Density perturbations in $f(R, \phi)$-gravity in general with an application to the (varying power)-law model*

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Density perturbations in the cosmic microwave background within general $f(R, \phi)$ models of gravity are presented. The general dynamical equations for the tensor and scalar modes in any $f(R, \phi)$ gravity model are provided. An application of the equations to the (varying power)-law modified gravity toy-model is then given. Based on the latest observations of the density perturbations in the sky, the model requires inflation to occur at an energy scale less than the GUT-scale (10^{14} GeV). The density perturbations obtained from observations are recovered naturally, with very high precision, and without fine tuning the model's parameters.

Keywords: Modified gravity; Inflation; Density perturbations; (Varying power)-law model.

1. Introduction

Some of the great problems of modern cosmology are the problem of dark energy, the problem of dark matter and the problem of the early Universe. A widely pursued axis of research to solve these problems consists in modifying general relativity (GR). The easiest way to modify GR is to replace the Ricci scalar R in the gravitational action by an arbitrary function of R and/or to add a possible extra degree of freedom for spacetime, besides the metric, in the form of a scalar field ϕ. Such modified GR models are known as $f(R, \phi)$ gravity models. In order to understand the motivation behind the peculiar (varying power)-law model, let us briefly draw a timeline of some of the well-known models of $f(R, \phi)$ modified gravity.

2. A brief history of $f(R, \phi)$ gravity

Here is a very simplified sketch of the various $f(R, \phi)$ models and their evolution in time.

*This talk is based on the published paper 1.

$$S \sim \int \mathrm{d}^4x\sqrt{-g}R \quad \text{Hilbert (1915).}$$

$$S \sim \int \mathrm{d}^4x\sqrt{-g}\left(\phi R - \frac{\omega(\phi)}{\phi}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right) \quad \begin{cases} \text{Thiry (1948), Jordan (1955),} \\ \text{Brans-Dicke (1961),} \\ \text{Bergman (1968),} \\ \text{Nordtveldt (1970),} \\ \text{Wagoner (1970),} \\ \text{O'Hanlon (1972),} \\ \text{Horndeski (1974, complicated),} \\ \text{Bekenstein (1977),} \\ \text{Barker (1978).} \end{cases}$$

$$S \sim \int \mathrm{d}^4x\sqrt{-g}f(R) \quad \text{Buchdahl (1970).}$$

$$S \sim \int \mathrm{d}^4x\sqrt{-g}\left(\alpha_0 + \alpha_1 R + \alpha_2(R^2 - 4R_{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right) \quad \text{Lovelock (1971).}$$

$$S \sim \int \mathrm{d}^4x\sqrt{-g}\left(R + \frac{R^2}{6M_P^2}\right) \quad \text{Starobinski (1980).}$$

$$S \sim \int \mathrm{d}^4x\sqrt{-g}\left(R + f_1R^2 + f_2R_{\mu\nu}R^{\mu\nu} + f_3R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right) \quad \text{Stelle's gravity (1980).}$$

$$S \sim \int \mathrm{d}^4x\sqrt{-g}\,(R + RF_1(\Box)R + R_{\mu\nu}F_2(\Box)R^{\mu\nu} + C_{\mu\nu\rho\rho}F_3(\Box)C^{\mu\nu\rho\sigma}) \begin{cases} \text{Biswas-} \\ \text{Gerwick-} \\ \text{Koivisto} \\ \text{(1980).} \end{cases}.$$

The tensor $C_{\mu\nu\rho\sigma}$ consists of the more complicated invariant $\frac{1}{3}R^2 - 2R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. These expressions arise from the study of the quantum trace anomaly. A pragmatic trend has appeared more recently, however, which consists in trying out an arbitrary function of $f(R)$ inside the Lagrangian such that it could a priori be Taylor-expanded in R: $f(R) = \alpha_0 + \alpha_1 R + \alpha_2 R^2 \ldots$ Such models are called $f(R)$ gravity. Below is a list of the most studied ones,

$$f(R) = R - \frac{c}{(R-\Lambda_1)^n} + b(R-\Lambda_2)^m \qquad \text{Nojiri \& Odintsov (2003),}$$

$$f(R) = R + \alpha\ln\left(\frac{R}{\mu^2}\right) + \beta R^m \quad \text{Nojiri \& Odintsov (2004).}$$

$$f(R) = R + \gamma R^{-n}\left(\frac{R}{\mu^2}\right)^m \quad \text{Meng \& Wang (2004).}$$

$$f(R) = \frac{\alpha R^{2n} - \beta R^n}{1 + \gamma R^n} \quad \text{Nojiri \& Odintsov (2007).}$$

The various powers in these models are constrained by solar-system observations one one hand, and the need to reproduce both the early expansion of the universe and the late-time cosmic acceleration on the other. Some models, however, require different (real and fractional) values for the powers whether one applies them to reproduce inflation or the late-time expansion. This suggests that one might as well let the power of the Ricci scalar in the gravitational action to vary in space and time, so that it would adjust itself depending on the epoch in the evolution of the Universe and on the cosmological environment. This leads to a (*varying power*)law model, in which the power of the Ricci scalar is a spacetime-dependent scalar field:

$$S \sim \int \mathrm{d}^4x\sqrt{-g}\left[R - \frac{\mu^2}{2}\left(\frac{R}{\mu^2}\right)^{\phi}\right]. \tag{1}$$

An important remark to make here is that, unlike $f(R)$ gravity models, the redefinitions $\partial_R f = \phi$ and $(2f - R\partial_R f)/3 = dV/d\phi$ do not transform the (varying power)-law model into GR with a scalar field. this is due to the high non-linearity of the model. On the other hand, a Weyl transformation $g_{\mu\nu} \to \phi^2 g_{\mu\nu}$, followed by the field redefintion $\phi^2 \sim \sigma^{-1}$, does transform the model into a Brans-Dicke like theory, in which matter also acquires a coupling with the scalar field σ, though.

3. Scalar modes in any $f(R,\phi)$ model

Start from the field equations of an $f(R,\phi)$ model.[2] Then, perturb the equations using $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, with $\bar{g}_{\mu\nu} = diag(-1, a^2, a^2, a^2)$ is the Friedmann-Lemaître-Robertson-Walker (FLRW) background metric and $|h_{\mu\nu}| \ll 1$, as well as $\phi = \bar{\phi} + \delta\phi$. Then, choose the Newtonian gauge $\mathrm{d}s^2 = -(1+2\Phi)\mathrm{d}t^2 + a^2(t)(1-2\Psi)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$. The dynamical equation for the gravitational potentials Ψ and Φ is,[1]

$$\ddot{\Phi} + \ddot{\Psi} + \left(H - \frac{2\ddot{\phi}}{\dot{\phi}} + \frac{3\dot{f}_{,R}}{f_{,R}}\right)(\dot{\Phi} + \dot{\Psi}) - \left(4\dot{H} - \frac{4\ddot{f}_{,R}}{f_{,R}} + \frac{6\dot{f}_{,R}}{f_{,R}}\frac{\ddot{\phi}}{\dot{\phi}}\right)\Phi$$
$$+\left[6\dot{H} + 6H^2 + H\left(\frac{4\dot{f}_{,R}}{f_{,R}} - \frac{2\ddot{\phi}}{\dot{\phi}}\right) - \frac{f - 2\Box f_{,R}}{f_R} - \frac{\ddot{f}_{,R}}{f_{,R}} + \frac{2\dot{f}_{,R}}{f_{,R}}\frac{\ddot{\phi}}{\dot{\phi}} - \frac{\dot{\phi}^2}{f_{,R}} - \frac{\nabla^2}{a^2}\right](\Phi + \Psi) = 0. \tag{2}$$

4. Tensor modes in any $f(R,\phi)$ model

In order to extract the tensor modes equation, perturb the FLRW background, $\mathrm{d}s^2 = -(1+2\Phi)\mathrm{d}t^2 + \left[a^2(t)(1-2\Psi)\delta_{ij} + h_{ij}\right]\mathrm{d}x^i\mathrm{d}x^j$ and impose the transverse gauge, $\partial_i h^i_j = 0$, with also $h^i_i = 0$. The transverse modes' equation is,[1]

$$\left[\partial_0^2 - \left(H - \frac{\dot{f}_{,R}}{f_{,R}}\right)\partial_0 + 4H^2 - \frac{f - 2\Box f_{,R}}{f_{,R}} - \frac{\nabla^2}{a^2}\right]h_{ij} = 0. \tag{3}$$

5. The (varying power)-law model

The action of the model is,[3]

$$S = \frac{M_P^2}{16\pi} \int \mathrm{d}^4x\sqrt{-g}\left[R - (\partial\phi)^2 - m^2\phi^2 - \frac{\mu^2}{2}\left(\frac{R}{\mu^2}\right)^{\phi}\right], \tag{4}$$

where M_P is the Planck mass and m, whose order of magnitude is not constrained by the model, is the scalar field's mass. The energy scale μ is determined by observation and it is found that it should be $\sim 10^{14}$ GeV. At the beginning of inflation $R \sim \mu^2$, so that the action approximates to $S \sim \int \mathrm{d}^4x\sqrt{-g}(R-(\partial\phi)^2)$. At late times, $R \ll \mu^2$, so that the action approximates to, $S \sim \int \mathrm{d}^4x\sqrt{-g}(R-(\partial\phi)^2-m^2\phi^2-\frac{1}{2}\mu^2(R/\mu^2)^{\phi})$, in which the last term could be identified with a cosmological constant, while the scalar field could be identified with dark matter. Thanks to the effective potential,

$$V(R,\phi) = \frac{m^2}{2}\phi^2 + \frac{\mu^2}{4}\left(\frac{R}{\mu^2}\right)^{\phi}, \tag{5}$$

of the scalar field, the mass of the latter is found to depend on the curvature of the environment.[3] In addition, the radiative corrections show that m_{eff} could be huge for very low curvatures.[3,4]

It turns out that this model easily finds a very nice application for investigating the early Universe.[1] The slow-roll parameters of the model are,

$$\epsilon \equiv \frac{|\dot{H}|}{H^2} \ll 1, \qquad \eta \equiv \frac{|\dot{\phi}|}{H} \ll 1. \tag{6}$$

In usual inflationary models, where the Hubble parameter is given by $H^2 \sim V(\phi)$, the analogue of the η-condition is $\dot{\phi}^2/H^2 \ll 1$, which, thanks to the scalar field's equation of motion, is just equivalent to the ϵ-condition in (6). However, in the model (4) the Hubble parameter is not simply given by $H^2 \sim V(\phi)$ and, therefore, no condition should *a priori* be imposed on $\dot{\phi}$. However, it turns out that the η-condition is automatically satisfied throughout all the duration of an inflation. The latter starts in this model at an energy scale μ and ends at another energy scale not very far off. The end of inflation occurs at that energy scale for which either both or one of the slow-roll parameters in (6) ceases to be small and negligible. It is found that both slow-roll conditions (6) are satisfied up to the scale $R/\mu^2 \sim 10^{-0.05}$. The number of e-folds N in this model can be deduced via the formula,[1]

$$N \simeq \frac{13}{4}\ln\left(\frac{\ln\rho_f}{\ln\rho_i}\right). \tag{7}$$

For example, if one assumes inflation to have ended at the energy scale of $\rho \simeq 10^{-0.05}$, then one should allow inflation to start at the energy scale of $\rho \simeq 10^{-10^{-8}}$, in which case one finds, $N \simeq 50$.[1] To have a bigger number of e-folds one needs to push back the starting of inflation to scales which are even closer to the μ-scale, if not exactly equal to μ.

5.1. *Scalar modes in the (varying power)-law model*

Using the scalar modes equations (2), one easily extracts for the model the following α_s, ν_s and β are given by,[1]

$$\alpha_s = \frac{4}{11} - \frac{3\epsilon}{22} + \frac{1501\eta}{968}, \quad \nu_s^2 = \frac{225}{121} + \frac{285\epsilon}{121} + \frac{145555\eta}{21296},$$
$$\beta = 1 + \frac{13\eta}{22}. \tag{8}$$

5.2. *Tensor modes in the (varying power)-law model*

Using the tensor modes equations (3), one easily extracts for the model the following α_t, ν_t,[1]

$$\alpha_t = \frac{1}{2} - \frac{\eta}{4} \quad \text{and} \quad \nu_t^2 = \frac{9}{4} + 3\epsilon - \frac{11\eta}{4}. \tag{9}$$

5.3. *The power spectra in the (varying power)-law model*

At Hubble-crossing one finds,[1]

$$r \simeq \eta_*^2, \tag{10}$$

where η_* is the slow-roll parameter in (6) evaluated at the Hubble crossing. The scalar and tensor tilts n_s and n_t can be computed from the scalar and tensor power spectra $\mathcal{P}_\mathcal{R}$ and $\mathcal{P}_h$, respectively, as,[1]

$$n_s - 1 = \left.\frac{d \ln \mathcal{P}_\mathcal{R}}{d \ln k}\right|_{k=k_*} = 2 + 2\alpha_s - 2\nu_s, \tag{11}$$

$$n_t = \left.\frac{d \ln \mathcal{P}_h}{d \ln k}\right|_{k=k_*} = 2 + 2\alpha_t - 2\nu_t. \tag{12}$$

One can compute the values of r, n_s and n_t in (10), (11) and (12), respectively, for various energy scales comprised between the energy scales $\rho \sim 10^{-10^{-8}}$ and $\rho \sim 10^{-0.05}$. One easily finds that the scale for the Hubble crossing that matches the best with observations is comprised between $\rho \sim 10^{-0.008}$ and $\rho \sim 10^{-0.01}$ for which the values of r, n_s, and n_t found using expressions (8) and (9) for α_s, ν_s, α_t and ν_t are: $r \in [0.00035, 0.00055]$, $n_s \in [0.965, 0.972]$, and $n_t \in [0.031, 0.040]$.[1]

The specific scale that agrees best with the observed values is found to be $\rho \sim 10^{-0.009}$. Indeed for this scale a computation using again expressions (8) and (9) for α_s, ν_s, α_t and ν_t gives the following results:[1]

$$r \simeq 0.00044, \quad n_s \simeq 0.969, \quad n_t \simeq 0.036. \tag{13}$$

Here, one uses the value of the scalar tilt to find the best match. In fact, the value of the ratio r is only known by its upper boundary value which is $r < 0.1$[5]. Notice that the tensor tilt here is positive in contrast to usual single-field inflationary models where it comes out negative. The other major difference is that the model predicts a very small value for the tensor-to-scalar ratio r.

One can also deduce the energy scale for inflation required by the model after making use of the relation between the scalar power spectrum $\mathcal{P}_{\mathcal{R}}$ and its amplitude A_S. The latest observations show that the amplitude of the scalar power spectrum at the Planck pivot scale k_* is $A_S \sim 2 \times 10^{-9}$.[5] This implies that the inflation scale according to the model should be around $\sim 10^{14}$ GeV.

6. Conclusion

The (varying power)-law model cannot be motivated from particle physics. It has only two free parameters, though, and does not require any fine-tuning. It might unify dark energy, dark matter and the early expansion of the Universe.

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