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Further considerations about the traversability of thin-shell wormholes

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Abstract Traversability in relation with tides in thin-shell wormholes is revisited to investigate the possibility of improving recently noted restrictive conditions for a safe travel across a wormhole throat. We consider wormholes mathematically constructed starting from background geometries which are solutions of scalar–tensor theories as dilaton gravity and Brans–Dicke gravity. The advantages of working within such frameworks are studied by examining the dependence of the extrinsic curvature and tides at the throat with the parameters determining the departure from pure relativity; the associated behaviour of tides in the smooth regions of the geometries is also analyzed. Other related but different approaches within pure relativity are discussed in the appendices.

1 Introduction

The recent study in [1] shows that the main difficulty with tides across the throat of thin-shell wormholes does not come from the contribution of the smooth parts of the geometries: for points along both the radial or an angular direction, the corresponding effects are finite and proportional to the spatial separation of these points, and could be controlled by an appropriate choice of the parameters defining the metrics at each side of the shell placed at the throat radius (see also Ref. [2]). Instead, the contribution coming from the jump in the extrinsic curvature across the shell presents some subtleties. For the radial tides this contribution is fixed and does not vanish for infinitely close points; hence the quotient of the relative acceleration and the separation between two points at different sides of the throat $\Delta a/\Delta x$ would, in principle, diverge when $\Delta x \rightarrow 0$. For the angular tides the problem is different: though the contribution of the curvature jump is proportional to the angular separation, it appears regulated by the traveling time across the shell, which, in a plain formal approach, is infinitely short. These results would seem to rule out the possibility of a safe passage through such configurations. However, in our previous work [1] we briefly introduced an alternative point of view, in the spirit of taking the formal results as a first approximation to more realistic situations in which shells have a non vanishing – though still very little – thickness.

Within this understanding of the problem, here we will investigate the conditions to, at least, reduce the jump of the components of the extrinsic curvature across the shell; when such reduction is possible, we will also evaluate the effects of these conditions on the curvature of the smooth regions. In particular, we will explore the possible advantages, if any exist, of considering wormhole constructions in the framework of gravity theories beyond relativity including a scalar field, as the so-called dilaton gravity and Brans–Dicke gravity.

Dilaton gravity (as it results from the low energy action of string theory) has been considered of interest for different reasons and at different scales. For example, the $3+1$ dimensional cosmological field equations make possible to propose a pre-big bang phase for the universe [3]; in relation with this, the quantization of string cosmological models has been thoroughly analysed (see, for instance, Refs. [4–6]). At a smaller scale, and also in four spacetime dimensions, black hole solutions have been studied in [7–10]. In a line even more related with ours, it was shown that, by means of an appropriate parameter choice, the dilaton field can reduce the amount of exotic matter in wormholes supported by charged thin shells [11].

On the other hand, given the usual objection against wormhole geometries in Einstein's gravity, i.e. the requirement of

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exotic matter, wormholes have been repeatedly considered in alternative theories; in fact, it was shown that the requirement of exotic matter could in some cases be avoided (see for example [12–14] or [15] and references therein). In particular, it was shown in [16] that, in Brans–Dicke gravity, Lorentzian wormholes of the Morris–Thorne type are compatible with matter which, apart from the Brans–Dicke scalar field, satisfies the energy conditions. An analogous result was found in [17,18] for spherical thin-shell configurations. Other related aspects of wormholes in Brans–Dicke or in other scalar–tensor theories were also discussed in Refs. [19–21].

The article is organized as follows: In Sect. 2 we review some aspects of tides in thin-shell wormholes as they were previously analyzed in [1], and we present the general expressions for the proper tidal acceleration experienced by moving objects traversing across the thin throat of the geometry. In Sect. 3 we apply the analysis to some specific solutions of dilaton gravity and of Brans–Dicke gravity, respectively. In Sect. 4 we summarize the results obtained for different scalar–tensor theories in comparison with General Relativity. Finally, we include two appendices dealing with different simple lines of analysis for other examples within the relativistic framework.

2 General aspects of tides in thin-shell wormholes

We restrict our analysis to static thin-shell wormholes constructed by pasting together two copies of the same geometry at the throat's timelike hypersurface defined by $r = r_0$; the line elements are

$$ds^2 = g^{\pm}_{00}\, dt^2 + g^{\pm}_{rr}\, dr^2_{\pm} + g^{\pm}_{\zeta\zeta}\, d\zeta^2 + g^{\pm}_{\varphi\varphi}\, d\varphi^2, \tag{1}$$

where the sign $\pm$ refers to each side of the throat with perpendicular radial coordinates $r_{\pm} \geqslant r_0$, respectively. The ζ coordinate represents the polar coordinate $\theta \in [0, \pi)$ in the case of spherical symmetry, or the axial coordinate $z \in \mathbb{R}$ when we consider a cylindrical geometry. In the second case the metric elements depend on $r_{\pm}$; in the first one, the coefficient $g^{\pm}_{\varphi\varphi}$ depends also on θ. As usual, $t \in \mathbb{R}$ and $\varphi \in [0, 2\pi)$. The normal coordinate associated to the radial direction is defined as

$$d\eta = \pm\sqrt{g^{\pm}_{rr}}\, dr_{\pm}, \tag{2}$$

i.e., $(\pm)\eta$ measures the perpendicular proper distance at the vicinities of the throat located at $\eta = 0$, for a static observer; the unit normal vector to the shell is correspondingly defined as $n_\mu = \partial_\mu \eta$, pointing from $-$ to $+$.

Tides acting on a body traversing a wormhole are best described by the covariant relative acceleration usually expressed as [22,23]

$$(\Delta a)^\mu = -g^{\rho\mu} R_{\rho\alpha\nu\beta} V^\alpha (\Delta x)^\nu V^\beta \tag{3}$$

where $R^{\mu}{}_{\alpha\nu\beta}$ is the Riemann tensor, V^μ is the four-velocity, and $(\Delta x)^\mu$ is a vector which stands for the small separation of two points in spacetime.[1] The components of the Riemann tensor can be put in the form[2]

$$\begin{aligned} R_{\mu\alpha\nu\beta} &= \Theta(-\eta) R^{-}_{\mu\alpha\nu\beta} + \Theta(\eta) R^{+}_{\mu\alpha\nu\beta} - \delta(\eta) \left[\kappa_{\alpha\beta}\, n_\mu n_\nu \right. \\ &\quad \left. +\kappa_{\mu\nu}\, n_\alpha n_\beta - \kappa_{\alpha\nu}\, n_\mu n_\beta - \kappa_{\mu\beta}\, n_\alpha n_\nu \right] \end{aligned} \tag{4}$$

where $R^{\mp}_{\mu\alpha\nu\beta}$ is the smooth tensor at each side of the shell and

$$\kappa_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial g^{+}_{\alpha\beta}}{\partial\eta} - \frac{\partial g^{-}_{\alpha\beta}}{\partial\eta}\right)\Big|_{r_0} \tag{5}$$

is the jump in the extrinsic curvature tensor at the shell. Given the symmetry of the problem, the components of the Riemann tensor which are relevant for our analysis are

$$R^{r0}{}_{r0} = -\frac{1}{4 g_{rr} g_{00}} \left\{2 g_{00,rr} - g_{00,r} \frac{(g_{00} g_{rr})_{,r}}{g_{00} g_{rr}}\right\}, \tag{6}$$

$$R^{\varphi 0}{}_{\varphi 0} = -\frac{g_{\varphi\varphi,r}\, g_{00,r}}{4 g_{rr} g_{00} g_{\varphi\varphi}}, \tag{7}$$

$$\begin{aligned} R^{\varphi r}{}_{\varphi r} &= R^{\varphi 0}{}_{\varphi 0} - \frac{1}{4 g_{rr} g_{\varphi\varphi}} \\ &\quad \times \left\{2 g_{\varphi\varphi,rr} - g_{\varphi\varphi,r} \frac{\left(g_{00} g_{\varphi\varphi} g_{rr}\right)_{,r}}{g_{00} g_{\varphi\varphi} g_{rr}}\right\} \end{aligned} \tag{8}$$

and analogous expressions substituting φ by z for cylindrically symmetric problems.

With these definitions, we can recall the result of [1] for the proper tidal acceleration of a radially extended object moving in the radial direction across the throat

$$\Delta a = \left[{R^{r0}{}_{r0}}^{-} + {R^{r0}{}_{r0}}^{+} \right]_{r_0} \frac{\Delta\tilde{\eta}}{2} + \mathcal{O}(\Delta\tilde{\eta}^2) - \kappa^0{}_0, \tag{9}$$

where $\kappa^0{}_0$ is the jump of the component $K^0{}_0$ of the extrinsic curvature across the infinitely thin shell, and $\Delta\tilde{\eta}$ is the proper radial separation between points of the object traversing the throat; this reduces to $\Delta\eta$ for a rest body, which is in the radial direction. The smooth regions contribute with a term proportional to $\Delta\tilde{\eta}$ which typically describes a tension exerted on the body. The jump in the extrinsic curvature, instead, adds a finite contribution which is fixed for a given geometry, so that it does not vanish for infinitely close points at different

[1] The acceleration is exclusively given by the derivatives of the metric function, in terms of the Riemann tensor, if the test body is not coupled to any other field.

[2] This corresponds to the standard result, as it is derived in [22], for the Riemann tensor in the context of the thin-shell formalism. The first derivatives of the metric carries two terms proportional to delta functions with different sign, arising from $\Theta'(\pm\eta) = \pm\delta(\eta)$, which cancel with each other.

sides of the throat; this implies not a tension but a compression, given that a point at each side of the throat is usually attracted towards it. This particular nature of tides reflects the discontinuous character of the gravitational field (i.e. of the first derivatives of the metric) at r_0; it suggests that such kind of quite generic wormhole geometry, though traversable in principle, presents the practical problem of great tides acting on a body extended across the throat.

In an analogous way, we can write down the result in [1] for the angular tide of a radially moving object, which is conveniently decomposed as a finite part plus a divergent one:

$$\Delta a_\perp = \Delta a_\perp^{finite} + \Delta a_\perp^{div}, \tag{10}$$

where

$$\Delta a_\perp^{finite} = \frac{\Delta x_\perp}{2}\left[{R^{\varphi 0}}_{\varphi 0}^{\;-} + \gamma^2\beta^2 \left({R^{\varphi 0}}_{\varphi 0}^{\;-} - {R^{\varphi r}}_{\varphi r}^{\;-} \right)\right]_{r_0} + \frac{\Delta x_\perp}{2}\left[{R^{\varphi 0}}_{\varphi 0}^{\;+} + \gamma^2\beta^2 \left({R^{\varphi 0}}_{\varphi 0}^{\;+} - {R^{\varphi r}}_{\varphi r}^{\;+} \right)\right]_{r_0} \tag{11}$$

and introducing the infinitely-short travelling proper time $\delta\tau$ of the object across the shell,

$$(\Delta a_\perp)^{div} = \Delta x_\perp \frac{\gamma\beta\, {\kappa^\varphi}_\varphi}{\delta\tau}. \tag{12}$$

Expressions with the coordinate z instead of φ hold for the axial tide in the case of a problem with symmetry along the z axis. Here, as usual in similar contexts, we use the velocity $V^{\hat\mu} = dx^{\hat\mu}/d\tau = (\gamma, \gamma\beta, 0, 0)$ as measured in an orthonormal frame $\{\vec{e}_{\hat\mu}\}$ at rest at the vicinities of the throat to define the parameters $\gamma = 1/\sqrt{1-\beta^2}$ and the positive radial speed β of the object as measured in the orthonormal frame. The divergent character of the result is apparent because of the proper time $\delta\tau$, which goes to zero for an infinitely thin shell, in the denominator. Differing from the case of radial tides, for tides in transverse directions to the radial one both contributions are proportional to the transverse extension $\Delta x_\perp$ of the object. For an angular tide, the finite term implies a compression produced by the curvature of the smooth parts of the geometry, while the divergent term corresponds to a stretching effect resulting from the jump in the extrinsic curvature produced by the flare-out at the shell. This serious difficulty could be avoided only in the case of a negligible velocity, or if the geometry is chosen so that it presents no curvature jump at the throat.

3 Scalar–tensor theories

Now we go beyond the analysis of simple corrections to the metric functions, by leaving the relativistic framework. We will consider two kinds of scalar–tensor gravity models: the low energy limit of bosonic string theory, yielding the so-called dilaton gravity, and the well known Brans–Dicke theory in which the gravitational constant is replaced by a field whose source is standard matter, and which together with it, determines the spacetime geometry. We will not perform a complete analysis of many cases, but we will restrict to two simple configurations useful to exemplify different aspects of the problem and possible suitable treatments. The central role in our analysis will be that of the extrinsic curvature tensor: these components are associated to the contribution to the radial tide which does not vanish for infinitely close points, and to the terms in the transverse tides which are formally divergent. Problems in the smooth part can be more easily controlled by suitably choosing certain parameters of the problem (see [1]). However we will also examine how the conditions on the extrinsic curvature are reflected in the behaviour of the tides in these regions.

3.1 Dilaton gravity

The dilaton spherically symmetric black hole in the Einstein frame is described by the metric [7–10]

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + h(r)(d\theta^2 + \sin^2\theta d\varphi^2), \tag{13}$$

with functions

$$f(r) = \left(1 - \frac{A}{r}\right)\left(1 - \frac{B}{r}\right)^{\frac{1-b^2}{1+b^2}},$$
$$h(r) = r^2\left(1 - \frac{B}{r}\right)^{\frac{2b^2}{1+b^2}}. \tag{14}$$

The constants A, B and the parameter b are related with the mass and electric charge of the black hole by

$$M = \frac{A}{2} + \left(\frac{1-b^2}{1+b^2}\right)\frac{B}{2},$$
$$Q = \sqrt{\frac{AB}{1+b^2}}. \tag{15}$$

The electromagnetic field tensor has non-null components $F_{tr} = -F_{rt} = Q/r^2$, and the dilaton field is given by

$$e^{2\phi} = \left(1 - \frac{B}{r}\right)^{2b/(1+b^2)}, \tag{16}$$

where the asymptotic value of the dilaton ϕ_0 is taken as zero. When $b = 0$, which corresponds to a uniform dilaton, the metric reduces to the Reissner–Nordström geometry, while for $b = 1$, one obtains $f(r) = 1 - 2M/r$, $h(r) = r^2\left[1 - Q^2/(Mr)\right]$. In what follows, we shall consider $0 \le b \le 1$. B and A are, respectively, the inner and outer horizons of the black hole; while the outer horizon is a regular event horizon for any value of b, the inner one is singular for any $b \neq 0$. For such metric, the relevant components of the Riemann tensor are:

$$R^{r0}{}_{r0} = -\frac{f''(r)}{2} \tag{17}$$

$$R^{\varphi 0}{}_{\varphi 0} = -\frac{f'(r)h'(r)}{4h(r)} \tag{18}$$

$$R^{\varphi r}{}_{\varphi r} = R^{\varphi 0}{}_{\varphi 0} - \frac{f(r)}{4h(r)}\left\{2h''(r) - \frac{h'^2(r)}{h(r)}\right\} \tag{19}$$

where we have adopted the notation of a prime for a derivative with respect to the radial coordinate.

If the flare-out condition $h'(r_0) > 0$ is fulfilled for $r > r_0$ (with r_0 a radius outside the larger horizon surface), two copies of the exterior region can be joined at a thin shell to define a Lorentzian wormhole. This thin-shell wormhole geometry was considered in [11], where the explicit construction was done according to the Einstein equations on the shell (Lanczos equations [24–29]). The induced surface stress tensor $S^i{}_j$ at the throat has the usual form: $8\pi S^i{}_j = \kappa\delta^i{}_j - \kappa^i{}_j$, with κ the trace of the extrinsic curvature jump (see [11] and references therein). The jump of the components of the extrinsic curvature at the shell (second fundamental form) is given by

$$\kappa^\theta{}_\theta = \kappa^\varphi{}_\varphi = \frac{h'(r_0)\sqrt{f(r_0)}}{h(r_0)} \tag{20}$$

and

$$\kappa^0{}_0 = \frac{f'(r_0)}{\sqrt{f(r_0)}}. \tag{21}$$

The latter results in a non-null energy density and transverse pressures at the throat, as analyzed extensively in [11]. The boundary condition for the scalar field is trivially satisfied for the kind of mathematical construction we are dealing with: from the field equation $\phi^{;\mu}{}_{;\mu} = -\frac{b}{2}e^{-2b\phi}F^2$, it is clear that no jump in the first derivative of ϕ can result given the finite character of $F_{\mu\nu}$ and ϕ at the joining surface (see footnote 6 below).

Let us analyze the tides in both the radial and the angular directions at the throat of a thin-shell wormhole. As we noted above, the central problem comes from the extrinsic curvature jump, as the contributions from smooth parts are finite and proportional to the separation between points. In the radial case this jump leads to a finite, non-vanishing tide between infinitely close points, while in the angular direction the curvature jump implies a divergent tide. In a practical approach, one would always be interested in reducing the corresponding extrinsic curvature jumps $\kappa^0{}_0$ and $\kappa^\varphi{}_\varphi$. Then we will evaluate these elements and examine the behaviour of the quotients ${\kappa^0{}_0}^{Dil}/{\kappa^0{}_0}^{RN}$ and ${\kappa^\varphi{}_\varphi}^{Dil}/{\kappa^\varphi{}_\varphi}^{RN}$, where "Dil" stands for "dilaton" and "RN" refers to the pure relativity limit $b \to 0$ corresponding to the Reissner–Nordström geometry. If we simplify the notation by writing $\frac{1-b^2}{1+b^2} = n$ and $\frac{2b^2}{1+b^2} = m$, for the jump associated to the tide in the radial direction we have

$$\begin{aligned}{\kappa^0{}_0}^{Dil} &= \frac{f'(r_0)}{\sqrt{f(r_0)}} = \frac{1}{r_0^2}\left(1 - \frac{A}{r_0}\right)^{-\frac{1}{2}}\left(1 - \frac{B}{r_0}\right)^{\frac{n}{2}-1} \\ &\times\left[A\left(1 - \frac{B}{r_0}\right) + nB\left(1 - \frac{A}{r_0}\right)\right]\end{aligned} \tag{22}$$

and

$$\frac{{\kappa^0{}_0}^{Dil}}{{\kappa^0{}_0}^{RN}} = \left(\frac{r_0 - B}{r_0}\right)^{\frac{n}{2}-\frac{1}{2}}\frac{(A + nB)r_0 - AB(n+1)}{(A+B)r_0 - 2AB}. \tag{23}$$

To understand this result let us consider two limits: a throat very near the outer horizon, so that $r_0 \to A$, and a throat radius very large compared with the horizon radius, for what we take $r_0 \to \infty$. For $r_0 \to A$, both in the dilaton as in the pure relativity cases the curvature jump involved in radial tides becomes very large, and we have

$$\frac{{\kappa^0{}_0}^{Dil}}{{\kappa^0{}_0}^{RN}} \to \left(\frac{A - B}{A}\right)^{\frac{n}{2}-\frac{1}{2}} > 1 \tag{24}$$

because $A - B < A$ and for $b > 0$ it is $n < 1$, and the dilaton would not improve the situation with radial tides; in the other limit, for $r_0 \to \infty$ we obtain

$$\frac{{\kappa^0{}_0}^{Dil}}{{\kappa^0{}_0}^{RN}} \to \frac{A + nB}{A + B} < 1. \tag{25}$$

Therefore the traversability, at least for radially extended objects, is improved in the dilaton framework if the throat is very far from the outer horizon. Apart from these two limits, one could wonder about the situation for intermediate regions. In the particular case $b = 1$, which corresponds to $n = 0$, we have

$$\frac{{\kappa^0{}_0}^{Dil\,(b=1)}}{{\kappa^0{}_0}^{RN}} = \left(\frac{r_0 - B}{r_0}\right)^{1/2}\frac{1}{1 + B((r_0 - 2A)/Ar_0)}, \tag{26}$$

and it is clear that for any $r_0 > 2A$ the quotient is smaller than unity. This is a sufficient condition, but not a necessary one: in fact, both the throat radius and the inner horizon radius could be scaled in terms of the outer horizon by writing $r_0 = \alpha_1 A$, $\alpha_1 > 1$ and $B = \alpha_2 A$, $\alpha_2 < 1$ to show that the condition to have a better situation with the dilaton field can be written

$$\frac{{\kappa^0{}_0}^{Dil\,(b=1)}}{{\kappa^0{}_0}^{RN}} = \left(\frac{\alpha_1 - \alpha_2}{\alpha_1}\right)^{1/2}\frac{\alpha_1}{\alpha_1 + \alpha_2(\alpha_1 - 2)} < 1. \tag{27}$$

Now, one may wonder about the possibility that the improvement of the radial tide across the wormhole throat is achieved at the price of worsening the situation in the smooth regions of the geometry. According to the general analysis of Sect. 2, answering this question requires to evaluate the behaviour of only one component of the Riemann tensor:

$$\frac{R^{r0}{}_{r0}{}^{Dil}}{R^{r0}{}_{r0}{}^{RN}} = \frac{(1-B/r_0)^n r_0(AB^2(1+n)(2+n) - B(1+n)(4A+Bn)r_0 + 2(A+Bn)r_0^2)}{2(B-r_0)^2(-3AB+(A+B)r_0)}. \tag{28}$$

We immediately see that in the convenient limit $r_0 \to \infty$ we have

$$\frac{R^{r0}{}_{r0}{}^{Dil}}{R^{r0}{}_{r0}{}^{RN}} \to \frac{A+nB}{A+B} < 1. \tag{29}$$

For any r_0, in the particular case $b = 1$ ($n = 0$) considered above we have

$$\frac{R^{r0}{}_{r0}{}^{Dil(b=1)}}{R^{r0}{}_{r0}{}^{RN}} = \frac{Ar_0}{Br_0 + A(-3B + r_0)} \tag{30}$$

and we find that for $\alpha_1 > 3$ we obtain

$$\left|\frac{R^{r0}{}_{r0}{}^{Dil(b=1)}}{R^{r0}{}_{r0}{}^{RN}}\right| = \frac{\alpha_1}{\alpha_1 + \alpha_2(\alpha_1 - 3)} < 1. \tag{31}$$

For these cases the improvement in the curvature jump associated to the radial tide across the throat is then achieved in correspondence with a reduction of the radial tide in the smooth parts of the geometry.

For the jump determining the formally divergent tide in the angular direction we have the analogous expressions

$$\kappa^{\varphi}{}_{\varphi}{}^{Dil} = \frac{h'(r_0)\sqrt{f(r_0)}}{h(r_0)} = \frac{1}{r_0^2}\left(1-\frac{A}{r_0}\right)^{\frac{1}{2}}\left(1-\frac{B}{r_0}\right)^{\frac{n}{2}-1} \times\left[2r_0\left(1-\frac{B}{r_0}\right) + mB\right] \tag{32}$$

and

$$\frac{\kappa^{\varphi}{}_{\varphi}{}^{Dil}}{\kappa^{\varphi}{}_{\varphi}{}^{RN}} = \left(\frac{r_0 - B}{r_0}\right)^{\frac{n}{2}-\frac{1}{2}} \frac{2(r_0-B)+mB}{2(r_0-B)}. \tag{33}$$

Now we can note that for any $n < 1$ and $m > 0$ both factors in the product above are larger than unity, as long as r_0 satisfies the natural condition $r_0 > A > B$. Hence no parameter choice in the dilaton model allows to improve the situation with the angular tide in relation with the pure relativistic framework.

3.2 Brans–Dicke gravity

As an example of the problem of tides across shells within the framework of Brans–Dicke scalar–tensor gravity, we consider the cylindrically symmetric wormholes connecting submanifolds of the form

$$ds^2 = -f(r)dt^2 + g(r)dr^2 + h(r)d\varphi^2 + k(r)dz^2, \tag{34}$$

where f, g, h and k are positive functions. Two copies of the outer region $r \geq r_0$ of this geometry joined at the hypersurface $r = r_0$ constitute a Lorentzian wormhole as long as the flare-out condition is satisfied. For cylindrical configurations we can adopt two definitions of such condition: the usual in which the area of a surface at the throat is minimum, or the circular notion proposed in [30], defined by the existence of a minimum of the function $h(r)$ at $r = r_0$ (see also [31]). To examine the radial and transverse (angular and axial) tides for a radially moving object we will need the following components of the Riemann tensor, generically written as

$$R^{r0}{}_{r0} = -\frac{1}{4f(r)g(r)}\left\{2f''(r) - f'(r)\frac{(f(r)g(r))'}{f(r)g(r)}\right\}, \tag{35}$$

$$R^{\varphi 0}{}_{\varphi 0} = -\frac{f'(r)h'(r)}{4f(r)g(r)h(r)}, \tag{36}$$

$$R^{\varphi r}{}_{\varphi r} = R^{\varphi 0}{}_{\varphi 0} - \frac{1}{4g(r)h(r)} \times\left\{2h''(r) - h'(r)\frac{(f(r)g(r)h(r))'}{f(r)g(r)h(r)}\right\}, \tag{37}$$

and analogous expressions with $k(r)$ instead of $h(r)$ for the components $R^{z0}{}_{z0}$ and $R^{zr}{}_{zr}$.

The Brans–Dicke equations on the shell are obtained, in the Jordan frame, from the field equation $\phi^{;\mu}{}_{;\mu} = 8\pi T/(2\omega+3)$ relating the second derivatives of ϕ with the trace of the stress tensor T, and the Einstein equations at the shell. The first leads to the boundary condition $\langle\phi_{,N}\rangle = 8\pi S/(2\omega+3)$ (see Refs. [32–34]), where N stands for the direction normal to the surface of the throat, S is the trace of the jump of the surface stress tensor, and ω is the Brans–Dicke constant.[3] The Lanczos equations relate the extrinsic curvature jump, the field ϕ and the surface matter content at the throat as $8\pi\left(S^i{}_j - \delta^i{}_j S/(2\omega+3)\right)\phi^{-1} = \kappa\delta^i{}_j - \kappa^i{}_j$. For the cylindrically symmetric solutions, the jump of the components of the extrinsic curvature across the shell are given by

$$\kappa^0{}_0 = \frac{f'(r_0)}{f(r_0)\sqrt{g(r_0)}}, \tag{38}$$

$$\kappa^{\varphi}{}_{\varphi} = \frac{h'(r_0)}{h(r_0)\sqrt{g(r_0)}}, \tag{39}$$

and

$$\kappa^z{}_z = \frac{k'(r_0)}{k(r_0)\sqrt{g(r_0)}}. \tag{40}$$

[3] The equivalence principle is satisfied as the scalar field does not exert any direct influence on the test particle; its only role is that of participant in the field equations that determine the geometry of spacetime [35].

We will carry out a detailed analysis for a class of geometries previously considered in [36,37] given by[4]

$$f(r) = g(r) = r^{2d(d-n)+\Omega(\omega)}, \quad (41)$$

$$h(r) = W^2 r^{2(n-d)}, \quad (42)$$

$$k(r) = r^{2d}. \quad (43)$$

Here $\Omega(\omega) = [\omega(n-1)+2n](n-1)$, with ω the Brans–Dicke constant (we shall assume $\omega > -3/2$ to avoid what could be interpreted as a negative effective gravitational constant; see [23]) and d and n are constants of integration: n is related to the departure from pure general relativity – see below – and d can be understood as a mass parameter; the case $d = 0$ within Einstein's gravity describes a conical geometry. The Brans–Dicke field has the behaviour

$$\phi = \phi_0 r^{1-n}. \quad (44)$$

This geometry can be found in Ref. [36],[5] and can be obtained as the zero current limit of the magnetic solution in Ref. [38]; see also [39]. The Einstein's gravity solutions (see for example [40] and also [41]) are obtained for $n = 1$. The usual areal flare-out condition demands $\left(\sqrt{h(r)k(r)}\right)' > 0$ at the throat, while the circular condition requires $\left(\sqrt{h(r)}\right)' > 0$. In the first case we would say that we have a throat at r_0 as long as $n > 0$, while in the second case the inequality $n > d \geq 0$ should hold. In what follows we will restrict to $0 \leq d < 1$ to avoid a circumference decrease for increasing r in the relativistic limit. The components of the jump of the extrinsic curvature at the shell (second fundamental form) take the form

$$\kappa^0{}_0 = \frac{2d(d-n) + \Omega(\omega, n)}{r_0^{d(d-n)+1+\Omega(\omega,n)/2}}, \quad (45)$$

$$\kappa^\varphi{}_\varphi = \frac{2(n-d)}{r_0^{d(d-n)+1+\Omega(\omega,n)/2}}, \quad (46)$$

$$\kappa^z{}_z = \frac{2d}{r_0^{d(d-n)+1+\Omega(\omega,n)/2}}. \quad (47)$$

The relativity limit $n = 1$ reduces to

$$\kappa^0{}_0 = \frac{2d(d-1)}{r_0^{d(d-1)+1}}, \quad (48)$$

$$\kappa^\varphi{}_\varphi = \frac{2(1-d)}{r_0^{d(d-1)+1}}, \quad (49)$$

$$\kappa^z{}_z = \frac{2d}{r_0^{d(d-1)+1}}. \quad (50)$$

The second fundamental form determines the presence of a surface energy density and pressures at the throat for the static wormhole configuration. The stress tensor components were analyzed in detail in [37]. Besides, as mentioned previously, the relation between the jump of the scalar field across the surface and the matter imposes the condition [37]

$$2\omega\frac{\phi'(r_0)}{\phi(r_0)} = \frac{f'(r_0)}{f(r_0)} + \frac{h'(r_0)}{h(r_0)} + \frac{k'(r_0)}{k(r_0)}, \quad (51)$$

which implies a constraint involving the throat radius r_0 and all the parameters that must be fulfilled by the wormhole construction from a given metric.[6] In this particular example the condition turns to give only the relation

$$\omega(1-n^2) = 2(d^2 + n^2 - nd) \quad (52)$$

which can be used to write, for example, ω in terms of the parameters n and d. We then obtain

$$\kappa^0{}_0 = \frac{4d(d-n) + 2n(n-1)}{(n+1)r_0^{1+(2d(d-n)+n(n-1))/(n+1)}}, \quad (53)$$

$$\kappa^\varphi{}_\varphi = \frac{2(n-d)}{r_0^{1+(2d(d-n)+n(n-1))/(n+1)}}, \quad (54)$$

$$\kappa^z{}_z = \frac{2d}{r_0^{1+(2d(d-n)+n(n-1))/(n+1)}}. \quad (55)$$

Note that for $d = 0$, which in the pure relativistic framework would correspond to a wormhole connecting two conical geometries, the component $\kappa^z{}_z$ determining the axial tide vanishes.

Now the question would be how does each component behave with the parameter n determining the departure from pure relativity. Does $n \neq 1$ reduce the curvature jump? The answer will, in principle, depend on the throat radius r_0. A straightforward way to perform the analysis could be to take $\kappa^0{}_0$ and $\kappa^\varphi{}_\varphi$ as functions of the parameter n and study the derivatives at $n = 1$: the relativity (Rel) limit would give the best conditions for traversability only if for $n = 1$ we find a radius r_0 such that there is a minimum of the elements of the curvature jump. If this is not the case, the sign of the first derivative of each component will indicate whether we must consider a choice of $n < 1$ or $n > 1$ within the Brans–Dicke (BD) gravity framework. This also seems to be the more

[4] To keep dimensions consistent, the radial coordinate is defined as $r = \rho/\rho^*$, where ρ^* can be understood as a "core" radius fixing the scale; in what follows we consider $\rho > \rho^*$, then $r \geqslant r_0 > 1$, and work with the dimensionless line element $ds^2 = (d\tilde{s}/\rho^*)^2$, where $z \in \mathbb{R}$ and W^2 is a dimensionless constant.

[5] There is a typo in Eq. (31) of that paper, where a factor $(n-1)$ multiplying ω is missing, as can be deduced by comparison with the immediately preceding equations.

[6] The jump of the normal derivative of ϕ comes from the fact that the field equations of Brans–Dicke gravity relate the second derivatives of the field to the trace of the energy-momentum tensor, which is singular at $r = r_0$. This is not the case in dilaton gravity, where the field equations establish a proportionality of the second derivatives of the scalar field with the trace of the electromagnetic field. Therefore no singular contribution appears associated to the existence of an infinitely thin matter layer, the normal derivative of the dilaton field is continuous across the shell, and this is automatically satisfied so no related restrictions result.

suitable approach for more general geometries in the same framework; however, for this example we can obtain a direct first insight in the dependence with n by writing quotients in the same way of the preceding section:

$$\frac{{\kappa^0}_0{}^{BD}}{{\kappa^0}_0{}^{Rel}} = \frac{[2d(d-n)+n(n-1)]\, r_0^{(d(n-1)(d+1)-!n(n-1))/(n+1)}}{d(d-1)(n+1)}, \tag{56}$$

$$\frac{{\kappa^\varphi}_\varphi{}^{BD}}{{\kappa^\varphi}_\varphi{}^{Rel}} = \frac{(d-n) r_0^{(d(n-1)(d+1)-n(n-1))/(n+1)}}{(d-1)}, \tag{57}$$

$$\frac{{\kappa^z}_z{}^{BD}}{{\kappa^z}_z{}^{Rel}} = r_0^{(d(n-1)(d+1)-n(n-1))/(n+1)}. \tag{58}$$

We immediately see that for large n the quotient of radial tides behaves in the form $\sim -nr_0^{-n}$, for the angular tide it is of the form $\sim nr_0^{-n}$, while for the axial tide it is of the form $\sim r_0^{-n}$. Because $r_0 > 1$, the ratios vanish in the limit $n \to \infty$ being the axial case the faster one. Hence, for all possible orientations, the tides are weaker in the case of a very large departure from pure relativity; the Brans–Dicke framework thus constitutes a better situation in what regards traversing the throat of this kind of wormhole geometry. The behaviour of the radial tide for a radially moving object in the smooth regions of the spacetime is inferred from the quotient

$$\frac{{R^{r0}}_{r0}{}^{BD}}{{R^{r0}}_{r0}{}^{Rel}} = \frac{(2d^2-2dn+(n-1)n) r_0^{2(d(n-1)(d+1)-n(n-1))/(n+1)}}{(d-1)d(n+1)}. \tag{59}$$

For large n this behaves as $\sim -nr_0^{-2n}$, so it vanishes for $n \to \infty$. The dependence of the angular tide in any of the two smooth submanifolds joined at the throat can be described by the quotient of the finite parts:

$$\frac{\Delta a_{\perp(\varphi)}^{finite}|^{BD}}{\Delta a_{\perp(\varphi)}^{finite}|^{Rel}} = \frac{(d-n)(d^2(2+4\gamma\beta)+(n-1)(n+(n-1)\gamma\beta)+d(\gamma\beta-n(2+3\gamma\beta)))}{(d-1)d(1+n)(d-1-\gamma\beta+2d\gamma\beta)} \times r_0^{2(d(n-1)(d+1)-n(n-1))/(n+1)}, \tag{60}$$

with the velocity dependent parameters β and γ defined as above. The large n behaviour has the form $\sim n^2 r_0^{-2n}$, and vanishes in the limit $n \to \infty$. For the tide along the symmetry axis we have

$$\frac{\Delta a_{\perp(z)}^{finite}|^{BD}}{\Delta a_{\perp(z)}^{finite}|^{Rel}} = \frac{(\gamma\beta+d^2(2+4\gamma\beta)+n(n-1-\gamma\beta+2n\gamma\beta)-d(\gamma\beta+n(2+5\gamma\beta)))}{(d-1)(n+1)(d-\gamma\beta+2d\gamma\beta)}\; r_0^{2(d(n-1)(d+1)-n(n-1))/(n+1)}. \tag{61}$$

For large n we find a behaviour $\sim nr_0^{-2n}$. Summarizing: for this example, in Brans–Dicke gravity the reduction of tides in the smooth regions of the geometry is in correspondence with the improvement of tides across the throat.

4 Discussion

If taken literally, previous results [1] about tides across the throat of thin-shell wormholes would rule out a safe travel from one side to the other: as a result of the jump of the extrinsic curvature at the throat, angular tides are formally divergent, and radial tides lead to a finite but non vanishing relative acceleration between two infinitely close points. However, an alternative more realistic interpretation of the formal results as an approximation to wormholes supported by exotic matter shells of little but non vanishing thickness is a reasonable possibility. Such point of view naturally leads to the problem of determining the conditions to reduce the jump of the components of the extrinsic curvature across the shell. We have examined the consequences of extending the traversability analysis to frameworks beyond relativity, as the scalar–tensor theories known as dilaton gravity and Brans–Dicke gravity. We have studied particular examples with spherical symmetry for the first, and with cylindrical symmetry for the second. We have considered certain limits and shown that in the dilaton example no general improvement is possible, as the dependence of angular tides with the departure from pure relativity is opposite to that of radial tides; however, in the case in which a reduction of the extrinsic curvature jump across the throat is possible, the suitable configurations also reduce the corresponding tide in the smooth parts of the geometry. In the Brans–Dicke cylindrical example, instead, a large departure from relativity seems to improve the situation with both transverse (i.e. angular and axial) and radial tides; moreover, this is in correspondence with a reduction of tides in the smooth regions of the geometry. An alternative analysis of other spherically and cylindrically symmetric examples within the relativistic framework can also be carried out. Indeed, in the appendices we consider the behaviour of tides in relation with certain parameters associated to the sources determining the geometries, both from the point of view of the extrinsic curvature jump, as also in an approximate more elementary study of the radial relative acceleration between points at different sides of the wormhole throat.

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Appendix A: Corrections in spherical and cylindrical symmetry

Let us begin studying the consequences of additional terms in the metric coefficients of geometries with a given symmetry. Take the spherically symmetric geometry

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{62}$$

so that from two copies of it we construct a symmetric wormhole with extrinsic curvature jump at the throat given by

$$\kappa^{\varphi}{}_{\varphi} = \frac{2}{r_0}\sqrt{f(r_0)} \tag{63}$$

and

$$\kappa^{0}{}_{0} = \frac{f'(r_0)}{\sqrt{f(r_0)}}. \tag{64}$$

If we introduce a correction to the geometry changing $f(r)$ to $f^*(r) = f(r) + q(r)$, we will have the new jumps

$$\kappa^{\varphi}{}_{\varphi}{}^{*} = \frac{2}{r_0}\sqrt{f(r_0) + q(r_0)} \tag{65}$$

and

$$\kappa^{0}{}_{0}{}^{*} = \frac{f'(r_0) + q'(r_0)}{\sqrt{f(r_0) + q(r_0)}}. \tag{66}$$

From this we immediately see that a negative $q(r)$ at r_0 improves the situation with the angular tide, while a positive addition to the original metric makes things worse in this direction. However, with the component $\kappa^0{}_0$ determining the radial tide the situation presents more possibilities: the case $q(r_0) > 0$ and $q'(r_0) < 0$ diminishes the radial tide, while the opposite happens for the reverse signs, and a particular analysis is required by the other two cases. A clear example is provided by the Reissner–Nordström (RN) wormhole [42], with the geometry at each side understood as a correction Q^2/r^2 added to the Schwarzschild (Sch) metric function $f(r) = 1 - 2M/r$:

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{67}$$

with

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \tag{68}$$

where Q is the electric charge. We then have

$$\kappa^{\varphi}{}_{\varphi}{}^{RN} = \frac{2}{r_0}\sqrt{f(r_0)} = \frac{2}{r_0}\sqrt{1 - \frac{2M}{r_0} + \frac{Q^2}{r_0^2}}, \tag{69}$$

and

$$\kappa^{0}{}_{0}{}^{RN} = \frac{f'(r_0)}{\sqrt{f(r_0)}} = \frac{2}{r_0^2}\left(\frac{Mr_0 - Q^2}{\sqrt{r_0^2 - 2Mr_0 + Q^2}}\right). \tag{70}$$

A direct inspection shows that the Schwarzschild case ($Q = 0$) gives a smaller jump $\kappa^{\varphi}{}_{\varphi}$ to which the angular tide is proportional. But the relation is reversed for $\kappa^0{}_0$, which is larger for $Q = 0$. Hence the addition of the charge improves the situation for the radial tide, but makes things worse for the angular tide. Note that, for a certain throat radius r_0, a decrease of $\kappa^{\varphi}{}_{\varphi}$ must necessarily come from a smaller value of $f(r_0)$, which in turn implies an increase of $\kappa^0{}_0$; this can only be avoided by an appropriate decrease of $f'(r_0)$.

A very similar analysis can be carried out in cylindrically symmetric problems. Let us study the example associated to a black string thin-shell wormhole [43]. In this case the throat connects two exterior submanifolds with metrics (see Ref. [44])

$$\begin{aligned} ds^2 = &-\left(\alpha^2 r^2 - \frac{4m}{\alpha r} + \frac{4\lambda^2}{\alpha^2 r^2}\right)dt^2 \\ &+\left(\alpha^2 r^2 - \frac{4m}{\alpha r} + \frac{4\lambda^2}{\alpha^2 r^2}\right)^{-1}dr^2 \\ &+r^2 d\varphi^2 + \alpha^2 r^2 dz^2, \end{aligned} \tag{71}$$

where m and λ are respectively the mass and charge per unit length, and $\alpha^2 = -\Lambda/3 > 0$ with $\Lambda < 0$ the cosmological constant. This geometry is singular at the axis of symmetry and depending on the values of these constants, the metric (71) can present an event horizon at $r = r_h$, which justifies the *black string* denomination. Both the singularity and the horizon are removed by the mathematical construction which pastes two copies of the outer region $r \geq r_0 > r_h$. The jump of the components of the extrinsic curvature across the shell at the throat at $r = r_0$ are

$$\kappa^0{}_0 = \frac{2}{\alpha r_0^2}\left(\frac{\alpha^4 r_0^4 + 2m\alpha r_0 - 4\lambda^2}{\sqrt{\alpha^4 r_0^4 - 4m\alpha r_0 + 4\lambda^2}}\right), \tag{72}$$

$$\kappa^\varphi{}_\varphi = \frac{2}{r_0}\sqrt{\alpha^2 r_0^2 - \frac{4m}{\alpha r_0} + \frac{4\lambda^2}{\alpha^2 r_0^2}}, \tag{73}$$

$$\kappa^z{}_z = \kappa^\varphi{}_\varphi. \tag{74}$$

We immediately find that the charge improves the situation with the radial tide by reducing the extrinsic component jump $\kappa^0{}_0$, but it enlarges the troublesome contributions to the transverse tides determined by the components $\kappa^\varphi{}_\varphi$ and $\kappa^z{}_z$.

In general, we can pose the problem as follows: write the metric coefficients as functions of r_0 and a parameter p, and study the behaviour of the derivatives $\partial\kappa^\alpha{}_\beta(r_0, p)/\partial p$; the best possible situation will be given in case that a parameter p exists such that all these derivatives (two in the spherical problem and three in the cylindrical one) have the same sign for r_0 an admissible throat radius. Then a suitable choice of p allows to reduce tides in the radial and the transverse directions. When such possibility does not exist, as with the charge in the Reissner–Nordström and black string examples, one must select which is the most convenient direction to reduce the corresponding tide. Of course, these considerations can easily be extended to more general metrics and to more than one parameter.

Appendix B: A fine tuning approach

The nature of the problems manifest in the infinitely thin shell limit suggests a somewhat different analysis; the point is that instead of assumptions about the dimensions of shells involved, or the search for ways to reduce the extrinsic curvature jumps, for negligible speeds we can try a fine tuning which, of course, is not always desirable as a rule. For radial tides on rest or moving objects the central point in our traversability analysis is the necessity of a surmountable relative acceleration between two infinitely close points separated by the wormhole throat. While this is automatically achieved in locally flat geometries or in those spacetimes with constant g_{00} and, thus, null acceleration, another possibility could be to admit two geometries connected at a radius r_0 such that just at each side we have a locally vanishing acceleration. The condition for a null acceleration at each side implies $g'_{00}(r_0) = 0$. The fact that near r_0 no problems arise associated to jumps of the acceleration is ensured by the assumption that the submanifolds joined at the wormhole throat are well behaved; as a practical rule, in order to avoid large, though smooth, changes in g'_{00} in the same vicinity, we could include the condition of a small second derivative g''_{00}. Of course, nothing in these considerations points to solving or at least improving the difficulties with angular tides; at this point we should just rely on a quasi static crossing through the throat.

Let us analyze some examples. In spherically symmetric backgrounds, such behaviour of the metric is not possible, for example, for finite radii in a Schwarzschild geometry. But consider, again, the thin-shell wormhole connecting two Reissner–Nordström geometries [42], now within this approach. In this case the condition $g'_{00}(r_0) = -f'(r_0) = 0$ is fulfilled at

$$r_0 = \frac{Q^2}{M}; \tag{75}$$

then a wormhole with such throat radius would present no tidal problems, at least at the minimal area surface. Note that if we insist in $|Q| < M$, which is usually required to avoid a naked singularity in the complete Reissner–Nordström geometry, this is not compatible with the condition of r_0 greater than the event horizon radius $r_h = M + \sqrt{M^2 - Q^2}$. Hence we should start this wormhole construction from two geometries including a naked singularity; however, this is not a problem within the context in which the regions $r < r_0$ are removed in the cut and paste mathematical construction.

In cylindrically symmetric backgrounds we have a similar situation: while the power-law behaviour of, for example, the well known Levi-Civita metric excludes the possibility to achieve the condition $g'_{00} = 0$, more general axisymmetric wormholes as those associated to Einstein–Maxwell spacetimes (see [31] and also [40,41]) could provide the freedom necessary to fulfil the conditions required, for certain values of the parameters. The pure electric case corresponding to a constant charge distribution along the symmetry axis and a radial electric field does not allow for $g'_{00} = 0$. However, let us consider the geometry associated to a constant current along the z axis giving an angular magnetic field, that is

$$ds^2 = r^{2m^2} G^2(r)(-dt^2 + dr^2) + r^2 G^2(r) d\varphi^2 + G^{-2} dz^2 \tag{76}$$

with

$$G(r) = k_1 r^m + k_2 r^{-m} \tag{77}$$

and m, k_1, k_2 constants such that $k_1 k_2 > 0$. This metric admits $g'_{00} = 0$ at a radius r_0 satisfying

$$r_0^{2m} = \frac{k_2(1 - m r_0)}{k_1(1 + m r_0)}; \tag{78}$$

given the requirement $k_1 k_2 > 0$, then there are two possibilities: $m < 0$, $-m r_0 < 1$ and $m > 0$, $m r_0 < 1$. If two copies of the outer part of such spacetime are joined at r_0, then the resulting thin-shell wormhole would have no traversability problems coming from radial strong tides at the throat. In general, one should also verify the flare-out condition at the radius r_0 (that is, that we have a minimal area per unit length or minimal radius surface at $r = r_0$); in this pure magnetic case the mostly adopted areal version of this condition is automatically fulfilled, as a direct inspection shows. Note, however, that besides the fact that the angular tides are not solved, differing from the gauge cosmic string example where $g_{zz} = 1$ ensured the absence of problems associated with tides in the direction parallel to the axis, here we have $g_{zz} = G^{-2}(r) \neq 1$; this implies a new difficulty which cannot be simultaneously avoided. In fact, we must choose between two directions which will be the safe one: to avoid dangerous tensions or pressures along the z axis we should demand (see below a strongly related calculation within the Brans–Dicke framework) $g'_{zz}(r_0^*) = 0$, which yields $r_0^* = (k_2/k_1)^{1/(2m)}$.

These considerations imply a fine tuning which is not always the best solution. Nevertheless, it could be right if we understand it as an operational simplification associated to a more realistic condition, i.e. the requirement of a sufficiently small acceleration a at each side of a shell of finite though little thickness ϵ, so that the quotient $\Delta a/\epsilon$ is admissible. This approximation would be in the line of the "traversability in practice" condition adopted in, for instance, Ref. [22] for wormholes which are not of the thin-shell class, where a maximum quotient g/l is admitted if an object can withstand a maximum tidal acceleration g between two points separated by a distance l.

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