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Thermodynamics of the $R_h = ct$ Universe: a simplification of cosmic entropy

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Abstract In the standard model of cosmology, the Universe began its expansion with an anomalously low entropy, which then grew dramatically to much larger values consistent with the physical conditions at decoupling, roughly 380,000 years after the Big Bang. There does not appear to be a viable explanation for this 'unnatural' history, other than via the *generalized* second law of thermodynamics (GSL), in which the entropy of the bulk, S_{bulk}, is combined with the entropy of the apparent (or gravitational) horizon, S_h. This is not completely satisfactory either, however, since this approach seems to require an inexplicable equilibrium between the bulk and horizon temperatures. In this paper, we explore the thermodynamics of an alternative cosmology known as the $R_h = ct$ universe, which has thus far been highly successful in resolving many other problems or inconsistencies in ΛCDM. We find that S_{bulk} is constant in this model, eliminating the so-called initial entropy problem simply and elegantly. The GSL may still be relevant, however, principally in selecting the arrow of time, given that $S_h \propto t^2$ in this model.

1 Introduction

A conjecture known as the 'past hypothesis' posits that the entropy of the observable Universe is increasing monotonically, and must therefore have been lower – significantly lower – at earlier times [1–4]. But this position appears to be at odds with the observed cosmic microwave background (CMB) which, in the context of standard ΛCDM, suggests that the Universe was close to thermal and chemical equilibrium, i.e., in a state of very high entropy, a mere $\sim 380,000$ years after the Big Bang. These two conflicting views give

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rise to what is commonly referred to as the 'cosmic initial entropy problem' (IEP) [5,6].

The Universe seems to be homogeneous on scales exceeding several hundred Mpc, so one may reasonably assume that no entropy is flowing between neighboring volumes across distances larger than this. And since physical processes, such as stellar evolution and black-hole accretion, appear (on balance) to be increasing the cosmic entropy locally, it is difficult to understand why we would be living in a portion of the Universe with anomalously low entropy today [7–11], if its entropy shortly after the Big Bang was as high as it appears to have been at decoupling (i.e., $z_{dec} \sim 1080$).

The problem arises because blackbody radiation in equilibrium contains the largest amount of entropy, with a volume density

$$s_{BB} = \frac{4}{3} a_{rad} T_{BB}^3, \tag{1}$$

in terms of the blackbody temperature T_{BB} and radiation constant a_{rad}. As the Universe expands, T_{BB} scales as $(1 + z) \propto a(t)^{-1}$, where $a(t)$ is the expansion factor in the Friedmann–Lemaître–Robertson–Walker (FLRW) metric (Eq. 36). At the same time, any given proper volume V scales as $a(t)^3$, so the blackbody entropy $S_{BB} = s_{BB} V$ must have remained constant [8]. Then how could the CMB have been created at z_{dec} with the same entropy it has today, when all the other indications are that the total entropy should have been much smaller back then? In the context of ΛCDM, solutions to the IEP must therefore simultaneously explain why the Universe was initially in a very low entropy state (as is seemingly required by the second law), and why the observed CMB acquired such high entropy. To be clear, a very low initial entropy on its own may not be a problem if, e.g., the Universe was created from 'nothing' and has evolved away from that a priori unique initial state to which it would never return [12–14].

This basic picture has undergone several refinements over the past two decades, and we shall summarize the key steps

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taken to better understand the origin of cosmic entropy in the next section. The principal goal of this paper, however, is to examine the IEP and related entropy issues in the context of an alternative FLRW cosmology known as the $R_h = ct$ universe, which is receiving support from many observational tests (see, e.g., Table 2 of Ref. [15]), and several compelling theoretical arguments, notably the Local Flatness Theorem in general relativity [16]. A complete description of this model and its observational and theoretical foundation may be found in Ref. [17]. In Sect. 2, we shall provide a more extended background of the IEP, and then summarize the inventory of cosmic entropy – as we know it today – in Sect. 3. We shall discuss the simplification of cosmic entropy when viewed from the perspective of $R_h = ct$ in Sect. 4, and then conclude in Sect. 5.

2 Background

2.1 The basic picture

Today, serious attempts at resolving the IEP tend to rely on the inclusion of a hypothesized gravitational entropy [10,18] which, at first, may seem counterintuitive. A concentration of particles in a kinetically-dominated system tend to diffuse until they reach a homogeneous distribution, corresponding to thermal equilibrium and maximum entropy. A smooth distribution in a gravitationally-dominated system, on the other hand, corresponds to low gravitational entropy, which increases as initially small clumps collapse gravitationally to eventually form black holes [19]. (The caveat here, however, is that this simple-minded consideration may be missing other contributions to gravitational entropy, as we shall discuss in § 3.A below.) Indeed, we shall see in Sect. 3 that supermassive black holes are among the largest contributors to the bulk entropy of the Universe [11]. A simple solution to the IEP may therefore be that the Universe began with a very low entropy at the Big Bang, which increased rapidly by z_{dec} to produce the observed CMB, and then continued to increase to values *even larger* than S_{BB} today, generated principally by the formation of supermassive black holes at the centers of most galaxies. It has been postulated that this gravitational entropy may increase even further if black holes eventually evaporate due to Hawking radiation [20–22].

But the thermodynamics of gravitational fields remains contentious [10,23,24] (see also Sect. 3.1), and black-hole entropy does not explain why the Universe would have been in a very low entropy state to begin with. Instead, the inflationary paradigm [25–27] is often invoked to explain why the initial entropy of the Universe might have been so low [5,28,29], based on the notion that its value was established during the slow-roll evolution of the inflaton potential. Unfortunately, this idea has its own detractors, given that it con-

tains a hidden fine-tuning of initial conditions implicit in the assumption of a pre-inflationary patch with exceedingly low entropy, producing an 'unnatural' and unlikely state for the inflaton field ϕ [30,31]. It has been noted, e.g., that it requires much less fine-tuning for the Universe to have been put in some state that evolved into the present conditions than to have undergone an early period of inflation [5].

Worse, as the precision of cosmological measurements continues to improve, the argument against such an 'unnatural' beginning continues to gain support. A recent study using the latest *Planck* data release [32] suggests that the primordial power spectrum $\mathcal{P}(k)$ has a hard cutoff, $k_{min} = (3.14 \pm 0.36) \times 10^{-4}$ Mpc^{-1} [33]. A zero value of k_{min} is therefore excluded at a confidence level exceeding $\sim 8\sigma$. But this is not what the slow-roll inflationary paradigm was counting on. In order to simultaneously solve the horizon problem and generate a near scale-free fluctuation spectrum consistent with the CMB observations, $\mathcal{P}(k)$ should have extended well below the measured k_{min} [34]. As a result, most (if not all) slow-roll inflationary models proposed thus far, fail to accommodate this minimum cutoff. Even extensions to the basic picture, incorporating kinetic-dominated or radiation-dominated pre-inflationary phases, have no impact on this conclusion. It therefore appears very unlikely that the very low entropy in the early ΛCDM Universe could have been due to inflation as we know it today.

An alternative explanation for the very low initial entropy has been around much longer, and began with Boltzmann himself [35,36], who proposed that the low-entropy Universe we live in started as a random fluctuation in an otherwise maximal entropy state. This is a highly unlikely event, of course, and the probability of it happening drops sharply as the size of the fluctuation increases but, given an infinite timeline, it is bound to happen sooner or later. This idea of an equilibrium state permitting a low entropy fluctuation is an accepted notion of why the ΛCDM Universe might have begun its expansion with an incredibly low entropy, but detractors question the size of the Universe in such a state.

Coupled with an anthropic hypothesis, the fluctuation theorem [37] allows one to quantify the relative probabilities of us living in a Universe of various sizes, showing that small low-entropy universes are exponentially more likely than large ones. But this is where the Boltzmann hypothesis breaks down. The Universe does not need to be this big for sentient beings to exist within it. We see trillions of stellar systems, many more than are necessary for life to emerge (we believe), so the anthropic principle cannot be a factor in this argument [38,39]. It appears that the Boltzmann equilibrium hypothesis is no more 'natural' than the initial conditions required for inflation to work.

It is fair to say that the IEP remains contentious for various reasons and is largely unsolved – at least without the inclusion of an exploratory new feature having to do with the entropy of

an apparent horizon (see Sect. 3.2 below). Meanwhile, gravitational entropy might explain why the CMB had such large entropy at z_{dec}, even though S_{BB} has remained constant, if the gravitational assembly of large structures and supermassive black holes continued to increase the Universe's entropy since then, which would thus be much greater than that of the blackbody radiation on its own. But neither the equilibrium models nor the inflationary paradigm can successfully account for the very low initial entropy without some criticism, much of it driven by our aversion towards a lack of 'naturalness.' If initial conditions arose randomly, with a uniform probability of microstates, we would expect our Universe to have been born in a state of maximum entropy (if the number of degrees of freedom is finite – more on this below), representing thermal equilibrium, not the exceedingly unlikely low-entropy fluctuation we appear to be in according to ΛCDM.

2.2 A more recent refinement

We have made no reference thus far to the issue of what is actually observable but, in retrospect, it makes little sense to discuss the evolution of entropy without invoking the observer who makes the measurements. The Universe may be infinite (which is suggested by the apparent spatial flatness inferred from cosmological observations), but each observer can 'see' only a finite volume within it. It is reasonable to assume, therefore, that the application of the first and second laws to cosmology ought to be physically meaningful only when talking about measurements made consistently by the same observer within their observable patch of the Universe.

Two developments in black-hole physics bolster this argument quite compellingly: black-hole complementarity [40] and the holographic principle [41–43]. The former postulates that consistent (and, if necessary, complementary) descriptions of physical situations can only include events within the 'horizon' of a single observer. In the case of black holes, the identification of a horizon within their static or stationary spacetimes is unambiguous – it is uniquely the *event* horizon. But the situation has not been so clear in the cosmic setting, as we shall see shortly. Nevertheless, the black-hole example does suggest that an analogous horizon must be utilized in cosmology as well. (We refer the reader to the Appendix for a detailed description and comparison of the apparent, particle and event horizons.)

One of the more important consequences of the holographic principle is the covariant entropy bound [44], which limits the entropy contained within any given, finite region. Ultimately, this limit is related to the area of a null bounding surface measured in Planck units. And since the area of an appropriately defined horizon is finite, the causally-connected volume within it can only contain a finite number of degrees of freedom. As noted earlier, this makes it very

unlikely for the early Universe to have fluctuated into an anomalously low-entropy state, but it opens up the possibility of 'generalizing' the second law by including a contribution – perhaps even a dominant contribution – from the horizon itself [45–48].

As we shall see in Sect. 3.2 below, however, identifying the appropriate cosmological horizon to use in such a generalization has not been straightforward. Experience with black-hole horizons reasonably suggested at first that one should use the cosmic 'event' horizon. This idea certainly seemed to work in de Sitter space, but it hasn't worked in other cosmologies. Some cosmological models don't even have an event horizon. Trial and error eventually revealed that the most likely bounding surface to employ in a generalized second law is actually the gravitational, or 'apparent' horizon (see Ref. [49] for a pedagogical description). As we now understand it, the event horizon works for de Sitter (and presumably also for black holes) only because its spacetime is static, and the event horizon therefore coincides with the apparent horizon in that model (see the Appendix). We shall return to this feature of the entropy problem shortly, but first we consider how much entropy the Universe actually possesses, and what its dominant contributions appear to be.

3 Inventory of cosmic entropy

3.1 Cosmic entropy in the bulk

Irreversible processes are apparently increasing at least some contributions to the cosmic entropy on all scales, from gravitational clustering in the largest volumes, to dissipative accretion through disks, fusion in the interior of stars and their explosive deaths, all the way down to planetary activity in the form of weather, chemical and biological processes [8,50].

The present cosmic entropy budget has been estimated by various workers [8,10,11,51–54] over the past several decades, and we shall here largely adopt their key results. As we go through these contributions, however, it is important to keep in mind that most of these estimates were based on a poorly defined measure of the 'observable' universe, which has undergone significant theoretical evolution with the continued refinement of how one should define what is actually 'observable' (see Sect. 2.2 above and Sect. 3.2 below). In the Appendix, we discuss what has typically been assumed in the past, and why the so-called 'particle' horizon is not a correct measure of the proper size of the observable Universe. Neither is the event horizon; indeed, some cosmological models don't even possess one. The correct proper distance [49,86] representing what an observer actually sees is $R_{\gamma, max} \sim \eta R_h$, where $R_h \equiv c/H$ is the radius of the gravitational (or apparent) horizon, written in terms of the redshift-dependent Hubble parameter $H(z)$. The constant η

Table 1 Entropy of the observable Universe

Contribution	Entropy density s $(k_B$ m$^{-3})$	Entropy S (k_B)
SMBHs	$(3.7\text{–}16.6) \times 10^{23}$	$(2.6\text{–}11.5) \times 10^{101}$
Stellar BHs		
($\lesssim 15\,M_\odot$)	$(1\text{–}64) \times 10^{16}$	$(0.7\text{–}44.2) \times 10^{94}$
CMB (S_{CMB})	$(1.478 \pm 0.003) \times 10^9$	$(1.02 \pm 0.02) \times 10^{87}$
Other	$\lesssim 10^9$	$\lesssim 6.9 \times 10^{86}$
Total (S_{bulk})	$(3.7\text{–}16.6) \times 10^{23}$	$(2.6\text{–}11.5) \times 10^{101}$
Speculative		
Massive Halo		
BHs ($\lesssim 10^5\,M_\odot$) $\sim 10^{25}$		$\sim (6.9 \pm 0.1) \times 10^{102}$

All entropy densities are taken from Ref. [11]. The total entropies are based on the updated estimate of the proper volume of the observable Universe in Eq. (3). The uncertain massive halo BHs [65] are not counted in the budget

is cosmology-dependent, but tends to be $\approx 1/2$ in the majority of cases. The notable exception is de Sitter space, for which $\eta = 1$.

It is important for us to stress here that this is a new, key result. Ironically, many other studies of horizon entropy (as we shall see shortly) have concluded that the 'correct' horizon to use is the apparent (or gravitational) horizon, not the particle or event horizons, though with limited fundamental motivation. The consensus has reached this point largely because the use of the other candidates simply does not produce sensible (or even correct) first and second laws of horizon thermodynamics [55–60]. That the thermodynamics associated with the event horizon is ill-defined was further argued in Refs. [61–63]. But here, for the first time, we are presenting a theoretical argument supporting the use of R_h for such purposes. It simply has to do with the fact that the proper size of the observable Universe is $R_{\gamma,\,max}$, which is indeed proportional to R_h. More strictly, one should use ηR_h when evaluating the horizon entropy and temperature rather than R_h, as we shall discuss in Sect. 3.2, but the proportionality constant η has no impact on the first and second laws.

The use of the particle horizon, R_p (Eq. 40), to represent the proper size of the observable Universe, instead of $R_{\gamma,\,max}$, introduces a non-negligible error in the estimated entropy budget (Table 1). According to Ref. [11], the particle horizon today is $R_p(t_0) = 46.9 \pm 0.4$ Glyr, assuming a Hubble constant $H_0 = 70.5 \pm 1.3$ km s^{-1} Mpc^{-1}. By comparison, $R_h(t_0)/2 \approx 6.9$ Glyr for the same cosmological parameters. The observable spherical volume estimated in previous works was therefore too big by a factor ~ 300. Here, we adopt the latest *Planck* optimized parameters [32], and estimate

$$R_{\gamma,\,max} \approx (5.8 \pm 0.04) \left(\frac{\eta}{0.4}\right) \text{ Glyr}, \tag{2}$$

for a Hubble constant $H_0 = 67.4 \pm 0.5$ km s^{-1} Mpc^{-1}. The value $\eta = 0.4$ represents the maximum proper distance traveled by photons reaching us in a ΛCDM background cosmology [64]. The corresponding proper spherical volume of the observable Universe, in the context of ΛCDM, is therefore

$$\begin{aligned} V_{obs} &= 817 \pm 17 \text{ Glyr}^3 \\ &= (6.9 \pm 0.1) \times 10^{77} \text{ m}^3. \end{aligned} \tag{3}$$

The cosmic entropy budget has contributions from many sources, most prominently from black holes, the CMB and the relic neutrino background. Though the various workers [8,10,11,51–54] who have attempted to estimate these quantities have not always agreed on their absolute magnitude, their importance on a relative scale has never been in doubt. Adopting the estimates from one of the most recent, and presumably most accurate, compilations, we show the dominant contributions in Table 1, together with their tentative sum. In this collection, the entropy densities (analogous to Eq. 1 for the CMB) are taken directly from Ref. [11], but the total entropies have been updated using our more physically motivated proper volume (Eq. 3) of the observable Universe.

According to Table 1, the largest contributions to the entropy of the observable Universe today are made by supermassive black holes (see Sect. 3.2 below). The entropy of the CMB, a blackbody distribution with a present temperature $T_\gamma \sim 2.7$ K, is at least $\sim 10^{14}$ times smaller. The entropy of any non-thermal radiation, including light produced by stars and the interstellar medium, is even smaller – by a factor $\sim 10^3$ [8,52,54].

But note that according to the discussion following Eq. (1), S_{CMB} has remained constant since decoupling (at $z_{dec} \sim 1080$), corresponding to cosmic time $\sim 380,000$ years. As we understand it, there were no black holes present prior to the onset of re-ionization at $z \sim 15$, so according to this simple-minded scenario, the total entropy (S_{bulk}) in the bulk must have experienced several phases of rapid, and dramatic, growth: first in reaching S_{CMB} at $\sim 380,000$ years, then increasing to S_{bulk} today, starting near the onset of Pop III star formation at $t \sim 200$–400 Myr.

How much confidence should we place in this conclusion, however? Gravitational entropy probably includes a contribution from the gravitational field itself, but its nature and size have remained a mystery. Penrose [66] proposed that it may be given by the Weyl curvature tensor, $W_{\alpha\beta\gamma\delta}$, which provides a measure of the curvature of spacetime (and hence the 'strength' of the gravitational field). In general relativity, $W_{\alpha\beta\gamma\delta}$ is the only part of the curvature that exists in free space, corresponding to solutions of the vacuum Einstein equations. Unlike the Ricci curvature $R_{\mu\nu}$, which vanishes in the absence of matter, the Weyl curvature may still be non-zero in vacuum if, e.g., gravitational waves are propagating

through the medium. It therefore seems to be a more appropriate choice for the representation of gravitational entropy, which may be non-zero even in 'empty' space, but Table 1 does not include such a contribution to the entropy budget because it is difficult to quantify.

Once matter fluctuations began to grow, and the spacetime became clumpy, $W_{\alpha\beta\gamma\delta}$ grew as well, as did the gravitational entropy. This process culminated with the large nonlinear overdensities we see today. In extreme cases, the clumping led to the formation of black holes, whose entropy is well known (Table 1). The story with gravitational entropy is therefore incomplete. It appears that we know the end point fairly well, but we have trouble quantifying the overall contribution of gravity to the total entropy budget prior to the formation of supermassive black holes. Was gravitational entropy in the early Universe much smaller than it is today, or was it merely present in a different form, i.e., gravitational fields as opposed to fully formed black holes? It is hard to say, but our analysis in Sect. 4 below may provide some helpful clues.

3.2 Entropy of the apparent horizon

The general discussion concerning cosmic entropy includes a growing realization (some would say 'acceptance') that a complete understanding of FLRW thermodynamics ought to include both the entropy in the bulk, as summarized in Sect. 3.1 above, and the entropy associated with the apparent (or gravitational) horizon, R_h. This horizon has both a temperature, T_h, and an entropy, S_h, extending the basic properties of black-hole horizons discovered in the 1970's [19,67,68]. The temperature is inferred using the Hamilton–Jacobi variant of the Parikh–Wilczek 'tunneling' approach [69], though different definitions of 'surface' gravity produce some variation on the actual value of T_h. The Kodama–Hayward version is noteworthy because it is based on a conserved current even when a timelike Killing vector is absent [70]. In addition, the Noether charge associated with the Kodama vector is the Misner–Sharp–Hernandez mass [71,72], which is commonly used to represent the total internal energy contained within a sphere of radius R_h [73].

Coupling the entropy in the bulk to the entropy of a horizon actually has a historical precedent dating back to Einstein himself [74]. It is well known that the Einstein-Hilbert action can be decomposed into a bulk term and surface term (see also Ref. [75]). Yet the field equations (and their solutions) may be derived exclusively from the variations of the bulk, without any recourse to the surface term. The reason for this, it turns out, is that the bulk and surface terms are actually not independent of each other [76]. The term 'holographic' (see Sect. 2.2 above) was thus coined because the information about the bulk action functional is encoded in the boundary action functional.

The earliest exploration of FLRW horizon thermodynamics was made in de Sitter space, first by Gibbons and Hawking [77], and later by Refs. [78–81], though in terms of the cosmic event horizon which, in de Sitter space, is static (see Eq. 41). Its temperature and entropy (commonly referred to as *Gibbons–Hawking* entropy) are, respectively,

$$T_e \equiv \frac{1}{k_B} \frac{\hbar H}{2\pi}, \tag{4}$$

and

$$S_e \equiv \frac{k_B c^3}{\hbar G} \frac{A_e}{4} = \frac{k_B c^5}{\hbar G} \frac{\pi}{H^2}, \tag{5}$$

analogous to those of a Schwarzschild event horizon, originally inferred using Euclidean field theory techniques [77]. In these expressions, k_B is the Boltzmann constant, H is the Hubble parameter (which is constant in de Sitter space), G is the gravitational constant, and

$$A_e \equiv 4\pi R_e^2 \tag{6}$$

is the area of the event horizon written in terms of its radius R_e. Note, however, that in de Sitter space the event and apparent horizons are degenerate since H (and therefore R_h) is constant. So for de Sitter, $R_e = R_h \equiv c/H$. As we shall see, the horizon area relevant to all FLRW cosmologies can therefore be written more commonly as

$$A_h \equiv 4\pi R_h^2. \tag{7}$$

A more physically motivated way of writing the entropy, highlighting the notion that it represents the number of units of "quantum area" that fit within A_h, is

$$S_h = k_B \frac{A_h}{4\ell_{Pl}^2}, \tag{8}$$

where

$$\ell_{Pl} \equiv \sqrt{\frac{G\hbar}{c^3}} \tag{9}$$

is the Planck length. If the cosmic spacetime today were de Sitter, the (event) horizon would produce thermal radiation with a characteristic temperature $T_e \sim 3 \times 10^{-30}$ K, and its entropy would be constant, given that the de Sitter spacetime curvature is independent of time.

A broader discussion relevant to the thermodynamical properties of the apparent horizon, the Kodama vector and other issues associated with FLRW in general, may be found in Ref. [82]. The Kodama–Hayward temperature of the apparent horizon is given by

$$T_h = \frac{1}{k_B} \frac{\hbar c}{8\pi R_h} (1 - 3w), \tag{10}$$

where $w \equiv p/\rho$ is the equation-of-state of the cosmic fluid, in terms of its total energy density ρ and total pressure p. One can easily confirm that Eq. (10) reduces to Eq. (4) in

the limit $w \rightarrow -1$. Correspondingly, it is not difficult to understand from Eq. (8) that the apparent horizon entropy in an expanding Universe increases if $\rho + p > 0$ (which produces a monotonically decreasing Hubble parameter $H[t]$), remains constant when $p = -\rho$ (i.e., de Sitter space, as we have seen), and decreases if $p < -\rho$, which produces a monotonically decreasing Hubble radius R_h and apparent horizon surface area A_h (see Eq. 7).

4 First and second laws of thermodynamics

Let us now see how the first and second laws of thermodynamics must be framed in FLRW in terms of the various physical concepts introduced in Sect. 3 above, and why cosmic entropy is simplified in the $R_h = ct$ universe compared to ΛCDM.

The total entropy within the visible Universe, whose proper size is $R_{\gamma,\,\mathrm{max}} = \eta R_h$, may be written

$$S = S_{\mathrm{bulk}} + S_h, \tag{11}$$

where S_{bulk} includes all of the elements in Table 1, and S_h is given in Eq. (8). As noted earlier, $\eta = 1$ for de Sitter space, but $\eta \lesssim 0.5$ for all other FLRW cosmologies expanding from an initial singularity at the Big Bang. In all cases, however, it is the physics associated with the gravitational (or *apparent*) horizon that determines the properties of S_{bulk} and S_h, not the particle or event horizons. As we shall see shortly, the distinction between $R_{\gamma,\,\mathrm{max}}$ and R_h in calculating S_{bulk} does not affect our analysis. The primary purpose of introducing the former radius (via the Appendix) as the proper size of the visible Universe is to provide greater justification for the relevance of R_h to the question of cosmic entropy.

In spherically-symmetric metrics such as FLRW, the properties of the apparent horizon may be determined using the previously introduced Misner–Sharp–Hernandez mass [49,71–73], which coincides with the Hawking-Hayward quasi-local energy in such systems [83,84]. Under such circumstances, it is given simply as

$$M_h \equiv \frac{4\pi}{3c^2} R_h^3 \rho. \tag{12}$$

It is not difficult to show [73,85] that a second useful relation is therefore

$$R_h = \frac{2GM_h}{c^2}, \tag{13}$$

and this in turn gives $R_h = c/H$ in a spatially flat Universe. As we shall confirm shortly, one thus finds that the first and second laws of thermodynamics applied to de Sitter space are satisfied automatically, since the internal energy $M_h c^2$, the volume $V_h \equiv 4\pi R_h/3$, and the horizon entropy S_h, are all constant, so dS_h, $dM_h c^2$ and dV_h vanish uniformly.

For the broader class of FLRW cosmologies, we follow Refs. [56–58] and define the so-called *work density*

$$W \equiv -\frac{1}{2} g_{ab} T^{ab}, \tag{14}$$

and the energy flux across the apparent horizon

$$\Psi_a \equiv T_a{}^b \partial_b[a(t)r] + W \partial_a[ar], \tag{15}$$

where g_{ab} are the FLRW metric coefficients in Eq. (36), T^{ab} is the energy-momentum tensor for a perfect fluid, $a(t)$ is the expansion factor and r is the comoving radius, and the indices 'a' and 'b' run over the values (0, 1). The work density represents the work done by a change in the horizon's radius, while the energy-supply vector, $A_h \Psi_a$, represents the total energy flow through that horizon. For a perfect fluid, it is not too difficult to show that

$$W = \frac{\rho - p}{2} \tag{16}$$

where, as usual, ρ and p are the total energy density and pressure.

The Einstein equations then lead to the expression

$$\nabla_a M_h c^2 = A_h \Psi_a + W \nabla_a V_h, \tag{17}$$

known as the 'unified first law' [56,57]. The interpretation of this equation is that the energy supply, $-A_h \Psi_a$, across the apparent horizon is the change in heat,

$$\nabla_a Q_h = -A_h \Psi_a = T_h \nabla_a S_h, \tag{18}$$

and that this heat goes into changing the internal energy

$$E_h \equiv M_h c^2, \tag{19}$$

and performing work due to the change in size of this horizon. In writing Eq. (18), we have used the idea that the Universe and its horizon entropy encode the positive heat out thermodynamic sign convention (see, e.g., Ref. [88]). The fact that the coefficient multiplying $\nabla_a V_h$ in the work term of Eq. (17) is not simply p is apparently due to the fact that $R_{\gamma,\,\mathrm{max}}$ is not a comoving radius [89].

As a basic test of the validity of these various ideas, let us follow the rather straightforward demonstration that Eq. (18) yields an evolution in the horizon entropy fully consistent with its definition in Eq. (8). From Eqs. (13) and (19), we see that a change in the horizon radius by an amount dR_h produces a change to the internal energy of

$$dE_h = \frac{c^4}{2G} dR_h. \tag{20}$$

Correspondingly, the work done during this change is

$$W \, dV_h = (1 - w) \frac{3c^4}{4G} dR_h \tag{21}$$

where, as always, $w \equiv p/\rho$ and we have assumed a spatially flat FLRW spacetime. Equations (17) and (18) therefore suggest that

$$T_{\rm h}\, dS_{\rm h} = \frac{c^4}{2G}\left(\frac{1-3w}{2}\right) dR_{\rm h}, \tag{22}$$

and one can trivially confirm that this expression is completely consistent with Eqs. (8) and (10), *regardless of the value of w*. The extension of Bekenstein and Hawking's work on black-hole horizon entropy [19,67,68] to the cosmological context therefore appears to be fully intact and completely consistent with Einstein's equations.

The corresponding thermodynamic quantities pertaining to the bulk satisfy the traditional Gibbs equation [90,91]

$$T_{\rm bulk}\, dS_{\rm bulk} = dE(R_{\gamma,\,\max}) + p\, dV(R_{\gamma,\,\max}). \tag{23}$$

But whereas Eq. (22) is satisfied by all cosmologies, we shall see shortly that the evolution of the physical entropy $S_{\rm bulk}$ in ΛCDM, based solely on Eq. (23), departs dramatically from the desired nondecreasing behavior, completely at odds with our discussion in Sects. 1–3 above. The growing consensus today is that one must therefore resort to a *generalized* second law (GSL), in which the geometrically defined $S_{\rm h}$ must be added to $S_{\rm bulk}$ in order to create a nondecreasing *total* entropy S, as given in Eq. (11). This approach is not universally accepted, however, due to its reliance on the 'local equilibrium assumption' (see below) which posits that the interior region and the apparent horizon have the same temperature, $T_{\rm bulk} = T_{\rm h}$ [92–95]. The principal goal of this paper is to demonstrate that this difficulty (perhaps one should say, 'unmotivated requirement') is completely absent in the $R_{\rm h} = ct$ universe.

4.1 First and second laws in ΛCDM

From the Gibbs Eq. (23) and the definition of $T_{\rm bulk}$ and $S_{\rm bulk}$, one can see that

$$T_{\rm bulk}\, dS_{\rm bulk} = \frac{c^4}{2G\eta^3}\, dR_{\rm h}\,(1+3w), \tag{24}$$

which becomes our first law for the bulk. The hurdle faced by ΛCDM (see Sects. 2 and 3) in accounting for the cosmic entropy evolution with time is therefore clearly apparent, since any acceleration of the universal expansion requires $\rho + 3p < 0$, i.e., $w < -1/3$. Clearly,

$$\frac{dS_{\rm bulk}}{dR_{\rm h}} < 0 \quad (\forall\ w < -1/3) \tag{25}$$

for such a model, regardless of how one interprets $T_{\rm bulk}$. The second law is therefore not satisfied for the bulk in the context of ΛCDM.

The GSL was introduced specifically to overcome this deficiency (Sect. 3.2), with the hope that $dS/dR_{\rm h} \geq 0$ (see Eq. 11), in spite of the fact that the bulk cosmic entropy in the standard model is apparently decreasing with time when the expansion is accelerated. The inspiration for this idea came by way of the GSL in black-hole thermodynamics [19], where the event horizon entropy added to the entropy outside the black hole never decreases.

Let use see what is required for the GSL to work in this fashion. It is straightforward to show from Eqs. (22) and (23) that the combined cosmic entropy in ΛCDM never decreases as long as

$$dS_{\rm h} + dS_{\rm bulk} = \frac{c^4}{2G}\, dR_{\rm h}\left(\frac{1-3w}{2T_{\rm h}} + \frac{1+3w}{\eta^3 T_{\rm bulk}}\right)$$
$$\geq 0. \tag{26}$$

That is, the GSL is satisfied as long as

$$w \leq \frac{1}{3}\frac{1+\alpha}{1-\alpha}, \tag{27}$$

where

$$\alpha \equiv \frac{2}{\eta^3}\left(\frac{T_{\rm h}}{T_{\rm bulk}}\right). \tag{28}$$

Evidently, the GSL is satisfied for any FLRW cosmology with an equation-of-state $w \leq 1/3$ in the cosmic fluid, as long as $\alpha < 1$. Otherwise, the Universe must be accelerating and $\alpha \gg 2$ to comply with the requirement that $dS \geq 0$ at all times. It is this tight coupling between $T_{\rm bulk}$ and $T_{\rm h}$ that has promoted the so-called 'local equilibrium assumption' introduced in the discussion following Eq. (23) above. More precisely, the constraint in Eq. (27) seems to require a proportionality between $T_{\rm h}$ and $T_{\rm bulk}$, though the proportionality constant is often assumed to be unity [92–95]. It needs to be acknowledged, however, that there is no known fundamental mechanism responsible for maintaining the bulk in thermal equilibrium with the horizon. Indeed, this condition cannot be satisfied at all if $T_{\rm bulk}$ is associated primarily with the CMB blackbody temperature $T_{\rm BB}$, as we shall see below.

4.2 First and second laws in $R_{\rm h} = ct$

The $R_{\rm h} = ct$ cosmology has been under development for over 15 years now, largely motivated by observational constraints, but more recently finding significant fundamental support from renewed interest in the FLRW metric itself [16]. A complete accounting of this model may be found in a recently released monograph [17]. A quick summary of its observational tests may also be found in Table 2 of Ref. [15]. Like ΛCDM, $R_{\rm h} = ct$ is a FLRW cosmology with a cosmic fluid comprised of matter (baryonic and dark), radiation and dark energy, though the latter is not a cosmological constant. The key new physical input that distinguishes it

from the standard model is the so-called zero active mass condition in general relativity,

$$\rho + 3p = 0, \tag{29}$$

sustained throughout the expansion. In other words, $R_h = ct$ is essentially ΛCDM, though constrained by the equation-of-state $w = -1/3$ over the entire cosmic evolution.

To understand why it is now timely to examine the issue of cosmic entropy in this cosmology, let us list some of the notable features that have allowed it to overcome many major hurdles and inconsistencies in the standard model. This is only a small sample; its successes extend well beyond this brief summary. First and foremost, $R_h = ct$ has no horizon problems. Whereas models with an early phase of deceleration, such as ΛCDM, must find ways of explaining how regions of the sky beyond their causal horizon achieved similar physical conditions, this issue is completely absent in $R_h = ct$ [96], which has always expanded at a constant rate. Inflation was introduced in part to address the CMB temperature horizon problem in the standard model [25,26,97], but has yet to find a completely self-consistent theoretical framework. The electroweak horizon problem appears to be worse. As of today, no viable explanation has been found for how opposite sides of the Universe seemingly acquired the same Higgs vacuum expectation value. But again, this problem is completely absent in $R_h = ct$ [98].

The $R_h = ct$ cosmology apparently also avoids the so-called trans-Planckian problem in ΛCDM [99–103]. Whereas quantum fluctuations in the standard model would have been seeded in the Bunch–Davies vacuum, well below the Planck scale, they emerged into the semi-classical Universe right at the Planck scale, at about the Planck time, in $R_h = ct$. Our current physical theories are not valid in the Planck regime, so it is currently a mystery exactly how these fluctuations evolved into the semi-classical Universe in the context of standard inflationary cosmology.

The third and final example we mention here has to do with the timeline in ΛCDM, which appears to be overly compressed at large z, based on the observation of high-redshift quasars and the so-called 'too-early' appearance of galaxies [104] and the accelerated rate of structure formation in the early Universe. In contrast, the time-redshift relation in $R_h = ct$ matches the rate of supermassive black-hole growth very well [105,106], and readily accounts for the appearance of galaxies [107] and large halos [108] at $z \gtrsim 10$.

One of the few remaining issues to consider is the nature and evolution of cosmic entropy in the $R_h = ct$ cosmology, which we now address. We see right away from Eqs. (24) and (29) why cosmic entropy is greatly simplified in this model compared to ΛCDM. Evidently,

$$S_{bulk} = \text{constant} \quad (w = -1/3) \tag{30}$$

throughout the cosmic expansion, independently of how we choose to evaluate T_{bulk}. There is no need to 'fix' an entropy problem by introducing a GSL, except to determine the arrow of time [5] which, in this model, is simply identified from the expansion or contraction of the apparent horizon. Insofar as S_{bulk} is concerned, however, there is no need to explain an anomalously low entropy at the beginning – because it wasn't low – nor is there any concern that it may actually be decreasing with time – since it is always constant – which would otherwise violate the second law. In this picture, the various contributions to the total entropy in the bulk shift relative to each other as the Universe expands, but always in such a way as to maintain a constant global value established at the beginning.

The GSL is not required in this model, but if we were to introduce it analogously to the procedure we followed in arriving at Eq. (26) for ΛCDM, then the total cosmic entropy (Eq. 11) would grow as

$$S = k_B \frac{\pi c^2}{\ell_{Pl}^2} t^2, \tag{31}$$

affirming the idea floated earlier that the arrow of time in this cosmology might be determined by the expansion of its apparent horizon.

In contrast to the standard model, one can see from this result that it is not necessary to speculatively constrain the evolution of T_{bulk} in $R_h = ct$. Ironically, however, it would actually be easier to demonstrate that $T_{bulk} \propto T_h$ in $R_h = ct$ than in ΛCDM, if one interprets T_{bulk} as being primarily T_{BB}. This is evident from Eq. (10) and the CMB temperature T_{BB} which scale, respectively, as

$$T_h(z) = \frac{1}{k_B} \frac{\hbar c}{4\pi \, R_h} = \frac{1}{k_B} \frac{\hbar c}{4\pi \, R_h(0)}(1 + z), \tag{32}$$

and

$$T_{BB}(z) = T_0(1 + z), \tag{33}$$

where $R_h(0)$ is the radius of the apparent horizon today and T_0 is the current CMB temperature. That is,

$$T_{bulk} = b \, T_h, \tag{34}$$

where

$$b \equiv k_B \frac{4\pi R_h(0) T_0}{\hbar c} \approx 2.1 \times 10^{30}. \tag{35}$$

We stress, however, that though these temperatures scale in proportion with the expansion of the Universe, Eq. (34) does not at all require that the bulk be in equilibrium with the apparent horizon, as is often assumed in the context of ΛCDM.

5 Impact on theories of the early Universe

Like ΛCDM, the $R_h = ct$ universe contains baryonic and dark matter (ρ_m), radiation (ρ_r) and dark energy (ρ_{de}). Unlike the standard model, however, these densities must exist in proportions consistent with the zero-active mass condition, $\rho + 3p = 0$, where $\rho = \rho_m + \rho_r + \rho_{de}$ and $p = p_r + p_m + p_{de}$. With $p_m = 0$, these imply that $p_r + p_{de} = -(\rho_m + \rho_r + \rho_{de})/3$. It is not difficult to convince oneself that this equation-of-state can be satisfied if $w_{de} \equiv p_{de}/\rho_{de} = -1/2$, with a fractional representation of $\rho_{de}/\rho = 1/3$ and $\rho_m/\rho = 2/3$ in the local universe ($z \sim 0$) and $\rho_{de}/\rho = 0.8$ and $\rho_r/\rho = 0.2$ as $z \to \infty$ (see, e.g., Refs. [17,109]. Quite clearly, dark energy could not be a cosmological constant in such a model, and one might go further and speculate that a fraction of dark energy present at the beginning eventually 'decayed' to create matter with a fractional representation of 1/3 towards the present day. Evidently, both dark matter and dark energy would thus be considered extensions to the standard model of particle physics.

We don't know yet, though, whether radiation and dark energy were present from the very beginning, or whether they were preceded by something else, perhaps a scalar field. But in order for this field to also satisfy the zero-active mass condition, it would have to have had an exponential potential [102] reminiscent of the category of minimally coupled fields explored in the 1980's, designed to produce so-called power-law inflation [110–113]. Unlike the other members of this class, however, this zero-active mass field actually would not have inflated, since $a(t) = (t/t_0)$.

But given these attributes, a more 'natural' candidate for the incipient dominant content of the Universe could very well have been cosmic strings [114] arising from phase transitions in such a field. The tension along a string is equal to its energy per unit length [115,116], so the equation-of-state of a chaotic ensemble of strings is simply $p = -\rho/3$, as required in $R_h = ct$. Though such strings are unlikely to be dynamically relevant today, they may have been dominant at early times [117]. The earliest manifestation of the $R_h = ct$ cosmology could thus have been a string-dominated universe, that eventually evolved into the more 'standard' scenario we see today.

The possible consequences of such a beginning are numerous and intriguing. Certainly, if $R_h = ct$ correctly describes the background spacetime of the Universe, the total entropy in the bulk has remained constant since the beginning, implying that a reasonable 'measurement' of S_{bulk} today would directly reveal an important attribute of the physical system during the string-dominated era – without us having to worry about the details concerning the transition from those earliest times to the subsequent more 'standard' evolution with $\rho = \rho_m + \rho_r + \rho_{de}$. For example, such a constraint would directly reveal whether entropy variations in the equation-of-state of the cosmic strings in the cosmological fluid, due to variations in the local density and gravitational radiation, could have led to perturbations of the background matter density, and the subsequent cosmic-string induced formation of structure [118]. Such ideas and possibilities will be explored elsewhere.

6 Conclusion

Prior to the introduction of the GSL, there was considerable confusion about how to handle cosmic entropy in the context of ΛCDM. The notion that S_{bulk} should have an anomalously low initial value was considered to be so 'unnatural' that some rather exotic (and equally 'unnatural') solutions were proposed. The idea that the Universe ought to have such a condition both at the beginning and at the 'end' was considered a viable possibility [2,119]. But all such models appear to be very *ad hoc*, actually exacerbating the 'unnaturalness' of the boundary conditions. The GSL at least obviates the need to patch the standard model in such a poorly motivated fashion, but it may suffer from an inconsistency of its own when applied to ΛCDM.

Busso [120] conjectured that the horizon enclosing the bulk should be a lightlike hypersurface, building on the argument of Fischler & Susskind [121], who used lightlike hypersurfaces to relate entropy and area (Eq. 8). Since then, however, several workers have established that neither the particle horizon (Eq. 40), nor the event horizon (Eq. 41), satisfy the laws of thermodynamics. Attention has thus been redirected onto the apparent (or gravitational) horizon to fulfill this role, and we describe in the Appendix a possible fundamental reason why it has to be this way. But the apparent horizon is generally not null – except in one unique cosmology: the $R_h = ct$ universe. This model is characterized by the zero active mass equation-of-state (Eq. 29), for which the radius R_h increases at lightspeed, i.e., $dR_h/dt = c$ – hence the eponymous naming of the model. Thus, the $R_h = ct$ cosmology uniquely satisfies the second law of thermodynamics in the bulk, and horizon thermodynamics on the null boundary hypersurface (at R_h) consistent with the proper size of the observable Universe.

The fact that S_{bulk} is constant in the $R_h = ct$ universe actually solves two problems in standard cosmology: (1) it eliminates the need for the Universe to have begun its expansion with an anomalously low entropy, and (2) it explains how the cosmic entropy could have been so large by the time of decoupling to produce the CMB. A fixed value for S_{bulk} is also more in line with the cosmological principle, which posits that the Universe is isotropic and homogeneous on large scales (certainly larger than ~ 300 Mpc). One should not expect to see a net inflow (outflow) of entropy into (out

of) a finite volume if the physical conditions are the same everywhere on each given time slice.

We thus see that the $R_h = ct$ cosmology may have resolved yet another difficulty or inconsistency in ΛCDM. Like several of the other problems preceding it, the IEP has been with us for several decades, stubbornly resisting attempts at reconciling the hypothesized expansion history in the standard model with expectations based on established physical theories. If the solution presented in this paper turns out to be correct, the answer is actually elegant and straightforward, requiring no 'unnatural' initial conditions, nor any forced thermodynamic equilibrium between the bulk and its apparent horizon. One might say the answer seems quite natural after all.

Appendix: Proper size of the observable Universe

Standard cosmology is based on the Friedmann–Lemaître–Robertson–Walker (FLRW) metric for a spatially homogeneous and isotropic three-dimensional space, expanding or contracting as a function of time:

$$ds^2 = c^2 dt^2 - a^2(t) \left[\frac{dr^2}{(1 - kr^2)} + r^2 d\Omega^2 \right], \qquad (36)$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta \, d\phi^2$. In these coordinates, t is the cosmic time, measured by a comoving observer (and is independent of position), $a(t)$ is the universal expansion factor, and r is the comoving radius. The geometric factor k is $+1$ for a closed universe, 0 for a flat universe, and -1 for an open universe. Most of the data suggest that the Universe is spatially flat, so we assume $k = 0$ throughout this paper.

It is not difficult to show [49,86] from Eq. (36) and the definition of proper distance, i.e., $R \equiv a(t)r$, that the null geodesic equation for a ray of light propagating radially may

be written

$$\frac{dR_\gamma}{dt} = c \left(\frac{R_\gamma}{R_h} \pm 1 \right), \qquad (37)$$

where $R_h \equiv c/H$ is the radius of the gravitational (or apparent) horizon, and the sign $\pm$ refers to either inward ($-$ sign) or outward ($+$ sign) propagation. The former corresponds to null geodesics actually reaching the observer at the origin of coordinates, and are therefore relevant in determining the proper size of the observable Universe [86]. As we shall see below, the outwardly propagating rays (with the $+$ sign in Eq. 37) define a 'particle' horizon which, however, cannot be used as a measure of the size of the observable Universe because these null geodesics are directed away from the observer and never return to their location.

From Eq. (37) for an inwardly directed ray (i.e., with a negative sign on the righthand side), it is clear that $dR_\gamma/dt = 0$ when $R_\gamma = R_h$. In other words, the spatial velocity of light measured in terms of the proper distance per unit cosmic time vanishes at the gravitational (or 'apparent') horizon. When $R_\gamma > R_h$, the photon's proper distance from us actually increases, even though the photon's velocity vector points towards the origin. The photon approaches the observer only when it is located within their apparent horizon at R_h. This separation at R_h of null geodesics approaching the observer from those receding is well known in general relativity, but it is even simpler to understand in the cosmic context due to the fact that FLRW is spherically symmetric.

But note that since R_h is a function of time, $\dot{R}_\gamma$ flips sign depending on whether R_γ overtakes R_h, or vice versa. For example, a photon emitted beyond $R_h(t_e)$ at some time $t_e < t$ (where t is the present), begins its trajectory receding from the observer, yet stops at $R_\gamma = R_h$, and reverses direction when R_h overtakes R_γ. Consequently, the path of the null geodesic $R_\gamma(t)$ depends on the cosmology, because the expansion history is solely responsible for the evolution in R_h. This dependence of $R_\gamma(t)$ on the evolution of R_γ/R_h affirms the nature of gravitational (or apparent) horizons discussed in the literature [73,87,89,122,123].

The proper size of our visible Universe hinges on the solution to Eq. (37) for the radius $R_h(t)$ associated with the chosen cosmology, starting at the Big Bang ($t = 0$) and ending at the observer's time. It is determined by the greatest proper distance achieved by those null geodesics that actually reach the observer at the origin of the coordinates. No matter what, however, Theorem 1 in Ref. [49] ensures that the proper size of the visible Universe at time t is always subject to the constraint

$$R_{\gamma,\max} \leq R_h(t), \qquad (38)$$

in any cosmology expanding monotonically with $\dot{H} \leq 0$.

Thus, no matter how R_h evolves in time, none of the light detected by the observer at time t originated from beyond

their gravitational (or apparent) horizon at that time. Indeed, detailed solutions to Eq. (37) for typical cosmological models show that expanding universes have a visibility limit restricted to about half of the current gravitational radius $R_h(t)$, or even somewhat less [86]. The only exception is de Sitter, for which $R_{\gamma,\,max} = R_h(t)$ (see below). One may therefore summarize these results with the following statement: the proper size of the observable Universe is

$$R_{\gamma,\,max} = \eta R_h(t), \tag{39}$$

where η ($\sim 1/2$) is a cosmology-dependent constant.

It is not difficult to understand the physical reason behind this outcome [86]. In all models other than de Sitter, there were no pre-existing detectable sources away from the origin of the observer's coordinates prior to the Big Bang. Thus, light reaching us at time t from the most distant sources was emitted only after the latter had reached their farthest detectable proper distance, i.e., $\sim R_h(t)/2$.

Equation (39) needs to be contrasted with typical past treatments of the cosmic entropy budget (see, e.g., Refs. [8,10,11,51,53,54]), in which the proper radius of the observable Universe was instead assumed to be the particle horizon,

$$R_p(t) \equiv a(t) \int_0^t \frac{c\, dt'}{a(t')}. \tag{40}$$

But we can clearly understand the distinction between the apparent (R_h), particle (R_p) and event,

$$R_e \equiv a(t) \int_t^\infty \frac{c\, dt'}{a(t')}, \tag{41}$$

horizons via the use of Eq. (37), and why the proper size of the observable Universe must be related to R_h through Eq. (39) rather than R_p and R_e.

Differentiating Eq. (40) with respect to t, one gets the null geodesic Eq. (37) with a '+' sign, which describes the propagation of a photon *away* from the observer. The solution in Eq. (40) therefore represents the maximum proper distance a particle traveled away from us during the time elapsed since the Big Bang. But this is different from the maximum proper distance a photon traveled in reaching us which, as we have seen, is given by Eq. (39). While $R_{\gamma,\,max}$ is bounded by the apparent horizon R_h, there is no limit to $R_p(t)$, since $\dot{R}_p$ is always greater than c, so R_p easily grows past R_h, especially at late times in ΛCDM, when the cosmological constant dominates the energy budget and the Universe enters a late de Sitter expansion. We never again see the photons receding from us, reaching proper distances $\sim R_p$, so regions of the Universe that far away are not observable. As noted earlier, null geodesics must actually *reach* us in order for us to see the photons traveling along them, providing information on their source.

In contrast, the 'event' horizon (Eq. 41) is defined to be the largest comoving distance from which light emitted now, at time t, can ever reach us in the asymptotic future. If we differentiate this equation with respect to t, we get Eq. (37) with a '$-$' sign, representing a photon propagating towards the observer. The physical meaning of R_e is thus similar to that of R_γ, except that the distance in Eq. (41) represents a horizon for null geodesics that will connect to us in our future, not today. This is the reason why the gravitational (or apparent) horizon is generally not an event horizon yet, though it may turn into one for some equations-of-state in the cosmic fluid, which influence the evolution of R_h.

We note, in this regard, that the apparent and event horizons coincide in the metrics of de Sitter, Schwarzschild and Kerr specifically because their spacetime curvatures are independent of time. As we discuss in Sect. 3.2, this appears to be the reason why the holographic principle involves R_e in those cases, whereas the correct horizon to use for cosmologies other than de Sitter is R_h (or, to be more precise, it should be $\eta R_h(t)$; see Sect. 3.1 above). In other words, it would be correct to say that the holographic principle should *always* be associated with the gravitational (or apparent) horizon R_h, rather than R_p or R_e, and it only appears to involve R_e when the apparent and event horizons coincide.

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