Torsion and non-metricity in gravity and holography

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Abstract

One of the most important achievements in theoretical physics in the last decades has been the development of the holographic principle. The important achievement of this principle is to illustrate that physical models have the potential to describe both quantum physics and gravity at the same time. In particular, the holographic principle states that a gravitational theory defined in a region of spacetime is equivalently described by a quantum field theory on the boundary of this region. Since the quantum theory is defined on the boundary of the gravitational theory, we study the correspondence of these theories by examining the boundary behavior of gravitational fields. We interpret gravity as an effect emergent of the geometry of spacetime. This geometry is usually characterized solely by curvature, and holography is routinely applied in this case to examine the boundary field theory. The essential new aspect of this thesis is that I consider holography based on geometries which feature non-trivial torsion and non-metricity. These field strengths characterize geometry similar to curvature. Motivated by holography, I focus in particular on the gravitational boundary behavior of geometries described by curvature, torsion and non-metricity.

If spacetime has a boundary, we generically need to introduce a Gibbons-Hawking-York (GHY) boundary term to render the variational principle well-defined. The first main result of this thesis is a universal GHY term that makes the Dirichlet problem well-defined in spacetimes which are allowed to have non-trivial curvature, torsion and non-metricity. I derive this universal GHY term for space-, time- and lightlike boundaries. As an important corollary, I observe that no GHY term is needed for actions which only depend on torsion and non-metricity. The method I develop allows to calculate GHY terms very efficiently compared to traditional approaches, even when considering theories with vanishing torsion and non-metricity. This enables us to calculate GHY terms for theories for which it was previously highly involved. Furthermore, the universality of my method allows to generalize existing results to spacetimes with non-vanishing torsion and non-metricity as well as to arbitrary dimensions. I illustrate this calculation for general relativity (GR), four-dimensional Chern-Simons modified gravity and Lovelock gravity. All of these examples correctly reproduce existing results for GHY terms in spacetimes with space- or timelike boundaries if torsion and non-metricity are constrained to vanish. I generalize these results to include torsion and non-metricity as well as lightlike boundaries. In the latter case only little is known about GHY terms. My method provides the first consistent approach that allows to

technically handle the involved derivation of lightlike GHY terms.

By means of the methods I have introduced for deriving the universal GHY term, I examine the boundary behavior of the geometrical trinity of general relativity. This trinity consists of three theories that equivalently describe the dynamics of GR by either curvature, torsion or non-metricity. The bulk actions of these theories differ by a boundary term $S^{\mathring{D}A}$. I generalize the geometrical trinity of general relativity to spacetimes with boundaries by including appropriate GHY terms. Careful analytic manipulation reveals that $S^{\mathring{D}A}$ is a difference of GHY terms. I show that its role is to render the variational principle correct for each way of expressing the bulk action. Since GHY terms are needed only for actions which depend on curvature, we conclude that the GHY term reminiscent of $S^{\mathring{D}A}$ must be eliminated in the curvature-free theories involved in the geometrical trinity. This provides the full equivalence of the actions in the geometrical trinity of general relativity in spacetimes with boundary and shows that the torsionful and non-metric sides do not require additional boundary terms.

I develop a new perspective on the geometrical trinity of general relativity by proving that all of its theories may be described by the Einstein action. I supplement the Einstein action by a boundary term which makes it covariant. The Einstein action is constructed using the Levi-Civita connection $\mathring{\omega}^{\mu}{}_{\nu}$ which is the connection of a theory with vanishing torsion and non-metricity. I show that $\mathring{\omega}^{\mu}{}_{\nu}$ may equally well be interpreted as a connection which has non-trivial torsion and non-metricity. From this point of view, the equivalence of the theories contained in the geometrical trinity of gravity becomes tautological as all of them are described by the exact same action. I show that the trinity of GR may be generalized to a geometrical trinity of gravity by realizing that every torsion-free, metric-compatible theory of gravity may be described equivalently in terms of torsion and non-metricity. I render the variational problem of these equivalent actions well-defined by discussing their GHY terms.

I obtain essential new results by applying the method of holographic renormalization to the torsionful equivalent of GR. This involves the construction of a Fefferman-Graham expansion for coframes in the vicinity of the boundary of spacetimes with negative cosmological constant. This coframe expansion reproduces the expansion of the metric tensor underlying the Fefferman-Graham theorem. It allows to apply holographic renormalization to the torsionful equivalent of the Schwarzschild black hole. I explicitly construct the counterterms which are needed to make its on-shell action finite. Furthermore, I prove that torsion holds thermodynamical information by calculating the free energy of the boundary field theory. This is the foundation of an encompassing understanding of holographic renormalization of gravitational theories defined on geometries with curvature, torsion and non-metricity.

The results presented in this thesis are based on the following publications.

[1] J. Erdmenger, B. Hess, I. Matthaiakakis, and R. Meyer. "Universal Gibbons-Hawking-York term for theories with curvature, torsion and non-metricity." In: *SciPost Phys.* 14 (2023). DOI: 10.21468/SciPostPhys.14.5.099. arXiv: 2211.02064 [hep-th]

[2] J. Erdmenger, B. Hess, I. Matthaiakakis, and R. Meyer. "Gibbons-Hawking-York boundary terms and the generalized geometrical trinity of gravity." In: *Phys. Rev. D* 110.6 (2024). DOI: 10.1103/PhysRevD.110.066002. arXiv: 2304.06752 [hep-th]

Section 3.1 contains an extended description of my normal vector field approach to space- and timelike hypersurfaces underlying both [1] and [2]. The derivation of the universal GHY term for space- and timelike boundaries contained in [1] is presented in section 4.1 of this thesis. I applied the results of [1] to the geometrical trinity of general relativity in [2] which I review in section 5.1. Section 5.3 contains the generalization of the geometrical trinity discussed in [2]. All results obtained in chapters 3 to 6 beyond the ones discussed above constitute my original results which are unpublished so far. This includes the generalization of all calculations to lightlike hypersurfaces in sections 3.2, 4.2 and chapter 5. Moreover, the unification of the geometrical trinity in the Einstein action in section 5.2 as well as the frame perspective on holographic renormalization in chapter 6 are my original results which I first present in this thesis.

Zusammenfassung

Das holographische Prinzip stellt eine der wichtigsten Errungenschaften der Physik in den vergangenen Jahrzehnten dar. Die Relevanz dieses Prinzips ist darin begründet, dass es zeigt, wie physikalische Modelle gleichzeitig zur Beschreibung von Quanten- und Gravitationsphysik verwendet werden können. Das holographische Prinzip besagt, dass eine Gravitationstheorie, die in einer Region der Raumzeit definiert ist, zu einer Quantenfeldtheorie auf dem Rand dieser Region äquivalent ist. Weil die Quantentheorie dabei auf dem Rand der Gravitationstheorie definiert ist, gründet die Erforschung ihrer Korrespondenz auf der Untersuchung der Randwerte der Gravitationsfelder. Gravitation interpretieren wir dabei als einen Effekt, der aus der Geometrie der Raumzeit erwächst. Diese Geometrie wird üblicherweise lediglich durch ihre Krümmung charakterisiert. Holographie wird in diesem Fall routiniert angewandt, um die auf dem Rand definierte Feldtheorie zu untersuchen. Der maßgebliche neue Aspekt dieser Dissertation ist, dass ich Holographie in Raumzeiten mit nicht-trivialer Torsion und Nicht-Metrizität diskutiere. Diese Feldstärken charakterisieren die Geometrie einer Raumzeit analog zu ihrer Krümmung. Motiviert durch das holographische Prinzip fokussiere ich mich auf das Randverhalten von Geometrien, die durch Krümmung, Torsion und Nicht-Metrizität beschrieben werden.

In Raumzeiten mit Rand muss im Allgemeinen ein Gibbons-Hawking-York-Randterm (GHY) eingeführt werden, um ein wohldefiniertes Variationsprinzip zu gewährleisten. Das erste wichtige Ergebnis dieser Arbeit ist ein universeller GHY-Term, der die Wohldefiniertheit des Dirichlet-Problems sicherstellt, wenn wir Raumzeiten mit nicht-trivialer Krümmung, Torsion und Nicht-Metrizität betrachten. Ich leite diesen universellen GHY-Term für raum-, zeit- und lichtartige Ränder her. Eine wichtige Konsequenz dieses Ergebnisses ist, dass kein GHY-Term zu Wirkungen hinzugefügt werden muss, wenn diese lediglich von Torsion und Nicht-Metrizität abhängen.

Die von mir entwickelte Methode erlaubt es, GHY-Terme im Vergleich zu traditionellen Herleitungen äußerst effizient zu berechnen. Dies gilt insbesondere auch dann, wenn Theorien mit verschwindender Torsion und Nicht-Metrizität betrachtet werden. Diese Effizienz ermöglicht es, GHY-Terme für Theorien herzuleiten, für welche die Berechnung zuvor hochgradig komplex war. Darüber hinaus erlaubt es die Universalität meiner Methode, bereits bekannte Ergebnisse auf Raumzeiten mit nicht-verschwindender Torsion und Nicht-Metrizität sowie auf beliebige Dimensionen zu verallgemeinern. Ich demonstriere diese Effizienz und Universalität am Beispiel der allgemeinen Relativi-

tätstheorie (ART), vier-dimensionaler Chern-Simons-modifizierter Gravitation sowie Lovelock-Gravitation. Für alle betrachteten Gravitationstheorien reproduzieren meine GHY-Terme bekannte Ergebnisse korrekt, wenn wir Raumzeiten mit verschwindender Torsion und Nicht-Metrizität betrachten und uns auf raum- oder zeitartige Ränder beschränken. Ich verallgemeinere diese Ergebnisse, sodass sie Raumzeiten mit Torsion und Nicht-Metrizität sowie lichtartige Ränder mit einschließen. Im letzteren Fall ist bisher nur wenig über GHY-Terme bekannt. Meine Methode stellt den ersten konsistenten Zugang dar, der es erlaubt, die komplexe Herleitung lichtartiger GHY-Terme technisch handzuhaben.

Mit Hilfe der Methoden, die ich zur Herleitung des universellen GHY-Terms eingeführt habe, untersuche ich das Randverhalten der geometrischen Trinität der allgemeinen Relativitätstheorie. Diese Trinität besteht aus drei Theorien, die zueinander äquivalent die Dynamik der ART entweder durch Krümmung oder durch Torsion oder durch Nicht-Metrizität beschreiben. Die Wirkungen dieser Theorien unterscheiden sich durch einen Randterm $S^{\mathring{D}A}$. Ich verallgemeinere die geometrische Trinität auf berandete Raumzeiten, indem ich die geeigneten GHY-Terme einführe. Eine analytische Rechnung zeigt, dass $S^{\mathring{D}A}$ eine Differenz von GHY-Termen ist. Auf Grundlage dieses Ergebnisses argumentiere ich, dass $S^{\mathring{D}A}$ die Funktion hat, ein wohldefiniertes Variationsprinzip für jede Formulierung der Wirkung sicherzustellen. Weil GHY-Terme nur für Wirkungen benötigt werden, die von der Krümmung abhängen, folgern wir, dass der aus $S^{\mathring{D}A}$ stammende GHY-Term in den krümmungsfreien Theorien der geometrischen Trinität eliminiert werden muss. Dadurch erhalten wir die volle Äquivalenz der Wirkungen der geometrischen Trinität der allgemeinen Relativitätstheorie in Raumzeiten mit Rand. Darüber hinaus zeigen wir, dass keine Notwendigkeit für das Hinzufügen weiterer Randterme zu den Seiten der geometrischen Trinität besteht, die Gravitation durch Torsion und Nicht-Metrizität modellieren.

Ich entwickle eine neue Perspektive auf die geometrische Trinität der allgemeinen Relativitätstheorie indem ich zeige, dass alle in dieser Trinität enthaltenen Theorien durch die Einstein-Wirkung beschrieben werden können. Durch das Hinzufügen eines Randterms stelle ich die Kovarianz dieser Wirkung her. Die Einstein-Wirkung wird auf Grundlage des Levi-Civita-Zusammenhangs $\mathring{\omega}^{\mu}{}_{\nu}$ konstruiert, welcher der Zusammenhang einer Theorie ist, in der Torsion und Nicht-Metrizität verschwinden. Ich zeige, dass $\mathring{\omega}^{\mu}{}_{\nu}$ mit gleichem Recht als ein Zusammenhang interpretiert werden kann, der eine Theorie mit nicht-trivialer Torsion und Nicht-Metrizität beschreibt. Die Äquivalenz der Theorien, die in der geometrischen Trinität der Gravitation enthalten sind, wird aus diesem Blickwinkel tautologisch, weil all diese Theorien durch ein und dieselbe Wirkung beschrieben werden. Ich zeige, dass die ART-Trinität zu einer geometrischen

Trinität der Gravitation verallgemeinert werden kann, indem ich feststelle, dass jede torsionsfreie, metrik-kompatible Gravitationstheorie eine äquivalente Beschreibung durch Torsion und Nicht-Metrizität zulässt. Die Wohldefiniertheit des Variationsprinzips in diesen äquivalenten Wirkungen stelle ich durch eine Diskussion der jeweiligen GHY-Terme sicher.

Ich leite essentielle neue Ergebnisse her, indem ich die Methode der holographischen Renormierung auf das torsionsbehaftete Äquivalent der ART anwende. Insbesondere konstruiere ich die Fefferman-Graham-Entwicklung der Ko-Repern in Nähe des Randes von Raumzeiten mit negativer kosmologischer Konstante, wobei Ko-Repern die geordneten Basen der Kotangetialräume bilden. Diese Entwicklung von Ko-Repern reproduziert die Entwicklung des metrischen Tensors, die dem Fefferman-Graham-Theorem zugrundeliegt. Sie erlaubt es, holographische Renormierung auf das torsionsbehaftete Äquivalent des Schwarzschildschen schwarzen Lochs anzuwenden. Ich konstruiere die Gegenterme, die benötigt werden, um einen endlichen Wert der Wirkung sicherzustellen, wenn diese auf der Schwarzschildschen Lösung ausgewertet wird. Darüber hinaus zeige ich durch Berechnung der freien Energie der korrespondierenden Feldtheorie, dass Torsion Informationen über die thermodynamischen Eigenschaften eines physikalischen Systems enthält. Diese Berechnungen bilden die Grundlage für ein umfassendes Verständnis der holographischen Renormierung von Graviationstheorien, die in Geometrien definiert sind, welche neben ihrer Krümmung durch Torsion und Nicht-Metrizität charakterisiert werden.

Die Ergebnisse, die ich in dieser Dissertation präsentiere, basieren auf den folgenden Veröffentlichungen.

[1] J. Erdmenger, B. Hess, I. Matthaiakakis und R. Meyer. „Universal Gibbons-Hawking-York term for theories with curvature, torsion and non-metricity“. In: *SciPost Phys.* 14 (2023). doi: 10.21468/SciPostPhys.14.5.099. arXiv: 2211.02064 [hep-th]

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Abschnitt 3.1 enthält eine erweiterte Beschreibung meines Zugangs zu raum- und zeitartigen Hyperflächen basierend auf einem Normalvektorfeld, der sowohl [1] als auch [2] zugrundeliegt. Die Herleitung des universellen GHY-Terms für raum- und zeitartige Ränder, die in [1] enthalten ist, präsentiere ich in Abschnitt 4.1 dieser Arbeit. Die Ergebnisse von [1] habe ich in [2] auf die geometrische Trinität der allgemeinen

Relativitätstheorie angewandt, was ich in Abschnitt 5.1 bespreche. Abschnitt 5.3 enthält die Verallgemeinerung der geometrischen Trinität, die ich in [2] besprochen habe. Alle übrigen Resultate, die ich zusätzlich zu den bereits erwähnten in den Kapiteln 3 bis 6 bespreche, stellen meine eigenen Ergebnisse dar, die ich bisher nicht veröffentlicht habe. Dies umfasst die Verallgemeinerung all meiner Berechnungen auf lichtartige Hyperflächen in den Abschnitten 3.2 und 4.2 sowie in Kapitel 5. Darüber hinaus sind die Vereinheitlichung der geometrischen Trinität in der Einstein-Wirkung in Abschnitt 5.2 sowie die Reper-Perspektive auf holographische Renormierung in Kapitel 6 Ergebnisse, die ich erstmalig in dieser Dissertation bespreche.

Contents

Introduction 1

The development of the theory of gravity more than three hundred years ago founded the modern way of research in physics [3]. In the 17th century, Isaac Newton derived the universal law of gravity which gave a mathematical description of gravitation, first published in the Philosophiæ Naturalis Principia Mathematica [4, 5]. While the precise mathematical form of Newton's gravitational force was mainly deduced from observations of celestial motions, it took more than two hundred years until Albert Einstein provided a fundamental explanation of Newton's theory [3, 6]. Einstein's theory of relativity is based on only a few fundamental observations. First, the theory of special relativity incorporates that the speed of light is constant according to Maxwell's theory of electrodynamics [7]. Combining this observation with the principle of relativity which states that each observer moving at constant velocity experiences the same laws of physics, Einstein concluded that the Newtonian concept of absolute time needs to be generalized. That is, both space and time intervals are measured differently by different observers. In other words, space and time are relative. Hermann Minkowski realized that it is most efficient to combine space and time in a model called *spacetime* which is characterized by a metric tensor that measures lengths and angles [8, 9].

A decade after Einstein published his theory of special relativity, he gave spacetime itself a dynamical role by allowing the metric tensor to take different values at different points in spacetime. The resulting general theory of relativity (GR) adds an additional assumption to the two observations underlying special relativity. This fundamental assumption of general relativity is that the principle of relativity extends to coordinate frames which are accelerated with respect to each other, and that the physical properties of spacetime experienced in a uniformly accelerated coordinate frame are indistinguishable from those in a gravitational field [10, 11]. In particular, this equivalence principle implies that the equations of special relativity hold locally in the physical system of any observer which is accelerated by gravitation [12, 13]. This generalizes the concept of uniform motion underlying the principle of relativity: Observers still follow straight lines in the spacetime underlying general relativity, but the geometry of spacetime causes their motions to be accelerated [14]. This generalizes the concept of straight lines to what is called a geodesic curve.

What remains open from this brief introduction to general relativity is the question of what causes the general relativistic geodesics to deviate from what we would naively call straight lines. Since the gravitation of an object is related to its mass, it is mass that is responsible for this deviation. That is, mass causes the metric tensor to differ from the Minkowski metric, and this deformation in turn causes geodesics to deviate from straight lines [13]. Hence, mass is accelerated by a non-trivial metric tensor and at the same time it changes the metric tensor by its motion. The metric tensor therefore changes in space and time, and is thus said to be dynamical.

The equations of motion of general relativity reduce to the motions described by Newton's gravitational force if observers move at small speed compared to the speed of light at non-cosmological distances [13]. Therefore, the predictions of GR differ substantially from Newton's only if one approaches the speed of light. The success of Einstein's theory is based on its explanation of observations in this limit which cannot be understood from Newton's theory alone. This includes the deflection of light by the mass of the Sun [13, 15, 16], which is the basis of the effect of gravitational lensing exploited in contemporary cosmology [17, 18]. Furthermore, general relativity explains the precession of the perihelion of planet Mercury, that is the movement of the point at which Mercury is closest to the Sun [19].

In Einstein's theory of gravity, the difference of the metric from the Minkowski metric is usually interpreted to imply a non-trivial curvature of spacetime [20]. That is, a spacetime described by the Minkowski metric is interpreted to be flat, while a spacetime endowed with a different metric tensor is said to be curved. However, this interpretation relies on assumptions which we elucidate in the following. From a mathematical point of view, there are two additional dynamical fields which describe the geometry of spacetime in a similar fashion as the metric tensor. First, this is the coframe, which is an ordered basis of the vector spaces defined at each point of spacetime. We call these vector spaces *tangent spaces*. As we will see, the relevance of the coframe arises from its property of connecting tangent spaces to the symmetry group which acts upon spacetime. Second, the connection field provides a prescription for how to connect the tangent spaces defined at nearby points in spacetime. It therefore allows us to compare vectors defined at different points which makes the connection one of the fundamental fields.

The three fundamental fields of a spacetime are thus the connection, the coframe and the metric tensor. Studying dynamics in spacetimes equipped with these fields amounts to examining their field strengths. The field strength of the connection is called curvature, the coframe's field strength is torsion, and the field strength of the metric is non-metricity. While curvature is a ubiquitous concept whenever geometry is studied in physics, torsion and non-metricity are usually less familiar, although being defined analogous to curvature. Let us therefore develop an intuition about these field strengths.

The standard geometric picture of curvature, torsion and non-metricity relies on the process of parallel transport. Parallel transport is induced by the connection which we introduced for connecting nearby points in spacetime. Using this connection, we consider the transport of vectors defined at some point in spacetime along a curve which contains this point. This curve is considered to be smooth, but may generically be non-geodesic. In this setup, a vector is said to be parallel transported if it is parallel to itself at nearby points with respect to the underlying geometry, where the latter is characterized by the connection. Curvature, torsion and non-metricity all allow for an interpretation in terms of parallel transport [20–22]. We discuss these interpretations in the following and depict the corresponding transport processes in figure 1.

First, consider a generic vector a defined at some point in spacetime. We parallel transport this vector along a closed curve such that we eventually end up at the point at which we started. However, the vector $\tilde{a}$ obtained by parallel transport of a is only equal to a if spacetime is considered to be flat. If spacetime is curved, the directions of $\tilde{a}$ and a will generically differ, and the angle they include is determined exactly by the curvature of the underlying spacetime. For obtaining a similar picture of torsion, we consider two vectors a and b defined at the same point and demand that the two vectors are not aligned with each other. Each of these vectors generates a curve to which the vector is tangent. Now consider the new vector $\tilde{a}$ to be the vector obtained by parallel transport of a along the curve generated by b, where we transport a for a distance which equals the length of b. Analogously, we parallel transport b for the length of a along the curve generated by a to obtain the vector $\tilde{b}$. In spacetimes with vanishing torsion, this procedure results in the four vectors a, b, $\tilde{a}$ and $\tilde{b}$ spanning a parallelogram. If torsion is non-vanishing, however, this parallelogram does not close. The amount by which the diagram does not close is directly determined by torsion. The simplest parallel transport perspective is obtained for non-metricity. To see that, we only need to parallel transport a vector a along any curve in spacetime. For vanishing non-metricity, this results in a vector $\tilde{a}$ which is generically defined at a different point but has the same length as a. This is no longer true if non-metricity is present. Thus, the change of the length of vectors under parallel transport is the pictorial interpretation of non-metricity.

Note that if we consider spacetimes which have torsion and non-metricity but no curvature, vectors stay parallel to themselves when we parallel transport them along closed curves. Therefore, such theories are called *teleparallel* [23–29]. In recent years, there has been an enormous interest in studying teleparallel theories of gravity [30–35]. But due to the success of Einstein's theory of general relativity, it is reasonable to ask

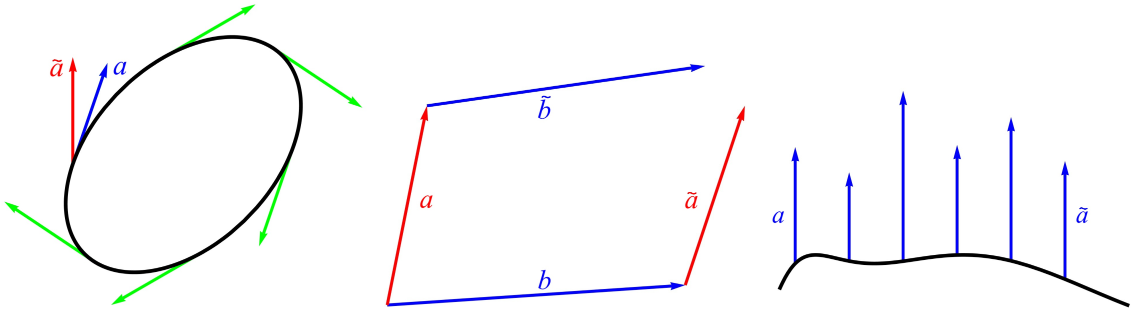

Fig. 1.: Parallel transport interpretation of geometrical field strengths. **Left:** The change in direction obtained by parallel transport of a along a closed non-geodesic curve is measured by curvature. **Middle:** Parallel transport of a along b and vice versa yields a non-closing diagram in presence of torsion. **Right:** Non-metricity causes vectors to change length under parallel transport. Figure inspired by [36].

why teleparallel theories are needed at all. There are many answers to this question.

First, there are yet quite a lot of open problems in cosmology. These arise from observations which cannot be explained by general relativity, hinting at the need for a theory beyond GR. Just as in the transition from Newton's to Einstein's theory of gravity, this modified theory of gravity is expected to differ from GR in extreme cases. For example, the rapid expansion of the universe in its early stage cannot be explained by GR without introducing the concept of inflation [37–46]. Likewise, the accelerated expansion of the universe at late times requires us to postulate the existence of dark energy [47, 48]. Moreover, if we leave GR unmodified, we are required to postulate the existence of dark matter to explain the observed motions of stars in the rotation of galaxies [49–55]. The need for all these concepts is widely accepted in cosmology, where they are part of the Standard Model [46, 56–58]. They are motivated by observations but cannot be explained by first principles from the underlying theory of general relativity. There is a clear need for a theory which provides a fundamental explanation of these phenomena, and we consider teleparallel extensions of GR to be a candidate for this fundamental theory [29, 59].

But even without going beyond GR, there is a second important argument for why we should study theories with torsion and non-metricity. In particular, the development of the geometrical trinity of general relativity [36, 60] as well as general teleparallel quadratic gravity [61] emphasized that teleparallel theories provide a new point of view on general relativity itself. We will refer to these teleparallel theories collectively as (S)TEGR. The central statement of the geometrical trinity of general relativity is that the dynamics of general relativity possess equivalent formulations in terms of either curvature or torsion or non-metricity. That is, general relativity may be described by means of curvature as developed by Einstein, but it may as well be described by (S)TEGR for which curvature is vanishing. Both descriptions are dynamically equivalent. Therefore, it is important to study theories with non-vanishing torsion and non-metricity in order to obtain new perspectives on general relativity.

Within general relativity, different maximally symmetric spacetimes are typically classified by the value of their cosmological constant. In Einstein's GR, this value immediately implies the value of scalar curvature by means of the equations of motion, such that the value of the cosmological constant and that of the scalar curvature are often used interchangeably. Depending on the sign of this constant, we call a spacetime de Sitter (dS), Minkowski or Anti-de Sitter (AdS) if it has positive, vanishing or negative cosmological constant, respectively. In two dimensions, we may imagine dS spaces as a sphere, Minkowski spaces as flat planes and AdS spaces as a saddle. While it is interesting to study gravity in these spacetimes in its own right, one of the most fascinating developments in theoretical physics in the last century is that gravitational dynamics in AdS spacetimes posses an a priori unexpected description in terms of field theory. Concretely, this is a quantum field theory which is invariant under conformal transformations (CFT), and its equivalence to gravity in AdS spacetimes was first conjectured by Juan Maldacena in 1997 [62].

In the AdS/CFT correspondence, the conformal field theory is defined on the boundary of the AdS spacetime [63, 64]. Therefore, the AdS/CFT correspondence is an example of the holographic principle which proposes that all information contained in a volume of spacetime is encoded on its boundary [65–68]. This correspondence does not only provide a duality between gravity and field theory. It furthermore maps strongly interacting field theories to weakly interacting theories of gravity which are mathematically tractable by means of perturbation theory. From this perspective, the AdS/CFT correspondence provides a powerful tool for studying strongly interacting field theories [69, 70].

The AdS/CFT correspondence is particularly relevant in the context of relativistic hydrodynamics. Hydrodynamics is an effective field theory which is used to describe fluctuations around the thermal equilibrium of a system at low energy and long wavelength. AdS/CFT provides a systematic approach to the calculation of transport properties of strongly coupled fluids on the conformal boundary [63, 64, 69–71]. In particular, the transport of energy, momentum and charge in hydrodynamics have been studied intensively in recent years [72–75] by means of gravity correlators determined by Kubo formulae as well as the fluid/gravity correspondence [76–78]. However, this research mostly involves gravitational theories which are described solely by curvature, while torsion and non-metricity are vanishing. For describing spin transport in strongly coupled fluids, it is essential to lift these constraints since the hydrodynamic spin current is determined by a torsionful connection within AdS/CFT. In a similar way, we expect the introduction of non-metricity into a theory of gravity to yield new transport phenomena in the corresponding strongly coupled fluid, related to a conserved current known as hypermomentum [79].

Studying strongly coupled fluids has become widely important for different branches of physics [80–87]. On the one hand, hydrodynamics has been found to be the most adequate transport theory to describe the properties of the quark gluon plasma in collider physics. On the other hand, relativistic hydrodynamics finds applications in condensed matter physics. The latter studies the dynamics of charges in solids which are determined by an electronic band structure. The properties of solids depend significantly on the form of this band structure. If at least two bands intersect in a distinct point, called a node, and all energy states underneath this node are occupied by electrons, the material is called semimetal. For semimetals, the appropriate theory for describing transport phenomena at nodes is again relativistic hydrodynamics. Studying relativistic hydrodynamics in semimetals is a subject of current research [88]. Since the fluids in semimetals are strongly coupled, we cannot use perturbation theory to study their transport behavior. Therefore, the AdS/CFT correspondence provides a groundbreaking tool for deriving the physical properties of semimetals through its connection to relativistic hydrodynamics.

If we consider an electron current in semimetals, not only the charge of the electron is transported, but so is its spin. Hence, in addition to charge, energy and momentum, we need to describe spin transport in hydrodynamics. As we argued above, this amounts to introducing torsion and non-metricity in the gravitational bulk of the AdS/CFT correspondence. This is the main motivation for me to study gravity with torsion and non-metricity in this thesis.

There are, however, questions of fundamental importance in gravity that need to be addressed before the AdS/CFT correspondence may be applied to the description of spin hydrodynamics. Since the CFT is defined on the boundary of the AdS space in which gravity propagates, it is essential to study the boundary terms of gravitational theories in order to establish AdS/CFT in this context. Giving a proper understanding of these boundary terms in generic theories of gravity which include curvature, torsion and non-metricity is the main achievement of my work which I present in this thesis. The presentation of this work is organized as follows.

We begin by explaining geometry with curvature, torsion and non-metricity in detail in chapter 2. This presentation will appear very mathematical to some physicists and very physical to some mathematicians. The spirit in which this introduction to differential geometry is written is the following. We provide all the mathematical background which is needed for understanding precisely what curvature, torsion and non-metricity are. We do not present proofs of the theorems we examine, since these are all contained in the canonical literature on mathematical physics. At the same time, we develop mathematical tools guided by physical observations. The realm of differential geometry is much broader than the presentation in chapter 2, but for the application of differential geometry to physics it is important to avoid over-constraining our description of nature by making too many assumptions. For this reason, we introduce precisely those levels of mathematical structure which we need in order to understand physical observations, but do not assume anything beyond that. In this spirit, we add layers of structure upon bare sets until the mathematical formalism is powerful enough to provide a language for describing physical reality.

The latter conceptual explanation needs an example to elucidate what we identify as the right amount of mathematical structure from a physics perspective. A prominent example is that classical physics is defined in three spatial and one time direction, which as a set can be thought of as $\mathbb{R}^4$. However, we can never observe the whole universe, we are only able to conduct physical experiments in the region of space and time in our immediate vicinity. Hence, we only assume that the mathematical description of our observable world is *locally* described by an $\mathbb{R}^4$, while globally the structure may be different. This line of thought results in what is called a *manifold*, and we will study manifolds and further structures constructed upon them in detail. This culminates in the definition of the frame bundle as a principal $\mathrm{GL}(m, \mathbb{R})$-bundle, and we examine the bundles associated with it. This level of structure will finally allow us to consistently define curvature, torsion and non-metricity, and we therefore need to understand it.

We use the precise intuition gained for geometries with curvature, torsion and non-metricity to study submanifolds in chapter 3. In particular, we introduce a new way of studying hypersurfaces which are manifolds that have one dimension less than the manifold in which they are embedded. This new perspective consists of defining hypersurfaces by means of a normal vector field. We compare this perspective to existing approaches for describing hypersurfaces and derive all the standard equations, such as the Gauß-Weingarten equations and the Ricci identities. However, we derive all these equations from one and only one decomposition, that of frames and coframes. This frame perspective on hypersurfaces is the main achievement of chapter 3. It constitutes the mathematical foundation of my publications [1, 2] in the case of space- and timelike hypersurfaces. Furthermore, we generalize the normal vector field approach to lightlike hypersurfaces and derive a frame decomposition perspective on these, which constitutes new, unpublished research.

The methods for studying hypersurfaces developed in chapter 3 are the foundation for the subsequent chapters in which we examine boundaries of manifolds. In particular, we interpret these boundaries as hypersurfaces and decompose every tensor into hypersurface tangent and normal contributions. In chapter 4, we derive this decomposition for curvature, torsion and non-metricity. By means of these decompositions, we derive the Gibbons-Hawking-York (GHY) terms in generic geometries. GHY terms are well known in general relativity, where they are introduced to render the variational problem well-defined. Generically, we introduce an action S for describing a physical system by considering variations of it. That is, enforcing the variation δS to vanish by means of Hamilton's principle allows to derive the equations of motion of a physical system. This principle only holds on manifolds with boundary if we supplement the action by a GHY term. Thus, the GHY term is a boundary term and it becomes manifest if we express the action solely in terms of boundary tangent fields.

Using this method, we derive GHY terms in chapter 4 for a wide realm of actions by decomposing curvature, torsion and non-metricity into boundary tangent and normal contributions. For space- and timelike boundaries, I published these results in [1], while the GHY terms of lightlike boundaries constitute original results first presented in this thesis. The main results of chapter 4 are the following. We find that actions which do not contain curvature never require us to introduce a GHY term. In other words, the variational principle is well-defined for any action constructed solely from torsion and non-metricity without the necessity of additional boundary terms. If, instead, the action depends on curvature, a GHY term is generically needed. The explicit form of this GHY term depends on the particular curvature dependence of the Lagrangian.

We present a universal method which allows to calculate the GHY term for a broad range of actions. This new approach represents an extraordinarily efficient method for calculating GHY terms. This efficiency allows us to calculate the GHY terms for various theories in a straightforward way, even if the derivation of GHY terms by standard calculations is highly involved or unknown altogether. We demonstrate the power of our method in several examples, ranging from efficient reproductions of literature results to theories for which the GHY term was previously unknown. In particular, we study Einstein-Hilbert gravity, four-dimensional Chern-Simons modified gravity and Lovelock gravity for both lightlike and non-lightlike boundaries. All calculations and derivations are extraordinarily compact compared to the literature approaches. This is due on the one hand to the efficiency of our method, and on the other hand to the usage of a differential form notation which is rarely used in the literature.

As a particular example for which a rigorous treatment of boundary terms was previously unknown, we examine the geometrical trinity of general relativity in chapter 5. As explained above, the geometrical trinity allows to express general relativity equivalently by means of either curvature or torsion or non-metricity. We re-derive this trinity in differential form language by transforming the Einstein-Hilbert action underlying general relativity to its teleparallel equivalent. The latter includes both torsion and non-metricity at the same time, so that we effectively reduce the geometrical trinity to an equivalence of general relativity and its general teleparallel equivalent (S)TEGR. The transformation from GR to (S)TEGR involves a boundary term $S^{\mathring{D}A}$ which is well-known but has not been examined sufficiently. I gave the first thorough interpretation of this term for space- and timelike boundaries in [2]. As a main result, the decomposition of $S^{\mathring{D}A}$ into boundary tangent and normal contributions reveals that it is a difference of GHY terms. Its role is therefore to render the variational principle correct, no matter if the action is written in terms of a curvature tensor which is constructed from a generic connection or the one which has vanishing torsion and non-metricity. We conclude that the GHY term needs to be removed when imposing the teleparallel constraint of vanishing curvature, since only curvature-related terms in the Lagrangian require us to introduce GHY terms. This discussion generalizes to lightlike boundaries as well, which is my original result first published in this thesis.

In section 5.2 I derive a new perspective on the geometrical trinity of general relativity. This is entirely new and, at the moment of writing, unpublished research. Using an ambiguity introduced by the (S)TEGR choice of connection, we encounter that all theories involved in the geometrical trinity are fully captured by one and only one action. This action is the Einstein action which Einstein originally used to derive the field equations of general relativity in 1915 [12]. While this action is not manifestly covariant, I derive the boundary term which restores its covariance for space-, time- and lightlike boundaries. The bulk Einstein action is quadratic in the Levi-Civita connection $\mathring{\omega}^{\mu}{}_{\nu}$. This connection is torsion-free and metric compatible. The interpretation of the Einstein action as a theory of general relativity is therefore immediate, while it is not obvious that it also describes teleparallel theories. Nevertheless, we derive the same action as a rewriting of the (S)TEGR action. I resolve this surprising equivalence by giving the Levi-Civita connection $\mathring{\omega}^{\mu}{}_{\nu}$ a teleparallel interpretation. That is, we rewrite

$$\mathring{\omega}^{\mu}{}_{\nu} = \omega^{\mu}{}_{\nu} - A^{\mu}{}_{\nu}\,, \tag{1.1}$$

where $\omega^{\mu}{}_{\nu}$ is a generic connection one-form and the deformation one-form $A^{\mu}{}_{\nu}$ is entirely determined by torsion and non-metricity. Hence, the right hand side of (1.1) generically captures effects of curvature, torsion and non-metricity. In particular, we may choose the connection $\omega^{\mu}{}_{\nu}$ to be the curvature-free connection of (S)TEGR, such that the right hand side of (1.1) describes only effects of torsion and non-metricity. That is, we have found an interpretation of the torsion-free, metric-compatible Levi-Civita connection in terms of torsion and non-metricity. This is the fundamental reason why the Einstein action describes both GR and its teleparallel equivalent at the same time.

This reinterpretation of the Levi-Civita connection and the Einstein action creates an entirely new perspective on the geometrical trinity of general relativity. It unifies all approaches to describing the dynamics of general relativity by GR, TEGR, STEGR, (S)TEGR, coincident general relativity (CGR) and theories beyond those. Furthermore, the reinterpretation of the Levi-Civita connection in torsionful and non-metric theories allows to assign teleparallel equivalents to any theory which is constructed upon the Levi-Civita connection. This generalizes the geometrical trinity of general relativity to constitute a geometrical trinity of gravity. I discuss this generalization and the role of boundary terms therein in detail in section 5.3. My discussion formalizes previous attempts to generalize the geometrical trinity of general relativity and adds a discussion of boundary terms.

In chapter 6, I develop the first systematic formalism for applying the method of holographic renormalization to teleparallel theories of gravity. We introduced torsion as the field strength of the coframe. In order to apply holographic renormalization to torsionful theories, it is therefore crucial to develop a frame perspective on the underlying formalism. I first develop this perspective for the Fefferman-Graham theorem which underlies holographic renormalization. I motivate that the Fefferman-Graham expansion may be understood as an expansion of the coframe at the boundary of AdS spacetimes. This reproduces the well-known near-boundary expansion of the metric in Fefferman-Graham coordinates, serving as a consistency check of my method. Building upon this frame perspective on the Fefferman-Graham expansion, we apply holographic renormalization to the five-dimensional Schwarzschild black hole in AdS space. In particular, we evaluate the on-shell TEGR action for this black hole and find that it is divergent as we approach the AdS boundary. Hence, we regularize by cutting off the radial integral and construct the counterterms needed to renormalize the action. Finally, we evaluate the free energy of the five-dimensional AdS Schwarzschild black hole in TEGR. The latter result demonstrates that torsion is able to hold information on the thermodynamic properties of a theory. This is the first time that a systematic approach to the application of holographic renormalization in teleparallel theories has been developed. The frame perspective on the Fefferman-Graham expansion as well as the holographic renormalization in teleparallel gravity are my original results first published in this thesis. These results serve as a guideline for the construction of the full frame perspective on holographic renormalization of teleparallel theories in future work.

2 Geometry with curvature, torsion and non-metricity

The core subject of the following chapters will be geometry with curvature, torsion and non-metricity. Therefore, we first develop the mathematical language for describing geometry in this chapter. As such, we consider mathematics to be the precise language which allows us to express physical observations in an unambiguous way. Just as any language has more words than you need to discuss about a certain topic, mathematics as a subject is more powerful and has way more branches than we need to describe our physically observable world. Our task as theoretical physicists is to pick these tools from the mathematical toolbox which are suitable to describe our physical observations. Doing so, we certainly need to provide enough mathematical structure to cover the features of our physical system. But we need to be careful which mathematical structures we assume on top of that, since an abundance of structure assumed yields prejudices about the physical nature which the latter might simply not obey. For example, school physics takes place in flat space. As we know since Einstein, gravity is modeled more accurately by considering spacetime to be curved. Hence, the assumption of a space being flat is simply too strong, it is too much mathematical structure. In this chapter, I therefore provide the minimal mathematical background sufficient to describe geometry, which is the foundation of much of modern physics. This foundational character is most immediately obvious for gravity, which in our modern way of describing gravitation is a direct interpretation of geometry. Nevertheless, since mathematics is only meant to provide the language for describing the physical systems which we want to consider in the following chapters, the introduction here is not meant as a language course. In particular, this means that we are going to provide an intuition about the mathematical backgrounds instead of presenting proofs. All of the contents of this chapter are common knowledge in mathematics, and we encourage the interested reader to study the topics of interest in greater detail by consulting some of the standard literature on set theory, topology and geometry [22, 89–92]. The presentation in this chapter is mainly based on a series of lectures given by Frederic Schuller [93], enriched with details from the lecture notes [94, 95] and the books [89, 90].

2.1. From logic to bundles

One of the most basic fields of mathematics is set theory. However, set theory is formulated axiomatically and needs a language on which it is founded. This language is the logic of propositions and predicates. For example, in set theory we want to make statements of the type "A is an element of B". But to understand this statement, we need to know what "being an element" means. This is defined in predicate logic, in which "being an element" is a predicate of two variables A and B called a *relation*. We do not aim at explaining set theory in more detail here, for sake of completeness we only point out that the remainder of this chapter is based on Zermelo-Fraenkel set theory including the axiom of choice (ZFC set theory).

The reason for starting this section with set theory is that this is the coarsest structure one could imagine to describe any physical system. Of course, sets themselves are not yet enough to describe any physics, e.g. the path of a classical particle, so we need to add more structure. But since sets are the weakest structure on which we built our mathematical formalism by adding more and more layers of structure, we point out a recurring theme that will be important on each layer in the following. In particular, each time we add a new layer of structure, we need to study the maps which preserve this structure. These maps are always called isomorphisms but often obtain special names. Let us apply this to set theory: Sets themselves have very little structure. In particular, the elements of a set are not ordered or grouped a priori. Thus, the only structure we have is the number of elements that are contained in a set. So maps which preserve the number of elements are the isomorphisms of set theory. Intuitively, these isomorphisms pair the elements of two different sets in a one-to-one fashion such that no elements remain unpaired. More abstractly, such a one-to-one pairing is given by a bijective map. Hence, we found bijections to be the isomorphisms of set theory. We often consider isomorphic structures being "the same", although strictly speaking they are not identical. For example, the sets $\{A, B, C\}$ and {Jack, Queen, King} are considered the same since they are isomorphic to each other. This principle will be more obvious when we add further structure on top of plain sets like a topology for instance. In fact, this is what we consider next.

Classical physics needs to be recovered by any physical theory which allows to take the appropriate limits. Hence, any mathematical formalism we set up in order to describe a physical system is supposed to be powerful enough to model classical physics. For that reason we need to have a notion of continuity at hand since the trajectories of particles in classical physics are always continuous. The weakest mathematical structure we can add on top of sets which is powerful enough to yield a notion of continuity is a topology. A topology is a choice of subsets of a set which are called open. This choice is not arbitrary, it needs to obey the following rules. For a set $\mathcal{M}$, the set $\mathcal{O}$ is a topology if and only if

1. $\mathcal{M} \in \mathcal{O}$,
2. $\varnothing \in \mathcal{O}$,
3. with two sets $U, V \in \mathcal{O}$, their intersection $U \cap V \in \mathcal{O}$ and union $U \cup V \in \mathcal{O}$ are contained[1] in $\mathcal{O}$.

Subsets of $\mathcal{M}$ are called open if they are contained in the chosen topology $\mathcal{O}$. The tupel $(\mathcal{M}, \mathcal{O})$ is called a topological space, where a *space* in mathematics is just a set with some additional structure.

Motivated by classical physics, we introduced a topology in order to be able to define a notion of continuity. Indeed such a definition is possible; a map $\phi : \mathcal{M} \longrightarrow \mathcal{N}$ is called continuous if the preimages of open sets in $\mathcal{O}_{\mathcal{N}}$ are open sets in $\mathcal{O}_{\mathcal{M}}$. This notion of continuity drastically depends on the choice of topology made for both sets. For physics, we usually choose what is called the standard topology for given sets. For example, if $(\mathcal{M}, \mathcal{O}_{\mathcal{M}})$ and $(\mathcal{N}, \mathcal{O}_{\mathcal{N}})$ are topological spaces, we can construct the so-called product topology on the Cartesian product of the sets $\mathcal{M} \times \mathcal{N}$ in the following way: A set $U \subset \mathcal{M} \times \mathcal{N}$ is contained in the product topology $\mathcal{O}_{\mathcal{M}\times\mathcal{N}}$ if for its elements $(m, n) \in \mathcal{M} \times \mathcal{N}$ there exist $S \in \mathcal{O}_{\mathcal{M}}$ containing m and $T \in \mathcal{O}_{\mathcal{N}}$ containing n such that $S \times T$ is a subset of U. The product topology $\mathcal{O}_{\mathcal{M}\times\mathcal{N}}$ is thus inherited from the topologies $\mathcal{O}_{\mathcal{M}}$ and $\mathcal{O}_{\mathcal{N}}$ in a straightforward manner. In a similar way, we define an induced topology on subsets. One of the most fundamental spaces in physics is $\mathbb{R}^d = \mathbb{R} \times \cdots \times \mathbb{R}$ on which the standard topology is the set of all open balls in $\mathbb{R}^d$. Choosing this standard topology on $\mathbb{R}^d$, we recover the familiar ε-δ-criterion for continuity. The usual understanding is that one always chooses the standard topology if nothing else is said.

Now that we added structure on top of sets, we need to study the isomorphisms of this structure again. These need to preserve all the structure which we have. Thus, the isomorphisms are bijective, continuous maps in order to preserve the structure of sets and their topologies. These isomorphisms of topological spaces are called *homeomorphisms*. Like in set theory, one often considers homeomorphic spaces as being the same. For example, a coffee cup is usually considered to be the same as a doughnut if they are only considered as topological spaces, since both are homeomorphic to each other. Beyond that, topology is a rich field of mathematics. From the point of view

[1]For two sets A and B, A being “contained” in B means that $A \in B$.

of physics it is particularly interesting to study properties of topological spaces which are invariant under homeomorphisms, called topological invariants. However, adding a little more structure to topological spaces will allow us to study topological invariants from the point of view of geometry by means of de Rham cohomology groups. This is the usual method for examining topological invariants in physics. For physics, there are some special topological spaces which are of immediate interest. These are the topological manifolds which we discuss next.

As we discussed when introducing topologies, every new physical theory needs to contain well-understood former theories in the appropriate limits. We will invoke this principle again to select these topological spaces which contain classical physics. In particular, classical physics[2] is defined in some $\mathbb{R}^d$, which is implicitly considered to be a topological space $(\mathbb{R}^d, \mathcal{O}_{\mathrm{St}.\mathbb{R}^d})$ using its standard topology $\mathcal{O}_{\mathrm{St}.\mathbb{R}^d}$. Hence, we could demand that physically relevant topological spaces need to be homeomorphic to $(\mathbb{R}^d, \mathcal{O}_{\mathrm{St}.\mathbb{R}^d})$, that is, structurally equivalent to $(\mathbb{R}^d, \mathcal{O}_{\mathrm{St}.\mathbb{R}^d})$. However, this would imply that we are unable to describe anything beyond physics in $(\mathbb{R}^d, \mathcal{O}_{\mathrm{St}.\mathbb{R}^d})$. Thus, we weaken the above condition and only demand that around every point $p \in \mathcal{M}$ physically relevant spaces look like an $\mathbb{R}^d$. This means that the physically relevant topological spaces are those which are locally homeomorphic to $(\mathbb{R}^d, \mathcal{O}_{\mathrm{St}.\mathbb{R}^d})$. Wrapping these thoughts up, we arrive at the following definition:

A d-dimensional (topological) manifold $(\mathcal{M}, \mathcal{O})$ is a topological space in which for every $p \in \mathcal{M}$ there exist a $U \in \mathcal{O}$ containing p and a homeomorphism $x : U \longrightarrow x(U) \subseteq \mathbb{R}^d$.

For completeness we mention that topological manifolds are also required to be paracompact and Hausdorff (T2). Both of these conditions are topological invariants which are, however, only implicitly relevant for the following discussion. Note that this definition of topological manifolds introduces a notion of dimension for the first time. This dimension $\dim \mathcal{M} := d$ is fixed by the space $\mathbb{R}^d$ which the manifold is locally homeomorphic to. Obviously, this does not include boundaries $\partial\mathcal{M}$. Including boundary effects will be the task of the following chapter, while we focus on introducing the relevant notions on manifolds without boundaries here.

Since topological manifolds are topological spaces, we can construct new manifolds from given ones in the same way as we did for topological spaces. Nevertheless, we need to verify that the result of these constructions still is a topological manifold. For example, the product topology $\mathcal{O}_{\mathcal{M}\times\mathcal{N}}$ we constructed above guarantees that

[2] We did not yet equip the set $\mathbb{R}^d$ with a metric. Hence, in this context $\mathbb{R}^d$ is not Euclidean flat space but the mere set. We will subsequently equip this set with more mathematical structure such that it is powerful enough to serve as a model for Euclidean flat space.

$(\mathcal{M} \times \mathcal{N}, \mathcal{O}_{\mathcal{M}\times\mathcal{N}})$ is indeed a topological manifold if $(\mathcal{M}, \mathcal{O}_{\mathcal{M}})$ and $(\mathcal{N}, \mathcal{O}_{\mathcal{N}})$ are topological manifolds. There exists an important generalization of such a product manifold which combines two topological manifolds as well. This generalization is called a manifold bundle.

The triple $(E, \pi, \mathcal{M})$ is called a bundle (of topological manifolds) if E and $\mathcal{M}$ are (topological) manifolds and $\pi : E \longrightarrow \mathcal{M}$ is a surjective, continuous map.

We will denote this bundle as $E \xrightarrow{\pi} \mathcal{M}$ and call E the total space, $\mathcal{M}$ the base space and $\pi : E \longrightarrow \mathcal{M}$ the projection map. Note that we may turn any surjective map into a continuous projection map by equipping the total space with a suitable topology, for instance the induced topology from $\mathcal{M}$. The preimage $F_p := \mathrm{preim}_\pi(\{p\})$ of a point $p \in \mathcal{M}$ in the base space is called the fibre F_p at this point. If this fibre is identical for every point $p \in \mathcal{M}$, that is $\forall p \in \mathcal{M} : \mathrm{preim}_\pi(\{p\}) = F$, the bundle is called *fibre bundle with (typical) fibre* F. An important special case arises when we consider product bundles $E \xrightarrow{\pi} \mathcal{M}$, where $E = \mathcal{M} \times F$ and $\pi = \mathrm{proj}_1$ is the projection to the first factor. This is trivially a fibre bundle with typical fibre F. We may thus imagine a product bundle as attaching the same fibre at every single point of the base space. For constructing further bundles, we may of course define subbundles and restricted bundles as a straightforward generalization of the notions for topological spaces and manifolds.

It is important to notice that the map $\pi : E \longrightarrow \mathcal{M}$ is a projection, and as such it is not invertible due to the lack of injectivity. Nevertheless, we will later see that there is an important physical interpretation of maps from the base to the total space $\sigma : \mathcal{M} \longrightarrow E$. Such maps are called *sections* if $\pi \circ \sigma = \mathrm{id}_{\mathcal{M}}$.

A bundle is of course a new layer of mathematical structure, and hence we need to study the isomorphisms which preserve this structure. This is a little more involved than before, since we now have more structure to preserve. As a prerequisite for defining the isomorphisms, we thus define bundle morphisms for two bundles $E \xrightarrow{\pi} \mathcal{M}$ and $E' \xrightarrow{\pi'} \mathcal{M}'$. The pair (f, g) of the two maps $f : E \longrightarrow E'$ and $g : \mathcal{M} \longrightarrow \mathcal{M}'$ is called a *bundle morphism* if $\pi' \circ f = g \circ \pi$. The use of this definition is understood best considering the commutative diagram:

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ \mathcal{M} & \xrightarrow{g} & \mathcal{M}' . \end{array}$$

From this diagram it is immediately obvious that a bundle isomorphism needs to be a bundle morphism in both directions. Hence, a bundle isomorphism is a pair (f, g) of invertible maps for which (f, g) and (f^{-1}, g^{-1}) are both bundle morphisms. Since the trivial way of constructing a bundle is the product bundle, we call bundles trivial if they are isomorphic to product bundles.

Just as for manifolds which we demanded to be homeomorphic to $\mathbb{R}^d$ only locally, it will be important later to consider bundles which are only locally bundle isomorphic to another bundle. The definition is analogous to that of topological manifolds. Specifically, a bundle $E \xrightarrow{\pi} \mathcal{M}$ is called *locally isomorphic to* $E' \xrightarrow{\pi'} \mathcal{M}'$ if for every $p \in \mathcal{M}$ there exists a $U \in \mathcal{O}_\mathcal{M}$ containing p such that the restricted bundle $\mathrm{preim}_\pi(U) \xrightarrow{\pi_{\mathrm{res}}} U$ with $\pi_{\mathrm{res}} := \pi|_{\mathrm{preim}_\pi(U)}$ is bundle isomorphic to $E' \xrightarrow{\pi'} \mathcal{M}'$. Following these lines, we weaken the triviality condition and call a bundle *locally trivial* if it is locally isomorphic to some product bundle. We will only consider locally trivial bundles in the following.

There is yet another important construction on bundles which is the pull-back. For its construction, we consider a bundle $E \xrightarrow{\pi} \mathcal{M}$ and a map $g : \mathcal{M}' \longrightarrow \mathcal{M}$. From only this construction we may construct a bundle $E' \xrightarrow{\pi'} \mathcal{M}'$ with base space $\mathcal{M}'$ called the *pull-back bundle*. The new information the pull-back bundle has are the total space E' and the projection π'. The total space is defined as $E' := \{(m', e) \in \mathcal{M}' \times E | \pi(e) = g(m')\}$ for which $\pi' := \mathrm{proj}_1$ obviously is a projection map. On top, the definition of E' immediately suggests to define a map $f : E' \longrightarrow E$ by means of $f := \mathrm{proj}_2$. This way, we constructed a bundle morphism (f, g). Moreover, sections $\sigma : \mathcal{M} \longrightarrow E$ pull back to the pull-back bundle as

$$\begin{aligned}\sigma' : \mathcal{M}' &\longrightarrow E' \\ m' &\longmapsto (m', \sigma(g(m')))\,.\end{aligned}$$

Bundles are the highest level structures which we need in order to understand geometry on an elementary level. In particular, the frame field as well as the affine connection are properly defined on the level of bundles. The field strengths of these fields are torsion and curvature, respectively, which is why bundles play an important role for understanding geometry with these fields strengths. However, the definition of these fields requires a deeper understanding of manifolds which we will introduce next. In particular, we will consider manifolds from the point of view of charts and atlases. The advantage of this a priori redundant formulation is that atlases allow to inherit further structure from $\mathbb{R}^d$ and thereby introduce a notion of differentiability on manifolds.

2.2. Charts, atlases and differentiable structures

In this section we revisit the definition of topological manifolds and exploit that there exists an open neighborhood of every point which is homeomorphic to a subset of $\mathbb{R}^d$. This way, we will construct a chart of the open neighborhood. Recall that we defined a d-dimensional topological manifold $(\mathcal{M}, \mathcal{O})$ as a topological space in which for every $p \in \mathcal{M}$ there exist a $U \in \mathcal{O}$ containing p and a homeomorphism $x : U \longrightarrow x(U) \subseteq \mathbb{R}^d$. We call any such pair (U, x) a *chart* of the manifold, where U is the chart domain and x is the chart map. A chart can therefore be interpreted as a local picture of the manifold in $\mathbb{R}^d$. Guided by our intuition about $\mathbb{R}^d$, we might want to describe the position of some object by means of coordinates. Coordinates of points $p \in \mathcal{M}$ only exist with respect to a particular chart (U, x), for which they are defined as the component functions of x. A component function x^i, where $i \in \{1, \dots, \dim \mathcal{M}\}$, is the projection of x to its i'th entry, that is $x^i \coloneqq \mathrm{proj}_i \circ x$ or explicitly

$$
\begin{aligned}
x^i : U &\longrightarrow x^i(U) \subseteq \mathbb{R} \\
p &\longmapsto \mathrm{proj}_i(x(p))\,.
\end{aligned}
$$

For describing physics on the entire manifold $\mathcal{M}$, we might wish to cover $\mathcal{M}$ by charts such that every point of $\mathcal{M}$ is contained in some chart. This constitutes an *atlas* $\mathcal{A}$ defined by the union

$$\bigcup_{(U,x)\in\mathcal{A}} U = \mathcal{M}$$

of its chart domains. Of course, some point $p \in \mathcal{M}$ will generically be contained in the domains of different charts (U, x) and (V, y). In this case, it is helpful to investigate how the charts are connected. We shrink to considering the maps in $U \cap V$. Then, the map $y \circ x^{-1} : x(U \cap V) \longrightarrow y(U \cap V)$ is called the *chart transition map* or the *coordinate change map*. The commutative diagram for this definition is

$$
\begin{array}{ccc}
 & U \cap V & \\
 {}^{x}\swarrow & & \searrow^{y} \\
\mathbb{R}^d \supseteq x(U \cap V) & \xrightarrow{\;\;y \circ x^{-1}\;\;} & y(U \cap V) \subseteq \mathbb{R}^d\,.
\end{array}
$$

The key feature of the chart transition map is that it can be considered as a map from $\mathbb{R}^d$ to $\mathbb{R}^d$ so that we can use all of the well-known analysis on $\mathbb{R}^d$. To a certain extent, we may even "forget" about the underlying manifold $\mathcal{M}$ and only consider physics in charts, studying the chart transition maps as coordinate changes. This is exactly the connection to classical physics which has been considered on an $\mathbb{R}^d$.

Two charts containing the same point might be of very different nature. While it is generically trivial to incorporate different chart domains, the consideration of differing chart maps can be more involved. Hence, we classify charts by the compatibility of their chart maps. In particular, two charts (U, x) and (V, y) are called C^n-compatible if either $U \cap V = \varnothing$ or the chart transition map $y \circ x^{-1}$ is C^n as a map from $\mathbb{R}^d$ to $\mathbb{R}^d$. For the sake of this thesis it suffices to consider $n \in \mathbb{N}$, where a map is called C^0 if it is continuous and C^n if it is n times differentiable. In particular, we will make extensive use of chart transition maps which are differentiable arbitrarily often, in which case these maps are called smooth or C^∞. Note that x and y being homeomorphisms implies that the chart transition map $y \circ x^{-1}$ is continuous and thus any chart transition map is C^0. The compatibility definitions of chart transition maps transfer to atlases: We call an atlas whose charts are pairwise C^n-compatible a C^n-atlas. A C^n-atlas is called maximal if all compatible charts are already contained in this atlas.

Having a C^n-atlas $\mathcal{A}$, some of the charts contained in $\mathcal{A}$ might actually be even C^{n+1}-compatible. Hence, we can add structure to an atlas by removing the charts which do not satisfy this additional compatibility criterion. In fact, starting with any maximal C^n-atlas for some $n \geq 1$, we may remove charts to conclude that it already contains a C^∞-atlas.

The such defined atlases are additional structure we added to topological manifolds. Hence, we assert a name to this structure. The triple $(\mathcal{M}, \mathcal{O}, \mathcal{A})$ is called a C^n-manifold if $(\mathcal{M}, \mathcal{O})$ is a topological manifold and $\mathcal{A}$ is a maximal C^n-atlas. We will mostly be concerned with smooth manifolds in the following, for which $\mathcal{A}$ is a C^∞-atlas. As for topological manifolds, one often simply denotes $\mathcal{M}$ as being a smooth manifold, implicitly understanding that $\mathcal{M}$ is a set equipped with a topology $\mathcal{O}$ and an atlas $\mathcal{A}$. Since every maximal C^n-atlas contains any atlas of higher degree of differentiability, we often simply speak of *differentiable manifolds* instead of C^n-manifolds.

Just as for the previous layers of structure, we wish to study the isomorphisms which preserve the structure of C^n-manifolds. The new structure of these manifolds is obviously the C^n-compatibility. Since we observed that any atlas is a C^0-atlas, the relevant new structure which isomorphisms should preserve is differentiability. But we only know how differentiability works in $\mathbb{R}^d$ and have no notion of differentiability for maps between generic manifolds yet. However, we have connected manifolds to $\mathbb{R}^d$ via charts in atlases, which allow to inherit the differentiable structure from $\mathbb{R}^d$. To this end, we consider two C^n-manifolds $(\mathcal{M}, \mathcal{O}_\mathcal{M}, \mathcal{A}_\mathcal{M})$, $(\mathcal{N}, \mathcal{O}_\mathcal{N}, \mathcal{A}_\mathcal{N})$ and a map $\phi : \mathcal{M} \longrightarrow \mathcal{N}$. Then ϕ is called n times differentiable at some point $p \in \mathcal{M}$ if the corresponding map $y \circ \phi \circ x^{-1}$ in some charts $(U, x) \in \mathcal{A}_\mathcal{M}$ and $(V, y) \in \mathcal{A}_\mathcal{N}$ is C^n as a map from $\mathbb{R}^d$ to $\mathbb{R}^d$. This is again understood more easily from the corresponding commutative diagram which is

$$\begin{array}{ccc} \mathcal{M} \supseteq U & \xrightarrow{\quad \phi \quad} & V \subseteq \mathcal{N} \\ {\scriptstyle x}\big\downarrow & & \big\downarrow{\scriptstyle y} \\ \mathbb{R}^d \supseteq x(U) & \xrightarrow{y \circ \phi \circ x^{-1}} & y(V) \subseteq \mathbb{R}^d \,. \end{array}$$

It is straightforward to show that this definition of differentiability is independent of the chosen charts and thus well-defined. Now that we constructed differentiable maps on C^n-manifolds, we may return to the construction of the isomorphisms which preserve the structure of these manifolds. Since a C^n-manifold is a triple $(\mathcal{M}, \mathcal{O}, \mathcal{A})$, the isomorphisms need to preserve all layers of structure in this triple. Therefore, the isomorphisms are bijective maps $\phi : \mathcal{M} \longrightarrow \mathcal{N}$ for which both ϕ and its inverse ϕ^{-1} are C^n. Such isomorphisms are called *diffeomorphisms*. Note that diffeomorphisms are continuous by construction of C^n-manifolds and thus also preserve the topological structure. If there exists a diffeomorphism between two C^k-manifolds, we call them diffeomorphic. As for the previous layers of structure, we often call two C^n-manifolds the same if they are diffeomorphic.

Coming back to our physics motivation of this mathematical introduction, we might want to model not only the position of particles moving through spacetime but to be able to calculate their velocities as well. Differentiable manifolds provide just enough structure to define the needed vector fields, since they have tangent spaces at each point of the manifold. Hence, we construct these spaces next.

2.3. Vectors, tensors and differential forms

Tangent spaces at points $p \in \mathcal{M}$ in a differentiable manifold are the key notions which allow us to introduce vectors and tensors on manifolds. They are thus essential for describing the dynamics of physical systems. Since tangent spaces are vector spaces, we first recap a few details of linear algebra before we apply them to differentiable manifolds.

The basic notion underlying a vector field is a *group*. Groups are tupels $(K, \circ)$ for which the map $\circ : K \times K \longrightarrow K$ is associative (`A`), has a neutral element (`N`) as well as an inverse (`I`). If the group operation $\circ$ is commutative (`C`) as well, we call the group *abelian*. On top of groups, we then construct an *(algebraic) field* $(K, +, \cdot)$ by demanding that both $(K, +)$ and $(K\backslash\{0\}, \cdot)$ are abelian groups and we may distribute (`D`) over both operations $+$ and $\cdot$ to connect them. We will later comment on structures for which the multiplication $\cdot$ is always associative `A` and has a neutral element `N`, but lacks the `C` or the `I` property which would make $(K, +, \cdot)$ a field. Such a structure is called a *ring*. We will be particularly concerned with commutative rings for which only the multiplicative inverse `I` is missing.

Having a field $(K, +, \cdot)$, we construct a *K-vector space* $(V, \oplus, \odot)$ where the addition $\oplus : V \times V \longrightarrow V$ again satisfies `CANI`. The multiplication $\odot : K \times V \longrightarrow V$ however fulfills `ADDU`, so it is associative (`A`), distributive (`D`) in both additions $+$ and $\oplus$ and inherits the unity element (`U`) of K. Note that the domain of $\odot$ is a Cartesian product of the sets underlying the field and the vector space. In this context, we call elements of the field *scalars* and the vector space multiplication $\odot$ *scalar multiplication*. Instead of constructing a vector space over a field, we could have used a ring $(K, +, \cdot)$. While the definition of the additional structure remains unchanged, we call the corresponding triples $(V, \oplus, \odot)$ *K-modules*. We will use these objects later when we discover structures relevant in physics which simply do not constitute a vector space. At this point, we need to point out that some intuitions we have for vector spaces simply do not translate to modules. For example, vector spaces may be equipped with a (Hamel) basis (ultimately due to the axiom of choice), while modules generically do not have a basis if they have no multiplicative inverse. A (Hamel) basis of a K-vector space $(V, \oplus, \odot)$ is a subset $B \subseteq V$ for which the elements e_i of B are linearly independent and allow to reconstruct vectors via $v = \sum\limits_{i=1}^{N} v^i e_i$, where $N \in \mathbb{N}\backslash\{0\}$ and the coefficients $v^i \in K$ are scalars. The dimension of the vector space is then defined as the cardinality of the basis, $\dim V := |B|$. When we required $v = \sum\limits_{i=1}^{N} v^i e_i$, the experienced reader might have wondered why we did not use Einstein's convention to sum over the indices. We will see that invoking Einstein's convention actually only works for linear maps, which we did not cover yet. Hence, we discuss linear maps next.

A map $\phi : V \longrightarrow W$ between two K-vector spaces $(V, \oplus, \odot)$ and $(W, \boxplus, \boxdot)$ is called *linear* if for all $\lambda \in K$ and $u,\ v \in V$ it fulfills $\phi(u \oplus v) = \phi(u) \boxplus \phi(v)$ and $\phi(\lambda \odot v) = \lambda \boxdot \phi(v)$. Note that by means of these properties it is always clear which addition or multiplication is meant by the context in which these operations appear. For this reason, we simply denote all additions by $+$ and all multiplications by $\cdot$ in the following. Moreover, we will be concerned with an abundance of linear maps so that we introduce the notation $\phi : V \overset{\sim}{\longrightarrow} W$ as an abbreviation which says that the map $\phi : V \longrightarrow W$ is linear. We will use this abbreviation later to indicate multilinearity as well if the domain V of the map is a Cartesian product. In this case, multiliniearity is equivalent to linearity in each entry of the domain. Linear maps are clearly the maps which preserve the new structure of vector spaces. Since isomorphisms need to preserve all of the structure we have, the vector space isomorphisms are bijective linear maps. Because the linear maps are so important, there exists a variety of names for different kinds of linear maps. The set of homomorphisms $\mathrm{Hom}(V,W) := \{\phi : V \xrightarrow{\sim} W\}$ of two K-vector spaces V and W can actually be made into a vector space over the same field K. This is done by pointwise inheriting the addition and multiplication from W. That is, we define the addition of two elements $\phi, \psi \in \mathrm{Hom}(V,W)$ as $(\phi+\psi)(v) := \phi(v)+\psi(v)$, while the multiplication by a scalar $\lambda \in K$ is given by $(\lambda\phi)(v) := \lambda\phi(v)$. Since both of these operations are defined at a point $v \in V$ in the domain of ϕ, we call them the *pointwise addition and scalar multiplication*. We will often use this principle for inheriting new operations from given ones.

From this vector space of homomorphisms, we derive multiple sub-vector spaces. In particular, $\mathrm{End}(V) := \mathrm{Hom}(V,V)$ are the endomorphisms and $\mathrm{Aut}(V) := \{\phi : V \xrightarrow{\sim} V \,|\, \phi \text{ invertible}\}$ are the automorphisms on V. The set $V^* := \mathrm{Hom}(V,K)$ equipped with the pointwise inherited addition and multiplication is called the *dual vectorspace*, where a field $(K,+,\cdot)$ may of course be considered as a K-vector space itself. The elements of the dual vectorspace are called *covectors*. Having a basis e_a on V, it simplifies a great number of calculations to choose the so-called *dual basis* ϵ^a on the dual vectorspace V^* by imposing $\epsilon^a(e_b) = \delta^a_b$, where the Kronecker delta δ^a_b is 1 if $a=b$ and 0 else. We will adapt the convention that the basis vectors of the dual vectorspace have upper indices.

Now that we defined the dual vectorspace V^*, let us return to the discussion of multilinear maps. In particular, we call the multilinear map

$$T : \underbrace{V^* \times \cdots \times V^*}_{p \text{ copies}} \times \underbrace{V \times \cdots \times V}_{q \text{ copies}} \xrightarrow{\sim} \mathbb{R}$$

a (p,q)-*tensor*. The set T^p_qV of (p,q)-tensors is again a vector space by pointwise inheriting addition and multiplication from $(K,+,\cdot)$. By definition, T^0_1V is nothing but the dual vectorspace V^*. We call elements of $T^0_1V = V^*$ *covectors*. The tensor space T^1_1V is actually isomorphic to $\mathrm{End}(V^*)$ as a vector space, so that we often do not distinguish between these two spaces. We would expect that T^1_1V is vector space isomorphic to $\mathrm{End}(V)$ as well, but this is in fact only true for finite dimensional vector spaces V since only in this case $(V^*)^*$ is isomorphic to V. For this reason, T^1_0V is only isomorphic to V for finite dimensional vector spaces. In short, this implies that covectors always map vectors to scalars, but vectors only map covectors to scalars in finite dimensional vector spaces. Thus, we will always implicitly confine to finite dimensional vector spaces if we construct such T^1_0V maps from vectors in the following.

It is sometimes helpful to express calculations in terms of tensor components instead of the full tensors. We construct those by considering the action of a tensor $T \in T^p_qV$

on the basis vectors e_a and ϵ^a of V and V^* to define

$$T^{a_1\ldots a_p}{}_{b_1\ldots b_q} := T(\epsilon^{a_1}, \ldots, \epsilon^{a_p}, e_{b_1}, \ldots, e_{b_q}) \,. \tag{2.1}$$

The multiplication $\cdot$ we defined to make $(T^p_q V, +, \cdot)$ a vector space is a scalar multiplication $K \times T^p_q V \longrightarrow T^p_q V$. We construct yet another product on tensor spaces denoted by

$$\begin{aligned} \otimes : T^p_q V \times T^r_s V &\longrightarrow T^{p+r}_{q+s} \\ (T, S) &\longmapsto T \otimes S \,. \end{aligned}$$

This *tensor product* is defined by

$$\begin{aligned} &(T \otimes S)(\omega_1, \ldots, \omega_{p+r}, v_1, \ldots, v_{q+s}) := \\ &T(\omega_1, \ldots, \omega_p, v_1, \ldots, v_q) \cdot S(\omega_{p+1}, \ldots, \omega_{p+r}, v_{q+1}, \ldots, v_{q+s}) \,. \end{aligned}$$

Using the tensor product, we may reconstruct a tensor $T \in T^p_q V$ from its components $T^{a_1\ldots a_p}{}_{b_1\ldots p_q}$ as

$$T = T^{a_1\ldots a_p}{}_{b_1\ldots b_q}\, e_{a_1} \otimes \cdots \otimes e_{a_p} \otimes \epsilon^{b_1} \otimes \cdots \otimes \epsilon^{b_q} \,. \tag{2.2}$$

Here, we used the Einstein summation convention for the first time. In this convention, we sum over all indices which appear once upstairs and once downstairs if nothing else is specified. Note that this convention requires maps to be linear, such that sums may be pulled out of their arguments. We may thus always use the convention in tensor calculus, since tensors are defined to be multilinear maps. We will adapt Einstein's summation convention in the following.

There are some tensors which will become particularly important later. These are the $T^0_n V$ tensors ω which are totally antisymmetric. We call these tensors *differential forms of rank n* or n-forms for short. For $n = 0$, the differential forms are simply the linear maps on the field[3], and for $n = 1$ the one-forms are given by the covectors. For $n > 1$, total antisymmetry is defined such that for every permutation π in the permutation group $\mathcal{S}^n$ we have

$$\omega(v_1, \ldots, v_n) = \operatorname{sgn}(\pi)\omega(v_{\pi(1)}, \ldots, v_{\pi(n)}) \,. \tag{2.3}$$

[3]Elements of the field are called zero-forms as well, since they correspond to the $n = 0$ case for the identity map.

The sign $\operatorname{sgn}(\pi)$ of the permutation π is defined to be 1 if π may be constructed by an even number of pairwise transpositions and -1 otherwise such that differential forms are totally antisymmetric as required. Due to this antisymmetry, the rank n of a differential form can at most be $\dim V$, in which case differential forms are also called *top forms*. Two different top forms only differ by a scalar. It is thus convenient to choose one top form and call it the *volume form* Vol on V. Then, $\mathrm{Vol}(v_1, \ldots, v_{\dim V})$ is the volume spanned by the vectors $v_1, \ldots, v_{\dim V}$. Using the volume form, we may also define determinants. The determinant is only defined for endomorphisms $\phi \in \mathrm{End}(V)$, and we recall that these are isomorphic to the $(1,1)$-tensors $T^1_1 V$ in finite dimensional vector spaces. For these objects, we define the *determinant* as

$$\det \phi := \frac{\mathrm{Vol}(\phi(e_1), \ldots, \phi(e_{\dim V}))}{\mathrm{Vol}(e_1, \ldots, e_{\dim V})}\,. \tag{2.4}$$

This definition is independent of the choice of basis $e_1, \ldots, e_{\dim V}$ of V and the choice of volume form. The independence of the choice of basis is the main reason for solely considering endomorphisms.

This introduction to multilinear algebra will be sufficient for developing the geometrical notions in the following chapters. However, we did not relate multilinear algebra to the manifolds we considered in the previous sections. Both topics are in fact closely related through the concept of tangent spaces. Introducing these spaces will be the main concern of the following section.

2.4. Tangent spaces to manifolds

We may imagine a tangent space as a vector space which is attached to each point of a manifold. However, "attaching" in this context implies that the manifold needs to be embedded in some higher-dimensional space like $\mathbb{R}^{2\dim\mathcal{M}}$. This may in fact be done by Whitney's embedding theorem [90]. Nevertheless, we do not need to use this machinery and may instead define tangent vectors directly within a manifold $\mathcal{M}$. To this end, consider some smooth curve $\gamma : \mathbb{R} \longrightarrow \mathcal{M}$ in the manifold, which we arrange such that $\gamma(0) = p$ without loss of generality. The intuition underlying tangent vectors is that they are tangent to the curve in p, the length of the vector being interpreted as the velocity. This intuition of a velocity of course requires some notion of differentiation again, which is only defined for powers of $\mathbb{R}$. Thus, we need to consider functions $f : \mathcal{M} \longrightarrow \mathbb{R}$ on $\mathcal{M}$ such that the composition $f \circ \gamma : \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable on $\mathbb{R}$. In order to preserve differentiability, we require f to be smooth, and the set of such smooth functions is denoted by $C^\infty(\mathcal{M}) \equiv C^\infty(\mathcal{M}, \mathbb{R})$. This set of smooth functions may be considered as a commutative ring $(C^\infty(\mathcal{M}), +, \bullet)$, where $+ : C^\infty(\mathcal{M}) \longrightarrow C^\infty(\mathcal{M})$ is the pointwise inherited addition from $\mathbb{R}$ and $\bullet : C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \longrightarrow C^\infty(\mathcal{M})$ is the multiplication of functions defined by $(f \bullet g)(p) := f(p) \cdot g(p)$, where $\cdot$ is the multiplication on $\mathbb{R}$. We will come back to this ring property later. At this point, we will instead exploit that $C^\infty(\mathcal{M})$ can also be equipped with the pointwise inherited scalar multiplication from $\mathbb{R}$ to construct the (infinite dimensional) $\mathbb{R}$-vector space $(C^\infty(\mathcal{M}), +, \cdot)$. Using that, we then define a *tangent vector* according to the above intuition as the linear map

$$
\begin{aligned}
X_{\gamma,p} : C^\infty(\mathcal{M}) &\stackrel{\sim}{\longrightarrow} \mathbb{R} \\
f &\longmapsto X_{\gamma,p} f := (f \circ \gamma)'(0)\,,
\end{aligned}
$$

where the prime denotes differentiation in $\mathbb{R}$.

Note that tangent vectors are only defined with respect to a specific curve γ containing $p \in \mathcal{M}$. Of course, there are generically many smooth curves which contain $p \in \mathcal{M}$ as $\gamma(0) = p$. We collect the tangent vectors of all these curves to construct the *tangent vector space*

$$
T_p\mathcal{M} := \{X_{\gamma,p} | \gamma \in C^\infty(\mathcal{M}) \text{ with } \gamma(0) = p\}\,.
$$

Again, we inherit addition and multiplication pointwise from $\mathbb{R}$ to make $T_p\mathcal{M}$ an $\mathbb{R}$-vector space. The vector space dimension of $T_p\mathcal{M}$ actually equals the manifold dimension of $\mathcal{M}$ which may be concluded in the following way. First, we consider very specific curves γ_a for $a \in \{1, \ldots, \dim \mathcal{M}\}$ that represent the coordinate axes in some chart (U, x), that is $(x^b \circ \gamma_a)(\lambda) = \delta^b_a(\lambda)$. The tangent vectors to these curves for arbitrary functions $f \in C^\infty(\mathcal{M})$ are

$$
X_{\gamma_a,p} f = \partial_a (f \circ x^{-1})(x(p))\,, \tag{2.5}
$$

where $\partial_a : C^\infty(\mathbb{R}^{\dim \mathcal{M}}) \stackrel{\sim}{\longrightarrow} C^\infty(\mathbb{R}^{\dim \mathcal{M}})$ is the partial derivative with respect to the a-th argument. In order to denote these tangent vectors independent of the chosen $f \in C^\infty(\mathcal{M})$, we define the new object

$$
\left(\frac{\partial}{\partial x^a}\right)_p f := \partial_a (f \circ x^{-1})(x(p))\,. \tag{2.6}
$$

Using this definition, the tangent vectors to the chart induced curves γ_a are

$$X_{\gamma_a,p} = \left(\frac{\partial}{\partial x^a}\right)_p . \tag{2.7}$$

Note that $\left(\frac{\partial}{\partial x^a}\right)_p$ is an element of $T_p\mathcal{M}$, that is, it maps from $C^\infty(\mathcal{M})$ into $\mathbb{R}$. Moreover, it is chart dependent since the curves needed to construct $\left(\frac{\partial}{\partial x^a}\right)_p$ were induced by charts. Although this is a very different object than the partial derivative ∂_a, we often abbreviate it by ∂_a in physics if and only if the context in which we use this symbol is clear. Note that by its definition it is obvious that the index in $\left(\frac{\partial}{\partial x^a}\right)_p$ needs to be understood as a downstairs index. Evaluating $\left(\frac{\partial}{\partial x^a}\right)_p$ for the chart map immediately yields $\left(\frac{\partial}{\partial x^a}\right)_p x^b = \delta^b_a$, which we may use to conclude that $\left(\frac{\partial}{\partial x^1}\right)_p, \ldots, \left(\frac{\partial}{\partial x^{\dim \mathcal{M}}}\right)_p$ is a basis of $T_p\mathcal{M}$. So, indeed, the vector space dimension of $T_p\mathcal{M}$ and the manifold dimension of $\mathcal{M}$ are equal. By considering chart transition maps from a chart (U, x) to another chart (V, y), we conclude that the components X^a of a vector $X = X^a \left(\frac{\partial}{\partial x^a}\right)_p$ in the chart induced basis transform by means of $\left(\frac{\partial}{\partial x^b}\right)_p y^a$.

Now that we identified $(T_p\mathcal{M}, +, \cdot)$ as an $\mathbb{R}$-vector space, we may adopt all of the machinery of the previous section. In particular, we wish to construct the dual basis of $\left(\frac{\partial}{\partial x^a}\right)_p$ on the dual vector space $T^*_p\mathcal{M} := (T_p\mathcal{M})^*$, called the *cotangent space*. To construct this basis, we first introduce the *gradient* as

$$\begin{aligned} \mathrm{d}_p : C^\infty(\mathcal{M}) &\xrightarrow{\sim} T^*_p\mathcal{M} \\ f &\longmapsto \mathrm{d}_p f \,, \end{aligned}$$

where we let the covector $\mathrm{d}_p f$ act on an arbitrary vector $X \in T_p\mathcal{M}$ to define

$$\mathrm{d}_p f(X) := X(f) . \tag{2.8}$$

Recalling that the component functions of chart maps x are elements of $C^\infty(\mathcal{M})$, we may evaluate the chart-induced covectors $\mathrm{d}_p x^a$ on the vector basis $\left(\frac{\partial}{\partial x^a}\right)_p$ to conclude that $\mathrm{d}_p x^1, \ldots, \mathrm{d}_p x^{\dim \mathcal{M}}$ constitute the dual basis on the covector space at the point $p \in \mathcal{M}$. The construction of vectors and covectors of course extends to tensor spaces $(T^r_s(T_p\mathcal{M}), +, \cdot)$ over the manifold as constructed in section 2.3.

If two smooth manifolds $\mathcal{M}$ and $\mathcal{N}$ are connected by means of a smooth map $\phi : \mathcal{M} \longrightarrow \mathcal{N}$, we may also connect the tangent spaces of these manifolds. To that effect, we define the *push-forward* induced by ϕ as

$$\begin{aligned}\phi_{*p} : T_p\mathcal{M} &\overset{\sim}{\longrightarrow} T_{\phi(p)}\mathcal{N}\\ X &\longmapsto \phi_{*p}(X)\,,\end{aligned}$$

given by its action on a $C^\infty(\mathcal{N})$-function f as

$$\phi_{*p}(X)(f) := X(f \circ \phi)\,. \tag{2.9}$$

The push-forward ϕ_{*p} is also called the derivative or differential of ϕ. Note that this is the first notion of a derivative for maps between arbitrary smooth manifolds we encounter. The definition of this derivative implicitly depends on the point $p \in \mathcal{M}$ since the vector X is an element of $T_p\mathcal{M}$ for some point p. Using this definition, it is straightforward to conclude that tangent vectors in $\mathcal{M}$ are pushed forward to tangent vectors in $\mathcal{N}$ as $\phi_{*p}(X_{\gamma,p}) = X_{\phi\circ\gamma,\phi(p)}$. So, in short, we found that vectors are pushed forward between manifolds. We can find a similar rule for covectors. For these, we define the *pull-back*

$$\begin{aligned}\phi_p^* : T^*_{\phi(p)}\mathcal{N} &\overset{\sim}{\longrightarrow} T^*_p\mathcal{M}\\ \omega &\longmapsto \phi_p^*(\omega)\end{aligned}$$

by means of its action on vectors $X \in T_p\mathcal{M}$ as

$$\phi_p^*(\omega)(X) = \omega(\phi_{*p}(X))\,. \tag{2.10}$$

Note that the smooth map $\phi : \mathcal{M} \longrightarrow \mathcal{N}$ is the same as before, but the pull-back maps between the cotangent spaces in the opposite direction. Thus, we say that covectors are pulled back.

So far, we only considered tangent spaces and their associated structures at one particular point $p \in \mathcal{M}$. If we want to define vector fields on the entire manifold we need to have a notion which is independent of the chosen point. This notion is the *tangent bundle*, the total space of which is simply defined as the unification of all tangent spaces, that is

$$T\mathcal{M} := \dot{\bigcup_{p\in\mathcal{M}}} T_p\mathcal{M}\,.$$

The dot on top of the unification symbol is used to indicate that the union is disjoint.

We construct a manifold bundle $T\mathcal{M} \xrightarrow{\pi} \mathcal{M}$ from this set by defining the projection

$$\begin{aligned} \pi : T\mathcal{M} &\longrightarrow \mathcal{M} \\ X &\longmapsto p\,, \end{aligned}$$

where p is the point for which $X \in T_p\mathcal{M}$. As indicated when we introduced manifold bundles, we may make this map continuous by equipping $T\mathcal{M}$ with the topology induced from $\mathcal{M}$, so that $T\mathcal{M} \xrightarrow{\pi} \mathcal{M}$ indeed is a manifold bundle, called the *tangent bundle*. In similar fashion as for the topology, we may construct a smooth atlas on $T\mathcal{M}$ from a smooth atlas on $\mathcal{M}$ to turn it into a smooth manifold and π into a smooth map.

Now consider a smooth section of the tangent bundle. Recall that we defined sections as maps $\sigma : \mathcal{M} \longrightarrow T\mathcal{M}$ which are compatible with the projection map in the sense $\pi \circ \sigma = \mathrm{id}_{\mathcal{M}}$. Analyzing the target and domain of sections on the tangent bundle, we note that evaluating a section at some point $p \in \mathcal{M}$ yields a vector $\sigma(p)$ in the tangent space at this point. Hence, sections of the tangent bundle associate a tangent vector to each point of the manifold. We will thus call them *vector fields* in the following. The set

$$\Gamma(T\mathcal{M}) := \{\sigma : \mathcal{M} \longrightarrow T\mathcal{M} | \pi \circ \sigma = \mathrm{id}_{\mathcal{M}}\} \tag{2.11}$$

of all vector fields on $\mathcal{M}$ can again be equipped with pointwise inherited addition and multiplication from $T_p\mathcal{M}$, but it is worth considering the scalar multiplication in more detail here.

For being able to consider vector fields which yield vectors that differ from one point of the tangent space to another, we need to scale vector fields with smooth functions instead of bare real numbers. That is, the scalar multiplication is

$$\begin{aligned} \cdot\ : C^\infty(\mathcal{M}) \times \Gamma(T\mathcal{M}) &\longrightarrow \Gamma(T\mathcal{M}) \\ (f, \sigma) &\longmapsto f \cdot \sigma \end{aligned}$$

defined by

$$(f \cdot \sigma)(p) := f(p) \cdot \sigma(p)\,, \tag{2.12}$$

the multiplication on the right hand side of the latter equation being the scalar multiplication on $T_p\mathcal{M}$. While all of this looks like a trivial repetition of the vector space definitions we had before, recall that $(C^\infty(\mathcal{M}), +, \bullet)$ was not a field but only a commutative ring. Thus, $(\Gamma(T\mathcal{M}), +, \cdot)$ is a $C^\infty(\mathcal{M})$-module, that is, a vector space structure over a ring. In other words, vector fields are not vectors. This has important consequences. For example, we already noted that modules over commutative rings generically do not have a basis. Hence, $(\Gamma(T\mathcal{M}), +, \cdot)$ does generically not have a basis and thus vector fields have no components. Therefore it is unavoidable to develop the component-free language introduced in this chapter to be able to handle vector fields. The only consistent way for expressing vector fields in components is to consider them in the tangent spaces at particular points of the manifold as we studied beforehand.

The best we may do to carry over the idea of a basis to bundles is to collect every single basis of every tangent space in a new bundle. To that end, we consider the ordered tupels $(e_1, \dots, e_{\dim \mathcal{M}}) \in T_p\mathcal{M} \times \cdots \times T_p\mathcal{M}$ which are called *frames* if the set $\{e_1, \dots, e_{\dim \mathcal{M}}\}$ constitutes a basis of $T_p\mathcal{M}$. We denote the set of all frames at a point $p \in \mathcal{M}$ by $L_p\mathcal{M}$. Then, we define the total space

$$L\mathcal{M} := \dot{\bigcup_{p\in\mathcal{M}}} L_p\mathcal{M} .$$

This is of course entirely analogous to the just defined tangent bundle, and we may again inherit a topology and an atlas from $\mathcal{M}$ to make $L\mathcal{M} \xrightarrow{\pi} \mathcal{M}$ a smooth bundle which is then called *frame bundle*. Analogously, the bundle projection is defined as

$$\begin{aligned} \pi : \quad L\mathcal{M} \quad &\longrightarrow \mathcal{M} \\ (e_1, \dots, e_{\dim \mathcal{M}}) &\longmapsto p , \end{aligned}$$

where p is the point for which $(e_1, \dots, e_{\dim \mathcal{M}}) \in T_p\mathcal{M} \times \cdots \times T_p\mathcal{M}$. The such defined frame bundle is of course associated to every tangent bundle. We will make this idea of associating a bundle to another one much more concrete when we introduce principal bundles in the following section. But let us return to the vector fields $\Gamma(T\mathcal{M})$ on $\mathcal{M}$ first to see which of the structural ideas we have gained for vectors carry over to vector fields. In particular, we may use linearity in $C^\infty(\mathcal{M})$ to define homomorphisms on $C^\infty(\mathcal{M})$-modules A and B as we did for vector spaces by means of

$$\mathrm{Hom}_{C^\infty(\mathcal{M})}(A, B) := \{\phi : A \xrightarrow{\sim} B | \phi \text{ is linear in } C^\infty(\mathcal{M})\} .$$

Equipping this set with addition and multiplication pointwise, the homomorphisms again constitute a $C^\infty(\mathcal{M})$-module.

As a special case of these homomorphisms we may consider the *dual module*

$$\Gamma(T\mathcal{M})^* := \mathrm{Hom}_{C^\infty(\mathcal{M})}(\Gamma(T\mathcal{M}), C^\infty(\mathcal{M}))\,,$$

where $\Gamma(T\mathcal{M})^*$ is isomorphic to $\Gamma(T^*\mathcal{M})$ and thus the elements of $\Gamma(T\mathcal{M})^*$ are called *covector fields*. These definitions of $C^\infty(\mathcal{M})$-modules for vector and covector fields may be extended to (p,q)-tensor fields T on $\mathcal{M}$ by demanding that

$$T:\ \underbrace{\Gamma(T^*\mathcal{M})\times\cdots\times\Gamma(T^*\mathcal{M})}_{p\text{ copies}}\times\underbrace{\Gamma(T\mathcal{M})\times\cdots\times\Gamma(T\mathcal{M})}_{q\text{ copies}}\xrightarrow{\sim} C^\infty(\mathcal{M})$$

are multilinear maps with respect to $C^\infty(\mathcal{M})$. We denote the set of all (p,q)-tensor fields by $T^p_q(\Gamma(T\mathcal{M}))$ and equip this set with pointwise inherited addition and multiplication from $C^\infty(\mathcal{M})$ to turn it into a $C^\infty(\mathcal{M})$-module. On this module, the tensor product generalizes to tensor fields as

$$\begin{aligned}\otimes:\ T^p_q(\Gamma(T\mathcal{M}))\times T^r_s(\Gamma(T\mathcal{M})) &\longrightarrow T^{p+r}_{q+s}(\Gamma(T\mathcal{M}))\\ (T,S) &\longmapsto T\otimes S\end{aligned}$$

defined by

$$\begin{aligned}&(T\otimes S)(\omega_1,\dots,\omega_{p+r},v_1,\dots,v_{q+s}) :=\\ &T(\omega_1,\dots,\omega_p,v_1,\dots,v_q)\bullet S(\omega_{p+1},\dots,\omega_{p+r},v_{q+1},\dots,v_{q+s})\,.\end{aligned}$$

The commutativity of the multiplication in $C^\infty(\mathcal{M})$ immediately implies

$$\begin{aligned}&(T\otimes S)(\omega_1,\dots,\omega_{p+r},v_1,\dots,v_{q+s}) =\\ &(S\otimes T)(\omega_{p+1},\dots,\omega_{p+r},\omega_1,\dots,\omega_p,v_{q+1},\dots,v_{q+s},v_1,\dots,v_q)\end{aligned} \tag{2.13}$$

which is important when considering differential forms. For (p,q)-tensor fields, we denote the set of n-forms by $\Omega^n(\mathcal{M})\subseteq T^0_n(\Gamma(T\mathcal{M}))$, which then obviously is a $C^\infty(\mathcal{M})$-module as well.

The elements of $\Omega^n(\mathcal{M})$ are totally antisymmetric as defined in section 2.3. Carrying this antisymmetry to the tensor product, it is convenient to define the *wedge product*

$$\begin{aligned}\wedge:\ \Omega^n(\mathcal{M})\times\Omega^m(\mathcal{M}) &\longrightarrow \Omega^{n+m}(\mathcal{M})\\ (\omega,\sigma) &\longmapsto \omega\wedge\sigma\end{aligned}$$

by means of

$$\omega \wedge \sigma(X_1, \dots, X_{n+m}) := \frac{1}{n!m!} \sum_{\pi \in \mathcal{S}(n+m)} \operatorname{sgn}(\pi)(\omega \otimes \sigma)(X_{\pi(1)}, \dots, X_{\pi(n+m)}) . \tag{2.14}$$

We may write down the wedge product explicitly without applying it to vectors as well. To that end, we use the commutativity of $C^\infty(\mathcal{M})$ in the tensor product defining the wedge product as we did in (2.13). For the wedge products of two one-forms ω and σ for instance, this yields $\omega \wedge \sigma = \omega \otimes \sigma - \sigma \otimes \omega$. Furthermore, the $C^\infty(\mathcal{M})$-commutativity (2.13) implies $\omega \wedge \sigma = (-1)^{nm} \sigma \wedge \omega$ for $\omega \in \Omega^n(\mathcal{M})$ and $\sigma \in \Omega^m(\mathcal{M})$.

The such defined wedge product combines with the pull-back in a well-behaved way, following a distributive law. To see this, we lift the definition of the pull-back induced by a map $\phi : \mathcal{M} \longrightarrow \mathcal{N}$ that we only introduced for covectors so far to differential forms $\omega \in \Omega^n(\mathcal{N})$. To that end, we note that we can consider the tensor fields $T^r_s(\Gamma(T\mathcal{M}))$ as sections of the bundle $T^r_s(T\mathcal{M}) \xrightarrow{\pi} \mathcal{M}$. Here, $T^r_s(T\mathcal{M})$ is the (disjoint) union of the tensor spaces $T^r_s(T_p\mathcal{M})$ of all points $p \in \mathcal{M}$, and π projects to the point p at which the tensor is defined. For short, we have that $T^r_s(\Gamma(T\mathcal{M}))$ is isomorphic to $\Gamma(T^r_s(T\mathcal{M}))$. Since n-forms are antisymmetric $(0, n)$-tensor fields, we use this isomorphism to define

$$\begin{aligned} \phi^* : \Omega^n(\mathcal{N}) &\longrightarrow \Omega^n(\mathcal{M}) \\ \omega &\longmapsto \phi^*\omega , \end{aligned}$$

where we use the interpretation of differential forms as sections of $T^0_n(T\mathcal{M}) \xrightarrow{\pi} \mathcal{M}$ to evaluate

$$\begin{aligned} \phi^*\omega : \mathcal{M} &\longrightarrow T^0_n(T\mathcal{M}) \\ p &\longmapsto \phi^*\omega(p) \end{aligned}$$

defined by

$$\begin{aligned} \phi^*\omega(p) : \quad (T_p\mathcal{M})^n &\longrightarrow \mathbb{R} \\ (X_1, \dots, X_n) &\longmapsto \phi^*\omega(p)(X_1, \dots, X_n) := \omega(\phi(p))(\phi_{*p}(X_1), \dots, \phi_{*p}(X_n)) . \end{aligned}$$

Note that the straightforward extension of the pull-back of a covector in some particular cotangent space $T^*_p\mathcal{M}$ to differential forms would use the push-forward of vector fields. While this would yield a much simpler result than the pointwise definition we just gave, the push-forward of vector fields is only well-defined if the map ϕ is a diffeomorphism. Thus, we are required to use the push-forward of vectors instead in order to generalize the pull-back for arbitrary smooth maps ϕ.

The pull-back of differential forms straightforwardly generalizes to $(0, n)$-tensor fields, but it has some particularly useful properties if we consider differential forms. For example, we may combine the pull-back with the definition of the wedge product to conclude

$$\phi^*(\omega \wedge \sigma) := \phi^*(\omega) \wedge \phi^*(\sigma) \,, \tag{2.15}$$

where $\omega \in \Omega^n(\mathcal{N})$ and $\sigma \in \Omega^m(\mathcal{N})$ are differential forms of arbitrary rank.

Lastly, we will make extensive use of the extension of the gradient of smooth functions defined in (2.8) to covector fields in $\Gamma(T^*\mathcal{M}) = \Omega^1(\mathcal{M})$. The resulting object

$$\begin{aligned} \mathrm{d} : C^\infty(\mathcal{M}) &\longrightarrow \Omega^1(\mathcal{M}) \\ f &\longmapsto \mathrm{d}f \end{aligned}$$

is called the *exterior derivative* and simply defined pointwise as

$$(\mathrm{d}f)(p) := \mathrm{d}_p f \,. \tag{2.16}$$

Noting that $\Omega^0(\mathcal{M}) = C^\infty(\mathcal{M})$, we may generalize the exterior derivative to differential forms of arbitrary rank n as

$$\begin{aligned} \mathrm{d} : \Omega^n(\mathcal{M}) &\longrightarrow \Omega^{n+1}(\mathcal{M}) \\ \omega &\longmapsto \mathrm{d}\omega \end{aligned}$$

by means of

$$\begin{aligned} &\mathrm{d}\omega(X_1, \ldots, X_{n+1}) := \\ &\sum_{i=1}^{n+1} (-1)^{i+1} X_i(\omega(X_1, \ldots, \not{X}_i, \ldots, X_{n+1})) \\ &+ \sum_{i=1}^{n+1} \sum_{j=i+1}^{n+1} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \not{X}_i, \ldots, \not{X}_j, \ldots, X_{n+1}) \,, \end{aligned} \tag{2.17}$$

where the second term is introduced to guarantee $C^\infty(\mathcal{M})$-linearity and the commutator of two vector fields $X, Y \in \Gamma(T\mathcal{M})$ is defined as[4]

$$[X,Y](f) = X(Y(f)) - Y(X(f))\,. \tag{2.18}$$

Although we only defined the action of vectors on $C^\infty(\mathcal{M})$ instead of the action of vector fields, this definition makes sense since we may utilize that $X(p) \in T_p\mathcal{M}$ to pointwise define

$$\begin{aligned} X : C^\infty(\mathcal{M}) &\overset{\sim}{\longrightarrow} C^\infty(\mathcal{M}) \\ f &\longmapsto X(f) \end{aligned}$$

by means of

$$\begin{aligned} X(f) : \mathcal{M} &\overset{\sim}{\longrightarrow} \mathbb{R} \\ p &\longmapsto (X(f))(p) := (X(p))(f)\,. \end{aligned}$$

Hence, we may interpret vector fields as $\mathbb{R}$-linear maps on $C^\infty(\mathcal{M})$ as well.

Combining the just defined exterior derivative with the pull-back induced by $\phi : \mathcal{M} \longrightarrow \mathcal{N}$, we observe

$$\phi^*(\mathrm{d}\omega) = \mathrm{d}(\phi^*\omega) \tag{2.19}$$

for differential forms $\omega \in \Omega^n(\mathcal{N})$ of arbitrary rank n. By use of the definitions we furthermore conclude that

$$\mathrm{d}(\omega \wedge \sigma) = \mathrm{d}\omega \wedge \sigma + (-1)^n \omega \wedge \mathrm{d}\sigma\,, \tag{2.20}$$

where n is the rank of $\omega \in \Omega^n(\mathcal{M})$ while the rank of $\sigma \in \Omega^m(\mathcal{M})$ is arbitrary. There are two further important results on the differentiation and integration of exterior derivatives which will become important. First, applying the exterior derivative to some $\omega \in \Omega^n(\mathcal{M})$ twice, we obtain $\mathrm{d}^2\omega := \mathrm{d}(\mathrm{d}\omega) = 0$. Differential forms which may be expressed as $\omega = \mathrm{d}\sigma$ for some $\sigma \in \Omega^{n-1}(\mathcal{M})$ are called exact, and if ω already fulfills $\mathrm{d}\omega = 0$ it is called closed. While all exact forms are closed, the two notions are even equal if the underlying manifold is $\mathcal{M} = \mathbb{R}$. Integrating exact forms is particularly

[4] The commutator of vector fields $[X, Y]$ is often denoted $\pounds_X Y$, called the *Lie derivative*. We choose to stick to the commutator notation to make the antisymmetry explicit. This will later relate $[X, Y]$ to a Lie bracket.

straightforward, since we may use *Stokes' theorem*

$$\int_{\mathcal{M}} \mathrm{d}\omega = \int_{\partial\mathcal{M}} \omega \,, \tag{2.21}$$

where $\partial\mathcal{M}$ is the boundary of $\mathcal{M}$. Although we did not discuss boundaries so far, they may be included to our discussion straightforwardly. To that end, we include points $p \in \mathcal{M}$ of topological spaces $(\mathcal{M}, \mathcal{O})$ to manifolds if their open neighborhood $U \in \mathcal{O}$ is only homeomorphic to the half space $\mathbb{R}^{d+}$.

To conclude this short introduction to differential forms, let us consider them in local coordinates. We again emphasize that the spaces of vector, covector and tensor fields generically do not have a global basis, being $C^\infty(\mathcal{M})$-modules. However, we have seen that the fields locally yield vectors, covectors and tensors stemming from vector spaces. Thus, we may express the fields in a basis locally. To find these expressions we choose a chart (U, x) in which we introduce the local chart-induced vector fields $\frac{\partial}{\partial x^a}$ as $\mathbb{R}$-linear maps on $C^\infty(\mathcal{M})$ by means of

$$\begin{aligned}\frac{\partial}{\partial x^a} : C^\infty(\mathcal{M}) &\xrightarrow{\sim} C^\infty(\mathcal{M}) \\ f &\longmapsto \frac{\partial}{\partial x^a} f := \partial_a(f \circ x^{-1}) \circ x \,.\end{aligned}$$

Since we defined differential forms $\omega \in \Omega^n(\mathcal{M})$ as totally antisymmetric $(0, n)$-tensor fields, we denote the local components as $\omega_{a_1 \ldots a_n} \equiv \omega\left(\frac{\partial}{\partial x^{a_1}}, \ldots, \frac{\partial}{\partial x^{a_n}}\right)$ and reconstruct the n-form as a tensor field locally by means of $\omega = \omega_{a_1 \ldots a_n} \mathrm{d}x^{a_1} \otimes \cdots \otimes \mathrm{d}x^{a_n}$. Due to the antisymmetry of differential forms, the component functions $\omega_{a_1 \ldots a_n} \in C^\infty(\mathcal{M})$ are antisymmetric. Using this antisymmetry, we may evaluate the tensor product in $\omega = \omega_{a_1 \ldots a_n} \mathrm{d}x^{a_1} \otimes \cdots \otimes \mathrm{d}x^{a_n}$ to conclude

$$\omega = \frac{1}{n!} \omega_{a_1 \ldots a_n} \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_n} \,. \tag{2.22}$$

Evaluating the exterior derivative using $\mathrm{d}(\mathrm{d}x^a) = 0$ yields

$$\mathrm{d}\omega = \frac{1}{n!} \frac{\partial}{\partial x^b} (\omega_{a_1 \ldots a_n}) \mathrm{d}x^b \wedge \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_n} \,. \tag{2.23}$$

This concludes the introduction to tangent spaces and tangent bundles. In the following section, we turn to the key structure for defining curvature, torsion and non-metricity which is the principal $\mathcal{G}$-bundle.

2.5. Principal $\mathcal{G}$-bundles

In this section, we will introduce the relevant structures on which we subsequently consistently define the fields underlying curvature, torsion and non-metricity. This structure is given by principal $\mathcal{G}$-bundles which come with associated bundles. We will see that the definition of a principal $\mathcal{G}$-bundle relies on Lie groups. While the study of Lie groups, their algebras and representations is a most relevant topic for several branches of physics, we only briefly introduce the notions necessary to follow the discussion of principal $\mathcal{G}$-bundles here.

A *Lie group* $(\mathcal{G}, \bullet)$ is a group for which $(\mathcal{G}, \mathcal{O}, \mathcal{A})$ is a smooth manifold for some topology $\mathcal{O}$ and some smooth atlas $\mathcal{A}$. Recall that we defined groups by requiring the group operation $\bullet$ to be associative and to have a neutral element as well as an inverse. For Lie groups, we furthermore require that the group operation

$$\begin{aligned} \bullet : \mathcal{G} \times \mathcal{G} &\longrightarrow \mathcal{G} \\ (G, H) &\longmapsto G \bullet H \end{aligned}$$

is a smooth map. Furthermore, the inversion map

$$\begin{aligned} {}^{-1} : \mathcal{G} &\longrightarrow \mathcal{G} \\ G &\longmapsto G^{-1} \end{aligned}$$

is required to be smooth as well. Maps $\phi : \mathcal{G} \longrightarrow \mathcal{G}'$ preserving the Lie group structure must therefore be smooth and be compatible with the group operation, that is $\phi(G \bullet H) = \phi(G) \bullet' \phi(H)$ for all $G, H \in \mathcal{G}$. Such structure preserving maps are called *Lie group homomorphisms*. An important example of a Lie group is the *general linear group* $(\mathrm{GL}(n, \mathbb{R}), \circ)$, where $\circ$ is just the composition of maps on the set

$$\mathrm{GL}(n, \mathbb{R}) := \{G : \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n | \det G \neq 0\} . \tag{2.24}$$

Note that evaluating the determinant is well-defined since $\mathrm{GL}(n, \mathbb{R})$ is a subset of the endomorphisms $\mathrm{End}(\mathbb{R}^n)$. Furthermore, recall that we identified the endomorphisms as an $\mathbb{R}$-vector space which is isomorphic to the $(1, 1)$-tensors $T^1_1\mathbb{R}^n$. Hence, we may examine the elements of the general linear group in terms of their components $G^\mu{}_\nu := G(\mathcal{E}^\mu, E_\nu)$, where E_ν is a basis of $\mathbb{R}^n$ with dual basis $\mathcal{E}^\mu$. Due to this identification with the endomorphisms on $\mathbb{R}^n$ the group $\mathrm{GL}(n, \mathbb{R})$ is called a *matrix group*. We will later use this to conclude that we may interpret basis transformations as actions of $\mathrm{GL}(n, \mathbb{R})$ on the frame bundle introduced in the previous section. These actions are what we define next.

There are two different ways for defining the action of a Lie group $(\mathcal{G}, \bullet)$ on some smooth manifold $\mathcal{M}$. First, we consider the *left $\mathcal{G}$-action* on $\mathcal{M}$ as the map

$$\begin{aligned} \triangleright \colon \mathcal{G} \times \mathcal{M} &\longrightarrow \mathcal{M} \\ (G, p) &\longmapsto G \triangleright p \end{aligned}$$

compatible with the group operation, that is, for all $p \in \mathcal{M}$ and $G, H \in \mathcal{G}$ the left $\mathcal{G}$-action fulfills $E \triangleright p = p$ and $G \triangleright (H \triangleright p) = (G \bullet H) \triangleright p$, where E is the unit element of the Lie group. If $\mathcal{M}$ is equipped with such a left $\mathcal{G}$-action, we call it a *left $\mathcal{G}$-space* and denote it as $\mathcal{M} \xrightarrow{\mathcal{G}\triangleright} \mathcal{M}$. Similarly, on *right $\mathcal{G}$-spaces* $\mathcal{M} \xrightarrow{\triangleleft\mathcal{G}} \mathcal{M}$ we have a *right $\mathcal{G}$-action* on $\mathcal{M}$ defined by

$$\begin{aligned} \triangleleft \colon \mathcal{G} \times \mathcal{M} &\longrightarrow \mathcal{M} \\ (G, p) &\longmapsto p \triangleleft G \end{aligned}$$

compatible with the group operation in the same sense as the left $\mathcal{G}$-action. Since the group axioms guarantee that the group operation has an inverse, we may inherit a right $\mathcal{G}$-action from a left $\mathcal{G}$-action on $\mathcal{M}$ by means of defining $p \triangleleft G := G^{-1} \triangleright p$, or vice versa inherit a left from a given right $\mathcal{G}$-action by means of $G \triangleright p := p \triangleleft G^{-1}$.

The actions of Lie groups $(\mathcal{G}, \bullet)$ on manifolds $\mathcal{M}$ induce important spaces. First, we consider the *orbit* of a point $p \in \mathcal{M}$ as all those points which can be reached by a left $\mathcal{G}$-action, that is

$$\mathcal{O}_p := \{q \in \mathcal{M} | \exists G \in \mathcal{G} : \, G \triangleright p = q\} \, .$$

Note that since the inverse G^{-1} of all group elements exist, we could have alternatively defined the orbit using the right $\mathcal{G}$-action. Moreover, the existence of the inverse implies that any point $q \in \mathcal{M}$ which lies in $\mathcal{O}_p$ has an orbit $\mathcal{O}_q$ in which $p \in \mathcal{M}$ lies, and hence $\mathcal{O}_p = \mathcal{O}_q$. This defines an equivalence relation, meaning that we say that points are in the same equivalence class $[p]$ if they lie on the same orbit $\mathcal{O}_p$. We may thus coarse grain our picture of Lie group actions on manifolds by only considering the orbits as elements of a space instead of the individual points. This *orbit space* is denoted $\mathcal{M}/\mathcal{G}$, and the idea of coarse graining is mathematically encoded in $\mathcal{M}/\mathcal{G}$ being a quotient space.

While the orbit contains all points which can be reached by movement by a $\mathcal{G}$-action, we will also need to consider those group elements which do not move the point $p \in \mathcal{M}$.

These group elements are collected in the *stabilizer*

$$S_p := \{G \in \mathcal{G} | G \triangleright p = p\} \,,$$

which is a subset of $\mathcal{G}$. By the definition of group actions, the stabilizer always contains the identity element. If no other element is contained in the stabilizer of all points $p \in \mathcal{M}$ we call the group action *free*. It is particularly useful to consider free $\mathcal{G}$-actions on manifolds $\mathcal{M}$ since for these each orbit is already diffeomorphic to the group itself, both considered as smooth manifolds. Hence, for free actions one is tempted to attach the group $\mathcal{G}$ to each point of the orbit space in order to get back the full manifold $\mathcal{M}$. This is in fact the idea underlying principal $\mathcal{G}$-bundles.

Concretely, we define a *principal $\mathcal{G}$-bundle* as a smooth bundle $P \overset{\pi}{\longrightarrow} \mathcal{M}$ for which P is a right $\mathcal{G}$-space with a free right $\mathcal{G}$-action $\triangleleft$ and the complete bundle is bundle isomorphic to $P \overset{[\cdot]}{\longrightarrow} P/\mathcal{G}$. The bundle projection

$$\begin{aligned} [\cdot] : P &\longrightarrow P/\mathcal{G} \\ p &\longmapsto [p] \end{aligned}$$

maps each element of $p \in P$ to its equivalence class $[p]$. Since the orbits $\mathcal{O}_p$ are isomorphic to $\mathcal{G}$ for all points $p \in P$ due to $\triangleleft$ being a free action, we immediately observe that $P \overset{[\cdot]}{\longrightarrow} P/\mathcal{G}$ is a fibre bundle with typical fibre $\mathcal{G}$. Let us study the structure preserving maps of these principal $\mathcal{G}$-bundles $P \overset{\pi}{\longrightarrow} \mathcal{M}$. Note that in the definition of principal $\mathcal{G}$-bundles we only required them to be bundle isomorphic to $P \overset{[\cdot]}{\longrightarrow} P/\mathcal{G}$. This is sufficient since the total spaces P of both bundles are identical, but in general we might consider principal $\mathcal{G}$-bundles with different total spaces. Since the total spaces are right $\mathcal{G}$-spaces, we may have different right actions on them, which may even stem from different groups. Thus, to relate the principal bundles we need to first relate the underlying groups by a Lie group homomorphism $\rho : \mathcal{G} \longrightarrow \mathcal{G}'$, where the total space P of one bundle is a right $\mathcal{G}$-space and the total space P' of the other bundle is a right $\mathcal{G}'$-space. We may understand this configuration using the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{f} & P' \\ {\scriptstyle \triangleleft \mathcal{G}}\uparrow & & \uparrow{\scriptstyle \triangleleft' \mathcal{G}'} \\ P & \xrightarrow{f} & P' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ \mathcal{M} & \xrightarrow{g} & \mathcal{M}' \,. \end{array}$$

From this diagram it is obvious that we require a bundle morphism (f, g) to not only fulfill $g \circ \pi = \pi' \circ f$ but in addition impose $f(p \lhd G) = f(p) \lhd' \rho(G)$ in order to call it a *principal bundle morphism* (f, g, ρ). Of course we are now working with smooth manifolds and projections, so we require the maps involved in principal bundle morphisms to be smooth as well.

Constructing a *principal bundle isomorphism* is then straightforward by requiring that $(f^{-1}, g^{-1}, \rho^{-1})$ exists and is a principal bundle morphism as well. This is in fact often trivial, since the existence of g^{-1} and ρ^{-1} already implies that f is invertible as well. Hence, if we consider bundles over the same base space $\mathcal{M}$ with the same Lie group $\mathcal{G}$ acting on the total spaces, any principal bundle morphism is an isomorphism already. This allows us to lift the notion of trivial bundles to principal $\mathcal{G}$-bundles in a straightforward manner. Recall that a trivial way of constructing a fibre bundle was to attach the fibre F at each point of the base space $\mathcal{M}$ by means of a Cartesian product, making $\mathcal{M} \times F \xrightarrow{\text{proj}_1} \mathcal{M}$ a product bundle. For principal bundles, we identified the fibres of $P \xrightarrow{[\cdot]} P/\mathcal{G}$ with the underlying Lie group $\mathcal{G}$. Hence, we call a principal $\mathcal{G}$-bundle trivial if it is principal bundle isomorphic to the principal $\mathcal{G}$-bundle $\mathcal{M} \times \mathcal{G} \xrightarrow{\text{proj}_1} \mathcal{M}$. Since the Lie group and the base space of $\mathcal{M} \times \mathcal{G} \xrightarrow{\text{proj}_1} \mathcal{M}$ are already identical to those of $P \xrightarrow{\pi} \mathcal{M}$, it suffices that there exists a smooth map $f : P \longrightarrow \mathcal{M} \times \mathcal{G}$ constituting a bundle morphism $(f, \text{id}_{\mathcal{M}})$ in order for $P \xrightarrow{\pi} \mathcal{M}$ to be trivial. This may be used to find yet another equivalent condition for a principal $\mathcal{G}$-bundle to be trivial. Namely, a principal $\mathcal{G}$-bundle $P \xrightarrow{\pi} \mathcal{M}$ is trivial if and only if there exists a section $\sigma : \mathcal{M} \longrightarrow P$. We will see that it is often convenient to use the triviality condition in the latter formulation.

The key property of principal $\mathcal{G}$-bundles which makes them useful for physics is that they allow us to understand the manifold bundles we considered so far in a new way by associating group actions to them. This is expressed by defining associated fibre bundles. The latter are associated to principal $\mathcal{G}$-bundles and may be understood as attaching some new fibre F to the base space and identifying all the points which are connected by actions of the group $\mathcal{G}$. As before, we identify points using equivalence classes. We make these ideas concrete by considering a principal G-bundle $P \xrightarrow{\pi} \mathcal{M}$ of which the total space is a right $\mathcal{G}$-space $P \xrightarrow{\lhd \mathcal{G}} P$. The new fibre is introduced as a smooth manifold F being a left $\mathcal{G}$-space $F \xrightarrow{\mathcal{G} \rhd} F$. Then, we define a new bundle by means of the quotient space $P_F := P \times F/\sim_{\mathcal{G}}$, where $\sim_{\mathcal{G}}$ is the equivalence relation which relates elements of both P and F connected by $\mathcal{G}$-actions. Recalling that left and right $\mathcal{G}$-actions are related by the inversion of group elements, it is straightforward to identify (p, f) and (p', f') if they are related as $p' = p \lhd G$ and $f' = G^{-1} \rhd f$ for some group element $G \in \mathcal{G}$. This construction yields equivalence classes $[(p, f)]$ as elements of P_F just as in the construction of orbit spaces. Inheriting the bundle projection from the underlying principal $\mathcal{G}$-bundle $P \xrightarrow{\pi} \mathcal{M}$ as

$$\begin{aligned} \pi_F : \quad P_F &\longrightarrow \mathcal{M} \\ [(p,f)] &\longmapsto \pi(p)\,, \end{aligned}$$

we finally constructed a fibre bundle $P_F \xrightarrow{\pi_F} \mathcal{M}$ with typical fibre F called the *associated fibre bundle*.

Let us consider a few examples to stress the importance of principal $\mathcal{G}$-bundles $P \xrightarrow{\pi} \mathcal{M}$ and their associated fibre bundles $P_F \xrightarrow{\pi_F} \mathcal{M}$ for physics. These arise from the frame bundle $L\mathcal{M} \xrightarrow{\pi} \mathcal{M}$ we constructed earlier. Recall that we identified the elements of $L\mathcal{M}$ with the frames, being ordered basis tupels $(e_1, \ldots, e_{\dim \mathcal{M}})$ at points $p \in \mathcal{M}$. We make this a principal $\mathrm{GL}(\dim \mathcal{M}, \mathbb{R})$-bundle by introducing the right action

$$\begin{aligned} \lhd\colon\ \mathrm{GL}(\dim \mathcal{M}, \mathbb{R}) \times L\mathcal{M} &\longrightarrow L\mathcal{M} \\ (G, (e_1, \ldots, e_{\dim \mathcal{M}})) &\longmapsto (e_1, \ldots, e_{\dim \mathcal{M}}) \lhd G := (e_m G^m{}_1, \ldots, e_m G^m{}_{\dim \mathcal{M}})\,, \end{aligned}$$

where we use that $\mathrm{GL}(\dim \mathcal{M}, \mathbb{R})$ is a matrix group and we may thus use the components $G^a{}_b$ of the endomorphisms $G \in \mathrm{GL}(\dim \mathcal{M}, \mathbb{R})$ as before. Hence, the frame bundle is a principal $\mathrm{GL}(\dim \mathcal{M}, \mathbb{R})$-bundle for which we may construct associated fibre bundles by attaching new fibres. Since the frame bundle consists of all the bases of the tangent spaces to a smooth manifold $\mathcal{M}$, it is straightforward to associate the components of vectors, covectors and tensors to this basis. We may do so by choosing different fibres.

First, in the spirit of investigating the $\dim \mathcal{M}$ vector components being elements of $\mathbb{R}$, choose $F = \mathbb{R}^{\dim \mathcal{M}}$ as the fibre. We define a left $\mathrm{GL}(\dim \mathcal{M}, \mathbb{R})$-action on this fibre by means of

$$\begin{aligned} \rhd\colon\ \mathrm{GL}(\dim \mathcal{M}, \mathbb{R}) \times \mathbb{R}^{\dim \mathcal{M}} &\longrightarrow \mathbb{R}^{\dim \mathcal{M}} \\ (G, f) &\longmapsto G \rhd f\,, \end{aligned}$$

modeling the transformation of vector components under a change of basis by componentwise requiring $(G \rhd f)^a := G^a{}_b f^b$. Then, clearly $L\mathcal{M}_{\mathbb{R}^d} \xrightarrow{\pi_{\mathbb{R}^d}} \mathcal{M}$ is an associated bundle to the frame bundle. Motivated by the comments on the construction, we obtain that $L\mathcal{M}_{\mathbb{R}^d} \xrightarrow{\pi_{\mathbb{R}^d}} \mathcal{M}$ is bundle isomorphic to the tangent bundle $T\mathcal{M} \xrightarrow{\pi_{T\mathcal{M}}} \mathcal{M}$, simply by constructing the map

$$\begin{aligned} u: \quad & L\mathcal{M}_{\mathbb{R}^d} \longrightarrow T\mathcal{M} \\ & [((e_1, \ldots, e_{\dim \mathcal{M}}), f)] \longmapsto f^a e_a \,. \end{aligned}$$

Hence, what we have achieved is that we gather a new understanding of vectors, that is, we may think of vectors in terms of their basis representation. But in addition it is now an immediate consequence of the associated bundle that the basis representation of vectors is independent of the choice of basis, and the transformation of frames and vector components is given in terms of group actions of $\mathrm{GL}(\dim \mathcal{M}, \mathbb{R})$.

We generalize this construction for vectors to tensors by considering the new fibre $F = \underbrace{\mathbb{R}^{\dim \mathcal{M}} \times \cdots \times \mathbb{R}^{\dim \mathcal{M}}}_{p \text{ times}} \times \underbrace{\mathbb{R}^{\dim \mathcal{M}*} \times \cdots \times \mathbb{R}^{\dim \mathcal{M}*}}_{q \text{ times}}$. The left $\mathrm{GL}(\dim \mathcal{M}, \mathbb{R})$-action is given componentwise as before by

$$(G \triangleright f)^{i_1,\ldots,i_p}{}_{j_1,\ldots,j_q} := f^{\tilde{i}_1,\ldots,\tilde{i}_p}{}_{\tilde{j}_1,\ldots,\tilde{j}_q} G^{i_1}{}_{\tilde{i}_1} \cdots G^{i_p}{}_{\tilde{i}_p} (G^{-1})^{\tilde{j}_1}{}_{j_1} \cdots (G^{-1})^{\tilde{j}_q}{}_{j_q} \,.$$

Just as for vectors before, $L\mathcal{M}_F \xrightarrow{\pi_F} \mathcal{M}$ is now bundle isomorphic to the (p, q)-tensor bundle. While this reproduces structures we already introduced in the previous sections, the concept of associated bundles is used to define new structures as well. An important example is the generalization of the latter bundle to (p, q)*-tensor densities of weight* k. While the fibre F is chosen as in the tensor case, we modify the left $\mathrm{GL}(\dim \mathcal{M}, \mathbb{R})$-action as

$$(G \triangleright f)^{i_1,\ldots,i_p}{}_{j_1,\ldots,j_q} := (\det G^{-1})^k f^{\tilde{i}_1,\ldots,\tilde{i}_p}{}_{\tilde{j}_1,\ldots,\tilde{j}_q} G^{i_1}{}_{\tilde{i}_1} \cdots G^{i_p}{}_{\tilde{i}_p} (G^{-1})^{\tilde{j}_1}{}_{j_1} \cdots (G^{-1})^{\tilde{j}_q}{}_{j_q}$$

for $k \in \mathbb{Z}$. Let us once again point out that the determinant is well-defined since $G \in \mathrm{GL}(\dim \mathcal{M}, \mathbb{R})$ is a endomorphism on $\mathbb{R}^{\dim \mathcal{M}}$.

This concludes our discussion of associated bundles to principal $\mathcal{G}$-bundles. Summarizing, associated bundles to the frame bundle may be used to recover all the structures we defined before. We understood this conceptionally by attaching new fibres to the underlying principal $\mathcal{G}$-bundle. Moreover, we have seen that associated bundles are used to introduce group actions to the bundles we met before, giving them more structure.

We will now use the concepts arising from principal $\mathcal{G}$-bundles to define a connection on them. By means of introducing a covariant exterior derivative, the connection will provide us with curvature and torsion of principal $\mathcal{G}$-bundles.

2.6. Equipping principal bundles with further structure

A connection can be thought of as a choice determining how to connect points in nearby fibres of principal $\mathcal{G}$-bundles. Since we found these fibres to be isomorphic to $\mathcal{G}$, the definition of a connection requires us to introduce further details on Lie groups and their algebras first.

Consider a Lie group $(\mathcal{G}, \bullet)$. From the representation theory of Lie groups we borrow the map

$$\begin{aligned} \mathrm{Ad}_G : \mathcal{G} &\longrightarrow \mathcal{G} \\ H &\longmapsto \mathrm{Ad}_G(H) := G \bullet H \bullet G^{-1} \end{aligned}$$

for all $G \in \mathcal{G}$. Pushing this map forward to the tangent spaces is particularly convenient at the identity element $E \in \mathcal{G}$ since E is invariant under the action of Ad_G. Thus, the push-forward $\mathrm{Ad}_{G*} : T_E\mathcal{G} \longrightarrow T_E\mathcal{G}$ is a general linear map on $T_E\mathcal{G}$ which may therefore be used to define the *Adjoint representation of* $(\mathcal{G}, \bullet)$ by means of

$$\begin{aligned} \mathrm{Ad} : \mathcal{G} &\longrightarrow \mathrm{GL}(T_E\mathcal{G}) \\ G &\longmapsto \mathrm{Ad}_{G*} \,. \end{aligned}$$

Recall that $\mathrm{GL}(\dim \mathcal{M}, \mathbb{R})$ is a matrix group and thus $T_E\mathrm{GL}(\dim \mathcal{M}, \mathbb{R})$ is matrix-valued as well. Therefore, we may write the push-forward of $A \in T_E\mathrm{GL}(\dim \mathcal{M}, \mathbb{R})$ induced by the Adjoint representation explicitly as $(\mathrm{Ad}_{G*}A)^\mu{}_\nu = G^\mu{}_\rho A^\rho{}_\sigma (G^{-1})^\sigma{}_\nu$ in this example. This Adjoint representation is particularly interesting since $T_E\mathcal{G}$ has further structural meaning in terms of Lie algebras.

The abstract definition of a Lie algebra is based on the set $\mathfrak{g} \subset \Gamma(T\mathcal{G})$ of so-called left invariant vector fields on $\mathcal{G}$. However, for practical purposes it is often more convenient to exploit that $\mathfrak{g}$ is vector space isomorphic to $T_E\mathcal{G}$. For the Adjoint representation introduced above, this isomorphism implies that Ad represents Lie groups on their Lie algebras. These Lie algebras $(T_E\mathcal{G}, +, \cdot, [\![\cdot, \cdot]\!])$ are built upon $\mathfrak{g} \cong T_E\mathcal{G}$ by equipping the set with a K-vector space structure $(T_E\mathcal{G}, +, \cdot)$ and an (abstract) Lie bracket

$$\begin{aligned} [\![\cdot, \cdot]\!] : T_E\mathcal{G} \times T_E\mathcal{G} &\xrightarrow{\sim} T_E\mathcal{G} \\ (A, B) &\longmapsto [\![A, B]\!] \,. \end{aligned}$$

In order to be called Lie bracket, $[\![\cdot, \cdot]\!]$ needs to be antisymmetric as $[\![A, B]\!] = -[\![B, A]\!]$

and fulfill the Jacobi identity

$$[\![A,[\![B,C]\!]]\!] + [\![B,[\![C,A]\!]]\!] + [\![C,[\![A,B]\!]]\!] = 0$$

for all $A, B, C \in T_E\mathcal{G}$. By means of this definition we are able to construct a Lie algebra $(T_E\mathcal{G}, +, \cdot, [\![\cdot,\cdot]\!])$ for a Lie group $(\mathcal{G}, \bullet)$. However, having an element of a Lie algebra we may actually construct a Lie group element from it as well. To that effect, note that Lie algebra elements $A \in T_E\mathcal{G}$ are elements of a tangent space. Recall that we defined such tangent vectors $X_{\gamma,E}$ as the unique tangent vectors to curves $\gamma : \mathbb{R} \longrightarrow \mathcal{G}$ arranged such that $\gamma(0) = E$. Hence, for each Lie algebra element $A \in T_E\mathcal{G}$ there exists a unique curve $\gamma^A : \mathbb{R} \longrightarrow \mathcal{G}$ for which A is the tangent vector at the point $E \in \mathcal{G}$. We use this curve to define the *exponential map*

$$\begin{aligned}\exp : T_E\mathcal{G} &\longrightarrow \mathcal{G}\\ A &\longmapsto \exp(A) := \gamma^A(1)\,.\end{aligned}$$

This provides the required map from Lie algebra elements to Lie group elements. Since vectors being elements of K-vector spaces may be scaled by $\lambda \in K$, the exponential map is immediately extended to $\exp(\lambda \cdot A) \equiv \gamma^A(\lambda)$.

Let us combine these insights on Lie algebras with the principal $\mathcal{G}$-bundles $P \xrightarrow{\pi} \mathcal{M}$ we discussed before. In particular, the exponential map is used to define a vector field $X^A \in \Gamma(TP)$ by means of

$$\begin{aligned}i : T_E\mathcal{G} &\longrightarrow \Gamma(TP)\\ A &\longmapsto X^A\,.\end{aligned}$$

This induces a vector $X^A(p) \equiv X^A_p \in T_pP$ at every point $p \in P$ defined by

$$\begin{aligned}X^A_p : C^\infty(P) &\longrightarrow \mathbb{R}\\ f &\longmapsto X^A_p f := [f(p \lhd \exp(\lambda A))]'(0)\,,\end{aligned}$$

where the prime denotes differentiation with respect to $\lambda \in \mathbb{R}$. To give these vector fields a structural interpretation, let us introduce the *vertical subspace* $V_pP \subset T_pP$ of the tangent space by means of

$$V_pP := \ker(\pi_{*p}) = \{X \in T_pP | \pi_{*p}(X) = 0\}\,.$$

The vertical subspace V_pP at a point $p \in P$ in the principal $\mathcal{G}$-bundle $P \xrightarrow{\pi} \mathcal{M}$ may be imagined as all the tangent vectors to the fibre $F_{\pi(p)} \equiv \mathrm{preim}_\pi(\pi(p))$. By definition, all vectors X^A_p for all $A \in T_E\mathcal{G}$ and $p \in P$ lie in the vertical subspace at that point, $X^A_p \in V_pP$. In other words, the push-forward $\pi_{*p}(X^A_p) = 0$ vanishes identically.

These prerequisites finally enable us to define the *connection one-form* ω on a principal $\mathcal{G}$-bundle $P \xrightarrow{\pi} \mathcal{M}$. This smooth one-form is Lie-algebra valued, meaning that the target space is the Lie algebra $T_E\mathcal{G}$ so that we have

$$\begin{aligned} \omega : \Gamma(TP) &\xrightarrow{\sim} T_E\mathcal{G} \\ X &\longmapsto \omega(X)\,. \end{aligned}$$

For such a smooth Lie algebra-valued one-form ω to be called a connection one-form it needs to fulfill two conditions. First, we require ω to be compatible with the group structure of the principal $\mathcal{G}$-bundle. Since ω is a one-form, it needs to be pulled back by the group action. Hence, we require for all $G \in \mathcal{G}$ that $[(\lhd\ G)^*\omega](X) = \mathrm{Ad}_{G^{-1}*}(\omega(X))$, where the Adjoint representation makes its appearance since we noted that it represents a group on its algebra. The second condition on connection one-forms relates to the elements of $\Gamma(TP)$ we already discussed before, namely X^A. Recall that their corresponding Lie algebra element was A, so that it is straightforward to require $\omega(X^A) = A$. This condition may equivalently expressed as $\omega \circ i = \mathrm{id}_{T_E\mathcal{G}}$ using the map $i : T_E\mathcal{G} \longrightarrow \Gamma(TP)$ we considered before for defining X^A. Since this second condition relates the connection one-form to the vertical subspace V_pP, it is useful to define the vertical component $\mathrm{ver} := i \circ \omega$ of a vector field $X \in \Gamma(TP)$ as $\mathrm{ver}(X) = i(\omega(X))$. The remaining part $\mathrm{hor}(X) := X - \mathrm{ver}(X)$ of X is called its horizontal component. From the condition $\omega \circ i = \mathrm{id}_{T_E\mathcal{G}}$ fulfilled by connection one-forms it is straightforward to conclude that $\omega(\mathrm{hor}(X)) = 0$. Thus, we define the *horizontal subspace* $H_pP \subset T_pP$ at each point $p \in P$ as $H_pP := \ker(\omega_p) \equiv \{X_p \in T_pP | \omega_p(X_p) = 0\}$. The existence of the unique decomposition $X_p = \mathrm{ver}_p(X_p) + \mathrm{hor}_p(X_p)$ is often denoted as the (inner) direct sum $V_pP \oplus H_pP = T_pP$. A choice of such behaved horizontal subspaces H_pP is called a *connection*, being fully equivalent to the choice of a connection 1-form on a principal bundle.

For applications of connections in physics it is instructive to consider some special cases. First, we note that the connection one-form is Lie algebra-valued. If this Lie algebra is the algebra of a matrix group, we may again explicit the Lie algebra indices and denote the connection one-form by $\omega^\mu{}_\nu$. This is the case for example on the frame bundle $L\mathcal{M} \xrightarrow{\pi} \mathcal{M}$ with Lie group $\mathrm{GL}(\dim \mathcal{M}, \mathbb{R})$ we considered before. Being a one-form, the connection one-form can furthermore be expressed in a basis in the tangent space at some point $p \in P$. Hence, it is always possible to denote the coefficients by $\Gamma_a := \omega_{p\,a}$. On a matrix-valued Lie algebra we may therefore use both types of indices and write $\Gamma^{\mu}_{a\nu}$. In $\mathrm{GL}(\dim \mathcal{M}, \mathbb{R})$ both types of indices take their values in $\{1, \dots, \dim \mathcal{M}\}$, and we will later examine when it makes sense to denote both indices by the same type of symbol. However, it is important to keep in mind that only the one-form index is a tensor index which is therefore often said to transform as a tensor. The indices μ, ν are obviously not of this type since we introduced them as indices in the Lie algebra.

This behavior of a connection one-form taking its values in a Lie algebra yields a special transformation behavior. For example, the globally defined connection on the principal bundle $P \xrightarrow{\pi} \mathcal{M}$ induces a one-form on the base manifold $\mathcal{M}$ which depending on context comes under the name *Yang Mills field*, *gauge field* or by abuse of nomenclature is called *connection one-form* as well. As we have seen in the discussion of manifolds, we should restrict this induction to some chart domain $U \in \mathcal{O}_{\mathcal{M}}$ and hence denote the resulting one-form by ω^U. Since ω is a one-form on P, the straightforward operation which is used to induce a one-form on $\mathcal{M}$ is a pull-back. Hence, we need a map $\sigma : U \longrightarrow P$, that is, a local section on the principal bundle[5]. Using this section, we define $\omega^U := \sigma^* \omega$. Since this is only a local connection one-form, it is instructive to study how the connection one-forms in the overlap of two charts U and V are related to each other. To do so, we introduce a unique gauge map $\Lambda : U \cap V \longrightarrow \mathcal{G}$ which relates the sections as $\sigma_V(m) = \sigma_U(m) \lhd \Lambda(m)$. Using this Lie group element $\Lambda(m) \in \mathrm{GL}(\dim \mathcal{M}, \mathbb{R})$ on the frame bundle, a lengthy calculation reveals that

$$\omega^{V\mu}{}_{\nu} = \Lambda^{-1\mu}{}_{\rho} \omega^{U\rho}{}_{\sigma} \Lambda^{\sigma}{}_{\nu} + \Lambda^{-1\mu}{}_{\rho} \mathrm{d}\Lambda^{\rho}{}_{\nu} \,. \tag{2.25}$$

If the principal bundle nature of the connection is disregarded, the latter equation is often introduced under the name *vielbein postulate*. But in the full approach we sketched here, we obtain the vielbein postulate as a consequence of representing the connection one-form ω on the base manifold.

Having a principal $\mathcal{G}$-bundle $P \xrightarrow{\pi} \mathcal{M}$ equipped with a connection one-form ω immediately induces further structure on the principal bundle. The most immediate implication is the *covariant exterior derivative* D. Since we understood the connection as a choice of horizontal subspace H_pP, we define the covariant exterior derivative

$$\begin{aligned} D : \Gamma(T^0_n TP) &\xrightarrow{\sim} \Gamma(T^0_{n+1} TP) \\ \phi &\longmapsto D\phi \end{aligned}$$

[5]Note that the existence of a local section means that the principal bundle is locally trivial. Hence, all of the constructions considered hereafter require us to work with locally trivial bundles.

by its action

$$D\phi(X_1,\dots,X_{n+1}) \coloneqq \mathrm{d}\phi(\mathrm{hor}(X_1),\dots,\mathrm{hor}(X_{n+1}))\,. \tag{2.26}$$

Note that we leave it entirely open which target space the $(0,n)$-tensor field ϕ has.

The covariant exterior derivative may act on the Lie algebra-valued one-form ω for instance, which yields the Lie algebra-valued two-form $\Omega:\ \Gamma(TP)\times\Gamma(TP)\xrightarrow{\sim} T_E\mathcal{G}$ defined by

$$\Omega \coloneqq D\omega\,, \tag{2.27}$$

called the *curvature* two-form. Expliciting the definition of the horizontal component of a vector field in the definition of the covariant exterior derivative, we may equivalently write

$$\Omega = \mathrm{d}\omega + \omega \curlywedge \omega\,, \tag{2.28}$$

where the wedge product of Lie algebra-valued one-forms ω is simply given by their Lie bracket as $(\omega\curlywedge\omega)(X,Y) \coloneqq [\![\omega(X),\omega(Y)]\!]$. If the Lie group $\mathcal{G}$ is a matrix group, the Lie bracket is given by the ordinary commutator $[a,b] \coloneqq a\circ b - b\circ a$ of matrices a,b so that we recover the standard wedge product as

$$\Omega^\mu{}_\nu = \mathrm{d}\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu\,. \tag{2.29}$$

Applying the covariant exterior derivative to the curvature two-form again we obtain

$$D\Omega = 0 \tag{2.30}$$

which is known as the *Bianchi identity for curvature*.

Just as we induced a Yang-Mills field ω^U on $U\subseteq\mathcal{M}$ by pulling back the connection one-form ω by means of a local section $\sigma:\ U\longrightarrow P$, we may pull back the curvature two-form to the base manifold as well. The resulting two-form $\Omega^U \coloneqq \sigma^*\Omega$ is called the *Yang-Mills field strength*, the *gauge field strength*, the *Riemann tensor* or by abuse of nomenclature denoted *curvature two-form* as well, depending on context. For constructing the explicit expression of the curvature two-form on the base manifold, recall that we already noted in (2.15) that the pull-back distributes over wedge products and commutes with exterior differentiation, see (2.19). Using the definition $\omega^U = \sigma^*\omega$ of the Yang-Mills field, we conclude

$$\Omega^U \equiv \sigma^*\Omega = \mathrm{d}\omega^U + \omega^U \mathbin{\curlywedge} \omega^U\,. \tag{2.31}$$

For matrix groups $\mathcal{G}$ this again simplifies to

$$\Omega^{U\mu}{}_\nu = \mathrm{d}\omega^{U\mu}{}_\nu + \omega^{U\mu}{}_\rho \wedge \omega^{U\rho}{}_\nu \tag{2.32}$$

utilizing the ordinary wedge product.

Having a connection on a principal $\mathcal{G}$-bundle, we may equip this bundle with yet another structure. Recall that we introduced the connection as a Lie algebra-valued one-form. Lie algebras $\mathfrak{g}$ may be represented on vector spaces similarly as we briefly discussed it for Lie groups $\mathcal{G}$. In particular, this representation on a finite-dimensional vector space V is given by the linear map $\rho : \mathfrak{g} \xrightarrow{\sim} \mathrm{End}(V)$ to the endomorphisms on V. Of course, this map is supposed to preserve the Lie bracket structure of the Lie algebra so that we require $\rho([\![a,b]\!]) = [\rho(a),\rho(b)] \equiv \rho(a)\circ\rho(b) - \rho(b)\circ\rho(a)$ for $a,b \in \mathfrak{g}$. Since such a linear representation space V is a vector space in particular, we might want to identify it with the tangent spaces of the base manifold $\mathcal{M}$. On the level of bundles, the tangent spaces correspond to the tangent bundle $T\mathcal{M}$ while the representation space may be used as the fibre of an associated bundle to create its total space P_V. Hence, we require that these two bundles are isomorphic to each other and introduce the V-valued one-form

$$\begin{aligned}\theta :\, \Gamma(TP) &\xrightarrow{\sim} V\\ X &\longmapsto \theta(X)\end{aligned}$$

to make this isomorphism explicit. θ is called the *soldering* or *solder form* if it is compatible with the structure we have on the principal $\mathcal{G}$-bundle already. For compatibility with the connection one-form we require $\theta \circ \mathrm{ver} = 0$, and for compatibility with the Lie group structure we invoke $[(\lhd\, G)^*\theta](X) = G^{-1} \rhd \theta(X)$ for all $G \in \mathcal{G}$ and $X \in \Gamma(TP)$.

Having a soldering form, we may again take its covariant exterior derivative to define the V-valued *torsion two-form*

$$T := D\theta\,. \tag{2.33}$$

Just as for curvature we use the definition of the covariant exterior derivative to explicitly write

$$T = \mathrm{d}\theta + \omega \curlywedge \theta\,, \tag{2.34}$$

where now the wedge product $\curlywedge$ is defined as the action of the Lie algebra-valued one-form ω on the soldering form θ taking values in a representation space of this Lie algebra. We use that the target space of θ is a vector space given by this representation space to see that the soldering form has one Lie-algebra index if we consider matrix groups. Thus, we may utilize the ordinary wedge product again to write

$$T^\mu = \mathrm{d}\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu \tag{2.35}$$

if $\mathcal{G}$ is a matrix group. By means of the explicit expression (2.34) of torsion on the principal bundle it is straightforward to conclude

$$DT = \Omega \curlywedge \theta\,. \tag{2.36}$$

This is called the *Bianchi identity for torsion* and takes the form

$$DT^\mu = \Omega^\mu{}_\nu \wedge \theta^\nu \tag{2.37}$$

for matrix groups.

In analogy to the connection one-form and the curvature two-form, we may induce a soldering form and a torsion two-form on the base manifold $\mathcal{M}$ by means of a local section $\sigma : U \longrightarrow P$. Defining $\theta^U := \sigma^*\theta$ and $T^U := \sigma^* T$, we use that the pull-back distributes over wedge products and commutes with the exterior derivative to find

$$T^U = \mathrm{d}\theta^U + \omega^U \curlywedge \theta^U\,. \tag{2.38}$$

For matrix groups, this simplifies as

$$T^{U\mu} = \mathrm{d}\theta^{U\mu} + \omega^{U\mu}{}_\nu \wedge \theta^{U\nu}\,. \tag{2.39}$$

We introduced the soldering form as an additional structure which can be added to principal bundles equipped with a connection. Thus, having a connection and curvature may be considered more fundamental than having a soldering form and torsion on principal bundles. However, a soldering form is not always a major new structure one needs to introduce to a bundle. For example, if we consider the frame bundle $L\mathcal{M} \overset{\pi}{\longrightarrow} \mathcal{M}$, being a principal $\mathrm{GL}(\dim \mathcal{M}, \mathbb{R})$-bundle, we find that it carries a soldering form already which is canonically defined. To see this, consider $V = \mathbb{R}^{\dim \mathcal{M}}$ as the representation space of the Lie algebra. On this vector space, we have local coordinates of vector fields at a point $p \in \mathcal{M}$ by using the coframe $\epsilon = (\epsilon^1, \ldots, \epsilon^{\dim \mathcal{M}})$ to define

$$\begin{aligned} u_\epsilon : T_p\mathcal{M} &\longrightarrow \mathbb{R}^{\dim \mathcal{M}} \\ X &\longmapsto u_\epsilon(X) := (\epsilon^1(X), \ldots, \epsilon^{\dim \mathcal{M}}(X)) \,. \end{aligned}$$

This immediately induces the soldering form

$$\theta_\epsilon := u_\epsilon \circ \pi_* \tag{2.40}$$

on the frame bundle $L\mathcal{M} \xrightarrow{\pi} \mathcal{M}$. While this soldering form on the frame bundle is given essentially by the coframe, we may use $\pi \circ \sigma = \mathrm{id}_\mathcal{M}$ for sections $\sigma : U \longrightarrow L\mathcal{M}$ to conclude that we may simply use the coframe ϵ as soldering form $\theta^U \equiv \sigma^*\theta$ on the base manifold $\mathcal{M}$. It is therefore common to denote the coframe by θ instead of ϵ on frame bundles.

To conclude the discussion of soldering forms, let us note that we introduced them as a way to connect a vector space being the representation space of a Lie algebra with a tangent space of the base space of a principal bundle. This isomorphism between the tangent bundle and the associated bundle $P_V \xrightarrow{\pi_V} \mathcal{M}$ allows to identify the two different indices we used so far if the Lie group was a matrix group, since the soldering form always allows us to map the indices to each other. This is often done for simplicity, but coming from the full principal bundle derivation it is important to keep in mind that we straightforwardly derived that the two types of indices behave differently under coordinate transformations for instance. This is the origin of some subtleties on which we will comment in the following chapters.

Now that we have equipped a principal bundle with a connection and a soldering form, we may introduce even more structure. Especially for applications in physics, we might want to measure lengths and angles for which we need a notion of a metric. Note that we do not need a metric for everything discussed so far; and if we only want to measure lengths, introducing a norm might be sufficient. In particular, we may discuss curvature and torsion no matter if the principal bundle is equipped with a metric or not.

For defining a metric on the total space P of a principal $\mathcal{G}$-bundle $P \xrightarrow{\pi} \mathcal{M}$, we consider $(0,2)$-tensor fields $g \in \Gamma(T^0_2 TP)$ which are explicitly given by

$$\begin{aligned} g:\ \Gamma(TP) \times \Gamma(TP) &\xrightarrow{\sim} C^\infty(P) \\ (X,Y) &\longmapsto g(X,Y)\,. \end{aligned}$$

We call such a $g \in \Gamma(T^0_2 TP)$ a *metric tensor field* or a *pseudo inner product* if it is symmetric as $g(X,Y) = g(Y,X)$ and non-degenerate, that is $g(X,Y) = 0$ for all $Y \in \Gamma(TP)$ only if $X = 0$. This non-degeneracy condition takes a more convenient form if we define the *musical map* $\flat(X) \equiv g(X,\cdot)$. This map

$$\begin{aligned} \flat:\ \Gamma(TP) &\xrightarrow{\sim} \Gamma(T^*P) \\ X &\longmapsto \flat(X) \end{aligned}$$

is defined by its action on vector fields as

$$\begin{aligned} \flat(X):\ \Gamma(TP) &\xrightarrow{\sim} C^\infty(P) \\ Y &\longmapsto g(X,Y)\,. \end{aligned}$$

Then, $g \in \Gamma(T^0_2 TP)$ is non-degenerate if and only if the musical map is an isomorphism. Hence, the inverse $\sharp \equiv \flat^{-1} : \Gamma(T^*P) \xrightarrow{\sim} \Gamma(TP)$ of the musical isomorphism always exists. We use this inverse to define

$$\begin{aligned} g^{-1}:\ \Gamma(T^*P) \times \Gamma(T^*P) &\xrightarrow{\sim} C^\infty(P) \\ (\omega,\sigma) &\longmapsto g^{-1}(\omega,\sigma) := \omega(\sharp(\sigma))\,. \end{aligned}$$

g^{-1} is often called the *inverse metric tensor field* although it is not a proper inverse in the sense $g^{-1} \circ g = \mathrm{id}_P$. The nomenclature is rather understood from local components, in which we have $(g^{-1})^{ac} g_{cb} = \delta^a_b$. Since in local components the metric tensor field has lower case indices while the inverse metric tensor field has upper case indices, it is common to denote the coefficients of the inverse unambiguously by means of $g^{ab} := (g^{-1})^{ab}$. Similar conventions are in place for the components of the musical isomorphism. We say that we *lower the index* of a vector $X \in TP$ by abbreviating $X_a := (\flat(X))_a = g_{ab} X^b$ and *raise the index* of a covector $\omega \in T^*P$ by denoting $\omega^a := (\sharp(\omega))^a = g^{ab}\omega_b$.

Just as for the connection and the soldering, the metric on P immediately induces a metric on the base space $\mathcal{M}$ of the principal bundle $P \xrightarrow{\pi} \mathcal{M}$ by pulling it back via a local section $\sigma : U \longrightarrow P$ as $g^U := \sigma^* g$. In analogy to the connection and the soldering, we may consider the covariant exterior derivative Dg of the metric. But unlike curvature and torsion, Dg is a $(0,3)$-tensor field instead of a differential form. Since it will turn out to be extraordinarily useful to solely work with differential forms, we use a trick to circumvent this issue. For this purpose, we consider the components $g_{ab} = g(e_a, e_b)$ of the metric tensor in a local frame. Recall that we identified these components as scalars, being zero-forms. Hence, taking the covariant exterior derivative $Q_{ab} := -Dg_{ab}$ yields a one-form called the *non-metricity* one-form, where we introduce the minus sign to match historical conventions. For a matrix group $\mathcal{G}$ underlying the principal bundle, we obtain

$$Q^U{}_{\mu\nu} := \sigma^* Q_{\mu\nu} = -\mathrm{d}g^U{}_{\mu\nu} + \omega^{U\sigma}{}_{\mu} g^U{}_{\sigma\nu} + \omega^{U\sigma}{}_{\nu} g^U{}_{\mu\sigma} \,. \tag{2.41}$$

The Bianchi identity for non-metricity is obtained by evaluating $DQ_{\mu\nu}$ which straightforwardly yields

$$DQ_{\mu\nu} = g_{\mu\rho}\Omega^\rho{}_\nu + g_{\nu\rho}\Omega^\rho{}_\mu \,. \tag{2.42}$$

The connection, soldering and metric introduced in this section constitute the most general kinematics necessary to describe a theory of gravity. It is, however, worth noticing that the dynamics of a theory of gravity may set their covariant exterior derivatives to zero by default. This is commonly done by choosing only one of the three covariant exterior derivatives to be non-vanishing, in which case the associated field is called dynamical. For example, the condition $Q_{\mu\nu} = -Dg_{\mu\nu} = 0$ is called *metric compatibility* or *metricity* and is assumed to hold in a broad range of theories like general relativity or Einstein-Cartan gravity. If we additionally impose *torsion freedom* as $T = D\theta = 0$, the connection is uniquely determined by these two conditions. This connection is called the *Levi-Civita connection* $\mathring{\omega}$, while its components are called the *Christoffel symbols*. We will use a circle to denote all tensors and maps which are derived from the Levi-Civita connection, such as the covariant exterior derivative $\mathring{D}$ and the curvature $\mathring{\Omega}$. We will neither impose metric compatibility nor torsion freedom in this thesis unless we explicitly specify it by using the circle notation.

This concludes the introduction to the geometrical nature of the physics we will develop in the following chapters. For making the connections to physics, we will introduce some commonly used conventions in the following section.

2.7. Conventions for using differential geometry in physics

Physics is usually considered in local charts (U, x) of a smooth manifold $\mathcal{M}$. The maps, tensors and transformations in these charts are inherited from the frame bundle $L\mathcal{M} \xrightarrow{\pi} \mathcal{M}$, which we identified as a principal $\mathrm{GL}(\dim \mathcal{M}, \mathbb{R})$-bundle. We follow this approach. Recall that we denoted inherited properties in charts by a superscript U in the latter subsection. Since the domain in which we work in the following is the chart domain U, we will suppress this superscript. Furthermore, we noted that we may only choose a basis locally at some point $p \in \mathcal{M}$, or at most in a chart domain by using the chart induced basis at every point $p \in U$. The reason for this restriction was that tensor fields were $C^\infty(\mathcal{M})$-modules and as such did not have a basis generically. Hence, we may unambiguously drop the subscript p on tensors, since choosing a basis always clarifies that we work with the tensor components of tensor fields. Furthermore, it is common in physics to consider the range of indices being $0, \dots, \dim \mathcal{M} - 1$ instead of $1, \dots, \dim \mathcal{M}$. This applies to tensor indices as well as matrix group indices of $\mathrm{GL}(\dim \mathcal{M}, \mathbb{R})$. We will only use Greek indices to denote both of these index types, since the frame bundle is always equipped with a soldering form canonically defined by the coframe.

3 Hypersurfaces

The main achievement of this thesis is to provide a proper understanding of boundary terms in gravitational theories featuring curvature, torsion and non-metricity. We will model these boundaries as hypersurfaces, which are manifolds of lower dimension that live inside the entire manifold. Making these ideas precise and gaining a thorough understanding of hypersurfaces is the aim of this chapter. We are going to subsequently use the results of this chapter to describe boundaries of manifolds in the following chapters. Space-, time- and lightlike hypersurfaces have been studied to different extent in literature before, and we mainly use [90, 96–98] here. However, this chapter will present a new approach to hypersurfaces. In particular, my approach defines hypersurfaces entirely through a vector field on $\mathcal{M}$ which we call the normal vector field. This approach is used to construct hypersurfaces in the differential form language developed in chapter 2. In particular, I will examine frames and coframes of manifolds and hypersurfaces in detail, which goes beyond the literature mentioned above. This is needed in order to discuss manifolds with torsion since torsion is defined as the covariant exterior derivative of the soldering form given by a coframe when we consider frame bundles. Obviously, my new approach to the description of hypersurfaces needs to reproduce the notions known in literature in the appropriate limits. Hence, I will discuss the relations to the literature definitions as well. This discussion will additionally contribute to a deeper understanding of hypersurfaces.

In general, hypersurfaces may be considered as $(m-p)$-dimensional manifolds Σ immersed or simply embedded in an m-dimensional smooth manifold $\mathcal{M}$ which we assume to be equipped with a metric g. The positive integer $p<m$ is called the *codimension* of the hypersurface, and we will consider hypersurfaces of codimension one unless we explicitly emphasize something else. We are going to study these hypersurfaces in terms of their normal vector field ζ. In particular, we identify vectors v to be tangent to Σ if $g(\zeta, v)=0$. Conversely, any vector field ξ fulfilling $g(\xi, v)=0$ for all vectors v tangent to Σ is a normal vector to that hypersurface. That is, every non-zero multiple of ζ creates the same hypersurface as ζ. We use this normal vector

field to characterize hypersurfaces as

$$\begin{aligned} &\text{timelike if} && g(\zeta,\zeta) > 0\,,\\ &\text{spacelike if} && g(\zeta,\zeta) < 0\,,\\ &\text{lightlike or null if} && g(\zeta,\zeta) = 0\,, \end{aligned} \tag{3.1}$$

where we only consider hypersurfaces with constant sign $\operatorname{sgn}(g(\zeta,\zeta))$ in the following. These three cases may not be treated at once; we will see that lightlike hypersurfaces in fact have a proper description in terms of $(m-2)$-dimensional hypersurfaces. Hence, we begin by examining the space- and timelike cases in section 3.1 before we return to lightlike hypersurfaces in section 3.2.

3.1. Space- and timelike hypersurfaces

We examine space- and timelike hypersurfaces in the normal vector field approach in this section. While we already used this as the foundation of [1] and [2], the following presentation elaborates on space- and timelike hypersurfaces in much more detail in order to gain a better understanding of the equations defining hypersurfaces. In particular, we will elucidate the connection of the normal vector field approach to the constant function definition greatly reviewed in [96] as well as the embedding approach elaborated in [97].

3.1.1. The normal vector field approach

For space- and timelike hypersurfaces Σ, the normal vector field ζ has in particular a non-vanishing pseudo inner product $g(\zeta,\zeta)$. Rescaling ζ by any nowhere vanishing $C^\infty(\mathcal{M})$-function yields another normal vector field of Σ, such that it is useful to define the unit normal vector field n as a reference. To do so, we introduce the normalization $N \coloneqq 1/\sqrt{|g(\zeta,\zeta)|}$ called the *lapse function* and define

$$n \coloneqq \varepsilon N \zeta\,. \tag{3.2}$$

The prefactor $\varepsilon \coloneqq \operatorname{sgn}(g(\zeta,\zeta))$ is $+1$ for timelike and -1 for spacelike hypersurfaces. ε is introduced in the definition of n as prefactor in order to give n the same direction as ζ, that is $g(n,\zeta) = N^{-1} > 0$. As required, the normal vector field n has the same likeness as ζ since $g(n,n) = \varepsilon = \operatorname{sgn}(g(\zeta,\zeta))$.

Note that in literature it is common to define a normal one-form instead of a normal vector field. To see why this is the case, we introduce the constant function approach to hypersurfaces. In this approach, hypersurfaces are considered as the collection of points $p \in \mathcal{M}$ for which some smooth function $f \in C^\infty(\mathcal{M})$ is constant, that is

$$\Sigma_\lambda := \{p \in \mathcal{M} | f(p) = \lambda\} , \tag{3.3}$$

where $\lambda \in \mathbb{R}$ is a constant. By definition, f only changes in directions normal to Σ_λ and thus the gradient one-form $\mathrm{d}f$ is normal to the hypersurface. We may then normalize the gradient $\mathrm{d}f$ just as we did with the normal vector field to create a unit normal. However, note that we defined the notions of being normal and tangent to a hypersurface using vector fields instead of one-forms. Hence, we usually need to consider the normal to a hypersurface to be a vector field. Some authors therefore first define the metric dual to the normal one-form by means of the inverse musical isomorphism $\sharp$ to create a vector field which subsequently is normalized by means of the metric. While it is possible to follow the latter approach, we choose to straightforwardly define hypersurfaces by means of a normal vector field which we need for defining tangent and normal vectors anyway. Conversely, this implies that we need to define the unit normal one-form $\tilde{n} := \flat(n)$ if we want to utilize the normal covector. This additional definition is not explicitly needed, but it is going to be useful for relating our results to those obtained in the constant function approach.

As we mentioned in the introduction to this section, hypersurfaces of codimension one may as well be defined by embedding an $(m-1)$-dimensional manifold $\hat{\Sigma}$ into the m-dimensional manifold $\mathcal{M}$. An embedding is a map $\Phi : \hat{\Sigma} \longrightarrow \Sigma \subset \mathcal{M}$ which is a homeomorphism. Being a homeomorphism, Φ induces a pull-back Φ^* as well as a push-forward Φ_*. Vector fields $\hat{v} \in \Gamma(T\hat{\Sigma})$ on $\hat{\Sigma}$ push forward to tangent vector fields $v \in \Gamma(T\Sigma)$ on $\Sigma \subset \mathcal{M}$ and p-forms $\omega \in \Omega^p(\Sigma)$ on $\mathcal{M}$ are pulled back to p-forms $\hat{\omega} \in \Omega^p(\hat{\Sigma})$. This becomes particularly straightforward if we consider tensors in local components. There, it is common to denote the push-forward by E^a_μ and the pull-back by e^μ_a, where Greek indices are related to the basis $\vartheta_\mu \in T_{\Phi(p)}\mathcal{M}$ on $\mathcal{M}$ and Latin indices are those of the $(m-1)$-dimensional hypersurface basis $\varphi_a \in T_p\hat{\Sigma}$. Hence, in a local basis, all tensor components may be pushed forward and pulled back by contraction with e^μ_a and E^a_μ.

Let us consider some important examples. First, one would intuitively expect that the pull-back of the normal vector field to the hypersurface vanishes. However, we are only able to pull back differential forms. Thus, the appropriate statement is $\Phi^*(\tilde{n}) = 0$, which in local components takes the form

$$e^\mu_a n_\mu = 0 . \tag{3.4}$$

The pull-back of the metric tensor field g induces a metric tensor field $\gamma := \Phi^*(g)$ on the hypersurface, which in local components reads

$$\gamma_{ab} = e^\mu_a e^\nu_b g_{\mu\nu} \,. \tag{3.5}$$

As a last example, we may utilize E^a_μ to extend vectors defined on $\hat{\Sigma}$ to vectors on $\mathcal{M}$. For example, the hypersurface basis $\varphi_a \in T_p\hat{\Sigma}$ induces the vector $E^a_\mu \varphi_a \in T_{\Phi(p)}\mathcal{M}$.

This summarizes the different possibilities of how to define hypersurfaces. Since all these definitions need to coincide, we need to reconstruct the above notions from our normal vector field approach. We formulated the embedding approach in local tensor components, so that for means of comparison we consider components in the normal vector field approach as well. To that end, we examine frames $\vartheta_\mu \in T_p\mathcal{M}$ in the tangent spaces of the m-dimensional manifold $\mathcal{M}$. We decompose this frame such that one of the vectors, say ϑ_{m-1} for instance, is aligned with the normal direction while the remaining basis vectors $(\vartheta_0, \ldots, \vartheta_{m-2})$ form a frame of the hypersurface Σ embedded in $\mathcal{M}$. From chapter 2 we know that this frame alignment may formally be achieved at every point in $\mathcal{M}$ by applying a $\mathrm{GL}(m, \mathbb{R})$ transformation to the frame bundle $L\mathcal{M} \xrightarrow{\pi} \mathcal{M}$. Writing this transformation in components as before, we may decompose

$$\vartheta_\mu = E^a_\mu \varphi_a + \frac{\varepsilon}{N} n_\mu \tilde{\varphi} \,, \tag{3.6}$$

where we choose the prefactor $\frac{\varepsilon}{N} n_\mu$ of the normal direction to match the standard convention and introduce the new, a priori undetermined coefficient E^a_μ.

Note that from the embedding perspective, we would like to interpret $E^a_\mu \varphi_a$ to be the extension of the hypersurface frame φ_a to $\mathcal{M}$. At the same time, we want to interpret $\tilde{\varphi}$ to be aligned with the normal direction. In the following calculation we adapt the frame to the hypersurface in this manner. In particular, we locally express the normal vector field in the decomposed basis (3.6) to find

$$n = n^\mu \vartheta_\mu = -\frac{1}{N} N^a \varphi_a + \frac{1}{N} \tilde{\varphi} \,, \tag{3.7}$$

where we introduce the components $N^a := -N n^\mu E^a_\mu$ of the *shift vector*. That is, $\tilde{\varphi}$ is not aligned with the normal direction. In order to obtain an improved decomposition of the frame, it is useful to decompose the dual coframe θ^μ as well. To that end, we write

$$\theta^\mu = e^\mu_a \tilde{\phi}^a + t^\mu \phi \,. \tag{3.8}$$

To connect the covectors involved in the latter decomposition with the vectors in the frame decomposition (3.6), we require them to be dual to the frame $(\varphi_a, \tilde{\varphi})$ such that

$$\tilde{\phi}^a(\varphi_b) = \delta^a_b\,, \qquad \phi(\tilde{\varphi}) = 1\,, \qquad \tilde{\phi}^a(\tilde{\varphi}) = 0 = \phi(\varphi_a)\,. \tag{3.9}$$

This is equivalent to expressing the coefficients in (3.8) as

$$e^\mu_a = \theta^\mu(\varphi_a)\,, \qquad t^\mu = \theta^\mu(\tilde{\varphi})\,. \tag{3.10}$$

Using this duality of frame and coframe, we exploit the linearity of covectors in the duality defining equation $\theta^\mu(\vartheta_\nu) = \delta^\mu_\nu$ to conclude

$$\delta^\mu_\nu = e^\mu_a E^a_\nu + \frac{\varepsilon}{N} t^\mu n_\nu\,. \tag{3.11}$$

Since we defined the unit normal vector such that $n^\mu n_\mu = \varepsilon$, it is useful to contract the latter equation with the components of the normal vector to obtain

$$n^\mu = \delta^\mu_\nu n^\nu = \frac{1}{N}(t^\mu - e^\mu_a N^a)\,. \tag{3.12}$$

The latter relation may be used to express n^μ in terms of t^μ, e^μ_a and N^a. But since the shift vector $N^a = -N n^\mu E^a_\mu$ explicitly depends on the normal vector, it is more useful to solve for t^μ as

$$t^\mu = N n^\mu + N^a e^\mu_a\,, \tag{3.13}$$

which allows to eliminate t^μ entirely.

Likewise, we may exploit that $\gamma_{ab} = e^\mu_a e^\nu_b g_{\mu\nu}$ is the induced metric on the hypersurface. Note that this is a new definition in the normal vector approach to hypersurfaces, and we will justify only later that γ_{ab} may indeed be interpreted as the hypersurface metric. To gain this object from contractions of (3.11), we multiply with $e^\rho_b g_{\rho\mu}$ and use (3.13) to obtain

$$e^\rho_b g_{\rho\nu} = e^\rho_b g_{\rho\mu}\delta^\mu_\nu = \gamma_{ab}(E^a_\nu + \frac{\varepsilon}{N} N^a n_\nu) + \varepsilon n_\nu e^\mu_b n_\mu\,. \tag{3.14}$$

We define γ^{ab} by $\gamma^{ac}\gamma_{cb} = \delta^a_b$, which we will later interpret as the inverse hypersurface metric. This inverse exists since we assume the hypersurface to be space- or timelike, so that $\gamma_{ab} = e^\mu_a e^\nu_b g_{\mu\nu}$ is non-degenerate. We will elaborate on cases in which γ_{ab} is degenerate in section 3.2. Using the inverse of γ_{ab}, we solve (3.14) for E^a_ν as

$$E^a_\nu = e^a_\nu - \frac{\varepsilon}{N} N^a n_\nu - \varepsilon n_\nu e^a_\mu n^\mu \,, \tag{3.15}$$

where we abbreviate $e^a_\nu := e^\rho_b g_{\rho\nu} \gamma^{ab}$. This result may be simplified even further. To see this, we use (3.13) and (3.15) to eliminate E^a_μ and t^μ from the unity decomposition (3.11) to obtain

$$\delta^\mu_\nu = e^\mu_a e^a_\nu + \varepsilon n^\mu n_\nu - \varepsilon n^\rho e^a_\rho e^\mu_a n_\nu \,. \tag{3.16}$$

Taking the trace of the latter equation on an m-dimensional manifold yields

$$m = \delta^\mu_\mu = (m-1) + 1 - \varepsilon n^\rho e^a_\rho e^\mu_a n_\mu \,, \tag{3.17}$$

from which we conclude $\varepsilon n^\rho e^a_\rho e^\mu_a n_\mu = 0$. But this is in turn used in

$$n_\mu e^\mu_a = n_\mu \delta^\mu_\nu e^\nu_a = n_\mu e^\mu_a (\varepsilon n^\rho e^a_\rho e^\mu_a n_\mu) \tag{3.18}$$

to obtain

$$n_\mu e^\mu_a = 0 \,. \tag{3.19}$$

Although the above calculation might appear lengthy, this result is important since it reproduces the expectation (3.4) we gained from the embedding approach to hypersurfaces. Thus, we may indeed interpret e^μ_a as the components of a pull-back to the hypersurface and using $\gamma_{ab} = e^\mu_a e^\nu_b g_{\mu\nu}$ as the hypersurface metric is justified. However, we have derived (3.19) only from the entirely arbitrary decompositions of frame (3.6) and coframe (3.8) by taking n_μ as one of the coefficients and examining the duality conditions of frame and coframe. Hence, the introduction of the coefficient n_μ in an arbitrary frame decomposition already implies that the coefficient e^μ_a in the coframe decomposition (3.8) induces a pull-back to the hypersurface. Furthermore, the vectors $e_a := e^\mu_a \vartheta_\mu$ are tangent to the hypersurface, as $g(e_a, n) = 0$ by means of (3.19). It is therefore common to use e_a as a frame on the hypersurface. This is reasonable since $\gamma_{ab} = g(e_a, e_b)$.

The relation $n_\mu e^\mu_a = 0$ considerably simplifies the previously derived results. We collect these simplified results and conclude

$$t^\mu = N n^\mu + N^a e^\mu_a \,, \tag{3.20a}$$

$$E^a_\mu = e^a_\mu - \frac{\varepsilon}{N} N^a n_\mu \,, \tag{3.20b}$$

$$\delta^\mu_\nu = e^\mu_a e^a_\nu + \varepsilon n^\mu n_\nu \,. \tag{3.20c}$$

These expressions may be inserted into the frame and coframe decompositions (3.6), (3.8) to obtain

$$\vartheta_\mu = e^a_\mu \varphi_a + \frac{\varepsilon}{N} n_\mu \varphi \,, \tag{3.21a}$$

$$\theta^\mu = e^\mu_a \phi^a + N n^\mu \phi \,, \tag{3.21b}$$

where we introduce

$$\phi^a := \tilde{\phi}^a + N^a \phi \,, \qquad \varphi := \tilde{\varphi} - N^a \varphi_a \,. \tag{3.22}$$

This is the adapted frame we aimed for. To see this, we verify that now

$$n = n^\mu \vartheta_\mu = \frac{1}{N} \varphi \tag{3.23}$$

proves that φ is indeed aligned with the normal vector[1], while its metric dual $\tilde{n} = \flat(n)$ is aligned with ϕ as

$$\tilde{n} = n_\mu \theta^\mu = \varepsilon N \phi \,. \tag{3.24}$$

For the remaining directions in the frame decomposition, we recall that we identified

$$e_a = e^\mu_a \vartheta_\mu = \varphi_a \tag{3.25}$$

to be tangent to the hypersurface. The normality condition (3.19) may thus be written as $g(e_a, n) = \frac{1}{N} g(\varphi_a, \varphi) = 0$. Furthermore, the frame transformation (3.22) keeps the duality (3.9) of frame and coframe invariant, that is

$$\phi^a(\varphi_b) = \delta^a_b \,, \qquad \phi(\varphi) = 1 \,, \qquad \phi^a(\varphi) = 0 = \phi(\varphi_a) \,. \tag{3.26}$$

Again, this is expected from the geometrical introduction in chapter 2 since adapting

[1] At this point, it seems to be straightforward to replace $\varphi = Nn$ in the frame decomposition. While this would immediately give the frame the intuition of an alignment with hypersurface and normal direction, we adhere to the prefactors as given in the previous calculation to match conventions in literature. These conventions are chosen such that $\varphi = Nn$ is the *normal evolution vector* if one considers foliations of manifolds [97]. Note, however, that the re-definition absorbing the prefactor N corresponds to a frame choice which entirely eliminates the lapse function N from all equations.

the frame is just a $\mathrm{GL}(\dim\mathcal{M},\mathbb{R})$ transformation on the frame bundle which we will examine in more detail shortly.

Before we do so, let us conclude the discussion of the frame and coframe decompositions by noting that all of the above alignments allow us to rewrite these decompositions as

$$\vartheta_\mu = e^a_\mu e_a + \varepsilon n_\mu n\,, \tag{3.27a}$$
$$\theta^\mu = e^\mu_a \epsilon^a + \varepsilon n^\mu \tilde{n}\,, \tag{3.27b}$$

where we use that $\epsilon^a := e^a_\mu \theta^\mu = \phi^a$ is the dual basis to e_a. In this form, the geometric interpretation of the frame and coframe as being adapted to the hypersurface becomes immediately manifest. Furthermore, frame and coframe decomposition have a very similar structure. Nevertheless, the form (3.27) of the decomposition does not preserve the duality conditions (3.26) of frame and coframe, and we will therefore refrain from using it in the following.

Let us now return to the discussion of the frame decomposition as a $\mathrm{GL}(m,\mathbb{R})$ transformation. Since the frame bundle $L\mathcal{M} \xrightarrow{\pi} \mathcal{M}$ is a principal $\mathrm{GL}(m,\mathbb{R})$-bundle, it is equipped with a right action $\triangleleft$. For any frame $\bar{\vartheta}_\mu$, we defined the action of a group element $\Lambda^\mu{}_\nu$ by $\vartheta_\mu := \bar{\vartheta}_\mu \triangleleft \Lambda = \bar{\vartheta}_\nu \Lambda^\nu{}_\mu$. Thus, the adapted frame (3.21) may be generated from an arbitrary frame by applying the gauge transformation

$$\begin{aligned} \Lambda^\mu{}_\nu &= e^a_\nu \delta^\mu_a + \frac{\varepsilon}{N} n_\nu \delta^\mu_{m-1}\,, \\ \Lambda^{-1\mu}{}_\nu &= e^\mu_a \delta^a_\nu + N n^\mu \delta^{m-1}_\nu\,. \end{aligned} \tag{3.28}$$

We use this transformation for transforming an arbitrary connection one-form $\bar{\omega}^\mu{}_\nu$ by means of (2.25), yielding

$$\bar{\omega}^\mu{}_\nu \triangleleft \Lambda = e^\mu_a e^b_\nu \omega^a{}_b + e^\mu_a \mathrm{d} e^a_\nu + \text{terms linear in } n^\mu \text{ or } n_\nu\,. \tag{3.29}$$

We denote the such generated connection one-form as $\omega^\mu{}_\nu := \bar{\omega}^\mu{}_\nu \triangleleft \Lambda$ for consistency with the frame notation (3.21). For eliminating the terms linear in n^μ or n_ν, we contract (3.29) with $e^a_\mu e^\nu_b$ and use the normality condition (3.19). This yields $e^a_\mu e^\nu_b \omega^\mu{}_\nu = \omega^a{}_b + e^\mu_b \mathrm{d} e^a_\mu$ which we solve for the hypersurface connection $\omega^a{}_b$ to finally obtain

$$\omega^a{}_b = e^a_\mu e^\nu_b \omega^\mu{}_\nu + e^a_\mu \mathrm{d} e^\mu_b\,. \tag{3.30}$$

This is the transformation of the connection to the hypersurface. For calculational purposes, it is useful to rewrite this as[2]

$$e^a_\mu D e^\mu_b = 0\,. \tag{3.31}$$

We may repeat this calculation for the metric in order to transform it to the hypersurface. Recall that the metric is a $(0,2)$-tensor field and the tensor bundle is an associated bundle to the frame bundle. This yields the by now familiar expression $\gamma_{ab} = e^\mu_a e^\nu_b g_{\mu\nu}$ for the hypersurface metric. However, it is useful to furthermore decompose the manifold metric g into hypersurface tangent and normal components. To that effect, we simply evaluate $g_{\mu\nu} := g(\vartheta_\mu, \vartheta_\nu)$ using the frame decomposition (3.21) to obtain

$$g_{\mu\nu} = e^a_\mu e^b_\nu \gamma_{ab} + \varepsilon n_\mu n_\nu\,. \tag{3.32}$$

Reconstructing the metric from its components as $g = g_{\mu\nu}\theta^\mu \otimes \theta^\nu$ by means of the frame decomposition (3.21), the components (3.32) imply[3]

$$g = \gamma_{ab}\phi^a \otimes \phi^b + \varepsilon N^2 \phi \otimes \phi\,. \tag{3.33}$$

The alignment of the adapted frame (3.21) with the hypersurface normal and tangent directions may be used to project all tensors to these directions. Expressing an arbitrary vector $A = A^\mu \vartheta_\mu$ and an arbitrary one-form $B = B_\mu \theta^\mu$ in the decomposed frame (3.21) yields

$$A = A^\mu e^a_\mu \varphi_a + \frac{\varepsilon}{N} A^\mu n_\mu \varphi = \mathrm{tang}(A) + \mathrm{norm}(A)\,, \tag{3.34a}$$

$$B = B_\mu e^\mu_a \phi^a + N B_\mu n^\mu \phi = \mathrm{tang}(B) + \mathrm{norm}(B)\,, \tag{3.34b}$$

which generalizes to tensors of higher rank by means of reconstructing them in terms of their components and the corresponding frames. Note that the decomposition of vectors and covectors into tangent and normal contributions is analogous to the decomposition of vector fields in vertical and horizontal parts which we interpreted as the introduction of a connection on a manifold in chapter 2. Here, we define the

[2]Since $(\varphi_0, \ldots, \varphi_{m-2}, \varphi)$ is a frame on $\mathcal{M}$, the covariant exterior derivative acts on hypersurfaces indices by contracting them with the hypersurface connection $\omega^a{}_b$. Hence, we have $De^\mu_a = \mathrm{d}e^\mu_a + \omega^\mu{}_\nu e^\nu_a - \omega^b{}_a e^\mu_b$.

[3]By means of (3.22), the metric decomposition (3.33) becomes $g = \gamma_{ab}(\tilde\phi^a + N^a\phi) \otimes (\tilde\phi^b + N^b\phi) + \varepsilon N^2 \phi \otimes \phi$ in the unadapted frame. While this is not relevant for the remainder of this thesis, the latter expression in the unadapted frame is particularly interesting for the Hamiltonian formulation of general relativity, where the shift vector N^a is used as a Lagrange multiplier for momentum conservation [97, 99–101].

tangent contribution to be $\text{tang}(A) \coloneqq A^\mu e^a_\mu \varphi_a$ while the normal component of A is $\text{norm}(A) \coloneqq \frac{\varepsilon}{N} A^\mu n_\mu \varphi$. Calling these components of A tangent and normal is reasonable since

$$\begin{aligned} g(\text{tang}(A), n) &= 0 & &\text{while} \quad g(\text{norm}(A), n) = g(A, n)\,, \\ g(\text{tang}(A), e_a) &= g(A, e_a) & &\text{while} \quad g(\text{norm}(A), e_a) = 0\,. \end{aligned} \tag{3.35}$$

This is what we intuitively expect: Contracting vector components with e^a_μ projects them to the hypersurface, while contraction with n_μ yields their normal contribution. This generalizes straightforwardly to the covectors in (3.34b) and, hence, to tensors of arbitrary rank. Note that this is also consistent with the interpretation of e^μ_a as a pull-back to the hypersurface.

Before we proceed by investigating the normal vector and the hypersurface basis by considering their field strengths, let us collect the results of this introduction to space- and timelike hypersurfaces in the normal vector field approach. We defined such hypersurfaces Σ by giving a smooth unit normal vector field n and called vector fields $v \in \Gamma(T\mathcal{M})$ tangent to Σ if $g(n, v) = 0$. We decomposed frames and coframes to single out the normal direction. Aligning one of the frame vectors with the unit normal lead us to the decomposition of the form

$$\vartheta_\mu = e^a_\mu \varphi_a + \frac{\varepsilon}{N} n_\mu \varphi\,, \tag{3.36a}$$

$$\theta^\mu = e^\mu_a \phi^a + N n^\mu \phi\,, \tag{3.36b}$$

where $\varphi_a = e_a = e^\mu_a \vartheta_\mu$ was interpreted as a hypersurface basis. We saw that e_a is indeed tangent to the hypersurface by deriving the normality condition

$$g(e_a, n) = e^\mu_a n_\mu = 0\,. \tag{3.37}$$

This in particular simplified the decomposition of unity as

$$\delta^\mu_\nu = e^\mu_a e^a_\nu + \varepsilon n^\mu n_\nu\,, \tag{3.38}$$

where we used the inverse of the hypersurface metric $\gamma_{ab} = g(\varphi_a, \varphi_b) = e^\mu_a e^\nu_b g_{\mu\nu}$ to raise Latin indices. This may as well be used to derive the decomposition of the metric,

$$g_{\mu\nu} = e^a_\mu e^b_\nu \gamma_{ab} + \varepsilon n_\mu n_\nu\,. \tag{3.39}$$

By analyzing the frame decomposition as a $\text{GL}(m, \mathbb{R})$ transformation on the frame bundle, we concluded that the hypersurface connection ${\omega^a}_b$ may be obtained from the manifold connection $\omega^{\mu}{}_{\nu}$ by means of

$$\omega^{a}{}_{b} = e^{a}_{\mu} e^{\nu}_{b} \omega^{\mu}{}_{\nu} + e^{a}_{\mu} \mathrm{d} e^{\mu}_{b} \qquad \Leftrightarrow \qquad e^{a}_{\mu} D e^{\mu}_{b} = 0 \,. \tag{3.40}$$

Finally, we noted that we may decompose any tensor into hypersurface normal and tangent contributions. In particular, we may project an index to the hypersurface by contracting it with e^{μ}_{a} such that e^{μ}_{a} acts as a pull-back to the hypersurface. We obtain the remaining pieces of the tensors from contracting indices with the unit normal n^{μ}. These results are the key foundations we need for the following sections and chapters.

Recall that we studied frames, connection one-forms and metric tensors by introducing their field strengths as covariant exterior derivatives in chapter 2. The new field defining space- and timelike hypersurfaces is the unit normal vector field n, which induces the hypersurface frame e_a via the frame decomposition. Thus, we will proceed to examine hypersurfaces by calculating the field strengths of these vector fields.

3.1.2. Field strengths

We introduced hypersurfaces by defining their unit normal vector field n. Since vector fields are identified with $(1,0)$-tensor fields on finite dimensional manifolds, their covariant exterior derivatives are $(1,1)$-tensor fields. Recall that we aimed to have a formalism based solely on differential forms, to which effect we need to utilize the same trick as in defining the non-metricity one-form in section 2.6. That is, we consider the covariant exterior derivative of the components n^{μ} of the normal vector field in a local frame. Hence, we define the one-form

$$K^{\mu} := D n^{\mu} \tag{3.41}$$

as the field strength of the normal vector. Using the decomposition of unity (3.38), we obtain the tangent and normal contributions of this one-form as

$$K^{\mu} = \delta^{\mu}_{\nu} D n^{\nu} = e^{\mu}_{a} e^{a}_{\nu} D n^{\nu} + \varepsilon n^{\mu} n_{\nu} D n^{\nu} \,. \tag{3.42}$$

Comparing to (3.34b), we observe that the latter result decomposes the covector K^{μ} into hypersurface tangent and normal contributions.

To further simplify this decomposition of K^{μ}, note that its normal component is fully determined by non-metricity. In particular, $\varepsilon = g(n, n)$ being constant implies

$$Q_{\mathbf{nn}} := n^{\mu} n^{\nu} Q_{\mu\nu} = 2 n_{\mu} D n^{\mu} \,. \tag{3.43}$$

For analyzing the hypersurface tangent component of K^μ, note that $K^a := e^a_\mu D n^\mu$ is the pull-back of K^μ to the hypersurface. For giving an interpretation of this one-form, it is useful to explicitly consider its component expression in a coframe. That is,

$$K^a = K^a{}_\mu \theta^\mu = K^a{}_b \phi^b + N e^a_\mu a^\mu \phi \,, \tag{3.44}$$

where we introduce $K^a{}_b := e^a_\mu e^\nu_b \nabla_\nu n^\mu$ and $a^\mu := n^\nu \nabla_\nu n^\mu$. It seems to put formalism to an extreme to assign a name to these components of the components of K^μ, while all the relevant information is already contained in K^μ. However, the new quantities $K^a{}_b$ and a^μ are relevant in terms of understanding the field strength of the normal vector. In particular, $K^a{}_b$ is known as the *extrinsic curvature* in standard general relativity. The extrinsic curvature gives information about how a hypersurface is embedded into a manifold. If for example the (intrinsic) curvature $\Omega^a{}_b$ of a hypersurface vanishes, it may nevertheless be embedded such that it has non-trivial extrinsic curvature. In the simplest case, a straight line embedded in $\mathbb{R}^2$ is not extrinsically curved, while a circle is. Note that both cases embed the intrinsically flat manifold $\mathbb{R}^1$, but the embedding function and thus the normal vectors differ. In this sense, the extrinsic curvature contributes to the characterization of the shape of a hypersurface. Further interesting interpretations may be assigned to $K^a{}_b$ which are summarized in [97]. This review also clarifies that a^μ is the acceleration of the Eulerian observer, that is an observer moving with velocity n.

We already noted that most literature uses the normal one-form instead of our vector field approach to hypersurfaces. For comparison, it is therefore useful to define its covariant exterior derivative

$$\tilde{K}_\mu := D n_\mu \tag{3.45}$$

in addition to K^μ. Both definitions differ only due to non-metricity as $\tilde{K}_\mu = K_\mu - n^\nu Q_{\mu\nu}$. Hence, the conceptual interpretation of $\tilde{K}_a := e^\mu_a D n_\mu$ as well as its components $\tilde{K}_{ab} := e^\mu_a e^\nu_b \nabla_\nu n_\mu$ and $\tilde{a}_\mu := n^\nu \nabla_\nu n_\mu$ remain the same as for K^μ. Due to this interpretation, we call both K^a and $\tilde{K}_a$ the *extrinsic curvature one-forms* of the hypersurface.

This concludes the discussion of the normal vector field strength, but it remains to calculate the field strength of the hypersurface frame $e_a = e^\mu_a \vartheta_\mu$. Just as for the metric tensor and the normal vector, we consider the one-form $D e^\mu_a$ instead of the covariant exterior derivative of e_a. Contracting with the decomposition of unity (3.38) as before, we obtain

$$De^\mu_a = \delta^\mu_\nu De^\nu_a = e^\mu_b e^b_\nu De^\nu_a + \varepsilon n^\mu n_\nu De^\nu_a \,. \tag{3.46}$$

We observe that the hypersurface tangent component of De^μ_a vanishes due to the transformation of the connection to the hypersurface (3.40). For the normal contribution, we use the normality condition (3.37) to conclude

$$De^\mu_a = -\varepsilon n^\mu \tilde{K}_a \,. \tag{3.47}$$

An analogous calculation for the hypersurface coframe $\epsilon^a = e^a_\mu \theta^\mu$ yields

$$De^a_\mu = -\varepsilon n_\mu K^a \,. \tag{3.48}$$

These expressions for the field strengths of the hypersurface frame and coframe are the differential form versions of the *Gauß-Weingarten equation*. Note that (3.48) implies that the field strength of the tangent coframe is completely fixed by the extrinsic curvature, that is, the field strength of the normal vector.

In summary, the field strengths of the unit normal n and its metric dual $\tilde{n}$ are

$$K^\mu = Dn^\mu = e^\mu_a K^a + \frac{\varepsilon}{2} n^\mu Q_{\mathbf{nn}} \qquad \text{and} \tag{3.49a}$$

$$\tilde{K}_\mu = Dn_\mu = e^a_\mu \tilde{K}_a - \frac{\varepsilon}{2} n_\mu Q_{\mathbf{nn}} = K_\mu - n^\nu Q_{\mu\nu} \,, \tag{3.49b}$$

respectively. These expressions also fix the field strength of the hypersurface frame e_a and its coframe ϵ^a as

$$De^\mu_a = -\varepsilon n^\mu \tilde{K}_a \,, \tag{3.50a}$$

$$De^a_\mu = -\varepsilon n_\mu K^a \,. \tag{3.50b}$$

Beyond these equations, we will need the second covariant exterior derivatives of the hypersurface frame as well as its unit normal. We note that the hypersurface curvature is given by $\Omega^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$ to obtain

$$D^2 e^\mu_a = \Omega^\mu{}_\nu e^\nu_a - \Omega^b{}_a e^\mu_b \,, \tag{3.51a}$$

$$D^2 e^a_\mu = -\Omega^\nu{}_\mu e^a_\nu + \Omega^a{}_b e^b_\mu \,, \tag{3.51b}$$

$$DK^\mu = D^2 n^\mu = \Omega^\mu{}_\nu n^\nu \,, \tag{3.51c}$$

$$D\tilde{K}_\mu = D^2 n_\mu = -\Omega^\nu{}_\mu n_\nu \,. \tag{3.51d}$$

The latter equations are the differential form version of the *Ricci identity*.

This discussion of the field strengths concludes the short introduction to space- and timelike hypersurfaces. Using a normal vector field approach to these hypersurfaces, we have been able to reproduce the expressions from the constant function and the embedding approaches. The particular new aspect of my approach is that I examine hypersurfaces based on a generic decomposition of frames and coframes which corresponds to a $\mathrm{GL}(m,\mathbb{R})$ transformation. This is advantageous because on the frame bundle, the transformation of every tensor is determined by this $\mathrm{GL}(m,\mathbb{R})$ transformation. Moreover, this perspective is important for studying manifolds with torsion which is the coframes' field strength.

In principle, the ideas developed in this section apply to lightlike hypersurfaces as well. However, the normal vector being lightlike yields some subtleties which make the treatment of lightlike hypersurfaces more involved. We are going to discuss this case next.

3.2. Lightlike hypersurfaces

Our method for defining lightlike hypersurfaces Σ is the same as for space- and timelike ones. That is, we consider a normal vector field ζ which now is lightlike. However, we will quickly encounter that this likeness of ζ has important consequences that make lightlike hypersurfaces substantially different from non-lightlike ones. In particular, we will see that there is no metric[4] on the $(m-1)$-dimensional hypersurface. We will develop the normal vector field approach to lightlike hypersurfaces in the following subsection before we turn to the discussion of the field strengths. Lightlike hypersurfaces have been discussed to a lesser extent than non-lightlike ones in the literature. I will in particular relate my normal vector field approach to the definitions in [96] and [98]. However, the results presented in this section reach substantially beyond the realm of [96, 98]. The normal vector field approach allows me to formulate a fully self-consistent definition of lightlike hypersurfaces. In this formulation, I will derive the decomposition of frames and coframes for lightlike hypersurfaces and adapt them to the tangent and non-tangent directions. This will induce the decompositions of the connection, the metric and the field strengths of the tangent and normal directions in geometries with curvature, torsion and non-metricity. These results are new and have not been published before. Moreover, we will gain detailed insights on the lightlike $(m-1)$-hypersurface as well as an immersed hypersurface of one dimension less

[4] In physics, it is often said that the metric is degenerate in this case. However, we defined a metric to be non-degenerate, so a degenerate metric is technically not a metric.

and consider the decompositions of frames, connections and metric tensors in both formalisms.

3.2.1. The normal vector field approach

For a lightlike hypersurface, the normal vector is lightlike and thus fulfills $g(\zeta, \zeta) = 0$. Unlike for non-lightlike hypersurfaces, $g(\zeta, \zeta)$ cannot be used to normalize the normal vector field. Instead, every multiple $\alpha\zeta$ for nowhere vanishing $\alpha \in C^\infty(\mathcal{M})$ is a normal vector field of Σ. Hence, it is more appropriate to consider an equivalence class $[\zeta]$ of normal vectors, and we will work with a nowhere vanishing representative of this equivalence class denoted by k.

This representative may be explicitly constructed in the literature approach which interprets hypersurfaces as those points of a manifold for which some function f is constant, that is

$$\Sigma_\lambda := \{p \in \mathcal{M} | f(p) = \lambda\} . \tag{3.52}$$

Defining a hypersurface like that, the function $f : \mathcal{M} \longrightarrow \mathbb{R}$ may only change in directions which are non-tangent to the hypersurface, such that $\mathrm{d}f$ is a non-tangent one-form. Compared to the normal vector field approach, $\mathrm{d}f$ could thus be interpreted as the normal one-form $\tilde{k}$ obtained from the normal vector field k by means of the musical isomorphism as $\tilde{k} := \flat(k)$. Calling vectors and covectors non-tangent or normal to lightlike hypersurfaces is, however, not equivalent to each other. It is therefore necessary to examine these notions in more detail.

Note that by our definition a vector field $v \in \Gamma(T\mathcal{M})$ is said to be tangent to a hypersurface if it fulfills $g(v, k) = 0$. Hence, in particular the normal vector field k itself is tangent to the hypersurface in addition to being normal to it. Nevertheless, we want to interpret the hypersurface Σ as an $(m-1)$-dimensional submanifold of $\mathcal{M}$. The normal vector being one of the vectors spanning this $(m-1)$-dimensional space, we need an additional vector l in order to create a frame decomposition on $\mathcal{M}$. In particular, l needs to be non-tangent to Σ, that is $g(k, l) \neq 0$. We fix the length of l such that $g(k, l) = \frac{1}{\varepsilon}$ is constant. Keeping the latter relation invariant if we choose a different representative $k \mapsto \alpha k$ of $[k]$ implies that we impose $l \mapsto \frac{1}{\alpha} l$ and, hence, l may be considered as an element of an equivalence class as well. It is therefore common to choose l to be lightlike such that $g(l, l) = 0$. Note that l is not aligned with k since $g(k, l) \neq 0$, so in particular it does not fulfill $g(v, l) = 0$ for all vectors tangent to the hypersurface. Hence, it is by construction neither normal nor tangent to the hypersurface, while the lightlike normal vector k is both.

This special behavior of the normal vector field k may also be seen from the frame decomposition. In analogy to the non-lightlike case, we decompose frames as

$$\vartheta_\mu = E^a_\mu \varphi_a + \varepsilon k_\mu \tilde{\varphi} \,. \tag{3.53}$$

Naively, we would like to align the basis vector $\tilde{\varphi}$ with the normal vector k as before. However, $g(k,k) = 0$ immediately implies

$$k = k^\mu \vartheta_\mu = k^\mu E^a_\mu \varphi_a + \varepsilon k^\mu k_\mu \tilde{\varphi} = k^\mu E^a_\mu \varphi_a \,, \tag{3.54}$$

so that an alignment is not possible. This is intuitively clear from the consideration above: If we want to interpret φ_a as the hypersurface basis vectors and k is tangent to the hypersurface, it needs to be expandable in the hypersurface basis. Thus, $\tilde{\varphi}$ is a vector which does not interfere with the hypersurface. Since this was the interpretation of the non-normal, non-tangent vector l, it is straightforward to align $\tilde{\varphi}$ with l. For this purpose, we calculate

$$l = l^\mu \vartheta_\mu = l^\mu E^a_\mu \varphi_a + \varepsilon l^\mu k_\mu \tilde{\varphi} = l^\mu E^a_\mu \varphi_a + \tilde{\varphi} \,. \tag{3.55}$$

Hence, the most straightforward way to align l with $\tilde{\varphi}$ is by defining $\varphi \coloneqq \tilde{\varphi} + l^\mu E^a_\mu \varphi_a$. This may be reinstated into the frame decomposition, yielding

$$\vartheta_\mu = (E^a_\mu - \varepsilon k_\mu l^\nu E^a_\nu)\varphi_a + \varepsilon k_\mu \varphi \,. \tag{3.56}$$

In analogy to the non-lightlike case, we would like to simplify the new coefficient of φ by means of the hypersurface metric and its inverse. However, this metric does not exist for lightlike hypersurfaces. This is most easily seen in the embedding picture.

Let us hence proceed by considering the embedding perspective on lightlike hypersurfaces. That is, we consider an $(m-1)$-dimensional manifold $\hat{\Sigma}$ and a smooth map $\Phi : \hat{\Sigma} \longrightarrow \Sigma \subseteq \mathcal{M}$ from this manifold into $\mathcal{M}$. We already explored embeddings in the previous section, in which case Φ is a homeomorphism. While it is often said that lightlike hypersurfaces are embedded in $\mathcal{M}$ as well, we generically need to consider a more general structure. For example, we have tangent vectors in all lightlike directions at the center of the light cone. Hence, it is impossible to construct an invertible push-forward from the $(m-1)$-dimensional manifold $\hat{\Sigma}$ to Σ, and Φ is therefore not an embedding. Instead, we will only require the push-forward Φ_{*p} induced by Φ at any point $p \in \mathcal{M}$ to be injective in order to preserve the vector structure of the $(m-1)$-dimensional manifold. Such a structure is called *immersion*. Note that every embedding is an immersion, and thus an immersion may be interpreted as a generalized notion. Since immersions are smooth maps, we may still push forward vectors by means of $\Phi_{*p} : T_p\hat{\Sigma} \longrightarrow T_{\Phi(p)}\mathcal{M}$ and pull back $(0, n)$-tensors using $\Phi_p^* : T_n^0(T_{\Phi(p)}^*\mathcal{M}) \longrightarrow T_n^0(T_p^*\hat{\Sigma})$, but these maps are not necessarily invertible. One particularly important example is the pull-back of the metric, which we define as

$$\gamma := \Phi^* g \tag{3.57}$$

in analogy to the non-lightlike case. There, we interpreted γ as the hypersurface metric. In the immersion perspective, lightlike hypersurfaces are in fact defined exactly by γ not being a metric. To see that this definition coincides with the one we gave in the normal vector field approach, we revisit the definitions of metrics and pull-backs. A metric was defined to be a symmetric, non-degenerate $(0, 2)$-tensor. Since the pull-back yields a symmetric $(0, 2)$-tensor, we must break non-degeneracy to obtain a non-metric tensor γ. By definition this implies that there is some non-vanishing vector field $v \in \Gamma(T\hat{\Sigma})$ such that $\gamma(v, w) = 0$ for all $w \in \Gamma(T\hat{\Sigma})$. Since $v \in \Gamma(T\hat{\Sigma})$ itself is a particular such vector field w, we may use the definition of the pull-back to conclude

$$0 = \gamma(v, v) = \Phi^* g(v, v) = g(\Phi_* v, \Phi_* v)\,, \tag{3.58}$$

that is, $\Phi_* v$ is a lightlike vector field. Thus, we have found that a lightlike vector field is tangent to the hypersurface, and we denote this vector field by $k = \Phi_* v$ for consistency with the normal vector field approach. But $\gamma(v, w) = 0$ does not only hold for $w = v$, it holds for all $w \in \Gamma(T\hat{\Sigma})$. That is, $g(k, \Phi_* w) = 0$ for all tangent vectors $\Phi_* w$, and thus k is a normal vector as well. Therefore, the condition that $\gamma = \Phi^* g$ is degenerate is equivalent to defining lightlike hypersurfaces by demanding the normal vector field to be lightlike.

There is a way to circumvent the subtleties involving lightlike hypersurfaces which became obvious in the immersion approach. In particular, it is often convenient to have a metric on a hypersurface. Recall that the non-degeneracy of the metric on $\hat{\Sigma}$ is only spoiled by the tangent lightlike vector v fulfilling $\gamma(v, w) = 0$ for all w. Hence, the basic idea is to mod out this lightlike vector from the hypersurface. Technically, this implies that we are considering yet another hypersurface $\check{\Sigma}$ of dimension $(m-2)$. This hypersurface is immersed in $\hat{\Sigma}$ by the smooth map $\hat{\Phi} : \check{\Sigma} \longrightarrow \hat{\Sigma}$ such that $\sigma := \hat{\Phi}^* \gamma$ is a non-degenerate metric. Note that $\Phi \circ \hat{\Phi}$ is a composition of smooth maps and thus immerses the $(m - 2)$-dimensional hypersurface $\check{\Sigma}$ directly in $\mathcal{M}$.

In the normal vector field approach, this double immersion appears much less technical. In particular, we decomposed frames as

$$\vartheta_\mu = E^a_\mu \varphi_a + \varepsilon k_\mu \tilde{\varphi} \tag{3.59}$$

and concluded that the normal vector takes the form $k = k^\mu E^a_\mu \varphi_a$ in this frame decomposition. Modding out the lightlike vector from the hypersurface tangent vectors is now straightforward. To that effect, we choose the frame such that the k direction is separated from the remaining φ_a vectors. That is, we align φ_{m-2} with k. Since $g(k,l) = \frac{1}{\varepsilon}$ is the only non-vanishing pseudo inner product involving k, the prefactor of the singled out direction needs to be εl_μ. Putting these things together, we set $E^{m-2}_\mu \varphi_{m-2} = \varepsilon l_\mu \tilde{\psi}$ such that the frame decomposition takes the form

$$\vartheta_\mu = E^A_\mu \varphi_A + \varepsilon l_\mu \tilde{\psi} + \varepsilon k_\mu \tilde{\varphi}\,, \tag{3.60}$$

where capital Latin indices take their values on the $(m-2)$-dimensional manifold $\check{\Sigma}$. Note that from a mathematical point of view, l and k are entirely equivalent in this $(m-2)$-dimensional perspective. For this point of view on lightlike hypersurfaces it is thus completely irrelevant if we started with k or l as the vector field which is both normal and tangent. This will reflect in all equations derived from the $(m-2)$ perspective which need to be symmetric under exchange of k and l.

Now we arrived at the point where we may proceed like in the non-lightlike case since we have a metric $\sigma_{AB} := e^\mu_A e^\nu_B g_{\mu\nu}$ on the hypersurface. Hence, we consider the coframe decomposition as

$$\theta^\mu = e^\mu_A \tilde{\phi}^A + u^\mu \Psi + v^\mu \phi\,, \tag{3.61}$$

where the duality of frames and coframes is used as

$$\tilde{\phi}^A(\varphi_B) = \delta^A_B\,, \qquad \phi(\tilde{\varphi}) = 1 = \Psi(\psi)\,, \tag{3.62}$$

while all the remaining pairings are vanishing, that is $\tilde{\phi}^A(\tilde{\varphi}) = \tilde{\phi}^A(\tilde{\psi}) = \phi(\varphi_A) = \phi(\tilde{\psi}) = \Psi(\varphi_A) = \Psi(\tilde{\varphi}) = 0$.

Just as in the non-lightlike case, there are constraints which relate the coefficients in the frame and coframe decompositions. To see that, we first evaluate the duality condition of frame and coframe to obtain

$$\delta^\mu_\nu = \theta^\mu(\vartheta_\nu) = e^\mu_A E^A_\nu + \varepsilon u^\mu l_\nu + \varepsilon v^\mu k_\nu\,. \tag{3.63}$$

For relating the coefficients in this expression, we need to use the relations $g(k,k) = 0 = g(l,l)$, $g(k,l) = \frac{1}{\varepsilon}$ and $\sigma_{AB} := e^\mu_A e^\nu_B g_{\mu\nu}$ we already discussed. To that end, we contract the decomposition of unity with k^ν, l^ν and $e^\rho_A g_{\rho\mu}$, yielding

$$\begin{aligned} k^\mu &= k^\nu \delta^\mu_\nu = k^\nu E^A_\nu e^\mu_A + u^\mu \,, \\ l^\mu &= l^\nu \delta^\mu_\nu = l^\nu E^A_\nu e^\mu_A + v^\mu \,, \\ e^\rho_A g_{\rho\nu} &= e^\rho_A g_{\rho\mu} \delta^\mu_\nu = \sigma_{AB} E^B_\nu + \varepsilon e^\mu_A u_\mu l_\nu + \varepsilon e^\mu_A v_\mu k_\nu \,. \end{aligned} \tag{3.64}$$

Since σ is a non-degenerate metric, we may invert it and define $e^A_\mu := \sigma^{AB} e^\nu_B g_{\mu\nu}$ in order to further simplify the above contractions. In particular, we solve them for u^μ, v^μ and E^A_μ to obtain

$$\begin{aligned} u^\mu &= k^\mu - k^\nu E^A_\nu e^\mu_A \,, \\ v^\mu &= l^\mu - l^\nu E^A_\nu e^\mu_A \,, \\ E^A_\mu &= e^A_\mu - \varepsilon e^A_\nu u^\nu l_\mu - \varepsilon e^A_\nu v^\nu k_\mu \,. \end{aligned} \tag{3.65}$$

These relations suffice to eliminate u^μ, v^μ and E^A_μ from the decomposition of unity. That is, inserting (3.65) into (3.63) yields

$$\delta^\mu_\nu = e^\mu_A e^A_\nu - \varepsilon e^\mu_A e^A_\rho k^\rho l_\nu - \varepsilon e^\mu_A e^A_\rho l^\rho k_\nu + \varepsilon k^\mu l_\nu + \varepsilon l^\mu k_\nu \,. \tag{3.66}$$

However, this may be simplified even further using the trace $\delta^\mu_\mu = \dim(\mathcal{M})$. Evaluating this trace by means of (3.66), we conclude $0 = \sigma^{AB} e^\mu_A e^\rho_B k_\mu l_\rho$, which in turn may be used in the contraction $k_\nu = k_\mu \delta^\mu_\nu$ to obtain $0 = k_\mu e^\mu_A e^A_\rho (\delta^\rho_\nu - \varepsilon k^\rho l_\nu)$. This is solved by $0 = \delta^\rho_\nu - \varepsilon k^\rho l_\nu$, but the trace of the latter expression yields $\dim(\mathcal{M}) = 1$ which is incompatible with the hypersurface formalism. Hence, we conclude $0 = k_\mu e^\mu_A e^A_\nu$. Contracting the latter equation with e^ν_B finally yields

$$0 = k_\mu e^\mu_A \,. \tag{3.67}$$

This is the normality condition we would have expected between the normal vector k and the $m-2$ remaining hypersurface vectors $e_A := e^\mu_A \vartheta_\mu$. In index-free notation, the normality condition reads $g(k, e_A) = 0$, so e_A is indeed tangent to the hypersurface. Furthermore, this justifies to consider e^μ_A as a pull-back to the hypersurface, connecting the normal vector field approach to the immersion approach again.

We could now repeat the same calculation for l, but from the symmetry under exchange of k and l it is immediately obvious that this calculation yields

$$0 = l_\mu e^\mu_A \,, \tag{3.68}$$

which in index-free notation is $g(l, e_A) = 0$. Hence, the vector field l is normal to the $(m-2)$-dimensional hypersurface just like k. We use these two normality conditions to finally simplify the decomposition of unity (3.66) as

$$\delta^\mu_\nu = e^\mu_A e^A_\nu + \varepsilon k^\mu l_\nu + \varepsilon l^\mu k_\nu \,. \tag{3.69}$$

Furthermore, we are now in the position to simplify the frame and coframe decompositions as

$$\vartheta_\mu = e^A_\mu \varphi_A + \varepsilon l_\mu \psi + \varepsilon k_\mu \varphi \,, \tag{3.70a}$$
$$\theta^\mu = e^\mu_A \phi^A + k^\mu \Psi + l^\mu \phi \,, \tag{3.70b}$$

where we introduce

$$\begin{aligned} \psi &:= \tilde{\psi} + k^\mu E^A_\mu \varphi_A \,, \\ \varphi &:= \tilde{\varphi} + l^\mu E^A_\mu \varphi_A \,, \\ \phi^A &:= \tilde{\phi}^A - k^\mu E^A_\mu \Psi - l^\mu E^A_\mu \phi \,. \end{aligned} \tag{3.71}$$

This redefinition leaves the duality conditions (3.62) of frames and coframes invariant, as we have

$$\phi^A(\varphi_B) = \delta^A_B \,, \qquad \phi(\varphi) = 1 = \Psi(\psi) \tag{3.72}$$

with all other pairings $\phi^A(\varphi) = \phi^A(\psi) = \phi(\varphi_A) = \phi(\psi) = \Psi(\varphi_A) = \Psi(\varphi) = 0$ vanishing. Note that the definition $\varphi := \tilde{\varphi} + l^\mu E^A_\mu \varphi_A$ is consistent with our previous $(m-1)$-dimensional hypersurface version $\varphi := \tilde{\varphi} + l^\mu E^a_\mu \phi_a$ since we aligned $E^{m-2}_\mu \varphi_{m-2} = \varepsilon l_\mu \tilde{\psi}$ and $g(l, l) = 0$.

The decomposition (3.70) of frames and coframes is the adapted decomposition we were aiming for. This adapted decomposition is an original result first presented in this thesis. To see that this is indeed correctly adapted to the tangent and non-tangent directions, we derive

$$\begin{aligned} k &= \psi \,, & l &= \varphi \,, & e_A &= \varphi_A \,, \\ \tilde{k} &= \frac{1}{\varepsilon}\phi \,, & \tilde{l} &= \frac{1}{\varepsilon}\Psi \,, & \epsilon^A &= \phi^A \end{aligned} \tag{3.73}$$

by a straightforward calculation, where $\epsilon^A := e^A_\mu \theta^\mu$ is the dual basis of e_A. Hence, we could give the decomposition of frames and coframes in the form

$$\vartheta_\mu = e^A_\mu e_A + \varepsilon l_\mu k + \varepsilon k_\mu l \,, \tag{3.74a}$$

$$\theta^\mu = e^\mu_A \epsilon^A + \varepsilon l^\mu \tilde{k} + \varepsilon k^\mu \tilde{l} \,. \tag{3.74b}$$

The latter expressions allow to immediately see the structure of the decompositions that we discussed before. However, they do not preserve the duality conditions (3.72) so that we will adhere to the form (3.70).

The alignment of frame and coframe with the tangent and normal directions may nevertheless be used to discuss the decomposition of vectors $A = A^\mu \vartheta_\mu \in T\mathcal{M}$ and covectors $B = B_\mu \theta^\mu \in T^*\mathcal{M}$ into hypersurface tangent and normal contributions. For the $(m-1)$-dimensional perspective on lightlike hypersurfaces we already discussed that these categories are not sufficient for a classification, since being normal or non-tangent does not coincide with each other. In terms of the geometrical introduction in chapter 2, this means that the tangent spaces $T\mathcal{M}$ may not be uniquely decomposed into a direct sum of tangent and normal subspaces. Instead, one may only decompose vectors and covectors into tangent and non-tangent contributions. That is,

$$A = A^\mu \vartheta_\mu = A^\mu e^A_\mu \varphi_A + \varepsilon A^\mu l_\mu \psi + \varepsilon A^\mu k_\mu \varphi = \operatorname{tang}(A) + \operatorname{nontang}(A) \,, \tag{3.75a}$$

$$B = B_\mu \theta^\mu = B_\mu e^\mu_A \phi^A + B_\mu k^\mu \Psi + B_\mu l^\mu \phi = \operatorname{tang}(B) + \operatorname{nontang}(B) \,, \tag{3.75b}$$

where the tangent contribution of vectors includes the $k = \psi$ direction, such that we obtain $\operatorname{tang}(A) = A^\mu e^A_\mu \varphi_A + \varepsilon A^\mu l_\mu \psi$ and $\operatorname{nontang}(A) = \varepsilon A^\mu k_\mu \varphi$. This is what we intuitively expect from the non-tangent part of a vector, it is aligned with $\varphi = l$ since l was explicitly constructed as a non-tangent vector. For covectors, we use that $\tilde{k}$ equals $\frac{1}{\varepsilon}\phi$ and thus the tangent contribution is $\operatorname{tang}(B) = B_\mu e^\mu_A \phi^A + B_\mu l^\mu \phi$, leaving us with the non-tangent part $\operatorname{nontang}(B) = B_\mu k^\mu \Psi$ aligned with $\tilde{l} = \frac{1}{\varepsilon}\Psi$.

Note that we obtain the non-tangent contributions of both vectors and covectors by contracting their components with the normal vector k. We thus obtain the frame independent formulation of the non-tangent parts as

$$\operatorname{nontang}(A) \coloneqq \varepsilon g(A,k)\, l \,, \qquad \operatorname{nontang}(B) \coloneqq \varepsilon g^{-1}(B,\tilde{k})\, \tilde{l} \,, \tag{3.76}$$

making the alignment with the non-tangent vector field l and its metric dual $\tilde{l}$ explicit. For the tangent contributions, an analogous consideration yields the index-free expressions

$$\begin{aligned} \operatorname{tang}(A) &= \sigma^{AB} g(A, e_B)\, e_A + \varepsilon g(A, l)\, k\,, \\ \operatorname{tang}(B) &= \sigma_{AB} g^{-1}(B, \epsilon^B)\, \epsilon^A + \varepsilon g^{-1}(B, \tilde{l})\, \tilde{k} \end{aligned} \tag{3.77}$$

which extend to vector and covector fields.

These decompositions become less complex in the $(m-2)$-dimensional perspective on hypersurfaces. Recall that both k and l fulfill $g(e_A, l) = 0$ and $g(e_A, k) = 0$ and may thus be considered as normal vectors. For this reason, it is useful to extend our definition of normal and tangent vectors to hypersurfaces of dimension $(m-p)$ for any p with $1 \leq p \leq m-1$. For such hypersurfaces, we generically have p normal vector fields $\zeta_1, \ldots, \zeta_p$. We already discussed that normal vector fields are considered as representatives of equivalence classes, and thus the set of normal vector fields is only non-trivial if we require that $g(\zeta_i, \zeta_j) \neq 0$ if $i \neq j$. Then, we call a vector field $v \in \Gamma(T\mathcal{M})$ tangent to the $(m-p)$-dimensional hypersurface if $g(v, \zeta_i) = 0$ for all p normal vector fields ζ_i. Conversely, any vector field ξ which obeys $g(v, \xi) = 0$ for all tangent vector fields v is called normal. Applying this to our $(m-2)$-dimensional perspective on lightlike hypersurfaces, we obtain a unique decomposition of vectors and covectors into hypersurface tangent and normal contributions. That is,

$$A = A^\mu \vartheta_\mu = A^\mu e^A_\mu \varphi_A + \varepsilon A^\mu l_\mu \psi + \varepsilon A^\mu k_\mu \varphi = \operatorname{tang}(A) + \operatorname{norm}(A)\,, \tag{3.78a}$$

$$B = B_\mu \theta^\mu = B_\mu e^\mu_A \phi^A + B_\mu k^\mu \Psi + B_\mu l^\mu \phi = \operatorname{tang}(B) + \operatorname{norm}(B)\,, \tag{3.78b}$$

but now the normal contributions contain both the k and the l directions. Hence, we have

$$\begin{aligned} \operatorname{tang}(A) &:= \sigma^{AB} g(A, e_B)\, e_A = A^\mu e^A_\mu \varphi_A\,, \\ \operatorname{tang}(B) &:= \sigma_{AB} g^{-1}(B, \epsilon^B)\, \epsilon^A = B_\mu e^\mu_A \phi^A \end{aligned} \tag{3.79}$$

for the tangent contribution while the normal directions are

$$\begin{aligned} \operatorname{norm}(A) &:= \varepsilon g(A, l)\, k + \varepsilon g(A, k)\, l = \varepsilon A^\mu l_\mu \psi + \varepsilon A^\mu k_\mu \varphi\,, \\ \operatorname{norm}(B) &:= \varepsilon g^{-1}(B, \tilde{l})\, \tilde{k} + \varepsilon g^{-1}(B, \tilde{k})\, \tilde{l} = B_\mu k^\mu \Psi + B_\mu l^\mu \phi\,. \end{aligned} \tag{3.80}$$

Thus, we always obtain the tangent contributions in the $(m-2)$ formalism by contracting tensor components with e^μ_A, while contractions with both normal vectors k and l yield normal contributions. Hence, we now discussed how any tensor may be projected to a lightlike hypersurface in the $(m-1)$ and $(m-2)$ formalisms and thereby obtain yet another justification for interpreting $\sigma_{AB} := e^\mu_A e^\nu_B g_{\mu\nu}$ as the $(m-2)$-dimensional hypersurface metric.

We may use this hypersurface metric and decompose the full metric on $\mathcal{M}$ into its contributions tangent and normal to the $(m-2)$-dimensional hypersurface. Note that in components, this is equivalent to its $(m-1)$ decomposition into tangent and non-tangent contributions. Hence, we use the adapted frame decomposition (3.70) to evaluate $g_{\mu\nu} = g(\vartheta_\mu, \vartheta_\nu)$ which yields

$$g_{\mu\nu} = e^A_\mu e^B_\nu \sigma_{AB} + \varepsilon l_\mu k_\nu + \varepsilon k_\mu l_\nu \,. \tag{3.81}$$

In order to derive the hypersurface connection, we need to examine the frame decomposition (3.70) as a $\mathrm{GL}(m,\mathbb{R})$ transformation. In analogy to the non-lightlike case, we interpret $\vartheta_\mu = \bar{\vartheta}_\mu \triangleleft \Lambda = \bar{\vartheta}_\nu \Lambda^\nu{}_\mu$ as the right action of $\Lambda \in \mathrm{GL}(m,\mathbb{R})$ on any frame $\bar{\vartheta}_\mu$ in the frame bundle. Comparing this right action with the frame decomposition (3.70) we read off

$$\begin{aligned} \Lambda^\mu{}_\nu &= e^A_\nu \delta^\mu_A + \varepsilon l_\nu \delta^\mu_{m-2} + \varepsilon k_\nu \delta^\mu_{m-1} \,, \\ \Lambda^{-1\mu}{}_\nu &= e^\mu_A \delta^A_\nu + k^\mu \delta^{m-2}_\nu + l^\mu \delta^{m-1}_\nu \,. \end{aligned} \tag{3.82}$$

We use this $\mathrm{GL}(m,\mathbb{R})$ transformation in the transformation law (2.25) of connections to obtain

$$\omega^\mu{}_\nu = e^\mu_A e^B_\nu \omega^A{}_B + e^\mu_A \mathrm{d} e^A_\nu + \text{terms linear in } k \text{ or } l \,. \tag{3.83}$$

Finally, we eliminate the terms linear in k or l by contracting with $e^A_\mu e^\nu_B$ and solve the resulting expression for the hypersurface connection to obtain

$$\omega^A{}_B = e^A_\mu e^\nu_B \omega^\mu{}_\nu + e^A_\mu \mathrm{d} e^\mu_B \,. \tag{3.84}$$

Just as in the non-lightlike case, it is useful for calculations to rewrite (3.84) as

$$e^A_\mu D e^\mu_B = 0 \,. \tag{3.85}$$

Let us comprehend the results of this derivation before we utilize them to examine field strengths.

We defined lightlike hypersurfaces $\Sigma \subset \mathcal{M}$ by a normal vector k which fulfills $g(k,k) = 0$. From this condition we already concluded that k is normal and tangent to Σ at the same time. Hence, there was a need for a non-tangent vector l which we saw to be non-normal as well, and we demanded $g(l,k) = \frac{1}{\varepsilon}$ and $g(l,l) = 0$. Aligning the decompositions of frames and coframes with the directions given by l, k and the hypersurface led us to

$$\vartheta_\mu = e^A_\mu \varphi_A + \varepsilon l_\mu \psi + \varepsilon k_\mu \varphi \,, \tag{3.86a}$$

$$\theta^\mu = e^\mu_A \phi^A + k^\mu \Psi + l^\mu \phi \,. \tag{3.86b}$$

From this adapted decomposition of frames and coframes, we immediately see that the unity $\delta^\mu_\nu = \theta^\mu(\vartheta_\nu)$ decomposes as

$$\delta^\mu_\nu = e^\mu_A e^A_\nu + \varepsilon k^\mu l_\nu + \varepsilon l^\mu k_\nu \,. \tag{3.87}$$

Additionally, we used the frame decomposition to conclude that the metric decomposes as

$$g_{\mu\nu} = e^A_\mu e^B_\nu \sigma_{AB} + \varepsilon k_\mu l_\nu + \varepsilon l_\mu k_\nu \,, \tag{3.88}$$

and we identified

$$\omega^A{}_B = e^A_\mu e^\nu_B \omega^\mu{}_\nu + e^A_\mu \mathrm{d} e^\mu_B \qquad \Leftrightarrow \qquad e^A_\mu D e^\mu_B = 0 \tag{3.89}$$

as the connection one-form on the lightlike hypersurface.

Having defined all fundamental vector fields which characterize a hypersurface as being lightlike, let us proceed by analyzing their field strengths.

3.2.2. Field strengths

Just as for non-lightlike hypersurfaces, we analyze the vectors e_A, k and l defining lightlike hypersurfaces by calculating their covariant exterior derivatives. Since the $(m-2)$-dimensional hypersurface $\check{\Sigma}$ is completely analogous to the space- and timelike case, we will keep this discussion short. Recall that both k and l fulfill $g(e_A, k) = 0$ and $g(e_A, l) = 0$, and thus both are normal vectors to the $(m-2)$-dimensional hypersurface. In the previous section we saw that all equations are symmetric under the exchange of k and l in the $(m-2)$-dimensional formalism. We will use this property to shorten the analysis of the field strengths and only explain how calculations are performed for the $(m-1)$-hypersurface normal vector k. The results for l follow from the symmetry argument, while they may of course be obtained by analogous calculations likewise.

We begin by asserting names to the field strengths of k, l and their dual covectors. Since k is the normal vector in the $(m-1)$ formalism, we denote its covariant exterior derivatives by the same symbols

$$K^\mu := Dk^\mu\,, \qquad \tilde{K}_\mu := Dk_\mu \tag{3.90}$$

as in the space- and timelike case. We note that $\tilde{K}_\mu = K_\mu - k^\nu Q_{\mu\nu}$ only differs from K_μ due to non-metricity. Analogously, we denote the covariant exterior derivatives of l by

$$L^\mu := Dl^\mu\,, \qquad \tilde{L}_\mu := Dl_\mu\,, \tag{3.91}$$

for which we have $\tilde{L}_\mu = L_\mu - l^\nu Q_{\mu\nu}$. For decomposing these field strengths, we contract them with the decomposition of unity (3.87) to obtain

$$K^\mu = \delta^\mu_\nu K^\nu = e^\mu_A K^A + \varepsilon k^\mu \mathcal{K} + \frac{\varepsilon}{2} l^\mu Q_{\mathbf{kk}}\,, \tag{3.92a}$$
$$\tilde{K}_\mu = \delta^\nu_\mu \tilde{K}_\nu = e^A_\mu \tilde{K}_A - \varepsilon k_\mu \mathcal{L} - \frac{\varepsilon}{2} l_\mu Q_{\mathbf{kk}}\,. \tag{3.92b}$$

In the latter equations we introduced some new differential forms which require a brief explanation. First, note that $K^A := e^A_\mu Dk^\mu$ and $\tilde{K}_A := e^\mu_A Dk_\mu$ are just the one-form extrinsic curvatures of the $(m-2)$-dimensional hypersurface. Hence, the tensor components of K^A and $\tilde{K}_A$ yield the extrinsic curvature as well as the acceleration of the Eulerian observer in the $(m-2)$ formalism. Second, the one-forms $\mathcal{K} := l_\mu Dk^\mu$ and $\mathcal{L} := k_\mu Dl^\mu = -\mathcal{K}$ appear in (3.92) only due to the $(m-2)$ form of this equation. That is, they may be considered the lightlike contribution to the $(m-1)$-dimensional extrinsic curvature and thus deserve a separate name. Since the latter contributions constitute the entire intrinsic $(m-1)$ piece of K^μ, we expect non-metricity to be responsible for the remaining part from analogy to (3.49). Indeed, we find that the remaining contribution is determined by $Q_{\mathbf{kk}} := k^\mu k^\nu Q_{\mu\nu} = 2k_\mu Dk^\mu$.

There is, however, an interpretation of K^μ beyond these properties. This interpretation arises if we consider the event horizon of a black hole, which is naturally a lightlike hypersurface. In this case, the normal vector k is called the Killing generator of the black hole horizon. In particular, the expression $K^\mu(k) = k^\nu \nabla_\nu k^\mu$ has a geometric meaning for Killing generators. Using the decomposition (3.92), we obtain

$$K^\mu(k) = e^\mu_A K^A(k) + \varepsilon k^\mu \mathcal{K}(k) + \frac{\varepsilon}{2} l^\mu Q_{\mathbf{kk}}(k)\,, \tag{3.93}$$

where usually one demands K^A to be a completely intrinsic hypersurface object by enforcing $K^A(k) = 0$. Despite this condition, the coefficient $\varkappa := \varepsilon\mathcal{K}(k) = \varepsilon l_\mu k^\nu \nabla_\nu k^\mu$ of the k^μ proportional contribution of (3.93) is called the *surface gravity* of the black hole. As [96] explains, the surface gravity is the force needed to hold an object stationary at the black hole horizon, counteracting the black hole's attraction. Precisely, it is the force applied by an observer placed at infinite distance. In standard Einstein-Hilbert gravity which we will discuss later, the surface gravity of any black hole may directly be calculated from its blackening factor and it already determines $K^\mu(k) = k^\nu \nabla_\nu k^\mu$ completely. However, (3.93) implies that this is no longer true in the presence of non-metricity since $Q_{\mathbf{kk}}$ contributes to $K^\mu(k)$ as well. This concludes the physical interpretation of the field strength K^μ of k and we turn to L^μ next. Using the symmetry of k and l, we obtain

$$L^\mu = \delta^\mu_\nu L^\nu = e^\mu_A L^A + \varepsilon l^\mu \mathcal{L} + \frac{\varepsilon}{2} k^\mu Q_{\mathbf{ll}} \,, \tag{3.94a}$$

$$\tilde{L}_\mu = \delta^\nu_\mu \tilde{L}_\nu = e^A_\mu \tilde{L}_A - \varepsilon l_\mu \mathcal{K} - \frac{\varepsilon}{2} k_\mu Q_{\mathbf{ll}} \,, \tag{3.94b}$$

where we introduce the one-form extrinsic curvatures $L^A := e^A_\mu D l^\mu$ and $\tilde{L}_A := e^\mu_A D l_\mu$ additional to the abbreviation $Q_{\mathbf{ll}} := l^\mu l^\nu Q_{\mu\nu} = 2 l_\mu D l^\mu$. The interpretation of all of these objects is analogous to those of the k derivatives in the $(m-2)$ formalism, corresponding to the symmetry of k and l. Hence, let us proceed by considering the covariant exterior derivative De^μ_A of the hypersurface frame e_A next.

Decomposing this expression by means of the unity decomposition (3.87) yields

$$De^\mu_A = \delta^\mu_\nu De^\nu_A = -\varepsilon k^\mu \tilde{L}_A - \varepsilon l^\mu \tilde{K}_A \,. \tag{3.95}$$

That is, De^μ_A is entirely determined by the extrinsic curvature one-forms. Note that this is analogous to the space- and timelike case, where we identified the corresponding result with the Gauß-Weingarten equation. Hence, we may interpret (3.95) as the lightlike version of the Gauß-Weingarten equation. For the metric inverse of e^μ_A, we obtain

$$De^A_\mu = -\varepsilon k_\mu L^A - \varepsilon l_\mu K^A \,. \tag{3.96}$$

To conclude this discussion of the field strengths of e_A, k and l, we derive the lightlike versions of the Ricci identity. Straightforwardly calculating the second covariant exterior derivatives yields

$$DK^\mu = D^2 k^\mu = \Omega^\mu{}_\nu k^\nu \,, \tag{3.97a}$$

$$D\tilde{K}_\mu = D^2 k_\mu = -\Omega^\nu{}_\mu k_\nu \,, \tag{3.97b}$$

$$DL^\mu = D^2 l^\mu = \Omega^\mu{}_\nu l^\nu \,, \tag{3.97c}$$

$$D\tilde{L}_\mu = D^2 l_\mu = -\Omega^\nu{}_\mu l_\nu \,, \tag{3.97d}$$

$$D^2 e^\mu_A = \Omega^\mu{}_\nu e^\nu_A - \Omega^B{}_A e^\mu_B \,, \tag{3.97e}$$

$$D^2 e^A_\mu = -\Omega^\nu{}_\mu e^A_\nu + \Omega^A{}_B e^B_\mu \,, \tag{3.97f}$$

where we introduce the $(m-2)$ hypersurface curvature $\Omega^A{}_B := \mathrm{d}\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B$. All of the decompositions derived in this section are my original results which have not been published before. They generalize the results of [96, 98] to geometries which allow for non-trivial torsion and non-metricity.

This concludes the brief introduction to lightlike hypersurfaces. Let us point out some conventions we have chosen in this chapter to see where generalizations of the discussed material are possible. For lightlike hypersurfaces, we chose the non-tangent and non-normal vector field l to be lightlike as well as k. This is the choice usually made without mentioning it, and it ultimately leads to the symmetry of k and l in the $(m-2)$-dimensional formalism. Moreover, we chose the $(m-2)$-dimensional lightlike hypersurface to have a non-degenerate metric tensor. We saw that this is equivalent to not having another lightlike vector that is tangent to the $(m-2)$-dimensional hypersurface. We may lift this assumption of course, in which case one might want to go further down the ladder and increase the codimension p of the immersed $(m-p)$-dimensional submanifold. The discussion in this section provides everything necessary to generalize the formalism in this way, if needed. Furthermore, we made a similar assumption for space- and timelike hypersurfaces. In this case, we assumed the $(m-1)$-dimensional hypersurface metric to be non-degenerate. This may be generalized as well, leading to a similar treatment as in the lightlike case. Lastly, we could consider hypersurfaces in the context of foliations, and even lift the assumption of $g(\zeta,\zeta)$ having constant sign. But since most of these generalizations are straightforward, we will proceed with space-, time- and lightlike hypersurfaces as we discussed them in this chapter. In particular, we will use the results we obtained to model the boundary of manifolds as a hypersurface in the following chapter. This will provide us with a formalism in which we are able to calculate Gibbons-Hawking-York boundary terms universally for theories featuring curvature, torsion and non-metricity.

Universal Gibbons-Hawking-York terms 4

The Gibbons-Hawking-York (GHY) term is well known in general relativity [102, 103]. This boundary term needs to be introduced to an action in order to make the variational principle well-defined on a manifold with boundary. Usually, an action S is defined such that its variation δS yields the equations of motion of a theory by means of Hamilton's principle. We would like to preserve this principle if we consider the same action S on a manifold with boundary, where variations δS generically include boundary contributions δS_{bdy}. In particular, we obtain $\delta S = \delta S_{\text{eom}} + \delta S_{\text{bdy}}$ on manifolds with boundary, where the equations of motion would be obtained by enforcing $\delta S_{\text{eom}} = 0$. However, Hamilton's principle requires the variation δS of the full action to vanish, while both δS_{eom} and δS_{bdy} do not vanish individually. We therefore need to refine an action if we consider it on a manifold with boundary in order to obtain the equations of motion from Hamilton's principle.

This is where the Gibbons-Hawking-York (GHY) term S_{GHY} enters the equations. Since the bulk dynamics of the theory is entirely described by S already, we require S_{GHY} to be a pure boundary term. We construct this boundary term such that it makes the variational principle well-defined if we add it to an action. That is, we consider a description of our physical system in terms of the action $S_{\text{full}} := S + S_{\text{GHY}}$ which coincides with S in the bulk. Then, the variational problem of S_{full} is well-defined if we construct the GHY term such that $\delta S_{\text{GHY}} = -\delta S_{\text{bdy}}$. In this case, we obtain $\delta S_{\text{full}} = \delta(S + S_{\text{GHY}}) = \delta S_{\text{eom}}$, so that we obtain the equations of motion by enforcing Hamilton's principle $\delta S_{\text{full}} = 0$ as required. Note that we only need to include a GHY term to a theory for those variations of S which include boundary terms. Otherwise, the problem of an ill-defined variational problem is not there in first place. The first aim of this chapter is thus to understand which fields included in actions require us to include a Gibbons-Hawking-York term if variations with respect to these fields are considered. Subsequently, we derive the generalized Gibbons-Hawking-York term for actions including curvature, torsion and non-metricity. This term is essential when considering actions on manifolds with boundary and has not been known in generic torsionful and non-metric geometries before I first derived it in [1].

In order to understand my method for deriving GHY terms, note that I introduced the boundary term S_{GHY} such that it fulfills $\delta S_{\text{GHY}} = -\delta S_{\text{bdy}}$ if the variation of the bulk action yields boundary contributions. The construction of a GHY term is thus particularly straightforward if δS_{bdy} is already the variation of a boundary term S_{bdy}. That is, the calculation of the GHY terms gets trivial if we may pull out the variation of the term δS_{bdy} and write it as a total variation of a boundary term. In this case, we simply obtain the GHY term as $S_{\text{GHY}} = -S_{\text{bdy}}$. Writing δS_{bdy} as the total variation of a boundary term may be involved, however. In particular, we need to express S_{bdy} solely in terms of boundary quantities. We obtain this expression by modeling the boundary as a hypersurface immersed in $\mathcal{M}$ and decomposing all tensors included in S into boundary tangent and non-tangent contributions. This will make the boundary contributions in S explicit and provide us with the straightforwardly constructed GHY term $S_{\text{GHY}} = -S_{\text{bdy}}$.

In principle, this calculation needs to be performed for each action S of interest individually, since the dependence on the theory's dynamical fields and thus the variation is not universal. I unify these calculations by constructing a universal Gibbons-Hawking-York term which renders the variational principle well-defined for a broad realm of actions. These original results of mine which I review in section 4.1 have been published in [1], where I first derived the universal Gibbons-Hawking-York term for space- and timelike hypersurfaces. Moreover, I discuss GHY terms on lightlike hypersurfaces in section 4.2. These results are unpublished so far. Let us first focus on the universal Gibbons-Hawking-York term on manifolds which have space- or timelike boundaries before we generalize the methods to include lightlike boundaries as well.

4.1. Universal Gibbons-Hawking-York terms for space- and timelike boundaries

In this section, we consider a manifold $\mathcal{M}$ on which a physical system is supposed to be described by an action S. Introducing a boundary $\partial\mathcal{M}$ to the manifold $\mathcal{M}$, we generically need to supplement the action with a GHY term as we have argued in the introduction to this chapter. In order to derive this GHY term, we consider the boundary $\partial\mathcal{M}$ to be a space- or timelike hypersurface. Then, the method for deriving a GHY term outlined in this chapter's introduction consist of writing the action S in terms of hypersurface tangent and normal tensors. Recall that we concluded that any tensor admits such a decomposition in section 3.1. In a local basis, this decomposition amounts to contracting the tensor indices with the components e^μ_a and n^μ of the hypersurface frame $e_a = e^\mu_a \vartheta_\mu$ and normal vector $n = n^\mu \vartheta_\mu$, respectively. In particular, we need to decompose the geometrical field strengths introduced on smooth manifolds in section 2.6 in this way. These field strengths are

$$\begin{aligned} &\text{curvature} && \Omega^\mu{}_\nu = D\omega^\mu{}_\nu = \mathrm{d}\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu\,, \\ &\text{torsion} && T^\mu = D\theta^\mu = \mathrm{d}\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu \quad \text{and} \\ &\text{non-metricity} && Q_{\mu\nu} = -Dg_{\mu\nu} = -\mathrm{d}g_{\mu\nu} + \omega^\rho{}_\mu g_{\rho\nu} + \omega^\rho{}_\nu g_{\mu\rho}\,, \end{aligned} \tag{4.1}$$

where the connection one-form $\omega^\mu{}_\nu$, the coframe θ^μ providing a soldering form and the components $g_{\mu\nu}$ of the metric tensor g are the dynamical fields of the theories which we consider. We have already investigated the decomposition of these dynamical fields into hypersurface tangent and normal contributions in section 3.1. Thus, we will proceed by decomposing curvature, torsion and non-metricity in hypersurface tangent and normal contributions next.

4.1.1. Decomposition of curvature, torsion and non-metricity

The most straightforward of the decompositions of curvature, torsion and non-metricity into hypersurface tangent and normal contributions is that of non-metricity. That is the case since we already encountered almost all of its contributions in the discussion of the field strengths of the hypersurface frame e_a and normal vector field n. All of these contributions are summarized in the decomposition of non-metricity which we obtain by contracting both of its indices with the decomposition (3.38) of unity. This yields

$$\begin{aligned} Q_{\mu\nu} &= \delta^\alpha_\mu \delta^\beta_\nu Q_{\alpha\beta} \\ &= e^a_\mu e^b_\nu (e^\alpha_a e^\beta_b Q_{\alpha\beta}) + \varepsilon e^a_\mu n_\nu (e^\alpha_a n^\beta Q_{\alpha\beta}) + \varepsilon n_\mu e^a_\nu (n^\alpha e^\beta_a Q_{\alpha\beta}) + n_\mu n_\nu (n^\alpha n^\beta Q_{\alpha\beta})\,. \end{aligned} \tag{4.2}$$

We thus determine the decomposition of non-metricity by calculating the terms which are denoted in parentheses in (4.2). From section 3.1.2 we immediately obtain

$$\begin{aligned} n^\mu n^\nu Q_{\mu\nu} &\equiv Q_{\mathbf{nn}} = 2n_\mu D n^\mu\,, \\ e^\mu_a n^\nu Q_{\mu\nu} &= n^\mu e^\nu_a Q_{\mu\nu} = K_a - \tilde{K}_a\,. \end{aligned} \tag{4.3}$$

Thus, the only contribution of the non-metricity decomposition (4.2) which is left to determine is $e^\mu_a e^\nu_b Q_{\mu\nu}$. We obtain this projection by straightforward use of the definition $Q_{\mu\nu} = -Dg_{\mu\nu}$ of non-metricity while recalling that $\gamma_{ab} = e^\mu_a e^\nu_b g_{\mu\nu}$ is the hypersurface metric. By means of the transformation (3.40) of the connection to the hypersurface we immediately conclude

$$e_a^{\mu} e_b^{\nu} Q_{\mu\nu} = Q_{ab} \,, \tag{4.4}$$

where we define the hypersurface non-metricity $Q_{ab} \coloneqq -D\gamma_{ab}$. Hence, we found that the projection of non-metricity to the hypersurface yields the hypersurface non-metricity. This is what we expect since the indices of non-metricity are tensor indices in contrast to the indices of torsion and curvature that take their values in a Lie algebra.

Indeed, the decompositions of these differential forms are slightly more involved. To obtain all contributions of its decomposition, we contract the torsion two-form T^{μ} with the decomposition (3.38) of unity. This yields

$$T^{\mu} = \delta^{\mu}_{\nu} T^{\nu} = e^{\mu}_a (e^a_{\nu} T^{\nu}) + \varepsilon n^{\mu} (n_{\nu} T^{\nu}) \,, \tag{4.5}$$

such that the calculation of the decomposition amounts to the evaluation of the projections in the parentheses in (4.5). By means of the definition $T^{\mu} = D\theta^{\mu}$ we immediately obtain

$$e^a_{\mu} T^{\mu} = D(e^a_{\mu} \theta^{\mu}) - De^a_{\mu} \wedge \theta^{\mu} \,. \tag{4.6}$$

Note that we identified $\phi^a = e^a_{\mu} \theta^{\mu}$ with the coframe of the hypersurface and thus denote $T^a \coloneqq D\phi^a = D(e^a_{\mu}\theta^{\mu})$ as the hypersurface torsion. Furthermore, we related De^a_{μ} to the extrinsic curvature one-form $K^a = e^a_{\mu} Dn^{\mu}$ by means of the Gauß-Weingarten equation (3.48) as $De^a_{\mu} = -\varepsilon n_{\mu} K^a$. Hence, the hypersurface tangent contribution of torsion is

$$e^a_{\mu} T^{\mu} = T^a + N K^a \wedge \phi \,, \tag{4.7}$$

where $\phi = \frac{1}{\varepsilon N} \tilde{n}$ is the coframe aligned with the normal covector $\tilde{n} = \flat(n)$.

For deriving the normal projection of torsion, we use its definition $T^{\mu} = D\theta^{\mu}$ again to see

$$n_{\mu} T^{\mu} = D(n_{\mu} \theta^{\mu}) - Dn_{\mu} \wedge \theta^{\mu} \,. \tag{4.8}$$

Hence, the normal contribution is determined by the normal covector $\tilde{n} = n_{\mu}\theta^{\mu} = \varepsilon N \phi$ and the decomposition (3.49) of $\tilde{K}_{\mu} = Dn_{\mu}$. Inserting this decomposition into (4.8), we conclude

$$n_\mu T^\mu = -\tilde{K}_a \wedge \phi^a + \varepsilon D(N\phi) + \frac{N}{2} Q_{\mathbf{nn}} \wedge \phi \,. \tag{4.9}$$

With this normal projection we determined the decomposition (4.5) of the torsion two-form completely as

$$\begin{aligned} e^a_\mu T^\mu &= T^a + N K^a \wedge \phi \,, \\ n_\mu T^\mu &= -\tilde{K}_a \wedge \phi^a + \varepsilon D(N\phi) + \frac{N}{2} Q_{\mathbf{nn}} \wedge \phi \,. \end{aligned} \tag{4.10}$$

Hence, the only field strength which we did not yet decompose into hypersurface tangent and normal contributions is the curvature two-form $\Omega^\mu{}_\nu$.

Contraction of $\Omega^\mu{}_\nu$ with the unity decomposition (3.38) yields

$$\begin{aligned} \Omega^\mu{}_\nu &= \delta^\mu_\alpha \delta^\beta_\nu \Omega^\alpha{}_\beta \\ &= e^\mu_a e^b_\nu (e^a_\alpha e^\beta_b \Omega^\alpha{}_\beta) + \varepsilon e^\mu_a n_\nu (e^a_\alpha n^\beta \Omega^\alpha{}_\beta) + \varepsilon n^\mu e^a_\nu (n_\alpha e^\beta_a \Omega^\alpha{}_\beta) + n^\mu n_\nu (n_\alpha n^\beta \Omega^\alpha{}_\beta) \,, \end{aligned} \tag{4.11}$$

and we need to calculate the terms in parentheses analogous to the decompositions of torsion and non-metricity. These terms are partially contained in the Ricci identities (3.51). For instance,

$$D^2 e^\mu_a = \Omega^\mu{}_\nu e^\nu_a - \Omega^b{}_a e^\mu_b \tag{4.12}$$

contains both $e^a_\mu e^\nu_b \Omega^\mu{}_\nu$ and $n_\mu e^\nu_a \Omega^\mu{}_\nu$ if we contract it with e^a_μ and n_μ, respectively. Hence, the calculation of the latter projections reduces to the decomposition of

$$\begin{aligned} e^a_\mu D^2 e^\mu_b &= D(e^a_\mu D e^\mu_b) - D e^a_\mu \wedge D e^\mu_b \,, \\ n_\mu D^2 e^\mu_a &= D(n_\mu D e^\mu_a) - D n_\mu \wedge D e^\mu_a \,. \end{aligned} \tag{4.13}$$

But these are exactly the covariant exterior derivatives which we evaluated in section 3.1.2. Hence, we may use all of the results obtained in this section and analyze the remaining projections analogously to conclude that the curvature projections are

$$\begin{aligned} e^a_\mu e^\nu_b \Omega^\mu{}_\nu &= \Omega^a{}_b - \varepsilon K^a \wedge \tilde{K}_b \,, & e^a_\mu n^\nu \Omega^\mu{}_\nu &= D K^a + \frac{\varepsilon}{2} K^a \wedge Q_{\mathbf{nn}} \,, \\ n_\mu e^\nu_a \Omega^\mu{}_\nu &= -D\tilde{K}_a + \frac{\varepsilon}{2} \tilde{K}_a \wedge Q_{\mathbf{nn}} \,, & n_\mu n^\nu \Omega^\mu{}_\nu &= \frac{1}{2} D Q_{\mathbf{nn}} + K^a \wedge \tilde{K}_a \,, \end{aligned} \tag{4.14}$$

where we recall that $\Omega^a{}_b = D\omega^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$ is the hypersurface curvature two-form.

We have now decomposed all field strengths into hypersurface tangent and normal contributions by means of the description of space- and timelike hypersurfaces we developed in section 3.1. This is already sufficient for calculating the Gibbons-Hawking-York term of a given action in an effective way as we saw in the introduction to this chapter. However, we will go one step further and use these results for calculating the Gibbons-Hawking-York terms for a broad range of actions at once. To that end, we linearize the action by introducing Lagrange multipliers in the following section, providing us with a universal Gibbons-Hawking-York term.

4.1.2. Universal Gibbons-Hawking-York term from Lagrange multipliers

In this section, we derive the universal Gibbons-Hawking-York (GHY) term for a family of actions on manifolds $\mathcal{M}$ with boundary $\partial\mathcal{M}$. We consider these manifolds to be equipped with a connection one-form $\omega^{\mu}{}_{\nu}$, a soldering form θ^{μ} locally defined by the coframes and a metric tensor field g. Hence, we work with a Lagrangian L which depends on these dynamical fields and their respective field strengths[1]. Recall that these field strengths are the curvature two-form $\Omega^{\mu}{}_{\nu}$, the torsion two-form T^{μ} and the non-metricity one-form $Q_{\mu\nu}$ given in (4.1). Therefore, we consider actions of the form

$$S[\omega^{\mu}{}_{\nu}, \theta^{\mu}, g_{\mu\nu}] = \int_{\mathcal{M}} L(\Omega^{\mu}{}_{\nu}, T^{\mu}, Q_{\mu\nu}) \,. \tag{4.15}$$

In order to gain insights from this general form of the action and nevertheless be able to use the hypersurface formalism for describing the boundary, we use a trick first employed by [104]. In this paper, the authors introduced auxiliary fields linearizing the Lagrangian in the field strengths. These auxiliary fields are used as Lagrange multipliers. In our differential form notation, we adapt the idea of [104] to define the action

$$\begin{aligned} &S_{\text{Lagr}}[\omega^{\mu}{}_{\nu}, \theta^{\mu}, g_{\mu\nu}, \varphi_{\mu}{}^{\nu}, \varrho^{\mu}{}_{\nu}, t_{\mu}, \tau^{\mu}, q^{\mu\nu}, \sigma_{\mu\nu}] \\ &= \int_{\mathcal{M}} [L(\varrho^{\mu}{}_{\nu}, \tau^{\mu}, \sigma_{\mu\nu}) + *\varphi_{\mu}{}^{\nu} \wedge (\Omega^{\mu}{}_{\nu} - \varrho^{\mu}{}_{\nu}) + *t_{\mu} \wedge (T^{\mu} - \tau^{\mu}) + *q^{\mu\nu} \wedge (Q_{\mu\nu} - \sigma_{\mu\nu})] \,, \end{aligned} \tag{4.16}$$

where derivatives of $\Omega^{\mu}{}_{\nu}$, T^{μ} and $Q_{\mu\nu}$ are linearized by the Bianchi identities (2.30),

[1] In order to obtain manifestly covariant actions, we only consider Lagrangians which depend on the connection one-form implicitly via the field strengths. For compactness of notation, we suppress the dependence of the Lagrangian on coframe and metric components in the following.

(2.37), (2.42) and thus included in this linearization. Since the integrand needs to be a m-form on an m-dimensional manifold, we require ${\varphi_\mu}^\nu$, ${\varrho^\mu}_\nu$, t_μ and τ^μ to be two-forms, while $q^{\mu\nu}$ and $\sigma_{\mu\nu}$ are one-forms. For the same reason, the *Hodge duality* denoted by $*$ is introduced in the integral. The Hodge duality is defined as

$$\begin{aligned} * : \Omega^p(\mathcal{M}) &\xrightarrow{\sim} \Omega^{m-p}(\mathcal{M}) \\ \omega &\longmapsto *\omega\,, \end{aligned}$$

and we call $*\omega$ the *Hodge dual* of ω. The target space of the Hodge duality being $\Omega^{m-p}(\mathcal{M})$ ensures that the integrand of (4.16) is a differential form of rank m such that its integral is well-defined. Since $*$ is a linear map, it is entirely determined by its action on coframes that is given by

$$\begin{aligned} \eta^{\mu_1\cdots\mu_p} &\equiv *(\theta^{\mu_1}\wedge\cdots\wedge\theta^{\mu_p}) \\ &:= \frac{\sqrt{|\det g|}}{(m-p)!}\varepsilon_{\sigma_1\ldots\sigma_p\nu_1\ldots\nu_{m-p}}g^{\mu_1\sigma_1}\cdots g^{\mu_p\sigma_p}\theta^{\nu_1}\wedge\cdots\wedge\theta^{\nu_{m-p}}\,. \end{aligned} \tag{4.17}$$

The totally antisymmetric ε-symbol is defined such that $\varepsilon_{0,1,\ldots,m-1}=1$. Furthermore, we introduce the determinant of the metric as[2]

$$\det g := (-1)^{\operatorname{ind} g}\varepsilon^{\mu_0\ldots\mu_{m-1}}g_{0\mu_0}\cdots g_{(m-1)\mu_{m-1}}\,, \tag{4.18}$$

in which $\operatorname{ind} g$ is the number of minus signs in the signature of g.

The variation of the action (4.16) with respect to the tensors denoted as Hodge duals yields the equations of motion

$$\Omega^\mu{}_\nu = \varrho^\mu{}_\nu\,, \qquad T^\mu = \tau^\mu\,, \qquad Q_{\mu\nu} = \sigma_{\mu\nu} \tag{4.19}$$

by enforcing Hamilton's principle $\delta S_{\text{Lagr}} = 0$. Hence, $*{\varphi_\mu}^\nu$, $*t_\mu$ and $*q^{\mu\nu}$ function as Lagrange multipliers, and considering the on-shell action by reinstating their equations of motion (4.19) into (4.16), the two actions S_{Lagr} and S are equivalent. This is why it is useful in first place to consider (4.16) to learn about the original action (4.15). The advantage of the Lagrange multiplier action (4.16) is that it is linear in curvature, torsion and non-metricity. In order to find the boundary contributions of S_{Lagr}, we

[2] Note that $\varepsilon^{\mu_1\ldots\mu_m} = (-1)^{\operatorname{ind} g}\varepsilon_{\mu_1\ldots\mu_m}$, from which the prefactor in the definition of $\det g$ originates. We furthermore emphasize that g is not an endomorphism, which is why we need to define its determinant explicitly at this point. Unlike the determinant of an endomorphism, $\det g$ is not invariant with respect to $\mathrm{GL}(m,\mathbb{R})$ transformations, it transforms as a scalar density instead. Recall that we studied this case in the framework of tensor density bundles in chapter 2.

thus straightforwardly insert the decompositions (4.2), (4.5) and (4.11) of these field strengths which we derived in the previous subsection. This yields

$$\begin{aligned}
*\varphi_\mu{}^\nu \wedge \Omega^\mu{}_\nu =& [e^\mu_a e^b_\nu * \varphi_\mu{}^\nu] \wedge (e^a_\alpha e^\beta_b \Omega^\alpha{}_\beta) + \varepsilon [e^\mu_a n_\nu * \varphi_\mu{}^\nu] \wedge (e^a_\alpha n^\beta \Omega^\alpha{}_\beta) \\
&+ \varepsilon [n^\mu e^a_\nu * \varphi_\mu{}^\nu] \wedge (n_\alpha e^\beta_a \Omega^\alpha{}_\beta) + [n^\mu n_\nu * \varphi_\mu{}^\nu] \wedge (n_\alpha n^\beta \Omega^\alpha{}_\beta)\,, \\
*t_\mu \wedge T^\mu =& [e^\mu_a * t_\mu] \wedge (e^a_\nu T^\nu) + \varepsilon [n^\mu * t_\mu] \wedge (n_\nu T^\nu)\,, \\
*q^{\mu\nu} \wedge Q_{\mu\nu} =& [e^a_\mu e^b_\nu * q^{\mu\nu}] \wedge (e^\alpha_a e^\beta_b Q_{\alpha\beta}) + \varepsilon [e^a_\mu n_\nu * q^{\mu\nu}] \wedge (e^\alpha_a n^\beta Q_{\alpha\beta}) \\
&+ \varepsilon [n_\mu e^a_\nu * q^{\mu\nu}] \wedge (n^\alpha e^\beta_a Q_{\alpha\beta}) + [n_\mu n_\nu * q^{\mu\nu}] \wedge (n^\alpha n^\beta Q_{\alpha\beta})\,,
\end{aligned} \tag{4.20}$$

in which we re-express the projected components of curvature, torsion and non-metricity by the decompositions we obtained in the previous subsection. However, we do not need to write this out in total at this point. To see this, recall that our motivation was to explicitly find the boundary contribution of the action (4.16). Hence, it suffices for our purposes to keep those terms of the decompositions which yield boundary contributions. By means of Stokes' theorem (2.21), the relevant terms are those containing derivatives of boundary fields. Hence, we collect these terms to conclude that the boundary relevant contributions of (4.20) are

$$\begin{aligned}
*\varphi_\mu{}^\nu \wedge \Omega^\mu{}_\nu =& [e^\mu_a e^b_\nu * \varphi_\mu{}^\nu] \wedge D\omega^a{}_b + \varepsilon [e^\mu_a n_\nu * \varphi_\mu{}^\nu] \wedge DK^a \\
&- \varepsilon [n^\mu e^a_\nu * \varphi_\mu{}^\nu] \wedge D\tilde{K}_a + \frac{1}{2} [n^\mu n_\nu * \varphi_\mu{}^\nu] \wedge DQ_{\mathbf{nn}} \\
&+ \text{terms irrelevant on } \partial\mathcal{M}\,, \\
*t_\mu \wedge T^\mu =& [e^\mu_a * t_\mu] \wedge D\phi^a + [n^\mu * t_\mu] \wedge D(N\phi) + \text{terms irrelevant on } \partial\mathcal{M}\,, \\
*q^{\mu\nu} \wedge Q_{\mu\nu} =& - [e^a_\mu e^b_\nu * q^{\mu\nu}] \wedge D\gamma_{ab} + \text{terms irrelevant on } \partial\mathcal{M}\,.
\end{aligned} \tag{4.21}$$

After an integration by parts, we may integrate these terms using Stokes' theorem (2.21). Note that the hypersurface on which we invoke Stokes' theorem is the boundary $\partial\mathcal{M}$ of the manifold, so that only the boundary tangent contributions are relevant in this integration. We denote this pull-back to the boundary as $\big|_{\partial\mathcal{M}}$ and thus obtain

$$\begin{aligned}
&\int_\mathcal{M} \left(*\varphi_\mu{}^\nu \wedge \Omega^\mu{}_\nu + *t_\mu \wedge T^\mu + *q^{\mu\nu} \wedge Q_{\mu\nu} \right) \\
&= \int_{\partial\mathcal{M}} \Big(\omega^a{}_b \wedge *\varphi_a{}^b + \varepsilon K^a \wedge *\varphi_{a\mathbf{n}} - \varepsilon \tilde{K}_a \wedge *\varphi^{\mathbf{n}a} + \frac{1}{2} Q_{\mathbf{nn}} \wedge *\varphi_{\mathbf{nn}} \\
&\qquad + \phi^a \wedge e^\mu_a * t_\mu + (-1)^m \gamma_{ab} e^a_\mu e^b_\nu * q^{\mu\nu} \Big) \Big|_{\partial\mathcal{M}} \\
&\qquad + \text{terms irrelevant on } \partial\mathcal{M}\,,
\end{aligned} \tag{4.22}$$

where we introduce the abbreviations

$$*\varphi_a{}^b := e^\mu_a e^b_\nu * \varphi_\mu{}^\nu\,, \quad *\varphi_{\mathbf{an}} := e^\mu_a n_\nu * \varphi_\mu{}^\nu\,, \quad *\varphi^{\mathbf{na}} := n^\mu e^a_\nu * \varphi_\mu{}^\nu\,, \quad *\varphi_{\mathbf{nn}} := n^\mu n_\nu * \varphi_\mu{}^\nu \tag{4.23}$$

for the projected Lagrange multipliers of curvature. As described in this chapter's introduction, we are now able to construct a generic boundary term which makes the variational principle of S_{Lagr} well-defined by means of the boundary action (4.22). However, the classical GHY term in general relativity is constructed as the special choice of this generic boundary term which solves the Dirichlet problem. For the manifolds $\mathcal{M}$ equipped with connection, coframe and metric we consider here, the Dirichlet boundary conditions are

$$\delta\omega^\mu{}_\nu\Big|_{\partial\mathcal{M}} = 0\,, \qquad \delta\theta^\mu\Big|_{\partial\mathcal{M}} = 0\,, \qquad \delta g_{\mu\nu}\Big|_{\partial\mathcal{M}} = 0\,. \tag{4.24}$$

It is intuitively clear that these conditions are fulfilled if we impose

$$\delta\omega^a{}_b = 0\,, \qquad \delta\phi^a = 0\,, \qquad \delta\gamma_{ab} = 0 \tag{4.25}$$

for the boundary connection, coframe and metric. A proof of the equivalence of these statements may be found in [20]. Hence, the GHY term only needs to cancel those contributions in (4.22) which do not vanish if we impose the Dirichlet boundary conditions (4.25) for the variational principle. Note that at this point it is straightforward to impose different boundary conditions if needed. The Gibbons-Hawking-York term of S_{Lagr} is therefore

$$S_{\text{GHY}} = -\int_{\partial\mathcal{M}} \left(\varepsilon K^a \wedge *\varphi_{\mathbf{an}} - \varepsilon \tilde{K}^a \wedge *\varphi_{\mathbf{na}} + \frac{1}{2} Q_{\mathbf{nn}} \wedge *\varphi_{\mathbf{nn}}\right)\Big|_{\partial\mathcal{M}}\,. \tag{4.26}$$

The latter result is, however, not yet the GHY term of the original action (4.15) which we wanted to calculate. In particular, we need to determine the Lagrange multipliers in order to calculate the GHY term for a given Lagrangian. We obtain these expressions from considering variations of the Lagrange multiplier action (4.16) with respect to the auxiliary field $\varrho^\mu{}_\nu$. A direct calculation using Hamilton's principle $\delta S_{\text{Lagr}} = 0$ yields $*\varphi_\mu{}^\nu \wedge \delta\varrho^\mu{}_\nu = \delta_{\varrho^\mu{}_\nu} L(\varrho^\mu{}_\nu, \tau^\mu, \sigma_{\mu\nu})$ which determines the on-shell value of $*\varphi_\mu{}^\nu$. Since we only need the projections, it is, however, more useful to consider the Lagrangian as being dependent on the projections $\varrho^{\mathbf{an}} := e^a_\mu n^\nu \varrho^\mu{}_\nu$, $\varrho_{\mathbf{na}} := n_\mu e^\nu_a \varrho^\mu{}_\nu$ and $\varrho_{\mathbf{nn}} := n_\mu n^\nu \varrho^\mu{}_\nu$, which is completely equivalent to a dependence on $\varrho^\mu{}_\nu$. Using this notation, the Lagrange multipliers are determined by

$$\begin{aligned} *\varphi_{a\mathbf{n}} \wedge \delta\varrho^{a\mathbf{n}} &= \varepsilon\delta_{\varrho^{a\mathbf{n}}} L(\varrho_{\mathbf{n}a}, \varrho^{a\mathbf{n}}, \varrho_{\mathbf{nn}}, \dots)\,, \\ *\varphi^{\mathbf{n}a} \wedge \delta\varrho_{\mathbf{n}a} &= \varepsilon\delta_{\varrho_{\mathbf{n}a}} L(\varrho_{\mathbf{n}a}, \varrho^{a\mathbf{n}}, \varrho_{\mathbf{nn}}, \dots)\,, \\ *\varphi_{\mathbf{nn}} \wedge \delta\varrho_{\mathbf{nn}} &= \ \ \delta_{\varrho_{\mathbf{nn}}} L(\varrho_{\mathbf{n}a}, \varrho^{a\mathbf{n}}, \varrho_{\mathbf{nn}}, \dots)\,. \end{aligned} \tag{4.27}$$

Hence, if we know the Lagrangian L of a theory, we may immediately calculate the Lagrange multipliers $*\varphi^{a\mathbf{n}}$, $*\varphi_{\mathbf{n}a}$ and $*\varphi_{\mathbf{nn}}$ by means of the straightforward variations derived in (4.27). Subsequently, we may reinstate these expressions into our result for the GHY term (4.26) to obtain the GHY term of any action S of the form (4.15). For this reason, we call (4.26) the *universal Gibbons-Hawking-York term* which solves the Dirichlet problem for any action constructed from curvature, torsion and non-metricity as (4.15).

Since it is one of the main results of this thesis, let me further elaborate on the universal GHY term (4.26). First, we already saw that the form of this term depends on the chosen Dirichlet boundary conditions, but my formalism allows to include different choices of boundary conditions likewise as I emphasized before. Second, the universal result (4.26) makes the calculation of GHY terms extraordinarily efficient. For a given Lagrangian, one only needs to evaluate the three variations (4.27) and insert the results into the universal GHY term. This becomes even more efficient if we consider manifolds which have vanishing non-metricity. In this case, we use $\tilde{K}^a = K^a$ as well as the Bianchi identity (2.42) to conclude that

$$S^{Q=0}_{\text{GHY}} = 2\varepsilon \int_{\partial\mathcal{M}} K^a \wedge *\varphi_{\mathbf{n}a}\Big|_{\partial\mathcal{M}} \tag{4.28}$$

is already the entire GHY term. This holds in particular for theories built solely upon curvature, for which my formalism provides a very efficient way of calculating GHY terms as well. But even in the presence of non-metricity, we observe that the result (4.26) for the universal GHY term only includes the Lagrange multipliers for curvature, while those of torsion and non-metricity do not contribute to the GHY term at all. That is, a GHY term is only needed for actions which involve curvature. Actions constructed solely from torsion or non-metricity do not require us to introduce a Gibbons-Hawking-York term. Connecting this to the discussion in this chapter's introduction, we found that the variational problem is well-defined for bulk actions involving only torsion and non-metricity even if we consider this action on a manifold with boundary.

It is important to notice that the range of Lagrangians for which we may use the universal result (4.26) for calculating the GHY term is limited by the equivalence of the Lagrange multiplier action (4.16) and the original one. While we noted that these are equivalent if we instate the equations of motions (4.19) of the Lagrange multipliers, there are cases in which this equivalence does not imply that (4.26) is the correct GHY term of S. To see this, we recall that the Lagrange multiplier formalism assumes that the variations of all fields are independent of each other. In particular, the variations of the Lagrange multiplies themselves are assumed to be independent of the variations of the auxiliary fields and curvature. But we determine the Lagrange multipliers $*\varphi_\mu{}^\nu$ for a given Lagrangian L from $*\varphi_\mu{}^\nu \wedge \delta\varrho^\mu{}_\nu = \delta_{\varrho^\mu{}_\nu} L(\varrho^\mu{}_\nu, ...)$ or the corresponding projected equations (4.27). Going on-shell thus expresses $\varphi_\mu{}^\nu$ in terms of a variation of the Lagrangian, which generically might depend on the remaining fields. This is important for the calculation of the GHY term since we constructed GHY terms such that they make the variational principle well-defined. Hence, we conclude that (4.26) is always the full GHY term for the Lagrange multiplier action (4.16), but it only provides the GHY term of the original action (4.15) if the relevant Lagrange multipliers $*\varphi_{a\mathbf{n}}$, $*\varphi^{\mathbf{n}a}$ and $*\varphi_{\mathbf{nn}}$ calculated by means of (4.27) are independent of all other fields. For short, this discussion implies that the processes of going on-shell in the Lagrange multiplier action and taking the variations which determine the GHY term do not commute.

In most cases, this will constrain the actions for which we calculate the GHY term by means of (4.26) to Lagrangians which are linear in the curvature two-form. In all other cases, we need to follow the calculation of this section and perform the decomposition of the fields explicitly without introducing Lagrange multipliers in order to derive the correct GHY term. We are going to consider examples which illustrate both methods in section 4.1.3. Note, however, that Lagrangians constructed only from torsion and non-metricity are not constrained by this argument since their Lagrange multipliers do not appear in the universal GHY term (4.26). Hence, an action built solely upon torsion and non-metricity never needs a GHY term in order for its variational principle to be well-defined. Furthermore, explicit calculations reveal that the Lagrange multiplier method yields the correct tensor structure of the GHY term for all examples which we consider in the remainder of this chapter. The results obtained from the Lagrange multiplier method differ from the explicit calculation at most by constant prefactors in each term. Hence, the Lagrange multiplier method provides an efficient way to estimate the tensor structure of the GHY term even in these cases, while the overall coefficients need to be calculated by an explicit calculation in general.

In this subsection, I derived the first systematic method which allows to calculate GHY terms of actions which depend on curvature, torsion and non-metricity if we consider manifolds with curvature, torsion and non-metricity with space- or timelike boundary. I concluded that the universal GHY term (4.26) may be used to further simplify this method unless the Lagrange multipliers $*\varphi_{a\mathbf{n}}$, $*\varphi^{\mathbf{n}a}$ and $*\varphi_{\mathbf{nn}}$ depend on the remaining fields. In the latter case, we may evaluate GHY terms using the methods of this section as well, following the calculation explicitly without introducing Lagrange multipliers. Let us consider examples for all these cases next in order to understand how my method is applied to concrete theories of interest and to prove consistency of my results with the GHY terms known in literature.

4.1.3. Examples for Gibbons-Hawking-York terms

The GHY term is best understood in general relativity [102, 103], so we will reproduce this example first by means of the method derived in section 4.1.2. Furthermore, we are going to generalize the GHY term of general relativity to include actions which admit torsion and non-metricity. We will do the same for four-dimensional Chern-Simons modified gravity, for which the GHY term has been derived in [105] first. For this theory of gravity, we will need to perform the derivation of the GHY term explicitly without introducing Lagrange multipliers, so that four-dimensional Chern-Simons modified gravity serves as an example for the explicit calculation. Lastly, we will derive the GHY term for Lovelock gravity in arbitrary dimensions. While this has been derived in literature for special cases [106, 107], the full GHY term of Lovelock gravity in differential form notation on manifolds with curvature and torsion is derived in this section for the very first time. This generalizes my previously derived results for five-dimensional Lovelock-Chern-Simons gravity in [1] as well.

Einstein-Hilbert gravity

On an m-dimensional manifold $\mathcal{M}$ without boundary, general relativity is fully described by the Einstein-Hilbert action

$$S^{\text{EH}} = \frac{1}{2\kappa}\int_{\mathcal{M}} \mathrm{d}^m x \sqrt{|\det g|}R\,, \tag{4.29}$$

where $\kappa = 8\pi G$ and R is the *Ricci scalar*. We utilize the formalism developed in section 4.1.2 for calculating the GHY term which needs to be added to the Einstein-Hilbert action (4.29) if we consider general relativity on a manifold with boundary $\partial\mathcal{M}$. To that end, we first need to rewrite the Einstein-Hilbert action in terms of differential forms. Recall that the Ricci scalar is obtained from the components of the Riemann tensor $R^{\mu}{}_{\nu\sigma\rho}$ by contracting $R \coloneqq g^{\nu\rho}R^{\mu}{}_{\nu\mu\rho}$. The components of the Riemann tensor are the tensor components of the curvature two-form in a frame ϑ_μ, that is $R^\mu{}_{\nu\sigma\rho} = \Omega^\mu{}_\nu(\vartheta_\sigma, \vartheta_\rho)$, such that $\Omega^\mu{}_\nu = \frac{1}{2} R^\mu{}_{\nu\sigma\rho} \theta^\sigma \wedge \theta^\rho$. Note that the integral measure $\mathrm{d}^m x := \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{m-1}$ is the volume form $\mathrm{dVol} = \theta^0 \wedge \cdots \wedge \theta^{m-1}$ in the chart-induced frame $\theta^\mu = \mathrm{d}x^\mu$. For writing the Einstein-Hilbert action in differential form notation, we furthermore use the relations[3]

$$\theta^{\mu_0} \wedge \cdots \wedge \theta^{\mu_{m-1}} = (-1)^{\operatorname{ind} g} \varepsilon^{\mu_0 \ldots \mu_{m-1}} \theta^0 \wedge \cdots \wedge \theta^{m-1}\,, \tag{4.30a}$$

$$\varepsilon_{\mu_1 \ldots \mu_p \mu_{p+1} \ldots \mu_m} \varepsilon^{\mu_1 \ldots \mu_p \nu_{p+1} \ldots \nu_m} = (-1)^{\operatorname{ind} g} p!(m-p)! \delta^{\nu_{p+1}}_{[\mu_{p+1}} \cdots \delta^{\nu_m}_{\mu_m]} \tag{4.30b}$$

proven in [108]. Combining these relations with the Hodge duality (4.17), we conclude that

$$\eta := *1 = \sqrt{|\det g|}\theta^0 \wedge \cdots \wedge \theta^{m-1} \tag{4.31}$$

is a top form that is related to the choice $\mathrm{dVol} = \theta^0 \wedge \cdots \wedge \theta^{m-1}$ of volume form by $\eta = \sqrt{|\det g|}\mathrm{dVol}$. By means of the same relations we use the Hodge dual $\eta^{\mu\nu} := *(\theta^\mu \wedge \theta^\nu)$ to verify that $\eta R = \eta_\mu{}^\nu \wedge \Omega^\mu{}_\nu$. Hence, we may rewrite the Einstein-Hilbert action (4.29) in terms of differential forms as

$$S^{\mathrm{EH}} = \frac{1}{2\kappa} \int_{\mathcal{M}} \eta_\mu{}^\nu \wedge \Omega^\mu{}_\nu\,. \tag{4.32}$$

This differential form notation of the Einstein-Hilbert action is well-known, see [109] for instance.

Introducing a boundary $\partial\mathcal{M}$ to our manifold, the action (4.32) needs to be supplemented by a GHY term. For calculating this GHY term, we compare the Einstein-Hilbert action (4.32) to the generic form (4.15) of an action to read of the Lagrangian

$$L^{\mathrm{EH}}(\Omega^\mu{}_\nu) = \frac{1}{2\kappa} \eta_\mu{}^\nu \wedge \Omega^\mu{}_\nu \tag{4.33}$$

of Einstein-Hilbert gravity. Following section 4.1.2, we introduce Lagrange multipliers $\varphi_\mu{}^\nu$ and auxiliary fields $\varrho^\mu{}_\nu$ resulting in the Lagrange multiplier action

$$S^{\mathrm{EH}}_{\mathrm{Lagr}} = \int_{\mathcal{M}} \left(\frac{1}{2\kappa} \eta_\mu{}^\nu \wedge \varrho^\mu{}_\nu - *\varphi_\mu{}^\nu \wedge (\Omega^\mu{}_\nu - \varrho^\mu{}_\nu) \right). \tag{4.34}$$

We need to calculate the Lagrange multipliers by means of (4.27) next. To that end, we decompose the Einstein-Hilbert Lagrangian (4.33) by means of the decomposition

[3] We antisymmetrize the indices of tensor components $A_{\mu_1 \ldots \mu_p}$ according to $A_{[\mu_1 \ldots \mu_p]} := \frac{1}{p!} \sum_{\sigma \in \mathcal{S}_p} \operatorname{sgn}(\sigma) A_{\sigma(\mu_1) \ldots \sigma(\mu_p)}$, where $\mathcal{S}_p$ is the group of permutations of p integers.

of unity (3.38) which yields

$$L^{\mathrm{EH}}(\varrho^{\mu}{}_{\nu}) = \frac{1}{2\kappa}\left((e^{\mu}_{a}e^{b}_{\nu}\eta_{\mu}{}^{\nu}) \wedge (e^{a}_{\alpha}e^{\beta}_{b}\varrho^{\alpha}{}_{\beta}) + \varepsilon\eta_{a\mathbf{n}} \wedge \varrho^{a\mathbf{n}} + \varepsilon\eta^{\mathbf{n}a} \wedge \varrho_{\mathbf{n}a}\right), \tag{4.35}$$

where we use the antisymmetry of $\eta^{\mu\nu}$ to simplify $n_{\mu}n_{\nu}\eta^{\mu\nu} = 0$ and introduce the abbreviations $\eta_{a\mathbf{n}} := e^{\mu}_{a}n_{\nu}\eta_{\mu}{}^{\nu}$ and $\eta^{\mathbf{n}a} := n^{\mu}e^{a}_{\nu}\eta_{\mu}{}^{\nu}$. Using this decomposed form of the Lagrangian, we obtain the Lagrange multipliers

$$*\varphi_{a\mathbf{n}} = \frac{1}{2\kappa}\eta_{a\mathbf{n}}\,, \qquad *\varphi^{\mathbf{n}a} = \frac{1}{2\kappa}\eta^{\mathbf{n}a}\,, \qquad *\varphi_{\mathbf{nn}} = 0 \tag{4.36}$$

by evaluating the variations in (4.27). We finally insert these Lagrange multipliers into the universal GHY term (4.26) to conclude

$$S^{\mathrm{EH}}_{\mathrm{GHY}} = \frac{\varepsilon}{2\kappa}\int_{\partial\mathcal{M}} (K^{a} + \tilde{K}^{a}) \wedge \eta_{\mathbf{n}a}\Big|_{\partial\mathcal{M}}\,, \tag{4.37}$$

where we use the antisymmetry of $\eta^{\mu\nu}$ for simplifying the GHY term. Thus, we have found (4.37) to be the GHY term for the Einstein-Hilbert action (4.32) in the presence of curvature, torsion and non-metricity. This simplifies if non-metricity vanishes, since we have $\tilde{K}^{a} = K^{a}$ in this case such that we are left with

$$S^{\mathrm{EH},Q=0}_{\mathrm{GHY}} = \frac{\varepsilon}{\kappa}\int_{\partial\mathcal{M}} K^{a} \wedge \eta_{\mathbf{n}a}\Big|_{\partial\mathcal{M}}\,. \tag{4.38}$$

For comparison with the traditional result for the GHY term in general relativity, we evaluate (4.38) in tensor components. To that end, we use the definition of the Hodge dual (4.17) as well as the relations (4.30) to obtain

$$\begin{aligned} &K^{a} \wedge \eta_{\mathbf{n}a}\Big|_{\partial\mathcal{M}} \\ &= \frac{\sqrt{|\det g|}}{(m-2)!}K^{a}{}_{b}n^{\mu}e^{\nu}_{a}\varepsilon_{\mu\nu\rho_{1}\ldots\rho_{m-2}}e^{\rho_{1}}_{b_{1}}\ldots e^{\rho_{m-2}}_{b_{m-2}}\phi^{b} \wedge \phi^{b_{1}} \wedge \cdots \wedge \phi^{b_{m-2}} \\ &= \sqrt{|\det\gamma|}K^{a}{}_{a}\mathrm{dVol}_{\partial\mathcal{M}}\,, \end{aligned} \tag{4.39}$$

where the volume element $\mathrm{dVol}_{\partial\mathcal{M}} = \phi^{0} \wedge \cdots \wedge \phi^{m-2}$ is usually written as $\mathrm{d}^{m-1}x$ in the chart-induced basis. Hence, the GHY term (4.38) of Einstein-Hilbert gravity reads

$$S^{\mathrm{EH},Q=0}_{\mathrm{GHY}} = \frac{\varepsilon}{\kappa}\int_{\partial\mathcal{M}} \mathrm{d}^{m-1}x\sqrt{|\det\gamma|}K^{a}{}_{a} \tag{4.40}$$

in components. This is the well-known result of York [102] which later was refined by Gibbons and Hawking [103]. However, note that we only assumed non-metricity to vanish in order to reproduce this result, while curvature and torsion may be non-vanishing. Hence, (4.40) already generalizes the results of [102, 103] to manifolds with non-trivial torsion. If additionally non-metricity is present, we instead need to consider (4.37) as the appropriate GHY term which we derived in full generality before.

We have been able to use the Lagrange multiplier formalism for deriving the GHY term of Einstein-Hilbert gravity in particular since the Lagrange multipliers (4.36) did not depend on curvature, torsion, non-metricity or the auxiliary fields ${\varrho_\mu}^\nu$. We will consider four-dimensional Chern-Simons modified gravity as a second example next, for which the Lagrange multipliers explicitly depend on these fields.

Four-dimensional Chern-Simons modified gravity

The action of four-dimensional Chern-Simons modified gravity [110] is a modification of the Einstein-Hilbert action we discussed in the previous example. In particular, the full action of this theory is $S^{\mathrm{CSMG}} = S^{\mathrm{EH}} + S^{\mathrm{CS}}$, being a sum of the four-dimensional Einstein-Hilbert action (4.32) and the Chern-Simons term

$$S^{\mathrm{CS}} = \frac{1}{8\kappa}\int_{\mathcal{M}} \mathrm{d}^4 x \sqrt{|\det g|}\, \chi * RR\,. \tag{4.41}$$

Here, χ is a background scalar field. The so-called Chern-Pontryagin scalar $*RR$ is given by

$$*RR := \frac{1}{2}\epsilon^{\mu\nu\sigma\rho} {R^\alpha}_{\beta\sigma\rho} {R^\beta}_{\alpha\mu\nu}\,, \tag{4.42}$$

where the Levi-Civita tensor $\epsilon^{\mu\nu\sigma\rho}$ differs from the totally antisymmetric symbol $\varepsilon^{\mu\nu\sigma\rho}$ by a factor $1/\sqrt{|\det g|}$, that is $\epsilon_{\mu\nu\sigma\rho} = \sqrt{|\det g|}\varepsilon_{\mu\nu\sigma\rho}$. The most straightforward way for expressing the four-dimensional Chern-Pontryagin scalar in differential form language is to realize that $\eta * RR = 2(-1)^{\operatorname{ind} g} {\Omega^\mu}_\nu \wedge {\Omega^\nu}_\mu$. This may be seen by combining the tensor components ${R^\mu}_{\nu\sigma\rho} = {\Omega^\mu}_\nu(\vartheta_\sigma, \vartheta_\rho)$ of the curvature two-form ${\Omega^\mu}_\nu = \frac{1}{2}{R^\mu}_{\nu\sigma\rho}\theta^\sigma \wedge \theta^\rho$ with the relations (4.30). Hence, the Chern-Simons contribution (4.41) of the action may be written as

$$S^{\mathrm{CS}} = \frac{1}{4\kappa}(-1)^{\operatorname{ind} g} \int_{\mathcal{M}} \chi\, {\Omega^\mu}_\nu \wedge {\Omega^\nu}_\mu\,. \tag{4.43}$$

Comparing the latter expression to the general form (4.15) of the action, we read off the Lagrangian

$$L^{\mathrm{CS}}({\Omega^\mu}_\nu) = \frac{1}{4\kappa}(-1)^{\operatorname{ind} g} \chi\, {\Omega^\mu}_\nu \wedge {\Omega^\nu}_\mu\,. \tag{4.44}$$

Let us now calculate the Lagrange multipliers $\varphi_{a\mathbf{n}}$, $\varphi^{\mathbf{n}a}$ and $\varphi_{\mathbf{nn}}$ by means of (4.27) to see if they are independent of the remaining fields. To that end, we decompose the Lagrangian using the unity decomposition (3.38) to obtain

$$L^{\mathrm{CS}}(\varrho^{\mu}{}_{\nu}) = \frac{1}{4\kappa}(-1)^{\operatorname{ind} g}\chi\left((e^a_\mu e^\nu_b \varrho^{\mu}{}_{\nu}) \wedge (e^b_\alpha e^\beta_a \varrho^{\alpha}{}_{\beta}) + 2\varepsilon\varrho^{a\mathbf{n}} \wedge \varrho_{\mathbf{n}a} + \varrho_{\mathbf{nn}} \wedge \varrho_{\mathbf{nn}}\right) . \quad (4.45)$$

Using this decomposition, (4.27) yields the Lagrange multipliers

$$*\varphi_{a\mathbf{n}} = \frac{1}{2\kappa}(-1)^{\operatorname{ind} g}\chi\varrho_{\mathbf{n}a} , \qquad *\varphi^{\mathbf{n}a} = \frac{1}{2\kappa}(-1)^{\operatorname{ind} g}\chi\varrho^{a\mathbf{n}} , \qquad *\varphi_{\mathbf{nn}} = \frac{1}{4\kappa}(-1)^{\operatorname{ind} g}\chi\varrho_{\mathbf{nn}} . \quad (4.46)$$

These Lagrange multipliers are in particular not independent of $\varrho^{\mu}{}_{\nu}$, and thus we may not use the Lagrange multiplier formalism of section 4.1.2. That is, we need to calculate the GHY term of (4.43) explicitly by means of decomposing the Lagrangian.

Inserting the curvature decomposition (4.14) on space- and timelike hypersurfaces into the Chern-Simons contribution (4.44) of the Lagrangian, we obtain

$$\begin{aligned} L^{\mathrm{CS}}(\Omega^{\mu}{}_{\nu}) =& \frac{1}{4\kappa}(-1)^{\operatorname{ind} g}\chi\left((e^a_\mu e^\nu_b \Omega^{\mu}{}_{\nu}) \wedge (e^b_\alpha e^\beta_a \Omega^{\alpha}{}_{\beta}) + 2\varepsilon(e^a_\mu n^\nu \Omega^{\mu}{}_{\nu}) \wedge (n_\alpha e^\beta_a \Omega^{\alpha}{}_{\beta})\right. \\ & \left. +(n_\mu n^\nu \Omega^{\mu}{}_{\nu}) \wedge (n_\alpha n^\beta \Omega^{\alpha}{}_{\beta})\right) \\ =& \frac{1}{4\kappa}(-1)^{\operatorname{ind} g}\chi\Big(\Omega^{a}{}_{b} \wedge \Omega^{b}{}_{a} - \varepsilon D(K^a \wedge D\tilde{K}_a) - \varepsilon D(\tilde{K}_a \wedge DK^a) \\ & + \frac{1}{4}DQ_{\mathbf{nn}} \wedge DQ_{\mathbf{nn}} + D(K^a \wedge \tilde{K}_a \wedge Q_{\mathbf{nn}})\Big) . \end{aligned} \quad (4.47)$$

We evaluate the integral of this Lagrangian by means of Stokes' theorem (2.21) to obtain

$$\begin{aligned} S^{\mathrm{CS}} =& \frac{1}{4\kappa}(-1)^{\operatorname{ind} g}\int_{\partial\mathcal{M}}\chi\Big(-\varepsilon K^a \wedge D\tilde{K}_a - \varepsilon\tilde{K}_a \wedge DK^a + \frac{1}{4}Q_{\mathbf{nn}} \wedge DQ_{\mathbf{nn}} \\ & + K^a \wedge \tilde{K}_a \wedge Q_{\mathbf{nn}}\Big)\Big|_{\partial\mathcal{M}} + \frac{1}{4\kappa}(-1)^{\operatorname{ind} g}\int_{\mathcal{M}}\chi\,\Omega^{a}{}_{b} \wedge \Omega^{b}{}_{a} . \end{aligned} \quad (4.48)$$

From this form of the action we directly read off

$$\begin{aligned} S^{\mathrm{CS}}_{\mathrm{GHY}} = \frac{1}{4\kappa}(-1)^{\operatorname{ind} g}\int_{\partial\mathcal{M}}\chi\Big(&\varepsilon K^a \wedge D\tilde{K}_a + \varepsilon\tilde{K}_a \wedge DK^a - \frac{1}{4}Q_{\mathbf{nn}} \wedge DQ_{\mathbf{nn}} \\ & - K^a \wedge \tilde{K}_a \wedge Q_{\mathbf{nn}}\Big)\Big|_{\partial\mathcal{M}} \end{aligned} \quad (4.49)$$

as the full GHY term for the Chern-Simons contribution (4.43) of the action. The GHY term of four-dimensional Chern-Simons modified gravity described by the entire action $S^{\text{CSMG}} = S^{\text{EH}} + S^{\text{CS}}$ is thus

$$\begin{aligned} S^{\text{CSMG}}_{\text{GHY}} &= S^{\text{EH}}_{\text{GHY}} + S^{\text{CS}}_{\text{GHY}} \\ &= \frac{\varepsilon}{2\kappa}(-1)^{\operatorname{ind} g} \int_{\partial\mathcal{M}} \Big((-1)^{\operatorname{ind} g} (K^a + \tilde{K}^a) \wedge \eta_{\mathbf{na}} + \frac{1}{2} K^a \wedge D\tilde{K}_a + \frac{1}{2}\tilde{K}_a \wedge DK^a \\ &\qquad\qquad - \frac{\varepsilon}{8} Q_{\mathbf{nn}} \wedge DQ_{\mathbf{nn}} - \frac{\varepsilon}{2} K^a \wedge \tilde{K}_a \wedge Q_{\mathbf{nn}} \Big)\Big|_{\partial\mathcal{M}} . \end{aligned} \tag{4.50}$$

This is the full GHY term of four-dimensional Chern-Simons modified gravity in the presence of curvature, torsion and non-metricity written in terms of differential forms. For manifolds on which both torsion and non-metricity are vanishing, the GHY term of this theory has already been calculated in [105] in tensor component notation. For comparison, we adopt the conventions of [105] and evaluate the Chern-Simons part (4.49) of the GHY term in components. Apart from vanishing torsion and non-metricity, [105] chooses $(-1)^{\operatorname{ind} g} = -1$ and $(-1)^{\operatorname{ind}\gamma} = +1$, such that $\varepsilon = (-1)^{\operatorname{ind} g + \operatorname{ind}\gamma} = -1$. We evaluate the pull-back of the Lagrangian to the boundary as

$$K^a \wedge D\tilde{K}_a\Big|_{\partial\mathcal{M}} = K^a{}_b \nabla_c \tilde{K}_{ad}\, \phi^b \wedge \phi^c \wedge \phi^d \,, \tag{4.51}$$

where we use the boundary covariant derivative $\nabla_c \tilde{K}_{ad} = \partial_c \tilde{K}_{ad} - \Gamma^b_{ca}\tilde{K}_{bd} - \Gamma^b_{cd}\tilde{K}_{ab}$ with $\Gamma^a_{cb} := \omega^a{}_b(\varphi_c)$. Since the boundary is considered as a three-dimensional manifold embedded in $\mathcal{M}$, $\phi^b \wedge \phi^c \wedge \phi^d$ is related to its volume form $\mathrm{dVol}_{\partial\mathcal{M}} = \phi^0 \wedge \phi^1 \wedge \phi^2$ as $\phi^b \wedge \phi^c \wedge \phi^d = (-1)^{\operatorname{ind}\gamma}\varepsilon^{bcd}\mathrm{dVol}_{\partial\mathcal{M}}$. Therefore, the GHY term (4.49) reduces to

$$S^{\text{CS}}_{\text{GHY}} = \frac{1}{2\kappa}\int_{\partial\mathcal{M}} \mathrm{d}^3x \sqrt{|\det\gamma|}\, \chi\, \epsilon^{bcd} K^a{}_b \nabla_c K_{ad} \tag{4.52}$$

for the conventions of [105]. This exactly reproduces the result of [105][4].

As a last example for how to use the formalism developed in section 4.1, let us consider Lovelock gravity on a manifold with boundary and derive its GHY term.

Lovelock gravity

Lovelock gravity originated as a generalization of the Einstein-Hilbert action including higher powers of curvature [111]. Following [112, 113], its action on an m-dimensional

[4] The global prefactors of both calculations match if we replace $\kappa \mapsto \frac{1}{2\kappa}$ in our derivation.

manifold $\mathcal{M}$ may be written as

$$S^{\mathrm{L}} = \sum_{p=0}^{\lfloor \frac{m-1}{2} \rfloor} \frac{\alpha_p}{m-2p} \mathcal{S}^p \,, \tag{4.53}$$

where the floor function $\lfloor x \rfloor$ yields the greatest integer less than or equal to x. The coefficients $\alpha_p \in \mathbb{R}$ may be chosen at will, and the p-th partial action $\mathcal{S}^p$ is given in differential form notation by

$$\mathcal{S}^p = \int_{\mathcal{M}} \epsilon_{\mu_1 \dots \mu_m} \Omega^{\mu_1 \mu_2} \wedge \dots \wedge \Omega^{\mu_{2p-1} \mu_{2p}} \wedge \theta^{\mu_{2p+1}} \wedge \dots \wedge \theta^{\mu_m} \,. \tag{4.54}$$

Here, we introduce the abbreviation $\Omega^{\mu\nu} := g^{\nu\rho} \Omega^{\mu}{}_{\rho}$. Note that this is nothing more but an abbreviation since the indices of $\Omega^{\mu}{}_{\nu}$ are Lie group indices which technically cannot be raised or lowered using the metric. We may rewrite the partial action (4.54) in a more compact form using the Hodge duals defined in (4.17). This definition immediately implies that $\mathcal{S}^p = (m-2p)!\, S^p$ with

$$S^p = \int_{\mathcal{M}} \eta_{\mu_1 \dots \mu_{2p}} \wedge \Omega^{\mu_1 \mu_2} \wedge \dots \wedge \Omega^{\mu_{2p-1} \mu_{2p}} \,, \tag{4.55}$$

such that the Lovelock action (4.53) becomes

$$S^{\mathrm{L}} = \sum_{p=0}^{\lfloor \frac{m-1}{2} \rfloor} \alpha_p (m-2p-1)!\, S^p \,. \tag{4.56}$$

Note that $S^0 = \int_{\mathcal{M}} \eta$ does not depend on curvature and hence does not require a GHY term for its variational principle to be well-defined. Hence, we have $S^0_{\mathrm{GHY}} = 0$. We consider $p = 1$ next, in which case the Lagrangian is linear in the curvature two-form. Since the Einstein-Hilbert action (4.32) is linear in curvature as well, it is useful to compare S^1 to S^{EH}. This is straightforward using the differential form expressions (4.32) and (4.55), from which we conclude $S^1 = 2\kappa S^{\mathrm{EH}}$. Therefore, the full Lovelock action may indeed be interpreted as a higher-curvature generalization of the Einstein-Hilbert action as we claimed before. From the correspondence of S^1 with S^{EH}, we obtain its GHY term as $S^1_{\mathrm{GHY}} = 2\kappa S^{\mathrm{EH}}_{\mathrm{GHY}}$, where we calculated the Einstein-Hilbert GHY term in the presence of curvature, torsion and non-metricity in (4.37).

Hence, the first non-trivial extension of Einstein-Hilbert gravity contained in the Lovelock action (4.56) is $S^2 = \int_{\mathcal{M}} \eta_{\mu\nu\rho\sigma} \wedge \Omega^{\mu\nu} \wedge \Omega^{\rho\sigma}$. Evaluating the Lagrangian in components by means of $\Omega^\mu{}_\nu = \frac{1}{2} R^\mu{}_{\nu\rho\sigma}\theta^\rho \wedge \theta^\sigma$, we obtain

$$S^2 = \int_{\mathcal{M}} \mathrm{dVol}\sqrt{|\det g|}\left(R^2 + R^{\mu\nu}{}_{\rho\sigma}R^{\rho\sigma}{}_{\mu\nu} + 2R^\mu{}_\nu R^{\nu\sigma}{}_{\sigma\mu} - R^\mu{}_\nu R^\nu{}_\mu - R^{\mu\sigma}{}_{\sigma\nu}R^{\nu\rho}{}_{\rho\mu}\right)\,, \tag{4.57}$$

where we introduce the components of Ricci tensor $R_{\mu\nu} \coloneqq R^\rho{}_{\mu\rho\nu}$ inducing the Ricci scalar $R \coloneqq g^{\mu\nu}R_{\mu\nu}$. While (4.57) still does not look very familiar, it simplifies considerably if non-metricity vanishes. From the Bianchi identity (2.42) we conclude that $\Omega^{\mu\nu} = -\Omega^{\nu\mu}$ for metric-compatible theories, and thus the component version of the action S^2 becomes

$$S^2 = \int_{\mathcal{M}} \mathrm{dVol}\sqrt{|\det g|}\left(R^2 + R^{\mu\nu}{}_{\rho\sigma}R^{\rho\sigma}{}_{\mu\nu} - 4R^\mu{}_\nu R^\nu{}_\mu\right)\,. \tag{4.58}$$

This is the Gauß-Bonnet action. Hence, S^2 is the straightforward extension of the Gauß-Bonnet action to manifolds with curvature, torsion and non-metricity. In four dimensions, S^2 is a topological term which yields the Euler number of a manifold. In arbitrary dimensions, the Lagrangians in S^p are called Euler densities [106]. We could now calculate the GHY term for Gauß-Bonnet gravity analogous to the calculation for four-dimensional Chern-Simons modified gravity we did before. However, we readily generalize this calculation and derive the GHY term of S^p for all p. To that end, we decompose the p-th partial action (4.55) into boundary tangent and normal contributions as

$$\begin{aligned} S^p = &\\ &\int_{\mathcal{M}} (e^{\mu_1}_{a_1}\dots e^{\mu_{2p}}_{a_{2p}}\eta_{\mu_1\dots\mu_{2p}}) \wedge (e^{a_1}_{\nu_1}e^{a_2}_{\nu_2}\Omega^{\nu_1\nu_2}) \wedge \dots \wedge (e^{a_{2p-1}}_{\nu_{2p-1}}e^{a_{2p}}_{\nu_{2p}}\Omega^{\nu_{2p-1}\nu_{2p}})\\ &+ p\varepsilon\eta_{\mathbf{n}a_1\dots a_{2p-1}} \wedge (n_\mu e^{a_1}_{\nu_1}(\Omega^{\mu\nu_1} - \Omega^{\nu_1\mu})) \wedge (e^{a_2}_{\nu_2}e^{a_3}_{\nu_3}\Omega^{\nu_2\nu_3}) \wedge \dots \wedge (e^{a_{2p-2}}_{\nu_{2p-2}}e^{a_{2p-1}}_{\nu_{2p-1}}\Omega^{\nu_{2p-2}\nu_{2p-1}})\,, \end{aligned} \tag{4.59}$$

where we use that any contraction of $\eta_{\mu_1\dots\mu_{2p}}$ with more than one unit normal n^μ vanishes due to the antisymmetry of $\eta_{\mu_1\dots\mu_{2p}}$. Moreover, this antisymmetry allowed to summarize all terms proportional to $\eta_{\mathbf{n}a_1\dots a_{2p-1}} \coloneqq n^\mu e^{\mu_1}_{a_1}\dots e^{\mu_{2p-1}}_{a_{2p-1}}\eta_{\mu\mu_1\dots\mu_{2p-1}}$ in the compact form (4.59). The antisymmetry of $\eta_{\mu_1\dots\mu_{2p}}$ may be used to simplify (4.59) even further. In particular, we have

$$e^{\mu_1}_{a_1}\dots e^{\mu_{2p}}_{a_{2p}}\eta_{\mu_1\dots\mu_{2p}}\Big|_{\partial\mathcal{M}} = 0 \tag{4.60}$$

on the boundary due to the definition (4.17) of the Hodge dual. That is, writing out $e^{\mu_1}_{a_1}\dots e^{\mu_{2p}}_{a_{2p}}\eta_{\mu_1\dots\mu_{2p}}\Big|_{\partial\mathcal{M}}$ in components yields a totally antisymmetric ε-symbol with m

indices which all are supposed to take their values on the $(m-1)$-dimensional boundary. Hence, the antisymmetry of the ε-symbol ultimately implies that $e^{\mu_1}_{a_1}\dots e^{\mu_{2p}}_{a_{2p}}\eta_{\mu_1\dots\mu_{2p}}\big|_{\partial\mathcal{M}}$ vanishes as we claimed in (4.60). We insert this relation into (4.59) and use the decomposition of curvature (4.14) to conclude that

$$S^p = -p\varepsilon\int_{\mathcal{M}}\eta_{\mathbf{n}a_1\dots a_{2p-1}}\wedge D(K^{a_1}+\tilde{K}^{a_1})\wedge(\Omega^{a_2a_3}-\varepsilon K^{a_2}\wedge\tilde{K}^{a_3})\wedge\dots$$
$$\dots\wedge(\Omega^{a_{2p-2}a_{2p-1}}-\varepsilon K^{a_{2p-2}}\wedge\tilde{K}^{a_{2p-1}})+\text{terms irrelevant on }\partial\mathcal{M}\,. \tag{4.61}$$

While it is extraordinarily involved to write this action as a boundary term if $Q_{\mu\nu}\neq 0$, the calculation simplifies in the metric-compatible case. There, we have $K^a=\tilde{K}^a$ and

$$\varepsilon_{a_1\dots a_q}D(K^{a_1}\wedge\dots\wedge K^{a_q}) = q\varepsilon_{a_1\dots a_q}DK^{a_1}\wedge\dots\wedge K^{a_q} \tag{4.62}$$

for arbitrary integers q. Note that we must include the prefactor $\frac{1}{q}$ which originates from (4.62) when writing (4.61) as a total derivative. Finally, we use the Bianchi identity (2.30) of curvature on $\partial\mathcal{M}$ to conclude that

$$\begin{aligned}S^p = -2p\varepsilon\int_{\mathcal{M}}\mathrm{d}\Big(\Big[&K^{a_1}\wedge\Omega^{a_2a_3}\wedge\dots\Omega^{a_{2p-2}a_{2p-1}}\\&-\varepsilon(p-1)\frac{1}{3}K^{a_1}\wedge K^{a_2}\wedge K^{a_3}\wedge\Omega^{a_4a_5}\wedge\dots\wedge\Omega^{a_{2p-2}a_{2p-1}}+\dots\\&+(-\varepsilon)^{p-1}\frac{1}{2p-1}K^{a_1}\wedge\dots\wedge K^{a_{2p-1}}\Big]\wedge\eta_{\mathbf{n}a_1\dots a_{2p-1}}\Big)\\&+\text{terms irrelevant on }\partial\mathcal{M}\,.\end{aligned} \tag{4.63}$$

For abbreviating the term in square brackets, we use the binomial theorem and introduce the binomial coefficient

$$\binom{p}{q} = \frac{p!}{q!(p-q)!} \tag{4.64}$$

to obtain

$$\begin{aligned}S^p = -2p\varepsilon\int_{\mathcal{M}}\mathrm{d}\Bigg(&\sum_{q=1}^{p}\binom{p-1}{q-1}\frac{(-\varepsilon)^{q-1}}{2q-1}K^{a_1}\wedge\dots\wedge K^{a_{2q-1}}\\&\wedge\Omega^{a_{2q}a_{2q+1}}\wedge\dots\wedge\Omega^{a_{2p-2}a_{2p-1}}\wedge\eta_{\mathbf{n}a_1\dots a_{2p-1}}\Bigg)\,.\end{aligned} \tag{4.65}$$

Hence, the Gibbons-Hawking-York term which makes the variational principle of the action S^p on manifolds with boundaries well-defined is

$$\begin{aligned} S^p_{\text{GHY}} = -2p \int_{\partial\mathcal{M}} \sum_{q=1}^{p} \binom{p-1}{q-1} \frac{(-\varepsilon)^q}{2q-1} K^{a_1} \wedge \cdots \wedge K^{a_{2q-1}} \\ \wedge \Omega^{a_{2q}a_{2q+1}} \wedge \cdots \wedge \Omega^{a_{2p-2}a_{2p-1}} \wedge \eta_{\mathbf{n}a_1\ldots a_{2p-1}}\Big|_{\partial\mathcal{M}} . \end{aligned} \tag{4.66}$$

For understanding this result, it is important to notice that an explicit calculation of the Hodge dual (4.17) reveals that $\eta_{\mathbf{n}a_1\ldots a_{2p-1}}\big|_{\partial\mathcal{M}}$ is in fact the corresponding Hodge dual on the boundary. That is, we have

$$\begin{aligned} \eta_{\mathbf{n}a_1\ldots a_{2p-1}}\Big|_{\partial\mathcal{M}} &= \gamma_{a_1b_1}\cdots\gamma_{a_{2p-1}b_{2p-1}} *_{\partial\mathcal{M}} (\phi^{b_1} \wedge \cdots \wedge \phi^{b_{2p-1}}) \\ &= \frac{\sqrt{|\det\gamma|}}{((m-1)-(2p-1))!} \varepsilon_{a_1\ldots a_{2p-1}b_1\ldots b_{m-2p}} \phi^{b_1} \wedge \cdots \wedge \phi^{b_{m-2p}} . \end{aligned} \tag{4.67}$$

We may write the result (4.66) for the GHY term of the p-th partial action more compact as

$$S^p_{\text{GHY}} = -2p \int_{\partial\mathcal{M}} \sum_{q=1}^{p} \binom{p-1}{q-1} \frac{(-\varepsilon)^q}{2q-1} \bigwedge_{m=1}^{2q-1} K^{a_m} \bigwedge_{n=q}^{p-1} \Omega^{a_{2n}a_{2n+1}} \wedge \eta_{\mathbf{n}a_1\ldots a_{2p-1}}\Big|_{\partial\mathcal{M}} , \tag{4.68}$$

which in components takes the form

$$\begin{aligned} S^p_{\text{GHY}} = -(2p)! \int_{\partial\mathcal{M}} \mathrm{dVol}_{\partial\mathcal{M}} \sqrt{|\det\gamma|} \sum_{q=1}^{p} \binom{p-1}{q-1} \frac{(-\varepsilon)^q}{2q-1} \frac{1}{2^{p-q}} \\ \prod_{m=1}^{2q-1} K^{a_m}{}_{[a_m} \prod_{n=q}^{p-1} R^{a_{2n}a_{2n+1}}{}_{a_{2n}a_{2n+1}]} . \end{aligned} \tag{4.69}$$

The GHY term given in different notations in (4.66), (4.68) and (4.69) is the full GHY term for the partial actions S^p we defined in (4.55) in the presence of curvature and torsion. By means of (4.56), this implies that the GHY term of the full m-dimensional Lovelock action S^{L} is

$$S^{\text{L}}_{\text{GHY}} = \sum_{p=0}^{\lfloor \frac{m-1}{2} \rfloor} \alpha_p (m-2p-1)!\, S^p_{\text{GHY}} . \tag{4.70}$$

This is, to my knowledge, the first time that the GHY term for the full Lovelock theory including torsion and curvature is given. It is my original result which has not been published before. However, there exist results for special cases of the theories considered here. First and foremost, the GHY term for all terms involved in Lovelock gravity has been proposed in [106] and revised in [107] using a dimensional continuation method. However, both only consider manifolds with curvature and the authors of [107] explicitly emphasize that their method does not extend to torsionful theories. Hence, our result is a generalization of these boundary terms which are sometimes called Gibbons-Hawking-Myers terms.

The authors of [107] gave the GHY terms explicitly for S^0, S^1 and S^2. In order to confirm that my results reproduce theirs in the limit of vanishing torsion, we thus calculate S^p_{GHY} for the first few p. Using the differential form version (4.68) of our result, these GHY terms are found to be

$$S^0_{\text{GHY}} = 0\,, \tag{4.71a}$$

$$S^1_{\text{GHY}} = 2\int_{\partial\mathcal{M}} \varepsilon K^a \wedge \eta_{\mathbf{n}a}\Big|_{\partial\mathcal{M}}\,, \tag{4.71b}$$

$$S^2_{\text{GHY}} = 4\int_{\partial\mathcal{M}} K^a \wedge \left(\varepsilon\Omega^{bc} - \frac{1}{3}K^b \wedge K^c\right) \wedge \eta_{\mathbf{n}abc}\Big|_{\partial\mathcal{M}}\,, \tag{4.71c}$$

$$\begin{aligned} S^3_{\text{GHY}} = 6\int_{\partial\mathcal{M}} K^{a_1} \wedge \Big(&\varepsilon\Omega^{a_2a_3} \wedge \Omega^{a_4a_5} - \frac{2}{3}K^{a_2} \wedge K^{a_3} \wedge \Omega^{a_4a_5} \\ &+ \frac{1}{5}\varepsilon K^{a_2} \wedge K^{a_3} \wedge K^{a_4} \wedge K^{a_5}\Big) \wedge \eta_{\mathbf{n}a_1a_2a_3a_4a_5}\Big|_{\partial\mathcal{M}} \end{aligned} \tag{4.71d}$$

in perfect agreement with the results of [107]. However, the advantage of my result is not only that it includes torsion, it furthermore does not require the introduction of background fields, parametric integrals or dimensional continuation in contrast to [107]. Note that S^2_{GHY} given in (4.71c) is the GHY term of Gauß-Bonnet gravity (4.57) in the presence of curvature and torsion since we identified S^2 with the Gauß-Bonnet action. For S^0 and $S^1 = 2\kappa S^{\text{EH}}$, we confirm the expectation $S^0_{\text{GHY}} = 0$ and $S^1_{\text{GHY}} = 2\kappa S^{\text{EH}}_{\text{GHY}}$ for which we argued in the beginning of this section.

Beyond the very general results of [106, 107], special cases of Lovelock gravity are routinely studied. Since the sum (4.56) which constitutes the Lovelock action only contains terms beyond Einstein-Hilbert gravity if $m \geq 5$, five-dimensional Lovelock gravity is often considered as the simplest non-trivial case. If the prefactors in this theory are chosen as $\alpha_0 = \kappa$, $\alpha_1 = 2\kappa$ and $\alpha_2 = \kappa$, the Lagrangian is that of a Chern-Simons theory and the theory is thus called five-dimensional Lovelock-Chern-Simons gravity. The latter theory was studied even in the presence of torsion in [114, 115] for instance, where an on-shell GHY term was calculated in [115]. Hence, the GHY term (4.70) derived in this section generalizes the results of [114, 115] for on-shell five-dimensional Lovelock-Chern-Simons gravity as well.

The general methods developed in section 4.1 may be applied to any action depending on curvature, torsion and non-metricity. The examples we considered in this subsection illustrate how the GHY term is calculated for such theories. This concludes the discussion of GHY terms for space- and timelike boundaries. We apply the methods used in section 4.1 to lightlike boundaries next.

4.2. Universal Gibbons-Hawking-York terms for lightlike boundaries

The method for deriving the GHY term for lightlike boundaries is the same as for space- and timelike boundaries which we discussed in section 4.1. That is, we first decompose curvature, torsion and non-metricity on lightlike hypersurfaces and subsequently interpret the boundary $\partial\mathcal{M}$ as a lightlike hypersurface. We study the Lagrange multiplier method on lightlike boundaries in section 4.2.2 and derive a universal lightlike GHY term from it. However, we already discussed in section 3.2 that the description of lightlike hypersurfaces is more involved than the space- and timelike cases, which is mainly due to the fact that the lightlike normal vector field k is both normal and tangent to lightlike hypersurfaces. Consequently, the derivation of lightlike GHY terms is more involved than the calculation of space- and timelike ones. Lightlike GHY terms have been studied in literature only very rarely, even if manifolds with vanishing torsion and non-metricity are considered. Hence, my results presented in this section yield entirely new insights on lightlike GHY terms. In particular, it is an entirely novel approach to apply my method for calculating GHY terms which we developed in the previous section to lightlike boundaries. My results for the such obtained universal lightlike GHY term are original and I have not published them so far. We start deriving this term by decomposing the field strengths of connection, coframe and metric in analogy to the space- and timelike case.

4.2.1. Decomposition of curvature, torsion and non-metricity

In section 3.2 we discussed lightlike hypersurfaces immersed in m-dimensional manifolds. One of the advantages of our normal vector approach to lightlike hypersurfaces was that the decompositions of tensors into boundary tangent and non-tangent contributions in $m-1$ dimensions was already contained in the $(m-2)$-dimensional decomposition. The latter, however, turned out to be more convenient for calculational purposes since we are able to define a non-degenerate metric only on the immersed $(m-2)$-dimensional hypersurface. On the latter hypersurface, we identified $e_A = e^\mu_A \vartheta_\mu$ with the hypersurface frame, while both k and l are vectors normal and non-tangent to this hypersurface. We will work in this framework in the following, but it is important to keep in mind that we need to describe the $(m-1)$-dimensional hypersurface in the end, since only this hypersurface may be used to model the manifold's boundary. Recall that k was both normal and tangent to the $(m-1)$-dimensional hypersurface, while l was non-normal and non-tangent. We will return to this special behavior of lightlike hypersurfaces later.

At this point, we use that both perspectives on lightlike hypersurfaces yield the same decomposition of tensors. Hence, we use the decompositions of the field strengths De^μ_A, $K^\mu = Dk^\mu$ and $L^\mu = Dl^\mu$ we already discussed in section 3.2.2 and derive the decompositions of curvature, torsion and non-metricity from them. Among the latter decompositions, the evaluation of non-metricity is particularly straightforward since most of its projections are contained in the terms discussed in section 3.2.2. From the results of this section we immediately read off

$$\begin{aligned} k^\mu k^\nu Q_{\mu\nu} &\equiv Q_{\mathbf{kk}} = 2k_\mu Dk^\mu\,, & e^\mu_A k^\nu Q_{\mu\nu} &= k^\mu e^\nu_A Q_{\mu\nu} = K_A - \tilde{K}_A\,,\\ l^\mu l^\nu Q_{\mu\nu} &\equiv Q_{\mathbf{ll}} = 2l_\mu Dl^\mu\,, & e^\mu_A l^\nu Q_{\mu\nu} &= l^\mu e^\nu_A Q_{\mu\nu} = L_A - \tilde{L}_A\,,\\ k^\mu l^\nu Q_{\mu\nu} &= l^\mu k^\nu Q_{\mu\nu} = \mathcal{K} + \mathcal{L}\,. \end{aligned} \tag{4.72}$$

These projections determine most of the terms in the decomposition of non-metricity, which we obtain by contracting $Q_{\mu\nu}$ with the lightlike unity decomposition (3.87). In particular, simplifying $Q_{\mu\nu} = \delta^\alpha_\mu \delta^\beta_\nu Q_{\alpha\beta}$ by means of non-metricity's symmetry yields

$$\begin{aligned} Q_{\mu\nu} =& e^A_\mu e^B_\nu (e^\alpha_A e^\beta_B Q_{\alpha\beta}) + 2\varepsilon e^A_\mu k_\nu (e^\alpha_A l^\beta Q_{\alpha\beta}) + 2\varepsilon e^A_\mu l_\nu (e^\alpha_A k^\beta Q_{\alpha\beta})\\ &+ \varepsilon^2 k_\mu k_\nu (l^\alpha l^\beta Q_{\alpha\beta}) + \varepsilon^2 l_\mu l_\nu (k^\alpha k^\beta Q_{\alpha\beta}) + 2\varepsilon^2 l_\mu k_\nu (k^\alpha l^\beta Q_{\alpha\beta})\,, \end{aligned} \tag{4.73}$$

so that the only projection left to determine is $e^\mu_A e^\nu_B Q_{\mu\nu}$. To that effect, we exploit the definitions $Q_{\mu\nu} = -Dg_{\mu\nu}$ and $\sigma_{AB} = e^\mu_A e^\nu_B g_{\mu\nu}$ of the non-metricity one-form and the hypersurface metric. Using the connection transformation law (3.89), we conclude

$$e^\mu_A e^\nu_B Q_{\mu\nu} = -D(e^\mu_A e^\nu_B g_{\mu\nu}) + g_{\mu\nu} e^\mu_A De^\nu_B + g_{\mu\nu} e^\nu_B De^\mu_A = Q_{AB}\,. \tag{4.74}$$

Hence, the projection of non-metricity to the $(m-2)$-dimensional hypersurface in all indices yields the hypersurface non-metricity $Q_{AB} := -D\sigma_{AB}$. This is what we expect, since the indices of the non-metricity one-form are tensor indices unlike those of curvature and torsion.

For determining the decomposition of the torsion two-form, we use its definition $T^\mu = D\theta^\mu$ and begin by projecting T^μ to the normal direction by contracting it with k. This yields

$$k_\mu T^\mu = D(k_\mu \theta^\mu) - Dk_\mu \wedge \theta^\mu\,. \tag{4.75}$$

We already discussed the terms involved in this decomposition. On the one hand, we decomposed $\tilde{K}_\mu = Dk_\mu$ in (3.92), on the other hand $\tilde{k} = k_\mu\theta^\mu$ is just the normal covector which we identified with $\frac{1}{\varepsilon}\phi$ in (3.73). Collecting these results, we obtain

$$k_\mu T^\mu = \frac{1}{\varepsilon}D\phi - \tilde{K}_A \wedge \phi^A + \mathcal{L} \wedge \phi + \frac{1}{2}Q_{\mathbf{kk}} \wedge \psi\,. \tag{4.76}$$

We may perform the same calculation for $l_\mu T^\mu$, but this calculation is simplified if we recall that all equations are symmetric with respect to the exchange of k and l. Hence, we immediately obtain

$$l_\mu T^\mu = \frac{1}{\varepsilon}D\psi - \tilde{L}_A \wedge \phi^A + \mathcal{K} \wedge \psi + \frac{1}{2}Q_{\mathbf{ll}} \wedge \phi\,, \tag{4.77}$$

where we use that $\psi = \varepsilon\tilde{l}$ and $\phi = \varepsilon\tilde{k}$ are aligned with $\tilde{l}$ and $\tilde{k}$, respectively, see (3.73). We calculate the hypersurface tangent projection $e^A_\mu T^\mu$ of the torsion two-form analogous to the normal projections by using the definition of T^μ to obtain

$$e^A_\mu T^\mu = D(e^A_\mu\theta^\mu) - De^A_\mu \wedge \theta^\mu\,, \tag{4.78}$$

where we know the decomposition of all the terms included in this projection from section 3.2. In particular, we found $e^A_\mu\theta^\mu = \phi^A$ in (3.73) and derived the decomposition of De^a_μ in (3.96) to be $De^A_\mu = -\varepsilon k_\mu L^A - \varepsilon l_\mu K^A$. Thus, the hypersurface tangent projection of the torsion two-form is

$$e^A_\mu T^\mu = T^A + L^A \wedge \phi + K^A \wedge \psi\,, \tag{4.79}$$

where we introduce the torsion two-form $T^A := D\phi^A$ of the $(m-2)$-dimensional hypersurface. Hence, we have collected all the projections needed to determine the decomposition of the torsion two-form

$$T^\mu = \delta^\mu_\nu T^\nu = e^\mu_A(e^A_\nu T^\nu) + \varepsilon k^\mu(l_\nu T^\nu) + \varepsilon l^\mu(k_\nu T^\nu) \tag{4.80}$$

which we obtain by means of the lightlike unity decomposition (3.87).

Using the unity decomposition (3.87) once more, we evaluate ${\Omega^\mu}_\nu = \delta^\mu_\alpha\delta^\beta_\nu{\Omega^\alpha}_\beta$ to obtain the curvature decomposition

$$\begin{aligned}{\Omega^\mu}_\nu =& e^\mu_A e^B_\nu(e^A_\alpha e^\beta_B{\Omega^\alpha}_\beta) + \varepsilon e^\mu_A k_\nu(e^A_\alpha l^\beta{\Omega^\alpha}_\beta) + \varepsilon e^\mu_A l_\nu(e^A_\alpha k^\beta{\Omega^\alpha}_\beta)\\ &+ \varepsilon k^\mu e^A_\nu(l_\alpha e^\beta_A{\Omega^\alpha}_\beta) + \varepsilon l^\mu e^A_\nu(k_\alpha e^\beta_A{\Omega^\alpha}_\beta) + \varepsilon^2 k^\mu k_\nu(l_\alpha l^\beta{\Omega^\alpha}_\beta)\\ &+ \varepsilon^2 l^\mu l_\nu(k_\alpha k^\beta{\Omega^\alpha}_\beta) + \varepsilon^2 k^\mu l_\nu(l_\alpha k^\beta{\Omega^\alpha}_\beta) + \varepsilon^2 l^\mu k_\nu(k_\alpha l^\beta{\Omega^\alpha}_\beta)\,,\end{aligned} \tag{4.81}$$

and we need to determine the projections in the parentheses of the latter equation analogous to the decompositions of torsion and non-metricity. These projections are contained in the generalized Ricci identity (3.97) of lightlike hypersurfaces. We will only discuss one example for this calculation explicitly, while all remaining projections implicitly contained in (3.97) are evaluated analogously. For this example, we obtain the three projections $e^B_\mu e^\nu_A \Omega^\mu{}_\nu$, $k_\mu e^\nu_A \Omega^\mu{}_\nu$ and $l_\mu e^\nu_A \Omega^\mu{}_\nu$ from contracting

$$D^2 e^\mu_A = \Omega^\mu{}_\nu e^\nu_A - \Omega^B{}_A e^\mu_B \tag{4.82}$$

with e^B_μ, k_μ and l_μ, respectively. We first consider the projection to the $(m-2)$-dimensional hypersurface in both indices for which we obtain

$$e^A_\mu e^\nu_B \Omega^\mu{}_\nu = e^A_\mu D^2 e^\mu_B + e^A_\mu \Omega^C{}_B e^\mu_C = D(e^A_\mu D e^\mu_B) - De^A_\mu \wedge De^\mu_B + \Omega^A{}_B \,. \tag{4.83}$$

Using the transformation law (3.89) of the connection one-form to hypersurfaces, we have $e^A_\mu De^\mu_B = 0$ and the first term on the right hand side of the latter projection vanishes. Hence, it remains to insert the decompositions of De^μ_B and De^A_μ which we constructed in (3.95) and (3.96), respectively. Collecting these results, the projection of curvature to the $(m-2)$-dimensional hypersurface is

$$e^A_\mu e^\nu_B \Omega^\mu{}_\nu = \Omega^A{}_B - \varepsilon K^A \wedge \tilde{L}_B - \varepsilon L^A \wedge \tilde{K}_B \,. \tag{4.84}$$

We proceed analogously to derive the normal projected component of the Ricci identity (4.82). By means of the normality condition $g(k, e_A) = k_\mu e^\mu_A = 0$ we have

$$k_\mu e^\nu_A \Omega^\mu{}_\nu = k_\mu D^2 e^\mu_A + k_\mu \Omega^B{}_A e^\mu_B = D(k_\mu De^\mu_A) - Dk_\mu \wedge De^\mu_A \,. \tag{4.85}$$

We derived the lightlike decomposition of $\tilde{K}_\mu = Dk_\mu$ in (3.92), involving the extrinsic curvature one-form $\tilde{K}_A = -k_\mu De^\mu_A$ as the tangent component. Furthermore, we use (3.95) for decomposing De^μ_A once more, yielding

$$k_\mu e^\nu_A \Omega^\mu{}_\nu = -D\tilde{K}_A + \varepsilon \tilde{K}_A \wedge \mathcal{L} + \frac{\varepsilon}{2} \tilde{L}_A \wedge Q_{\mathbf{kk}} \tag{4.86}$$

as the decomposition of $k_\mu e^\nu_A \Omega^\mu{}_\nu$. Finally, the Ricci identity (4.82) induces the decomposition of $l_\mu e^\nu_A \Omega^\mu{}_\nu$. We may obtain this projection immediately from the $k_\mu e^\nu_A \Omega^\mu{}_\nu$ decomposition by using the symmetry of k and l in the $(m-2)$-dimensional formalism. This yields

$$l_\mu e^\nu_A \Omega^\mu{}_\nu = -D\tilde{L}_A + \varepsilon \tilde{L}_A \wedge \mathcal{K} + \frac{\varepsilon}{2} \tilde{K}_A \wedge Q_{\mathbf{ll}} \tag{4.87}$$

with which we derived all the decompositions implied by the Ricci identity (4.82).

We use the remaining Ricci identities for lightlike hypersurfaces that are summarized in (3.97) analogously to calculate all projections of the curvature two-form. This yields, in summary,

$$\begin{aligned}
e^A_\mu e^\nu_B \Omega^\mu{}_\nu &= \Omega^A{}_B - \varepsilon K^A \wedge \tilde{L}_B - \varepsilon L^A \wedge \tilde{K}_B \,, \\
k_\mu e^\nu_A \Omega^\mu{}_\nu &= -D\tilde{K}_A + \varepsilon \tilde{K}_A \wedge \mathcal{L} + \frac{\varepsilon}{2} \tilde{L}_A \wedge Q_{\mathbf{kk}} \,, \\
e^A_\mu k^\nu \Omega^\mu{}_\nu &= DK^A + \varepsilon K^A \wedge \mathcal{K} + \frac{\varepsilon}{2} L^A \wedge Q_{\mathbf{kk}} \,, \\
l_\mu e^\nu_A \Omega^\mu{}_\nu &= -D\tilde{L}_A + \varepsilon \tilde{L}_A \wedge \mathcal{K} + \frac{\varepsilon}{2} \tilde{K}_A \wedge Q_{\mathbf{ll}} \,, \\
e^A_\mu l^\nu \Omega^\mu{}_\nu &= DL^A + \varepsilon L^A \wedge \mathcal{L} + \frac{\varepsilon}{2} K^A \wedge Q_{\mathbf{ll}} \,, \\
k_\mu l^\nu \Omega^\mu{}_\nu &= D\mathcal{L} + L^A \wedge \tilde{K}_A - \frac{\varepsilon}{4} Q_{\mathbf{ll}} \wedge Q_{\mathbf{kk}} \,, \\
l_\mu k^\nu \Omega^\mu{}_\nu &= D\mathcal{K} + K^A \wedge \tilde{L}_A - \frac{\varepsilon}{4} Q_{\mathbf{kk}} \wedge Q_{\mathbf{ll}} \,, \\
k_\mu k^\nu \Omega^\mu{}_\nu &= \frac{1}{2} DQ_{\mathbf{kk}} + K^A \wedge \tilde{K}_A - \frac{\varepsilon}{2} (\mathcal{K} - \mathcal{L}) \wedge Q_{\mathbf{kk}} \,, \\
l_\mu l^\nu \Omega^\mu{}_\nu &= \frac{1}{2} DQ_{\mathbf{ll}} + L^A \wedge \tilde{L}_A - \frac{\varepsilon}{2} (\mathcal{L} - \mathcal{K}) \wedge Q_{\mathbf{ll}} \,.
\end{aligned} \tag{4.88}$$

With these projections, we have finally collected all the decompositions needed for calculating the GHY term of an action constructed from curvature, torsion and non-metricity on manifolds with a lightlike boundary. We thus proceed by deriving the universal GHY term for these actions, which serves as a guideline as well for those cases in which the Lagrange multiplier formalism is not applicable.

4.2.2. Universal Gibbons-Hawking-York term from Lagrange multipliers

In order to examine general properties of the GHY term on manifolds with light-like boundaries, we consider generic theories built from curvature, torsion and non-metricity. That is, we investigate actions of the form

$$S[\omega^\mu{}_\nu, \theta^\mu, g_{\mu\nu}] = \int_\mathcal{M} L(\Omega^\mu{}_\nu, T^\mu, Q_{\mu\nu}) \tag{4.89}$$

and do not specify the explicit form of the Lagrangian $L(\Omega^\mu{}_\nu, T^\mu, Q_{\mu\nu})$. In equivalence to the space- and timelike case discussed in section 4.1.2, we introduce the Lagrange multiplier two-forms ${\varphi_\mu}^\nu$ and t_μ as well as the one-form $q^{\mu\nu}$ to define the action

$$\begin{aligned}&S_{\mathrm{Lagr}}[{\omega^\mu}_\nu, \theta^\mu, g_{\mu\nu}, {\varphi_\mu}^\nu, {\varrho^\mu}_\nu, t_\mu, \tau^\mu, q^{\mu\nu}, \sigma_{\mu\nu}] \\ &= \int_{\mathcal{M}} [L({\varrho^\mu}_\nu, \tau^\mu, \sigma_{\mu\nu}) + *{\varphi_\mu}^\nu \wedge ({\Omega^\mu}_\nu - {\varrho^\mu}_\nu) + *t_\mu \wedge (T^\mu - \tau^\mu) + *q^{\mu\nu} \wedge (Q_{\mu\nu} - \sigma_{\mu\nu})] \,.\end{aligned} \tag{4.90}$$

The two-forms ${\varrho^\mu}_\nu$, τ^μ and the one-form $\sigma_{\mu\nu}$ are auxiliary fields introduced such that the equations of motion of the Lagrange multipliers are

$$\Omega^\mu{}_\nu = {\varrho^\mu}_\nu \,, \qquad T^\mu = \tau^\mu \,, \qquad Q_{\mu\nu} = \sigma_{\mu\nu} \,. \tag{4.91}$$

While everything in the Lagrange multiplier formalism for lightlike GHY terms is equivalent to the space- and timelike cases so far, the decomposition of the Lagrange multiplier action must be evaluated with respect to the unity decomposition (3.87) on lightlike hypersurfaces. That is, we interpret the boundary $\partial\mathcal{M}$ as a lightlike hypersurface and insert the lightlike decompositions (4.73), (4.80) and (4.81) of non-metricity, torsion and curvature into the action (4.90). In particular, the contraction of these field strengths with their Lagrange multipliers decomposes as

$$\begin{aligned}*{\varphi_\mu}^\nu \wedge {\Omega^\mu}_\nu =&(e^\mu_A e^B_\nu * {\varphi_\mu}^\nu) \wedge (e^A_\alpha e^\beta_B {\Omega^\alpha}_\beta) + \varepsilon(e^\mu_A k_\nu * {\varphi_\mu}^\nu) \wedge (e^A_\alpha l^\beta {\Omega^\alpha}_\beta) \\ &+ \varepsilon(e^\mu_A l_\nu * {\varphi_\mu}^\nu) \wedge (e^A_\alpha k^\beta {\Omega^\alpha}_\beta) + \varepsilon(k^\mu e^A_\nu * {\varphi_\mu}^\nu) \wedge (l_\alpha e^\beta_A {\Omega^\alpha}_\beta) \\ &+ \varepsilon(l^\mu e^A_\nu * {\varphi_\mu}^\nu) \wedge (k_\alpha e^\beta_A {\Omega^\alpha}_\beta) + \varepsilon^2(k^\mu k_\nu * {\varphi_\mu}^\nu) \wedge (l_\alpha l^\beta {\Omega^\alpha}_\beta) \\ &+ \varepsilon^2(l^\mu l_\nu * {\varphi_\mu}^\nu) \wedge (k_\alpha k^\beta {\Omega^\alpha}_\beta) + \varepsilon^2(k^\mu l_\nu * {\varphi_\mu}^\nu) \wedge (l_\alpha k^\beta {\Omega^\alpha}_\beta) \\ &+ \varepsilon^2(l^\mu k_\nu * {\varphi_\mu}^\nu) \wedge (k_\alpha l^\beta {\Omega^\alpha}_\beta) \,, \\ *t_\mu \wedge T^\mu =&(e^\mu_A * t_\mu) \wedge (e^A_\nu T^\nu) + \varepsilon(k^\mu * t_\mu) \wedge (l_\nu T^\nu) + \varepsilon(l^\mu * t_\mu) \wedge (k_\nu T^\nu) \,, \\ *q^{\mu\nu} \wedge Q_{\mu\nu} =&(e^A_\mu e^B_\nu * q^{\mu\nu}) \wedge (e^\alpha_A e^\beta_B Q_{\alpha\beta}) + 2\varepsilon(e^A_\mu k_\nu * q^{\mu\nu}) \wedge (e^\alpha_A l^\beta Q_{\alpha\beta}) \\ &+ 2\varepsilon(e^A_\mu l_\nu * q^{\mu\nu}) \wedge (e^\alpha_A k^\beta Q_{\alpha\beta}) + \varepsilon^2(k_\mu k_\nu * q^{\mu\nu}) \wedge (l^\alpha l^\beta Q_{\alpha\beta}) \\ &+ \varepsilon^2(l_\mu l_\nu * q^{\mu\nu}) \wedge (k^\alpha k^\beta Q_{\alpha\beta}) + 2\varepsilon^2(l_\mu k_\nu * q^{\mu\nu}) \wedge (k^\alpha l^\beta Q_{\alpha\beta})\end{aligned} \tag{4.92}$$

on lightlike hypersurfaces. Recall that we decompose these contractions since we need to single out the boundary terms contained in the Lagrangian. We obtain these boundary contributions by means of Stokes' theorem (2.21). Hence, only those terms in the lightlike projections of curvature, torsion and non-metricity containing derivatives yield non-trivial boundary terms. Using the decompositions we derived in section 4.2.1, the relevant terms are thus

$$\begin{aligned}
*\varphi_\mu{}^\nu \wedge \Omega^\mu{}_\nu = & * \varphi_A{}^B \wedge \Omega^A{}_B + \varepsilon * \varphi_{A\mathbf{k}} \wedge DL^A + \varepsilon * \varphi_{A\mathbf{l}} \wedge DK^A \\
& - \varepsilon * \varphi^{\mathbf{k}A} \wedge D\tilde{L}_A - \varepsilon * \varphi^{\mathbf{l}A} \wedge D\tilde{K}_A + \frac{\varepsilon^2}{2} * \varphi_{\mathbf{kk}} \wedge DQ_{\mathbf{ll}} \\
& + \frac{\varepsilon^2}{2} * \varphi_{\mathbf{ll}} \wedge DQ_{\mathbf{kk}} + \varepsilon^2 * \varphi_{\mathbf{kl}} \wedge D\mathcal{K} + \varepsilon^2 * \varphi_{\mathbf{lk}} \wedge D\mathcal{L} \\
& + \text{terms irrelevant on } \partial\mathcal{M} ,
\end{aligned} \tag{4.93a}$$

$$\begin{aligned}
*t_\mu \wedge T^\mu = & (e^\mu_A * t_\mu) \wedge T^A + \varepsilon(k^\mu * t_\mu) \wedge D\Psi + \varepsilon(l^\mu * t_\mu) \wedge D\phi \\
& + \text{terms irrelevant on } \partial\mathcal{M} ,
\end{aligned} \tag{4.93b}$$

$$*q^{\mu\nu} \wedge Q_{\mu\nu} = (e^A_\mu e^B_\nu * q^{\mu\nu}) \wedge Q_{AB} + \text{terms irrelevant on } \partial\mathcal{M} , \tag{4.93c}$$

where we introduce the abbreviations

$$\begin{aligned}
*\varphi_A{}^B &:= e^\mu_A e^B_\nu * \varphi_\mu{}^\nu , & *\varphi_{A\mathbf{k}} &:= e^\mu_A k_\nu * \varphi_\mu{}^\nu , & *\varphi_{A\mathbf{l}} &:= e^\mu_A l_\nu * \varphi_\mu{}^\nu , \\
*\varphi^{\mathbf{k}A} &:= k^\mu e^A_\nu * \varphi_\mu{}^\nu , & *\varphi^{\mathbf{l}A} &:= l^\mu e^A_\nu * \varphi_\mu{}^\nu , & *\varphi_{\mathbf{kk}} &:= k^\mu k_\nu * \varphi_\mu{}^\nu , \\
*\varphi_{\mathbf{ll}} &:= l^\mu l_\nu * \varphi_\mu{}^\nu , & *\varphi_{\mathbf{kl}} &:= k^\mu l_\nu * \varphi_\mu{}^\nu , & *\varphi_{\mathbf{lk}} &:= l^\mu k_\nu * \varphi_\mu{}^\nu
\end{aligned} \tag{4.94}$$

for the Lagrange multipliers of curvature.

We reinstate the decompositions (4.93) into the Lagrange multiplier action (4.90) to obtain the boundary relevant contribution of S_{Lagr} as

$$\begin{aligned}
S_{\text{Lagr}} = \int_{\partial\mathcal{M}} \Big(& \omega^A{}_B \wedge *\varphi_A{}^B + \varepsilon L^A \wedge *\varphi_{A\mathbf{k}} + \varepsilon K^A \wedge *\varphi_{A\mathbf{l}} - \varepsilon \tilde{L}_A \wedge *\varphi^{\mathbf{k}A} - \varepsilon \tilde{K}_A \wedge *\varphi^{\mathbf{l}A} \\
& + \frac{\varepsilon^2}{2} Q_{\mathbf{ll}} \wedge *\varphi_{\mathbf{kk}} + \frac{\varepsilon^2}{2} Q_{\mathbf{kk}} \wedge *\varphi_{\mathbf{ll}} + \varepsilon^2 \mathcal{K} \wedge *\varphi_{\mathbf{kl}} + \varepsilon^2 \mathcal{L} \wedge *\varphi_{\mathbf{lk}} + \phi^A \wedge (e^\mu_A * t_\mu) \\
& + \Psi \wedge (k^\mu * t_\mu) + \phi \wedge (l^\mu * t_\mu) + (-1)^m \sigma_{AB} \wedge (e^A_\mu e^B_\nu * q^{\mu\nu}) \Big) \Big|_{\partial\mathcal{M}} \\
& + \text{terms irrelevant on } \partial\mathcal{M} ,
\end{aligned} \tag{4.95}$$

where we have used Stokes' theorem (2.21). For obtaining the GHY term induced by this boundary integral, we impose the Dirichlet boundary conditions

$$\delta\omega^\mu{}_\nu \Big|_{\partial\mathcal{M}} = 0 , \qquad \delta\theta^\mu \Big|_{\partial\mathcal{M}} = 0 , \qquad \delta g_{\mu\nu} \Big|_{\partial\mathcal{M}} = 0 . \tag{4.96}$$

We may express these boundary conditions in terms of the boundary fields as proven in [20], but for this purpose we need to recall that the lightlike normal vector field k is tangent to the $(m-1)$-dimensional boundary as well. Hence, we need to impose boundary conditions on $\phi = \varepsilon\tilde{k}$ as well, and the Dirichlet boundary conditions become

$$\delta\omega^A{}_B = 0\,, \qquad \delta\phi^A = 0\,, \qquad \delta\phi = 0\,, \qquad \delta\sigma_{AB} = 0\,. \tag{4.97}$$

Furthermore, we recall that $\Psi = \varepsilon\tilde{l}$ is non-tangent to the $(m-1)$-dimensional boundary and thus we have $\Psi\big|_{\partial\mathcal{M}} = 0$. Collecting these results, we finally read off the GHY term from (4.95) and conclude

$$\begin{aligned} S_{\text{GHY}} = -\varepsilon \int_{\partial\mathcal{M}} \Big(& L^A \wedge *\varphi_{A\mathbf{k}} + K^A \wedge *\varphi_{A\mathbf{l}} - \tilde{L}_A \wedge *\varphi^{\mathbf{k}A} - \tilde{K}_A \wedge *\varphi^{\mathbf{l}A} \\ & + \frac{\varepsilon}{2} Q_{\mathbf{ll}} \wedge *\varphi_{\mathbf{kk}} + \frac{\varepsilon}{2} Q_{\mathbf{kk}} \wedge *\varphi_{\mathbf{ll}} + \varepsilon\mathcal{K} \wedge *\varphi_{\mathbf{kl}} + \varepsilon\mathcal{L} \wedge *\varphi_{\mathbf{lk}} \Big)\Big|_{\partial\mathcal{M}} . \end{aligned} \tag{4.98}$$

This is the universal GHY term for the Lagrange multiplier action (4.90) considered on manifolds with lightlike boundaries.

We obtain the GHY term of the generic action (4.26) from the Lagrange multiplier result (4.98) by calculating the on-shell values of the Lagrange multipliers $\varphi_\mu{}^\nu$. These are obtained from variations of the action (4.90) with respect to the auxiliary fields $\varrho^\mu{}_\nu$. To see this, we enforce Hamilton's principle in the variation $\delta_{\varrho^\mu{}_\nu} S_{\text{Lagr}} = 0$ which yields the constraint $*\varphi_\mu{}^\nu \wedge \delta\varrho^\mu{}_\nu = \delta_{\varrho^\mu{}_\nu} L(\varrho^\mu{}_\nu)$. Analogous to the non-lightlike case, it will be useful for calculational purposes to decompose the Lagrangian into boundary tangent and non-tangent contributions. We therefore need the projected version of the latter constraint as well. Therefore, we use the unity decomposition (3.87) on lightlike hypersurfaces to decompose the Lagrange multiplier action (4.90). Subsequently invoking Hamilton's principle yields

$$\begin{aligned} &*\varphi_A{}^B \wedge \delta\varrho^A{}_B = \delta_{\varrho^A{}_B} L\,, && *\varphi_{A\mathbf{k}} \wedge \delta\varrho^{A\mathbf{l}} = \frac{1}{\varepsilon}\delta_{\varrho^{A\mathbf{l}}} L\,, && *\varphi_{A\mathbf{l}} \wedge \delta\varrho^{A\mathbf{k}} = \frac{1}{\varepsilon}\delta_{\varrho^{A\mathbf{k}}} L\,, \\ &*\varphi^{\mathbf{k}A} \wedge \delta\varrho_{\mathbf{l}A} = \frac{1}{\varepsilon}\delta_{\varrho_{\mathbf{l}A}} L\,, && *\varphi^{\mathbf{l}A} \wedge \delta\varrho_{\mathbf{k}A} = \frac{1}{\varepsilon}\delta_{\varrho_{\mathbf{k}A}} L\,, && *\varphi_{\mathbf{kk}} \wedge \delta\varrho_{\mathbf{ll}} = \frac{1}{\varepsilon^2}\delta_{\varrho_{\mathbf{ll}}} L\,, \\ &*\varphi_{\mathbf{ll}} \wedge \delta\varrho_{\mathbf{kk}} = \frac{1}{\varepsilon^2}\delta_{\varrho_{\mathbf{kk}}} L\,, && *\varphi_{\mathbf{lk}} \wedge \delta\varrho_{\mathbf{kl}} = \frac{1}{\varepsilon^2}\delta_{\varrho_{\mathbf{kl}}} L\,, && *\varphi_{\mathbf{kl}} \wedge \delta\varrho_{\mathbf{lk}} = \frac{1}{\varepsilon^2}\delta_{\varrho_{\mathbf{lk}}} L\,. \end{aligned} \tag{4.99}$$

We define the abbreviations of the projections of $\varrho^\mu{}_\nu$ analogous to those of $\varphi^\mu{}_\nu$ in (4.94).

Using the constraints (4.99), we are able to calculate the on-shell values of the Lagrange multipliers for a given action of the form (4.89). Provided that these Lagrange multipliers do not depend on the remaining fields as we discussed in the space- and timelike case, we insert their on-shell values into the GHY term (4.98). Hence, we have found (4.98) to be the universal GHY term of the generic action (4.89) describing theories which depend on curvature, torsion and non-metricity on manifolds with lightlike boundaries. This is the first time ever that this result is found.

The universal GHY term (4.98) of manifolds with lightlike boundaries may be seen as a straightforward generalization of the result (4.26) for space- and timelike boundaries. Indeed, all the extrinsic curvatures appear in (4.98) in addition to the twice normal projected components of non-metricity. This is exactly what we found in the space- and timelike case. From this analogy to the non-lightlike case, we expect that only Lagrangians containing curvature require us to introduce a GHY term. This is indeed what we find for lightlike boundaries in (4.98) as well. Note that the calculation of the lightlike universal GHY term (4.98) by means of evaluating the Lagrange multiplier constraints (4.99) requires us to calculate more variations of a given Lagrangian but is no more complicated than the non-lightlike case otherwise. The efficiency of deriving GHY terms thus transfers from the space- and timelike to the lightlike case. This is remarkable due to the asymmetry which exists in the research on lightlike and non-lightlike GHY terms. Therefore, the methods developed in this chapter advance the knowledge about lightlike GHY terms even further than for space- and timelike boundaries.

There is, however, a subtle conceptional difference of the lightlike and the non-lightlike GHY terms. This difference concerns the role of ε. For non-lightlike boundaries, ε allowed us to fix the likeness of the boundary to be space- or timelike and thereby introduced relative signs to the GHY term (4.26). In contrast to that, ε is the relative orientation of the normal vectors k and l in the lightlike case, defined by $g(k, l) = \frac{1}{\varepsilon}$. That is, if we change either the sign of k or l, we simultaneously map ε to $-\varepsilon$. Therefore, we find the power of ε in the universal lightlike GHY term (4.98) to match the power of $k^\mu l^\nu$ such that a change of sign in either k or l leaves the GHY term unaffected.

Let us illustrate how the universal result (4.98) provides an effective method for the calculation of lightlike GHY terms by considering specific theories of gravity in the following subsection.

4.2.3. Examples for lightlike Gibbons-Hawking-York terms

In this subsection, we derive the lightlike GHY term for the same examples which we discussed for space- and timelike boundaries in section 4.1.3. This enables us to compare the lightlike GHY terms calculated in this section with their non-lightlike counterparts. We need to study this analogy in order to verify that the lightlike results are reasonable, since lightlike GHY terms have only been studied for very few theories in literature so far, see [116, 117] for examples. The lightlike GHY terms I derive in this section are thus entirely new. Moreover, I have not published these results so far, they are original results presented in this thesis for the first time. Let us begin by deriving the lightlike GHY term for the simplest non-trivial case which is Einstein-Hilbert gravity.

Einstein-Hilbert gravity

The action of Einstein-Hilbert gravity on an m-dimensional manifold $\mathcal{M}$ with no boundary is

$$S^{\mathrm{EH}} = \frac{1}{2\kappa}\int_{\mathcal{M}} \mathrm{d}^m x\sqrt{|\det g|}R = \frac{1}{2\kappa}\int_{\mathcal{M}} \eta_\mu{}^\nu \wedge \Omega^\mu{}_\nu\,, \tag{4.100}$$

where we derived the differential form version of the action by rewriting (4.29) as (4.32) in section 4.1.3. We may re-construct the component version of the Einstein-Hilbert action from its differential form expression by writing the curvature two-form in terms of its local tensor components $R^\mu{}_{\nu\sigma\rho} := \Omega^\mu{}_\nu(\vartheta_\sigma, \vartheta_\rho)$ as $\Omega^\mu{}_\nu = \frac{1}{2}R^\mu{}_{\nu\sigma\rho}\theta^\sigma \wedge \theta^\rho$. From these tensor components we obtain the Ricci scalar by contracting $R = g^{\nu\rho}R^\mu{}_{\nu\mu\rho}$. Reading off the Einstein-Hilbert Lagrangian $L^{\mathrm{EH}}(\Omega^\mu{}_\nu)$ from the action (4.100), we use the decomposition of curvature (4.81) on lightlike hypersurfaces to obtain

$$\begin{aligned} L^{\mathrm{EH}}(\Omega^\mu{}_\nu) =& \frac{1}{2\kappa}\eta_\mu{}^\nu \wedge \Omega^\mu{}_\nu \\ =& \frac{1}{2\kappa}\Big(\eta_A{}^B \wedge (e^A_\alpha e^\beta_B \Omega^\alpha{}_\beta) + \varepsilon\eta_{A\mathbf{k}} \wedge (e^A_\alpha l^\beta \Omega^\alpha{}_\beta) + \varepsilon\eta_{A\mathbf{l}} \wedge (e^A_\alpha k^\beta \Omega^\alpha{}_\beta) \\ &+ \varepsilon\eta^{\mathbf{k}A} \wedge (l_\alpha e^\beta_A \Omega^\alpha{}_\beta) + \varepsilon\eta^{\mathbf{l}A} \wedge (k_\alpha e^\beta_A \Omega^\alpha{}_\beta) + \varepsilon^2\eta_{\mathbf{kk}} \wedge (l_\alpha l^\beta \Omega^\alpha{}_\beta) \\ &+ \varepsilon^2\eta_{\mathbf{ll}} \wedge (k_\alpha k^\beta \Omega^\alpha{}_\beta) + \varepsilon^2\eta_{\mathbf{kl}} \wedge (l_\alpha k^\beta \Omega^\alpha{}_\beta) + \varepsilon^2\eta_{\mathbf{lk}} \wedge (k_\alpha l^\beta \Omega^\alpha{}_\beta)\Big)\,, \end{aligned} \tag{4.101}$$

where we introduce the abbreviations

$$\begin{aligned} \eta_A{}^B &:= e^\mu_A e^B_\nu \eta_\mu{}^\nu\,, & \eta_{A\mathbf{k}} &:= e^\mu_A k_\nu \eta_\mu{}^\nu\,, & \eta^{\mathbf{l}A} &:= l^\mu e^A_\nu \eta_\mu{}^\nu\,, \\ \eta_{\mathbf{kk}} &:= k^\mu k_\nu \eta_\mu{}^\nu\,, & \eta_{A\mathbf{l}} &:= e^\mu_A l_\nu \eta_\mu{}^\nu\,, & \eta_{\mathbf{kl}} &:= k^\mu l_\nu \eta_\mu{}^\nu\,, \\ \eta_{\mathbf{ll}} &:= l^\mu l_\nu \eta_\mu{}^\nu\,, & \eta^{\mathbf{k}A} &:= k^\mu e^A_\nu \eta_\mu{}^\nu\,, & \eta_{\mathbf{lk}} &:= l^\mu k_\nu \eta_\mu{}^\nu\,. \end{aligned} \tag{4.102}$$

For calculating the Lagrange multipliers $\varphi_\mu{}^\nu$, we consider the auxiliary field Lagrangian $L^{\mathrm{EH}}(\varrho^\mu{}_\nu)$ in the decomposition (4.101) and invoke the constraints (4.99) to conclude

$$\begin{aligned} *\varphi_A{}^B &= \frac{1}{2\kappa}\eta_A{}^B\,, & *\varphi_{A\mathbf{k}} &= \frac{1}{2\kappa}\eta_{A\mathbf{k}}\,, & *\varphi_{A\mathbf{l}} &= \frac{1}{2\kappa}\eta_{A\mathbf{l}}\,, \\ *\varphi^{\mathbf{k}A} &= \frac{1}{2\kappa}\eta^{\mathbf{k}A}\,, & *\varphi^{\mathbf{l}A} &= \frac{1}{2\kappa}\eta^{\mathbf{l}A}\,, & *\varphi_{\mathbf{kk}} &= \frac{1}{2\kappa}\eta_{\mathbf{kk}}\,, \\ *\varphi_{\mathbf{ll}} &= \frac{1}{2\kappa}\eta_{\mathbf{ll}}\,, & *\varphi_{\mathbf{lk}} &= \frac{1}{2\kappa}\eta_{\mathbf{lk}}\,, & *\varphi_{\mathbf{kl}} &= \frac{1}{2\kappa}\eta_{\mathbf{kl}}\,. \end{aligned} \tag{4.103}$$

We obtain the lightlike GHY term from the latter Lagrange multipliers by inserting their on-shell values (4.103) into the universal expression (4.98) of the lightlike GHY term. This yields

$$S_{\mathrm{GHY}}^{\mathrm{EH}} = -\frac{\varepsilon}{2\kappa}\int_{\partial\mathcal{M}} \Big((L^A + \tilde{L}^A)\wedge\eta_{A\mathbf{k}} + (K^A + \tilde{K}^A)\wedge\eta_{A\mathbf{l}} + \varepsilon(\mathcal{K}-\mathcal{L})\wedge\eta_{\mathbf{kl}}\Big)\Big|_{\partial\mathcal{M}}\,, \tag{4.104}$$

where we have used the antisymmetry of $\eta^{\mu\nu} = *(\theta^\mu\wedge\theta^\nu)$ to simplify the integral. We may simplify this even further by analyzing the Hodge dual (4.17) in $\eta^{\mu\nu} = *(\theta^\mu\wedge\theta^\nu)$. For the projection $\eta_{A\mathbf{k}}$ we have

$$\eta_{A\mathbf{k}}\Big|_{\partial\mathcal{M}} = \frac{\sqrt{|\det g|}}{(m-2)!} e^\mu_A k^\nu \varepsilon_{\mu\nu\sigma_1\dots\sigma_{m-2}} e^{\sigma_1}_{A_1}\dots e^{\sigma_{m-2}}_{A_{m-2}} \phi^{A_1}\wedge\dots\wedge\phi^{A_{m-2}}\,. \tag{4.105}$$

Due to the contraction with e^μ_A, we observe that in total $(m-1)$ indices of the totally antisymmetric ε-symbol take their values on the $(m-2)$-dimensional manifold. Thus, we have $\eta_{A\mathbf{k}}\big|_{\partial\mathcal{M}} = 0$. For the second contraction of $\eta^{\mu\nu}$ appearing in (4.104), we obtain

$$\eta_{A\mathbf{l}}\Big|_{\partial\mathcal{M}} = \frac{\sqrt{|\det g|}}{(m-3)!} e^\mu_A l^\nu \varepsilon_{\mu\nu\sigma_1\dots\sigma_{m-2}} \varepsilon l^{\sigma_1} e^{\sigma_2}_{A_2}\dots e^{\sigma_{m-2}}_{A_{m-2}} \phi\wedge\phi^{A_2}\wedge\dots\wedge\phi^{A_{m-2}}\,, \tag{4.106}$$

where we use that $\phi = \varepsilon\tilde{k}$ is tangent to Σ. But two of the indices of $\varepsilon_{\mu\nu\sigma_1\dots\sigma_{m-2}}$ in (4.106) are contracted with the components of the same vector l. Using the total antisymmetry of the ε-symbol, this implies that $\eta_{A\mathbf{l}}\big|_{\partial\mathcal{M}} = 0$ vanishes as well. Hence, (4.104) simplifies considerably and yields

$$S_{\mathrm{GHY}}^{\mathrm{EH}} = \frac{\varepsilon^2}{2\kappa}\int_{\partial\mathcal{M}} (\mathcal{K}-\mathcal{L})\wedge\eta_{\mathbf{lk}}\Big|_{\partial\mathcal{M}} \tag{4.107}$$

as our final result for the lightlike GHY term of Einstein-Hilbert gravity on manifolds with curvature, torsion and non-metricity in differential form notation. To my knowledge, this is the first time that the lightlike GHY term of Einstein-Hilbert gravity has been derived in such generality.

For understanding this result, it is instructive to further analyze $\eta_{\mathbf{lk}}\big|_{\partial\mathcal{M}}$. By means of the Hodge duality (4.17) we have

$$\begin{aligned}\eta_{\mathbf{lk}}\Big|_{\partial\mathcal{M}} &= \frac{\sqrt{|\det g|}}{(m-2)!} l^\mu k^\nu \varepsilon_{\mu\nu\sigma_1\dots\sigma_{m-2}} e^{\sigma_1}_{A_1}\dots e^{\sigma_{m-2}}_{A_{m-2}} \phi^{A_1}\wedge\dots\wedge\phi^{A_{m-2}}\\ &= \frac{\sqrt{|\det \sigma|}}{(m-2)!} \varepsilon_{A_1\dots A_{m-2}} \phi^{A_1}\wedge\dots\wedge\phi^{A_{m-2}}\,.\end{aligned} \tag{4.108}$$

But this is just the Hodge dual of 1 on the $(m-2)$-dimensional hypersurface, which is related to its volume element $\mathrm{dVol}_{m-2} = \phi^0 \wedge \cdots \wedge \phi^{m-3}$ as $\eta_{\mathbf{lk}}\big|_{\partial\mathcal{M}} = *_{(m-2)}1 = \sqrt{|\det\sigma|}\mathrm{dVol}_{m-2}$. Using this identity, it is straightforward to evaluate the lightlike GHY term of Einstein-Hilbert gravity (4.107) in components, for which we obtain

$$\begin{aligned} S^{\mathrm{EH}}_{\mathrm{GHY}} =& \frac{\varepsilon^2}{2\kappa}\int_{\partial\mathcal{M}} \mathrm{dVol}_{\partial\mathcal{M}}\sqrt{|\det\sigma|}\,(\mathcal{K}(l)-\mathcal{L}(l)) \\ =& \frac{\varepsilon^2}{2\kappa}\int_{\partial\mathcal{M}} \mathrm{dVol}_{\partial\mathcal{M}}\sqrt{|\det\sigma|}\,l^\mu(\mathcal{K}_\mu-\mathcal{L}_\mu)\,, \end{aligned} \tag{4.109}$$

where $\mathrm{dVol}_{\partial\mathcal{M}} = \phi\wedge\phi^0\wedge\cdots\wedge\phi^{m-3} = \phi\wedge\mathrm{dVol}_{m-2}$ is the boundary volume form. From section 3.2.2 we recall that $\mathcal{K} = -\mathcal{L}$ in the metric-compatible case $Q_{\mu\nu} = 0$. Thus, the GHY terms simplifies to

$$S^{\mathrm{EH},Q=0}_{\mathrm{GHY}} = -\frac{\varepsilon^2}{\kappa}\int_{\partial\mathcal{M}} \mathrm{dVol}_{\partial\mathcal{M}}\sqrt{|\det\sigma|}\mathcal{L}(l) \tag{4.110}$$

if we consider a metric-compatible theory.

For interpreting the result (4.110), recall that we identified $\varkappa = \varepsilon\mathcal{K}(k)$ as the surface gravity of a black hole in section 3.2.2. Likewise, one might call $\lambda \coloneqq \varepsilon\mathcal{L}(l) = \varepsilon k_\mu l^\nu \nabla_\nu l^\mu$ the l-surface gravity of the black hole. In the few publications which derived the lightlike GHY term of Einstein-Hilbert gravity, the standard surface gravity $\varkappa$ of black holes appears in the result instead of the l-surface gravity λ, see [116, 117]. To see that our lightlike GHY term (4.110) reproduces the results of [116, 117], one needs to carefully compare the respective attempts to lightlike hypersurfaces. In particular, we defined hypersurfaces by means of a normal vector field in section 3, while [116, 117] both use the constant function approach for describing hypersurfaces. The constant function approach yields a normal covector which defines the hypersurface, although this is not explicitly analyzed in [116] and [117] since both papers only use tensor components. Recall that we discussed this in detail in section 3.2.

There is, however, a subtlety involved in going from the vector field to the covector field approach which is special for lightlike hypersurfaces. One could define $\tilde{l} = \flat(l)$ to be the covector tangent to the hypersurface if k is its normal vector, which is reasonable since $\tilde{l}$ is the covector dual of k. Indeed, we found $\varepsilon\tilde{l}(k) = 1$ using the identifications in (3.73). However, we used the metric to define the notions of being tangent or normal to a hypersurface for vector fields. In this respect, it is appropriate to use the metric dual $\tilde{k} = \flat(k)$ instead of the covector dual for defining the direction which is both normal and tangent to the hypersurface. This is the choice we made here. If we instead chose $\tilde{l}$ to define the normal covector direction of hypersurfaces, we would have obtained the surface gravity term $-\varepsilon\mathcal{K}(k)$ instead of the l-surface gravity $\varepsilon\mathcal{L}(l)$ in the component version (4.110) of the lightlike GHY term. This choice has no physical consequences due to the symmetry of k and l in the $(m-2)$-dimensional formalism. However, our treatment of hypersurfaces in differential form notation in chapter 3 allows to define all of these notions entirely self-consistently, and in particular to distinguish the terms *normal* and *non-tangent* for lightlike hypersurfaces. This clarity partially gets lost in component notation, where the difference of metric dual and covector dual is often disregarded. Hence, our result (4.110) is the entirely analogous expression of the GHY term found in [116, 117] which is consistent with the definitions of lightlike hypersurfaces we gave in section 3.2.

Note that all of this discussion is only necessary since we needed to derive the component expression (4.110) of the lightlike Einstein-Hilbert GHY term (4.107) in order to compare to [116, 117]. The differential form result (4.107) is entirely valid no matter if we choose the covector dual $\tilde{l}$ or the metric dual $\tilde{k} = \flat(k)$ of the normal vector field k to be tangent to hypersurfaces[5]. We find this universality of the differential form GHY term since the integral in (4.107) still contains the pull-back to the boundary, and this pull-back makes the choice of $\tilde{k}$ or $\tilde{l}$ as a tangent one-form explicit. Hence, the result (4.107) is the lightlike GHY term of Einstein-Hilbert gravity independently of this choice. This is the first time that the lightlike Einstein-Hilbert GHY term has been calculated in the general differential form notation which is independent of the choice of a tangent one-form. Furthermore, (4.107) does not only generalize the results of [116, 117] to differential forms, it additionally is the full lightlike GHY term on manifolds which are allowed to have torsion and non-metricity in addition to curvature. It is a major advantage of the formalism developed in the previous sections that we obtained this general lightlike GHY term of Einstein-Hilbert gravity by a comparably simple calculation.

Four-dimensional Chern-Simons modified gravity

The action $S^{\mathrm{CSMG}} = S^{\mathrm{EH}} + S^{\mathrm{CS}}$ of four-dimensional Chern-Simons modified gravity is the sum of the four-dimensional Einstein-Hilbert action (4.100) discussed in the previous example and the Chern-Simons modification

$$S^{\mathrm{CS}} = \frac{1}{8\kappa}\int_{\mathcal{M}} d^4x\,\chi * RR = \frac{1}{4\kappa}(-1)^{\operatorname{ind} g}\int_{\mathcal{M}} \chi\,\Omega^{\mu}{}_{\nu}\wedge\Omega^{\nu}{}_{\mu}\,. \tag{4.111}$$

[5]Note that $\eta_{A\mathbf{k}}\big|_{\partial\mathcal{M}} = 0$ and $\eta_{A\mathbf{l}}\big|_{\partial\mathcal{M}} = 0$ still hold in this case, but the arguments in the proofs of these equations need to be interchanged if the opposite choice is made.

The Lagrange multipliers $\varphi_\mu{}^\nu$ of curvature are calculated by $*\varphi_\mu{}^\nu \wedge \delta\varrho^\mu{}_\nu = \delta_{\varrho^\mu{}_\nu} L(\varrho^\mu{}_\nu)$, which by means of the Lagrangian $L^{\mathrm{CS}}(\Omega^\mu{}_\nu) = \frac{1}{4\kappa}(-1)^{\mathrm{ind}\, g}\chi\, \Omega^\mu{}_\nu \wedge \Omega^\nu{}_\mu$ yields

$$*\varphi_\mu{}^\nu = \frac{1}{2\kappa}(-1)^{\mathrm{ind}\, g}\chi\varrho^\nu{}_\mu\,. \tag{4.112}$$

We observe that $*\varphi_\mu{}^\nu$ clearly depends on the auxiliary field $\varrho^\mu{}_\nu$, and thus we need to calculate the lightlike GHY term of four-dimensional Chern-Simons modified gravity explicitly without using the Lagrange multiplier method. This explicit calculation is performed by first decomposing the Lagrangian into boundary tangent and non-tangent contributions and subsequently integrating the decomposed Lagrangian by means of Stokes' theorem.

The decomposition of the Lagrangian is obtained by inserting the lightlike curvature decomposition (4.81) into (4.111) which yields

$$\begin{aligned} S^{\mathrm{CS}} = \frac{1}{4\kappa}(-1)^{\mathrm{ind}\, g}\int_{\mathcal{M}}\chi\Big(&(e^A_\alpha e^\beta_B\Omega^\alpha{}_\beta)\wedge(e^\mu_A e^B_\nu\Omega^\nu{}_\mu) + 2\varepsilon(e^A_\alpha l^\beta\Omega^\alpha{}_\beta)\wedge(e^\mu_A k_\nu\Omega^\nu{}_\mu)\\ &+ 2\varepsilon(e^A_\alpha k^\beta\Omega^\alpha{}_\beta)\wedge(e^\mu_A l_\nu\Omega^\nu{}_\mu) + 2\varepsilon^2(k_\alpha k^\beta\Omega^\alpha{}_\beta)\wedge(l^\mu l_\nu\Omega^\nu{}_\mu)\\ &+ \varepsilon^2(l_\alpha k^\beta\Omega^\alpha{}_\beta)\wedge(k^\mu l_\nu\Omega^\nu{}_\mu) + \varepsilon^2(k_\alpha l^\beta\Omega^\alpha{}_\beta)\wedge(l^\mu k_\nu\Omega^\nu{}_\mu)\Big)\,, \end{aligned} \tag{4.113}$$

where we used the symmetry $\Omega^\mu{}_\nu \wedge \Omega^\nu{}_\mu = \Omega^\nu{}_\mu \wedge \Omega^\mu{}_\nu$ of the Lagrangian to simplify (4.113). Recall that in (4.88) we summarized all of the projections of curvature which are contained in (4.113). Inserting these projections into (4.113) and re-grouping the terms containing derivatives yields

$$\begin{aligned} S^{\mathrm{CS}} = \frac{1}{4\kappa}(-1)^{\mathrm{ind}\, g}\int_{\mathcal{M}}\chi\Big(&-2\varepsilon DK^A\wedge D\tilde{L}_A - 2\varepsilon DL^A\wedge D\tilde{K}_A\\ &+\varepsilon^2 D\mathcal{K}\wedge D\mathcal{K} + \varepsilon^2 D\mathcal{L}\wedge D\mathcal{L}\\ &+\frac{\varepsilon^2}{2}DQ_{\mathbf{kk}}\wedge DQ_{\mathbf{ll}} - \frac{\varepsilon^3}{2}D\left[(\mathcal{K}-\mathcal{L})\wedge Q_{\mathbf{kk}}\wedge Q_{\mathbf{ll}}\right]\\ &+\varepsilon^2 D\left[K^A\wedge\tilde{K}_A\wedge Q_{\mathbf{ll}} + L^A\wedge\tilde{L}_A\wedge Q_{\mathbf{kk}}\right]\\ &+2\varepsilon^2 D\left[K^A\wedge\tilde{L}_A\wedge\mathcal{K} + L^A\wedge\tilde{K}_A\wedge\mathcal{L}\right]\Big)\\ &+\text{terms irrelevant on } \partial\mathcal{M}\,. \end{aligned} \tag{4.114}$$

Finally, we apply Stokes' theorem (2.21) to obtain the lightlike GHY term of four-dimensional Chern-Simons modified gravity

$$
\begin{aligned}
S_{\mathrm{GHY}}^{\mathrm{CS}} = \frac{\varepsilon}{4\kappa}(-1)^{\operatorname{ind} g} \int_{\partial\mathcal{M}} \chi\Big(& K^A \wedge D\tilde{L}_A + \tilde{L}_A \wedge DK^A + L^A \wedge D\tilde{K}_A + \tilde{K}_A \wedge DL^A \\
& - \varepsilon\mathcal{K} \wedge D\mathcal{K} - \varepsilon\mathcal{L} \wedge D\mathcal{L} - \frac{\varepsilon}{4} Q_{\mathbf{kk}} \wedge DQ_{\mathbf{ll}} - \frac{\varepsilon}{4} Q_{\mathbf{ll}} \wedge DQ_{\mathbf{kk}} \\
& + \frac{\varepsilon^2}{2}(\mathcal{K} - \mathcal{L}) \wedge Q_{\mathbf{kk}} \wedge Q_{\mathbf{ll}} - \varepsilon K^A \wedge \tilde{K}_A \wedge Q_{\mathbf{ll}} \\
& - \varepsilon L^A \wedge \tilde{L}_A \wedge Q_{\mathbf{kk}} - 2\varepsilon K^A \wedge \tilde{L}_A \wedge \mathcal{K} \\
& - 2\varepsilon L^A \wedge \tilde{K}_A \wedge \mathcal{L}\Big)\Big|_{\partial\mathcal{M}} .
\end{aligned}
\tag{4.115}
$$

The lightlike GHY term (4.115) is my original result presented in this thesis for the first time. While (4.115) appears rather non-compact, this GHY term simplifies considerably if non-metricity is vanishing. Additional to $Q_{\mu\nu} = 0$ we have $\mathcal{L} = -\mathcal{K}$, $\tilde{K}_A = K_A$ and $\tilde{L}_A = L_A$ in the metric-compatible case. Inserting these identifications into the GHY term (4.115) yields

$$
\begin{aligned}
S_{\mathrm{GHY}}^{\mathrm{CS},Q=0} = \frac{\varepsilon}{2\kappa}(-1)^{\operatorname{ind} g} \int_{\partial\mathcal{M}} \chi\Big(& K^A \wedge DL_A + L^A \wedge DK_A - \varepsilon\mathcal{K} \wedge D\mathcal{K} \\
& - 2\varepsilon K^A \wedge L_A \wedge \mathcal{K}\Big)\Big|_{\partial\mathcal{M}} .
\end{aligned}
\tag{4.116}
$$

Even this simple form of the lightlike GHY term of four-dimensional Chern-Simons modified gravity has not been found before. We note that it is a direct generalization of the result (4.49) we obtained on manifolds with space- and timelike boundaries. Hence, the discussion of (4.49) applies to the lightlike case as well.

As a last example for applying our method for calculating GHY terms on manifolds with lightlike boundaries, let us investigate Lovelock gravity in arbitrary dimensions.

Lovelock gravity

We describe Lovelock gravity by the action

$$
S^{\mathrm{L}} = \sum_{p=0}^{\lfloor \frac{m-1}{2} \rfloor} \alpha_p (m-2p-1)!\, S^p
\tag{4.117}
$$

on an m-dimensional manifold, being a sum of the partial actions

$$
S^p = \int_{\mathcal{M}} \eta_{\mu_1\ldots\mu_{2p}} \wedge \Omega^{\mu_1\mu_2} \wedge \cdots \wedge \Omega^{\mu_{2p-1}\mu_{2p}} .
\tag{4.118}
$$

As we discussed in section 4.1.3, these partial actions are related to the standard differential form notation (4.54) by $\mathcal{S}^p = (m-2p)!\, S^p$. Since the Lovelock action is constituted by the sum (4.117) of the partial actions (4.118), the calculation of its lightlike GHY terms amounts to the derivation of the lightlike GHY term of S^p. The first step for obtaining this GHY term is to decompose the partial action (4.118) using the lightlike decomposition of curvature (4.81). However, let us derive some simplifying equations which will make this decomposition more compact beforehand.

First, we note that

$$e^{\mu_1}_{A_1} \dots e^{\mu_{2p}}_{A_{2p}} \eta_{\mu_1 \dots \mu_{2p}} \Big|_{\partial\mathcal{M}} = 0\,. \tag{4.119}$$

We see this by a similar argument as we used in the discussion of (4.105) for the lightlike Einstein-Hilbert GHY term. That is, the Hodge duality in the definition (4.17) of $\eta_{\mu_1 \dots \mu_{2p}}$ involves a totally antisymmetric ε-symbol with m indices which all take values on the $(m-1)$-dimensional boundary. For the explicit discussion of this argument we also refer to the space- and timelike case discussed in section 4.1.3 which applies here mutatis mutandis. Likewise, the relations $\eta_{\mathbf{k}A}|_{\partial\mathcal{M}} = 0$ and $\eta_{\mathbf{l}A}|_{\partial\mathcal{M}} = 0$ found in the calculation of the lightlike GHY term of Einstein-Hilbert gravity straightforwardly generalize to

$$\eta_{\mathbf{k}A_1 \dots A_{2p-1}} \Big|_{\partial\mathcal{M}} = 0\,, \qquad \eta_{\mathbf{l}A_1 \dots A_{2p-1}} \Big|_{\partial\mathcal{M}} = 0\,. \tag{4.120}$$

Furthermore, we have $\eta_{\mathbf{kk}A_1 \dots A_{2p-2}} = 0 = \eta_{\mathbf{ll}A_1 \dots A_{2p-2}}$ because two of the antisymmetric indices of $\eta_{\mu_1 \dots \mu_{2p}}$ are contracted with the components of the same vector. Thus, the only relevant boundary contributions of (4.118) originate from

$$\eta_{\mathbf{lk}A_1 \dots A_{2p-2}} := l^\mu k^\nu e^{\sigma_1}_{A_1} \dots e^{\sigma_{2p-2}}_{A_{2p-2}} \eta_{\mu\nu\sigma_1 \dots \sigma_{2p-2}}\,. \tag{4.121}$$

Using this simplification, we finally insert the lightlike decomposition (4.81) of curvature into the partial action (4.118) to obtain

$$\begin{aligned} S^p = p\varepsilon^2 \int_{\mathcal{M}} \eta_{\mathbf{lk}A_1 \dots A_{2p-2}} \wedge \Big(& k_\mu l_\nu (\Omega^{\mu\nu} - \Omega^{\nu\mu}) \wedge (e^{A_1}_{\sigma_1} e^{A_2}_{\sigma_2} \Omega^{\sigma_1\sigma_2}) \wedge \dots \\ & \wedge (e^{A_{2p-3}}_{\sigma_{2p-3}} e^{A_{2p-2}}_{\sigma_{2p-2}} \Omega^{\sigma_{2p-3}\sigma_{2p-2}}) + (p-1)(k_\mu e^{A_1}_{\sigma_1} (\Omega^{\mu\sigma_1} - \Omega^{\sigma_1\mu})) \\ & \wedge (l_\nu e^{A_2}_{\sigma_2} (\Omega^{\nu\sigma_2} - \Omega^{\sigma_2\nu})) \wedge (e^{A_3}_{\sigma_3} e^{A_4}_{\sigma_4} \Omega^{\sigma_3\sigma_4}) \wedge \dots \wedge (e^{A_{2p-3}}_{\sigma_{2p-3}} e^{A_{2p-2}}_{\sigma_{2p-2}} \Omega^{\sigma_{2p-3}\sigma_{2p-2}}) \Big) \\ & + \text{terms irrelevant on } \partial\mathcal{M}\,. \end{aligned} \tag{4.122}$$

This decomposed Lagrangian contains the projections of curvature we derived in (4.88). We may thus straightforwardly insert (4.88) into the decomposed Lagrangian. It is, however, highly involved to write the resulting boundary relevant term as a total derivative, which is needed in order to apply Stokes' theorem (2.21). Therefore, we readily simplify this calculation by considering vanishing non-metricity, such that $\mathcal{K} = -\mathcal{L}$, $\tilde{K}^A = K^A$ and $\tilde{L}^A = L^A$. By means of these identifications the decomposed action (4.122) takes the form

$$\begin{aligned} S^p = -2p\varepsilon^2 \int_{\mathcal{M}} \eta_{\mathbf{lk}A_1\ldots A_{2p-2}} \wedge \Big(& D\mathcal{K} \wedge (\Omega^{A_1A_2} - 2\varepsilon K^{A_1} \wedge L^{A_2}) \wedge \ldots \\ & \wedge (\Omega^{A_{2p-3}A_{2p-2}} - 2\varepsilon K^{A_{2p-3}} \wedge L^{A_{2p-2}}) \\ & + 2(p-1)(DK^{A_1} + \varepsilon K^{A_1} \wedge \mathcal{K}) \wedge (DL^{A_2} - \varepsilon L^{A_2} \wedge \mathcal{K}) \\ & \wedge (\Omega^{A_3A_4} - 2\varepsilon K^{A_3} \wedge L^{A_4}) \wedge \cdots \wedge (\Omega^{A_{2p-3}A_{2p-2}} - 2\varepsilon K^{A_{2p-3}} \wedge L^{A_{2p-2}}) \Big) \\ & + \text{terms irrelevant on } \partial\mathcal{M} \,, \end{aligned} \tag{4.123}$$

where we used that $\epsilon_{AB}(K^A \wedge L^B + L^A \wedge K^B) = 2\epsilon_{AB} K^A \wedge L^B$ due to the contraction with an antisymmetric tensor.

It is technically highly involved but in principle straightforward to prove that the terms given in (4.123) may be rewritten as total exterior derivatives. We explicitly construct this proof in appendix A and insert its results into (4.123) to obtain

$$\begin{aligned} S^p = -2p\varepsilon^2 \int_{\mathcal{M}} \eta_{\mathbf{lk}A_1\ldots A_{2p-2}} \wedge D\Big[& \mathcal{K} \wedge (\Omega^{A_1A_2} - 2\varepsilon K^{A_1} \wedge L^{A_2}) \wedge \ldots \\ & \wedge (\Omega^{A_{2p-3}A_{2p-2}} - 2\varepsilon K^{A_{2p-3}} \wedge L^{A_{2p-2}}) \\ & + (p-1)(K^{A_1} \wedge DL^{A_2} - L^{A_1} \wedge DK^{A_2}) \sum_{q=2}^{p} \binom{p-2}{q-2} \frac{(-2\varepsilon)^{p-q}}{p-q+2} \\ & \bigwedge_{m=1}^{p-q} K^{A_{2m+1}} \wedge L^{A_{2m+2}} \bigwedge_{n=p-q+1}^{p-2} \Omega^{A_{2n+1}A_{2n+2}} \Big] \\ & + \text{terms irrelevant on } \partial\mathcal{M} \,. \end{aligned} \tag{4.124}$$

For coherence of notation, we rewrite

$$\begin{aligned} & (\Omega^{A_1A_2} - 2\varepsilon K^{A_1} \wedge L^{A_2}) \wedge \cdots \wedge (\Omega^{A_{2p-3}A_{2p-2}} - 2\varepsilon K^{A_{2p-3}} \wedge L^{A_{2p-2}}) \\ & = \sum_{q=1}^{p} \binom{p-1}{q-1} (-2\varepsilon)^{p-q} \bigwedge_{m=1}^{p-q} K^{A_{2m-1}} \wedge L^{A_{2m}} \bigwedge_{n=p-q+1}^{p-1} \Omega^{A_{2n-1}A_{2n}} \end{aligned} \tag{4.125}$$

in (4.124). Using that, we finally read off the lightlike GHY term of the p-th partial action (4.118) of Lovelock gravity as

$$\begin{aligned} S^p_{\mathrm{GHY}} = 2p\varepsilon^2 \int_{\mathcal{M}} \Big[\mathcal{K} \sum_{q=1}^{p} \binom{p-1}{q-1} (-2\varepsilon)^{p-q} \bigwedge_{m=1}^{p-q} K^{A_{2m-1}} \wedge L^{A_{2m}} \bigwedge_{n=p-q+1}^{p-1} \Omega^{A_{2n-1}A_{2n}} \\ + (p-1)(K^{A_1} \wedge DL^{A_2} - L^{A_1} \wedge DK^{A_2}) \sum_{q=2}^{p} \binom{p-2}{q-2} \frac{(-2\varepsilon)^{p-q}}{p-q+2} \\ \bigwedge_{m=1}^{p-q} K^{A_{2m+1}} \wedge L^{A_{2m+2}} \bigwedge_{n=p-q+1}^{p-2} \Omega^{A_{2n+1}A_{2n+2}} \Big] \wedge \eta_{\mathbf{lk}A_1 \ldots A_{2p-2}} \Big|_{\partial\mathcal{M}} . \end{aligned} \tag{4.126}$$

We obtained the full Lovelock action as a sum of the partial actions in (4.117). Hence, the full lightlike GHY term of Lovelock gravity is obtained from (4.126) as

$$S^{\mathrm{L}}_{\mathrm{GHY}} = \sum_{p=0}^{\lfloor \frac{m-1}{2} \rfloor} \alpha_p (m-2p-1)!\, S^p_{\mathrm{GHY}} . \tag{4.127}$$

This is the first time ever that the full lightlike GHY term of Lovelock gravity has been calculated. This result is unpublished so far. Note that the GHY term (4.127) even contains contributions of torsion in addition to those of curvature if one needs to consider manifolds with torsion. Although the result reads rather complex on first glance, it is straightforward to evaluate (4.126) for given p. For the first few cases, we obtain

$$S^0_{\mathrm{GHY}} = 0 , \tag{4.128a}$$

$$S^1_{\mathrm{GHY}} = 2\varepsilon^2 \int_{\partial\mathcal{M}} \mathcal{K} \wedge \eta_{\mathbf{lk}} \Big|_{\partial\mathcal{M}} , \tag{4.128b}$$

$$\begin{aligned} S^2_{\mathrm{GHY}} = \\ 2\varepsilon^2 \int_{\partial\mathcal{M}} \Big((K^A \wedge DL^B - L^A \wedge DK^B) + 2\mathcal{K} \wedge (\Omega^{AB} - 2\varepsilon K^A \wedge L^B) \Big) \wedge \eta_{\mathbf{lk}AB} \Big|_{\partial\mathcal{M}} , \end{aligned} \tag{4.128c}$$

$$\begin{aligned} S^3_{\mathrm{GHY}} = 2\varepsilon^2 \int_{\partial\mathcal{M}} \Big((K^A \wedge DL^B - L^A \wedge DK^B) \wedge (3\Omega^{CD} - 4\varepsilon K^C \wedge L^D) + 3\mathcal{K} \wedge \\ (4\varepsilon^2 K^A \wedge L^B \wedge K^C \wedge L^D - 4\varepsilon K^A \wedge L^B \wedge \Omega^{CD} + \Omega^{AB} \wedge \Omega^{CD}) \Big) \wedge \eta_{\mathbf{lk}ABCD} \Big|_{\partial\mathcal{M}} . \end{aligned} \tag{4.128d}$$

To see that these results are reasonable, let us explicitly give the according partial actions as well. From (4.118) we immediately obtain

$$S^0 = \int_{\mathcal{M}} \eta\,, \tag{4.129a}$$

$$S^1 = \int_{\mathcal{M}} \eta_{\mu\nu} \wedge \Omega^{\mu\nu}\,, \tag{4.129b}$$

$$S^2 = \int_{\mathcal{M}} \eta_{\mu\nu\rho\sigma} \wedge \Omega^{\mu\nu} \wedge \Omega^{\rho\sigma}\,, \tag{4.129c}$$

$$S^3 = \int_{\mathcal{M}} \eta_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6} \wedge \Omega^{\mu_1\mu_2} \wedge \Omega^{\mu_3\mu_4} \wedge \Omega^{\mu_5\mu_6}\,. \tag{4.129d}$$

Let us briefly interpret these actions and their corresponding GHY terms (4.128).

First, we note that S^0 does not depend on curvature. In section 4.2.2 we found that only actions containing curvature need to be supplemented with a GHY term, and thus we obtain $S^0_{\text{GHY}} = 0$ in (4.128a). For $p = 1$, we compare the action S^1 to Einstein-Hilbert gravity (4.100) which yields $S^1 = 2\kappa S^{\text{EH}}$. We therefore need to recover $S^1_{\text{GHY}} = 2\kappa S^{\text{EH}}_{\text{GHY}}$. Indeed, comparison of S^1_{GHY} in (4.128b) with the lightlike Einstein-Hilbert GHY term (4.107) in the metric-compatible case confirms $S^1_{\text{GHY}} = 2\kappa S^{\text{EH}}_{\text{GHY}}$. Concerning S^2, recall that we identified the action (4.129c) with the Gauß-Bonnet action if non-metricity is vanishing. Hence, (4.128c) is the lightlike GHY term for Gauß-Bonnet gravity even if torsion is non-vanishing in addition to curvature. Note that S^2_{GHY} may be understood as a generalization of the space- and timelike Gauß-Bonnet GHY term (4.71c). Hence, the first Euler density action beyond these cases is S^3, for which the lightlike GHY term is given by (4.128d). The actions S^0, S^1, S^2 and S^3 we discussed explicitly here are sufficient to determine the lightlike GHY term (4.127) of the full Lovelock gravity on manifolds which have up to seven dimensions. For higher-dimensional manifolds, the lightlike GHY terms are straightforwardly evaluated using the general result (4.126).

The examples discussed in this section illustrate how lightlike GHY terms may be calculated for any action considered on manifolds with curvature, torsion and non-metricity. We emphasize again that lightlike GHY terms are known for very few theories, even if curvature is the only relevant field strength. Furthermore, the calculation of GHY terms for theories beyond Einstein-Hilbert gravity was a highly involved problem so far. This becomes obvious from reading the original papers to which we compared the results of this chapter. The new methods for calculating GHY terms I developed in this chapter are extraordinarily simple in comparison, having a greater realm of applicability at the same time. This does not only hold for the lightlike case, but for space- and timelike GHY terms likewise. The efficiency of the methods developed in this chapter allows to investigate the boundary behavior of theories for which a thorough boundary discussion was not possible before. As an example of that, we use the GHY terms discussed in this chapter for studying the boundary terms in the geometrical trinity of gravity in the following chapter.

5 The geometrical trinity of gravity

Albert Einstein's theory of general relativity is the most successful model for describing the observations in gravity we have [6, 20, 118]. From a differential geometric point of view, general relativity is based on the Levi-Civita connection which has vanishing torsion and non-metricity. The success of general relativity raises the question of how useful it is to consider more general connections for describing the measurable reality. Among many other answers to this question, the development of the *geometrical trinity of general relativity* shed light on that field. This theory was first introduced in [36, 60], and its main statement is that there are three dynamically equivalent descriptions of general relativity (GR). While Einstein modeled gravity in GR by Riemannian curvature, the *teleparallel equivalent of general relativity* (TEGR) uses torsion to describe gravity. Furthermore, there exists a dynamically equivalent description of GR in terms of non-metricity called the *symmetric teleparallel equivalent of general relativity* (STEGR) [21]. Hence, the trinity of GR, TEGR and STEGR describes general relativity by means of three different theories. The correspondence of these theories is called the geometrical trinity of general relativity [29].

We re-derive the three equivalent descriptions of general relativity in section 5.1 using differential form notation. I published this derivation in [2] which was the first time that the geometrical trinity was discussed in differential form notation[1]. The main achievement I published in [2] is a thorough discussion of the boundary terms which arise in the geometrical trinity if we consider GR, TEGR and STEGR on a manifold with boundary. I review this discussion of boundary terms in section 5.1 using the methods for describing hypersurfaces developed in chapter 3. This discussion in particular includes the GHY terms that we derived in chapter 4, which have not been discussed explicitly in the context of the geometrical trinity before. In section 5.3, we study possible generalizations of the correspondence which constitute a *geometrical trinity of gravity*.

As an important generalization, I derive the analogs of all results published in [2] for manifolds which have lightlike boundaries. This generalization to lightlike boundaries is unpublished research so far. In addition to these results, I develop a new perspective

[1] This unifies discussions of the individual theories involved in the geometrical trinity which have been developed before in differential form notation. See [119] for an early work, [120] for a discussion of TEGR for particular choices of frame and connection and [61] which briefly comments on the differential form approach to the full trinity.

on the geometrical trinity of general relativity based on an ambiguity of the connection one-form that exists in both TEGR and STEGR. This new perspective allows to introduce a unified approach to the geometrical trinity of general relativity originating from different interpretations of the Einstein action. I discuss this point of view in section 5.2 for the first time, and this unified approach is yet unpublished.

5.1. The geometrical trinity of general relativity

The connection of general relativity (GR) is the unique[2] connection which is torsion-free and metric-compatible. We call this connection the *Levi-Civita connection* and denote it by $\mathring{\omega}^{\mu}{}_{\nu}$. Analogously, we denote all tensors constructed by the Levi-Civita connection using a circle. Recall that we defined the field strengths

$$\begin{aligned} &\text{curvature} && \Omega^{\mu}{}_{\nu} = D\omega^{\mu}{}_{\nu} = \mathrm{d}\omega^{\mu}{}_{\nu} + \omega^{\mu}{}_{\rho} \wedge \omega^{\rho}{}_{\nu}\,, \\ &\text{torsion} && T^{\mu} = D\theta^{\mu} = \mathrm{d}\theta^{\mu} + \omega^{\mu}{}_{\nu} \wedge \theta^{\nu} \quad \text{and} \\ &\text{non-metricity} && Q_{\mu\nu} = -Dg_{\mu\nu} = -\mathrm{d}g_{\mu\nu} + \omega^{\rho}{}_{\mu} g_{\rho\nu} + \omega^{\rho}{}_{\nu} g_{\mu\rho} \end{aligned} \tag{5.1}$$

in chapter 2. Hence, the Levi-Civita connection is the connection which fulfills

$$0 = \mathring{T}^{\mu} = \mathrm{d}\theta^{\mu} + \mathring{\omega}^{\mu}{}_{\nu} \wedge \theta^{\nu} \qquad \text{and} \qquad 0 = \mathring{Q}_{\mu\nu} = -\mathrm{d}g_{\mu\nu} + \mathring{\omega}^{\rho}{}_{\mu} g_{\rho\nu} + \mathring{\omega}^{\rho}{}_{\nu} g_{\mu\rho}\,. \tag{5.2}$$

These defining equations of the Levi-Civita connection enable us to re-express the exterior derivatives of the coframe and the metric components as

$$\mathrm{d}\theta^{\mu} = -\mathring{\omega}^{\mu}{}_{\nu} \wedge \theta^{\nu} \qquad \text{and} \qquad \mathrm{d}g_{\mu\nu} = \mathring{\omega}^{\rho}{}_{\mu} g_{\rho\nu} + \mathring{\omega}^{\rho}{}_{\nu} g_{\mu\rho} \tag{5.3}$$

which will be useful for simplifying equations in the following.

Since GR is our reference theory of gravity, it is useful to quantify how much another theory of gravity differs from GR. This difference is encoded in the connection since we introduced torsion-freedom as well as the metricity condition as constraints on the connection one-form in (5.2). Therefore, the difference in geometries is measured by the *deformation one-form* [119]

$$A^{\mu}{}_{\nu} := \omega^{\mu}{}_{\nu} - \mathring{\omega}^{\mu}{}_{\nu} \tag{5.4}$$

which subtracts the Levi-Civita contributions from the full connection one-form $\omega^{\mu}{}_{\nu}$.

[2] In this context, the term *unique* means that the Levi-Civita connection is unambiguously fixed if we choose a coframe and a metric tensor.

Note that $\omega^\mu{}_\nu$ may be used to describe manifolds which have non-vanishing curvature, torsion and non-metricity, while $\mathring{\omega}^\mu{}_\nu$ is the connection of geometries which are modeled solely by curvature. Hence, the deformation one-form $A^\mu{}_\nu$ subtracts the pure curvature-related contributions from the full connection and is thus entirely described by torsion and non-metricity. To see that, we insert the simplifications (5.3) derived for the Levi-Civita connection into the definitions (5.1) of torsion and non-metricity to conclude that

$$Q_{\mu\nu} = A_{\mu\nu} + A_{\nu\mu} \qquad \text{and} \qquad T^\mu = A^\mu{}_\nu \wedge \theta^\nu \,, \tag{5.5}$$

where we abbreviate $A_{\mu\nu} := g_{\mu\rho} A^\rho{}_\nu$. Although the symmetric contribution of the deformation one-form is nothing but non-metricity, its antisymmetric part $(A_{\mu\nu} - A_{\nu\mu}) \wedge \theta^\nu = 2T_\mu - Q_{\mu\nu} \wedge \theta^\nu$ depends on non-metricity as well. The torsional contribution to the deformation one-form is called *contortion*, such that $A^\mu{}_\nu$ is the contortion one-form in metric-compatible theories. Analogously, $A^\mu{}_\nu$ is called *disformation one-form* if torsion-free theories are considered [60]. This decomposition of the deformation one-form into torsional and non-metric contributions was implicitly described in [121] using the interior product.

While (5.5) enables us to construct torsion and non-metricity from a given deformation one-form, we may construct the full curvature two-form $\Omega^\mu{}_\nu = \mathrm{d}\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu$ from the so-called *Riemannian curvature two-form* $\mathring{\Omega}^\mu{}_\nu = \mathrm{d}\mathring{\omega}^\mu{}_\nu + \mathring{\omega}^\mu{}_\rho \wedge \mathring{\omega}^\rho{}_\nu$ and the deformation one-form as well. To see this, we insert $\omega^\mu{}_\nu = A^\mu{}_\nu + \mathring{\omega}^\mu{}_\nu$ obtained from (5.4) into the definition (5.1) of the curvature two-form to conclude

$$\Omega^\mu{}_\nu = \mathring{\Omega}^\mu{}_\nu + \mathring{D} A^\mu{}_\nu + A^\mu{}_\rho \wedge A^\rho{}_\nu \,. \tag{5.6}$$

Note that this decomposes the full curvature $\Omega^\mu{}_\nu$ into its Riemannian part $\mathring{\Omega}^\mu{}_\nu$ and the deformation contribution $A^\mu{}_\rho \wedge A^\rho{}_\nu$ which is solely determined by torsion and non-metricity. In addition to these contributions, we obtain the exterior covariant derivative $\mathring{D} A^\mu{}_\nu$ of the deformation one-form with respect to the Levi-Civita connection. Recall that the Einstein-Hilbert action is the integral of (5.6) contracted with $\eta_\mu{}^\nu$, so that $\mathring{D} A^\mu{}_\nu$ is a boundary term in GR by means of Stokes' theorem (2.21).

Let us concretize this argument by inserting the curvature decomposition (5.6) into the Einstein-Hilbert action. This yields

$$S^{\mathrm{EH},\mathring{\Omega}} = \frac{1}{2\kappa} \int_{\mathcal{M}} \eta_\mu{}^\nu \wedge \mathring{\Omega}^\mu{}_\nu + S^{\mathrm{EH},\mathring{\Omega}}_{\mathrm{GHY}} \tag{5.7a}$$

$$= \frac{1}{2\kappa} \int_{\mathcal{M}} \eta_\mu{}^\nu \wedge (\Omega^\mu{}_\nu - A^\mu{}_\rho \wedge A^\rho{}_\nu) - \frac{1}{2\kappa} \int_{\mathcal{M}} \eta_\mu{}^\nu \wedge \mathring{D} A^\mu{}_\nu + S^{\mathrm{EH},\mathring{\Omega}}_{\mathrm{GHY}} \,, \tag{5.7b}$$

where we defined the Hodge dual $\eta^{\mu\nu} = *(\theta^\mu \wedge \theta^\nu)$ in (4.17). Note that we introduced a GHY term to the Einstein-Hilbert action (5.7) to describe the dynamics on manifolds with boundaries. We calculated this GHY term for space- and timelike boundaries in (4.38) as well as for lightlike boundaries in (4.110).

For deriving the geometrical trinity of general relativity on manifolds with boundary, we need to recall some of the results for GHY terms we obtained in chapter 4. In particular, we found that Lagrangians do not require us to introduce a GHY term in order for their variational problem to be well-defined if they are solely constructed from torsion and non-metricity. Since the deformation one-form is independent of curvature, the contribution $-\frac{1}{2\kappa}\int_{\mathcal{M}} \eta_\mu{}^\nu \wedge A^\mu{}_\rho \wedge A^\rho{}_\nu$ of the action (5.7b) has a well-defined variational problem without adding a GHY term. In contrast to that, the curvature contribution of (5.7b) needs to be supplemented by a GHY term. Since this contribution is the Einstein-Hilbert action of the full curvature two-form $\Omega^\mu{}_\nu$, its GHY term is $S^{\text{EH},\Omega}_{\text{GHY}}$. In sections 4.1.3 and 4.2.3, we found that the GHY term $S^{\text{EH},\Omega}_{\text{GHY}}$ of the full curvature Einstein-Hilbert action explicitly differs from its Riemannian counterpart $S^{\text{EH},\mathring{\Omega}}_{\text{GHY}}$. Hence, the Riemannian GHY term $S^{\text{EH},\mathring{\Omega}}_{\text{GHY}}$ that is contained in (5.7b) is not the correct GHY term for the bulk action

$$S^{\text{bulk}} = \frac{1}{2\kappa}\int_{\mathcal{M}} \eta_\mu{}^\nu \wedge (\Omega^\mu{}_\nu - A^\mu{}_\rho \wedge A^\rho{}_\nu) \tag{5.8}$$

which is constructed upon the full curvature two-form $\Omega^\mu{}_\nu$. This mismatch of GHY terms is resolved by the term

$$S^{\mathring{D}A} := -\frac{1}{2\kappa}\int_{\mathcal{M}} \eta_\mu{}^\nu \wedge \mathring{D}A^\mu{}_\nu \tag{5.9}$$

contained in (5.7b) in addition to the bulk action (5.8) and the GHY term $S^{\text{EH},\mathring{\Omega}}_{\text{GHY}}$. To see how $S^{\mathring{D}A}$ accounts for the apparent mismatch, we examine (5.9) in detail next.

As a first step, we note that GHY terms are boundary terms and we should thus write $S^{\mathring{D}A}$ as a boundary term as well. Since we defined the Levi-Civita connection in (5.2) by $\mathring{D}\theta^\mu = 0$ and $\mathring{D}g_{\mu\nu} = 0$, we immediately conclude that the covariant exterior derivative of the Hodge dual $\eta^{\mu_1\cdots\mu_p}$ with respect to the Levi-Civita connection vanishes for all integers p. That is, by means of the definition (4.17) of the Hodge duality, we obtain

$$\mathring{D}\eta^{\mu_1\cdots\mu_p} = 0\,. \tag{5.10}$$

For the action $S^{\mathring{D}A}$ defined in (5.9), $\mathring{D}\eta_\mu{}^\nu = 0$ allows us to use Stokes' theorem (2.21) to conclude

$$S^{\mathring{D}A} = -\frac{1}{2\kappa}\int_{\partial\mathcal{M}} A^{\mu}{}_{\nu} \wedge \eta_{\mu}{}^{\nu}\Big|_{\partial\mathcal{M}} . \tag{5.11}$$

Hence, $S^{\mathring{D}A}$ is indeed a boundary term. In order to compare this boundary term to the Einstein-Hilbert GHY term, recall that we derived GHY terms from a decomposition into boundary tangent and non-tangent contributions in chapter 4. Hence, we need an analogous decomposition of $S^{\mathring{D}A}$ as well. This decomposition differs for lightlike and non-lightlike boundaries as it did for GHY terms. Hence, we investigate these cases separately, starting with space- and timelike boundaries.

5.1.1. Decomposition of $S^{\mathring{D}A}$ on space- and timelike boundaries

We obtain all terms contained in the decomposition of $A^{\mu}{}_{\nu}$ into normal and tangent contributions on space- or timelike hypersurfaces by evaluating $A^{\mu}{}_{\nu} = \delta^{\mu}_{\alpha}\delta^{\beta}_{\nu}A^{\alpha}{}_{\beta}$ using the unity decomposition (3.38). This yields

$$A^{\mu}{}_{\nu} = e^{\mu}_{a}e^{b}_{\nu}(e^{a}_{\alpha}e^{\beta}_{b}A^{\alpha}{}_{\beta}) + \varepsilon e^{\mu}_{a}n_{\nu}(e^{a}_{\alpha}n^{\beta}A^{\alpha}{}_{\beta}) + \varepsilon n^{\mu}e^{a}_{\nu}(n_{\alpha}e^{\beta}_{a}A^{\alpha}{}_{\beta}) + n^{\mu}n_{\nu}(n_{\alpha}n^{\beta}A^{\alpha}{}_{\beta}) , \tag{5.12}$$

such that we determine the entire decomposition of $A^{\mu}{}_{\nu}$ by calculating the projections which are denoted in parentheses in (5.12). We derive these projections in the following calculation by means of the definition (5.4) of the deformation one-form. In particular, it suffices to derive the decomposition of the connection because this decomposition immediately induces the decomposition of the deformation one-form by means of $A^{\mu}{}_{\nu} \equiv \omega^{\mu}{}_{\nu} - \mathring{\omega}^{\mu}{}_{\nu}$. Hence, we proceed by calculating the projections of the full and the Levi-Civita connections which we subsequently use to obtain the deformation decomposition.

We already derived the tangent projection $e^{a}_{\mu}e^{\nu}_{b}\omega^{\mu}{}_{\nu}$ of the connection one-form implicitly in (3.40) by examining its transformation to hypersurfaces. In particular, the tangent projections of the full and the Levi-Civita connections are contained in (3.40) as

$$\omega^{a}{}_{b} = e^{a}_{\mu}\mathrm{d}e^{\mu}_{b} + e^{a}_{\mu}e^{\nu}_{b}\omega^{\mu}{}_{\nu} \qquad \text{and} \qquad \mathring{\omega}^{a}{}_{b} = e^{a}_{\mu}\mathrm{d}e^{\mu}_{b} + e^{a}_{\mu}e^{\nu}_{b}\mathring{\omega}^{\mu}{}_{\nu} . \tag{5.13}$$

For obtaining the projections $e^{a}_{\mu}n^{\nu}\omega^{\mu}{}_{\nu}$ and $e^{a}_{\mu}n^{\nu}\mathring{\omega}^{\mu}{}_{\nu}$, we exploit the definition $K^{a} = e^{a}_{\mu}Dn^{\mu}$ of the extrinsic curvature one-form. Evaluated for the full and the Levi-Civita connection, this definition is

$$K^a = e^a_\mu \mathrm{d}n^\mu + e^a_\mu n^\nu \omega^\mu{}_\nu \qquad \text{and} \qquad \mathring{K}^a = e^a_\mu \mathrm{d}n^\mu + e^a_\mu n^\nu \mathring{\omega}^\mu{}_\nu \tag{5.14}$$

which contains the desired projections. Analogously, the definition $\tilde{K}_a = e^\mu_a D n_\mu$ implies

$$\tilde{K}_a = e^\mu_a \mathrm{d}n_\mu - e^\mu_a n_\nu \omega^\nu{}_\mu \qquad \text{and} \qquad \mathring{K}_a = e^\mu_a \mathrm{d}n_\mu - e^\mu_a n_\nu \mathring{\omega}^\nu{}_\mu \,, \tag{5.15}$$

where we used that the metric compatibility of the Levi-Civita connection implies $\mathring{\tilde{K}}_a = \mathring{K}_a$. The extrinsic curvatures (5.15) hence include the projections $n_\mu e^\nu_a \omega^\mu{}_\nu$ and $n_\mu e^\nu_a \mathring{\omega}^\mu{}_\nu$ of the connection one-forms. Finally, we obtain the normal projected connection components $n_\mu n^\nu \omega^\mu{}_\nu$ and $n_\mu n^\nu \mathring{\omega}^\mu{}_\nu$ from the non-metricity projections

$$Q_{\mathbf{nn}} = 2n_\mu \mathrm{d}n^\mu + 2n_\mu n^\nu \omega^\mu{}_\nu \qquad \text{and} \qquad 0 = \mathring{Q}_{\mathbf{nn}} = 2n_\mu \mathrm{d}n^\mu + 2n_\mu n^\nu \mathring{\omega}^\mu{}_\nu \,. \tag{5.16}$$

Note that $\mathring{Q}_{\mathbf{nn}}$ vanishes as a direct consequence of the metric compatibility of the Levi-Civita connection. Hence, we have found that all the projections of the connection one-form are contained in the definitions of the hypersurface connection, the extrinsic curvature one-forms and non-metricity.

The projections of the connection one-form induce the deformation projections by means of $A^\mu{}_\nu \equiv \omega^\mu{}_\nu - \mathring{\omega}^\mu{}_\nu$. Hence, by inserting the connection projections, we obtain

$$\begin{aligned} e^a_\mu e^\nu_b A^\mu{}_\nu &= A^a{}_b \,, & e^a_\mu n^\nu A^\mu{}_\nu &= K^a - \mathring{K}^a \,, \\ n_\mu e^\nu_a A^\mu{}_\nu &= -\tilde{K}_a + \mathring{K}_a \,, & n_\mu n^\nu A^\mu{}_\nu &= \frac{1}{2} Q_{\mathbf{nn}} \,. \end{aligned} \tag{5.17}$$

We use these projections in the decomposition (5.12) of the deformation one-form to conclude that the boundary action (5.11) decomposes as

$$S^{\mathring{D}A} = -\frac{1}{2\kappa} \int_{\partial\mathcal{M}} \left(A^a{}_b \wedge \eta_a{}^b + \varepsilon (K^a - \mathring{K}^a) \wedge \eta_{a\mathbf{n}} + \varepsilon (-\tilde{K}^a + \mathring{K}^a) \wedge \eta_{\mathbf{n}a} + \frac{1}{2} Q_{\mathbf{nn}} \wedge \eta_{\mathbf{nn}} \right) \bigg|_{\partial\mathcal{M}} . \tag{5.18}$$

This result simplifies considerably if we use the antisymmetry of $\eta^{\mu\nu} = *(\theta^\mu \wedge \theta^\nu)$. In particular, we found that this antisymmetry implies $\eta^{ab} \big|_{\partial\mathcal{M}} = e^a_\mu e^b_\nu \eta^{\mu\nu} \big|_{\partial\mathcal{M}} = 0$ in section 4.1.3 since $m \coloneqq \dim \mathcal{M}$ indices of the totally antisymmetric symbol take their values on the $(m-1)$-dimensional boundary. Furthermore, the antisymmetry of $\eta^{\mu\nu}$ immediately implies $\eta_{\mathbf{nn}} \equiv n_\mu n_\nu \eta^{\mu\nu} = 0$ and $\eta_{a\mathbf{n}} = -\eta_{\mathbf{n}a} = -n^\mu e^\nu_a \eta_{\mu\nu}$, such that (5.18) becomes

$$S^{\mathring{D}A} = -\frac{\varepsilon}{\kappa}\int_{\partial\mathcal{M}} \mathring{K}^a \wedge \eta_{\mathbf{n}a}\Big|_{\partial\mathcal{M}} + \frac{\varepsilon}{2\kappa}\int_{\partial\mathcal{M}} (K^a + \tilde{K}^a)\wedge \eta_{\mathbf{n}a}\Big|_{\partial\mathcal{M}} \,. \tag{5.19}$$

In this form, we are finally able to compare $S^{\mathring{D}A}$ to the GHY term of Einstein-Hilbert gravity. We adapt this GHY term from (4.37) and (4.38) for the full and the Levi-Civita connection, respectively, and compare the boundary action (5.19) to these results. This calculation reveals that $S^{\mathring{D}A}$ is nothing but a difference of GHY terms, that is,

$$S^{\mathring{D}A} = -S^{\mathrm{EH},\mathring{\Omega}}_{\mathrm{GHY}} + S^{\mathrm{EH},\Omega}_{\mathrm{GHY}} \,. \tag{5.20}$$

This solves the mismatch of GHY terms we encountered in (5.7). To see this, we insert (5.20) into (5.7b) to obtain

$$\begin{aligned} S^{\mathrm{EH},\mathring{\Omega}} &= \frac{1}{2\kappa}\int_{\mathcal{M}} \eta_\mu{}^\nu \wedge \mathring{\Omega}^\mu{}_\nu + S^{\mathrm{EH},\mathring{\Omega}}_{\mathrm{GHY}} \\ &= \frac{1}{2\kappa}\int_{\mathcal{M}} \eta_\mu{}^\nu \wedge (\Omega^\mu{}_\nu - A^\mu{}_\rho \wedge A^\rho{}_\nu) + S^{\mathrm{EH},\Omega}_{\mathrm{GHY}} \,, \end{aligned} \tag{5.21}$$

in which both expressions of the Einstein-Hilbert action on a manifold with boundary are supplemented by the correct GHY term to make the variational problem well-defined.

The actions (5.20) and (5.21) are the main results of this subsection. They render the GHY term of the Einstein-Hilbert action $S^{\mathrm{EH},\mathring{\Omega}}$ correct regardless if it is expressed by means of the Riemannian or the full curvature two-form. The interpretation of the boundary term $S^{\mathring{D}A}$ as a difference of GHY terms we found in (5.20) was not known before I first published it in [2]. In fact, GHY terms have not been properly discussed in the context of the geometrical trinity at all, whereas my results stress the importance of including the appropriate GHY terms when the geometrical trinity is considered on a manifold with boundary. The action (5.21) is the starting point for transitioning form GR to its teleparallel and symmetric teleparallel equivalents. Before we explicitly discuss this transition, let us first give an interpretation of $S^{\mathring{D}A}$ in terms of GHY terms if manifolds with lightlike boundaries are considered.

5.1.2. Decomposition of $S^{\mathring{D}A}$ on lightlike boundaries

In order to understand the relation of $S^{\mathring{D}A}$ to the GHY term of the Einstein-Hilbert action on m-dimensional manifolds with lightlike boundary, we need to derive the decomposition of $A^\mu{}_\nu$ into boundary tangent and non-tangent contributions analogously to the non-lightlike case. For this purpose, we insert the lightlike unity decomposition (3.87) into $A^\mu{}_\nu = \delta^\mu_\alpha \delta^\beta_\nu A^\alpha{}_\beta$ to obtain

$$\begin{aligned} A^\mu{}_\nu =& e^\mu_A e^B_\nu (e^A_\alpha e^\beta_B A^\alpha{}_\beta) + \varepsilon e^\mu_A k_\nu (e^A_\alpha l^\beta A^\alpha{}_\beta) + \varepsilon e^\mu_A l_\nu (e^A_\alpha k^\beta A^\alpha{}_\beta) \\ &+ \varepsilon k^\mu e^A_\nu (l_\alpha e^\beta_A A^\alpha{}_\beta) + \varepsilon l^\mu e^A_\nu (k_\alpha e^\beta_A A^\alpha{}_\beta) + \varepsilon^2 k^\mu k_\nu (l_\alpha l^\beta A^\alpha{}_\beta) \\ &+ \varepsilon^2 l^\mu l_\nu (k_\alpha k^\beta A^\alpha{}_\beta) + \varepsilon^2 k^\mu l_\nu (l_\alpha k^\beta A^\alpha{}_\beta) + \varepsilon^2 l^\mu k_\nu (k_\alpha l^\beta A^\alpha{}_\beta) \,. \end{aligned} \tag{5.22}$$

The projections in parentheses in the latter equation determine the lightlike decomposition of $A^\mu{}_\nu$. Analogous to the non-lightlike case, we use $A^\mu{}_\nu \equiv \omega^\mu{}_\nu - \mathring{\omega}^\mu{}_\nu$ for deriving the projections in (5.22) by evaluating the corresponding projections of the full and the Levi-Civita connection.

In order to obtain the contribution of the connection in the direction tangent to the $(m-2)$-dimensional hypersurface, we use the connection transformation law (3.89) which is

$$\omega^A{}_B = e^A_\mu \mathrm{d}e^\mu_B + e^A_\mu e^\nu_B \omega^\mu{}_\nu \,, \qquad \mathring{\omega}^A{}_B = e^A_\mu \mathrm{d}e^\mu_B + e^A_\mu e^\nu_B \mathring{\omega}^\mu{}_\nu \tag{5.23}$$

for the full and the Levi-Civita connection, respectively. Thus, the tangent projection $e^A_\mu e^\nu_B A^\mu{}_\nu$ of the deformation one-form is the hypersurface deformation

$$e^A_\mu e^\nu_B A^\mu{}_\nu = A^A{}_B := \omega^A{}_B - \mathring{\omega}^A{}_B \,. \tag{5.24}$$

We obtain the projections in which only one of the indices of $A^\mu{}_\nu$ is contracted with e^μ_A from the extrinsic curvature one-forms. For the full connection $\omega^\mu{}_\nu$, their definitions evaluate as

$$\begin{aligned} K^A &= e^A_\mu \mathrm{d}k^\mu + e^A_\mu k^\nu \omega^\mu{}_\nu \,, & \tilde{K}_A &= e^\mu_A \mathrm{d}k_\mu - e^\mu_A k_\nu \omega^\nu{}_\mu \,, \\ L^A &= e^A_\mu \mathrm{d}l^\mu + e^A_\mu l^\nu \omega^\mu{}_\nu \,, & \tilde{L}_A &= e^\mu_A \mathrm{d}l_\mu - e^\mu_A l_\nu \omega^\nu{}_\mu \,. \end{aligned} \tag{5.25}$$

Note that the evaluation of these extrinsic curvature one-forms for the Levi-Civita connection simplifies as we have $\mathring{\tilde{K}}_A = \mathring{K}_A$ and $\mathring{\tilde{L}}_A = \mathring{L}_A$ due to metric compatibility. Hence, it suffices to consider

$$\mathring{K}^A = e^A_\mu \mathrm{d}k^\mu + e^A_\mu k^\nu \mathring{\omega}^\mu{}_\nu \qquad \text{and} \qquad \mathring{L}^A = e^A_\mu \mathrm{d}l^\mu + e^A_\mu l^\nu \mathring{\omega}^\mu{}_\nu \,, \tag{5.26}$$

from which we read off the corresponding projections of the deformation one-form as

$$\begin{aligned} e^A_\mu k^\nu A^\mu{}_\nu &= K^A - \mathring{K}^A \,, & k_\mu e^\nu_A A^\mu{}_\nu &= -\tilde{K}_A + \mathring{K}_A \,, \\ e^A_\mu l^\nu A^\mu{}_\nu &= L^A - \mathring{L}^A \,, & l_\mu e^\nu_A A^\mu{}_\nu &= -\tilde{L}_A + \mathring{L}_A \,. \end{aligned} \tag{5.27}$$

In order to determine the entire lightlike decomposition (5.22) of the deformation one-form it only remains to calculate those projections in which both indices of $A^\mu{}_\nu$ are contracted with the normal directions of the $(m-2)$-dimensional hypersurface. We obtain these projections from

$$\begin{aligned} \mathcal{K} &= l_\mu \mathrm{d}k^\mu + l_\mu k^\nu \omega^\mu{}_\nu \,, \qquad & Q_{\mathbf{kk}} &= 2k_\mu \mathrm{d}k^\mu + 2k_\mu k^\nu \omega^\mu{}_\nu \,, \\ \mathcal{L} &= k_\mu \mathrm{d}l^\mu + k_\mu l^\nu \omega^\mu{}_\nu \,, \qquad & Q_{\mathbf{ll}} &= 2l_\mu \mathrm{d}l^\mu + 2l_\mu l^\nu \omega^\mu{}_\nu \,. \end{aligned} \tag{5.28}$$

For the Levi-Civita connection, $\mathring{Q}_{\mu\nu} = 0$ implies $\mathring{\mathcal{L}} = -\mathring{\mathcal{K}}$ such that the expressions (5.28) are captured simply by

$$\begin{aligned} \mathring{\mathcal{K}} &= l_\mu \mathrm{d}k^\mu + l_\mu k^\nu \mathring{\omega}^\mu{}_\nu \,, \\ 0 = k_\mu \mathrm{d}k^\mu + k_\mu k^\nu \mathring{\omega}^\mu{}_\nu \qquad &\text{and} \qquad 0 = l_\mu \mathrm{d}l^\mu + l_\mu l^\nu \mathring{\omega}^\mu{}_\nu \,. \end{aligned} \tag{5.29}$$

Hence, the normal projections of the deformation one-form are

$$\begin{aligned} k_\mu l^\nu A^\mu{}_\nu &= \mathcal{L} + \mathring{\mathcal{K}} \,, \qquad & k_\mu k^\nu A^\mu{}_\nu &= \frac{1}{2} Q_{\mathbf{kk}} \,, \\ l_\mu k^\nu A^\mu{}_\nu &= \mathcal{K} - \mathring{\mathcal{K}} \,, \qquad & l_\mu l^\nu A^\mu{}_\nu &= \frac{1}{2} Q_{\mathbf{ll}} \,. \end{aligned} \tag{5.30}$$

We have thus completed the calculation of all projections of $A^\mu{}_\nu$ involved in its decomposition (5.22).

Before we insert the decomposition (5.22) into the boundary action (5.11), we note that most of the thereby obtained terms are vanishing due to the antisymmetry of $\eta^{\mu\nu} = *(\theta^\mu \wedge \theta^\nu)$. As we derived in section 4.2.3, the projections

$$\begin{aligned} \eta^{AB}\big|_{\partial\mathcal{M}} = e^A_\mu e^B_\nu \eta^{\mu\nu}\big|_{\partial\mathcal{M}} = 0 \,, \qquad \eta^{A\mathbf{k}}\big|_{\partial\mathcal{M}} = e^A_\mu k_\nu \eta^{\mu\nu}\big|_{\partial\mathcal{M}} = 0 \,, \\ \eta^{A\mathbf{l}}\big|_{\partial\mathcal{M}} = e^A_\mu l_\nu \eta^{\mu\nu}\big|_{\partial\mathcal{M}} = 0 \end{aligned} \tag{5.31}$$

all vanish on the boundary. The reason for that are either $m \equiv \dim \mathcal{M}$ antisymmetric indices taking values on the $(m-1)$-dimensional hypersurface or two antisymmetric indices which are contracted with the components of the same vector. Moreover, the antisymmetry of $\eta^{\mu\nu}$ implies that $\eta_{\mathbf{kk}} = 0 = \eta_{\mathbf{ll}}$. Hence, the only relevant projection of $\eta^{\mu\nu}$ which is not vanishing on the boundary is $\eta_{\mathbf{lk}} = l_\mu k_\nu \eta^{\mu\nu} = -\eta_{\mathbf{kl}}$. Using these simplifications, we finally insert the decomposition (5.22) of the deformation one-form $A^\mu{}_\nu$ into the boundary action (5.11) and use the projections (5.30) to obtain

$$S^{\mathring{D}A} = -\frac{\varepsilon^2}{\kappa} \int_{\partial\mathcal{M}} \mathring{\mathcal{K}} \wedge \eta_{\mathbf{lk}}\big|_{\partial\mathcal{M}} + \frac{\varepsilon^2}{2\kappa} \int_{\partial\mathcal{M}} (\mathcal{K} - \mathcal{L}) \wedge \eta_{\mathbf{lk}}\big|_{\partial\mathcal{M}} \,. \tag{5.32}$$

Comparing this boundary action to the lightlike GHY term (4.107) of Einstein-Hilbert gravity, we conclude that $S^{\mathring{D}A}$ is the difference

$$S^{\mathring{D}A} = -S_{\text{GHY}}^{\text{EH},\mathring{\Omega}} + S_{\text{GHY}}^{\text{EH},\Omega} \tag{5.33}$$

of GHY terms. This is the exact same result which we obtained in (5.20) for space- and timelike boundaries. Hence, the discussion of $S^{\mathring{D}A}$ for space- and timelike boundaries generalizes to the lightlike case as well. In particular, (5.33) guarantees that the Einstein-Hilbert action (5.7) has a well-defined variational problem, no matter if we express it using the Riemannian or the full curvature. Inserting (5.33) into the Einstein-Hilbert action (5.7b), we obtain

$$\begin{aligned} S^{\text{EH},\mathring{\Omega}} &= \frac{1}{2\kappa}\int_{\mathcal{M}} \eta_\mu{}^\nu \wedge \mathring{\Omega}^\mu{}_\nu + S_{\text{GHY}}^{\text{EH},\mathring{\Omega}} \\ &= \frac{1}{2\kappa}\int_{\mathcal{M}} \eta_\mu{}^\nu \wedge (\Omega^\mu{}_\nu - A^\mu{}_\rho \wedge A^\rho{}_\nu) + S_{\text{GHY}}^{\text{EH},\Omega}\,, \end{aligned} \tag{5.34}$$

from which we explicitly see that both expressions are supplemented by the correct GHY term.

This concludes the separate discussion of the boundary term $S^{\mathring{D}A}$ for lightlike and non-lightlike boundaries. Both cases yield the same formal expressions (5.33), (5.34) for the boundary term $S^{\mathring{D}A}$ and the full Einstein-Hilbert action. Therefore, we proceed by deriving the geometrical trinity of general relativity on manifolds with boundaries of any desired likeness based on (5.34).

5.1.3. Boundary refined geometrical trinity of general relativity

In general relativity, we consider the Einstein-Hilbert action

$$S^{\text{EH},\mathring{\Omega}} = \frac{1}{2\kappa}\int_{\mathcal{M}} \eta_\mu{}^\nu \wedge (\Omega^\mu{}_\nu - A^\mu{}_\rho \wedge A^\rho{}_\nu) + S_{\text{GHY}}^{\text{EH},\Omega} \tag{5.35}$$

for the special choice of connection $\omega^\mu{}_\nu = \mathring{\omega}^\mu{}_\nu$. Choosing the Levi-Civita connection, the only non-vanishing field strength of GR is Riemannian curvature. That is, the GR choice of connection implies

$$\Omega^\mu{}_\nu = \mathring{\Omega}^\mu{}_\nu\,, \qquad T^\mu = \mathring{T}^\mu = 0\,, \qquad Q_{\mu\nu} = \mathring{Q}_{\mu\nu} = 0\,, \tag{5.36}$$

which we used to define the Levi-Civita connection in (5.2). Note that for $\omega^\mu{}_\nu = \mathring{\omega}^\mu{}_\nu$ the deformation one-form $A^\mu{}_\nu = \omega^\mu{}_\nu - \mathring{\omega}^\mu{}_\nu$ vanishes by definition, so that the action (5.35) trivially reproduces the standard expression (5.7a) of the Einstein-Hilbert action in differential form notation.

The choice $\omega^{\mu}{}_{\nu} = \mathring{\omega}^{\mu}{}_{\nu}$ is, however, only one particular choice of connection we could make. If we want to describe the dynamics of GR by only one field strength, we could instead choose a connection for which either torsion or non-metricity are non-vanishing while the curvature two-form vanishes identically. In these cases, the Einstein-Hilbert action (5.35) still describes a theory equivalent to GR in which gravity is, however, modeled by torsion or non-metricity. Such theories with vanishing curvature are called *teleparallel* since vectors are still aligned to themselves after parallel transport[3] along a closed curve in $\mathcal{M}$. Therefore, the GR equivalent theory in which gravity is interpreted as the manifold's torsion is called the *teleparallel equivalent of general relativity* (TEGR). Taking into account the symmetry of $Q_{\mu\nu}$ with respect to exchange of its indices, the GR equivalent theory of gravity which describes gravity by means of non-metricity is called the *symmetric teleparallel equivalent of general relativity* (STEGR).

In summary, the geometrical trinity of general relativity is thus described by the Einstein-Hilbert action (5.35) for which we choose the connection one-form such that the field strengths are

$$\begin{aligned} &\Omega^{\mu}{}_{\nu} \neq 0\,, && T^{\mu} = 0\,, && Q_{\mu\nu} = 0 && \text{for GR,} \\ &\Omega^{\mu}{}_{\nu} = 0\,, && T^{\mu} \neq 0\,, && Q_{\mu\nu} = 0 && \text{for TEGR,} \\ &\Omega^{\mu}{}_{\nu} = 0\,, && T^{\mu} = 0\,, && Q_{\mu\nu} \neq 0 && \text{for STEGR.} \end{aligned} \tag{5.37}$$

This summarizes the traditional form [36, 60] of the geometrical trinity of general relativity. We generalize this form of the geometrical trinity in two ways. First, we discuss the boundary terms involved in the action for the three choices (5.37). Second, we effectively interpret the geometrical trinity as a duality. The latter is straightforward, since we described all the effects of torsion and non-metricity solely by means of the deformation one-form $A^{\mu}{}_{\nu}$. Hence, instead of discussing TEGR and STEGR separately, we discuss both theories at the same time by demanding that the curvature two-form vanishes while $A^{\mu}{}_{\nu} \neq 0$. We call this summarized case (S)TEGR, such that the geometrical trinity of general relativity (5.35) is the statement of equivalence of the two theories obtained by imposing

$$\begin{aligned} &\Omega^{\mu}{}_{\nu} \neq 0\,, && A^{\mu}{}_{\nu} = 0\,, && \text{for GR,} \\ &\Omega^{\mu}{}_{\nu} = 0\,, && A^{\mu}{}_{\nu} \neq 0\,, && \text{for (S)TEGR.} \end{aligned} \tag{5.38}$$

[3]Formally, parallel transport is defined using a horizontal lift of the curve $\gamma : \mathbb{R} \longrightarrow \mathcal{M}$ to the frame bundle. This lift connects the horizontal subspaces introduced in the definition of the connection. Intuitively, one may understand parallel transport as transporting a vector on a manifold while keeping it parallel to itself with respect to the underlying geometry.

Just as we denote all objects constructed from the Levi-Civita connection $\mathring{\omega}^\mu{}_\nu$ by circles, we use a bullet for denoting objects constructed from the gauge choice $\overset{\bullet}{\omega}{}^\mu{}_\nu$ of (S)TEGR which is defined by $\overset{\bullet}{\Omega}{}^\mu{}_\nu = 0$.

While (5.38) is effectively a reduction of the geometrical trinity to a duality, it may be interpreted as an extension at the same time. This may be understood from $A^\mu{}_\nu$ containing contributions of both torsion and non-metricity in general instead of assuming that only one of the two is non-vanishing. Hence, we are able to describe mixtures of torsion and non-metricity by studying (S)TEGR. This property was independently found in a tensor component description in [61]. Realizing that the Einstein-Hilbert action (5.35) only contains a quadratic contribution of the deformation one-form, [61] introduced the name *general teleparallel quadratic gravity* for the effective duality (5.38). Let us return to the Einstein-Hilbert action (5.35) and analyze the boundary terms needed in GR and (S)TEGR. Naively inserting the choices (5.38) for the connection one-form into (5.35) yields the actions

$$\begin{aligned} S^{\mathrm{GR}} &= S^{\mathrm{EH},\mathring{\Omega}}\Big|_{A^\mu{}_\nu=0} = \frac{1}{2\kappa}\int_{\mathcal{M}} \eta_\mu{}^\nu \wedge \mathring{\Omega}^\mu{}_\nu + S^{\mathrm{EH},\mathring{\Omega}}_{\mathrm{GHY}}\,, \\ S^{\mathrm{(S)TEGR}} &= S^{\mathrm{EH},\mathring{\Omega}}\Big|_{\Omega^\mu{}_\nu=0} = -\frac{1}{2\kappa}\int_{\mathcal{M}} \eta_\mu{}^\nu \wedge \overset{\bullet}{A}{}^\mu{}_\rho \wedge \overset{\bullet}{A}{}^\rho{}_\nu + S^{\mathrm{EH},\overset{\bullet}{\Omega}}_{\mathrm{GHY}} \end{aligned} \tag{5.39}$$

which are supposed to describe GR and (S)TEGR on manifolds with boundary. In order to see that this naive substitution is incorrect, we recall some of the results for GHY terms we derived in chapter 4.

We introduced GHY terms to make the variational principle well-defined. That is, we considered bulk actions S for which the variation $\delta S = \delta S_{\mathrm{eom}} + \delta S_{\mathrm{bdy}}$ includes a non-vanishing boundary term δS_{bdy}. For these actions, we defined the GHY term S_{GHY} such that $\delta S_{\mathrm{GHY}} = -\delta S_{\mathrm{bdy}}$ and thus Hamilton's principle $0 = \delta(S + S_{\mathrm{GHY}}) = \delta S_{\mathrm{eom}}$ yields the equations of motion. In contrast to that, consider an action $\mathcal{S}$ for which the variation $\delta\mathcal{S} = \delta S_{\mathrm{eom}}$ does not include a boundary contribution. Adding a GHY term to $\mathcal{S}$ makes the variational principle ill-defined. To see this, we note that applying Hamilton's principle to $\delta(\mathcal{S} + S_{\mathrm{GHY}}) = \delta S_{\mathrm{eom}} - \delta S_{\mathrm{bdy}}$ does not yield the equations of motion. Now, recall that only actions including curvature require us to add a GHY term if the theory is considered on a manifold with boundary. We derived this important result in chapter 4. But the bulk action of (S)TEGR in (5.39) depends solely on the deformation one-form which is in particular independent of curvature. Hence, the bulk (S)TEGR action is an action of type $\mathcal{S}$ and the addition of a GHY term to such an action makes the variational principle ill-defined. Imposing the gauge choice $\omega^\mu{}_\nu = \overset{\bullet}{\omega}{}^\mu{}_\nu$ which transforms GR to (S)TEGR, we therefore need to eliminate the GHY term $S^{\mathrm{EH},\Omega}_{\mathrm{GHY}}$ as well. In short, the GHY term $S^{\mathrm{EH},\Omega}_{\mathrm{GHY}}$ belongs to the curvature contribution of the action (5.35) and thus needs to be eliminated when we cancel the curvature dependent term. Hence, we need to refine the (S)TEGR action obtained in (5.39) for manifolds with boundary. We conclude that

$$S^{\mathrm{(S)TEGR}} = -\frac{1}{2\kappa}\int_{\mathcal{M}} \eta_\mu{}^\nu \wedge \mathring{A}^\mu{}_\rho \wedge \mathring{A}^\rho{}_\nu \tag{5.40}$$

is the well-defined action of the (symmetric) teleparallel equivalent of general relativity, no matter if the underlying manifold does or does not have a boundary.

The (S)TEGR action (5.40) coincides with the differential form action found in [119] for manifolds with no boundary. For comparison with the actions of TEGR and STEGR given in [26, 60], we need to derive the component version of (5.40). For this purpose, we introduce the tensor components $A^\mu{}_{\nu\rho} = A^\mu{}_\nu(\vartheta_\rho)$ of the deformation one-form to obtain the component expression of (5.40) as

$$S^{\mathrm{(S)TEGR}} = -\frac{1}{2\kappa}\int_{\mathcal{M}} \mathrm{dVol}_{\mathcal{M}}\sqrt{|\det g|}\,g^{\sigma\nu}(\mathring{A}^\mu{}_{\rho\mu}\mathring{A}^\rho{}_{\sigma\nu} - \mathring{A}^\mu{}_{\rho\nu}\mathring{A}^\rho{}_{\sigma\mu})\,. \tag{5.41}$$

This reproduces the well-known bulk expressions of TEGR and STEGR in 3 + 1 dimensions described in [26, 60] if we recall that $A^\mu{}_\nu$ is contortion in the metric-compatible case and disformation in the torsion-free case.

Furthermore, the integrand

$$\eta_\mu{}^\nu \wedge \mathring{A}^\mu{}_\rho \wedge \mathring{A}^\rho{}_\nu = \mathrm{dVol}_{\mathcal{M}}\sqrt{|\det g|}\,g^{\sigma\nu}(\mathring{A}^\mu{}_{\rho\mu}\mathring{A}^\rho{}_{\sigma\nu} - \mathring{A}^\mu{}_{\rho\nu}\mathring{A}^\rho{}_{\sigma\mu}) \tag{5.42}$$

of the boundary refined (S)TEGR action (5.40) is equivalent to the components of

$$T_\mu \wedge \star T^\mu \tag{5.43}$$

in (3+1)-dimensional TEGR, see [122] and [26]. That is, in 3+1 dimensions we found that the TEGR action (5.40) is reminiscent of a gauge theory. One might thus interpret (5.40) as a gauge theory of general relativity. The reason why the latter argument only holds in 3+1 dimensions is that the map $\star$ defining the gauge Lagrangian (5.43) does not denote the Hodge duality defined in (4.17). Instead, it is a generalized Hodge dual we need to introduce for torsion on bundles having a soldering form. While the component expression of this generalized Hodge dual was derived for 3+1 dimensions in [122], such an expression is not known in arbitrary dimensions so far [123]. In contrast to that, our result (5.40) is formulated in differential form notation in arbitrary dimensions, while we do not need to introduce the generalized Hodge dual $\star$.

The differential form (S)TEGR action (5.40) may therefore be used for deriving an expression of the torsional gauge Lagrangian (5.43) in arbitrary dimensions.

We conclude this section by summarizing the results in order to obtain a boundary refined version of the geometrical trinity of general relativity which has a well-defined variational principle. The action of general relativity may be described in two equivalent ways. The first way is to choose the connection to be the Levi-Civita connection $\mathring{\omega}^\mu{}_\nu$. This is the connection which is torsion-free and metric-compatible, such that the Einstein-Hilbert action is solely constituted by Riemannian curvature. Denoting this gauge choice by means of the deformation one-form as $\mathring{A}^\mu{}_\nu = 0$, GR is defined by

$$\Omega^\mu{}_\nu = \mathring{\Omega}^\mu{}_\nu \,, \qquad A^\mu{}_\nu = 0 \,. \tag{5.44}$$

We insert this choice into the Einstein-Hilbert action (5.35) to conclude that GR is entirely described by the action

$$S^{\mathrm{EH},\mathring{\Omega}} = \frac{1}{2\kappa}\int_{\mathcal{M}} \eta_\mu{}^\nu \wedge \mathring{\Omega}^\mu{}_\nu + S^{\mathrm{EH},\mathring{\Omega}}_{\mathrm{GHY}} \,. \tag{5.45}$$

The second choice of connection we considered in this section is constructed such that the curvature two-form vanishes. Note that this (S)TEGR connection $\overset{\bullet}{\omega}{}^\mu{}_\nu$ is not unique. In fact, we need the remaining gauge freedom in order to obtain TEGR and STEGR by further constraining $\overset{\bullet}{\omega}{}^\mu{}_\nu$. The deformation one-form of the (S)TEGR connection is $\overset{\bullet}{A}{}^\mu{}_\nu = \overset{\bullet}{\omega}{}^\mu{}_\nu - \mathring{\omega}^\mu{}_\nu$, such that the gauge choice of (S)TEGR is equivalent to constraining

$$\Omega^\mu{}_\nu = 0 \,, \qquad A^\mu{}_\nu = \overset{\bullet}{A}{}^\mu{}_\nu \,. \tag{5.46}$$

The careful treatment of boundary terms in this section revealed that the action

$$S^{\mathrm{(S)TEGR}} = -\frac{1}{2\kappa}\int_{\mathcal{M}} \eta_\mu{}^\nu \wedge \overset{\bullet}{A}{}^\mu{}_\rho \wedge \overset{\bullet}{A}{}^\rho{}_\nu \tag{5.47}$$

entirely describes (S)TEGR on a manifold with boundary, ensuring that the variational principle is well-defined.

The results of this section reproduce the bulk actions which are usually interpreted as the equivalent descriptions of general relativity through curvature, torsion and non-metricity. However, the interpretation of these results is ambiguous due to the (S)TEGR gauge choice $\omega^\mu{}_\nu = \overset{\bullet}{\omega}{}^\mu{}_\nu$. We discuss this ambiguity in detail in the following section.

5.2. A unifying perspective on the geometrical trinity of general relativity

Let us take one step back and consider the (S)TEGR action (5.39) in the form

$$S^{\text{(S)TEGR}} = S^{\text{EH},\mathring{\Omega}}\Big|_{\Omega^\mu{}_\nu=0} = -\frac{1}{2\kappa}\int_{\mathcal{M}} \eta_\mu{}^\nu \wedge \overset{\bullet}{A}{}^\mu{}_\rho \wedge \overset{\bullet}{A}{}^\rho{}_\nu + S^{\text{EH},\overset{\bullet}{\Omega}}_{\text{GHY}} \tag{5.48}$$

again. We found this action by choosing the (S)TEGR connection $\overset{\bullet}{\omega}{}^\mu{}_\nu$ such that $\Omega^\mu{}_\nu = \overset{\bullet}{\Omega}{}^\mu{}_\nu = 0$, see (5.46). Recalling the definition $\Omega^\mu{}_\nu = \mathrm{d}\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu$ of the curvature two-form, the (S)TEGR gauge choice implies

$$\overset{\bullet}{\omega}{}^\mu{}_\rho \wedge \overset{\bullet}{\omega}{}^\rho{}_\nu = -\mathrm{d}\overset{\bullet}{\omega}{}^\mu{}_\nu\,. \tag{5.49}$$

This defining equation of the (S)TEGR connection induces an ambiguity to actions using the gauge choice $\omega^\mu{}_\nu = \overset{\bullet}{\omega}{}^\mu{}_\nu$. To see this, we notice that $\overset{\bullet}{A}{}^\mu{}_\rho \wedge \overset{\bullet}{A}{}^\rho{}_\nu$ constituting the (S)TEGR bulk action in (5.48) contains the combination $\overset{\bullet}{\omega}{}^\mu{}_\rho \wedge \overset{\bullet}{\omega}{}^\rho{}_\nu$ of the (S)TEGR connection. Concretely, the definition $\overset{\bullet}{A}{}^\mu{}_\nu = \overset{\bullet}{\omega}{}^\mu{}_\nu - \mathring{\omega}^\mu{}_\nu$ of the (S)TEGR deformation implies

$$\overset{\bullet}{A}{}^\mu{}_\rho \wedge \overset{\bullet}{A}{}^\rho{}_\nu = \overset{\bullet}{\omega}{}^\mu{}_\rho \wedge \overset{\bullet}{\omega}{}^\rho{}_\nu - \overset{\bullet}{\omega}{}^\mu{}_\rho \wedge \mathring{\omega}^\rho{}_\nu - \mathring{\omega}^\mu{}_\rho \wedge \overset{\bullet}{\omega}{}^\rho{}_\nu + \mathring{\omega}^\mu{}_\rho \wedge \mathring{\omega}^\rho{}_\nu\,. \tag{5.50}$$

Inserting this decomposition into the action (5.48), we find that the (S)TEGR action includes the term

$$S^{\overset{\bullet}{\omega}} = -\frac{1}{2\kappa}\int_{\mathcal{M}} \eta_\mu{}^\nu \wedge \overset{\bullet}{\omega}{}^\mu{}_\rho \wedge \overset{\bullet}{\omega}{}^\rho{}_\nu = \frac{1}{2\kappa}\int_{\mathcal{M}} \mathrm{d}\overset{\bullet}{\omega}{}^\mu{}_\nu \wedge \eta_\mu{}^\nu\,, \tag{5.51}$$

where we used (5.49) for rewriting the connection. Due to $\mathring{D}\eta^{\mu\nu} = 0$ we have

$$\mathrm{d}\eta_\mu{}^\nu = \mathring{\omega}^\rho{}_\mu \wedge \eta_\rho{}^\nu - \mathring{\omega}^\nu{}_\rho \wedge \eta_\mu{}^\rho\,, \tag{5.52}$$

by means of which we write the action (5.51) as

$$S^{\overset{\bullet}{\omega}} = \frac{1}{2\kappa}\int_{\partial\mathcal{M}} \overset{\bullet}{\omega}{}^\mu{}_\nu \wedge \eta_\mu{}^\nu\Big|_{\partial\mathcal{M}} + \frac{1}{2\kappa}\int_{\mathcal{M}} \overset{\bullet}{\omega}{}^\mu{}_\nu \wedge \left(\mathring{\omega}^\rho{}_\mu \wedge \eta_\rho{}^\nu - \mathring{\omega}^\nu{}_\rho \wedge \eta_\mu{}^\rho\right), \tag{5.53}$$

where we used Stokes' theorem (2.21). Using the decomposition (5.50) in the original (S)TEGR action and reinstating the result (5.53) for $S^{\overset{\bullet}{\omega}}$ we obtain

$$S^{\text{(S)TEGR}} = -\frac{1}{2\kappa}\int_{\mathcal{M}} \eta_\mu{}^\nu \wedge \mathring{\omega}^\mu{}_\rho \wedge \mathring{\omega}^\rho{}_\nu + \frac{1}{2\kappa}\int_{\partial\mathcal{M}} \overset{\bullet}{\omega}{}^\mu{}_\nu \wedge \eta_\mu{}^\nu\Big|_{\partial\mathcal{M}} + S^{\text{EH},\overset{\bullet}{\Omega}}_{\text{GHY}}\,. \tag{5.54}$$

Expressing (S)TEGR as (5.54), we find that its action contains an additional boundary term beyond $S_{\text{GHY}}^{\text{EH},\overset{\bullet}{\Omega}}$. For understanding how the new boundary term $\frac{1}{2\kappa}\int_{\partial\mathcal{M}} \overset{\bullet}{\omega}^{\mu}{}_{\nu} \wedge \eta_{\mu}{}^{\nu}\big|_{\partial\mathcal{M}}$ combines with the GHY term $S_{\text{GHY}}^{\text{EH},\overset{\bullet}{\Omega}}$, we need to decompose it into boundary tangent and non-tangent contributions. Let us consider this decomposition for space- and timelike hypersurfaces first. By means of the decomposition of unity (3.38) we have

$$\frac{1}{2\kappa}\int_{\partial\mathcal{M}} \overset{\bullet}{\omega}^{\mu}{}_{\nu} \wedge \eta_{\mu}{}^{\nu}\Big|_{\partial\mathcal{M}} = \frac{1}{2\kappa}\int_{\partial\mathcal{M}} \Big(\varepsilon(e^{a}_{\mu} n^{\nu}\overset{\bullet}{\omega}^{\mu}{}_{\nu}) \wedge \eta_{a\mathbf{n}} + \varepsilon(n_{\mu} e^{\nu}_{a}\overset{\bullet}{\omega}^{\mu}{}_{\nu}) \wedge \eta^{\mathbf{n}a}\Big)\Big|_{\partial\mathcal{M}}, \quad (5.55)$$

where we used the simplifications $\eta_{\mathbf{nn}} = 0 = \eta^{ab}\big|_{\partial\mathcal{M}}$ derived in section 5.1.1. The remaining projections $e^{a}_{\mu} n^{\nu}\overset{\bullet}{\omega}^{\mu}{}_{\nu}$ and $n_{\mu} e^{\nu}_{a}\overset{\bullet}{\omega}^{\mu}{}_{\nu}$ of the (S)TEGR connection are determined by extrinsic curvature. Applying the corresponding calculation in (5.14) and (5.15) to the (S)TEGR connection, we rewrite (5.55) as

$$\frac{1}{2\kappa}\int_{\partial\mathcal{M}} \overset{\bullet}{\omega}^{\mu}{}_{\nu} \wedge \eta_{\mu}{}^{\nu}\Big|_{\partial\mathcal{M}} = -\frac{\varepsilon}{2\kappa}\int_{\partial\mathcal{M}} \Big(\overset{\bullet}{K}{}^{a} + \overset{\bullet}{\tilde{K}}{}^{a} - e^{a}_{\mu}(\mathrm{d}n^{\mu} + g^{\mu\nu}\mathrm{d}n_{\nu})\Big) \wedge \eta_{\mathbf{n}a}\Big|_{\partial\mathcal{M}}. \quad (5.56)$$

For interpreting this term, we recall from (4.37) that the GHY term $S_{\text{GHY}}^{\text{EH},\overset{\bullet}{\Omega}}$ is given by

$$S_{\text{GHY}}^{\text{EH},\overset{\bullet}{\Omega}} = \frac{\varepsilon}{2\kappa}\int_{\partial\mathcal{M}} (\overset{\bullet}{K}{}^{a} + \overset{\bullet}{\tilde{K}}{}^{a}) \wedge \eta_{\mathbf{n}a}\Big|_{\partial\mathcal{M}} \quad (5.57)$$

for the (S)TEGR connection. Hence, we conclude that

$$\frac{1}{2\kappa}\int_{\partial\mathcal{M}} \overset{\bullet}{\omega}^{\mu}{}_{\nu} \wedge \eta_{\mu}{}^{\nu}\Big|_{\partial\mathcal{M}} = -S_{\text{GHY}}^{\text{EH},\overset{\bullet}{\Omega}} + \frac{\varepsilon}{2\kappa}\int_{\partial\mathcal{M}} e^{a}_{\mu}(\mathrm{d}n^{\mu} + g^{\mu\nu}\mathrm{d}n_{\nu}) \wedge \eta_{\mathbf{n}a}\Big|_{\partial\mathcal{M}}, \quad (5.58)$$

which we finally insert into the (S)TEGR action (5.54) to obtain

$$S^{\text{(S)TEGR}} = -\frac{1}{2\kappa}\int_{\mathcal{M}} \eta_{\mu}{}^{\nu} \wedge \mathring{\omega}^{\mu}{}_{\rho} \wedge \mathring{\omega}^{\rho}{}_{\nu} + \frac{\varepsilon}{2\kappa}\int_{\partial\mathcal{M}} e^{a}_{\mu}(\mathrm{d}n^{\mu} + g^{\mu\nu}\mathrm{d}n_{\nu}) \wedge \eta_{\mathbf{n}a}\Big|_{\partial\mathcal{M}}. \quad (5.59)$$

We postpone both the discussion of the covariance of this action as well as the thorough interpretation of its boundary term to subsection 5.2.1 in order to readily include lightlike boundaries.

The action (5.59) is equivalent to the full (S)TEGR action (5.48) including the GHY term $S_{\text{GHY}}^{\text{EH},\overset{\bullet}{\Omega}}$. Therefore, we have found that (S)TEGR may be described solely by the Levi-Civita connection which is torsion-free and metric-compatible. This is remarkable because the bulk contribution of (S)TEGR in the form (5.48) was fully described by torsion and non-metricity. Note that this ambiguity originates from the choice (5.49) of the connection which we used to replace $\overset{\bullet}{\omega}^{\mu}{}_{\rho} \wedge \overset{\bullet}{\omega}^{\rho}{}_{\nu}$ by $-\mathrm{d}\overset{\bullet}{\omega}^{\mu}{}_{\nu}$. This ambiguity of the

(S)TEGR connection was already found in [124] in a different context. The derivation above may as well be read in the inverse direction. That is, we may re-express the GHY term in the (S)TEGR gauge as a bulk term by combining (5.58) with (5.54). Let us emphasize again that the comparison of the two equivalent expressions (5.48) and (5.59) of the (S)TEGR action implies that gravity in (S)TEGR is modeled either by torsion and non-metricity or by the torsion-free, metric-compatible Levi-Civita connection. Since this is counterintuitive, let us explore the action (5.59) in more detail.

To begin with, we mention that it was already noticed by Einstein [12] that (5.59) yields the correct equations of motion of general relativity. In fact, we may re-express the Einstein-Hilbert action in the form

$$S^{\mathrm{EH},\mathring{\Omega}} = \frac{1}{2\kappa}\int_{\mathcal{M}} \eta_\mu{}^\nu \wedge \mathring{\Omega}^\mu{}_\nu + S^{\mathrm{EH},\mathring{\Omega}}_{\mathrm{GHY}} \tag{5.60}$$

by a calculation along the same lines from which we obtained (5.59) as an expression of the (S)TEGR action. That is, we use the definition $\Omega^\mu{}_\nu = \mathrm{d}\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu$ of the curvature two-form in the Einstein-Hilbert action (5.60) and insert $\mathring{D}\eta^{\mu\nu} = 0$ to conclude

$$S^{\mathrm{EH},\mathring{\Omega}} = -\frac{1}{2\kappa}\int_{\mathcal{M}} \eta_\mu{}^\nu \wedge \mathring{\omega}^\mu{}_\rho \wedge \mathring{\omega}^\rho{}_\nu + \frac{\varepsilon}{2\kappa}\int_{\partial\mathcal{M}} e^a_\mu(\mathrm{d}n^\mu + g^{\mu\nu}\mathrm{d}n_\nu)\wedge \eta_{\mathbf{n}a}\Big|_{\partial\mathcal{M}}. \tag{5.61}$$

In the latter form, the Einstein-Hilbert action is called *Einstein action*. Hence, the result (5.59) for the (S)TEGR action exactly reproduces the Einstein-Hilbert action with which we have started, $S^{\mathrm{(S)TEGR}} = S^{\mathrm{EH},\mathring{\Omega}}$.

Naively, one might conclude that there is no new information in the (S)TEGR action. However, recall that the theories described by $S^{\mathrm{(S)TEGR}}$ and $S^{\mathrm{EH},\mathring{\Omega}}$ describe gravitational dynamics by means of a very different field content. In GR, gravity is modeled by curvature while torsion and non-metricity vanish. In contrast to that, (S)TEGR is a theory with non-trivial torsion and non-metricity, while curvature is vanishing. The fact that these seemingly utterly different theories may be described by the exact same action implies that the field content modeled by this action is not unambiguous. Since this field content is constituted solely by the Levi-Civita connection $\mathring{\omega}^\mu{}_\nu$, we need to study $\mathring{\omega}^\mu{}_\nu$ in more detail. For this purpose, we solve the definition of the deformation one-form (5.4) for the Levi-Civita connection to obtain

$$\mathring{\omega}^\mu{}_\nu = \omega^\mu{}_\nu - A^\mu{}_\nu\,. \tag{5.62}$$

While the latter equation is trivial mathematically, it has important physical consequences. In the (S)TEGR gauge choice, (5.62) implies that the Levi-Civita connection may be expressed through the (S)TEGR connection and the deformation one-form, where the latter encodes contributions of torsion and non-metricity. This is the interpretation of $\mathring{\omega}^{\mu}{}_{\nu}$ which allows us to find the Einstein action (5.59) as the action which describes dynamics either on curved or on flat manifolds with non-trivial torsion and non-metricity.

This provides a new perspective on the geometrical trinity of general relativity, because we found all of its equivalent theories to be described by the Einstein action, while the seemingly different dynamics rely on the ambiguous interpretation of the Levi-Civita connection $\mathring{\omega}^{\mu}{}_{\nu}$. This new perspective on the (S)TEGR action, its GHY term and the geometrical trinity of general relativity is the main result of this section. I did not publish this result before. The importance of the Einstein action for the individual theories contained in the geometrical trinity was studied to different extent in the literature. Einstein already discussed this action in GR, and its importance as a GR action in view of the geometrical trinity was recently stressed in [125] again. In the realm of STEGR, the Einstein action was found in a subgroup of theories called *coincident GR*, see [21]. Finally, [126] recently hinted at the importance of the Einstein action in TEGR. My results derived in this section unify all these approaches and explain why the Einstein action possesses these different interpretations.

This discussion qualitatively generalizes to the (S)TEGR action on manifolds with lightlike boundaries, although we note that the boundary term in the (S)TEGR action (5.59) explicitly depends on the likeness of the boundary and therefore needs to be altered. Just as for space- and timelike boundaries, we note that the original (S)TEGR action (5.48) may be interpreted as containing a boundary term in $\eta_{\mu}{}^{\nu} \wedge \overset{\bullet}{A}{}^{\mu}{}_{\rho} \wedge \overset{\bullet}{A}{}^{\rho}{}_{\nu}$. This is due to the (S)TEGR gauge choice $\overset{\bullet}{\omega}{}^{\mu}{}_{\rho} \wedge \overset{\bullet}{\omega}{}^{\rho}{}_{\nu} = -\mathrm{d}\overset{\bullet}{\omega}{}^{\mu}{}_{\nu}$. The derivation of the boundary term contained in $\eta_{\mu}{}^{\nu} \wedge \overset{\bullet}{A}{}^{\mu}{}_{\rho} \wedge \overset{\bullet}{A}{}^{\rho}{}_{\nu}$ proceeds analogous to the space- and timelike case and we rewrite the (S)TEGR action as

$$S^{\text{(S)TEGR}} = -\frac{1}{2\kappa}\int_{\mathcal{M}} \eta_{\mu}{}^{\nu} \wedge \mathring{\omega}^{\mu}{}_{\rho} \wedge \mathring{\omega}^{\rho}{}_{\nu} + \frac{1}{2\kappa}\int_{\partial\mathcal{M}} \overset{\bullet}{\omega}{}^{\mu}{}_{\nu} \wedge \eta_{\mu}{}^{\nu}\Big|_{\partial\mathcal{M}} + S^{\text{EH},\overset{\bullet}{\Omega}}_{\text{GHY}} \tag{5.63}$$

analogous to (5.54). In order to compare the boundary term $\frac{1}{2\kappa}\int_{\partial\mathcal{M}} \overset{\bullet}{\omega}{}^{\mu}{}_{\nu} \wedge \eta_{\mu}{}^{\nu}\Big|_{\partial\mathcal{M}}$ with the GHY term $S^{\text{EH},\overset{\bullet}{\Omega}}_{\text{GHY}}$ on lightlike boundaries, we decompose it into boundary tangent and non-tangent contributions. For this purpose, we insert the lightlike decomposition of unity (3.87) into this boundary term to obtain

$$\begin{aligned}
&\frac{1}{2\kappa}\int_{\partial\mathcal{M}} \overset{\bullet}{\omega}^{\mu}{}_{\nu}\wedge\eta_{\mu}{}^{\nu}\Big|_{\partial\mathcal{M}} = \\
&\frac{1}{2\kappa}\int_{\partial\mathcal{M}} \Big((e^{A}_{\mu}e^{\nu}_{B}\overset{\bullet}{\omega}^{\mu}{}_{\nu})\wedge\eta_{A}{}^{B} + \varepsilon(e^{A}_{\mu}k^{\nu}\overset{\bullet}{\omega}^{\mu}{}_{\nu})\wedge\eta_{A\mathbf{l}} + \varepsilon(e^{A}_{\mu}l^{\nu}\overset{\bullet}{\omega}^{\mu}{}_{\nu})\wedge\eta_{A\mathbf{k}} \\
&\qquad + \varepsilon(k_{\mu}e^{\nu}_{A}\overset{\bullet}{\omega}^{\mu}{}_{\nu})\wedge\eta^{\mathbf{l}A} + \varepsilon(l_{\mu}e^{\nu}_{A}\overset{\bullet}{\omega}^{\mu}{}_{\nu})\wedge\eta^{\mathbf{k}A} + \varepsilon^{2}(k_{\mu}l^{\nu}\overset{\bullet}{\omega}^{\mu}{}_{\nu})\wedge\eta_{\mathbf{lk}} \\
&\qquad + \varepsilon^{2}(l_{\mu}k^{\nu}\overset{\bullet}{\omega}^{\mu}{}_{\nu})\wedge\eta_{\mathbf{kl}} + \varepsilon^{2}(l_{\mu}l^{\nu}\overset{\bullet}{\omega}^{\mu}{}_{\nu})\wedge\eta_{\mathbf{kk}} + \varepsilon^{2}(k_{\mu}k^{\nu}\overset{\bullet}{\omega}^{\mu}{}_{\nu})\wedge\eta_{\mathbf{ll}}\Big)\Big|_{\partial\mathcal{M}}.
\end{aligned} \tag{5.64}$$

In section 5.1.2, we concluded that the only non-vanishing projection of $\eta^{\mu\nu}$ in the lightlike boundary term (5.64) is $\eta_{\mathbf{lk}}\big|_{\partial\mathcal{M}} = -\eta_{\mathbf{kl}}\big|_{\partial\mathcal{M}}$. The corresponding projections of the (S)TEGR connection are contained in the definitions of $\mathcal{K}$ and $\mathcal{L}$ which we examined in (5.28). By means of these projections, we evaluate the boundary term (5.64) to obtain

$$\frac{1}{2\kappa}\int_{\partial\mathcal{M}} \overset{\bullet}{\omega}^{\mu}{}_{\nu}\wedge\eta_{\mu}{}^{\nu}\Big|_{\partial\mathcal{M}} = -\frac{\varepsilon^{2}}{2\kappa}\int_{\partial\mathcal{M}}\Big(\overset{\bullet}{\mathcal{K}} - \overset{\bullet}{\mathcal{L}} - l_{\mu}\mathrm{d}k^{\mu} + k_{\mu}\mathrm{d}l^{\mu}\Big)\wedge\eta_{\mathbf{lk}}\Big|_{\partial\mathcal{M}}. \tag{5.65}$$

This includes the GHY term (4.107) of Einstein-Hilbert gravity for lightlike boundaries in the (S)TEGR gauge. Comparison to (4.107) reveals that the boundary term (5.65) may be rewritten as

$$\frac{1}{2\kappa}\int_{\partial\mathcal{M}} \overset{\bullet}{\omega}^{\mu}{}_{\nu}\wedge\eta_{\mu}{}^{\nu}\Big|_{\partial\mathcal{M}} = -S^{\mathrm{EH},\overset{\bullet}{\Omega}}_{\mathrm{GHY}} + \frac{\varepsilon^{2}}{2\kappa}\int_{\partial\mathcal{M}}(l_{\mu}\mathrm{d}k^{\mu} - k_{\mu}\mathrm{d}l^{\mu})\wedge\eta_{\mathbf{lk}}\Big|_{\partial\mathcal{M}}. \tag{5.66}$$

In analogy to the space- and timelike case, we reinstate the latter result into (5.63) to rewrite the (S)TEGR action (5.48) on manifolds with lightlike boundary as

$$S^{\mathrm{(S)TEGR}} = -\frac{1}{2\kappa}\int_{\mathcal{M}}\eta_{\mu}{}^{\nu}\wedge\overset{\circ}{\omega}^{\mu}{}_{\rho}\wedge\overset{\circ}{\omega}^{\rho}{}_{\nu} + \frac{\varepsilon^{2}}{2\kappa}\int_{\partial\mathcal{M}}(l_{\mu}\mathrm{d}k^{\mu} - k_{\mu}\mathrm{d}l^{\mu})\wedge\eta_{\mathbf{lk}}\Big|_{\partial\mathcal{M}}. \tag{5.67}$$

Note that due to $k_{\mu}\mathrm{d}l^{\mu} = -l^{\mu}\mathrm{d}k_{\mu}$, (5.67) is completely analogous to the result (5.59) we obtained for space- and timelike boundaries. The discussion of (5.59) therefore applies for the lightlike case (5.67) as well. In particular, (5.67) is the Einstein action which we may obtain from rewriting the Einstein-Hilbert action on manifolds with lightlike boundaries. The bulk term contained in (5.67) is consistently the same as in the non-lightlike case. Thus, the interpretation of this bulk term as a model for either GR or (S)TEGR is identical to the discussion of the non-lightlike case. There are two aspects which remain to be discussed in order to obtain a complete interpretation of the Einstein action in the context of the geometrical trinity of general relativity. First, this is the invariance of the Einstein action with respect to $\mathrm{GL}(m,\mathbb{R})$ transformations and second, this is an interpretation of the boundary term. We proceed by investigating both of these aspects.

5.2.1. Covariance of the Einstein action and its boundary term

In this section, we examine the role of the boundary term $S^{\mathrm{E}}_{\mathrm{cov}}$ of the Einstein action. For later reference, we decompose the Einstein action into its contribution $S^{\mathrm{E}}_{\mathrm{bulk}}$ in the bulk and its boundary term $S^{\mathrm{E}}_{\mathrm{cov}}$ such that $S^{\mathrm{E}} = S^{\mathrm{E}}_{\mathrm{bulk}} + S^{\mathrm{E}}_{\mathrm{cov}}$. The main result of this subsection is that S^{E} is covariant[4] because $S^{\mathrm{E}}_{\mathrm{cov}}$ accounts for the boundary terms which make $S^{\mathrm{E}}_{\mathrm{bulk}}$ non-covariant. We examine the transformation behavior of each term separately in the following in order to proof this statement.

The bulk contribution $S^{\mathrm{E}}_{\mathrm{bulk}}$ of the Einstein action is given by

$$S^{\mathrm{E}}_{\mathrm{bulk}} := -\frac{1}{2\kappa}\int_{\mathcal{M}} \eta_{\mu}{}^{\nu} \wedge \mathring{\omega}^{\mu}{}_{\rho} \wedge \mathring{\omega}^{\rho}{}_{\nu}\,, \tag{5.68}$$

while the boundary contribution differs for lightlike and non-lightlike boundaries. On space- and timelike hypersurfaces, (5.59) implies

$$S^{\mathrm{E}}_{\mathrm{cov}} = \frac{\varepsilon}{2\kappa}\int_{\partial\mathcal{M}} e^{a}_{\mu}(\mathrm{d}n^{\mu} + g^{\mu\nu}\mathrm{d}n_{\nu}) \wedge \eta_{\mathbf{n}a}\Big|_{\partial\mathcal{M}}\,, \tag{5.69}$$

while we obtain

$$S^{\mathrm{E}}_{\mathrm{cov}} = \frac{\varepsilon^{2}}{2\kappa}\int_{\partial\mathcal{M}} (l_{\mu}\mathrm{d}k^{\mu} - k_{\mu}\mathrm{d}l^{\mu}) \wedge \eta_{\mathbf{lk}}\Big|_{\partial\mathcal{M}} \tag{5.70}$$

on lightlike hypersurfaces from (5.67). We denote both boundary terms by the same symbol $S^{\mathrm{E}}_{\mathrm{cov}}$ by abuse of notation since the considered boundary geometry makes it unambiguous which of the boundary terms we need to choose.

For interpreting (5.69) and (5.70), we first notice that $S^{\mathrm{E}}_{\mathrm{cov}}$ does not depend on the connection and is thus irrelevant for the variational principle. Therefore, it may *not* be understood as a GHY term, while at the same time it does not require us to introduce another GHY term which would cancel its variation. Instead, this term makes the Einstein action $S^{\mathrm{E}} = S^{\mathrm{E}}_{\mathrm{bulk}} + S^{\mathrm{E}}_{\mathrm{cov}}$ invariant with respect to $\mathrm{GL}(m,\mathbb{R})$ transformations. The most straightforward argument proving this covariance is that we have found S^{E} as a rewriting of the manifestly covariant Einstein-Hilbert action including its GHY term. The proof of covariance may be given explicitly as well, and we proceed by constructing it in differential form notation. A lengthy derivation in [125] showed that the bulk Einstein action is covariant up to a boundary term, which was recently argued

[4] We call terms *covariant* if they are invariant with respect to $\mathrm{GL}(m,\mathbb{R})$ transformations. For manifestly covariant terms, this invariance is obvious.

in [126] as well. We obtain this result very efficiently in the differential form formalism which we developed in this thesis. To see that, we consider an arbitrary $\mathrm{GL}(m,\mathbb{R})$ transformation $\Lambda^{\mu}{}_{\nu}$. In accordance to the transformation properties we derived for connection one-forms and associated bundles in sections 2.5 and 2.6, we denote the transformed tensors by

$$\begin{aligned}\bar{\omega}^{\mu}{}_{\nu} &= \Lambda^{-1\mu}{}_{\rho}\omega^{\rho}{}_{\sigma}\Lambda^{\sigma}{}_{\nu} + \Lambda^{-1\mu}{}_{\rho}\mathrm{d}\Lambda^{\rho}{}_{\nu}\,,\\ \bar{\eta}_{\mu}{}^{\nu} &= \Lambda^{\rho}{}_{\mu}\Lambda^{-1\nu}{}_{\sigma}\eta_{\rho}{}^{\sigma}\,.\end{aligned} \tag{5.71}$$

For evaluating the transformed Einstein action $\bar{S}^{\mathrm{E}}_{\mathrm{bulk}}$, we need to recall that $\mathring{D}\eta_{\mu}{}^{\nu} = 0$ for the Levi-Civita covariant exterior derivative. Furthermore, we use $0 = \mathrm{d}\delta^{\mu}_{\nu}$ with $\delta^{\mu}_{\nu} = \Lambda^{-1\mu}{}_{\rho}\Lambda^{\rho}{}_{\nu}$ to derive that

$$\mathrm{d}\Lambda^{-1\mu}{}_{\nu} = -\Lambda^{-1\mu}{}_{\rho}\Lambda^{-1\sigma}{}_{\nu}\mathrm{d}\Lambda^{\rho}{}_{\sigma}\,. \tag{5.72}$$

By means of these simplifications, we insert (5.71) into the transformed bulk Einstein action $\bar{S}^{\mathrm{E}}_{\mathrm{bulk}}$ to obtain

$$\bar{S}^{\mathrm{E}}_{\mathrm{bulk}} = S^{\mathrm{E}}_{\mathrm{bulk}} - \frac{1}{2\kappa}\int_{\partial\mathcal{M}} \Lambda^{-1\rho}{}_{\nu}\mathrm{d}\Lambda^{\mu}{}_{\rho}\wedge\eta_{\mu}{}^{\nu}\Big|_{\partial\mathcal{M}}\,. \tag{5.73}$$

Hence, we verified that the bulk Einstein action (5.68) is invariant with respect to $\mathrm{GL}(m,\mathbb{R})$ transformations up to the boundary term

$$S^{\Lambda} := -\frac{1}{2\kappa}\int_{\partial\mathcal{M}} \Lambda^{-1\rho}{}_{\nu}\mathrm{d}\Lambda^{\mu}{}_{\rho}\wedge\eta_{\mu}{}^{\nu}\Big|_{\partial\mathcal{M}}\,. \tag{5.74}$$

In order to compare this result to the boundary term $S^{\mathrm{E}}_{\mathrm{cov}}$ of the Einstein action, we need to decompose (5.74) on lightlike and non-lightlike boundaries separately. For the space- and timelike case, we found in section 5.1.1 that only the projection $\eta_{\mathbf{n}a}\big|_{\partial\mathcal{M}} = -\eta_{a\mathbf{n}}\big|_{\partial\mathcal{M}}$ of $\eta^{\mu\nu} = *(\theta^{\mu}\wedge\theta^{\nu})$ contributes on hypersurfaces. By means of the unity decomposition (3.38), we thus obtain

$$S^{\Lambda} = -\frac{\varepsilon}{2\kappa}\int_{\partial\mathcal{M}} (n_{\mu}g^{\nu\sigma}e^{a}_{\sigma} - e^{a}_{\mu}n^{\nu})\Lambda^{-1\rho}{}_{\nu}\mathrm{d}\Lambda^{\mu}{}_{\rho}\wedge\eta_{\mathbf{n}a}\Big|_{\partial\mathcal{M}} \tag{5.75}$$

on space- and timelike boundaries. For the lightlike case, section 5.1.2 revealed that $\eta_{\mathbf{lk}}\big|_{\partial\mathcal{M}} = -\eta_{\mathbf{kl}}\big|_{\partial\mathcal{M}}$ is the only non-vanishing projection of $\eta^{\mu\nu}$. Inserting the lightlike decomposition of unity (3.87) into (5.74), we thus obtain

$$S^{\Lambda} = -\frac{\varepsilon^2}{2\kappa}\int_{\partial\mathcal{M}} (k_{\mu}l^{\nu} - l_{\mu}k^{\nu})\Lambda^{-1\rho}{}_{\nu}\mathrm{d}\Lambda^{\mu}{}_{\rho}\wedge\eta_{\mathbf{lk}}\Big|_{\partial\mathcal{M}}\,. \tag{5.76}$$

This boundary term appears in the transformation of the boundary action $S^{\mathrm{E}}_{\mathrm{cov}}$ as well. Concretely, applying the $\mathrm{GL}(m,\mathbb{R})$ transformation $\Lambda^{\mu}{}_{\nu}$ to (5.69) and (5.70), we obtain

$$\bar{S}^{\mathrm{E}}_{\mathrm{cov}} = S^{\mathrm{E}}_{\mathrm{cov}} - S^{\Lambda} \tag{5.77}$$

for both lightlike and non-lightlike boundaries. Combining (5.77) with (5.74), we conclude that

$$\bar{S}^{\mathrm{E}} = S^{\mathrm{E}}\,. \tag{5.78}$$

This result clarifies two aspects. First, we explicitly see that the Einstein action including the boundary term $S^{\mathrm{E}}_{\mathrm{cov}}$ is invariant under $\mathrm{GL}(m,\mathbb{R})$ transformations. Second, (5.77) reveals that the role of the boundary terms (5.69) and (5.70) is exactly to make S^{E} fully covariant. This covariance is essential for the interpretation of the Einstein action as a model for physical systems. While it was found for instance in [125, 126] that the bulk action is invariant under diffeomorphisms up to boundary terms, my results discussed in this subsection show how to restore $\mathrm{GL}(m,\mathbb{R})$ invariance completely. This is my original result which I did not publish before. Combined with the remaining results I derived in section 5.2 so far, this sheds new light on the geometrical trinity of general relativity. That is, from the Einstein action and its boundary term we may entirely understand the geometrical trinity of general relativity as well as general teleparallel quadratic gravity. We have seen that this action resembles GR if we interpret $\mathring{\omega}^{\mu}{}_{\nu}$ as the Levi-Civita connection, while we rewrite $\mathring{\omega}^{\mu}{}_{\nu} = \overset{\bullet}{\omega}{}^{\mu}{}_{\nu} - \overset{\bullet}{A}{}^{\mu}{}_{\nu}$ to obtain a theory of (S)TEGR. Finally, we concluded the discussion of the Einstein action perspective on the geometrical trinity of general relativity by proving that its boundary term makes the Einstein action fully invariant under $\mathrm{GL}(m,\mathbb{R})$ transformations. Therefore, the Einstein action provides a unified approach to the geometrical trinity of general relativity. This unifying perspective is one of the main results of this chapter.

5.2.2. Component expression of the Einstein action

For calculational purposes, we will need the expression of the Einstein action (5.68) in tensor components. For obtaining this expression, we introduce the tensor components of the relevant tensors as

$$\Gamma^{\mu}_{\rho\nu} := \omega^{\mu}{}_{\nu}(\vartheta_{\rho})\,, \qquad c^{\mu}{}_{\nu\rho} := \mathrm{d}\theta^{\mu}(\vartheta_{\nu},\vartheta_{\rho})\,, \qquad T^{\mu}{}_{\nu\rho} := T^{\mu}(\vartheta_{\nu},\vartheta_{\rho})\,, \qquad Q_{\mu\nu\rho} := Q_{\mu\nu}(\vartheta_{\rho})\,. \tag{5.79}$$

By means of the definitions $T^\mu = \mathrm{d}\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu$ and $Q_{\mu\nu} = -\mathrm{d}g_{\mu\nu} + \omega^\rho{}_\mu g_{\rho\nu} + \omega^\rho{}_\nu g_{\mu\rho}$ of torsion and non-metricity, we re-express their tensor components (5.79) as

$$T^\mu{}_{\nu\rho} = c^\mu{}_{\nu\rho} + \Gamma^\mu_{\nu\rho} - \Gamma^\mu_{\rho\nu}\,, \qquad Q_{\mu\nu\rho} = -\partial_\rho g_{\mu\nu} + \Gamma^\sigma_{\rho\mu} g_{\sigma\nu} + \Gamma^\sigma_{\rho\nu} g_{\sigma\mu} \tag{5.80}$$

and we furthermore introduce the abbreviations $T_{\mu\nu\rho} := g_{\mu\sigma} T^\sigma{}_{\nu\rho}$ and $c_{\mu\nu\rho} := g_{\mu\sigma} c^\sigma{}_{\nu\rho}$.

It is now useful to evaluate the combination

$$-T_{\mu\nu\rho} + T_{\nu\rho\mu} + T_{\rho\mu\nu} - Q_{\mu\nu\rho} + Q_{\nu\rho\mu} + Q_{\rho\mu\nu} \tag{5.81}$$

of torsion and non-metricity by means of (5.80). Reordering the terms in this combination, we obtain the expression

$$\begin{aligned}\Gamma^\sigma_{\mu\nu} =& \frac{1}{2} g^{\sigma\rho}(c_{\mu\nu\rho} - c_{\nu\rho\mu} - c_{\rho\mu\nu}) + \frac{1}{2} g^{\sigma\rho}(-\partial_\rho g_{\mu\nu} + \partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu}) \\ &+ \frac{1}{2} g^{\sigma\rho}(-T_{\mu\nu\rho} + T_{\nu\rho\mu} + T_{\rho\mu\nu}) + \frac{1}{2} g^{\sigma\rho}(-Q_{\mu\nu\rho} + Q_{\nu\rho\mu} + Q_{\rho\mu\nu})\,.\end{aligned} \tag{5.82}$$

These are the components $\Gamma^\sigma_{\mu\nu} = \omega^\sigma{}_\nu(\vartheta_\mu)$ of any connection one-form $\omega^\mu{}_\nu$. Thus, from evaluating (5.82) we obtain the coefficients of every connection we previously discussed in this section. For example, we defined the Levi-Civita connection $\mathring{\omega}^\mu{}_\nu$ as the connection which has vanishing torsion and non-metricity. Inserting $T_{\mu\nu\rho} = 0 = Q_{\mu\nu\rho}$ into (5.82), we thus obtain the components of the Levi-Civita connection as

$$\mathring{\Gamma}^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(c_{\mu\nu\rho} - c_{\nu\rho\mu} - c_{\rho\mu\nu}) + \frac{1}{2} g^{\sigma\rho}(-\partial_\rho g_{\mu\nu} + \partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu})\,. \tag{5.83}$$

This is the standard expression of the Levi-Civita connection components in frames which do not obey $\mathrm{d}\theta^\mu = 0$, see [127, 128] for instance. Combining (5.83) with (5.82), we explicitly verify that the components $A^\mu{}_{\nu\rho} := A^\mu{}_\nu(\vartheta_\rho)$ of the deformation one-form are independent of curvature,

$$A^\sigma{}_{\nu\mu} = \Gamma^\sigma_{\mu\nu} - \mathring{\Gamma}^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(-T_{\mu\nu\rho} + T_{\nu\rho\mu} + T_{\rho\mu\nu}) + \frac{1}{2} g^{\sigma\rho}(-Q_{\mu\nu\rho} + Q_{\nu\rho\mu} + Q_{\rho\mu\nu})\,. \tag{5.84}$$

These tensor component results will be useful for the calculation of (S)TEGR on-shell actions in chapter 6.

We conclude the discussion of the geometrical trinity of general relativity by comparing the Einstein action approach to the perspective developed in section 5.1. There are two main advantages of the formulation of the (S)TEGR action by means of the deformation one-form. First, due to the relation (5.84) of $A^\mu{}_\nu$ to torsion and non-metricity, it is unambiguous that the action $-\frac{1}{2\kappa}\int_{\mathcal{M}} \eta_\mu{}^\nu \wedge A^\mu{}_\rho \wedge A^\rho{}_\nu$ describes a theory in which gravity is understood as torsion and non-metricity of the underlying manifold. Second, this action is manifestly covariant. Formally, it induces the bulk contribution of the Einstein action if we fix the gauge $\overset{\bullet}{\omega}{}^\mu{}_\nu = 0$. Nevertheless, we found that the Einstein action describes geometries with $\overset{\bullet}{\omega}{}^\mu{}_\nu \neq 0$ as well since we may rewrite $\mathring{\omega}^\mu{}_\nu$ as $\overset{\bullet}{\omega}{}^\mu{}_\nu - \overset{\bullet}{A}{}^\mu{}_\nu$, giving the action a (symmetric) teleparallel interpretation. In section 5.2.1 we concluded that the Einstein action is invariant with respect to $\mathrm{GL}(m, \mathbb{R})$ transformations as well. The main advantage of the new Einstein action perspective on the geometrical trinity of general relativity is that all of its equivalent theories are described by the very same action. From this perspective, the equivalence of GR and (S)TEGR becomes tautological.

5.3. The geometrical trinity of gravity

As a last facet of the geometrical trinity of general relativity we discuss how it may be generalized to generic theories of gravity based on Riemannian curvature. For space- and timelike hypersurfaces, I first discussed this in [2]. I generalize these results to lightlike hypersurfaces in this section which is unpublished so far. My results generalize those of [129] which found that every metric theory of gravity has a teleparallel equivalent. Also see [130] for a recent discussion of Gauß-Bonnet terms in the geometrical trinity of gravity.

The basic idea underlying this generalization is that the decomposition (5.6) of curvature into its Riemannian contributions and terms originating from torsion and non-metricity may be used to re-interpret the Riemannian curvature in any theory. To see that, we consider a generic action of the form

$$S^{\mathring{\Omega}} = \int_{\mathcal{M}} L(\mathring{\Omega}^\mu{}_\nu) + S^{\mathring{\Omega}}_{\mathrm{GHY}}\,, \tag{5.85}$$

where the GHY term may be obtained from (4.26) for space- and timelike boundaries, whereas (4.98) yields $S^{\mathring{\Omega}}_{\mathrm{GHY}}$ for lightlike boundaries. In order to study this action systematically in its generic form (5.85), we introduce Lagrange multipliers and auxiliary fields as in sections 4.1.2 and 4.2.2. This yields

$$\begin{aligned} S^{\mathring{\Omega}}_{\mathrm{Lagr}} &= \int_{\mathcal{M}} \Big(L(\mathring{\varrho}^\mu{}_\nu) + {*\varphi_\mu{}^\nu} \wedge (\mathring{\Omega}^\mu{}_\nu - \mathring{\varrho}^\mu{}_\nu)\Big) + S^{\mathring{\Omega}}_{\mathrm{GHY}} \\ &= \int_{\mathcal{M}} \Big(L(\mathring{\varrho}^\mu{}_\nu) + {*\varphi_\mu{}^\nu} \wedge (\Omega^\mu{}_\nu - \mathring{D}A^\mu{}_\nu - A^\mu{}_\rho \wedge A^\rho{}_\nu - \mathring{\varrho}^\mu{}_\nu)\Big) + S^{\mathring{\Omega}}_{\mathrm{GHY}} \end{aligned} \tag{5.86}$$

by means of the curvature decomposition (5.6). The contribution $\mathring{D}A^\mu{}_\nu$ of this Lagrangian will generically yield boundary terms due to

$$-\int_{\mathcal{M}} *\varphi_{\mu}{}^{\nu} \wedge \mathring{D} A^{\mu}{}_{\nu} = -\int_{\partial\mathcal{M}} A^{\mu}{}_{\nu} \wedge *\varphi_{\mu}{}^{\nu}\Big|_{\partial\mathcal{M}} - \int_{\mathcal{M}} A^{\mu}{}_{\nu} \wedge \mathring{D} * \varphi_{\mu}{}^{\nu} \tag{5.87}$$

which we need to compare to $S_{\text{GHY}}^{\mathring{\Omega}}$ for understanding the boundary contributions of (5.86). For this comparison, we decompose

$$S_{\text{Lagr}}^{A*\varphi} := -\int_{\partial\mathcal{M}} A^{\mu}{}_{\nu} \wedge *\varphi_{\mu}{}^{\nu}\Big|_{\partial\mathcal{M}} \tag{5.88}$$

into hypersurface tangent and normal contributions. This decomposition differs for lightlike and non-lightlike boundaries, and we examine the space- and timelike case first.

Inserting the projections (5.17) associated to the decomposition (5.12) of the deformation one-form into (5.88), we obtain

$$\begin{aligned} S_{\text{Lagr}}^{A*\varphi} = -\int_{\partial\mathcal{M}} \Big(& A^{a}{}_{b} \wedge *\varphi_{a}{}^{b} + \varepsilon(K^{a} - \mathring{K}^{a}) \wedge *\varphi_{\mathbf{n}a} \\ & + \varepsilon(-\tilde{K}^{a} + \mathring{K}^{a}) \wedge *\varphi_{\mathbf{n}a} + \frac{1}{2} Q_{\mathbf{nn}} \wedge *\varphi_{\mathbf{nn}} \Big)\Big|_{\partial\mathcal{M}}, \end{aligned} \tag{5.89}$$

which we compare to the universal GHY term (4.26) to conclude

$$S_{\text{Lagr}}^{A*\varphi} = -\int_{\partial\mathcal{M}} A^{a}{}_{b} \wedge *\varphi_{a}{}^{b}\Big|_{\partial\mathcal{M}} - S_{\text{GHY}}^{\mathring{\Omega}} + S_{\text{GHY}}^{\Omega}. \tag{5.90}$$

For lightlike boundaries, we analogously use the decomposition of the deformation one-form (5.22) and the according projections of $A^{\mu}{}_{\nu}$ to obtain

$$\begin{aligned} S_{\text{Lagr}}^{A*\varphi} = -\int_{\partial\mathcal{M}} \Big(& A^{A}{}_{B} \wedge *\varphi_{A}{}^{B} + \varepsilon(K^{A} - \mathring{K}^{A}) \wedge *\varphi_{A\mathbf{l}} + \varepsilon(-\tilde{K}^{A} + \mathring{K}^{A}) \wedge *\varphi_{\mathbf{l}A} \\ & + \varepsilon(L^{A} - \mathring{L}^{A}) \wedge *\varphi_{A\mathbf{k}} + \varepsilon(-\tilde{L}^{A} + \mathring{L}^{A}) \wedge *\varphi_{\mathbf{k}A} + \varepsilon^{2}(\mathcal{L} + \mathring{\mathcal{K}}) \wedge *\varphi_{\mathbf{lk}} \\ & + \varepsilon^{2}(\mathcal{K} - \mathring{\mathcal{K}}) \wedge *\varphi_{\mathbf{kl}} + \frac{\varepsilon^{2}}{2} Q_{\mathbf{kk}} \wedge *\varphi_{\mathbf{ll}} + \frac{\varepsilon^{2}}{2} Q_{\mathbf{ll}} \wedge *\varphi_{\mathbf{kk}} \Big)\Big|_{\partial\mathcal{M}}. \end{aligned} \tag{5.91}$$

Comparing this decomposition to the universal lightlike GHY term (4.98) yields

$$S_{\text{Lagr}}^{A*\varphi} = -\int_{\partial\mathcal{M}} A^{A}{}_{B} \wedge *\varphi_{A}{}^{B}\Big|_{\partial\mathcal{M}} - S_{\text{GHY}}^{\mathring{\Omega}} + S_{\text{GHY}}^{\Omega}. \tag{5.92}$$

This decomposition is analogous to the result (5.90) on space- and timelike boundaries. We thus proceed by using (5.90) for all cases, keeping in mind that the range of the boundary indices is only $0, \ldots, m-3$ if the boundary of the m-dimensional manifold is considered to be lightlike. Inserting the decomposed boundary term (5.90) into the Lagrange multiplier action (5.86), we obtain

$$\begin{aligned} S^{\mathring{\Omega}}_{\text{Lagr}} = &\int_{\mathcal{M}} \Big(L(\mathring{\varrho}^{\mu}{}_{\nu}) + *\varphi_{\mu}{}^{\nu} \wedge (\Omega^{\mu}{}_{\nu} - A^{\mu}{}_{\rho} \wedge A^{\rho}{}_{\nu} - \mathring{\varrho}^{\mu}{}_{\nu}) \Big) + S^{\Omega}_{\text{GHY}} \\ &- \int_{\mathcal{M}} A^{\mu}{}_{\nu} \wedge \mathring{D} * \varphi_{\mu}{}^{\nu} - \int_{\partial\mathcal{M}} A^{a}{}_{b} \wedge *\varphi_{a}{}^{b}\Big|_{\partial\mathcal{M}} . \end{aligned} \tag{5.93}$$

Hence, we conclude that the contribution $\mathring{D}A^{\mu}{}_{\nu}$ renders the GHY term correct for the expression of the Lagrange multiplier action by means of the full curvature two-form $\Omega^{\mu}{}_{\nu}$.

Before generalizing to generic actions, let us first discuss the case in which the contribution stemming from $\mathring{D}A^{\mu}{}_{\nu}$ is only a difference of GHY terms. This is the case in which $0 = -\int_{\mathcal{M}} A^{\mu}{}_{\nu} \wedge \mathring{D} * \varphi_{\mu}{}^{\nu} - \int_{\partial\mathcal{M}} A^{a}{}_{b} \wedge *\varphi_{a}{}^{b}\big|_{\partial\mathcal{M}}$ so that only the first line of (5.93) is non-vanishing. In this case, we evaluate the action on the field equations of the Lagrange multipliers to obtain

$$S^{\mathring{\Omega}} = \int_{\mathcal{M}} L(\Omega^{\mu}{}_{\nu} - A^{\mu}{}_{\rho} \wedge A^{\rho}{}_{\nu}) + S^{\Omega}_{\text{GHY}} . \tag{5.94}$$

From this action, it is straightforward to impose the teleparallel gauge $\Omega^{\mu}{}_{\nu} = 0$ for the connection. Analogous to the GR case discussed in section 5.1.3, we need to eliminate the GHY term when imposing this gauge in order to render the variational principle well-defined. Hence, the (symmetric) teleparallel equivalent of $S^{\mathring{\Omega}}$ is

$$S^{\text{(S)TE}\mathring{\Omega}} = \int_{\mathcal{M}} L(-\overset{\bullet}{A}{}^{\mu}{}_{\rho} \wedge \overset{\bullet}{A}{}^{\rho}{}_{\nu}) . \tag{5.95}$$

We thus formally obtain the teleparallel equivalent of the Riemannian action (5.85) by replacing $\mathring{\Omega}^{\mu}{}_{\nu} \mapsto -\overset{\bullet}{A}{}^{\mu}{}_{\rho} \wedge \overset{\bullet}{A}{}^{\rho}{}_{\nu}$ and subtracting $S^{\mathring{\Omega}}_{\text{GHY}}$ from the action.

These ideas generalize to the case in which the second line of (5.93) is not vanishing. Generically it is involved to obtain the (symmetric) teleparallel equivalent of the Riemannian action (5.85) by going on-shell in the Lagrange multipliers. However, the boundary decomposition (5.90) clarifies that the terms proportional to $\mathring{D}A^{\mu}{}_{\nu}$ always render the GHY term correct. Therefore, the following procedure yields the (symmetric) teleparallel equivalent of any action constructed only by Riemannian curvature on a manifold with boundary:

1. Ensure that an appropriate GHY term $S^{\mathring{\Omega}}_{\text{GHY}}$ is added to the action to make the variational principle well-defined. This GHY term may be constructed following the method of chapter 4.

2. Use the decomposition $\mathring{\Omega}^\mu{}_\nu = \Omega^\mu{}_\nu - \mathring{D}A^\mu{}_\nu - A^\mu{}_\rho \wedge A^\rho{}_\nu$ we obtained in (5.6) to rewrite Riemannian curvature by the full curvature two-form and contributions of torsion and non-metricity.

3. Constrain the full curvature two-form $\Omega^\mu{}_\nu$ to vanish by choice of the (symmetric) teleparallel connection fulfilling $\overset{\bullet}{\omega}^\mu{}_\rho \wedge \overset{\bullet}{\omega}^\rho{}_\nu = -\mathrm{d}\overset{\bullet}{\omega}^\mu{}_\nu$.

4. Render the variational principle of the resulting action well-defined by subtracting $S^{\Omega}_{\mathrm{GHY}}$ from the (symmetric) teleparallel action.

By means of this procedure, we obtain a (symmetric) teleparallel equivalent for every metric-compatible theory with vanishing torsion. This agrees with the results of [129], generalizing the latter to manifolds with boundary. Our results furthermore generalize the geometrical trinity of general relativity which we discussed in the previous sections. In particular, we derived a correspondence with broader applicability which we call the *geometrical trinity of gravity*. Note that the actions of the (symmetric) teleparallel equivalents in this trinity always include powers of $\overset{\bullet}{A}^\mu{}_\rho \wedge \overset{\bullet}{A}^\rho{}_\nu$. This originates from the defining equation $\mathring{\Omega}^\mu{}_\nu = \Omega^\mu{}_\nu - \mathring{D}A^\mu{}_\nu - A^\mu{}_\rho \wedge A^\rho{}_\nu$ which allows to transition from a Riemannian theory to its (symmetric) teleparallel equivalent. Hence, these (symmetric) teleparallel theories always admit an equivalent description in terms of the Levi-Civita connection analogous to the discussion in section 5.2 because of the ambiguity introduced by the gauge choice $\overset{\bullet}{\omega}^\mu{}_\rho \wedge \overset{\bullet}{\omega}^\rho{}_\nu = -\mathrm{d}\overset{\bullet}{\omega}^\mu{}_\nu$.

Let me briefly summarize the results found in this chapter to conclude the discussion of the geometrical trinity of gravity. The fundamental tensor upon which the geometrical trinity is built is the deformation one-form $A^\mu{}_\nu = \omega^\mu{}_\nu - \mathring{\omega}^\mu{}_\nu$. This one-form measures how much the geometry described by the connection one-form $\omega^\mu{}_\nu$ is deformed from a geometry described by the Levi-Civita connection $\mathring{\omega}^\mu{}_\nu$ which has vanishing torsion and non-metricity. In turn, the deformation one-form is independent of the manifold's curvature. By a straightforward calculation we decomposed the curvature two-form $\Omega^\mu{}_\nu$ into a Riemannian contribution as well as additional contributions of the deformation one-form which implement torsion and non-metricity. We inserted this decomposition into the Einstein-Hilbert action as $\Omega^\mu{}_\nu = \mathring{\Omega}^\mu{}_\nu + \mathring{D}A^\mu{}_\nu + A^\mu{}_\rho \wedge A^\rho{}_\nu$ and subsequently rewrote the contribution stemming from $\mathring{D}A^\mu{}_\nu$ as the boundary term

$$S^{\mathring{D}A} = -\frac{1}{2\kappa}\int_{\partial\mathcal{M}} A^\mu{}_\nu \wedge \eta_\mu{}^\nu\Big|_{\partial\mathcal{M}} = -S^{\mathrm{EH},\mathring{\Omega}}_{\mathrm{GHY}} + S^{\mathrm{EH},\Omega}_{\mathrm{GHY}}\,, \tag{5.96}$$

where an algebraic manipulation revealed that we may interpret $S^{\mathring{D}A}$ as a difference of GHY terms. Hence, the boundary action $S^{\mathring{D}A}$ renders the variational principle well-defined, no matter if we express the Einstein-Hilbert action by means of the

Riemannian curvature $\mathring{\Omega}^{\mu}{}_{\nu}$ or the full curvature two-form $\Omega^{\mu}{}_{\nu}$. General relativity may therefore be equivalently described by either expression of the action

$$\begin{aligned} S^{\mathrm{EH},\mathring{\Omega}} &= \frac{1}{2\kappa}\int_{\mathcal{M}} \eta_{\mu}{}^{\nu}\wedge\mathring{\Omega}^{\mu}{}_{\nu} + S^{\mathrm{EH},\mathring{\Omega}}_{\mathrm{GHY}} \\ &= \frac{1}{2\kappa}\int_{\mathcal{M}} \eta_{\mu}{}^{\nu}\wedge\left(\Omega^{\mu}{}_{\nu} - A^{\mu}{}_{\rho}\wedge A^{\rho}{}_{\nu}\right) + S^{\mathrm{EH},\Omega}_{\mathrm{GHY}}\,. \end{aligned} \tag{5.97}$$

The (symmetric) teleparallel equivalent of general relativity is obtained by imposing the (S)TEGR gauge $\overset{\bullet}{\omega}{}^{\mu}{}_{\rho}\wedge\overset{\bullet}{\omega}{}^{\rho}{}_{\nu} = -\mathrm{d}\overset{\bullet}{\omega}{}^{\mu}{}_{\nu}$ defined such that the curvature two-form vanishes. We argued that the GHY term needs to be eliminated in this gauge in order to obtain a well-defined variational principle. This yields the (S)TEGR action

$$S^{\mathrm{(S)TEGR}} = -\frac{1}{2\kappa}\int_{\mathcal{M}} \eta_{\mu}{}^{\nu}\wedge\overset{\bullet}{A}{}^{\mu}{}_{\rho}\wedge\overset{\bullet}{A}{}^{\rho}{}_{\nu}\,, \tag{5.98}$$

and its equivalence with the Einstein-Hilbert action is called the geometrical trinity of general relativity. While it was already known that the boundary action $S^{\mathring{D}A}$ appears in the transition from GR to (S)TEGR, the interpretation of $S^{\mathring{D}A}$ as a difference of GHY terms is my original result. This yields the boundary refined version of the geometrical trinity of general relativity which we discussed here.

Reaching beyond the boundary refinement of the geometrical trinity of general relativity, I developed a new perspective on the trinity in section 5.2. This perspective is based on the ambiguity of bulk and boundary terms introduced by the (S)TEGR gauge choice $\overset{\bullet}{\omega}{}^{\mu}{}_{\rho}\wedge\overset{\bullet}{\omega}{}^{\rho}{}_{\nu} = -\mathrm{d}\overset{\bullet}{\omega}{}^{\mu}{}_{\nu}$ which allowed to rewrite the (S)TEGR action including its GHY term as

$$S^{\mathrm{E}} = -\frac{1}{2\kappa}\int_{\mathcal{M}} \eta_{\mu}{}^{\nu}\wedge\mathring{\omega}^{\mu}{}_{\rho}\wedge\mathring{\omega}^{\rho}{}_{\nu} + S^{\mathrm{E}}_{\mathrm{cov}}\,. \tag{5.99}$$

We proved that the boundary term $S^{\mathrm{E}}_{\mathrm{cov}}$ depending on the boundary geometry is needed to make the action (5.99) invariant with respect to $\mathrm{GL}(m,\mathbb{R})$ transformations. In this form, the (S)TEGR action (5.99) is the Einstein action which may as well be obtained by rewriting the Einstein-Hilbert action describing GR. Hence, the Einstein action allows for a unifying perspective on all theories included in the geometrical trinity of general relativity. For understanding this new perspective, we noted that the Levi-Civita connection $\mathring{\omega}^{\mu}{}_{\nu}$ which describes the dynamics in (5.99) admits two different interpretations. On the one hand, $\mathring{\omega}^{\mu}{}_{\nu}$ is the connection of a theory with vanishing torsion and non-metricity, such that its dynamics is constituted solely by Riemannian curvature. On the other hand, $\mathring{\omega}^{\mu}{}_{\nu}$ may be rewritten by means of the definition of the deformation one-form as $\mathring{\omega}^{\mu}{}_{\nu} = \omega^{\mu}{}_{\nu} - A^{\mu}{}_{\nu}$. Hence, if we choose a (S)TEGR connection $\omega^\mu{}_\nu = \overset{\bullet}{\omega}{}^\mu{}_\nu$, the Levi-Civita connection is determined solely by torsionful and non-metric contributions while the curvature two-form vanishes.

Lastly, we found that all of these results generalize to more complicated actions constructed from the Levi-Civita connection. That is, any such action may equivalently be interpreted as a theory whose dynamics is described by torsion and non-metricity. In different terms, any action constructed from the Levi-Civita connection possesses a (symmetric) teleparallel equivalent. We developed a method for constructing this equivalent and found that the contributions originating from $\mathring{D}A^\mu{}_\nu$ always render the variational principle of the corresponding theories well-defined by including the appropriate GHY terms.

A frame perspective on holographic renormalization 6

In this chapter, we apply the formalism for describing hypersurfaces developed in chapter 3 to the timelike boundary of Anti-de Sitter (AdS) spaces. AdS spaces are spaces which have a constant, negative cosmological constant. In general relativity, this implies that the scalar curvature of an AdS space is negative[1]. For us, AdS spaces are of unparalleled importance due to their role in explicit realizations of the *holographic principle* [65–68]. Being inspired by the Bekenstein-Hawking result [131–133] for the entropy of black holes, the conjecture of the holographic principle is that the information contained within a region of a manifold $\mathcal{M}$ is entirely encoded on its boundary. Here, a region of a manifold is a submanifold $\mathcal{N} \subseteq \mathcal{M}$ which is often called a volume since it is required to have the same dimension as $\mathcal{M}$. The holographic principle may be understood as a dictionary which relates the physical quantities in a region $\mathcal{N} \subseteq \mathcal{M}$ to physical quantities on its boundary $\partial\mathcal{N}$. The importance of AdS manifolds in the context of the holographic principle arises from their duality to conformal field theories (CFT) in one dimension less.

Concretely, the *AdS/CFT correspondence* predicts that gravitational theories on an AdS manifold permit an equivalent description in terms of a CFT defined on the boundary of this manifold [62–64]. The AdS/CFT correspondence provides the best understood implementation of the holographic principle. Since this correspondence is conjectured and does not allow for a full proof at this time, we need to perform explicit tests to support its viability. Typical tests involve the calculation of correlation functions of boundary fields by means of bulk dynamics [69]. Subsequently, one compares the resulting correlation functions to the corresponding value obtained from a direct field theory derivation in cases in which the latter is known. However, physical quantities in the bulk are typically divergent upon restriction to the boundary in correspondence to the divergences one encounters in CFT computations. Soon after the AdS/CFT correspondence has been first proposed, the authors of [134–136] therefore developed a systematic method for dealing with these divergences. This method is known as *holographic renormalization*.

[1] As we have seen in chapter 5, scalar curvature is only one of three ways to encode geometric information of a manifold. Alternatively, we could encode the negative cosmological constant in the so-called torsion or non-metricity scalars, both of which are obtained from the $\eta_\mu{}^\nu \wedge A^\mu{}_\rho \wedge A^\rho{}_\nu$ term of the (S)TEGR action.

In this chapter, I develop a new perspective on holographic renormalization based on the frame decomposition on hypersurfaces. This perspective is my unpublished original research. In particular, I apply the method of holographic renormalization to the AdS Schwarzschild black hole in TEGR to illustrate how teleparallel theories of gravity may be renormalized in the context of the AdS/CFT correspondence. This serves as a proof of concept demonstrating that torsion can hold information about the boundary thermodynamics. My results serve as a starting point for developing a comprehensive understanding of holographic renormalization in (symmetric) teleparallel gravity entirely in the future. Since holographic renormalization is based on the *Fefferman-Graham theorem*, I begin by developing the frame perspective on this theorem.

6.1. The Fefferman-Graham frame and its expansion

The methodology of holographic renormalization relies on the Fefferman-Graham theorem. One of the implications of this theorem is that the metric of a negatively curved manifold may be expanded at the boundary of this manifold in a certain way [137, 138]. For obtaining this expansion it is crucial to choose the correct coordinates which are called the Fefferman-Graham coordinates. From the differential geometric setup in chapter 2 we know that such coordinates may only be chosen in a chart in which they appear as the components of the chart projection map. These coordinates x^α induce the frame $\frac{\partial}{\partial x^\alpha}$ and its associated coframe $\mathrm{d}x^\alpha$. In the context of the Fefferman-Graham theorem, the choice of Fefferman-Graham coordinates may therefore be considered as a choice of frame. Since we need to expand this frame in the vicinity of the AdS boundary, it is useful to align one of its vectors with the normal direction. Hence, the decomposition of frames and coframes on hypersurfaces we developed in chapter 3 is useful in the context of holographic renormalization as well. We thus proceed by applying this formalism to the Fefferman-Graham coordinates in order to construct Fefferman-Graham frames and coframes.

The Fefferman-Graham coordinates x^α are constructed to describe the negatively curved manifold called *Anti-de Sitter* (AdS) space. This manifold has an asymptotic timelike boundary. The Fefferman-Graham coordinates x^α are decomposed into boundary coordinates x^a and the coordinate $z := x^{m-1}$ parametrizing the normal direction such that the boundary of the m-dimensional manifold is located at $z \to 0$. The chart-induced frame and coframe tangent to the boundary are thus given by $\frac{\partial}{\partial x^a}$ and $\mathrm{d}x^a$, while $\frac{\partial}{\partial z}$ and $\mathrm{d}z$ are normal to the AdS boundary. Adapting the decomposition of frame and coframe (3.36) into boundary tangent and normal contributions as

$$\vartheta_\mu = e^a_\mu \varphi_a + \frac{\varepsilon}{N} n_\mu \varphi\,, \qquad \theta^\mu = e^\mu_a \phi^a + N n^\mu \phi\,, \tag{6.1}$$

we identify

$$\varphi_a = \frac{\partial}{\partial x^a}\,, \qquad \phi^a = \mathrm{d}x^a\,, \qquad \varphi = \frac{\partial}{\partial z}\,, \qquad \phi = \mathrm{d}z \tag{6.2}$$

for the chart-induced frame. This choice respects the conditions (3.26) which relate frame and coframe, and we choose $\varepsilon = 1$ in the following in order to implement that the AdS boundary is timelike. Starting from (6.1) and (6.2), we may transform from the chart-induced to any frame which is appropriate for describing the considered physical system by choosing e^μ_a and Nn^μ. We discuss this choice for the chart-induced frame first before we generalize to generic frames.

The chart-induced frame. We decompose the frame $\frac{\partial}{\partial x^\alpha}$ and its coframe $\mathrm{d}x^\alpha$ induced by the Fefferman-Graham coordinates x^α into boundary tangent and normal contributions as

$$\frac{\partial}{\partial x^\alpha} = \delta^a_\alpha \frac{\partial}{\partial x^a} + \delta^z_\alpha \frac{\partial}{\partial z}\,, \qquad \mathrm{d}x^\alpha = \delta^\alpha_a \mathrm{d}x^a + \delta^\alpha_z \mathrm{d}z\,, \tag{6.3}$$

where we abbreviate $\delta^\alpha_z := \delta^\alpha_{\dim\mathcal{M}-1}$ and $\delta^z_\alpha := \delta^{\dim\mathcal{M}-1}_\alpha$ throughout this chapter. Comparing (6.3) to the generic frame decomposition (6.1) with (6.2), we decompose the chart-induced frame by choosing $e^\alpha_a = \delta^\alpha_a$ and $Nn^\alpha = \delta^\alpha_z$. For representing the metric in the chart-induced frame, we recall that $g(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial z}) = 0$ due to the definition of tangent and normal directions. Therefore, we obtain

$$g = \gamma_{ab} \mathrm{d}x^a \otimes \mathrm{d}x^b + N^2 \mathrm{d}z \otimes \mathrm{d}z\,, \tag{6.4}$$

where we used $\gamma_{ab} = g(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b})$ and $N = \sqrt{g(\frac{\partial}{\partial z}, \frac{\partial}{\partial z})}$.

The *Fefferman-Graham theorem* [137, 138] implies that the metric (6.4) may be written in the form

$$g = \frac{L^2}{z^2}(G_{ab}\mathrm{d}x^a \otimes \mathrm{d}x^b + \mathrm{d}z \otimes \mathrm{d}z)\,, \tag{6.5}$$

where in m-dimensional AdS spaces G_{ab} admits the expansion

$$G_{ab} = \sum_{i=0}^{\infty} G_{ab}^{(2i)}(x) z^{2i} + \tilde{G}_{ab}(x) \ln(z) z^{m-1} \tag{6.6}$$

as we approach the AdS boundary at $z \to 0$. Here, $\tilde{G}_{ab}(x) = 0$ for odd dimensional manifolds $\mathcal{M}$ and a dependence on x is meant to be a dependence on the boundary coordinates x^a only. Comparing the metric in the Fefferman-Graham form to the chart-induced expression (6.4), we obtain

$$G_{ab} = \frac{z^2}{L^2}\gamma_{ab} \qquad \text{and} \qquad N = \frac{L}{z}\,. \tag{6.7}$$

From (6.7) we immediately see that G_{ab} is not a well-defined metric on the boundary, while rescaling G_{ab} by $\frac{L^2}{z^2}$ yields a well-defined boundary metric. The boundary metric is therefore only well-defined up to an overall positive function which appears multiplicatively. This function may be removed by a conformal rescaling which implies that our boundary is well-defined up to conformal transformations. This defines an equivalence class at the boundary, called a *conformal structure* [134–136]. We obtain a boundary metric from a conformal structure by evaluating $\frac{L^2}{z^2} G_{ab}\big|_{z\to 0}$, which by means of the Fefferman-Graham expansion (6.6) yields $\frac{L^2}{z^2} G_{ab}^{(0)}(x)$ as the components of a boundary metric. We conclude the discussion of the chart-induced frame by reinstating the result (6.7) for the lapse function N into the frame defining equation $Nn^\alpha = \delta^\alpha_z$. This yields the components $n^\alpha = \frac{z}{L}\delta^\alpha_z$ of the normal vector in the chart-induced frame. These components will be useful for later calculations.

The chart-induced frame discussed here is a useful choice for various theories. In GR, this choice simplifies the Levi-Civita connection (5.83) since the tensor components $c^\alpha{}_{\beta\gamma}$ of $\mathrm{d}(\mathrm{d}x^\alpha)$ vanish. Furthermore, this frame choice is useful for (S)TEGR. To see that, note that in the simple gauge choice $\overset{\bullet}{\omega}{}^\mu{}_\nu = 0$ for the (S)TEGR connection the torsion two-form becomes $T^\mu = \mathrm{d}\theta^\mu$. Working with the chart-induced frame thus yields a theory of STEGR which describes gravity solely by non-metricity. However, the chart-induced frame is not the most fundamental choice for describing a theory of TEGR in which torsion is the only non-vanishing field strength. Working in the (S)TEGR gauge $\overset{\bullet}{\omega}{}^\mu{}_\nu = 0$, we obtain a theory of TEGR if we choose the metric components to be constant as $\mathrm{d}g_{\mu\nu} = 0$ such that the non-metricity one-form $Q_{\mu\nu} = -\mathrm{d}g_{\mu\nu}$ in this gauge vanishes. We will therefore elaborate on generic frames next before we impose $\overset{\bullet}{\omega}{}^\mu{}_\nu = 0$ and $\mathrm{d}g_{\mu\nu} = 0$ to study the most fundamental theory of TEGR.

Generic frames and the vielbein frame. Generic frames ϑ_μ may be obtained from the chart-induced frame by means of $\mathrm{GL}(m, \mathbb{R})$ transformations since the frame bundle is a principal $\mathrm{GL}(m, \mathbb{R})$-bundle. By means of combining (6.1) with (6.2), we obtain the expressions

$$\begin{aligned} \vartheta_\mu &= e^a_\mu \frac{\partial}{\partial x^a} + \frac{1}{N} n_\mu \frac{\partial}{\partial z} \,, \\ \theta^\mu &= e^\mu_a \mathrm{d}x^a + N n^\mu \mathrm{d}z \end{aligned} \tag{6.8}$$

for generic frames ϑ_μ and their coframes θ^μ. Expanding the metric in the decomposed coframe, we obtain

$$g = g_{\mu\nu} \theta^\mu \otimes \theta^\nu = \gamma_{ab} \mathrm{d}x^a \otimes \mathrm{d}x^b + N^2 \mathrm{d}z \otimes \mathrm{d}z \,, \tag{6.9}$$

where $\gamma_{ab} = e^\mu_a e^\nu_b g_{\mu\nu}$ and $N = \sqrt{g(\frac{\partial}{\partial z}, \frac{\partial}{\partial z})} = \frac{L}{z}$. The identifications (6.7) for the Fefferman-Graham metric imply

$$G_{ab} = \frac{z^2}{L^2} \gamma_{ab} = (\frac{z}{L} e^\mu_a)(\frac{z}{L} e^\nu_b) g_{\mu\nu} \,. \tag{6.10}$$

The Fefferman-Graham expansion (6.6) of G_{ab} may thus be understood as an expansion of $\frac{z}{L} e^\mu_a$ and the metric components $g_{\mu\nu} = g(\vartheta_\mu, \vartheta_\nu)$ in generic frames.

Note that we obtained the chart-induced frame by choosing $e^\mu_a = \delta^\mu_a$, in which case the only non-trivial expansion is that of the metric components. However, this is just a special property of the chart-induced frame. If we instead consider a frame with constant metric components $g_{\mu\nu}$ like the vielbein frame for instance, there is no other possibility but to interpret the Fefferman-Graham expansion as an expansion of $\frac{z}{L} e^\mu_a$. For simplicity of notation, we introduce $h^\mu_a := \frac{z}{L} e^\mu_a$, such that the coframe (6.8) becomes

$$\theta^\mu = \frac{L}{z} (h^\mu_a \mathrm{d}x^a + n^\mu \mathrm{d}z) \tag{6.11}$$

analogous to the expression (6.5) of the Fefferman-Graham metric. We call (6.11) the *Fefferman-Graham coframe* and conjecture that its coefficients h^μ_a behave as

$$h^\mu_a = \sum_{i=0}^{\infty} h^{(i)\mu}{}_a(x) z^{2i} + \tilde{h}^\mu_a(x) \sqrt{\ln(z) z^{m-1}} \tag{6.12}$$

in the vicinity of the AdS boundary located at $z \to 0$. For reconstructing the expansion (6.6) of the Fefferman-Graham metric from the coframe expansion (6.12) by means of (6.10), we impose $\tilde{h}^\mu_a(x) h^{(i)\nu}{}_b g_{\mu\nu} = 0$ as well as $\tilde{h}^\mu_a(x) = 0$ for manifolds of odd dimension m. This yields

$$G_{ab} = h^\mu_a h^\nu_b g_{\mu\nu} = \sum_{i=0}^{\infty}\Big(\sum_{j=0}^{i} h^{(j)\mu}{}_a(x) h^{(i-j)\nu}{}_b(x) g_{\mu\nu}\Big) z^{2i} + \tilde{h}^\mu_a(x)\tilde{h}^\nu_b(x) g_{\mu\nu}\ln(z) z^{m-1} \tag{6.13}$$

after re-organizing the double sum. Hence, we recover the Fefferman-Graham expansion (6.6) of the metric by identifying

$$G^{(2i)}_{ab}(x) = \sum_{j=0}^{i} h^{(j)\mu}{}_a(x) h^{(i-j)\nu}{}_b(x) g_{\mu\nu} \qquad \text{and} \qquad \tilde{G}_{ab}(x) = \tilde{h}^\mu_a(x)\tilde{h}^\nu_b(x) g_{\mu\nu}\,. \tag{6.14}$$

Assuming that the conjectured coframe expansion (6.12) holds is therefore reasonable, while an explicit proof along the lines of [137] will need to be constructed in future work. In analogy to obtaining $\frac{L^2}{z^2} G_{ab}\big|_{z\to 0} = \frac{L^2}{z^2} G^{(0)}_{ab}$ as a boundary metric, we obtain a well-defined boundary coframe as $\frac{L}{z} h^b_a\big|_{z\to 0} = \frac{L}{z} h^{(0)b}{}_a$.

We conclude the frame perspective on the Fefferman-Graham theorem by calculating the contribution $\mathrm{d}\theta^\mu$ of the torsion two-form by means of the $\mathrm{GL}(m,\mathbb{R})$ transformation implicitly included in (6.11). This is frequently needed since the tensor components $c^\mu{}_{\nu\rho} = \mathrm{d}\theta^\mu(\vartheta_\nu, \vartheta_\rho)$ contribute to the Levi-Civita connection derived in (5.83). To that end, we interpret (6.11) as a coframe transformation $\theta^\mu = \Lambda^\mu{}_\alpha \mathrm{d}x^\alpha$ with $\Lambda^\mu{}_\alpha = \frac{L}{z}(h^\mu_a \delta^a_\alpha + n^\mu \delta^z_\alpha)$. Using $\mathrm{d}(\mathrm{d}x^\alpha) = 0$, we obtain

$$\mathrm{d}\theta^\mu = \mathrm{d}\Lambda^\mu{}_\alpha \wedge \mathrm{d}x^\alpha \tag{6.15}$$

which implies $c^\mu{}_{\nu\rho} = (\Lambda^{-1})^\alpha{}_\rho \partial_\nu \Lambda^\mu{}_\alpha - (\Lambda^{-1})^\alpha{}_\nu \partial_\rho \Lambda^\mu{}_\alpha$ for the tensor components, where the inverse $\mathrm{GL}(m,\mathbb{R})$ transformation is given by $(\Lambda^{-1})^\alpha{}_\mu = \frac{z}{L}(h^a_\mu \delta^\alpha_a + n_\mu \delta^\alpha_z)$.

As we emphasized in the introduction to this section, an important application of the Fefferman-Graham theorem arises in the context of the AdS/CFT correspondence. There, the Fefferman-Graham theorem is used for constructing counterterms for on-shell actions which are divergent at the AdS boundary [134–136]. This systematic construction called *holographic renormalization* is usually performed for chart-induced frames on curved manifolds. We will use the frame perspective on the Fefferman-Graham theorem constructed in this section to apply the method of holographic renormalization to the AdS Schwarzschild black hole in the teleparallel equivalent of general relativity next. This serves as an example of how holographic renormalization may be used in theories in the presence of non-vanishing torsion. The case of TEGR is a special example in the realm of these theories since its equivalence to GR allows us to directly compare our results to the textbook examples of holographic renormalization in GR [69].

6.2. Holographic renormalization of the TEGR Schwarzschild black hole

Black holes take a crucial role in the context of the AdS/CFT correspondence because they allow to study boundary field theories at finite temperature. This is due to their emitted Hawking radiation which assigns thermodynamic properties to black holes and thus introduces thermodynamics in holography as well. We are therefore examining a black hole solution in the context of TEGR in this section in order to see how torsion provides thermodynamic information in holography. In particular, we utilize metric solutions in general relativity to derive torsionful solutions from them by means of the equivalence of TEGR and GR. The first solution to the field equations of general relativity was found by Karl Schwarzschild in 1916 [139, 140]. Extending Scharzschild's result to manifolds of arbitrary dimensions m, the metric

$$g = -f(r)\mathrm{d}t \otimes \mathrm{d}t + \frac{1}{f(r)}\mathrm{d}r \otimes \mathrm{d}r + r^2 \sum_{i=1}^{m-2} \prod_{j=1}^{i-1} \sin(\varphi_j)^2 \mathrm{d}\varphi_i \otimes \mathrm{d}\varphi_i \tag{6.16}$$

solves the field equations of general relativity in the coordinates (t, φ_i, r). Here, φ_i summarizes angular contributions for $i \in \{1, \ldots, m-2\}$, and these angular coordinates take values in $\varphi_1 \in [0, 2\pi[$ and $\varphi_i \in [0, \pi[$ for all other i. The chart-induced frame is thus

$$\mathrm{d}x^\alpha = \delta_0^\alpha \mathrm{d}t + \sum_{i=1}^{m-2} \delta_i^\alpha \mathrm{d}\varphi_i + \delta_{m-1}^\alpha \mathrm{d}r\,. \tag{6.17}$$

The prefactor $f(r)$ in the Schwarzschild metric (6.16) is called the *blackening factor*. For an m-dimensional asymptotical AdS space, the blackening factor takes the form

$$f(r) = k - \frac{2\mu}{r^{m-3}} + \frac{r^2}{L^2}\,, \tag{6.18}$$

where $k = 1$ for spherical, $k = 0$ for flat and $k = -1$ for hyperbolic boundary geometries [69].

The asymptotic boundary is located at $r \to \infty$ in Schwarzschild coordinates. For obtaining Fefferman-Graham coordinates x^α, we thus need to invert the radial coordinate when defining z. Comparing the Schwarzschild metric (6.16) to the Fefferman-Graham metric (6.5), we therefore define z by

$$-\frac{L}{z}\mathrm{d}z = \frac{\mathrm{d}r}{\sqrt{f(r)}} \tag{6.19}$$

such that the asymptotic boundary is located at $z \to 0$. Integrating (6.19), we obtain the Fefferman-Graham coordinate z as

$$z = \exp\left(-\int \frac{\mathrm{d}r}{L\sqrt{f(r)}}\right) \tag{6.20}$$

which is solved in $m = 5$ dimensions by[2]

$$z(r) = \frac{c}{\sqrt{2\frac{r^2}{L^2} + k + 2\frac{r}{L}\sqrt{f(r)}}} \tag{6.21}$$

with $c \geq 0$. This coordinate transformation is inverted as

$$r(z) = \frac{z}{c}\sqrt{2\mu + \frac{L^2}{4}\left(\frac{c^2}{z^2} - k\right)^2} \,. \tag{6.22}$$

By means of these relations, we may compare the remaining coefficients of the Schwarzschild metric (6.16) to the Fefferman-Graham metric (6.5) to obtain

$$\frac{L^2}{z^2}G_{ab} = -\delta^0_a\delta^0_b f(r(z)) + r(z)^2 \sum_{i=1}^{m-2} \delta^i_a\delta^i_b \prod_{j=1}^{i-1} \sin(\varphi_j)^2 \,. \tag{6.23}$$

The Schwarzschild black hole may as well be understood from a frame perspective. Since we aim at describing the Schwarzschild black hole in TEGR, we choose $\overset{\bullet}{\omega}{}^{\mu}{}_{\nu} = 0$ for simplicity. In this gauge, the simplest choice for the metric components $g_{\mu\nu} = g(\vartheta_\mu, \vartheta_\nu)$ which yields a theory of TEGR is the Minkowski metric whose components are $g_{\mu\nu} = -\delta^0_\mu\delta^0_\nu + \sum_{i=1}^{m-1} \delta^i_\mu\delta^i_\nu$. The frame ϑ_μ in which $g_{\mu\nu}$ are the components of the Minkowski metric is the vielbein frame, and we will work in this frame for the remainder of this section. For explicitly constructing the vielbein frame which describes the Schwarzschild black hole, we need to fix the coefficients h^μ_a and n^μ in (6.11) such

[2] The technical reason for choosing $m = 5$ is that computer algebra systems have problems processing $\frac{1}{\sqrt{f(r)}}$ for the AdS blackening factor (6.18). Evaluating the integral (6.20) by computer algebra yields results which do not actually solve the differential equation (6.19). The algebraic evaluation of (6.20) is involved, however. It yields the result (6.21) in $m = 5$ dimensions.

that

$$g = g_{\mu\nu}\theta^\mu \otimes \theta^\mu = \frac{L^2}{z^2}(h^\mu_a h^\nu_b g_{\mu\nu}\mathrm{d}x^a \otimes \mathrm{d}x^b + \mathrm{d}z \otimes \mathrm{d}z) \tag{6.24}$$

reproduces the Schwarzschild metric (6.16) in Fefferman-Graham coordinates. A convenient choice is

$$\begin{aligned} h^\mu_a &= \frac{z}{L}\left(\delta^\mu_0\delta^0_a\sqrt{f(r(z))} + r(z)\sum_{i=1}^{m-2}\delta^\mu_i\delta^i_a\prod_{j=1}^{i-1}\sin(\varphi_j)\right), \\ n^\mu &= -\delta^\mu_z\,, \end{aligned} \tag{6.25}$$

for which the vielbein coframe (6.11) becomes

$$\theta^\mu = \delta^\mu_0\sqrt{f(r(z))}\mathrm{d}t + r(z)\sum_{i=1}^{m-2}\delta^\mu_i\prod_{j=1}^{i-1}\sin(\varphi_j)\mathrm{d}\varphi_i - \delta^\mu_z\frac{L}{z}\mathrm{d}z\,. \tag{6.26}$$

By means of the coordinate transformation (6.19) this reproduces the vielbein which is routinely used to study the Schwarzschild black hole in four-dimensional TEGR [124, 141, 142]. In particular, the Schwarzschild coframe (6.26) constructed from convenient choices of h^μ_a and n^μ solves the equations of motion of TEGR which have been shown to be equivalent to those of GR, see [26] for a comprehensive proof[3]. Evaluating the vielbein (6.25) in the vicinity of the boundary, we obtain

$$\begin{aligned} h^\mu_a =& \delta^\mu_0\delta^0_a\frac{c}{L}\left[\frac{1}{2} + \frac{k}{2}\frac{z^2}{c^2} - \frac{6\mu}{L^2}\frac{z^4}{c^4} - \frac{10k\mu}{L^2}\frac{z^6}{c^6} + \mathcal{O}\left(\frac{z^8}{c^8}\right)\right] \\ &+ \sum_{i=1}^{m-2}\delta^\mu_i\delta^i_a\prod_{j=1}^{i-1}\sin(\varphi_j)\frac{1}{c}\left[\frac{1}{2} - \frac{k}{2}\frac{z^2}{c^2} + \frac{2\mu}{L^2}\frac{z^4}{c^4} + \frac{2k\mu}{L^2}\frac{z^6}{c^6} + \mathcal{O}\left(\frac{z^8}{c^8}\right)\right] \end{aligned} \tag{6.27}$$

for $m = 5$ dimensional AdS manifolds. The vielbein expansion (6.27) is an explicit example for the Fefferman-Graham expansion of coframes (6.12). This concludes the frame perspective on the Fefferman-Graham theorem for the AdS Schwarzschild black hole in TEGR. We use these results to apply the method of holographic renormaliza-

[3] Following the derivation in [26, 143], the TEGR equations of motion are

$$\partial_\beta(\det(\theta)S_\mu{}^{\alpha\beta}) - \kappa\det(\theta)j_\mu{}^\alpha = \kappa\det(\theta)\Theta_\mu{}^\alpha\,,$$

where $S_\mu{}^{\alpha\beta} = -\frac{\kappa}{\det(\theta)}\frac{\partial L}{\partial(\partial_\beta\theta^\mu{}_\alpha)} = A^{\alpha\beta}{}_\mu - \theta_\mu{}^\beta T^{\gamma\alpha}{}_\gamma + \theta_\mu{}^\alpha T^{\gamma\beta}{}_\gamma$ is called the superpotential, $j_\mu{}^\alpha = -\frac{1}{\det(\theta)}\frac{\partial L}{\partial\theta^\mu{}_\alpha} = \frac{1}{\kappa}\theta_\mu{}^\beta S_\nu{}^{\gamma\alpha}T^\nu{}_{\gamma\beta} - \frac{1}{\det(\theta)}\theta_\mu{}^\alpha L + \frac{1}{\kappa}\Gamma^\nu_{\beta\mu}S_\nu{}^{\alpha\beta}$ is the gauge current and $\Theta_\mu{}^\alpha = -\frac{1}{\det(\theta)}\frac{\delta L_{\text{matter}}}{\delta\theta^\mu{}_\alpha} = -\frac{1}{\det(\theta)}\left(\frac{\partial L_{\text{matter}}}{\partial\theta^\mu{}_\alpha} - \partial_\beta\frac{\partial L_{\text{matter}}}{\partial(\partial_\beta\theta^\mu{}_\alpha)}\right)$ is the matter energy-momentum tensor. In the latter expressions, $\theta^\mu{}_\alpha$ are the components of the vielbein coframe θ^μ in a chart-induced coframe $\mathrm{d}x^\alpha$ and $\det(\theta)$ is their determinant. All quantities mentioned in this footnote are considered in (S)TEGR gauge using the Weitzenböck connection with vanishing non-metricity [23].

tion [134–136] to the AdS Schwarzschild black hole coframe next.

In the AdS/CFT correspondence, the gravitational action evaluated on solutions of the equations of motion is equivalent to the generating functional of correlation functions in the boundary CFT. To make this equivalence precise, one has to deal with the divergences that appear upon substituting the solution of the equations of motion into the action. These divergences are reminiscent of the UV singularities that are expected to appear in a generic CFT functional and must be renormalized. The method for removing the singularities on the gravity side is called holographic renormalization and we adapt it to the frame formulation of TEGR and the AdS Schwarzschild solution in this section. In order to describe a theory in an m-dimensional AdS space, we introduce the cosmological constant

$$\Lambda = -\frac{(m-1)(m-2)}{2L^2}\,, \tag{6.28}$$

where $L > 0$ is defined by the latter equation as the *AdS radius*. In the presence of a cosmological constant, the (S)TEGR action (5.59) becomes

$$S^{\text{(S)TEGR}} = -\frac{1}{2\kappa}\int_{\mathcal{M}} (\eta_{\mu}{}^{\nu} \wedge \mathring{\omega}^{\mu}{}_{\rho} \wedge \mathring{\omega}^{\rho}{}_{\nu} + 2\Lambda\eta) + \frac{\varepsilon}{2\kappa}\int_{\partial\mathcal{M}} e^{a}_{\mu}(\mathrm{d}n^{\mu} + g^{\mu\nu}\mathrm{d}n_{\nu}) \wedge \eta_{\mathbf{n}a}\Big|_{\partial\mathcal{M}}\,. \tag{6.29}$$

Note that the boundary term of the latter action vanishes in the vielbein frame choice (6.25).

We have several possibilities for calculating the on-shell TEGR Lagrangian of the AdS Schwarzschild black hole at this point. Transferring the standard calculation in the context of holographic renormalization to the frame perspective, we obtain the on-shell action by inserting the Fefferman-Graham expansion (6.27) of the coframe into the action (6.29). We may, however, evaluate the action for the exact expression (6.26) of the Fefferman-Graham coframe instead, where the coordinate transformation $r(z)$ is given in (6.22). The advantage of this method is that it gives an exact result which does not utilize series expansions. As a third possibility, we may as well evaluate the on-shell action in Schwarzschild coordinates first and transform the result to Fefferman-Graham coordinates by means of (6.21) and (6.22). We follow the latter approach to stress the simplicity of the on-shell Lagrangian in Schwarzschild coordinates. An explicit calculation shows that all three approaches yield the same result as expected due to the covariance of the action.

For obtaining the on-shell Lagrangian in Schwarzschild coordinates, we insert the AdS Schwarzschild coframe (6.26) alongside the differential coordinate transformation (6.19) into (6.15) to obtain the coefficients $c^\mu{}_{\nu\rho} = \mathrm{d}\theta^\mu(\vartheta_\nu, \vartheta_\rho)$. Since the metric in the vielbein frame fulfills $\mathrm{d}g_{\mu\nu} = 0$ by construction, the coefficients $c^\mu{}_{\nu\rho}$ entirely determine the Levi-Civita connection $\mathring{\omega}^\mu{}_\nu$ by means of (5.83). This suffices to evaluate the action (6.29) which becomes

$$S^{\mathrm{TEGR}}_{\mathrm{Schwarz}} = \frac{1}{\kappa}\int_{\mathcal{M}} \mathrm{d}^4x \wedge \mathrm{d}r \sin(\varphi_1)^2 \sin(\varphi_2)\Big(12\frac{r^2}{L^2} + 3k + \cot(\varphi_1)^2\Big)\,, \tag{6.30}$$

where we abbreviate $\mathrm{d}^4x := \mathrm{d}t\wedge\mathrm{d}\varphi_1\wedge\mathrm{d}\varphi_2\wedge\mathrm{d}\varphi_3$. For transforming this on-shell action to Fefferman-Graham coordinates, we straightforwardly replace $\mathrm{d}r$ by means of (6.19) and introduce $r(z)$ obtained in (6.22). This yields

$$\begin{aligned} S^{\mathrm{TEGR}}_{\mathrm{Schwarz}} = -\frac{1}{4\kappa}\int_{\mathcal{M}} \mathrm{d}^4x \wedge \frac{\mathrm{d}z}{c}\sin(\varphi_1)^2\sin(\varphi_2)\Big(3L^2\frac{c^5}{z^5} - L^2(3k - \cot(\varphi_1)^2)\frac{c^3}{z^3} \\ + (k^2L^2 + 8\mu)(3k - \cot(\varphi_1)^2)\frac{z}{c} - \frac{3}{L^2}(k^2L^2+8\mu)^2\frac{z^3}{c^3}\Big)\,. \end{aligned} \tag{6.31}$$

We emphasize again that this on-shell action in Fefferman-Graham coordinates may be obtained from all the possible calculations discussed above.

The Lagrangian in (6.31) is clearly divergent as we approach the AdS boundary at $z \to 0$, analogous to (6.30) being divergent at $r \to \infty$. Hence, we need to renormalize the action. To that end, we first regularize by cutting off the z-integral in (6.31) at $z = \epsilon$, which yields

$$\begin{aligned} S^{\mathrm{TEGR}}_{\mathrm{Schwarz,reg}} = \frac{1}{16\kappa}\int_{z=\epsilon} \mathrm{d}^4x \sin(\varphi_1)^2\sin(\varphi_2)\Big(3L^2\frac{c^4}{\epsilon^4} - 2L^2(3k - \cot(\varphi_1)^2)\frac{c^2}{\epsilon^2}\Big) \\ + \text{finite contributions}\,, \end{aligned} \tag{6.32}$$

where we integrate over the hypersurface at $z = \epsilon$. We cancel the divergences in the regularized action (6.32) by adding terms to the original action (6.31) which are constructed to have the exact opposite boundary divergence. These terms are called *counterterms* and we construct them solely from boundary tensors in order to preserve diffeomorphism invariance of the bulk action.

For constructing the counterterms we recall that

$$\frac{L}{z}h^a_b\Big|_{z=\epsilon} = \frac{L}{\epsilon}h^{(0)a}{}_b = \Big(\delta^a_0\delta^0_b\frac{1}{2} + \sum_{i=1}^{3}\delta^a_i\delta^i_b\prod_{j=1}^{i-1}\sin(\varphi_j)\frac{L}{2}\Big)\frac{c}{\epsilon} \tag{6.33}$$

is a vielbein on the cutoff hypersurface which we evaluated by means of the coframe expansion (6.27). For simplicity of notation, we introduce $H^a_b := \frac{L}{\epsilon}h^{(0)a}{}_b$ such that

$H^a = H^a_b \mathrm{d}x^b$ constitutes a coframe on the cutoff hypersurface. The determinant

$$\det(H^a_b) = \frac{L^3}{16}\sin(\varphi_1)^2\sin(\varphi_2)\frac{c^4}{\epsilon^4} \tag{6.34}$$

of H^a_b contributes to the volume form on the cutoff hypersurface as $\zeta := *_{z=\epsilon}1 = \det(H^a_b)\mathrm{d}^4x$. It is thus crucial for the construction of counterterms. The analog of the bulk Lagrangian on the cutoff hypersurface is

$$\zeta_a{}^b \wedge \mathring{\omega}^a{}_c \wedge \mathring{\omega}^c{}_b = -\zeta\frac{8}{L^2}\cot(\varphi_1)^2\frac{\epsilon^2}{c^2} \tag{6.35}$$

for the five-dimensional AdS Schwarzschild black hole. Here, $\zeta^{ab} := *_{z=\epsilon}\left(H^a \wedge H^b\right)$ is the Hodge dual on the cutoff hypersurface and $\mathring{\omega}^a{}_b$ is the Levi-Civita connection determined by the coframe H^a via (5.83). By means of this coframe, we evaluate the GHY-like term

$$\mathring{K}^a \wedge \eta_{\mathbf{n}a}\Big|_{z=\epsilon} = -\zeta\frac{4}{L}\Big(1 + k\frac{\epsilon^2}{c^2} + \mathcal{O}\Big(\frac{\epsilon^4}{c^4}\Big)\Big) \tag{6.36}$$

on the cutoff hypersurface. Collecting these results, the divergence (6.32) of the regularized action is canceled by adding the counterterm

$$S^{\text{TEGR}}_{\text{Schwarz,ct}} = -\frac{1}{2\kappa}\int_{z=\epsilon}\left(\frac{18}{L}\zeta - \frac{L}{2}\zeta_a{}^b \wedge \mathring{\omega}^a{}_c \wedge \mathring{\omega}^c{}_b + 3\mathring{K}^a \wedge \eta_{\mathbf{n}a}\Big|_{z=\epsilon}\right), \tag{6.37}$$

such that

$$S^{\text{TEGR}}_{\text{Schwarz,ren}} = \lim_{\epsilon\to 0}(S^{\text{TEGR}}_{\text{Schwarz,reg}} + S^{\text{TEGR}}_{\text{Schwarz,ct}}) \tag{6.38}$$

is the renormalized action of the TEGR Schwarzschild black hole.

The counterterm (6.37) is the main result of this section. It is what we conceptionally expect from holographic renormalization in general relativity [69, 134]. There, the volume divergence is canceled by the volume element of the cutoff hypersurface which we find as $\frac{18}{L}\zeta$ in (6.37). Furthermore, the bulk Lagrangian in general relativity consists of the Ricci scalar, and the boundary term needed for renormalizing the five-dimensional action involves the Ricci scalar of the cutoff hypersurface. In our formalism, this corresponds to the term $-\frac{L}{2}\zeta_a{}^b \wedge \mathring{\omega}^a{}_c \wedge \mathring{\omega}^c{}_b$ which is the analog of the bulk Lagrangian on the cutoff hypersurface. Lastly, we find that the GHY-like term $3\mathring{K}^a \wedge \eta_{\mathbf{n}a}\big|_{z=\epsilon}$ on the cutoff hypersurface is needed for consistently renormalizing the action. Again, it is well-known for holographic renormalization in general relativity that the GHY term must be included in the action to obtain the correct renormalized result.

Motivated by the results of general relativity, one might however wonder why we do not find an anomalous term in the renormalized action. In particular, the conformal anomaly is typically found in odd bulk dimensions in the form of terms which are proportional to $\ln(\epsilon)$ in the regularized action, see [134]. This is analogous to terms of order $\frac{1}{z}$ appearing in the original action for which the radial integral has not been performed. This argument suffices to understand why we do not observe an anomaly here. In particular, evaluating the TEGR action on the AdS Schwarzschild solution in five dimensions we obtained the Lagrangian in (6.31) as a series of odd powers of z. However, there are no terms of order $\frac{1}{z}$ in (6.31) and thus we do not obtain anomalous contributions to the regularized action. Hence, we found that the absence of an anomaly is clearly a feature of the AdS Schwarzschild solution, while applying holographic renormalization to a different solution is generically expected to yield anomalous terms on odd-dimensional manifolds.

My results presented in this section may be used as a guideline for considering generic solutions and arbitrary dimensions in future work. Constructing the counterterms for renormalizing these cases may in general involve additional boundary terms. Note, for instance, that the boundary coframe H^a is proportional to $\frac{L}{\epsilon}$ and thus its determinant $\det(H^a_b)$ is always proportional to $\left(\frac{L}{\epsilon}\right)^{m-1}$ on an m-dimensional manifold. From the analogy to holographic renormalization in GR, we expect terms of higher power in $\mathring{\omega}^a{}_b$ to account for these additional divergent contributions. For example, it is straightforward to implement $\zeta_a{}^b{}_c{}^d \wedge \mathring{\omega}^a{}_e \wedge \mathring{\omega}^e{}_b \wedge \mathring{\omega}^c{}_f \wedge \mathring{\omega}^f{}_d$ as the next non-trivial counterterm. We conclude the discussion of the frame perspective on holographic renormalization by using the renormalized action to derive the holographic thermodynamics of the five-dimensional AdS Schwarzschild black hole in the following section.

6.3. Black hole thermodynamics in TEGR

One of the main applications of holographic renormalization arises in the context of black hole thermodynamics. In particular, black holes emit Hawking radiation, and we use this radiation to assign a temperature to black holes. This temperature makes its appearance in the action of a theory if we evaluate it on the black hole solution. We call this the gravitational on-shell action hereafter, and we have already seen that the on-shell action needs to be renormalized in order to obtain a finite value. In AdS/CFT, the gravitational on-shell action corresponds to the thermal partition function on the conformal boundary. Hence, black holes and holographic renormalization are essential for describing the boundary thermodynamics in the AdS/CFT correspondence. We will make these qualitative statements concrete in this section based on derivations in [69, 70].

The temperature of a black hole is obtained from its blackening factor $f(r)$ [131–133]. To see this, we examine the blackening factor as well as its derivative at the black hole horizon. In Schwarzschild coordinates, the radius r_{h} of the black hole is the largest solution of $f(r) = 0$. We immediately see that at r_{h} the radial contributions of the metric (6.16) and the coframe (6.26) diverge, so that Schwarzschild coordinates may only be used for $r > r_{\mathrm{h}}$. For simplicity we assume that the first derivative $f'(r)$ of the blackening factor is non-vanishing at the black hole horizon, such that $f(r)$ may be non-trivially expanded at r_{h} as

$$f(r) = f'(r_{\mathrm{h}})(r - r_{\mathrm{h}}) + \mathcal{O}((r - r_{\mathrm{h}})^2)\,. \tag{6.39}$$

The following calculation extends to black holes for which the blackening factor fulfills $f'(r_{\mathrm{h}}) = 0$ if we consider the latter expansion to the first non-vanishing order.

In the vicinity of the black hole horizon, the metric (6.16) behaves as

$$g = -f'(r_{\mathrm{h}})(r - r_{\mathrm{h}})\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f'(r_{\mathrm{h}})(r - r_{\mathrm{h}})} + r_{\mathrm{h}}^2\mathrm{d}\Omega^2 + \mathcal{O}((r - r_{\mathrm{h}})^2) \tag{6.40}$$

which we obtain by inserting (6.39) into (6.16). It is useful to apply the coordinate transformation

$$r = r_{\mathrm{h}} + \frac{f'(r_{\mathrm{h}})}{4}\rho^2\,, \qquad t = \frac{2\mathrm{i}}{|f'(r_{\mathrm{h}})|}\phi \tag{6.41}$$

to the expanded metric (6.40) to see that it has the form

$$g = \rho^2\mathrm{d}\phi^2 + \mathrm{d}\rho^2 + r_{\mathrm{h}}^2\mathrm{d}\Omega^2 + \mathcal{O}((r - r_{\mathrm{h}})^2)\,. \tag{6.42}$$

Thus, the contribution of the metric in the direction of t and r takes the form of a two-dimensional Minkowski metric in spherical coordinates at the black hole boundary. This implies that ϕ is periodic as $\phi \sim \phi + 2\pi$ to avoid a conical singularity, which implies a periodicity $t \sim t + \frac{4\pi\mathrm{i}}{|f'(r_{\mathrm{h}})|}$ via the coordinate transformation (6.41). However, Euclidean time $\mathrm{i}t$ in field theories is already periodic with the period given by the inverse of the temperature T of the field theory. Thus, the near-horizon expansion (6.40) of the black hole metric ultimately implies

$$T = \frac{|f'(r_\mathrm{h})|}{4\pi} \tag{6.43}$$

which is the Hawking temperature of the black hole [133].

Hence, we obtain the Hawking temperature of any black hole by determining $f'(r_\mathrm{h})$. We apply this to the AdS Schwarzschild black hole described by the blackening factor (6.18), where we do not fix a dimension m yet in order for our results to have a broad applicability. We recast the derivative

$$f'(r) = \frac{2(m-3)}{r^{m-4}}\mu + \frac{2r}{L^2} \tag{6.44}$$

of the AdS Schwarzschild blackening factor (6.18) in terms of $f(r)$ by solving (6.18) for μ. This yields

$$\mu = \frac{1}{2} r^{m-3}\left(-f(r) + k + \frac{r^2}{L^2}\right), \tag{6.45}$$

which we insert into (6.44) to obtain

$$f'(r_\mathrm{h}) = \frac{L^2 k(m-3) + (m-1) r_\mathrm{h}^2}{L^2 r_\mathrm{h}} \tag{6.46}$$

at the black hole horizon. This result is straightforwardly solved for r_h to yield

$$r_\mathrm{h} = \frac{L^2 f'(r_\mathrm{h})}{2(m-1)} + \sqrt{\left(\frac{L^2 f'(r_\mathrm{h})}{2(m-1)}\right)^2 - \frac{m-3}{m-1}L^2 k} \tag{6.47}$$

which relates the radius r_h of the AdS Schwarzschild black hole to its Hawking temperature by inserting (6.43).

This relation of the black hole radius and its Hawking temperature is fundamental for describing thermodynamics in AdS/CFT. In particular, the GKPW prescription [63, 64] which is basic for AdS/CFT implies that the free energy F of the field theory is related to the gravitational on-shell action as

$$F = S_\mathrm{ren} T\,. \tag{6.48}$$

Hence, we only need the renormalized gravitational on-shell action for determining the free energy of the dual field theory.

For the five-dimensional AdS Schwarzschild black hole, we already constructed the counterterms needed for renormalization in the previous section. Hence, it only remains to evaluate the integrals contained in (6.38). This yields the renormalized on-shell action[4]

$$S^{\mathrm{TEGR}}_{\mathrm{Schwarz,ren}} = \frac{2\pi^2}{TL^2} r_{\mathrm{h}}^2 \left((1+3k)L^2 + 6r_{\mathrm{h}}^2 \right) . \tag{6.49}$$

Using the relation (6.47) to re-express r_{h} by means of the Hawking temperature, the AdS/CFT relation (6.48) of free energy and gravitational action implies

$$F = \frac{L^2\pi^2}{2} \left(L\pi T + \sqrt{-2k + L^2\pi^2T^2} \right)^2 \left(1 + 3L\pi T \left(L\pi T + \sqrt{-2k + L^2\pi^2T^2} \right) \right) . \tag{6.50}$$

This is the desired result which determines the free energy by means of temperature.

While the latter calculation is mostly a textbook calculation in AdS/CFT, the important new ingredient is that we derive the CFT free energy from torsion in TEGR. Therefore, (6.50) proves that torsion can and does hold information about thermodynamics. This is already remarkable in pure gravity without applying the results to AdS/CFT. In gravity it was previously unclear how to obtain the black hole temperature in TEGR, since the GHY term is essential to black hole thermodynamics in GR while it is absent in TEGR[5]. The results of this chapter thus prove that torsion holds information about the thermodynamics of black holes in gravity and holography. These are my original results presented in this thesis for the first time.

The results of this chapter may be used as a starting point for constructing the full method of holographic renormalization in TEGR as well as in more general (symmetric) teleparallel theories of gravity. For this purpose, we need to solve the TEGR equations of motion order by order for the expansion (6.12) of the coframe. This is an involved calculation since the TEGR equations of motion are considerably more complex than Einstein's field equations, although being equivalent to the latter [26, 143]. The method outlined here is sufficient, however, to construct the counterterms which renormalize the on-shell action for specific solutions of these equations of motion. This broad applicability of my method generalizes the results of a first paper which studied holographic renormalization in a torsionful theory [115] which is only

[4] This is the first point at which we need to choose Euclidean time for evaluating the time integral and comparing it to the free energy obtained by the AdS/CFT calculation in (6.48). All other results presented in this chapter are written in a form in which they are invariant under the transition from Lorentzian to Riemannian metric signature.

[5] See [144] for a first discussion of this problem without a rigorous treatment of boundary terms. Also note that [145] recently evaluated Euclidean on-shell actions in teleparallel gravity for black holes without discussing renormalization.

applicable in the case of finite Fefferman-Graham expansions [114]. In contrast to that, the methodology I developed in this chapter applies to general actions which describe theories of teleparallel gravity.

7 Conclusion and outlook

My research presented in this thesis is based on a simple mathematical observation. That is, we interpret the observable universe as a differentiable manifold $\mathcal{M}$ and model gravity on the frame bundle of this manifold. The frame bundle is a principal $\mathrm{GL}(\dim \mathcal{M}, \mathbb{R})$-bundle which we equip with the necessary structure to describe physical observations. In particular, we consider a soldering form canonically provided by a coframe θ^μ, a connection one-form $\omega^\mu{}_\nu$ and a metric tensor field g as the dynamical fields of a theory of gravity. The covariant exterior derivative D induced by the connection allows to define the field strengths of the fundamental fields which are torsion $T^\mu = D\theta^\mu$, curvature $\Omega^\mu{}_\nu = D\omega^\mu{}_\nu$ and non-metricity $Q_{\mu\nu} = -Dg_{\mu\nu}$. These fields constitute the differential geometric foundation of gravity as discussed in chapter 2.

In chapter 3 I examined hypersurfaces in this geometry from a vector field approach. This perspective corresponds to alternative approaches existing in literature, including definitions which utilize constant functions or immersions. The particular new aspect I introduced in chapter 3 is the implementation of hypersurfaces in gravity via a decomposition of frames and coframes. This reaches beyond the hypersurface formalisms discussed in literature. We need these decompositions to study torsion on hypersurfaces, because torsion is the field strength of the coframe. It turned out useful to adapt the frame and coframe decompositions to the hypersurface to reduce the number of free parameters. For this purpose, we aligned one of the basis vectors with the normal direction to obtain the adapted frame decomposition. For space- and timelike hypersurfaces, adapting the decompositions of frames and coframes resulted in (3.36), while the corresponding lightlike result is (3.86). We found that these decompositions are immensely important because they correspond to a $\mathrm{GL}(\dim \mathcal{M}, \mathbb{R})$-transformation. This transformation allows to decompose every structure we have on the frame bundle and its associated bundles. In particular, the decompositions of frame and coframe allowed to decompose unity, the metric tensor and the connection one-form.

The decompositions of the fundamental dynamical fields induce the decomposition of each tensor into hypersurface tangent and normal contributions. We first introduced this decomposition in sections 3.1.2 and 3.2.2 for the extrinsic curvatures which partially characterize the shape of the boundary. In sections 4.1.1 and 4.2.1, we applied this formalism to the field strengths of the fundamental dynamical fields to obtain the decompositions of curvature, torsion and non-metricity. The latter decompositions are fundamental for studying actions which are constructed in terms of curvature, torsion and non-metricity if we consider them on manifolds with boundary. In particular, we identified boundaries with hypersurfaces, such that the decomposition of the field strengths made the boundary terms of an action manifest by means of Stokes' theorem. These boundary terms immediately induced the GHY terms which are needed to make the variational problem well-defined on a manifold with boundary.

We examined general properties of GHY terms in chapter 4 by deriving a universal equation that allows to calculate the GHY terms for a broad range of actions in an extraordinarily efficient manner. In particular, it suffices to evaluate a few variations of a given Lagrangian in order to determine its corresponding GHY term by means of my universal GHY result, given in (4.26) and (4.98) for non-lightlike and lightlike boundaries, respectively. These universal GHY terms are constructed from the extrinsic curvature components as well as projections of non-metricity into the normal directions. Their explicit form implies that actions constructed solely from torsion and non-metricity have a well-defined variational principle and do not require additional boundary terms.

My new method for constructing GHY terms on manifolds with curvature, torsion and non-metricity is particularly efficient, which we have seen in various examples in sections 4.1.3 and 4.2.3. Specifically, for Einstein-Hilbert, four-dimensional Chern-Simons modified and Lovelock gravity, we have derived the GHY terms which are needed to make the variational problem well-defined. These results reproduce the expressions known in literature upon imposing the appropriate constraints on dimensionality, field content and boundary likeness. My results generalize the corresponding GHY terms known in literature in precisely these aspects by lifting constraints on dimension, field content and boundary likeness. In particular, all GHY terms derived in chapter 4 account for a well-defined variational problem on manifolds which have non-trivial torsion in addition to curvature. Furthermore, we derived the lightlike analogs of all of these GHY terms, which were previously only known for four-dimensional Einstein-Hilbert gravity on manifolds with vanishing torsion and non-metricity. The efficiency as well as the generality of the derivation of GHY terms as compared to literature approaches in all of these cases is the strength of my method.

In chapter 5 we applied the tools of boundary analysis developed in view of GHY terms to the geometrical trinity of general relativity. We re-derived this trinity in differential form notation, demonstrating that it may be reduced to a duality of theories which are constructed either solely upon curvature or upon torsion and non-metricity. The resulting theories are general relativity (GR) and the (symmetric) teleparallel equivalent of general relativity ((S)TEGR), respectively. These theories are known to be dynamically equivalent, and (S)TEGR encompasses all dynamically equivalent subcases of teleparallel theories such as TEGR, STEGR or GTEGR. The deformation one-form $A^{\mu}{}_{\nu} = \omega^{\mu}{}_{\nu} - \mathring{\omega}^{\mu}{}_{\nu}$ serves as the fundamental tensor underlying the geometrical trinity. This tensor subtracts the Levi-Civita contributions from the full connection, thereby ensuring that it only depends on torsion and non-metricity.

Introducing the deformation one-form immediately enabled us to rewrite the Einstein-Hilbert action $S^{\mathrm{EH},\mathring{\Omega}}$ including its GHY term $S^{\mathrm{EH},\mathring{\Omega}}_{\mathrm{GHY}}$ in terms of the full curvature two-form $\Omega^{\mu}{}_{\nu}$, see (5.7). This rewriting introduced a new boundary term $S^{\mathring{D}A}$ which is known as the (S)TEGR boundary term. One of the main results of chapter 5 is that we gained a comprehensive understanding of this boundary term. In particular, by means of decomposing $S^{\mathring{D}A}$ into boundary tangent and non-tangent contributions we obtained $S^{\mathring{D}A} = -S^{\mathrm{EH},\mathring{\Omega}}_{\mathrm{GHY}} + S^{\mathrm{EH},\Omega}_{\mathrm{GHY}}$ for non-lightlike as well as for lightlike boundaries in (5.20) and (5.33), respectively. That is, the (S)TEGR boundary term is just a difference of the GHY terms of Einstein-Hilbert gravity, once expressed using the full curvature two-form and once its torsion-free, metric-compatible analog. We concluded that $S^{\mathring{D}A}$ renders the variational problem of the Einstein-Hilbert action well-defined, regardless of which of the latter curvature two-forms we use for expressing it.

Starting from the decomposed form (5.7) of the Einstein-Hilbert action, we examined how the teleparallel limit $\Omega^{\mu}{}_{\nu} = 0$ needs to be taken in order to preserve the variational principle on manifolds with boundaries. In a first argument, we considered the case in which the bulk teleparallel Lagrangian is proportional to $\eta_{\mu}{}^{\nu} \wedge \dot{A}^{\mu}{}_{\rho} \wedge \dot{A}^{\rho}{}_{\nu}$, see action (5.47). This is the standard case studied in literature. I have contributed to this approach by including the boundary terms to the teleparallel limit. In particular, I argued that the GHY term $S^{\mathrm{EH},\Omega}_{\mathrm{GHY}}$ needs to be eliminated when imposing $\Omega^{\mu}{}_{\nu} = 0$ since a GHY term would make the variational problem ill-defined when added to the curvature-free bulk action (5.47). This argument extends the geometrical trinity of general relativity to manifolds with boundary, which is the second main result of chapter 5.

The third main result of chapter 5 is my new perspective on the geometrical trinity of general relativity, which reduces the duality even further to a tautology. I achieved this by unifying all of the trinity's theories in the Einstein action. In particular, we found that every action describing one of the trinity's theories may be rewritten as the Einstein action $-\frac{1}{2\kappa}\int_{\mathcal{M}} \eta_{\mu}{}^{\nu} \wedge \mathring{\omega}^{\mu}{}_{\rho} \wedge \mathring{\omega}^{\rho}{}_{\nu}$ supplemented by a boundary term $S^{\mathrm{E}}_{\mathrm{cov}}$ which depends on the boundary likeness. I proved that this boundary term renders the action covariant while preserving the well-definedness of the variational principle. The Einstein action is capable of describing all the theories contained in the geometrical trinity due to different interpretations of the Levi-Civita connection. This may be seen from the definition of the deformation one-form which implies that $\mathring{\omega}^{\mu}{}_{\nu} = \overset{\bullet}{\omega}^{\mu}{}_{\nu} - \overset{\bullet}{A}^{\mu}{}_{\nu}$ may be utilized to re-express the Levi-Civita connection $\mathring{\omega}^{\mu}{}_{\nu}$ in terms of the (S)TEGR connection $\overset{\bullet}{\omega}^{\mu}{}_{\nu}$. The Levi-Civita connection being defined as the connection of a theory with vanishing torsion and non-metricity, we may on the one hand straightforwardly interpret the Einstein action as a theory in which gravity is modeled by curvature. On the other hand, because the right hand side of $\mathring{\omega}^{\mu}{}_{\nu} = \overset{\bullet}{\omega}^{\mu}{}_{\nu} - \overset{\bullet}{A}^{\mu}{}_{\nu}$ only contains torsion and non-metricity while curvature is vanishing, we may as well use the Levi-Civita connection and thus the Einstein action for describing teleparallel theories of gravity. This approach provides a unified perspective on the geometrical trinity of general relativity, which is my original result that I first presented in this thesis. From this perspective, the equivalence of GR and its teleparallel equivalents becomes trivial since all of these theories are described by the exact same action. Furthermore, it implies that any modified theory of gravity with vanishing torsion and non-metricity may be equivalently described in terms of teleparallel theories. This results in the generalized geometrical trinity of gravity, which I discussed in section 5.3 including a thorough examination of the boundary terms.

Finally, I established a connection between teleparallel gravity and the AdS/CFT correspondence in chapter 6. In particular, I developed a frame perspective on holographic renormalization, providing the foundation for the systematic investigation of the AdS/CFT correspondence with torsion. Holographic renormalization is based on the Fefferman-Graham theorem, so the first step for developing holographic renormalization of torsionful theories is to include frames in the Fefferman-Graham expansion. For this purpose, we conjecture that the coframes (6.11) posses an expansion (6.12) in the vicinity of the AdS boundary if we choose Fefferman-Graham coordinates. This coframe expansion correctly reproduces the well-known Fefferman-Graham expansion of the metric tensor.

As an example, we verified that the coframe which corresponds to the five-dimensional AdS Schwarzschild solution obeys an expansion of the conjectured form (6.12). Using this solution, we explicitly constructed the counterterms needed to apply holographic renormalization to TEGR in the appearance of the Einstein action. We argued that the explicit form of these counterterms given in (6.37) is expected from holographic renormalization in standard AdS/CFT. We used the renormalized action to derive the free energy of the five-dimensional TEGR Schwarzschild black hole in asymptotically AdS spacetimes in (6.50). These results of chapter 6 prove that torsion holds information about the thermodynamics of black holes. Furthermore, they provide the first systematic approach for generalizing the AdS/CFT correspondence such that it includes torsion and non-metricity. These original results of mine may be generalized to apply the method of holographic renormalization to any teleparallel theory of gravity.

The generalization of the AdS/CFT correspondence to geometries which include torsion and non-metricity is an important project for future work. In particular, the generalized correspondence will allow to describe spin and hypermomentum currents in the boundary field theory [76–78, 80–87]. Considering the hydrodynamic limit of this strongly coupled field theory, the results of this thesis allow to describe spin and hypermomentum transport in semimetals [146–148] as we discussed in chapter 1. This is particularly interesting not only because this project connects different branches of physics, but also because the calculation of transport coefficients in semimetals provides in principle experimentally accessible quantities originating from the AdS/CFT approach to hydrodynamics.

There are further immediate implications in holography originating from the results of this thesis. These implications arise from the derivation of the lightlike GHY terms in section 4.2. In the AdS/CFT correspondence, the quantum complexity of field theory states is calculated on the gravity side using a Wheeler-DeWitt patch which has a lightlike boundary [149]. Lightlike hypersurfaces typically play an important role in alternative approaches to the holographic calculation of quantum complexity as well [150, 151]. Therefore, I expect the lightlike GHY terms derived in section 4.2 to be important for holographic calculations of complexity in the presence of torsion or non-metricity. Furthermore, these GHY terms are important for complexity calculations based on more complicated Lagrangians like the ones discussed in subsection 4.2.3. These lightlike GHY terms are my original results first published in this thesis.

In addition, lightlike hypersurfaces are equally relevant in gravity. Consider, for example, black holes which have an event horizon that we describe as a lightlike hypersurface [96]. It is therefore essential to include the correct lightlike boundary terms for studying black holes in modified theories of gravity. The lightlike GHY term derived in section 4.2 is hence an essential result in view of modified gravity. Including the appropriate GHY term to gravitational actions is particularly important for studying black hole thermodynamics as I outlined in section 6.3 [131–133]. In particular, it is well-known that the on-shell Einstein-Hilbert action vanishes in the bulk if we consider Schwarzschild black holes in spacetimes with vanishing cosmological constant [152]. Thus, black hole thermodynamics in this case is attributed solely to the GHY term. Although the on-shell bulk action is generically non-vanishing if we consider more complicated solutions, the GHY terms are equally important in these cases. Only if we consider spacetimes with vanishing curvature must the bulk action contain all thermodynamical information. We have explicitly verified that it does contain this thermodynamical information as a main result of section 6.3. We need to extend this work in the future to study black hole thermodynamics in modified gravity and holography.

Concerning the geometrical trinity of general relativity, we need to clearly distinguish what the dynamical equivalence of the three theories implies and what it does not imply. The descriptions in terms of curvature, torsion and non-metricity being equivalent provides a new perspective on general relativity itself. It is the common interpretation of general relativity that this theory introduces gravity as an effect of curvature. But it is equally valid to introduce gravity in GR as an effect of torsion or non-metricity, and there are special choices of the action which make the dynamics of these theories mere rewritings of one another. The geometrical trinity of general relativity thus implies that we may think of gravity in GR in terms of either curvature, torsion or non-metricity [36, 60]. Besides shedding new light on GR itself, this also implies that torsion and non-metricity may have physical effects of the same strength as curvature. This was not clear before TEGR and STEGR had been introduced, see [109] for instance. Stressing the strength and importance of torsion and non-metricity is thus the main achievement of the geometrical trinity of general relativity.

What we must not expect from the geometrical trinity of general relativity is new physics. This trinity is specifically constructed to not yield physics beyond GR, since all its constituents are equivalent to each other. For resolving open problems in cosmology, we therefore need to study theories which go beyond the models that have been examined before. More than the inclusion of theories of higher order in the field strengths, this involves a mixture of them. Since the geometrical trinity of gravity implies that we may attribute equal strength of gravitational effects to curvature, torsion and non-metricity, it is straightforward to construct theories in which two or all three of them are non-vanishing. These are the models in which new physical phenomena are to be expected.

As an important example, this involves the understanding of dark matter. There are several candidates for dark matter being discussed, all of which go back to the observation of rotation curves of spiral galaxies [49–55]. For explaining these rotation curves, we need an additional source of gravity besides the curvature of spacetime that we attribute to the visible matter through Einstein's theory of general relativity. This additional source of gravity is usually thought of as some matter that we cannot see, therefore called dark matter [46, 56–58]. However, we have found that torsion and non-metricity may be used as sources of gravity as well. It is therefore important to study theories that include more than just one field strength as alternatives to the dark matter hypothesis [29, 59]. As a first step, this involves developing a comprehensive understanding of the matter couplings of torsion and non-metricity.

It contributes to the success of general relativity that we have an intuitive picture of how curvature implies a gravitational attraction of massive objects. That is, the mass of these objects curves spacetime in a similar way as a person standing on a trampoline curves the latter. I am convinced that we need to develop similar intuitions for torsion and non-metricity in order for them to be widely accepted as sources of gravity. As a first attempt, we may return to the parallel transport arguments discussed in chapter 1. We noticed that non-metricity is changing the lengths of vectors when we parallel transport them. In chapter 2 we then saw that the velocities of curves are the most fundamental vectors which exist on manifolds. Therefore, we may interpret non-metricity as changing velocities under parallel transport and thus consider non-metricity as an effective acceleration due to the geometry of a manifold. This is my own argument for how non-metricity may be interpreted as a source of gravity, but we clearly need to further develop such interpretations and provide an analog for torsion as well. After all, this is how we connect the mathematical formalism of differential geometry to physical observations in our universe.

In recent years, a wide variety of higher curvature [153–156] or teleparallel theories of gravity [30–35] have been studied, only a few of which I have discussed in this thesis. It will be a task of fundamental importance in the next years to test these theories by experiments in order to find out which of them are appropriate for describing the observations in our universe. From a theoretical point of view, it is useful to avoid over-constraining theories in first place in order to derive universal results that allow us to examine the observable universe regardless of which constraints experimental data will imply. Note that in this respect we have studied hypersurfaces and GHY terms without fixing a dimension, an action or any of the fundamental fields in chapters 3 and 4. This makes the results of this thesis universal and widely applicable. We need to revisit existing results in various branches of modified gravity from this unifying point of view to distinguish those properties which are specific to certain theories from those which are universal features of manifolds with non-trivial curvature, torsion and non-metricity. This distinction is needed in order to construct physical experiments which test specific properties of spacetime that help to identify those theories which most accurately describe our universe.

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Derivation of the lightlike Lovelock GHY term

A

In this appendix we prove that the decomposed action (4.123) may be rewritten as a boundary term. The most straightforward way to see this is to write the first summand of (4.123) as a total derivative. Since $\mathcal{K}$ is a one-form, we obtain

$$\begin{aligned}
&\eta_{\mathbf{lk}A_1\dots A_{2p-2}} \wedge D\mathcal{K} \wedge (\Omega^{A_1A_2} - 2\varepsilon K^{A_1} \wedge L^{A_2}) \wedge \dots \\
&\qquad\qquad \wedge (\Omega^{A_{2p-3}A_{2p-2}} - 2\varepsilon K^{A_{2p-3}} \wedge L^{A_{2p-2}}) \\
&= \eta_{\mathbf{lk}A_1\dots A_{2p-2}} \wedge D\Big(\mathcal{K} \wedge (\Omega^{A_1A_2} - 2\varepsilon K^{A_1} \wedge L^{A_2}) \wedge \dots \\
&\qquad\qquad \wedge (\Omega^{A_{2p-3}A_{2p-2}} - 2\varepsilon K^{A_{2p-3}} \wedge L^{A_{2p-2}})\Big) \\
&\quad + (p-1)\eta_{\mathbf{lk}A_1\dots A_{2p-2}} \wedge \mathcal{K} \wedge D(\Omega^{A_1A_2} - 2\varepsilon K^{A_1} \wedge L^{A_2}) \wedge \dots \\
&\qquad\qquad \wedge (\Omega^{A_3A_4} - 2\varepsilon K^{A_3} \wedge L^{A_4}) \wedge (\Omega^{A_{2p-3}A_{2p-2}} - 2\varepsilon K^{A_{2p-3}} \wedge L^{A_{2p-2}}) \,,
\end{aligned} \tag{A.1}$$

where we used the antisymmetry of $\eta_{\mathbf{lk}A_1\dots A_{2p-2}}$ to observe that we obtain the last term on the right hand side $(p-1)$ times. This term is simplified by the Bianchi identity of curvature (2.30), due to which we have $D\Omega^{A_1A_2} = 0$. Hence, we obtain

$$\begin{aligned}
&(p-1)\eta_{\mathbf{lk}A_1\dots A_{2p-2}} \wedge \mathcal{K} \wedge D(\Omega^{A_1A_2} - 2\varepsilon K^{A_1} \wedge L^{A_2}) \wedge \dots \\
&\qquad\qquad \wedge (\Omega^{A_3A_4} - 2\varepsilon K^{A_3} \wedge L^{A_4}) \wedge (\Omega^{A_{2p-3}A_{2p-2}} - 2\varepsilon K^{A_{2p-3}} \wedge L^{A_{2p-2}}) \\
&= (p-1)\eta_{\mathbf{lk}A_1\dots A_{2p-2}} \wedge \mathcal{K} \wedge (-2\varepsilon DK^{A_1} \wedge L^{A_2} + 2\varepsilon K^{A_1} \wedge DL^{A_2}) \wedge \dots \\
&\qquad\qquad \wedge (\Omega^{A_3A_4} - 2\varepsilon K^{A_3} \wedge L^{A_4}) \wedge (\Omega^{A_{2p-3}A_{2p-2}} - 2\varepsilon K^{A_{2p-3}} \wedge L^{A_{2p-2}}) \,.
\end{aligned} \tag{A.2}$$

This result combines nicely with the second term in the action decomposition (4.123). To see this, we rewrite this second summand using

$$\begin{aligned}
&2(p-1)\eta_{\mathbf{lk}A_1\dots A_{2p-2}} \wedge (DK^{A_1} + \varepsilon K^{A_1} \wedge \mathcal{K}) \wedge (DL^{A_2} - \varepsilon L^{A_2} \wedge \mathcal{K}) \\
&\qquad\qquad \wedge (\Omega^{A_3A_4} - 2\varepsilon K^{A_3} \wedge L^{A_4}) \wedge \dots \wedge (\Omega^{A_{2p-3}A_{2p-2}} - 2\varepsilon K^{A_{2p-3}} \wedge L^{A_{2p-2}}) \\
&= (p-1)\eta_{\mathbf{lk}A_1\dots A_{2p-2}} \wedge \Big(DK^{A_1} \wedge DL^{A_2} - DL^{A_1} \wedge DK^{A_2} \\
&\qquad\qquad + 2\varepsilon\mathcal{K} \wedge DK^{A_1} \wedge L^{A_2} - 2\varepsilon\mathcal{K} \wedge K^{A_1} \wedge DL^{A_2}\Big) \\
&\qquad\qquad \wedge (\Omega^{A_3A_4} - 2\varepsilon K^{A_3} \wedge L^{A_4}) \wedge \dots \wedge (\Omega^{A_{2p-3}A_{2p-2}} - 2\varepsilon K^{A_{2p-3}} \wedge L^{A_{2p-2}}) \,.
\end{aligned} \tag{A.3}$$

We observe that the addition of (A.2) to the latter terms cancels the contribution which is proportional to $2\varepsilon\mathcal{K}\wedge DK^{A_1}\wedge L^{A_2} - 2\varepsilon\mathcal{K}\wedge K^{A_1}\wedge DL^{A_2}$. The remaining contributions of (A.3) may be rewritten as total derivatives since we have

$$\begin{aligned}
&\eta_{\mathbf{lk}A_1\ldots A_{2p-2}}\wedge D(K^{A_1}\wedge DL^{A_2}\wedge K^{A_3}\wedge L^{A_4}\wedge\cdots\wedge K^{A_{2p-3}}\wedge L^{A_{2p-2}})\\
&=(p-1)\eta_{\mathbf{lk}A_1\ldots A_{2p-2}}\wedge DK^{A_1}\wedge DL^{A_2}\wedge K^{A_3}\wedge L^{A_4}\wedge\cdots\wedge K^{A_{2p-3}}\wedge L^{A_{2p-2}}\\
&\quad-\eta_{\mathbf{lk}A_1\ldots A_{2p-2}}\wedge K^{A_1}\wedge \Omega^{A_2}{}_B\wedge L^B\wedge K^{A_3}\wedge L^{A_4}\wedge\cdots\wedge K^{A_{2p-3}}\wedge L^{A_{2p-2}}\,.
\end{aligned}\tag{A.4}$$

Here we used $D^2L^A=\Omega^A{}_B\wedge L^B$ which yields a term that is irrelevant at the boundary. Analogously, we obtain

$$\begin{aligned}
&\eta_{\mathbf{lk}A_1\ldots A_{2p-2}}\wedge D(L^{A_1}\wedge DK^{A_2}\wedge K^{A_3}\wedge L^{A_4}\wedge\cdots\wedge K^{A_{2p-3}}\wedge L^{A_{2p-2}})\\
&=(p-1)\eta_{\mathbf{lk}A_1\ldots A_{2p-2}}\wedge DL^{A_1}\wedge DK^{A_2}\wedge K^{A_3}\wedge L^{A_4}\wedge\cdots\wedge K^{A_{2p-3}}\wedge L^{A_{2p-2}}\\
&\quad-\eta_{\mathbf{lk}A_1\ldots A_{2p-2}}\wedge L^{A_1}\wedge \Omega^{A_2}{}_B\wedge K^B\wedge K^{A_3}\wedge L^{A_4}\wedge\cdots\wedge K^{A_{2p-3}}\wedge L^{A_{2p-2}}\,.
\end{aligned}\tag{A.5}$$

It renders this proof a little more involved that we cannot straightforwardly apply the latter results to

$$\begin{aligned}
&(p-1)\eta_{\mathbf{lk}A_1\ldots A_{2p-2}}\wedge\Big(DK^{A_1}\wedge DL^{A_2}-DL^{A_1}\wedge DL^{A_2}\Big)\\
&\qquad\wedge(\Omega^{A_3A_4}-2\varepsilon K^{A_3}\wedge L^{A_4})\wedge\cdots\wedge(\Omega^{A_{2p-3}A_{2p-2}}-2\varepsilon K^{A_{2p-3}}\wedge L^{A_{2p-2}})\,,
\end{aligned}\tag{A.6}$$

which we recall to be the only term left in the decomposed action (4.123) that we did not yet write as a total derivative. This is non-trivial since the second line of (A.6) is a polynomial in $K^A\wedge L^B$. In particular, we obtain different prefactors when writing the terms of this polynomial as a total derivative by means of (A.4) and (A.5). This is most easily solved by adapting the wedge product notation we introduced for the GHY term of Lovelock gravity on manifolds with space- and timelike boundaries in section 4.1.3. Using this notation and the binomial theorem, we have

$$\begin{aligned}
&(\Omega^{A_3A_4}-2\varepsilon K^{A_3}\wedge L^{A_4})\wedge\cdots\wedge(\Omega^{A_{2p-3}A_{2p-2}}-2\varepsilon K^{A_{2p-3}}\wedge L^{A_{2p-2}})\\
&=\sum_{q=2}^{p}\binom{p-2}{q-2}(-2\varepsilon)^{p-q}\bigwedge_{m=1}^{p-q}K^{A_{2m+1}}\wedge L^{A_{2m+2}}\bigwedge_{n=p-q+1}^{p-2}\Omega^{A_{2n+1}A_{2n+2}}\,.
\end{aligned}\tag{A.7}$$

In this form, it is straightforward to apply (A.4) and (A.5) in order to write (A.6) as a total derivative. Carfully evaluating the prefactors yields

$$\begin{aligned}
&\eta_{\mathbf{lk}A_1\dots A_{2p-2}} \wedge DK^{A_1} \wedge DL^{A_2} \wedge \\
&\qquad (\Omega^{A_3A_4} - 2\varepsilon K^{A_3} \wedge L^{A_4}) \wedge \dots \wedge (\Omega^{A_{2p-3}A_{2p-2}} - 2\varepsilon K^{A_{2p-3}} \wedge L^{A_{2p-2}}) \\
&\quad = \eta_{\mathbf{lk}A_1\dots A_{2p-2}} \wedge D\Big(K^{A_1} \wedge DL^{A_2} \sum_{q=2}^{p} \binom{p-2}{q-2} \frac{(-2\varepsilon)^{p-q}}{p-q+2} \\
&\qquad \bigwedge_{m=1}^{p-q} K^{A_{2m+1}} \wedge L^{A_{2m+2}} \bigwedge_{n=p-q+1}^{p-2} \Omega^{A_{2n+1}A_{2n+2}}\Big)
\end{aligned} \tag{A.8}$$

and

$$\begin{aligned}
&\eta_{\mathbf{lk}A_1\dots A_{2p-2}} \wedge DL^{A_1} \wedge DK^{A_2} \wedge \\
&\qquad (\Omega^{A_3A_4} - 2\varepsilon K^{A_3} \wedge L^{A_4}) \wedge \dots \wedge (\Omega^{A_{2p-3}A_{2p-2}} - 2\varepsilon K^{A_{2p-3}} \wedge L^{A_{2p-2}}) \\
&\quad = \eta_{\mathbf{lk}A_1\dots A_{2p-2}} \wedge D\Big(L^{A_1} \wedge DK^{A_2} \sum_{q=2}^{p} \binom{p-2}{q-2} \frac{(-2\varepsilon)^{p-q}}{p-q+2} \\
&\qquad \bigwedge_{m=1}^{p-q} K^{A_{2m+1}} \wedge L^{A_{2m+2}} \bigwedge_{n=p-q+1}^{p-2} \Omega^{A_{2n+1}A_{2n+2}}\Big) .
\end{aligned} \tag{A.9}$$

Hence, we now rewrote all boundary relevant terms contained in the action decomposition (4.123) in terms of total derivatives. Collecting these results and inserting them into (4.123) yields

$$\begin{aligned}
S^p = -2p\varepsilon^2 \int_{\mathcal{M}} &\eta_{\mathbf{lk}A_1\dots A_{2p-2}} \wedge D\Big[\mathcal{K} \wedge (\Omega^{A_1A_2} - 2\varepsilon K^{A_1} \wedge L^{A_2}) \wedge \dots \\
&\wedge (\Omega^{A_{2p-3}A_{2p-2}} - 2\varepsilon K^{A_{2p-3}} \wedge L^{A_{2p-2}}) \\
&+ (p-1)(K^{A_1} \wedge DL^{A_2} - L^{A_1} \wedge DK^{A_2}) \sum_{q=2}^{p} \binom{p-2}{q-2} \frac{(-2\varepsilon)^{p-q}}{p-q+2} \\
&\bigwedge_{m=1}^{p-q} K^{A_{2m+1}} \wedge L^{A_{2m+2}} \bigwedge_{n=p-q+1}^{p-2} \Omega^{A_{2n+1}A_{2n+2}}\Big] \\
&+ \text{terms irrelevant on } \partial\mathcal{M} .
\end{aligned} \tag{A.10}$$

The latter result recasts the decomposed action (4.123) as a total covariant exterior derivative as desired. We therefore use this result in (4.124) for deriving the lightlike GHY term of Lovelock gravity.

Bibliography

[1] J. Erdmenger, B. Hess, I. Matthaiakakis, and R. Meyer. "Universal Gibbons-Hawking-York term for theories with curvature, torsion and non-metricity." In: *SciPost Phys.* 14 (2023). doi: 10.21468/SciPostPhys.14.5.099. arXiv: 2211.02064 [hep-th] (mentioned on pp. iii, vii, 7, 8, 54, 81, 82, 92).

[2] J. Erdmenger, B. Hess, I. Matthaiakakis, and R. Meyer. "Gibbons-Hawking-York boundary terms and the generalized geometrical trinity of gravity." In: *Phys. Rev. D* 110.6 (2024). doi: 10.1103/PhysRevD.110.066002. arXiv: 2304.06752 [hep-th] (mentioned on pp. iii, vii, viii, 7, 9, 54, 123, 129, 146).

[3] S. W. Hawking and W. Israel, eds. *Three Hundred Years of Gravitation.* Cambridge University Press, 1987 (mentioned on p. 1).

[4] I. Newton. *Philosophiæ Naturalis Principia Mathematica.* England, 1687 (mentioned on p. 1).

[5] S. Chandrasekhar. *Newton's Principia for the Common Reader.* Clarendon Press, 2003 (mentioned on p. 1).

[6] S. Weinberg. *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity.* New York: John Wiley and Sons, 1972 (mentioned on pp. 1, 123).

[7] A. Einstein. "Zur Elektrodynamik bewegter Körper." In: *Annalen der Physik* 322.10 (1905), pp. 891–921. doi: 10.1002/andp.19053221004 (mentioned on p. 1).

[8] H. Minkowksi. "Raum und Zeit." In: *Jahresberichte der Deutschen Mathematiker-Vereinigung.* B. G. Teubner, 1909 (mentioned on p. 1).

[9] H. Minkowski and A. Sommerfeld. "Raum und Zeit." In: *Das Relativitätsprinzip: Eine Sammlung von Abhandlungen.* Wiesbaden: Vieweg+Teubner Verlag, 1923, pp. 54–71. doi: 10.1007/978-3-663-19510-8_5 (mentioned on p. 1).

[10] A. Einstein. *Grundgedanken und Methoden der Relativitätstheorie, in ihrer Entwicklung dargestellt.* 1920 (mentioned on p. 1).

[11] A. Einstein. “Lichtgeschwindigkeit und Statik des Gravitationsfeldes.” In: *Annalen der Physik* 38 (1912), pp. 355–369 (mentioned on p. 1).

[12] A. Einstein. “Zur Allgemeinen Relativitätstheorie.” In: *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)* 1915 (1915). [Addendum: Sitzungsber.Preuss.Akad.Wiss.Berlin (Math.Phys.) 1915, 799–801 (1915)], pp. 778–786 (mentioned on pp. 1, 9, 139).

[13] A. Einstein. “Die Grundlagen der allgemeinen Relativitätstheorie.” In: *Annalen der Physik* IV. Folge. 49 (1916), pp. 769–822 (mentioned on pp. 1, 2).

[14] A. Einstein. “Die Feldgleichungen der Gravitation.” In: *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften* (1915), pp. 844–847 (mentioned on p. 1).

[15] A. Einstein. “Über den Einfluß der Schwerkraft auf die Ausbreitung des Lichtes.” In: *Annalen der Physik* 35 (1911), pp. 898–908 (mentioned on p. 2).

[16] F. W. Dyson, A. S. Eddington, and C. Davidson. “IX. A determination of the deflection of light by the Sun’s gravitational field, from observations made at the total eclipse of May 29, 1919.” In: *Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character* 220.571-581 (1920), pp. 291–333. doi: 10.1098/rsta.1920.0009 (mentioned on p. 2).

[17] T. Sauer. “Nova Geminorum 1912 and the origin of the idea of gravitational lensing.” In: *Archive for History of Exact Sciences* 62.1 (June 2007), 1–22. doi: 10.1007/s00407-007-0008-4 (mentioned on p. 2).

[18] N. Aghanim et al. “Planck 2018 results. VIII. Gravitational lensing.” In: *Astron. Astrophys.* 641 (2020). doi: 10.1051/0004-6361/201833886. arXiv: 1807.06210 [astro-ph.CO] (mentioned on p. 2).

[19] A. Einstein. “Erklärung der Perihelbewegung des Merkur aus der allgemeinen Relativitätstheorie.” In: *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften* (1915), pp. 831–839 (mentioned on p. 2).

[20] R. M. Wald. *General relativity*. University of Chicago press, 2010 (mentioned on pp. 2, 3, 89, 109, 123).

[21] J. Beltrán Jiménez, L. Heisenberg, and T. Koivisto. “Coincident General Relativity.” In: *Phys. Rev. D* 98.4 (2018). doi: 10.1103/PhysRevD.98.044048. arXiv: 1710.03116 [gr-qc] (mentioned on pp. 3, 123, 140).

[22] M. Nakahara. *Geometry, topology and physics.* Second Edition. CRC Press, 2003. doi: 10.1201/9781315275826 (mentioned on pp. 3, 13).

[23] R. Weitzenböck. *Invariantentheorie.* Groningen: P. Noordhoff, 1923 (mentioned on pp. 3, 161).

[24] A. Einstein. "Riemann-Geometrie mit Aufrechterhaltung des Begriffes des Fernparallelismus." In: *Preussische Akademie der Wissenschaften, Phys.-math. Klasse* (1928), 217–221 (mentioned on p. 3).

[25] K. Hayashi and T. Shirafuji. "New general relativity." In: *Phys. Rev. D* 19 (1979). Ed. by J.-P. Hsu and D. Fine. [Addendum: Phys.Rev.D 24, 3312–3314 (1982)], pp. 3524–3553. doi: 10.1103/PhysRevD.19.3524 (mentioned on p. 3).

[26] R. Aldrovandi and J. Pereira. *Teleparallel Gravity: An Introduction.* Fundamental Theories of Physics. Springer Netherlands, 2012 (mentioned on pp. 3, 135, 161, 168).

[27] Y. Akrami et al. *Modified Gravity and Cosmology. An Update by the CANTATA Network.* Ed. by E. N. Saridakis, R. Lazkoz, V. Salzano, P. Vargas Moniz, S. Capozziello, J. Beltrán Jiménez, M. De Laurentis, and G. J. Olmo. Springer, 2021. doi: 10.1007/978-3-030-83715-0. arXiv: 2105.12582 [gr-qc] (mentioned on p. 3).

[28] M. Hohmann. "Teleparallel Gravity." In: *Lect. Notes Phys.* 1017 (2023), pp. 145–198. doi: 10.1007/978-3-031-31520-6_4. arXiv: 2207.06438 [gr-qc] (mentioned on p. 3).

[29] L. Heisenberg. "Review on f(Q) gravity." In: *Phys. Rept.* 1066 (2024). doi: 10.1016/j.physrep.2024.02.001. arXiv: 2309.15958 [gr-qc] (mentioned on pp. 3, 4, 123, 177).

[30] A. Lymperis. "Late-time cosmology with phantom dark-energy in f(Q) gravity." In: *JCAP* 11 (2022), p. 018. doi: 10.1088/1475-7516/2022/11/018. arXiv: 2207.10997 [gr-qc] (mentioned on pp. 3, 177).

[31] A. Pradhan, A. Dixit, and M. Zeyauddin. "Reconstruction of ΛCDM model from f(T) gravity in viscous-fluid universe with observational constraints." In: *Int. J. Geom. Meth. Mod. Phys.* 21.01 (2024). doi: 10.1142/S0219887824500270 (mentioned on pp. 3, 177).

[32] S. A. Narawade, S. P. Singh, and B. Mishra. "Accelerating cosmological models in f(Q) gravity and the phase space analysis." In: *Phys. Dark Univ.* 42 (2023). doi: 10.1016/j.dark.2023.101282. arXiv: 2303.06427 [gr-qc] (mentioned on pp. 3, 177).

[33] S. A. Narawade, S. H. Shekh, B. Mishra, W. Khyllep, and J. Dutta. "Modelling the accelerating universe with f(Q) gravity: observational consistency." In: *Eur. Phys. J. C* 84.8 (2024), p. 773. doi: 10.1140/epjc/s10052-024-13150-5. arXiv: 2303.01985 [gr-qc] (mentioned on pp. 3, 177).

[34] O. Sokoliuk, S. Arora, S. Praharaj, A. Baransky, and P. K. Sahoo. "On the impact of f(Q) gravity on the large scale structure." In: *Mon. Not. Roy. Astron. Soc.* 522.1 (2023), pp. 252–267. doi: 10.1093/mnras/stad968. arXiv: 2303.17341 [astro-ph.CO] (mentioned on pp. 3, 177).

[35] S. H. Shekh, A. Pradhan, S. P. Gaikwad, and K. R. Mule. "Exploring Dynamic Dark Energy Models in f(T) Gravity: A Comparative Study of NHDE, THDE, and BHDE." In: *Int. J. Theor. Phys.* 63.12 (2024), p. 301. doi: 10.1007/s10773-024-05833-2 (mentioned on pp. 3, 177).

[36] L. Heisenberg. "A systematic approach to generalisations of General Relativity and their cosmological implications." In: *Phys. Rept.* 796 (2019). doi: 10.1016/j.physrep.2018.11.006. arXiv: 1807.01725 [gr-qc] (mentioned on pp. 4, 123, 133, 176).

[37] A. H. Guth. "The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems." In: *Phys. Rev. D* 23 (1981). Ed. by L.-Z. Fang and R. Ruffini, pp. 347–356. doi: 10.1103/PhysRevD.23.347 (mentioned on p. 4).

[38] A. D. Linde. "A New Inflationary Universe Scenario: A Possible Solution of the Horizon, Flatness, Homogeneity, Isotropy and Primordial Monopole Problems." In: *Phys. Lett. B* 108 (1982). Ed. by L.-Z. Fang and R. Ruffini, pp. 389–393. doi: 10.1016/0370-2693(82)91219-9 (mentioned on p. 4).

[39] S. W. Hawking. "The Development of Irregularities in a Single Bubble Inflationary Universe." In: *Phys. Lett. B* 115 (1982), p. 295. doi: 10.1016/0370-2693(82)90373-2 (mentioned on p. 4).

[40] E. W. Kolb. *The Early Universe*. Vol. 69. Taylor and Francis, May 2019. doi: 10.1201/9780429492860 (mentioned on p. 4).

[41] A. R. Liddle. "The Early universe." In: *ASP Conf. Ser.* 126 (1997), p. 31. arXiv: astro-ph/9612093 (mentioned on p. 4).

[42] A. Riotto. "Inflation and the theory of cosmological perturbations." In: *ICTP Lect. Notes Ser.* 14 (2003). Ed. by G. Dvali, A. Perez-Lorenzana, G. Senjanovic, G. Thompson, and F. Vissani, pp. 317–413. arXiv: hep-ph/0210162 (mentioned on p. 4).

[43] S. Tsujikawa. "Introductory review of cosmic inflation." In: *2nd Tah Poe School on Cosmology: Modern Cosmology*. Apr. 2003. arXiv: hep-ph/0304257 (mentioned on p. 4).

[44] D. Baumann. "TASI Lectures on Inflation." In: *Theoretical Advanced Study Institute in Elementary Particle Physics: Physics of the Large and the Small*. 2011, pp. 523–686. doi: 10.1142/9789814327183_0010. arXiv: 0907.5424 [hep-th] (mentioned on p. 4).

[45] Y. Akrami et al. "Planck 2018 results. X. Constraints on inflation." In: *Astron. Astrophys.* 641 (2020). doi: 10.1051/0004-6361/201833887. arXiv: 1807.06211 [astro-ph.CO] (mentioned on p. 4).

[46] N. Deruelle, J.-P. Uzan, and P. de Forcrand-Millard. *Relativity in Modern Physics*. Oxford University Press, Aug. 2018. doi: 10.1093/oso/9780198786399.001.0001 (mentioned on pp. 4, 176).

[47] E. J. Copeland, M. Sami, and S. Tsujikawa. "Dynamics of dark energy." In: *Int. J. Mod. Phys. D* 15 (2006), pp. 1753–1936. doi: 10.1142/S021827180600942X. arXiv: hep-th/0603057 (mentioned on p. 4).

[48] T. M. C. Abbott et al. "DES Y3 + KiDS-1000: Consistent cosmology combining cosmic shear surveys." In: *Open J. Astrophys.* 6 (2023). doi: 10.21105/astro.2305.17173. arXiv: 2305.17173 [astro-ph.CO] (mentioned on p. 4).

[49] V. C. Rubin and W. K. Ford Jr. "Rotation of the Andromeda Nebula from a Spectroscopic Survey of Emission Regions." In: *Astrophys. J.* 159 (1970), pp. 379–403. doi: 10.1086/150317 (mentioned on pp. 4, 176).

[50] V. C. Rubin. "Dark matter in spiral galaxies." In: *Scientific American* 248.6 (1983), pp. 96–109 (mentioned on pp. 4, 176).

[51] T. S. Van Albada, J. N. Bahcall, K Begeman, and R Sancisi. "Distribution of dark matter in the spiral galaxy NGC 3198." In: *Astrophysical Journal, Part 1 (ISSN 0004-637X), vol. 295, Aug. 15, 1985, p. 305-313.* 295 (1985), pp. 305–313 (mentioned on pp. 4, 176).

[52] T. Van Albada and R Sancisi. "Dark matter in spiral galaxies." In: *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences* 320.1556 (1986), pp. 447–464 (mentioned on pp. 4, 176).

[53] S. M. Kent. "Dark matter in spiral galaxies. I-Galaxies with optical rotation curves." In: *Astronomical Journal (ISSN 0004-6256), vol. 91, June 1986, p. 1301-1327.* 91 (1986), pp. 1301–1327 (mentioned on pp. 4, 176).

[54] S. M. Kent. "Dark matter in spiral galaxies. II-Galaxies with HI rotation curves." In: *Astronomical Journal (ISSN 0004-6256), vol. 93, April 1987, p. 816-832.* 93 (1987), pp. 816–832 (mentioned on pp. 4, 176).

[55] G. Bertone and D. Hooper. "History of dark matter." In: *Rev. Mod. Phys.* 90.4 (2018). doi: 10.1103/RevModPhys.90.045002. arXiv: 1605.04909 [astro-ph.CO] (mentioned on pp. 4, 176).

[56] M. Tegmark et al. "Cosmological parameters from SDSS and WMAP." In: *Phys. Rev. D* 69 (2004). doi: 10.1103/PhysRevD.69.103501. arXiv: astro-ph/0310723 (mentioned on pp. 4, 176).

[57] G. Hinshaw et al. "Nine-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Parameter Results." In: *Astrophys. J. Suppl.* 208 (2013), p. 19. doi: 10.1088/0067-0049/208/2/19. arXiv: 1212.5226 [astro-ph.CO] (mentioned on pp. 4, 176).

[58] N. Aghanim et al. "Planck 2018 results. VI. Cosmological parameters." In: *Astron. Astrophys.* 641 (2020). [Erratum: Astron.Astrophys. 652, C4 (2021)]. doi: 10.1051/0004-6361/201833910. arXiv: 1807.06209 [astro-ph.CO] (mentioned on pp. 4, 176).

[59] R. Ferraro and F. Fiorini. "Modified teleparallel gravity: Inflation without inflaton." In: *Phys. Rev. D* 75 (2007). doi: 10.1103/PhysRevD.75.084031. arXiv: gr-qc/0610067 (mentioned on pp. 4, 177).

[60] J. Beltrán Jiménez, L. Heisenberg, and T. S. Koivisto. "The Geometrical Trinity of Gravity." In: *Universe* 5.7 (2019). doi: 10.3390/universe5070173. arXiv: 1903.06830 [hep-th] (mentioned on pp. 4, 123, 125, 133, 135, 176).

[61] J. Beltrán Jiménez, L. Heisenberg, D. Iosifidis, A. Jiménez-Cano, and T. S. Koivisto. "General teleparallel quadratic gravity." In: *Physics Letters B* 805 (2020). doi: https://doi.org/10.1016/j.physletb.2020.135422 (mentioned on pp. 4, 123, 134).

[62] J. Maldacena. "The Large-N Limit of Superconformal Field Theories and Supergravity." In: *International Journal of Theoretical Physics* 38.4 (1999), pp. 1113–1133. doi: 10.1023/a:1026654312961 (mentioned on pp. 5, 153).

[63] S. Gubser, I. R. Klebanov, and A. M. Polyakov. "Gauge theory correlators from noncritical string theory." In: *Phys. Lett. B* 428 (1998), pp. 105–114. doi: 10.1016/S0370-2693(98)00377-3. arXiv: hep-th/9802109 (mentioned on pp. 5, 153, 167).

[64] E. Witten. “Anti-de Sitter space and holography.” In: *Adv. Theor. Math. Phys.* 2 (1998), pp. 253–291. doi: 10.4310/ATMP.1998.v2.n2.a2. arXiv: hep-th/9802150 (mentioned on pp. 5, 153, 167).

[65] G. ’t Hooft. “Dimensional reduction in quantum gravity.” In: *Conf. Proc. C* 930308 (1993), pp. 284–296. arXiv: gr-qc/9310026 (mentioned on pp. 5, 153).

[66] C. R. Stephens, G. ’t Hooft, and B. F. Whiting. “Black hole evaporation without information loss.” In: *Class. Quant. Grav.* 11 (1994), pp. 621–648. doi: 10.1088/0264-9381/11/3/014. arXiv: gr-qc/9310006 (mentioned on pp. 5, 153).

[67] L. Susskind. “The World as a Hologram.” In: *J. Math. Phys.* 36 (1995). doi: 10.1063/1.531249. arXiv: hep-th/9409089 (mentioned on pp. 5, 153).

[68] J. D. Bekenstein. “Information in the holographic universe.” In: *Scientific American* 289.2 (2003), pp. 58–65 (mentioned on pp. 5, 153).

[69] M. Ammon and J. Erdmenger. *Gauge/gravity duality: Foundations and applications.* Cambridge University Press, 2015 (mentioned on pp. 5, 153, 159, 164, 166).

[70] S. A. Hartnoll, A. Lucas, and S. Sachdev. *Holographic quantum matter.* 2016. arXiv: 1612.07324 [hep-th] (mentioned on pp. 5, 166).

[71] E. Witten. “Anti-de Sitter space, thermal phase transition, and confinement in gauge theories.” In: *Adv. Theor. Math. Phys.* 2 (1998). Ed. by L. Bergstrom and U. Lindstrom, pp. 505–532. doi: 10.4310/ATMP.1998.v2.n3.a3. arXiv: hep-th/9803131 (mentioned on p. 5).

[72] C. Charmousis, B. Goutéraux, B. Soo Kim, E. Kiritsis, and R. Meyer. “Effective holographic theories for low-temperature condensed matter systems.” In: *Journal of High Energy Physics* 11 (Nov. 2010). doi: 10.1007/JHEP11(2010)151 (mentioned on p. 5).

[73] C. P. Herzog, P. Kovtun, S. Sachdev, and D. T. Son. “Quantum critical transport, duality, and M theory.” In: *Phys. Rev. D* 75 (8 Apr. 2007). doi: 10.1103/PhysRevD.75.085020 (mentioned on p. 5).

[74] P. K. Kovtun, D. T. Son, and A. O. Starinets. “Viscosity in Strongly Interacting Quantum Field Theories from Black Hole Physics.” In: *Phys. Rev. Lett.* 94 (11 Mar. 2005). doi: 10.1103/PhysRevLett.94.111601 (mentioned on p. 5).

[75] J. Erdmenger, I. Matthaiakakis, R. Meyer, and D. R. Fernández. "Strongly coupled electron fluids in the Poiseuille regime." In: *Phys. Rev. B* 98 (19 Nov. 2018). DOI: 10.1103/PhysRevB.98.195143 (mentioned on p. 5).

[76] V. E. Hubeny, S. Minwalla, and M. Rangamani. "The fluid/gravity correspondence." In: *Black holes in higher dimensions.* 2012, pp. 348–383. arXiv: 1107.5780 [hep-th] (mentioned on pp. 5, 175).

[77] S. Bhattacharyya, V. E. Hubeny, S. Minwalla, and M. Rangamani. "Nonlinear Fluid Dynamics from Gravity." In: *JHEP* 02 (2008), p. 045. DOI: 10.1088/1126-6708/2008/02/045. arXiv: 0712.2456 [hep-th] (mentioned on pp. 5, 175).

[78] M. Rangamani. "Gravity and hydrodynamics: lectures on the fluid-gravity correspondence." In: *Classical and Quantum Gravity* 26.22 (Oct. 2009). DOI: 10.1088/0264-9381/26/22/224003 (mentioned on pp. 5, 175).

[79] F. W. Hehl, G. D. Kerlick, and P. von der Heyde. "On hypermomentum in general relativity III. Coupling hypermomentum to geometry." In: *Zeitschrift fuer Naturforschung A* 31.7 (1976), pp. 823–827 (mentioned on p. 6).

[80] J. Erdmenger, M. Haack, M. Kaminski, and A. Yarom. "Fluid dynamics of R-charged black holes." In: *Journal of High Energy Physics* 2009.01 (Jan. 2009). DOI: 10.1088/1126-6708/2009/01/055 (mentioned on pp. 6, 175).

[81] M. Ammon, M. Kaminski, R. Koirala, J. Leiber, and J. Wu. "Quasinormal modes of charged magnetic black branes & chiral magnetic transport." In: *JHEP* 04 (2017). DOI: 10.1007/JHEP04(2017)067. arXiv: 1701.05565 [hep-th] (mentioned on pp. 6, 175).

[82] Y. Ferreiros, Y. Kedem, E. J. Bergholtz, and J. H. Bardarson. "Mixed Axial-Torsional Anomaly in Weyl Semimetals." In: *Phys. Rev. Lett.* 122 (5 Feb. 2019). DOI: 10.1103/PhysRevLett.122.056601 (mentioned on pp. 6, 175).

[83] M. Ammon, S. Grieninger, J. Hernandez, M. Kaminski, R. Koirala, J. Leiber, and J. Wu. "Chiral hydrodynamics in strong external magnetic fields." In: *JHEP* 04 (2021). DOI: 10.1007/JHEP04(2021)078. arXiv: 2012.09183 [hep-th] (mentioned on pp. 6, 175).

[84] C. Cartwright, M. G. Amano, M. Kaminski, J. Noronha, and E. Speranza. "Convergence of hydrodynamics in a rotating strongly coupled plasma." In: *Phys. Rev. D* 108.4 (2023). DOI: 10.1103/PhysRevD.108.046014. arXiv: 2112.10781 [hep-th] (mentioned on pp. 6, 175).

[85] M. Ammon, D. Arean, M. Baggioli, S. Gray, and S. Grieninger. "Pseudo-spontaneous $U(1)$ symmetry breaking in hydrodynamics and holography." In: *JHEP* 03 (2022). doi: 10.1007/JHEP03(2022)015. arXiv: 2111.10305 [hep-th] (mentioned on pp. 6, 175).

[86] M. Hongo, X.-G. Huang, M. Kaminski, M. Stephanov, and H.-U. Yee. "Spin relaxation rate for heavy quarks in weakly coupled QCD plasma." In: *JHEP* 08 (2022). doi: 10.1007/JHEP08(2022)263. arXiv: 2201.12390 [hep-th] (mentioned on pp. 6, 175).

[87] Z.-Y. Xian, S. Danz, D. R. Fernández, I. Matthaiakakis, C. Tutschku, R. L. Klees, J. Erdmenger, R. Meyer, and E. M. Hankiewicz. "Hall viscosity and hydrodynamic inverse Nernst effect in graphene." In: *Phys. Rev. B* 107.20 (2023). doi: 10.1103/PhysRevB.107.L201403. arXiv: 2207.10528 [cond-mat.mes-hall] (mentioned on pp. 6, 175).

[88] J. A. Sulpizio et al. "Visualizing Poiseuille flow of hydrodynamic electrons." In: *Nature* 576.7785 (Dec. 2019), pp. 75–79. doi: 10.1038/s41586-019-1788-9 (mentioned on p. 6).

[89] Y. Choquet-Bruhat, C. DeWitt-Morette, and M. Dillard-Bleick. *Analysis, manifolds and physics*. Revised Edition. North-Holland, 1982 (mentioned on p. 13).

[90] J. M. Lee. *Introduction to Smooth Manifolds*. Springer New York, 2003. doi: 10.1007/978-0-387-21752-9 (mentioned on pp. 13, 25, 53).

[91] G. Rudolph and M. Schmidt. *Differential Geometry and Mathematical Physics: Part I. Manifolds, Lie Groups and Hamiltonian Systems*. Springer Dordrecht, 2012. doi: 10.1007/978-94-007-5345-7 (mentioned on p. 13).

[92] G. Rudolph and M. Schmidt. *Differential Geometry and Mathematical Physics: Part II. Fibre Bundles, Topology and Gauge Fields*. Springer Dordrecht, 2017. doi: 10.1007/978-94-024-0959-8 (mentioned on p. 13).

[93] F. P. Schuller. *Geometric anatomy of theoretical physics*. Lecture Series. https://www.fau.tv/course/id/242. Friedrich-Alexander University Erlangen-Nürnberg. 2013 (mentioned on p. 13).

[94] S. Rea. *Fredric Schuller's Geometric anatomy of theoretical physics*. Lecture Notes. https://github.com/sreahw/schuller-geometric. 2018 (mentioned on p. 13).

[95] S. Waldmann. *Differential Geometry*. Lecture Notes. Julius-Maximilians University Würzburg. 2023 (mentioned on p. 13).

[96] E. Poisson. *A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics*. Cambridge University Press, 2004. doi: 10.1017/CBO9780511606601 (mentioned on pp. 53, 54, 66, 77, 79, 175).

[97] E. Gourgoulhon. *3+1 formalism and bases of numerical relativity*. 2007. arXiv: gr-qc/0703035 [GR-QC] (mentioned on pp. 53, 54, 59, 61, 64).

[98] E. Gourgoulhon and J. L. Jaramillo. "A 3+1 perspective on null hypersurfaces and isolated horizons." In: *Phys. Rept.* 423 (2006), pp. 159–294. doi: 10.1016/j.physrep.2005.10.005. arXiv: gr-qc/0503113 (mentioned on pp. 53, 66, 79).

[99] R. L. Arnowitt, S. Deser, and C. W. Misner. "Dynamical Structure and Definition of Energy in General Relativity." In: *Phys. Rev.* 116 (5 Dec. 1959). doi: 10.1103/PhysRev.116.1322 (mentioned on p. 61).

[100] R. L. Arnowitt, S. Deser, and C. W. Misner. *The Dynamics of General Relativity*. First published in [101]. 2008. doi: 10.1007/s10714-008-0661-1. arXiv: gr-qc/0405109 (mentioned on p. 61).

[101] L. Witten, ed. *Gravitation: An Introduction to Current Research*. John Wiley & Sons, Inc., 1962 (mentioned on pp. 61, 194).

[102] J. W. York. "Role of Conformal Three-Geometry in the Dynamics of Gravitation." In: *Phys. Rev. Lett.* 28 (16 Apr. 1972), pp. 1082–1085. doi: 10.1103/PhysRevLett.28.1082 (mentioned on pp. 81, 92, 94, 95).

[103] G. W. Gibbons and S. W. Hawking. "Action integrals and partition functions in quantum gravity." In: *Phys. Rev. D* 15 (10 May 1977), pp. 2752–2756. doi: 10.1103/PhysRevD.15.2752 (mentioned on pp. 81, 92, 94, 95).

[104] N. Deruelle, M. Sasaki, Y. Sendouda, and D. Yamauchi. "Hamiltonian formulation of f(Riemann) theories of gravity." In: *Prog. Theor. Phys.* 123 (2010), pp. 169–185. doi: 10.1143/PTP.123.169. arXiv: 0908.0679 [hep-th] (mentioned on p. 86).

[105] D. Grumiller, R. B. Mann, and R. McNees. "Dirichlet boundary value problem for Chern-Simons modified gravity." In: *Phys. Rev. D* 78 (2008). doi: 10.1103/PhysRevD.78.081502. arXiv: 0803.1485 [gr-qc] (mentioned on pp. 92, 97).

[106] R. C. Myers. "Higher Derivative Gravity, Surface Terms and String Theory." In: *Phys. Rev. D* 36 (1987), p. 392. doi: 10.1103/PhysRevD.36.392 (mentioned on pp. 92, 99, 101, 102).

[107] O. Miskovic and R. Olea. “Counterterms in Dimensionally Continued AdS Gravity.” In: *JHEP* 10 (2007), p. 028. doi: 10.1088/1126-6708/2007/10/028. arXiv: 0706.4460 [hep-th] (mentioned on pp. 92, 101, 102).

[108] B. Hess. “Fluid/gravity correspondence for post-Riemannian spacetimes: Metric-affine black holes for hydrodynamics.” Master thesis. Julius-Maximilians-Universität Würzburg, 2020 (mentioned on p. 93).

[109] A. Trautman. “Einstein-Cartan theory.” In: *Encyclopedia of Mathematical Physics*. Ed. by D. Gottesman, J. Francoise, G. Naber, and S. Tsou. Vol. 2. Oxford: Elsevier, 2006, pp. 189–195 (mentioned on pp. 93, 176).

[110] R. Jackiw and S. Y. Pi. “Chern-Simons modification of general relativity.” In: *Phys. Rev. D* 68 (2003). doi: 10.1103/PhysRevD.68.104012. arXiv: gr-qc/0308071 (mentioned on p. 95).

[111] D. Lovelock. “The Einstein Tensor and Its Generalizations.” In: *Journal of Mathematical Physics* 12.3 (Mar. 1971), pp. 498–501. doi: 10.1063/1.1665613 (mentioned on p. 97).

[112] X. O. Camanho. “Lovelock gravity, black holes and holography.” PhD thesis. https://inspirehep.net/files/6290541fe1cf1b5b5137cd16ff99a23e. University of Santiago de Compostela, Mar. 2013 (mentioned on p. 97).

[113] J. D. Edelstein. “Lovelock theory, black holes and holography.” In: *Springer Proc. Math. Stat.* 60 (2014). Ed. by A. García-Parrado, F. C. Mena, F. Moura, and E. Vaz, pp. 19–36. doi: 10.1007/978-3-642-40157-2_2. arXiv: 1303.6213 [gr-qc] (mentioned on p. 97).

[114] M. Bañados, O. Mišković, and S. Theisen. “Holographic currents in first order Gravity and finite Fefferman-Graham expansions.” In: *Journal of High Energy Physics* 2006.06 (June 2006). doi: 10.1088/1126-6708/2006/06/025. arXiv: hep-th/0604148 (mentioned on pp. 102, 169).

[115] A. D. Gallegos and U. Gürsoy. “Holographic spin liquids and Lovelock Chern-Simons gravity.” In: *JHEP* 11 (2020), p. 151. doi: 10.1007/JHEP11(2020)151. arXiv: 2004.05148 [hep-th] (mentioned on pp. 102, 168).

[116] L. Lehner, R. C. Myers, E. Poisson, and R. D. Sorkin. “Gravitational action with null boundaries.” In: *Phys. Rev. D* 94.8 (2016). doi: 10.1103/PhysRevD.94.084046. arXiv: 1609.00207 [hep-th] (mentioned on pp. 111, 114, 115).

[117] K. Parattu, S. Chakraborty, B. R. Majhi, and T. Padmanabhan. "A Boundary Term for the Gravitational Action with Null Boundaries." In: *Gen. Rel. Grav.* 48.7 (2016). doi: 10.1007/s10714-016-2093-7. arXiv: 1501.01053 [gr-qc] (mentioned on pp. 111, 114, 115).

[118] S. W. Hawking and G. F. R. Ellis. *The Large Scale Structure of Space-Time.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1973. doi: 10.1017/CBO9780511524646 (mentioned on p. 123).

[119] J. M. Nester and H.-J. Yo. "Symmetric teleparallel general relativity." In: *Chin. J. Phys.* 37 (1999), p. 113. arXiv: gr-qc/9809049 (mentioned on pp. 123, 124, 135).

[120] G. Kofinas and E. N. Saridakis. "Teleparallel equivalent of Gauss-Bonnet gravity and its modifications." In: *Phys. Rev. D* 90 (2014). doi: 10.1103/PhysRevD.90.084044. arXiv: 1404.2249 [gr-qc] (mentioned on p. 123).

[121] F. W. Hehl, J. McCrea, E. W. Mielke, and Y. Ne'eman. "Metric-affine gauge theory of gravity: field equations, Noether identities, world spinors, and breaking of dilation invariance." In: *Physics Reports* 258.1 (1995), pp. 1–171. doi: 10.1016/0370-1573(94)00111-F (mentioned on p. 125).

[122] T. G. Lucas and J. G. Pereira. "Hodge Dual for Soldered Bundles." In: *J. Phys. A* 42 (2009). doi: 10.1088/1751-8113/42/3/035402. arXiv: 0811.2066 [gr-qc] (mentioned on p. 135).

[123] P. Huang and F.-F. Yuan. "Generalized Hodge dual for torsion in teleparallel gravity." In: *Int. J. Geom. Meth. Mod. Phys.* 13.09 (2016). doi: 10.1142/S0219887816501061. arXiv: 1408.6412 [gr-qc] (mentioned on p. 135).

[124] M. Krššák. "Holographic Renormalization in Teleparallel Gravity." In: *Eur. Phys. J. C* 77.1 (2017), p. 44. doi: 10.1140/epjc/s10052-017-4621-3. arXiv: 1510.06676 [gr-qc] (mentioned on pp. 139, 161).

[125] C. G. Boehmer and E. Jensko. "Modified gravity: A unified approach." In: *Phys. Rev. D* 104.2 (2021). doi: 10.1103/PhysRevD.104.024010. arXiv: 2103.15906 [gr-qc] (mentioned on pp. 140, 142, 144).

[126] M. Krššák. *Einstein Gravity from Einstein Action: Counterterms and Covariance.* June 2024. arXiv: 2406.08452 [gr-qc] (mentioned on pp. 140, 143, 144).

[127] V. I. Arnold and B. A. Khesin. *Topological Methods in Hydrodynamics.* 2nd ed. Vol. 19. Applied Mathematical Sciences. Springer Cham, 2021. DOI: 10.1007/978-3-030-74278-2 (mentioned on p. 145).

[128] L. Ciambelli and R. G. Leigh. “Weyl Connections and their Role in Holography.” In: *Phys. Rev. D* 101.8 (2020). DOI: 10.1103/PhysRevD.101.086020. arXiv: 1905.04339 [hep-th] (mentioned on p. 145).

[129] A. Baldazzi, O. Melichev, and R. Percacci. “Metric-Affine Gravity as an effective field theory.” In: *Annals Phys.* 438 (2022). DOI: 10.1016/j.aop.2022.168757. arXiv: 2112.10193 [gr-qc] (mentioned on pp. 146, 149).

[130] F. Bajardi, D. Blixt, and S. Capozziello. “The Gauss–Bonnet topological scalar in the geometric trinity of gravity.” In: *Int. J. Geom. Meth. Mod. Phys.* 20.Supp01 (2023). DOI: 10.1142/S0219887824500427. arXiv: 2308.03632 [gr-qc] (mentioned on p. 146).

[131] J. D. Bekenstein. “Black holes and entropy.” In: *Physical Review D* 7.8 (1973) (mentioned on pp. 153, 166, 175).

[132] S. W. Hawking. “Gravitational radiation from colliding black holes.” In: *Phys. Rev. Lett.* 26 (1971), pp. 1344–1346. DOI: 10.1103/PhysRevLett.26.1344 (mentioned on pp. 153, 166, 175).

[133] S. W. Hawking. “Particle Creation by Black Holes.” In: *Commun. Math. Phys.* 43 (1975). Ed. by G. W. Gibbons and S. W. Hawking. [Erratum: Commun.Math.Phys. 46, 206 (1976)], pp. 199–220. DOI: 10.1007/BF02345020 (mentioned on pp. 153, 166, 167, 175).

[134] M. Henningson and K. Skenderis. “The Holographic Weyl anomaly.” In: *JHEP* 07 (1998). DOI: 10.1088/1126-6708/1998/07/023. arXiv: hep-th/9806087 (mentioned on pp. 153, 156, 158, 162, 164, 165).

[135] M. Bianchi, D. Z. Freedman, and K. Skenderis. “Holographic renormalization.” In: *Nucl. Phys. B* 631 (2002). DOI: 10.1016/S0550-3213(02)00179-7. arXiv: hep-th/0112119 (mentioned on pp. 153, 156, 158, 162).

[136] K. Skenderis. “Lecture notes on holographic renormalization.” In: *Class. Quant. Grav.* 19 (2002). Ed. by B. de Wit and S. Vandoren, pp. 5849–5876. DOI: 10.1088/0264-9381/19/22/306. arXiv: hep-th/0209067 (mentioned on pp. 153, 156, 158, 162).

[137] C. Fefferman and C. R. Graham. "Conformal invariants." In: *"Elie Cartan et les Mathematiques d'Aujourd'hui", Asterisque, hors serie* (1985), pp. 95–116 (mentioned on pp. 154, 155, 158).

[138] C. Fefferman and C. R. Graham. *The Ambient Metric*. Princeton University Press, 2012 (mentioned on pp. 154, 155).

[139] K. Schwarzschild. "Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie." In: *Sitzungsberichte der königlich preussischen Akademie der Wissenschaften* (1916), pp. 189–196 (mentioned on p. 159).

[140] K. Schwarzschild. "Über das Gravitationsfeld einer Kugel aus inkompressibler Flüssigkeit nach der Einsteinschen Theorie." In: *Sitzungsberichte der königlich preußischen Akademie der Wissenschaften zu Berlin* (1916), pp. 424–434 (mentioned on p. 159).

[141] M. Krššák and J. G. Pereira. "Spin Connection and Renormalization of Teleparallel Action." In: *Eur. Phys. J. C* 75.11 (2015), p. 519. doi: 10.1140/epjc/s10052-015-3749-2. arXiv: 1504.07683 [gr-qc] (mentioned on p. 161).

[142] E. D. Emtsova, M. Krššák, A. N. Petrov, and A. V. Toporensky. "On the Schwarzschild solution in TEGR." In: *J. Phys. Conf. Ser.* 2081.1 (2021). doi: 10.1088/1742-6596/2081/1/012017 (mentioned on p. 161).

[143] V. C. de Andrade, L. C. T. Guillen, and J. G. Pereira. "Gravitational energy momentum density in teleparallel gravity." In: *Phys. Rev. Lett.* 84 (2000), pp. 4533–4536. doi: 10.1103/PhysRevLett.84.4533. arXiv: gr-qc/0003100 (mentioned on pp. 161, 168).

[144] G. Paci. "Black Hole Thermodynamics and Boundary Terms in Teleparallel Gravity." Master thesis. University of Bologna, 2021 (mentioned on p. 168).

[145] J. Beltrán Jiménez and T. S. Koivisto. "Euclidean teleparallel relativity and black hole partition functions." In: (Dec. 2024). arXiv: 2412.13946 [gr-qc] (mentioned on p. 168).

[146] K. Landsteiner and Y. Liu. "The holographic Weyl semi-metal." In: *Phys. Lett.* B753 (2016), pp. 453–457. doi: 10.1016/j.physletb.2015.12.052. arXiv: 1505.04772 [hep-th] (mentioned on p. 175).

[147] K. Landsteiner. "Notes on Anomaly Induced Transport." In: *Acta Phys. Polon.* B47 (2016). doi: 10.5506/APhysPolB.47.2617. arXiv: 1610.04413 [hep-th] (mentioned on p. 175).

[148] Y. Liu and Y.-W. Sun. "Topological hydrodynamic modes and holography." In: *Phys. Rev. D* 105.8 (2022). DOI: 10.1103/PhysRevD.105.086017. arXiv: 2005.02850 [hep-th] (mentioned on p. 175).

[149] A. R. Brown, D. A. Roberts, L. Susskind, B. Swingle, and Y. Zhao. "Holographic Complexity Equals Bulk Action?" In: *Phys. Rev. Lett.* 116 (19 May 2016). DOI: 10.1103/PhysRevLett.116.191301 (mentioned on p. 175).

[150] J. Couch, W. Fischler, and P. H. Nguyen. "Noether charge, black hole volume, and complexity." In: *Journal of High Energy Physics* 2017.3 (Mar. 2017). DOI: 10.1007/JHEP03(2017)119 (mentioned on p. 175).

[151] A. Belin, R. C. Myers, S.-M. Ruan, G. Sárosi, and A. J. Speranza. "Complexity equals anything II." In: *JHEP* 01 (2023). DOI: 10.1007/JHEP01(2023)154. arXiv: 2210.09647 [hep-th] (mentioned on p. 175).

[152] S Carlip. "Black hole thermodynamics." In: *One Hundred Years of General Relativity: From Genesis and Empirical Foundations to Gravitational Waves, Cosmology and Quantum Gravity.* World Scientific, 2017, pp. 415–465 (mentioned on p. 175).

[153] T. P. Sotiriou and V. Faraoni. "f(R) Theories Of Gravity." In: *Rev. Mod. Phys.* 82 (2010), pp. 451–497. DOI: 10.1103/RevModPhys.82.451. arXiv: 0805.1726 [gr-qc] (mentioned on p. 177).

[154] S. Akbarpour, A. S. Sefiedgar, and N. Rashidi. "Non-equilibrium thermodynamics in f(R, T) gravity and dark energy." In: *Phys. Scripta* 99.12 (2024). DOI: 10.1088/1402-4896/ad8fdf (mentioned on p. 177).

[155] J. G. de Lima Júnior, P. H. R. S. Moraes, E. Brito, and J. A. S. Fortunato. "The Palatini formalism of the $f(R, \mathcal{L}_m, T)$ theory of gravity." In: (Nov. 2024). arXiv: 2411.15615 [gr-qc] (mentioned on p. 177).

[156] P. S. Ens and A. F. Santos. "Barrow holographic dark energy: a path to reconstructing $f(R, T)$ gravity." In: (Dec. 2024). arXiv: 2412.09189 [gr-qc] (mentioned on p. 177).