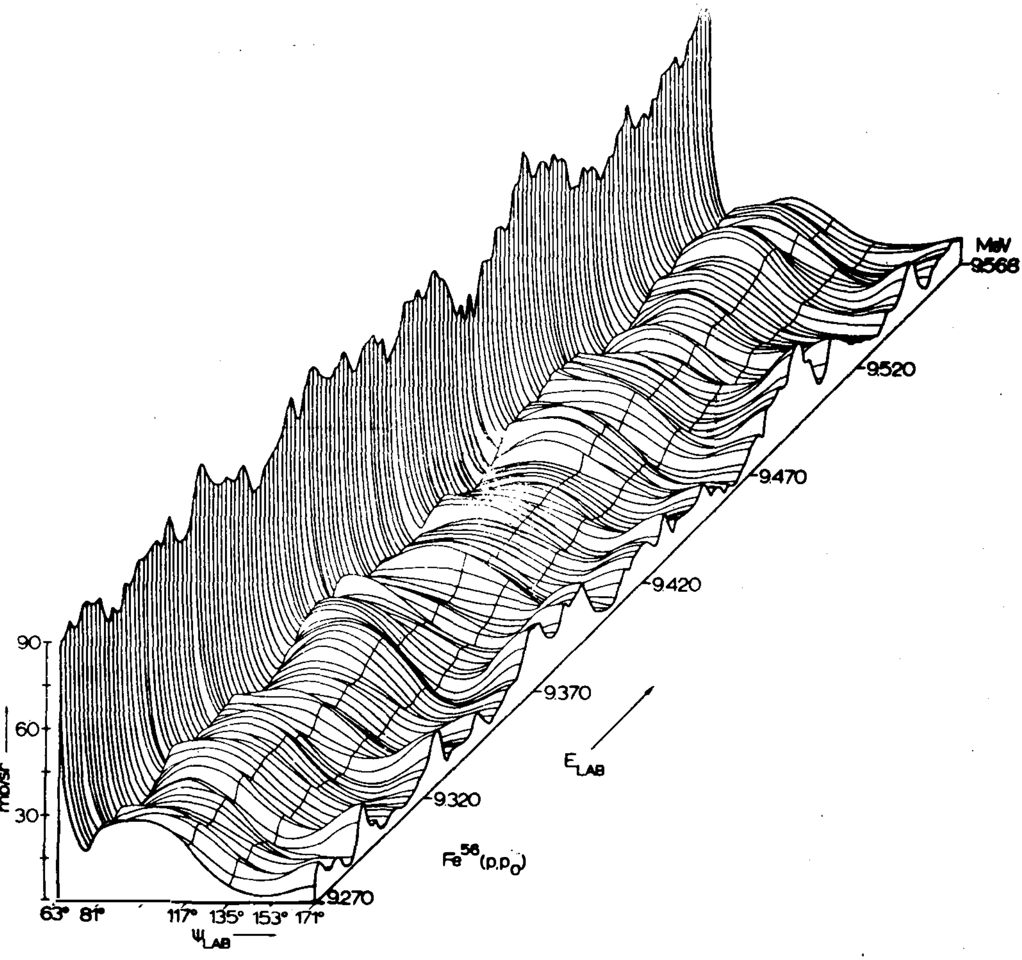


Fig. 1 : $d\sigma/d\Omega$ for $p+^{56}Fe \rightarrow p+^{56}Fe$ measured in steps of 2-5 keV around 9.4 MeV [taken from Ref. 15].

$$\langle\sigma\rangle = \langle|A^C + A^F|^2\rangle$$
$$= |A^C|^2 + \langle|A^F|^2\rangle \equiv \sigma^C + \sigma^F \qquad (34)$$

since the interference term averages to zero. In terms of these quantities, one can show that[14)]

$$C_{+h} = \frac{(\sigma^F)^2 + 2\sigma^F\sigma^C}{(\sigma^F + \sigma^C)^2} \qquad (35)$$

which in our case (A^F interfering with larger A^C) is approximately $C = 2\sigma^F/\sigma^C$. Comparing (35) and (33), one deduces σ^F/σ^C from the data. This information, plus the independent determination of Γ from the average width of fluctuations in energy, allows an estimate of the level density

$$\rho \simeq \Gamma^{-1} N \simeq \Gamma^{-1}\,\sigma^F(0^\circ)/\sigma^C(0^\circ).$$

5. Ericson fluctuations in hadron physics

Historically, Ericson fluctuations were searched for by Allaby et al. [16)], who studied

$$p + p \longrightarrow p + p \qquad (36)$$

at 16.9 GeV/c at centre-of-mass angles 67° to 90°. They found the cross-section fell off smoothly. Another relevant experiment was that of Akerlof et al. [17)] who studied reaction (36) at fixed $\theta_{CM} = 90^\circ$, varying E in small steps. Once again, no evidence for fluctuations was found. These negative results discouraged further searches for Ericson fluctuations in hadron physics.

On the other hand, the motivation given earlier for Ericson fluctuations is very general and should apply also to hadrons. But not necessarily to all hadron reactions; if the pp channel lacks resonances, no fluctuations need occur there ! So the early searches may simply have had the bad luck to look in the wrong place.

It is important then to continue the search for Ericson fluctuations, because :

i) in principle if one believes in a statistical model at all, then it must be possible to find fluctuations.

ii) the study of fluctuations could provide a practical tool giving information on $\rho(E)$ at energies where resonances overlap and a direct count cannot be made. This approach would supplement the only existing method - Hagedorn's analysis of the $p_\perp$ distribution.

Thus motivated, we proceed to look for fluctuations in channels where resonances exist [18]. We shall borrow the whole mathematical apparatus of the nuclear analysis, always remembering that it will be less convincing for hadrons because whereas $\Gamma_{nuclear}$ was very small on the scale on which the nuclear level density varies ($\approx$ 1 MeV), $\Gamma_{hadron} \approx m_\pi$ is the same as the scale on which the hadron level density varies.

In practice, one has the choice of studying reactions such as backward $K^- p \to K^- p$ where A^C is small (here we expect large fluctuations, but the data are usually quite incomplete) or reactions where A^C is large (smaller fluctuations, but better data). We shall focus first on the later type of case as exemplified by

$$\pi + p \longrightarrow \pi + p \,. \tag{37}$$

The coherent term $|A^C|^2$ is dominant, at least near the forward and backward directions, and is therefore easy to estimate from the experimental cross-section.

To obtain a crude estimate for the number of overlapping resonances

$$N(m) \simeq \Gamma(m) \rho(m) \tag{38}$$

we take

$$\Gamma(m) \simeq m_\pi \tag{39}$$

and

$$\rho(m) = \frac{c}{m^{7/2}}\, e^{m/kT_0}\,. \tag{40}$$

Equation (40) is the Hagedorn level density with $a = -3$ and with an extra $m^{-\frac{1}{2}}$ which occurs because only that fraction of intermediate states with $J_z = J_z$(initial) contributes to each amplitude. We use Hagedorn and Ranft's value [4]

$$kT_0 = 160\,\mathrm{MeV} \tag{41}$$

and

$$c = \frac{m_\pi^{5/2}}{3} \tag{42}$$

which is roughly consistent with the numerical studies of Hamer and Frautschi [8].

If the elastic amplitude were made up entirely of direct channel resonances, one would estimate

$$\left|A^F(0^\circ)\right| \simeq \frac{\mathrm{Im}\,A^C(0^\circ)}{\sqrt{N}} \tag{43}$$

in which case

$$\frac{d\sigma^F}{dt}(0^\circ) = \frac{1}{N}\,\frac{d\sigma^C}{dt}(0^\circ)\,x^2 \tag{44}$$

with x^2 of order 1. However, according to Freund and Harari [19], only a fraction

$$y = \frac{\sigma^{tot}(E) - \sigma^{tot}(\infty)}{\sigma^{tot}(E)} \qquad (45)$$

of the forward amplitude couples to direct channel resonances. So the amplitude A^F must be multiplied by this small fraction and, in what follows, we take $x^2 = y^2$.

There is accurate elastic πN data of the type we require (E varied over a considerable range in small, equally spaced steps at fixed θ or momentum transfer) only at 0^o and 180^o. At 0^o the data are in the form of total cross-section measurements; for the purposes of our analysis we converted these data into $\mathrm{Im}\,A\,(0^o)$ by means of the optical theorem and squared to form an effective "$d\sigma(0^o)/dt$". The data [20)-24)] for $\pi^- p \rightarrow \pi^- p$ and $\pi^+ p \rightarrow \pi^+ p$ are displayed in Figs. 2-4.

Also displayed is $d\sigma^F/dt$, calculated at 0^o by means of Eqs. (38)-(45). We recall that $d\sigma^F/dt$ takes the same value at 180^o as 0^o, and we give it the same value for $\pi^+ p$ and $\pi^- p$ elastic scattering. Thus $d\sigma^F/dt$ is the same for all cases in Figs. 2-4.

The peaks and dips in the data of Figs. 2-4 are traditionally interpreted in terms of individual (high J) resonances interfering with a smooth background. We are proposing a somewhat different interpretation in terms of numerous overlapping resonances with fluctuating level density and coupling strengths. To show whether the peaks and dips are commonly due to fluctuations in more than one J^P state, as required by our interpretation, comparable data at angles other than 0^o and 180^o will be needed. Lacking such information, the best we can do at present is to show that our interpretation is consistent with the existing data.

It is immediately clear from Figs. 2-4 that our interpretation is qualitatively consistent with the data. Through interference terms, the fluctuations in $d\sigma/dt$ are expected to be of order 10%

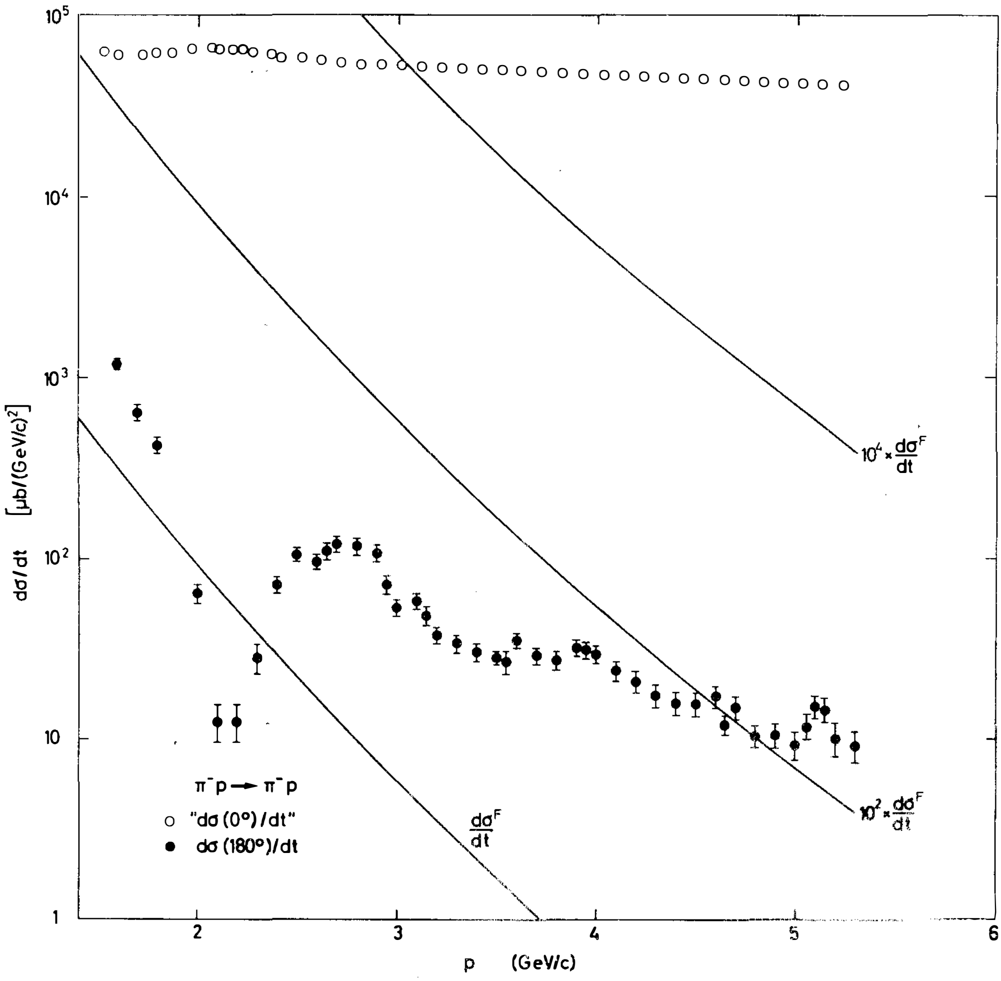


Fig. 2 : Comparison of data on $\pi^- p \to \pi^- p$ at 180° and 0° with the theoretical cross-section $d\sigma^F/dt$ at which fluctuations would reach 100%. The experimental values for $d\sigma(180^{\circ})/dt$ are from Ref. 20), "$d\sigma(0^{\circ})/dt$" is calculated from data of Refs. 21) and 22), and $d\sigma^F/dt$ is calculated from Eqs. (38)-(45) of the text.

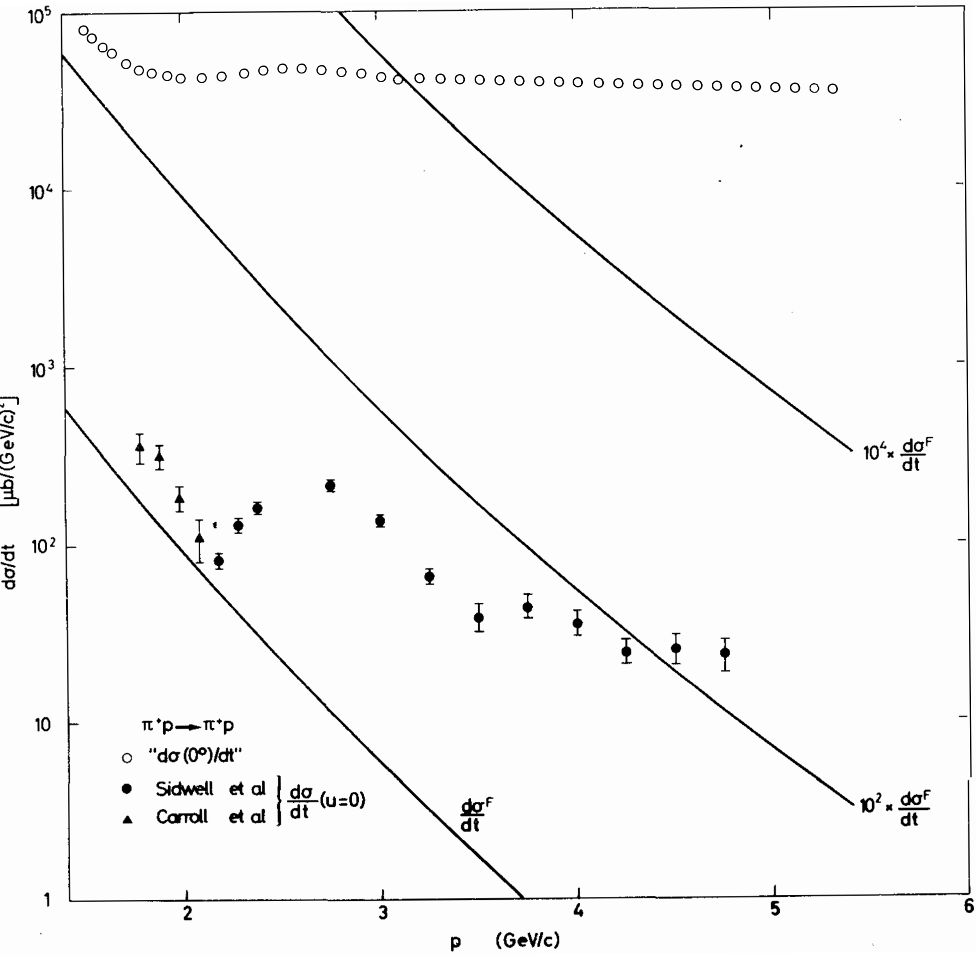


Fig. 3 : Comparison of data on $\pi^+ p \to \pi^+ p$ at $u = 0$ and 0^o with $d\sigma^F(0^o)/dt$. The experimental values for $d\sigma(u=0)/dt$ are from Refs. 23) and (24), and "$d\sigma(0^o)/dt$" is calculated from data of Refs. 21) and 22).

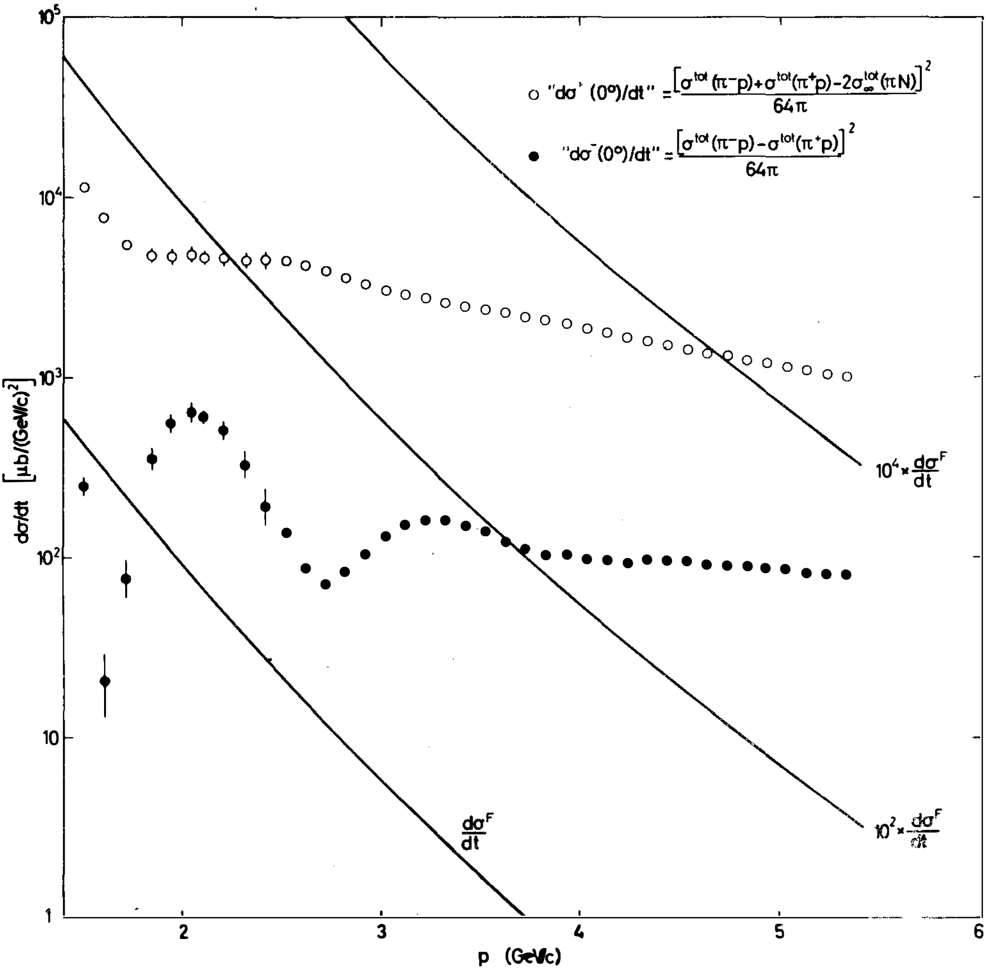


Fig. 4 : π N elastic scattering : comparision of "$d\sigma(0^{\circ})/dt$" for $I = 1$ exchange and $I = 0$ Regge exchange [calculated from data of Refs. 21) and 22)] with $d\sigma^F/dt$.

when $d\sigma/dt = 10^2 d\sigma^F/dt$, and of order 1% when $d\sigma/dt = 10^4 d\sigma^F/dt$. The experimental fluctuations are indeed of this order of magnitude; they are greater for the smaller cross-sections and die away more slowly for $d\sigma(180^\circ)/dt \sim s^{-2.5}$ than for "$d\sigma(0^\circ)/dt$" $\sim$ constant.

To show that our interpretation is quantitatively consistent with the data, we must study the correlation function, generalized to the present case where the non-fluctuating part of the cross-section varies strongly with E. An appropriate generalization is

$$C_{exp} = \left\langle \left(\frac{\sigma(E) - \sigma^S(E)}{\sigma^S(E)} \right)^2 \right\rangle \tag{46}$$

where $\sigma^S(E)$ is a smooth curve drawn through the data. The curve is subjective, but constrained to have the same average as $\sigma(E)$. In practice we always used a simple monotonically decreasing form for $\sigma^S(E)$. The resulting values of C_{exp} for elastic $\pi^- p$ and $\pi^+ p$ scattering, backward and forward, are displayed in Table III.

Also displayed in Table III are theoretical values for the correlation function. These are computed using the appropriate generalization of Eq. (35),

$$C_{th} = \left\langle \frac{(\sigma^F(E))^2 + 2\,\sigma^F(E)\,\sigma^C(E)}{(\sigma^F(E) + \sigma^C(E))^2} \right\rangle , \tag{47}$$

with $\sigma^F(E)$ given by Eqs. (38)-(45) and $\sigma^C(E)$ estimated from the (smoothed-out) data.

The agreement in Table III between C_{th} thus computed and C_{exp} is quite good. The important result here is not the agreement in over-all magnitude, which could easily be adjusted by varying the imperfectly known parameters c, T_o, and Γ. Rather, it is the agreement with the non-adjustable features of C_{th} - that C_{th} is less for larger $d\sigma/dt$, and decreases rapidly with energy - which establishes the consistency of our interpretation.

TABLE III : ANALYSIS OF FLUCTUATIONS IN πN SCATTERING

Reaction	Momentum Range (p in GeV/c)	C_{exp}	C_{th}
$\pi^- p \rightarrow \pi^- p \quad (180^o)$	1.7 - 3.4 3.5 - 5.3 1.7 - 5.3	0.31 0.04 0.17	0.48 0.04 0.27
$\pi^+ p \rightarrow \pi^+ p \quad (u = 0)$	1.75 - 5.25	0.15	0.23
$\frac{\sigma^{tot}(\pi^+ p) - \sigma^{tot}(\pi^- p)}{2}$	1.5 - 3.4 3.5 - 5.3 1.5 - 5.3	0.41 0.00 0.23	0.44 0.00 0.24
$\frac{\sigma^{tot}(\pi^+ p) + \sigma^{tot}(\pi^- p) - 2\sigma^{tot}(\infty)}{2}$	1.5 - 3.9	0.02	0.02

We now turn to reactions with small A^C. Recall that we did not consider such reactions immediately because the data are generally less complete; nevertheless they are important because statistical ideas appear in their purest form here.

To find examples of small A^C we look at reactions where only exotic exchanges are present, such as

$$K^- + p \rightarrow K^- + p \tag{48}$$

and

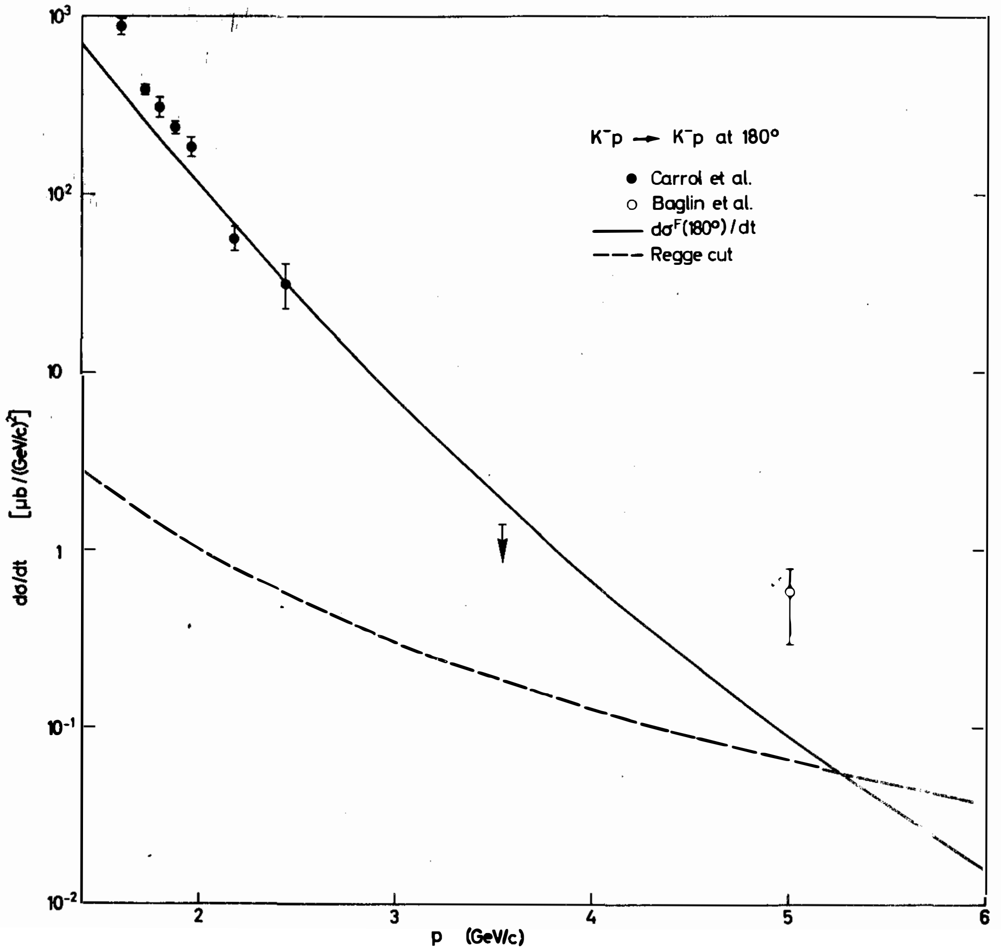
$$\bar{p} + p \rightarrow \bar{p} + p \tag{49}$$

at small u. Here A^C consists of Regge cut contributions plus possible exotic Regge pole exchanges, both of which may be quite small. Indeed, $d\sigma/du$ typically falls like s^{-10} (or, as we shall suggest below, $e^{-b\sqrt{s}}$) in such cases.

For the particular case $K^-p \rightarrow K^-p$ at $u = 0$, Michael [25] has estimated that the Regge cut contribution is less than the experimental $d\sigma/du$ at laboratory momenta less than $\approx$ 5 GeV/c. Above 5 GeV/c the Regge cut term, falling only as s^{-3} compared to the more rapid decrease of the lower energy cross-section, is expected to dominate. The existing data [26]–[28] and Michael's estimate for the cut contribution are shown in Fig. 5.

Michael's interest was in finding the Regge cut term above 5 GeV/c. Our main interest is in the region below 5 GeV/c where this form of coherent term is relatively very small. Of course, the data may still be dominated by $|A^C|^2 \sim s^{-10}$ corresponding to an exotic Regge pole exchange with $\alpha \approx -4$. But it is also possible that the dominant term is $|A^F|^2 \sim e^{-b\sqrt{s}}$, i.e., the incoherent part of the sum over direct channel resonances with a Hagedorn spectrum.

This possibility can be tested in several ways :



<u>Fig. 5</u> : Comparison of data on $K^-p \rightarrow K^-p$ at 180^o with $d\sigma^F/dt$. The data are from Refs. 26), 27) and 28).

i) one can predict $d\sigma^F(180^o)/dt$ in the same way as for πN scattering, making small changes appropriate for the slightly different kinematics and level density in the K^-p channel. The predicted $d\sigma^F(180^o)/dt$, as shown in Fig. 5, is quite close to the data.

ii) the crucial test would be to measure $d\sigma(180^o)/dt$ with good accuracy at small intervals over a broad range of energies, as in the backward π^-p measurements of Kormanyos et al.[20], and look for fluctuations. Large fluctuations are expected in our theory since $d\sigma^F/dt$ is so close to $d\sigma/dt$. It must be admitted that no fluctuations are visible in the present data below 2.5 GeV/c, but closely spaced measurements over a broader range of momenta are needed to settle the question.

Similar comments apply to a number of other exotic reactions, such as backward $\bar{p}p \rightarrow \bar{p}p$ and backward $\bar{p}p \rightarrow \bar{K}K$.

In summary, we have given a quantitative framework for estimating when a reaction is dominated by statistical factors, and at what level Ericson fluctuations are expected. These considerations should aid in testing statistical models and, hopefully, will help us understand a broad range of data in the intermediate energy range.

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