Analytic Properties of Feynman Integrals for Scattering Amplitudes

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1. Introduction

Scattering amplitudes are key quantities in quantum field theory since on the one hand they are in principle accessible from theoretical calculations and on the other hand closely linked to physical quantities like the cross section and can be measured by high energy physics experiments, the most prominent of the current time being the LHC. The evaluation of scattering amplitudes, however, is in general very complex and for most theories only possible in a perturbative expansion of Feynman integrals.

In the past decade there has been a lot of progress in the development of new methods for the calculation of Feynman integrals [1, 2] and some of them strongly rely on special analytic properties of Feynman integrals. One such property is the uniform transcendental weight [3], which we will introduce in the second chapter. Feynman integrals having the uniform transcendentality property turn out to fulfill simple systems of differential equations, which can often easily be solved as a Laurent series in ϵ, the parameter of dimensional regularization [3].

Other quantities that are closely related to Feynman integrals are the leading singularities of a given Feynman integrand. Leading singularities can be obtained by replacing the integral paths along the real axis by contour integrals around the poles of the integrand. Being relatively easy to compute, leading singularities turn out to be a very useful tool to analyze analytic properties of Feynman integrals. For example, they are very effective for searching integrals with uniform transcendental weight. One conjecture is that Feynman integrals where all leading singularities are numerical constants are also functions of uniform transcendental weight [3] and in this thesis we are going to test this conjecture for several two- and three-loop integrals.

To test this conjecture we make a systematic analysis of integrands for different diagram topologies. Furthermore we want to follow the idea of [4], where the integrands of several amplitudes for planar $\mathcal{N} = 4$ Super Yang-Mills theory have been written in a basis of integrands with constant leading singularities. We try to extend this approach by searching integrand bases for diagrams of non-supersymmetric theories such as QCD and without restricting to planar diagrams. Integrands with constant leading singularities can be written as sums of so called dlog forms with constant coefficients and deriving these dlog forms was also one part of this thesis.

The main effort we had make to achieve these goals was to find methods for the computation of leading singularities and dlog forms. We therefore used two different approaches. The first is an algorithm that we used as a basis to write a MATHEMATICA program, which is able to compute the leading singularities for several two- and three loop diagrams by repeatedly taking the residues of rational functions. The second approach is to use generalized one-loop integrands with known leading singularities to take them as building blocks for multi-loop diagrams. This method enables in a simple manner to verify diagrams that were analyzed with the algorithm.

For the application of the analysis we chose different diagrams of increasing complexity. We restricted ourselves on diagrams with four massless external momenta and massless propagators and performed the calculation in four dimensions. The relevant integrand families in two loops for this kinetmatics are the planar and the non-planar double box. In three loops there are 9 relevant integrand families, from which we analyzed the two planar cases, denoted as diagram A and diagram E. One key step before applying the algorithm on the integrands was to find a good parametrization of the integration variables. In our case the spinor-helicity variables turned out to be particularly well suited.

The thesis is organized as follows. In the second chapter we will explain some basic methods of loop integral computations and also introduce the concepts of uniform transcendental weight functions, leading singularities, and dlog-forms. The third chapter will describe the two different methods for the computation of leading singularities and dlog forms. In chapter four we will present the results of a systematic analysis of the two- and three-loop four-point diagrams. We will compare the integrands that we found to have constant leading singularities to the corresponding integrals and test if they fulfill the uniform transcendental weight property. In chapter five we present a brief summary and a conclusion for our results.

2. Loop-Integral Methods

2.1. D-Dimensional Integrals

The calculation of scattering amplitudes at loop level is associated with several difficulties, one of them is that many loop integrals diverge. The standard way to make sense of loop integrals nevertheless is to use renormalization, which means that we allow some physical parameters like the coupling constant and the masses of particles to be infinite in such a way that they compensate the divergences of the integrals so that the physical results of the calculations are finite. In order to make sense of calculations containing divergent integrals we have to choose a regularization scheme. So we introduce a parameter, let us call it Λ, such that taking the limit $\Lambda \to a$, where a is a potentially infinite constant, we recover the definition of the original divergent integrals and for values $\Lambda \neq a$ the integrals remain finite in general, with possible exceptions. A very popular regularization scheme is dimensional regularization, where we generalize four-dimensional integrals to any integer dimension D and finally define our integrals also for real or even complex values of D by analytic continuation.

Taking an example from [1] we show how to generalize a four-dimensional integral to a D-dimensional integral and consider the one-propagator integral

$$\int \frac{d^D k}{i\pi^{D/2}} \frac{1}{(-k^2 + m^2 - i0)^a}, \tag{2.1}$$

where a is an arbitrary power of the propagator and $i0$ an infinitesimal small imaginary constant, which defines the correct integration contour around the poles at $k^2 = m^2 - i0$ for Feynman propagators. By Wick rotation, which we won't explain here (see for example [5]), we can relate this integral, which is defined for Lorentz vectors k, to Euclidean vectors k_E

$$\int \frac{d^D k_E}{\pi^{D/2}} \frac{1}{(k_E^2 + m^2)^a}, \tag{2.2}$$

where we can drop the $i0$-term, since we do not have a pole anymore as long as $m^2 > 0$.

Now using the relation, which is known as Schwinger parametrization

$$\frac{1}{x^a} = \frac{1}{\Gamma(a)} \int_0^\infty d\alpha \alpha^{a-1} e^{-\alpha x}, \tag{2.3}$$

we can evaluate the integral for arbitrary integer D as

$$\int \frac{d^D k_E}{\pi^{D/2}} \frac{1}{(k_E^2 + m^2)^a} = \frac{1}{\Gamma(a)} \int_0^\infty d\alpha \alpha^{a-1} \int \frac{d^D k_E}{\pi^{D/2}} e^{-\alpha(k_E^2+m^2)} \tag{2.4}$$

$$= \frac{1}{\Gamma(a)} \int_0^\infty d\alpha \alpha^{a-1} \left(\int_{-\infty}^\infty \frac{dk_E}{\sqrt{\pi}} e^{-\alpha(k_E^2+m^2)} \right)^D \tag{2.5}$$

$$= \frac{1}{\Gamma(a)} \int_0^\infty \frac{d\alpha}{\alpha} \alpha^{a-D/2} e^{-\alpha m^2} \tag{2.6}$$

$$= \frac{\Gamma(a - D/2)}{\Gamma(a)} \frac{1}{(m^2)^{a-D/2}}. \tag{2.7}$$

The decisive step in this derivation is the one from (2.4) to (2.5) where we rewrote the integral in D dimensions into a D-fold product of one dimensional integrals. This step, however, is only possible for integer D, so the derivation is also only valid for integer values of D. However, what we can do now is to take the result in equation (2.7) and take it as the definition of equation (2.1) for arbitrary complex values of D. This specific analytic continuation to complex values of D is not unique but the physical quantities we obtain by the calculations do not depend on the choice of the analytic continuation just as they do not depend on the choice of the regularization scheme.

In this way we can define arbitrary loop integrals in D-dimensions. We will state the result for a one-loop integral using Feynman parametrization (see also [1]):

$$F_n = \int \frac{d^D x_0}{i\pi^{D/2}} \prod_{j=1}^n \frac{1}{(-(x_0 - x_j)^2 + m_j^2)^{a_j}} \tag{2.8}$$

$$= \frac{\Gamma(a - D/2)}{\prod_{i=1}^n \Gamma(a_i)} \int_0^\infty \left(\prod_{i=1}^n d\alpha_i \alpha_i^{a_i - 1} \right) \frac{\delta(\sum_{i=1}^n c_i \alpha_i - 1) U^{a-D}}{(V + U \sum_{i=1}^n m_i^2 \alpha_i)^{a-D/2}}, \tag{2.9}$$

where x_0 is the loop momentum, $x_1, x_2, ..., x_n$ are sums of external momenta, $a = \sum_i^n a_i$, $U = \sum_{i=1}^n \alpha_i$, and $V = \sum_{i<j} \alpha_i \alpha_j (-(x_i - x_j)^2)$. The parameters c_i can be freely chosen with the only restriction that at least one is nonzero. The result will not depend on the particular choice of the c_i. It is possible to generalize this formula to more than one loop, with almost the same formula but with generalized definitions of U and V.

Using the formula (2.9) we can for example calculate the bubble integral with massless internal lines (see figure 2.1)

$$\mathcal{I}_2 = \int \frac{d^D k_1}{i\pi^{D/2}} \frac{1}{k_1^2 (k_1 - p_1 - p_2)^2}. \tag{2.10}$$

Here the external lines p_1, p_2, p_3, and p_4 are all massless which is equivalent to one massive external line on each side, since any massive four-momentum can be written as a sum of two massless four-momenta.

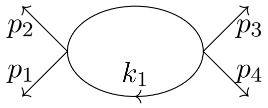


Figure 2.1.: Bubble integral

We will rewrite this with the variables of equation (2.9) and make a generalization by allowing arbitrary powers a_1 and a_2 of the propagators:

$$\mathcal{I}_2 = \int \frac{d^D x_0}{i\pi^{D/2}} \frac{1}{(-(x_0 - x_1)^2)^{a_1}(-(x_0 - x_2)^2)^{a_2}}, \tag{2.11}$$

where $x_0 = k_1$, $x_1 = 0$ and $x_2 = p_1 + p_2$. The only Lorentz invariant quantity the result can depend on is $s := (p_1 + p_2)^2$ and on dimensional grounds it is already clear that the result will be $s^{D/2-a}$ times a function that does not depend on s. After a short calculation we get the useful formula [6]

$$\mathcal{I}_2 = (-s)^{D/2-a} \frac{\Gamma(a - D/2)\Gamma(D/2 - a_1)\Gamma(D/2 - a_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(D - a)}. \tag{2.12}$$

2.2. Integration by Parts Identities

It turns out that there are many algebraic relations between different Feynman integrals and since the evaluation of most Feynman integrals is rather difficult, it is very efficient to use these relations to obtain further Feynman integrals from the ones we already know. A very rich set of such relations can be obtained via the integration by parts identities (IBP), which relate different integrals of a given integral family [7]. An integral family is the set of all integrals with a given propagator structure, but allowing arbitrary integer powers of the propagators.

As an example we consider the integral family containing the massless box using the labeling defined in figure 2.2 (a similar example can be found in [6]),

$$\mathcal{I}_{a_1,a_2,a_3,a_4} = \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{(-k^2)^{a_1}(-(k-p_1)^2)^{a_2}(-(k-p_1-p_2)^2)^{a_3}(-(k+p_4)^2)^{a_4}}. \tag{2.13}$$

Now we use the fact that if we differentiate the integral with respect to the loop momentum k before integration the integral vanishes. So

$$0 = \int \frac{d^D k}{i\pi^{D/2}} \frac{\partial}{\partial k^\mu} \left(v^\mu \frac{1}{(-k^2)^{a_1}(-(k-p_1)^2)^{a_2}(-(k-p_1-p_2)^2)^{a_3}(-(k+p_4)^2)^{a_4}} \right), \tag{2.14}$$

where v^μ is an arbitrary vector built from the loop momentum or the external momenta. It is always possible to express the integrals obtained by taking the derivative as a linear combination of integrals of the integral family. Different values of v^μ will create different integration by parts identities and one could for example try to find

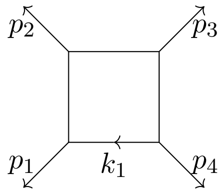


Figure 2.2.: Family of one-loop box diagrams

vectors v^μ that relate only integrals with powers of the propagators not bigger than one (examples for this can be found e. g. in [8]). In this case, however, we simply choose $v = k$ which generates the following IBP identity

$$(D - 2a_1 - a_2 - a_3 - a_4 - sa_3Y_3^+ - a_2Y_2^+Y_1^- - a_3Y_3^+Y_1^- - a_4Y_4^+Y_1^-) \times I_{a_1,a_2,a_3,a_4} = 0, \tag{2.15}$$

where $Y_i^\pm I_{a_1,a_2,a_3,a_4} = I_{a_1\pm\delta_{1i},a_2\pm\delta_{2i},a_3\pm\delta_{3i},a_4\pm\delta_{4i}}$. By choosing different vectors v or using the diagram symmetry we obtain further IBP-relations, so that we can express any integral of that family in terms of $\mathcal{I}_{1,1,1,1}, \mathcal{I}_{1,0,1,0}$, and $\mathcal{I}_{0,1,0,1}$. The choice of these three integrals as basis integrals is of course not unique and we could also have taken three different integrals. Normally one would choose integrals that are particularly easy to calculate or have some preferred properties such as UV-finiteness. Any integrals that form a basis for a given integral family are called master integrals. Smirnov [9] showed that the number of master integrals for any integral family is always finite.

A few examples of such reductions for our example are

$$\mathcal{I}_{1,1,1,0} = \frac{2(D-3)}{(D-4)s}\mathcal{I}_{1,0,1,0}, \tag{2.16}$$

$$\mathcal{I}_{1,1,1,2} = \frac{(D-5)}{t}\mathcal{I}_{1,1,1,1} - \frac{4(D-5)(D-3)}{(D-6)s^2t}\mathcal{I}_{1,0,1,0}, \tag{2.17}$$

$$\mathcal{I}_{1,0,0,0} = 0. \tag{2.18}$$

The first shows that the triangle with two massless external legs is directly related to the bubble integral. The second relation is an example for the reduction of an integral with propagator powers bigger than one. The last one might be a bit surprising if we compare it to the original definition of the integral but this is a special property of dimensional regularization, where all integrals that do not depend on any scale are zero. For more details on such subtleties of dimensional regularization see for example [10].

The IBP-procedure is implemented with the Laporta algorithm in free available computer programs such as AIR [11], FIRE [12], LITERED [13] and REDUZE [14]. For this master thesis a combination of FIRE and LITERED was used.

2.3. Differential Equations

It turns out that we can obtain further relations between Feynman integrals if we also consider derivatives with respect to external variables. We will show how these derivatives can be used together with the IBP identities to setup a linear system of differential equations for the master integrals.

Since the massless box integral of the last section is a Lorentz scalar, its value can only depend on scalar products of the external momenta. There are only two independent scalar products, which are also known as Mandelstam variables and we define them as $s = (p_1 + p_2)^2$ and $t = (p_1 + p_4)^2$. So it is useful to consider derivatives with respect to s or t. Since the integrand in (2.13) is written in terms of p_1, p_2, and p_4 rather than s and t we have to use the chain rule. Here some care has to be taken, since the components of p_1, p_2, and p_4 are not independent and must fulfill the conditions $p_1^2 = p_2^2 = p_4^2 = (-p_1 + p_2 + p_4)^2 = 0$ to ensure that all external particles are massless. What we have to do to get a consistent calculation is to ensure that the derivatives ∂_s and ∂_t commute with the boundary conditions (see also [6]). So we start making the ansatz

$$\partial_s = (\alpha_1 p_1 + \alpha_2 p_2 + \alpha_4 p_4) \cdot \partial_{p_2} \tag{2.19}$$

and make sure that $\partial_s p_1^2 = \partial_s (p_1 + p_2 + p_4)^2 = 0$ and also $\partial_s (p_1 + p_2)^2 = \partial_s s = 1$. These equations can be solved by choosing the parameters in (2.19) to be

$$\alpha_1 = \frac{1}{2s}, \tag{2.20}$$

$$\alpha_2 = \frac{-2s - t}{2s(s+t)}, \tag{2.21}$$

$$\alpha_4 = \frac{1}{2(s+t)}. \tag{2.22}$$

Now we can do the same for ∂_t but now we must be careful that if we use ∂_s and ∂_t simultaneously that also $\partial_s (p_1 + p_4)^2 = 0$ and $\partial_t (p_1 + p_2)^2 = 0$. The first is already fulfilled by having chosen to make the ansatz (2.19) containing ∂_{p_2} rather than to ∂_{p_1}. So consequently the correct ansatz for ∂_t is

$$\partial_t = (\beta_1 p_1 + \beta_2 p_2 + \beta_4 p_4) \cdot \partial_{p_4} \tag{2.23}$$

and imposing the corresponding boundary conditions we find

$$\beta_1 = \frac{1}{2t}, \tag{2.24}$$

$$\beta_2 = \frac{1}{2(s+t)}, \tag{2.25}$$

$$\beta_4 = \frac{-s - 2t}{2t(s+t)}. \tag{2.26}$$

With these formulas at hand we can calculate the derivatives of the three master integrals with respect to s and t. The result can again be expressed in terms of integrals of the same family which can then be reexpressed in terms of the three master integrals using the IBP-relations. Thus the derivatives of the master integrals with respect to external variables are again linear combinations of the master integrals. By introducing the vector

$$\vec{f} = \begin{pmatrix} \mathcal{I}_{0,1,0,1} \\ \mathcal{I}_{1,0,1,0} \\ \mathcal{I}_{1,1,1,1} \end{pmatrix} \tag{2.27}$$

we can set up the following system of differential equations,

$$\partial_s \vec{f} = A_s \vec{f} \tag{2.28}$$

$$\partial_t \vec{f} = A_t \vec{f} \tag{2.29}$$

with

$$A_s = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{\epsilon}{s} & 0 \\ \frac{4\epsilon-2}{st(s+t)} & \frac{2-4\epsilon}{s^2(s+t)} & -\frac{s+t+t\epsilon}{s^2+ts} \end{pmatrix}, \tag{2.30}$$

$$A_t = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{\epsilon}{s} & 0 \\ \frac{4\epsilon-2}{st(s+t)} & \frac{2-4\epsilon}{s^2(s+t)} & -\frac{s+t+t\epsilon}{s^2+ts} \end{pmatrix}, \tag{2.31}$$

where we have set $D = 4-2\epsilon$. A useful cross check for these matrices can be obtained by using $\partial_s \partial_t \vec{f} = \partial_t \partial_s \vec{f}$ which is equivalent to $A_s \cdot A_t - A_s \cdot A_t + \partial_t A_s - \partial_s A_t = 0$. Another crosscheck is to calculate

$$sA_s + tA_t = \begin{pmatrix} -\epsilon & 0 & 0 \\ 0 & -\epsilon & 0 \\ 0 & 0 & -\epsilon-2 \end{pmatrix} \tag{2.32}$$

where the diagonal entries should correspond to the scaling dimension of the basis integrals, which is true in our case. We can simplify the differential equations by normalizing the basis integrals appropriately so that they will only depend on the dimensionless variable $x = \frac{t}{s}$. Henn [3] showed that for a more sophisticated choice of basis the differential equations can simplify in a way that they can be solved easily order by order in an ϵ-power expansion. For the current integral family such a basis is [6]

$$g_1 = c\epsilon(-s)^{\epsilon} t \mathcal{I}_{0,1,0,2}, \tag{2.33}$$

$$g_2 = c\epsilon(-s)^{\epsilon} s \mathcal{I}_{1,0,2,0}, \tag{2.34}$$

$$g_3 = c\epsilon^2(-s)^{\epsilon} st \mathcal{I}_{1,1,1,1}, \tag{2.35}$$

where $c = e^{\epsilon\gamma_E}$ and γ_E is the Euler-Mascheroni constant. The factors of ϵ have been included to make all integrals finite in the limit $\epsilon \to 0$ and the powers of s make the integral dimensionless so that they depend only on $x = t/s$. The factors like $s\,t$ for g_4 is connected with the leading singularity of the integrand, which will be introduced in section 2.7. Using this basis the differential equation gets the following convenient form,

$$\partial_x \vec{g}(x,\epsilon) = \epsilon \left(\frac{a}{x} + \frac{b}{1+x} \right) \vec{g}(x,\epsilon), \tag{2.36}$$

where

$$a = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 2 & 1 \end{pmatrix}. \tag{2.37}$$

With this equation we can now solve the integrals $\vec{g}$ order by order in ϵ using the ansatz

$$\vec{g}(x) = \sum_{k=0}^{\infty} \epsilon^k \vec{g}^{(k)}(x). \tag{2.38}$$

The differential equation (2.36) relates the coefficient functions $g^{(k)}$ of different orders by

$$g^{(0)} = \text{const.}, \tag{2.39}$$

$$\partial_x \vec{g}(x)^{(k)} = \left(\frac{a}{x} + \frac{b}{1+x} \right) g^{(k-1)}(x), \text{ for } k \geq 1. \tag{2.40}$$

So we can find the solution for $\vec{g}$ by integrating order by order in ϵ and the only open question now is how we can determine the integration constants for each integration. Here we must include additional knowledge about the integrals, for instance the integrals g_1 and g_2 can directly be solved using equation (2.12). We also use the fact that the integrals cannot have a pole at $x \to -1$, which corresponds to $u = -s - t = (p_1 + p_3)^2 \to 0$, since a planar diagram can only have poles if neighbored legs are collinear. These informations suffice to fix all integration constants.

The solution can be written in terms of polylogarithms, which are defined as

$$\text{Li}_1 = -\log(1-x), \quad \text{Li}_{n+1}(x) = \int_0^x \frac{\text{Li}_n(t)}{t} \text{d}t, \tag{2.41}$$

and also have the power series representation

$$\text{Li}_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n} = x + \frac{x^2}{2^n} + \frac{x^3}{3^n} + \dots \tag{2.42}$$

So the first few orders in the ϵ-expansion of g_3 are

$$g_3(x) = 4 + \epsilon(-2\log x) + \epsilon^2 \left(-\frac{4\pi^2}{3} \right) + \epsilon^3 \bigg(\frac{7\pi^2}{6} \log x + \frac{1}{3} \log^3 x - \pi^2 \log(1+x) \tag{2.43}$$

$$- \log^2(x) \log(1+x) - 2\log(x)\text{Li}_2(-x) + 2\text{Li}_3(-x) - \frac{34}{3}\zeta_3 \bigg) + \mathcal{O}(\epsilon^4).$$

Note that since we chose $g_3 \propto \epsilon^2 \mathcal{I}_{1,1,1,1}$ the ϵ-expansion of the box integral will start with a term $\frac{1}{\epsilon^2}$, which is due to an infrared divergency that is caused by contributions to the integral in the region where $k^\mu \to 0$ in equation (2.13).

2.4. Uniform Transcendentality and Pure Functions

In the last section we saw that if we choose our basis integral in a clever way the resulting differential equations simplify a lot and can be solved much easier and the question is now how we can find such functions. The key property of these functions can be described with the concept of the degree of transcendentality $\mathcal{T}(f)$ of a function [3]. In the last section we saw that in order to calculate the m-th element of the ϵ-expansion of any basis function g_i we needed m iterated integrals. Now $\mathcal{T}(f)$ is precisely defined as the number of iterated integrals needed to define the function f, which means that e. g. $\mathcal{T}(g_i^{(m)}) = m$ for $g_i^{(m)}$ defined in equation (2.38). This also means that e. g. $\mathcal{T}(\log(x)) = 1$ and $\mathcal{T}(\mathrm{Li}_k(x)) = k$. Besides $\mathcal{T}(f_1 f_2) = \mathcal{T}(f_1) + \mathcal{T}(f_2)$ and algebraic factors are assigned 0 degree of transcendentality. Numerical constants such as π and ζ_n are assigned the value of the corresponding function from which they can be derived. So since $\zeta_n = \mathrm{Li}_n(1)$ we have $\mathcal{T}(\zeta_n) = n$ and because of $\zeta_2 = \frac{\pi^2}{6}$ we can conclude that $\mathcal{T}(\pi) = 1$. We are now interested in functions with uniform transcendentality which is defined as a function that can be written as a sum of terms having all the same degree of transcendentality. If the function also satisfies

$$\mathcal{T}\left(\frac{d}{dx} f(x)\right) = \mathcal{T}(f(x)) - 1, \tag{2.44}$$

then the function f is called a pure function. So with this definition we can see that if we would multiply a pure function with an algebraic function of x the resulting function would still be of uniform transcendentality but not a pure function anymore, since the derivative is also applied on the algebraic function. The functions $g_i^{(k)}(x)$ of the last section are all pure functions and if we define $\mathcal{T}(\epsilon) = -1$ then also the whole functions $g_i(x)$ can be called a pure functions.

So to find out if a given Feynman integral is a pure function it has turned out to be very useful to calculate the *leading singularities* of the corresponding integrand. The leading singularities can be calculated by analytically continuing the integrand to complex momenta and then take contour integrals around the poles of the integrand. Since this can be done by simply calculating the residues at these poles this is most times much easier than taking the integral over the real axis to calculate the original Feynman integral. A conjecture that will be tested in this thesis is that functions where all leading singularities are kinematic independent constants correspond to integrals that are pure functions [3].

2.5. Integral Reduction

For one-loop integrals it is possible to express any n-point Feynman integral as the sum of four basis integrals and a rational function [15]. The four basis integrals are tadpole, bubble, triangle, and the box with an arbitrary number of external legs of the original diagram at each vertex.

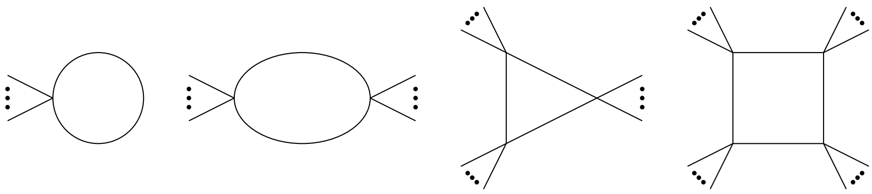


Figure 2.3.: Basis of one-loop integrals: tadpole, bubble, triangle, and box

So given a generic n-point one-loop integral with a possible numerator $\mathcal{N}(k)$ we have the decomposition (see also [1]),

$$I_n[\mathcal{N}(k)] = \sum_{j_4} c_{4,j_4} I_4^{(j_4)} + \sum_{j_3} c_{3,j_3} I_3^{(j_3)} + \sum_{j_2} c_{2,j_2} I_2^{(j_2)} + \sum_{j_1} c_{1,j_1} I_1^{(j_1)} + \mathcal{R} + \mathcal{O}(\epsilon), \quad (2.45)$$

where I_4 is a box, I_3 a triangle, I_2 a bubble, and I_1 a tadpole integral and $\mathcal{R}$ is a rational term. The external legs of these four basis integrals are sums of adjacent external momenta of the original integral I_n and since there are several possibilities to distribute the n external legs to the legs of each of the four basis integrals, we have to take corresponding sums and each summand gets its own coefficient c_{i,j_i}. The decomposition is only valid to order $\mathcal{O}(\epsilon^0)$. If we wanted a decomposition that is valid to all orders in ϵ we would have to include the scalar pentagon integral as well [8].

To realize the reduction we have to perform three different steps for integrals with $n \geq 5$ with a possible numerator.

- Reduce the n-point integral with numerator $\mathcal{N}(k)$ and $n \geq 5$ to a linear combination of n-point integrals with a scalar numerator (which means independent of the loop momentum) and integrals with less propagators.
- Reduce all scalar integrals with $n \geq 5$ legs to integrals with less propagators.
- Reduce all integrals with $n \leq 4$ having a k-dependent numerator to scalar integrals with $n \leq 4$.

We explain the reduction process in more detail for the special case where all internal propagators are massless, even though the reduction process also works in the case of arbitrary massive propagators. In addition we restrict our external momenta to be strictly four-dimensional, as well as all vectors in the numerator that are multiplied with the loop momentum.

We consider the integral

$$I_n[\mathcal{N}(k)] = \int \frac{d^D k}{(2\pi)^D} \frac{\mathcal{N}(k)}{k^2(k-p_1)^2(k-p_1-p_2)^2 \cdots (k-p_1-\cdots-p_n)^2}, \tag{2.46}$$

and to show the first step it is enough to show that any product $k \cdot v$ can be written as a liner combination of inverse propagators and a constant. For $n \geq 5$ we have in general four linear independent external momenta. Note that because of momentum conservation there are only 3 linear independent external momenta for $n = 4$. So for $n \geq 5$ we may choose four external momenta as a basis and thus can express $k \cdot v$, where v is an arbitrary four-dimensional vector, in terms of scalar products of k with external momenta. The next observation is that any scalar product $k \cdot p_i$ can be expressed as a linear combination of inverse propagators plus terms independent of the loop momentum:

$$k \cdot p_i = -\frac{1}{2}(k-p_1-\cdots-p_i)^2 + \frac{1}{2}(k-p_1-\cdots-p_{i-1})^2 + (p_1-\cdots-p_{i-1}) \cdot p_i + \frac{1}{2}p_i^2. \tag{2.47}$$

This means that any integral $I_n[(k \cdot v)^j]$ with $n \geq 5$ can be reduced to integrals $I_{n-1}[(k \cdot v)^{j-1}]$ and integrals $I_n[(k \cdot v)^{j-1}]$. So repeating this step the integral $I_n[(k \cdot v)^j]$ can finally be written as a linear combination of $I_n[1]$ and integrals with fewer propagators, where the coefficients are only dependent on external momenta.

To reduce scalar integrals with $n \geq 5$ we follow [8] and make use of Gram determinants, which are defined as

$$G\begin{pmatrix} p_1, \ldots, p_l \\ q_1, \ldots, q_l \end{pmatrix} = \det_{(i,j) \in l \times l} (2p_i \cdot q_j) \tag{2.48}$$

Now one can make use of the fact that the Gram determinant vanishes if either the p_i-vectors or the q_i-vectors are linear dependent. So in four dimensions any Gram determinant with $l \geq 5$ will vanish. So by expanding

$$G\begin{pmatrix} k, p_1, p_2, p_3, p_4 \\ p_5, p_1, p_2, p_3, p_4 \end{pmatrix} = 0 \tag{2.49}$$

we get a relation between a scalar numerator for $n \geq 6$ and numerators that cancel propagators and thus lead to integrals with less propagators. To reduce the scalar pentagon we consider

$$G\begin{pmatrix} k, p_1, p_2, p_3, p_4 \\ k, p_1, p_2, p_3, p_4 \end{pmatrix} = \mathcal{O}(\epsilon), \tag{2.50}$$

which is only zero for $D = 4$. Thus the scalar pentagon can only be reduced up order $\mathcal{O}(\epsilon)$.

Finally, we need to discuss the reduction of integrals with $n \leq 4$ and k-dependent numerators to scalar integrals with $n \leq 4$. The argument here is similar to the one we used for the reduction of k-dependent numerators with $n \geq 5$ with the difference that in this case there are not enough external momenta to construct a four-dimensional basis. However, in these cases it can be argued that scalar products of k with vectors that are perpendicular to the external momenta lead to numerators that vanish after integration [1]. Note that the reduction process to integrals with $n \leq 4$ and k-dependent numerators was purely on the integrand level, whereas the reduction to scalar integrals with $n \leq 4$ is only true after integration. So which ansatz we choose for our Feynman integral depends whether we want an equality on the integrand level or the integral level.

Integral reduction can also be applied for more than one loop, even though the number of irreducible integrals is in this case much larger than in the one-loop case. One example are the planar two-loop diagrams with massless internal propagators. Here each diagram with eleven or less propagators can be reduced. In contrast to the one-loop case, there are also diagrams with loop variable dependent numerators, since in the two-loop case not all possible scalar products can be written as linear combinations of inverse propagators plus a constant as in the one-loop case. A more detailed description of the reduction of the planar double box with massless propagators can be found e. g. in [8].

2.6. Unitarity Cuts

2.6.1. Optical Theorem

Unitarity cuts are a useful tool that can be helpful in different ways to calculate Feynman loop integrals. Cutting a propagator of an integral means replacing this propagator by a delta function with the corresponding inverse propagator as its argument.

One application of unitarity cuts is a variation of the optical theorem, which gives a relation between the imaginary part of the loop amplitude and the integral over tree amplitudes [1],

$$2\mathrm{Im}A^{(\text{1-loop})} = \int d^4k\delta^{(+)}(k^2)\delta^{(+)}((k-P_L)^2)A_L^{(\text{tree})}A_R^{(\text{tree})}, \tag{2.51}$$

where we had to use $\delta^{(+)}(k^2) := \delta(k^2)\Theta(k^0)$, because of a subtlety that is related to causality [1].

The shaded gray regions in figure 2.4 define arbitrary tree-level diagrams for a given number of external legs. The imaginary part of the amplitude is proportional to the discontinuity of the amplitude across the branch cut of interest.

This relation can be very useful if we use it along with the ansatz of equation (2.45) to determine the coefficients $c_{n,j}$. So in order to calculate any one-loop amplitude $A^{(\text{1-loop})}$ we know that we can write it as

$$A^{(\text{1-loop})} = \sum_{j_4} c_{4,j_4} I_4^{(j_4)} + \sum_{j_3} c_{3,j_3} I_3^{(j_3)} + \sum_{j_2} c_{2,j_2} I_2^{(j_2)} + \sum_{j_1} c_{1,j_1} I_1^{(j_1)} + \mathcal{R} + \mathcal{O}(\epsilon). \tag{2.52}$$

Now we cut the amplitude into two pieces in all relevant ways. This corresponds for the left-hand side of equation (2.52) to integrate over the corresponding tree-level amplitudes and for the right hand side to calculate the branch cut discontinuities of the $I_i^{(j_i)}$ by replacing the two propagators, where the diagram is cut, by delta functions. Integrals $I_i^{(j_i)}$ where the corresponding propagator is not present will vanish on that cut, so that for every cut we get linear relations between some of the coefficients c_{i,j_i} and integrals over tree amplitudes from the left hand side. The only part we cannot obtain with this method is the rational part $\mathcal{R}$ of the amplitude.

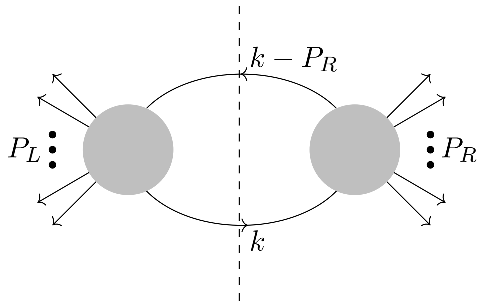


Figure 2.4.: Cutting a one-loop amplitude into two tree amplitudes and using the optical theorem

The non-rational part of the amplitude is thus also called 'cut-constructable'.

2.6.2. Generalized Cuts

Using the optical theorem for a one-loop amplitude we would always cut two propagators. However, sometimes it is helpful to cut more than two propagators. Because of the expansion in equation (2.45) the maximal number of propagators we can cut in a one-loop amplitude is four, since otherwise the right hand side of the expansion always vanishes. One method to determine the coefficients for a general one-loop amplitude using generalized cuts is to match the amplitude and the expansion in boxes, triangles, bubbles, tadpoles on the integrand level. Since the integral reduction with $n \leq 4$ with a numerator to scalar integrals is only possible on the level of the integral and not of the integrand we have to make the more general ansatz (see also [1])

$$\begin{aligned} A_n^{(\text{1-loop integrand})}(k) = & \sum_{1\leq i_1<i_2<i_3<i_4\leq n} \frac{d_{i_1i_2i_3i_4}(k)}{d_{i_1}d_{i_2}d_{i_3}d_{i_4}} + \sum_{1\leq i_1<i_2<i_3\leq n} \frac{c_{i_1i_2i_3}(k)}{d_{i_1}d_{i_2}d_{i_3}} \\ & + \sum_{1\leq i_1<i_2\leq n} \frac{b_{i_1i_2}(k)}{d_{i_1}d_{i_2}} + \sum_{1\leq i_1\leq n} \frac{a_{i_1}(k)}{d_{i_1}}, \end{aligned} \tag{2.53}$$

where the numerators also depend on the scalar products of the loop momentum with vectors that are perpendicular to the external momenta of the corresponding box, triangle, bubble, or tadpole. Even though after integration only the part independent of the loop momenta will survive we need them in the ansatz to determine the coefficients. The strategy is now to start with the maximal number of four cuts, where all but one of the box integrands vanish to determine the numerators $d_{i_1i_2i_3i_4}(k)$ of the boxes. We continue to cut one propagator less, so that we get contributions of boxes and triangles. By also using the already known box numerators $d_{i_1 i_2 i_3 i_4}(k)$ this allows us to determine the numerators $c_{i_1 i_2 i_3}(k)$ of the triangle. We proceed in the same way to get also the bubble numerators and the tadpole numerators. Note that in this procedure all we needed to know are the tree-level amplitudes and the four basic scalar integrals to determine the whole amplitude without the rational part. Especially no further integration was needed in comparison to the previous method, where only two propagators were cut, even though also there the integration can sometimes be avoided.

Another difference to the method where we cut only two propagators is that we also have to allow complex momenta since the solution of setting four propagators to zero is in general complex. Using complex momenta makes also a difference for the tree amplitudes in the left hand side of equation (2.53) since amplitudes like the three-point tree amplitude for gluons vanishes for real momenta but not for general complex momenta.

2.7. Leading Singularities

Unitarity cuts are not only useful to calculate the coefficients of the amplitude in the ways described in the previous section, but can be also used to analyze analytic properties of Feynman integrals in general. We introduced unitarity cuts by replacing propagators with delta functions. Another possible interpretation is to understand them as taking contour integrals around the poles of the propagators (see also [4]). Taking the four cuts of the propagators P_i can be written as taking four contour integrals around the poles of the propagators P_i,

$$\frac{1}{(2\pi i)^4} \prod_{i=1}^{4} \oint_{P_i=0} \frac{dP_i}{P_i} \frac{1}{J(k^*)} \times R(k^*). \tag{2.54}$$

Here R is the rest of the integrand, which has to be evaluated at k^*, the solution to $P_1 = P_2 = P_3 = P_4 = 0$. J is the Jacobian of the variable transformation from k^μ to P_i defined as

$$J = \det\left(\frac{\partial(P_1, P_2, P_3, P_4)}{\partial(k_0, k_1, k_2, k_3)}\right), \tag{2.55}$$

which also has to be evaluated at $k = k^*$. Different orderings can lead to a different sign, which is related to the orientation of the contour. Signs of residues, however, will not be important for the analysis we are going to do.

The only difference between calculating unitarity cuts and calculating the integral along the real axis is the path on which we integrate. So it is reasonable to assume that the quantities we get from the unitarity cuts and the Feynman integrals have similar properties and since calculating unitarity cuts is most times much easier than determining the integral over the real axis it can be useful to calculate unitarity cuts in order to learn something about the properties of the Feynman integral. If we take the residues for each integration variable we obtain the quantity that is referred to as a leading singularity of the integrand, which we already mentioned in section 2.4. Sometimes the number of propagators is smaller than the number of integration variables as for example in the case of the triangle and also in most multi-loop diagrams, so one might wonder if there are enough poles where we can take residues. In fact since the propagators are quadratic in the momenta taking the residue at the pole of one propagator creates new factors in the denominator so that we can take also the residues of these new factors. Leading singularities of this type are called composite leading singularities. We will see detailed examples for leading singularities in the next chapters.

2.8. Dlog Forms

A particularly nice way to expose the leading singularities of a rational integrand is to write it as a sum of dlog form. As an example from [16] the integrand

$$f(x,y) = \frac{dx \wedge dy}{xy(x+y+1)} \tag{2.56}$$

can also be rewritten as

$$f(x,y) = d\log\frac{x}{1+x+y} \wedge d\log\frac{y}{1+x+y}, \tag{2.57}$$

where

$$d\log f(x,y) \equiv \frac{\partial f(x,y)}{\partial x}\frac{dx}{f(x,y)} + \frac{\partial f(x,y)}{\partial y}\frac{dy}{f(x,y)}. \tag{2.58}$$

Using this definition it is easy to verify that equation (2.57) is equivalent to equation (2.56). Finding a dlog form like in (2.57) for a given form on the other hand is not always simple and we will discuss methods to derive dlog forms for a given integrands in the next chapter. However, for some integrands there exists no dlog form at all since for example no algebraic transformation of $\Omega(x) = dx$ or $\Omega(x) = dx/x^2$ can bring these two cases to a form like $c\, d\log(f(x))$, where c is a numeric constant. One can show that any function having a double pole or pole of higher order in any variable cannot be written in a dlog form. Also $\Omega(x) = dx$ has a double pole which is not so obvious, because the double pole in this case is at infinity which can be seen by transforming $x \to 1/y$ resulting in $\Omega(x) = dx \to -dy/y^2$.

Leading singularities are directly related to dlog forms. If we can write a function as

$$\Omega = \sum_i c_i d\log g_{i1} \wedge d\log g_{i2} \wedge \cdots \wedge d\log g_{ij}, \tag{2.59}$$

then the leading singularities of Ω are precisely c_i, provided that all dlog forms in the sum of (2.59) are linearly independent.

Another nice feature of dlog forms is that taking generalized cuts is trivial, since we do not have to calculate any Jacobian factors.

Dlog-forms also play an important role in $\mathcal{N} = 4$ Super Yang-Mills theory, where all planar amplitudes could be reformulated with a dual formulation using on-shell diagrams and the positive Grassmannian, which implies that all integrands can be written as dlog forms. This reformulation is connected to the geometric concept of the amplituhedron, which, however, is defined in momentum twistor variables, which can only be used for planar diagrams. Since we do not need momentum twistor variables to express diagrams as dlog forms, they turn out to be a useful tool for the analysis of non-planar diagrams in $\mathcal{N} = 4$ Super Yang-Mills theory and also for the investigation of the question if the concept of the amplituhedron is also valid in the non-planar case [17].

3. Computing Dlog Forms and Leading Singularities

3.1. Example for Deriving a Dlog Form

As a simple example for demonstrating how we can actually rewrite a given integrand as a dlog form we again consider the example of [16]:

$$f(x,y) = \frac{dx \wedge dy}{xy(x+y+1)}. \tag{3.1}$$

One way to proceed is to make a partial fraction decomposition of $f(x,y)$ with respect to one of the integration variables.

Choosing y in this example we get

$$f(x,y) = \frac{dx \wedge dy}{xy(1+x)} - \frac{dx \wedge dy}{x(1+x)(1+x+y)}. \tag{3.2}$$

In this case the partial fraction decomposition brings it into a convenient form, because for both summands the denominators are linear in y and the numerators do not depend on y. Later we will see an example where this is not the case. Here we can write all y-dependencies in the dlog factors with prefactors being functions of the remaining variables, in this case only x. So

$$f(x,y) = \frac{1}{x(1+x)} dx \wedge d\log(y) - \frac{1}{x(1+x)} dx \wedge d\log(1+x+y), \tag{3.3}$$

and now we can repeat this procedure for the other variables, in this case x, which gives us the full decomposition in dlog forms as

$$\begin{aligned} f(x,y) = {} & d\log(x) \wedge d\log(y) - d\log(1+x) \wedge d\log(y) \\ & - d\log(x) \wedge d\log(1+x+y) + d\log(1+x) \wedge d\log(1+x+y). \end{aligned} \tag{3.4}$$

This can also be written in a more compact form by summarizing the terms like

$$f(x,y) = (d\log(x) - d\log(1+x)) \wedge (d\log(y) - d\log(1+x+y)) \tag{3.5}$$

$$= d\log\frac{x}{1+x} \wedge d\log\frac{y}{1+x+y}. \tag{3.6}$$

So we can see that the dlog form is not unique. While the equivalence of the dlog forms (3.4) and (3.6) can be easily seen by summarizing and expanding, there are dlog forms where the equivalence is less obvious like for example the equivalence of (3.6) and (2.57).

Next, we look at an example where the partial fraction decomposition leads to square root terms. If we start with the integrand

$$h(x,y) = \frac{dx \wedge dy}{y(y^2 - x)} \tag{3.7}$$

and try to make a partial fraction decomposition for y we leave the space of rational functions getting

$$h(x,y) = -\frac{dx \wedge dy}{xy} + \frac{dx \wedge dy}{2x(-\sqrt{x}+y)} + \frac{dx \wedge dy}{2x(\sqrt{x}+y)}. \tag{3.8}$$

In this case we can still get to a solution, because after stripping off the dlog terms the remaining functions are pure rational again, so we get

$$h(x,y) = d\log(x) \wedge [-d\log(y) + \frac{1}{2} d\log(-\sqrt{x}+y) + \frac{1}{2} d\log(\sqrt{x}+y)] \tag{3.9}$$

$$= -d\log(x) \wedge d\log(y) + \frac{1}{2} d\log(x) \wedge d\log(-x+y^2). \tag{3.10}$$

So in this case we could actually derive a dlog form despite the square roots and at the end we could even summarize the terms in such a way that the square roots vanished. But in more complicated examples with more variables this may not be the case, since we can only make partial fraction decomposition of functions that are rational in the according variable. But clearly we could have gotten a dlog form of (3.7) much easier if we just would have chosen to do the first partial fraction decomposition with respect to x instead of y. So sometimes it is possible to avoid these square roots by just taking the right order of choosing the integration variables. In section 3.3.3 we will consider examples that cannot be solved like (3.7) and present methods that enable the calculation of leading singularities also in some of the more involved cases.

3.2. One-loop Dlog Forms

Next, we want to apply our method to integrands that are also relevant in physics by starting with the one-loop diagrams. We restrict ourselves to four-point integrands with massless external momenta and massless propagators and want to discuss the bubble, triangle, and box diagram. The corresponding integrands for these three loops are

$$d\mathcal{I}_2 = d^4k_1 \frac{1}{k_1^2(k_1-p_1-p_2)^2}, \tag{3.11}$$

$$d\mathcal{I}_3 = d^4k_1 \frac{s}{k_1^2(k_1-p_1)^2(k_1-p_1-p_2)^2}, \tag{3.12}$$

$$d\mathcal{I}_4 = d^4k_1 \frac{st}{k_1^2(k_1-p_1)^2(k_1-p_1-p_2)^2(k_1+p_4)^2}, \tag{3.13}$$

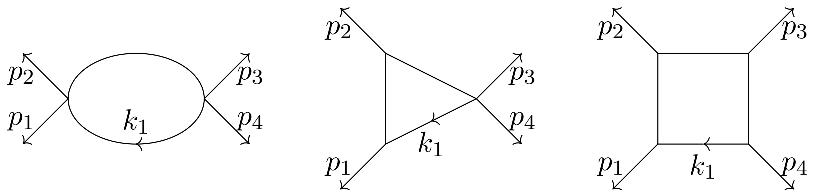


Figure 3.1.: The bubble, triangle and box one-loop diagrams

where the numerators are chosen such that the integrands are dimensionless. The variables $s = (p_1 + p_2)^2$, $t = (p_2 + p_3)^2$ and $u = (p_1 + p_3)^2$ are the usual Mandelstam variables and they are related by momentum conservation as $s + t + u = 0$, so that we actually need only two of them.

If we now want to derive dlog forms for these integrands, working with the four components of k_1 as free variables would immediately lead to square root terms, so it is crucial to switch variables and it turns out that spinor-helicity variables work well in our case. An introduction to the spinor helicity formalism can be found e. g. in [18]. So following [16] we rewrite k_1 as a linear combination of four basis vectors constructed from $p_1 = \lambda_1\tilde{\lambda}_1$ and $p_2 = \lambda_2\tilde{\lambda}_2$ as

$$k_1 = \alpha_1\lambda_1\tilde{\lambda}_1 + \alpha_2\lambda_2\tilde{\lambda}_2 + \alpha_3\lambda_1\tilde{\lambda}_2 + \alpha_4\lambda_2\tilde{\lambda}_1. \tag{3.14}$$

The Jacobian arising due to the variable transformation is the determinant of the four by four matrix with components

$$J_i{}^{\mu} = \frac{\partial k_1^{\mu}}{\partial \alpha_i}, \quad \mu = 0, ..., 3, \quad i = 1, ..., 4. \tag{3.15}$$

It turns out to be easier to calculate the determinant of JgJ^T with g being the metric tensor g =diag$(1, -1, -1, -1)$:

$$(JgJ^T)_{ij} = J_i{}^{\mu} g_{\mu\nu} J_j{}^{\nu} = J_i{}^{\mu} J_{j\mu} = J_i \cdot J_j \tag{3.16}$$

The only products not being zero are $J_1 \cdot J_2 = k_1 \cdot k_2 = \frac{s}{2}$, $J_3 \cdot J_4 = \frac{1}{2}\langle 12\rangle[12] = -\frac{s}{2}$, and the two components from the symmetric exchange of i and j. This leads to

$$\det(JgJ^T) = \frac{1}{16s^4} = \det(J)\det(g)\det(J^T) = -\det(J)^2 \tag{3.17}$$

and finally

$$\det(J) = \pm\mathrm{i}\frac{s^2}{4}. \tag{3.18}$$

Since we are only interested in leading singularities up to a numerical factor, in the following we will ignore the factor $i/4$ from the Jacobian.

Even though the parametrization in (3.14) simplifies the integrands a lot, we will have dependencies on terms like $\langle ij\rangle$, which can be avoided by using a slightly modified representation

$$k_1 = \alpha_1\lambda_1\tilde{\lambda}_1 + \alpha_2\lambda_2\tilde{\lambda}_2 + \frac{\langle 23\rangle}{\langle 13\rangle}\alpha_3\lambda_1\tilde{\lambda}_2 + \frac{\langle 13\rangle}{\langle 23\rangle}\alpha_4\lambda_2\tilde{\lambda}_1, \tag{3.19}$$

leaving the Jacobian the same as before. With this parametrization our integrands will only depend on α_i and the two Mandelstam variables s and t.

3.2.1. Bubble

First we try to apply the dlog decomposition procedure to the bubble diagram, where using (3.19) the integrand reads

$$\mathcal{I}_2 = \frac{d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3 \wedge d\alpha_4}{(\alpha_1\alpha_2 - \alpha_3\alpha_4)(1 - \alpha_1 + \alpha_1\alpha_2 - \alpha_2 - \alpha_3\alpha_4)}. \tag{3.20}$$

If we now put α_2, α_3 and α_4 into dlog-factors, we get

$$\mathcal{I}_2 = d\alpha_1 \wedge d\log(-1 + \alpha_1 + \alpha_2) \wedge d\log(\alpha_3) \wedge \tag{3.21}$$
$$[d\log(\alpha_1\alpha_2 - \alpha_3\alpha_4) - d\log(1 - \alpha_1 - \alpha_2 + \alpha_1\alpha_2 - \alpha_3\alpha_4)]. \tag{3.22}$$

The remaining variable α_1 cannot be transformed into a dlog factor since the remaining function is a constant not dependent on α_1 any more and thus leading to a double pole at infinity. So the bubble integrand is an example where no dlog form exists.

3.2.2. Triangle

Unlike the bubble, the triangle can actually be brought into a dlog form. Writing k_1 as in equation (3.19) the integrand reads

$$\mathcal{I}_3 = \frac{d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3 \wedge d\alpha_4}{(\alpha_1\alpha_2 - \alpha_3\alpha_4)(-\alpha_2 + \alpha_1\alpha_2 - \alpha_3\alpha_4)(1 - \alpha_1 - \alpha_2 + \alpha_1\alpha_2 - \alpha_3\alpha_4)}. \tag{3.23}$$

Already in this simple case, the order in which we put the variables into the dlog forms matters. After putting α_1 into dlog factors we have

$$\mathcal{I}_3 = -\frac{1}{\alpha_2\alpha_3\alpha_4} d\log\left(\alpha_1\alpha_2 - \alpha_2 - \alpha_3\alpha_4\right) \tag{3.24}$$
$$+ \frac{1}{\alpha_2\left(\alpha_2^2 - \alpha_2 + \alpha_3\alpha_4\right)} d\log\left(\alpha_1\alpha_2 - \alpha_3\alpha_4\right) \tag{3.25}$$
$$+ \frac{\alpha_2 - 1}{\alpha_3\alpha_4\left(\alpha_2^2 - \alpha_2 + \alpha_3\alpha_4\right)} d\log\left(\alpha_2\alpha_1 - \alpha_1 - \alpha_2 - \alpha_3\alpha_4 + 1\right), \tag{3.26}$$

so we can see, that we cannot simply put α_2 into dlog factors, since this would lead to square root terms.

If we choose α_3 or α_4 next, however, we won't get any problems for the rest of the procedure.

The full dlog form for the triangle is

$$\begin{aligned}\mathcal{I}_3 =& d\log\left(\alpha_4\right) \wedge d\log\left(\alpha_2\right) \wedge d\log\left(\alpha_3\right) \wedge d\log\left(-\alpha_2 + \alpha_1\alpha_2 - \alpha_3\alpha_4\right) \\ &+ d\log\left(\alpha_4\right) \wedge d\log\left(\alpha_2\right) \wedge d\log\left(-\alpha_2 + \alpha_2^2 + \alpha_3\alpha_4\right) \wedge d\log\left(1 - \alpha_1 - \alpha_2 + \alpha_1\alpha_2 - \alpha_3\alpha_4\right) \\ &- d\log\left(\alpha_4\right) \wedge d\log\left(\alpha_2\right) \wedge d\log\left(\alpha_3\right) \wedge d\log\left(1 - \alpha_1 - \alpha_2 + \alpha_1\alpha_2 - \alpha_3\alpha_4\right) \\ &- d\log\left(\alpha_4\right) \wedge d\log\left(\alpha_2\right) \wedge d\log\left(-\alpha_2 + \alpha_2^2 + \alpha_3\alpha_4\right) \wedge d\log\left(\alpha_1\alpha_2 - \alpha_3\alpha_4\right).\end{aligned} \tag{3.27}$$

Bern et al. [16] found also a dlog form consisting of only a single term,

$$d\mathcal{I}_3 = d\log(\alpha_1\alpha_2-\alpha_3\alpha_4)\wedge d\log(\alpha_1\alpha_2-\alpha_3\alpha_4-\alpha_2)\wedge d\log(\alpha_1\alpha_2-\alpha_3\alpha_4-\alpha_1-\alpha_2+1)\wedge d\log\alpha_3, \tag{3.28}$$

or written in the original variables

$$d\mathcal{I}_3 = d\log k_1^2 \wedge d\log(k_1-p_1)^2 \wedge d\log(k_1-p_1-p_2)^2 \wedge d\log[(k_1-p_1)\cdot(k_1^*-p_1)], \tag{3.29}$$

where $k_1^* = \beta\lambda_2\tilde{\lambda}_1 + \lambda_1\tilde{\lambda}_1$ is one of the solutions to $k_1^2 = (k_1-p_1)^2 = (k_1-p_1-p_2)^2 = 0$.

3.2.3. Box

The box can also be written as a dlog form using the repeated partial fraction decomposition procedure which leads to a sum of 48 dlog forms. In this case it becomes obvious that our procedure does not lead to the most compact ways of writing integrands as dlog forms, since it is also possible to write the box with just a single dlog form as [16]

$$d\mathcal{I}_4 = d\log\frac{k_1^2}{(k_1-k_1^*)^2} \wedge d\log\frac{(k_1-p_1)^2}{(k_1-k_1^*)^2} \wedge d\log\frac{(k_1-p_1-p_2)^2}{(k_1-k_1^*)^2} \wedge \frac{(k_1+p_4)^2}{(k_1-k_1^*)^2}, \tag{3.30}$$

where $k_1^* = -\frac{\langle 14\rangle}{\langle 24\rangle}\lambda_2\tilde{\lambda}_1 + \lambda_1\tilde{\lambda}_1$ is one of the two solutions for k_1 to $k_1^2 = (k_1-p_1)^2 = (k_1-p_1-p_2)^2 = (k_1+p_4)^2 = 0$.

3.3. Algorithm for Automated Computation

From the method we applied in the last section an algorithm can be extracted, so that a computer program can be written that automatically derives the dlog forms for a given diagram or calculates the leading singularities. To derive the results that are presented in this thesis MATHEMATICA [19] was used to implement an algorithm for the calculation of leading singularities and dlog forms.

The algorithm, however, is not always successful and there are two reasons why the algorithm can fail. The first is the appearance of square roots in intermediate steps and the second is when a term in an intermediate step gets so large that a specific MATHEMATICA function is not able to handle this term in a reasonable time. In both cases, however, it was possible to improve the algorithm in such a way that these problems could be avoided in some special cases so that diagrams where the algorithm in its original form failed was eventually successful. Some of these improvements will be discussed in this section.

3.3.1. Choosing the Right Parametrization

One critical step that decides if the algorithm is successful or not in many cases is a good choice for the parametrization of the loop momenta. Since the standard Lorentz vectors almost immediately lead to expressions that are quadratic in all integration variables the algorithm will fail in most cases if we do not perform a variable transformation. For this thesis we concentrated on four-dimensional massless four-point diagrams. For these diagrams the spinor helicity variables turned out to be a particularly good choice. So for a loop momentum k_i we would use the parametrization, which we already used in previous examples,

$$k_i = \alpha_{i,1}\lambda_m\tilde{\lambda}_m + \alpha_{i,2}\lambda_n\tilde{\lambda}_n + \frac{\langle nj\rangle}{\langle mj\rangle}\alpha_{i,3}\lambda_m\tilde{\lambda}_n + \frac{\langle mj\rangle}{\langle nj\rangle}\alpha_{i,4}\lambda_n\tilde{\lambda}_m, \tag{3.31}$$

where $m, n, j \in \{1, 2, 3, 4\}$ and $m \neq n \neq j$. Note that we can make a different choice of m and n for each loop momentum k_i giving us many different possible parametrizations in a multi-loop diagram which can be quite helpful, because if the algorithm fails with one parametrization it might be successful with another parametrization. For a two-loop diagram such as the non-planar double box there are 36 different possible parametrizations of (3.31) and in this case the algorithm was successful in 21 of the 36 cases and failed in the other cases. In the successful cases the calculation time was roughly the same in all but two cases, where the program was more than an order of magnitude slower, so that for higher loop calculations choosing the wrong parametrization can make solving the diagram in a reasonable time impossible.

Analyzing diagrams with more external legs than the number of spacetime dimensions D one can also simply use the parametrization

$$k = \sum_{i=1}^{D} \alpha_i p_i. \tag{3.32}$$

Possible applications are e. g. four-point diagrams in two dimensions or five-point diagrams in four dimensions.

Another very promising parametrization are momentum twistor variables (see e. g. [4]), which can, however, only be used for planar diagrams.

3.3.2. Basic Algorithm

Assuming we have a Feynman integrand in an appropriate parametrization, we now want to list the explicit steps the algorithm performs to either calculate all leading singularities or derive a dlog form, which implies the knowledge of all the leading singularities. In both cases a successful calculation also implies that the integrand has only logarithmic singularities, which means that there are no poles of order two or higher in any parametrization that is related to the original loop variables by an algebraic variable transformation. So we start with the integrand

$$f(\alpha_1, ..., \alpha_k) = \frac{d\alpha_1 \wedge ... \wedge d\alpha_k \text{numerator}}{\text{denominator}}, \tag{3.33}$$

where $\alpha_1, ..., \alpha_k$ are integration variables and both numerator and denominator are multivariate polynomials in the variables $\alpha_1, ..., \alpha_k$. f may also depend on external variables like the Mandelstam variables s and t, which will be treated as constants.

The basic algorithm to calculate a dlog form for f is the following:

Algorithm 1 : Calculate dlog forms

1: Write both the numerator and the denominator of $f(\alpha_1, ..., \alpha_k)$ as products of irreducible polynomials[1] in the variables $\alpha_1, ..., \alpha_k$ and cancel common factors.

2: If any factor in the denominator of f containing any variable α_i has power bigger than one the algorithm stops and f has no dlog form.
For example $f = \frac{d\alpha_1 \wedge d\alpha_2}{\alpha_1(\alpha_1-\alpha_2)^2}$
has no dlog form.

3: If for any variable α_i the polynomial degree in the numerator is as big or bigger than in the denominator, the algorithm stops and f has no dlog form because of poles at infinity.
For example $f = \frac{d\alpha_1 \wedge d\alpha_2 \alpha_2}{\alpha_1(\alpha_1-\alpha_2)}$ has a pole at infinity for $\alpha_2 \to \infty$.

4: Choose a variable α_i that is linear in all denominator factors. If this is not possible, the algorithm stops without a result.
For example for $f = \frac{d\alpha_1 \wedge d\alpha_2}{\alpha_1(1+\alpha_1^2+\alpha_2^2)}$ the algorithm fails.

5: Perform a partial fraction decomposition with respect to α_i:

$$f(\alpha_1, ..., \alpha_k) = \sum_j \frac{d\alpha_i \wedge g_j(\alpha_1, ..., \hat{\alpha}_i, ..., \alpha_k)}{a_j \alpha_i - b_j},$$

where $\hat{\alpha}_i$ means that α_i is omitted and $(a_j \alpha_i - b_j)$ is an irreducible polynomial in $\alpha_1, ..., \alpha_k$. The terms a_j and b_j are polynomials in $\alpha_1, ..., \alpha_k$ but independent of α_i.

6: Put the denominators in dlog terms by writing

$$f(\alpha_1, ..., \alpha_k) = \sum_j d\log(a_j \alpha_i - b_j) \wedge g_j(\alpha_1, ..., \hat{\alpha}_i, ..., \alpha_k) \frac{1}{a_j}.$$

So g_j/a_j are the residues of f at the poles in α_i.

7: Repeat from step 1 for all $g_j(\alpha_1, ..., \hat{\alpha}_i, ..., \alpha_k)/a_j$ if $k \geq 2$.

If we do not need a dlog form and only want to know the leading singularities, we can omit all dlog factors in the algorithm.

3.3.3. Handling Terms with Quadratic Factors

Since the algorithm fails if there is no integration variable α_i that is linear in all denominator factors many integrands cannot be treated by this algorithm. We already discussed the possibility of choosing another parametrization of the loop variables. With this strategy, however, we observed cases where the calculation was successful

[1] Irreducible polynomials are polynomials that cannot be written as a product of two non-constant polynomials. For example $\alpha_1^2 - \alpha_2^2$ is not irreducible because it can be written as the product of the two irreducible polynomials $\alpha_1 + \alpha_2$ and $\alpha_1 - \alpha_2$

for all but the last two or four integration variables and then the algorithm stopped because of one specific term having a denominator factor being a polynomial that is quadratic in all remaining integration variables. Now changing the parametrization and completely restart the whole calculation seems not to be the most efficient way to handle this situation. The idea to improve the algorithm is to perform the variable transformation directly on the problematic terms. So suppose $T(k_1, y)$ is such a problematic term that depends on k_1 and other integration variables indicated by y. Now let us assume we have parametrized the loop momentum as

$$k_1 = \alpha_1 \lambda_1 \tilde{\lambda}_1 + \alpha_2 \lambda_2 \tilde{\lambda}_2 + \frac{\langle 23 \rangle}{\langle 13 \rangle} \alpha_3 \lambda_1 \tilde{\lambda}_2 + \frac{\langle 13 \rangle}{\langle 23 \rangle} \alpha_4 \lambda_2 \tilde{\lambda}_1, \tag{3.34}$$

so that $T(k_1, y)$ becomes a function of $\alpha_1, ..., \alpha_4$ and y, which we denote as $T_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4, y)$. Imagine now that T_1 is a function that cannot be handled by the algorithm, but on the other hand the parametrization

$$k_1 = \beta_1 \lambda_1 \tilde{\lambda}_1 + \beta_2 \lambda_3 \tilde{\lambda}_3 + \frac{\langle 32 \rangle}{\langle 12 \rangle} \beta_3 \lambda_1 \tilde{\lambda}_3 + \frac{\langle 12 \rangle}{\langle 32 \rangle} \beta_4 \lambda_3 \tilde{\lambda}_1, \tag{3.35}$$

inserted in $T(k_1, y)$ would lead to a function $T_2(\beta_1, \beta_2, \beta_3, \beta_4, y)$ that can be handled by the algorithm. The question is how we can directly change from $T_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4, y)$ to $T_2(\beta_1, \beta_2, \beta_3, \beta_4, y)$ without using the term $T(k_1, y)$. The reason for not using $T(k_1, y)$ is that this function is not available for the algorithm in intermediate steps of the calculation. The solution is to use the transformation:

$$\alpha_1 = \beta_1 + \frac{t\beta_2}{s} + \frac{t\beta_3}{s} + \beta_4, \tag{3.36}$$

$$\alpha_2 = -\beta_2 - \frac{t\beta_2}{s}, \tag{3.37}$$

$$\alpha_3 = \beta_2 + \frac{t\beta_2}{s} + \beta_3 + \frac{t\beta_3}{s}, \tag{3.38}$$

$$\alpha_4 = -\frac{t\beta_2}{s} - \beta_4, \tag{3.39}$$

and also change $d^4\alpha$ to $Jd^4\beta$ with J being the Jacobian determinant of the variable transformation. The transformation rules can be obtained by taking the scalar product of k_1 with an appropriate vector in both parametrizations (3.34) and (3.35) leading to e. g. $\alpha_1 = \frac{s}{2} k_1 \cdot (\lambda_2 \tilde{\lambda}_2) = \beta_1 + \frac{t\beta_2}{s} + \frac{t\beta_3}{s} + \beta_4$ and similar for the other transformations. So for a term $T_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ with the property that all variables α_i are quadratic in at least one of the denominator factors, we then would try to make all possible variable transformations of the type we just explained and hope that after at least one of these transformations we get an expression with an integration variable that is linear in all denominator factors. This type of transformation was a very useful improvement for the algorithm and rendered possible many diagrams that failed before.

Even though the type of transformation we just discussed is useful in many cases there are still many cases where no such transformation leads to a term where the algorithm can proceed. However, if we have calculated residues of all integration variables but two and we have to deal with a term where both integration variables are quadratic in exactly one denominator factor the problem can actually always be solved. One example where such a transformations can be successfully applied is the triangle with three massive external legs. So suppose we have the term

$$R = \frac{N(x,y)dx \wedge dy}{(Ax^2 + By^2 + Cxy + Dx + Ey + F)G(x,y)}, \tag{3.40}$$

where A, B, C, D, and E are constants independent of x and y, $G(x,y)$ is a polynomial in x and y, which can be written as product of factors that are all linear in x and y, and $N(x,y)$ is an arbitrary numerator. We now have to find a variable transformation that removes either x^2 or y^2. This can be achieved using the transformation

$$x = \tilde{x} + \frac{-C - \sqrt{-4AB + C^2}}{2A} y, \tag{3.41}$$

which does not lead to a Jacobian factor, and so we get

$$R = \frac{2AN(x(\tilde{x}), y)d\tilde{x} \wedge dy}{H(\tilde{x}, y)G(x(\tilde{x}), y)}, \tag{3.42}$$

with

$$H(\tilde{x}, y) = 2AF + 2AD\tilde{x} + 2A^2\tilde{x}^2 - CDy + 2AEy + \sqrt{-4AB + C^2}(-Dy - 2A\tilde{x}y), \tag{3.43}$$

so that we can now easily take residues of the poles in y.

3.3.4. Systematic Analysis of Numerators

One important application of the algorithm is to determine possible numerators for a given denominator structure that have constant leading singularities. To do this we need a slight extension of the algorithm that is described in section 3.3.2. So the integrand will have the following structure

$$f(y_1, y_2, ...) = \frac{n_1 t_1 + n_2 t_2 + ... + n_N t_N}{\text{denominator}}, \tag{3.44}$$

where $y_1, y_2, ...$ are again the integration variables, $n_1, ..., n_N$ are free parameters which may depend on external variables but not on integration variables, and $t_1, ..., t_N$ as well as the denominator are polynomial functions of the integration variables. To calculate the leading singularities we use our basic algorithm with the only difference that whenever we find a higher order pole or a pole at infinity, instead of stopping the algorithm we fix a minimal set of the free parameters n so that the undesired pole vanishes. The integrand

$$\frac{(n_1 + n_2 x)dx}{x(x-1)^2} \tag{3.45}$$

for example has a double pole at $x = 1$. However, if we fix n_2 by $n_2 = -n_1$ the integrand becomes

$$\frac{(n_1 - n_1 x)dx}{x(x-1)^2} = \frac{n_1 dx}{x(x-1)} \tag{3.46}$$

and thus the double pole vanishes. The leading singularities are n_1 and $-n_1$. So in this trivial example we have found that with the numerator $N = 1 - x$ the integrand

$$\frac{N dx}{x(x-1)^2} \tag{3.47}$$

has constant leading singularities. In chapter 4 we will use this kind of analysis to find numerators with constant leading singularities for several different diagrams.

3.4. Deriving Dlog Forms Using Building Blocks

The method we described so far is also suited to derive dlog forms in multi-loop diagrams. In some cases it is also possible to use another method, where we combine results from lower loop orders and thus get much more compact results. To use this method we first need some building blocks from lower loops, which we can use for higher loops.

3.4.1. Generalized Box

One very important building block is the generalized box [16]

$$\frac{d^4 k J}{F_1 F_2 F_3 F_4} = d\log\frac{F_1}{F^*} \wedge d\log\frac{F_2}{F^*} \wedge d\log\frac{F_3}{F^*} \wedge d\log\frac{F_4}{F^*}, \tag{3.48}$$

where J is the Jacobian determinant of the variable transformation $(l_0, l_1, l_2, l_3) \to (F_1, F_2, F_3, F_4)$ evaluated at $F_i = 0$ with $i = 1, 2, 3, 4$, which can most conveniently calculated using equation (3.16). Similar to equation (3.30), $F^* = (k - k^*)$, where k^* is again one of the solutions for k at $F_i = 0$ with $i = 1, 2, 3, 4$. Note that equation (3.48) is only correct if the left-hand side has only logarithmic singularities. One special case of this generalized box that can be quite useful is the box with massive external lines.

$$d^4 k_1 \frac{st}{k_1^2 (k_1 - P_1)^2 (k_1 - P_1 - P_2)^2 (k_1 + P_4)^2}, \tag{3.49}$$

The case where one has four massive external lines the Jacobian can only be expressed with a square root which make further calculations quite complicated. If one external particle is massless, the Jacobian J of (3.48) is $J = (p_1 + P_2)^2 (P_4 + P_1)^2 - P_2^2 P_4^2$, where p_1 is massless and P_2, P_3, P_4 are massive [16]. If one more external particle is massless the Jacobian simplifies even more, but we have to distinguish the two cases: In the first case where the two massless lines are neighbors the Jacobian is just $J = (p_1 + p_2)^2 (P_4 + p_1)^2$ and in the second case, where the massless lines are diagonally opposite the Jacobian is $J = (P_2 + p_1)^2 (P_4 + p_1)^2 - P_2^2 P_4^2 = (2P_2 \cdot q)(2P_2 \cdot \bar{q})$, with $q = \lambda_1 \tilde{\lambda}_3$ and $\bar{q} = \lambda_3 \tilde{\lambda}_1$.

Figure 3.2.: Planar and non-planar double box

3.4.2. Planar Double Box

As a simple example we will calculate the dlog form of the scalar double box (see also figure 3.2)

$$d\mathcal{I}^{(\mathrm{p})} = \frac{d^4k_1 d^4k_2}{k_1^2(k_1+p_1)^2(k_1-p_2)^2(k_1+k_2-p_2-p_3)^2 k_2^2(k_2-p_3)^2(k_2+p_4)^2} \tag{3.50}$$

Since the diagram is composed of two boxes with two neighbored massless external lines we can start by using the building block described in the last section. The (outgoing) external lines for the k_1 loop are $p_1, p_2, -k_2+p_3$ and k_2+p_4. Thus the Jacobian is $J = (p_1+p_2)^2(p_2-k_2+p_3)^2$ and we can write equation (3.50) as

$$d\mathcal{I}^{(\mathrm{p})} = \mathrm{dlog}\frac{k_1^2}{(k_1-k_1^*)^2} \wedge \mathrm{dlog}\frac{(k_1+p_2)^2}{(k_1-k_5^*)^2} \wedge \mathrm{dlog}\frac{(k_1-p_2)^2}{(k_1-k_1^*)^2} \wedge \mathrm{dlog}\frac{(k_1+k_2-p_2-p_3)^2}{(k_1-k_1^*)^2} \tag{3.51}$$

$$\wedge \frac{1}{(p_1+p_2)^2(p_2-k_2+p_3)^2} \frac{d^4k_2}{k_2^2(k_2-p_3)^2(k_2+p_4)^2} \tag{3.52}$$

In this case the Jacobian gives us a rest term, which we can immediately solve, since it nothing else than the integrand of the massless box. Thus the full dlog form reads

$$d\mathcal{I}^{(\mathrm{p})} = \mathrm{dlog}\frac{(p_2-k_2+p_3)^2}{(k_1-k_1^*)^2} \wedge \mathrm{dlog}\frac{(k_1+p_2)^2}{(k_1-k_1^*)^2} \wedge \mathrm{dlog}\frac{(k_1-p_2)^2}{(k_1-k_1^*)^2} \wedge \mathrm{dlog}\frac{(k_1+k_2-p_2-p_3)^2}{(k_1-k_1^*)^2} \tag{3.53}$$

$$\wedge\, \mathrm{dlog}\frac{(p_2-k_2+p_3)^2}{(k_2-k_2^*)^2} \wedge \mathrm{dlog}\frac{k_2^2}{(k_2-k_2^*)^2} \wedge \mathrm{dlog}\frac{(k_2-p_3)^2}{(k_2-k_2^*)^2} \wedge \mathrm{dlog}\frac{(k_2+p_4)^2}{(k_2-k_2^*)^2}. \tag{3.54}$$

3.4.3. Non-planar Double Box

The non-planar double box is an example, where no dlog form exists for a scalar numerator. Using a suitable numerator, we need another one-loop building block to construct the whole dlog form for the diagram. First, we show how a scalar numerator leads to a double pole, then we will motivate a well-suited numerator and finally we will present the dlog form for the non-planar double box with this numerator. The integrand of the non-planar double box is

$$d\mathcal{I}^{(\text{np})} = \frac{N d^4k_1 d^4k_2}{k_1^2(k_1+p_1)^2(k_1-p_3-p_4)^2 k_2^2(k_1+k_2)^2(k_1+k_2-p_4)^2(k_2+p_3)^2} \tag{3.55}$$

and we immediately see that we can again start using the building block of the box of the k_2-loop. The resultant Jacobian in this case is

$$J = (k_1-p_3)^2(k_1-p_4)^2 - (k_1-p_3-p_4)^2(k_1)^2 = (2k_1\cdot q)(2k_1\cdot\bar{q}), \tag{3.56}$$

where $q = \lambda_3\tilde{\lambda}_4$ and $q = \lambda_4\tilde{\lambda}_3$. The remaining integrand is thus

$$d\tilde{\mathcal{I}}^{(\text{np})} = \frac{N d^4k_1 d^4k_2}{k_1^2(k_1+p_1)^2(k_1-p_3-p_4)^2(2k_1\cdot q)(2k_1\cdot\bar{q})}, \tag{3.57}$$

and in this case we do not end up with another generalized box. To reveal the double pole for the case that N does not depend on the loop momenta let's switch to spinor helicity variables:

$$k_1 = \alpha_1\lambda_3\tilde{\lambda}_3 + \alpha_2\lambda_4\tilde{\lambda}_4 + \frac{\langle 41\rangle}{\langle 31\rangle}\alpha_3\lambda_3\tilde{\lambda}_4 + \frac{\langle 31\rangle}{\langle 41\rangle}\alpha_4\lambda_4\tilde{\lambda}_3. \tag{3.58}$$

The integrand in these variables reads

$$-\frac{N d\alpha_1\wedge d\alpha_2\wedge d\alpha_3\wedge d\alpha_4}{\alpha_3\alpha_4\left(\alpha_3\alpha_4-\alpha_1\alpha_2\right)\left(-\alpha_2\alpha_1+\alpha_1+\alpha_2+\alpha_3\alpha_4-1\right)s^2} \tag{3.59}$$

$$\times\frac{1}{\left(\left(-\alpha_2\alpha_1+\alpha_1+\alpha_3\alpha_4+\alpha_4\right)s+\left(\alpha_1-\alpha_2-\alpha_3+\alpha_4\right)t\right)}. \tag{3.60}$$

We start by taking the residues on $\alpha_3 = 0$ and $\alpha_4 = 0$ giving us:

$$\frac{N d\alpha_1\wedge d\alpha_2}{\alpha_1\alpha_2\left(\alpha_2\alpha_1-\alpha_1-\alpha_2+1\right)s^2\left(\alpha_1(-s)+\alpha_2\alpha_1 s-\alpha_1 t+\alpha_2 t\right)} \tag{3.61}$$

Now we take another residue at $\alpha_1 = 0$ and we finally see the double pole in the α_2-variable:

$$-\frac{N d\alpha_2}{\left(\alpha_2-1\right)\alpha_2^2 s^2 t}. \tag{3.62}$$

So we have shown that for a scalar Numerator N no dlog exists for the non-planar double box. If we allow N to depend on k_1 we can remove the double pole and in this case N must be proportional to α_2 or expressed with the original variables proportional to $k_1\cdot p_3 = \frac{s}{2}\alpha_2$. This numerator, however, only removes this special double pole and with different cuts we would expose another double pole. This leads to the question how we can find numerators having only logarithmic singularities with a given denominator. In section 4.2 we present a systematic analysis of possible numerators for the non-planar double box. At this point, however, we want to take a simpler approach to get an educated guess for a well suited numerator. For this purpose we go back to the integrand after we removed the first loop and this time we write the integrand in a non-factored way:

$$\tilde{\mathcal{I}}^{(\mathrm{np})} = \frac{N}{k_1^2(k_1+p1)^2(k_1-p_3-p_4)^2[(k_1-p_3)^2(k_1-p_4)^2-(k_1-p_3-p_4)^2(k_1)^2]} \tag{3.63}$$

The integrand in this form is quite complicated, but significantly simplifies if we consider only leading singularities around at $k_1^2 = 0$ or $(k_1 - p_3 - p_4)^2 = 0$. This is equivalent of computing the leading singularities of the integrand

$$\tilde{\mathcal{I}}^{(\mathrm{np})} = \frac{N}{k_1^2(k_1+p1)^2(k_1-p_3-p_4)^2(k_1-p_3)^2(k_1-p_4)^2}, \tag{3.64}$$

where it is now easy to determine a numerator that allows us to write this integrand in a dlog form, since in both cases where we take either $N = (k_1 - p_3)^2$ or $N = (k_1 - p_4)^2$ we are left with four propagators that form a scalar box. Even though this does not prove that these numerators are also well suited for the more complicated integrand in equation (3.63) they serve well as a first guess and in this case they turn out to be correct even in the more complicated case.

To get the full dlog form for the non-planar double box we can use the algorithm in section 3.3.2 leading to a very long expression. However, Bern et al. [16] found a very compact expression for the term in equation (3.57) and $N = su(k_1 - p_4)^2$:

$$\tilde{\mathcal{I}}^{(\mathrm{np})} = d\log\frac{k_1^2}{k_1\cdot\bar{q}} \wedge d\log\frac{(k_1+p_1)^2}{k_1\cdot\bar{q}} \wedge d\log\frac{(k_1+p_1+p_2)^2}{k_1\cdot q} \wedge d\log\frac{(k_1-k_1^*)^2}{k_1\cdot q}, \tag{3.65}$$

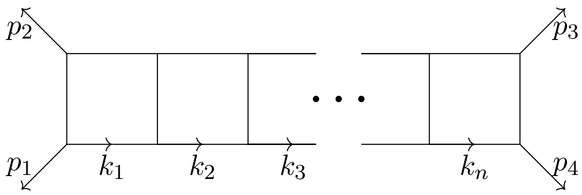
with $k_1^* = -\frac{\langle 34\rangle}{\langle 31\rangle}\lambda_1\tilde{\lambda}_4$, which is the solution to $k_1^2 = (k_1+p1)^2 = (k_1-p_3-p_4)^2 = 2k_1\cdot q = 0$. Combining this result with the dlog-form of the k_2-loop we have the full dlog form of the non-planar double box for this specific numerator.

3.5. Leading Singularities in Multi-loop Diagrams

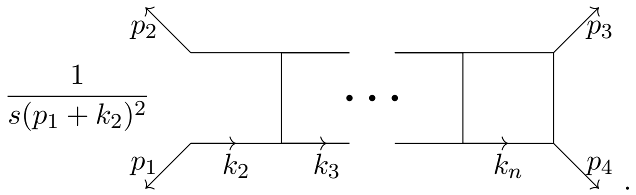
In this section we want to use the building-block method for the derivation of the last section but this time to calculate the leading singularities only. So the only difference in this case is that we do not need to know explicitly the dlog forms of the building blocks we use, but just the leading singularities of that building block and a proof that it only has logarithmic singularities which is equivalent of proofing that this building block can be written in a dlog form.

3.5.1. The n-loop Ladder

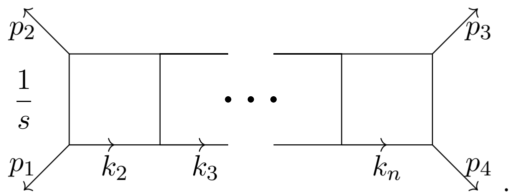
We begin with the n-point ladder which is a generalization of the planar double box to n loops.



The only building block we need for this example here is the box with two adjacent massive and two adjacent massless external legs. The internal momenta are massless as in all other examples. So the leading singularity of the k_1 box is $\frac{1}{(p_1+p_2)^2(p_1+k_2)^2}$ and using $(p_1+p_2)^2 = s$ we are left with the diagram



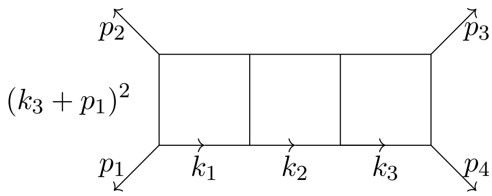
Since $\frac{1}{(p_1+k_1)^2}$ is exactly the missing propagator between p_1 and p_2 in the diagram we can complete the diagram.



Repeating this process $n-1$ times we will get another factor $\frac{1}{s}$ each time until only the k_n-box with four massless external lines and leading singularity $\frac{1}{st}$ is left. So the leading singularity of the n-loop ladder is $\frac{1}{s^n t}$.

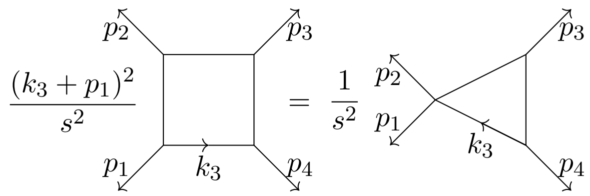
3.5.2. Three Loop Ladder with Numerator

The next example is the three loop ladder with a numerator, which does not cancel a propagator.



Since the k_1- and the k_2-loop are both unaffected by the numerator we can again use the box-building-blocks for the two left boxes leaving us the k_3-box with the numerator

$(k_3+p_1)^2$. This numerator, however, cancels a propagator, leaving us with a triangle with leading singularity $\frac{1}{s}$:



So the leading singularity of the whole diagram is $\frac{1}{s^3}$.

3.5.3. Triangle Building Blocks

So far we only considered the triangle with two massless external legs and one massive external leg. Many subtopologies of the planar double box or the three loop ladder, which we discuss in more detail in the next chapter, however, consist of triangles with two or three massive external legs.

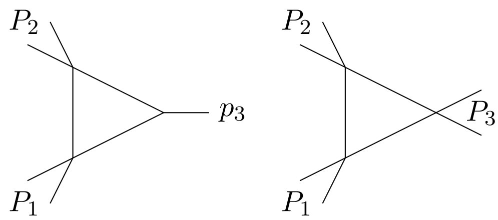


Figure 3.3.: Triangle with two and three massive external momenta. Massive external legs are indicated by a pair of two massless external lines.

The leading singularities of the triangle with two massive external masses $T_{2\text{m}}$ and the triangle with three external masses $T_{3\text{m}}$ with the external momenta defined as in figure 3.3 are

$$LS(T_{2\text{m}}) = \frac{1}{2(p_3\cdot P_2)} = -\frac{1}{2(p_3\cdot P_1)}, \tag{3.66}$$

$$LS(T_{3\text{m}}) = \frac{1}{2\sqrt{(P_1\cdot P_2)^2 - P_1^2P_2^2}}. \tag{3.67}$$

The leading singularity of $T_{3\text{m}}$ has a square root, which makes calculating the remaining leading singularities difficult if we use it as building block, so if possible it is better to use other building blocks for a given diagram. Even though the leading singularity of $T_{2\text{m}}$ is simpler, we can see that unlike in the case of the box with two adjacent massless legs the leading singularity in this case is not like a propagator. So using it as a building block, the rest of the diagram will contain building blocks with a numerator structure we have not discussed so far. One such new building block has the following propagator structure,

$$T_{\text{generic}} = \frac{d^4k}{2[p_1 \cdot (k + P_2)]k^2(k - P_3)^2}, \tag{3.68}$$

where p_1 is a massless external leg while P_2 and P_3 may have masses. There may be further external legs with arbitrary masses not contributing to the propagators. The leading singularity in this case is

$$LS(T_{\text{generic}}) = \frac{1}{2(p_1 \cdot P_2)}. \tag{3.69}$$

Sometimes we also need the leading singularity of the same integrand with an additional propagator

$$B_{\text{generic}} = \frac{d^4k}{2[p_1 \cdot (k + P_2)](k + P_2)^2k^2(k - P_3)^2}, \tag{3.70}$$

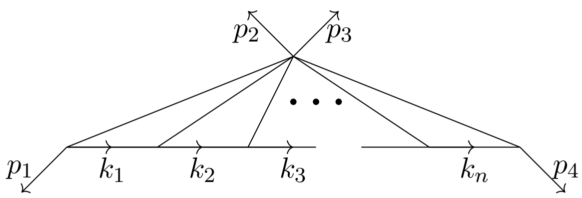
in which case the leading singularity is

$$LS(B_{\text{generic}}) = \frac{1}{(p_1 + P_2)^2(P_2 + P_3)^2 - P_2^2(p_1 + P_2 + P_3)^2}. \tag{3.71}$$

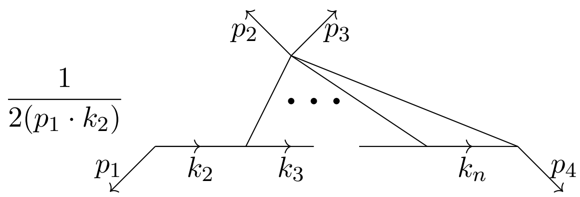
The leading singularities of T_{generic} and B_{generic} are the same as for the triangle and the box with one massless external leg p_1.

3.5.4. Iterative Triangles

Here we want to discuss two examples where we use the building blocks we defined in the last subsection. The first example is an aggregation of triangles:



As a first step we need the building block of a triangle with two massive external legs leaving us with the following diagram:



The k_2-loop now has the structure of T_{gen} with the propagators

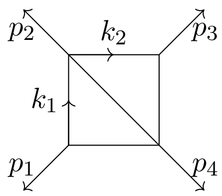
$$\frac{d^4k_2}{2(p_1 \cdot k_2)k_2^2(k_2 - k_3)^2}. \tag{3.72}$$

Since the leading singularity of this building block is $\frac{1}{2(p_1 \cdot k_3)}$ we observe a self-repeating pattern so by repeating the pattern until k_n we end up with

$$\frac{d^4k_n}{2(p_1 \cdot k_n)k_n^2(k_n - p_4)^2}, \tag{3.73}$$

which has leading singularity $\frac{1}{2(p_1 \cdot p_4)} = \frac{1}{t}$, which is thus also the leading singularity of the whole diagram.

Another interesting diagram is the planar double box where two opposite propagators are canceled.



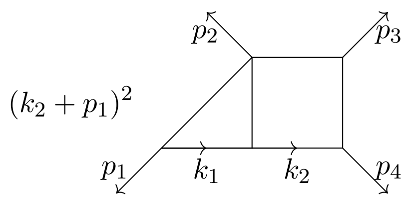
The leading singularity of the k_1-loop is $\frac{1}{2p_1 \cdot (k_2+p_2)}$. So again we have the propagator structure of T_{gen} for the k_2-loop:

$$\frac{d^4k_2}{2[p_1 \cdot (k_2 + p_2)]k_2^2(k - p_3)^2} \tag{3.74}$$

and thus the leading singularity of the whole diagram is $\frac{1}{2(p_1 \cdot p_3)} = \frac{1}{-s-t} = \frac{1}{u}$, which is remarkable since we will later show that this is the only subtopology of the planar double box not having a leading singularity that is a just a product of s and t.

3.5.5. Diagrams with Mixed Leading Singularities

So far we only considered examples where all leading singularities were the same up to numerical constants. In general, however, diagrams can have different leading singularities so that we cannot normalize the diagram in such a way that the diagram has constant leading singularities. In these cases is is always possible to find a linear combination of diagrams that have again constant leading singularities. One example for this case is the planar double box where we removed one propagator in the k_1-loop and have a numerator in the k_2 loop.



The leading singularity of the k_1-triangle is $\frac{1}{2p_1 \cdot k_2}$ and the remaining k_2-integrand

$$\begin{aligned} &\frac{d^4 k_2 (k_2^2 + 2p_1 \cdot k_2)}{2(p_1 \cdot k_2) k_2^2 (k_2 - p_4)^2 (k_2 + p_1 + p_2)^2} \\ =&\frac{d^4 k_2}{2(p_1 \cdot k_2)(k_2 - p_4)^2 (k_2 + p_1 + p_2)^2} + \frac{d^4 k_2}{k_2^2 (k_2 - p_4)^2 (k_2 + p_1 + p_2)^2}, \end{aligned} \tag{3.75}$$

where we expanded the numerator so that we could write it as a sum of two diagrams where the numerators canceled. These two remaining terms, however, have different leading singularities. The first has leading singularity $\frac{1}{u}$ and the second $\frac{1}{s}$. So we can conclude that for a numerator $(k_2 + p_1)^2$ we get mixed leading singularities. If we only want numerators with constant leading singularities, which means that we have to consider linear combinations of different numerators N_i and for the present case we would find

$$N_1 = u[(k_2 + p_1)^2 - k_2^2] \quad \text{and} \tag{3.76}$$

$$N_2 = s\, k_2^2 \tag{3.77}$$

to be such numerators.

4. Results

In this chapter we will present a systematic analysis of the leading singularities of planar and non-planar four-point two-loop diagrams and planar four-point three-loop diagrams, each with massless external momenta and massless propagators. The strategy is to start with a parent diagram that has the maximal number of propagators of the corresponding loop order. In the two-loop case there are two independent parent diagrams, the planar and the non-planar double box, whereas in three loops there are 9 diagrams (see e. g. [16]) from which we will only consider the two planar diagrams, which we call diagram A and E, following the notation of [16] and [20].

We do not have to consider diagrams with propagators that are independent of the loop momentum separately, since such a diagram is always the same as the corresponding diagram without this propagator up to a constant. We also do not consider diagrams consisting of a triangle with one massless external leg and the other two legs being propagator lines that connect to the rest of the diagram as indicated by figure 4.1, where p_1 is the external leg and k_1 and $k_1 - p_1$ the momenta of the propagator that are connected to the rest of the diagram.

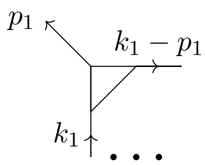


Figure 4.1.: Diagrams containing this subdiagram will not be considered

The reason for not considering such diagrams is that they are always trivially related by IBP-relations to integrals where the propagator k_1^2 or $(k_1 - p_1)^2$ is removed.

So starting with a parent diagram we make an ansatz for the numerator containing all possible products of inverse propagators and irreducible numerators up to a given power of the loop momenta.

Since in four point kinematics, we have only three linearly independent external momenta p_1, p_2, p_3, we cannot express scalar products of loop momenta with an arbitrary constant vector only as a linear combination of scalar products of the loop momentum with external momenta. An additional vector that forms a basis together with the external momenta is $\omega_\mu = \epsilon_{\mu\nu\rho\sigma} p_1^\nu p_2^\rho p_3^\sigma$. However, since integrals proportional to ω_μ vanish after integration, we will not consider numerators containing the term $k_i \cdot \omega$. We also do not have to consider numerators containing higher powers of $k_i \cdot \omega$ because $(k_i \cdot \omega)^2$ can be expressed in terms of k_i^2 and $k_i \cdot p_j$, with $j \in \{1, 2, 3\}$.

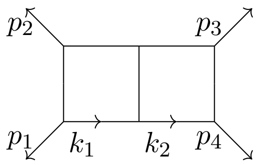


Figure 4.2.: Planar double box

4.1. Planar Double Box

We will now present a systematic analysis of possible numerators for the planar double box having constant leading singularities. We will show that all integrands correspond to uniform transcendental weight functions after integration. Furthermore, we will analyze additional properties of these integrals such as infrared finiteness and use these integrals as a start to construct a new basis.

We have four massless outgoing external momenta p_1, p_2, p_3, p_4, but due to momentum conservation only three are independent, so we choose p_1, p_2, p_3 and we have $p_4 = -p_1 - p_2 - p_3$. The only independent Lorentz-invariant variables that can be formed from these momenta are $s = (p_1 + p_2)^2$ and $t = (p_2 + p_3)^2$.

Using the labeling indicated in figure 4.2 we have the integrand

$$d\mathcal{I}^{(\mathrm{p})} = \frac{d^D k_1 d^D k_2 N(k_1, k_2)}{k_1^2 (k_1 + p_1)^2 (k_1 + p_1 + p_2)^2 k_2^2 (k_2 + p_1 + p_2)^2 (k_2 + p_1 + p_2 + p_3)^2 (k_1 - k_2)^2}, \tag{4.1}$$

with an arbitrary numerator $N(k_1, k_2)$. Building all scalar products between the internal momenta and the external momenta and between the internal momenta among each other we get $k_1^2, k_1 \cdot p_1, k_1 \cdot p_2, k_1 \cdot p_3, k_2^2, k_2 \cdot p_1, k_1 \cdot p_2, k_1 \cdot p_3$, and $k_1 \cdot k_2$. Seven of these scalar products can be written as linear combinations of inverse propagators, while two are irreducible. We could use these scalar products as basis terms for a general numerator ansatz, however, to simplify the calculation we instead choose the seven inverse propagators and change the two irreducible scalar products with two further propagator like terms. So we define the following integrand family,

$$\begin{aligned} J_{a_1,\ldots,a_9} = & \frac{d^D k_1 d^D k_2}{[-k_1^2]^{a_1}[-(k_1+p_1)^2]^{a_2}[-(k_1+p_1+p_2)^2]^{a_3}[-k_2^2]^{a_5}} \\ & \times \frac{[-(k_1+p_1+p_2+p_3)^2]^{-a_4}[-(k_2+p_1)^2]^{-a_6}}{[-(k_2+p_1+p_2)^2]^{a_7}[-(k_1+p_1+p_2+p_3)^2]^{a_8}[-(k_1-k_2)^2]^{a_9}}, \end{aligned} \tag{4.2}$$

which allows us to write for example the scalar double box integrand as $J_{1,1,1,0,1,0,1,1,1}$. Lowering indices of this integrand is equivalent to canceling propagators or introducing numerators.

Since we will also consider the integrals of the planar double box family, we define

$$I_{a_1,\ldots,a_9} := \frac{e^{2\epsilon\gamma_E}}{(i\pi^{D/2})^2} \int J_{a_1,\ldots,a_9}, \tag{4.3}$$

similar to the notation in [3].

4.1.1. Integrand Basis

Our first goal is to find all possible numerators for the planar double box, where the integrand has only constant leading singularities or equivalently can be written in a dlog form. For this purpose we make the following general ansatz for the numerator in equation (4.1)

$$
\begin{aligned}
N = & n_1 + k_1^2 n_2 + (p_1+k_1)^2 n_3 + (p_1+p_2+k_1)^2 n_4 \\
& + (p_1+p_2+p_3+k_1)^2 n_5 + k_2^2 n_6 + (p_1+k_2)^2 n_7 \\
& + (p_1+p_2+k_2)^2 n_8 + (p_1+p_2+p_3+k_2)^2 n_9 + (k_1-k_2)^2 n_{10} \\
& + k_1^2 k_2^2 n_{11} + k_1^2 (p_1+k_2)^2 n_{12} + k_1^2 (p_1+p_2+k_2)^2 n_{13} \\
& + k_1^2 (p_1+p_2+p_3+k_2)^2 n_{14} + (p_1+k_1)^2 k_2^2 n_{15} \\
& + (p_1+k_1)^2 (p_1+k_2)^2 n_{16} + (p_1+k_1)^2 (p_1+p_2+k_2)^2 n_{17} \\
& + (p_1+k_1)^2 (p_1+p_2+p_3+k_2)^2 n_{18} + (p_1+p_2+k_1)^2 k_2^2 n_{19} \\
& + (p_1+p_2+k_1)^2 (p_1+k_2)^2 n_{20} + (p_1+p_2+k_1)^2 (p_1+p_2+k_2)^2 n_{21} \\
& + (p_1+p_2+k_1)^2 (p_1+p_2+p_3+k_2)^2 n_{22} \\
& + (p_1+p_2+p_3+k_1)^2 k_2^2 n_{23} \\
& + (p_1+p_2+p_3+k_1)^2 (p_1+k_2)^2 n_{24} \\
& + (p_1+p_2+p_3+k_1)^2 (p_1+p_2+k_2)^2 n_{25} \\
& + (p_1+p_2+p_3+k_1)^2 (p_1+p_2+p_3+k_2)^2 n_{26}.
\end{aligned}
\tag{4.4}
$$

Having in mind that for one-loop diagrams the bubble integrand has no dlog form we restrict our ansatz to integrands where each loop momentum scales not less than $1/(k_i^2)^3$ for $k_i \to \infty$. Since in the denominator both loop momenta scale as $(k_i^2)^4$ for $k_i \to \infty$ we build the numerator as a linear combination of products $P_i P_j$ that do not scale more than k_i^2 for both loop momenta. This ansatz can be extended by also including terms like $(k_i \cdot p_j)^2$ without violating our power constraint, however, in the case of the planar double box we found that such factors will always lead to double poles and thus do not contribute to the integrand basis we are going to construct.

We can also write this ansatz with the notation defined in equation (4.2) as

$$
J^{(\mathrm{p})} = \sum_i n_i J_{a_{1,i},\ldots,a_{15,i}}
\tag{4.5}
$$

where $a_j \leq 1$ for $j \in \{1,2,3,5,7,8,9\}$, $a_j \leq 0$ for $j \in \{4,6\}$, $a_1+a_2+a_3+a_4+a_9 \geq 3$, and $a_5+a_6+a_7+a_8+a_9 \geq 3$.

To obtain the leading singularities we calculate in $D = 4$ dimensions and use the algorithm of section 3.3.2 and also use the methods we described in section 3.3.4.

Requiring only logarithmic singularities will lead to the following constraints on the parameters n_i:

$$n_{26} = 0, \quad n_{16} = 0, \quad n_{24} = -n_{18}$$

After imposing these constraints we find a set of leading singularities from which we can choose maximal 23 that are linear independent:

$$\begin{aligned}
&-\frac{n_5 t + n_9 t - n_1}{s^2 t}, \frac{n_5 t - n_1}{s^2 t}, \frac{n_5 - n_{18} s}{s^2}, \frac{n_7 t - n_1}{s^2 t}, \\
&-\frac{n_8 s - n_1}{s^2 t}, -\frac{n_3 - n_{17} s}{s^2}, -\frac{n_{18} t^2 + n_5 t + n_7 t + n_{10} t - n_1}{s^2 t}, \\
&\frac{n_1}{s^2 t}, -\frac{n_{15} s - n_3}{s^2}, \frac{-n_3 t - n_7 t + n_1}{s^2 t}, \frac{-n_{21} s^2 + n_4 s + n_8 s - n_1}{s^2 t}, \\
&\frac{-n_{22} st + n_4 s + n_9 t - n_1}{s^2 t}, -\frac{n_4}{st}, -\frac{n_{20} t - n_4}{st}, \\
&\frac{n_{19} s^2 t + n_{20} s^2 t + n_{18} st^2 - n_{23} st^2 - n_4 s(s+t) + n_5 t(s+t)}{s^2 t(s+t)}, \\
&\frac{n_2 s - n_1}{s^2 t}, -\frac{n_{12} s^2(-t) - n_{13} s^2 t - n_{18} st^2 + n_{25} st^2 + n_2 s(s+t) - n_5 t(s+t)}{s^2 t(s+t)}, \\
&\frac{n_{25} st - n_8 s - n_5 t + n_1}{s^2 t}, \frac{-n_{12} st + n_2 s + n_7 t - n_1}{s^2 t}, \\
&-\frac{-n_{14} st + n_2 s + n_9 t - n_1}{s^2 t}, \frac{-n_{11} s^2 + n_2 s + n_6 s - n_1}{s^2 t}, \frac{n_{23} s - n_5}{s^2}, -\frac{n_6 s - n_1}{s^2 t}.
\end{aligned} \tag{4.6}$$

Note that the number of linear independent logarithmic singularities is the same as the number of free parameters n_i which are left after imposing the constraint. So we can easily find values in terms of s and t for the parameters n_i to get a set of 23 linear independent numerators all having constant leading singularities not depending on s or t.

The full list of integrands using the notation of equation (4.2) is given in table 4.1.2. In figure 4.3 we present the diagrams in a graphical form, where irreducible numerators are indicated by wavy lines in the corresponding loop.

The correctness of the leading singularities can also be verified for all but the last integrand using the graphical methods of section 3.5.

4.1.2. Testing Uniform Transcendental Weight

In this section we want to check if all integrands with constant logarithmic singularities are uniform transcendental weight (UT) functions after integration in the case of the planar double box.

To do so we use integration by parts (IBP) relations to express them in terms of a basis where the UT-property is already proven [3].

$$f_1 = -\epsilon^2 t(-s)^{2\epsilon} t I_{0,2,0,0,0,0,0,1,2} \tag{4.7}$$
$$f_2 = \epsilon^2 t(-s)^{2\epsilon} s I_{0,0,2,0,1,0,0,0,2} \tag{4.8}$$
$$f_3 = \epsilon^3 t(-s)^{2\epsilon} s I_{0,1,0,0,1,0,1,0,2} \tag{4.9}$$
$$f_4 = -\epsilon^2 t(-s)^{2\epsilon} s^2 I_{2,0,1,0,2,0,1,0,0} \tag{4.10}$$
$$f_5 = \epsilon^3 t(-s)^{2\epsilon} st I_{1,1,1,0,0,0,0,1,2} \tag{4.11}$$
$$f_6 = -\epsilon^4 t(-s)^{2\epsilon} (s+t) I_{0,1,1,0,1,0,0,1,1} \tag{4.12}$$
$$f_7 = -\epsilon^4 t(-s)^{2\epsilon} s^2 t I_{1,1,1,0,1,0,1,1,1} \tag{4.13}$$
$$f_8 = -\epsilon^4 t(-s)^{2\epsilon} s^2 I_{1,1,1,0,1,-1,1,1,1} \tag{4.14}$$

A way to proof the uniform transcendental weight property of these integrals is to show that they satisfy the relation

$$\partial_x \vec{f} = \epsilon \left(\frac{a}{x} + \frac{b}{1+x} \right) \vec{f} \tag{4.15}$$

where $x = t/s$, $\vec{f} = (f_1, f_2, ..., f_8)^T$, and a, b are matrices, which do not depend on x or ϵ. Using IBP-relations the matrices can be determined as

$$a = \begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{2} & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & -2 & 0 & 0 \\ -3 & -3 & 0 & 0 & 4 & 12 & -2 & 0 \\ \frac{9}{2} & 3 & -3 & -1 & -4 & -18 & 1 & 1 \end{pmatrix}, \tag{4.16}$$

and

$$b = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{3}{2} & 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 3 & 6 & 6 & 2 & -4 & -12 & 2 & 2 \\ -\frac{9}{2} & -3 & 3 & -1 & 4 & 18 & -1 & -1 \end{pmatrix}. \tag{4.17}$$

With this differential equation, the requirement that the integrals are finite for $x = -1$ and the knowledge of the trivial propagator-type integrals f_2 and f_4, the other integrals can be solved order by order in ϵ and the special form of the differential equation (4.15) guarantees the uniform transcendental weight property of the integrals to all orders in ϵ.

Using IBP-relations we now relate the integrals

$$g_i := \frac{e^{2\epsilon\gamma_E}\epsilon^{-4}(-s)^{2\epsilon}}{(i\pi^{D/2})^2}\int j_i \tag{4.18}$$

to the basis f_i and we find

$$g_i = M_{ij} f_j \tag{4.19}$$

with

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -\frac{3}{4} & -\frac{3}{4} & 0 & 0 & 1 & 3 & 0 & 0 \\ -\frac{3}{4} & -\frac{3}{4} & 0 & 0 & 1 & 3 & 0 & 0 \\ -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{3}{4} & -\frac{3}{4} & 0 & 0 & 1 & 3 & 0 & 0 \\ -\frac{3}{4} & -\frac{3}{4} & 0 & 0 & 1 & 3 & 0 & 0 \\ 0 & \frac{1}{4} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{3}{2} & -3 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -\frac{3}{2} & -3 & -1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 & -\frac{1}{3} & -1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 & -\frac{1}{3} & -1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 & -\frac{1}{3} & -1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 & -\frac{1}{3} & -1 & 0 & 0 \\ \frac{9}{4} & -\frac{1}{2} & -\frac{9}{2} & -\frac{3}{2} & -\frac{4}{3} & -7 & \frac{1}{2} & \frac{3}{2} \end{pmatrix} \tag{4.20}$$

Since the transformations matrix M does not depend on x or ϵ the integrals g_i are UT-functions. M is a rank 8 matrix and thus we can select 8 integrals g_i to form another UT-basis, which we will do in the following subsections by choosing integrals having additional useful properties.

4.1.3. Finite Integrals

Because of the massless propagators all integrals of the form $\epsilon^{-4}(-s)^{-2\epsilon}g_i$ are potentially infrared divergent. Since the integrals g_i without the prefactor are all finite, the order of divergence is at most four. If the first terms of the ϵ-expansion of any linear combination of g_i vanishes, the order of divergence reduces.

The question is now, whether there are linear combinations of integrals of the form $\epsilon^{-4}(-s)^{-2\epsilon}g_i$ that are of order ϵ^0 and thus correspond to a finite integral .

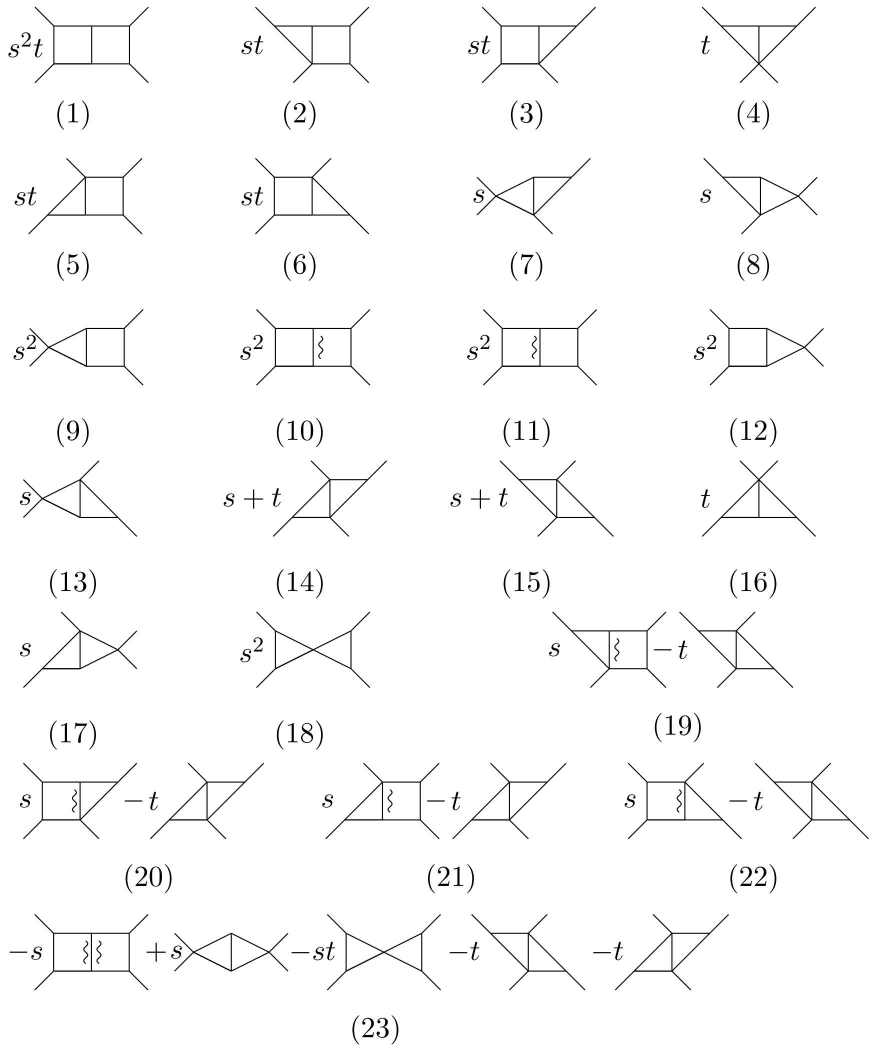


Figure 4.3.: Planar double box integrands with constant leading singularities. A wavy line in a loop indicates a numerator depending on the corresponding loop variable.

$j_1 = s^2 t J_{1,1,1,0,1,0,1,1,1}$	$j_2 = st J_{0,1,1,0,1,0,1,1,1}$	$j_3 = st J_{1,1,1,0,0,0,1,1,1}$
$j_4 = t J_{0,1,1,0,0,0,1,1,1}$	$j_5 = st J_{1,1,0,0,1,0,1,1,1}$	$j_6 = st J_{1,1,1,0,1,0,0,1,1}$
$j_7 = s J_{1,0,1,0,0,0,1,1,1}$	$j_8 = s J_{0,1,1,0,1,0,1,0,1}$	$j_9 = s^2 J_{1,0,1,0,1,0,1,1,1}$
$j_{10} = s^2 J_{1,1,1,0,1,-1,1,1,1}$	$j_{11} = s^2 J_{1,1,1,-1,1,0,1,1,1}$	$j_{12} = s^2 J_{1,1,1,0,1,0,1,0,1}$
$j_{13} = s J_{1,0,1,0,1,0,0,1,1}$	$j_{14} = (s+t) J_{1,1,0,0,0,0,1,1,1}$	$j_{15} = (s+t) J_{0,1,1,0,1,0,0,1,1}$
$j_{16} = t J_{1,1,0,0,1,0,0,1,1}$	$j_{17} = s J_{1,1,0,0,1,0,1,0,1}$	$j_{18} = s^2 J_{1,1,1,0,1,0,1,1,0}$
$j_{19} = s J_{0,1,1,0,1,-1,1,1,1} + t J_{0,1,1,0,1,0,0,1,1}$	$j_{20} = t J_{1,1,0,0,0,0,1,1,1} + s J_{1,1,1,-1,0,0,1,1,1}$	
$j_{21} = t J_{1,1,0,0,0,0,1,1,1} + s J_{1,1,0,0,1,-1,1,1,1}$	$j_{22} = t J_{0,1,1,0,1,0,0,1,1} + s J_{1,1,1,-1,1,0,0,1,1}$	
$j_{23} = -t J_{0,1,1,0,1,0,0,1,1} + s J_{1,0,1,0,1,0,1,0,1} - t J_{1,1,0,0,0,0,1,1,1} - s J_{1,1,1,-1,1,-1,1,1,1} - st J_{1,1,1,0,1,0,1,1,0}$		

Table 4.1.: Planar double box integrands with constant leading singularities

The divergent pole occurs when an external momentum is collinear to the momenta of the adjacent propagators. This pole, however, vanishes if the corresponding corner cut vanishes.

Taking a corner cut means setting the two propagators next to an external momentum zero which is equivalent to demanding the loop momenta of these propagators to be proportional to the external momentum. So in practice we make the following replacement if the loop momentum is k and the external momentum in that corner is p:

$$\frac{d^4 k f(k)}{k^2 (k+p)^2} \rightarrow dx f(xp). \tag{4.21}$$

To find finite integrals as linear combinations of g_i we make a general ansatz $\tilde{n}_i g_i$ with some rational numbers $\tilde{n}_i$ and determine these, so that the four corner-cuts vanish.

As a result we get two linear independent solutions

$$b_1 = g_1 + g_2 + g_3 + g_4 + g_5 + g_6 + g_{10} + g_{11} - g_{14} \tag{4.22}$$
$$- g_{15} + g_{16} - g_{18} + g_{19} + g_{20} + g_{21} + g_{22}, \tag{4.23}$$
$$b_2 = g_{23}.$$

These two integrals will form the first two basis vectors of the new basis we are going to describe.

As a next step we will discuss the cases where we take less than four corner cuts at the same time and see if it will reduce the power of the infrared pole.

It turns out that taking just one corner cut will not reduce the degree of divergence in any sense. When taking two corner cuts at the same time, the first two terms in the ϵ-expansion vanish in two cases, so that the resulting integrals have quadratic divergences. We get these two cases when we take the corner cuts at p_1 and p_3 or p_2 and p_4. If we take corner cuts of pairs of corners with the same loop momentum the term $1/\epsilon^3$ will vanish but not the term $1/\epsilon^4$. Also for adjacent corners with different loop momenta we still have a $1/\epsilon^4$ divergence. So only cutting diagonal corners successfully reduces the order of divergence and we get for both combinations five linear independent solutions each and when we combine them, we still have five solutions.

So we can add three more integrals to the basis we are going to construct. We choose

$$\begin{aligned} b_3 &= g_2 + g_4 + g_{19} \\ b_4 &= g_{13} \\ b_5 &= g_{15} \end{aligned} \tag{4.24}$$

4.1.4. New Basis

To complete our basis we finally choose three more integrals that have the property to only depend on s or t. From the seven integrals that depend only on s and the two that depend only on t we can choose g_4, g_9 and g_{18}, since they are linearly independent to the rest of the basis. So our final basis is the following:

$$\begin{aligned} b_1 &= g_1 + g_2 + g_3 + g_4 + g_5 + g_6 + g_{10} + g_{11} - g_{14} \\ &\quad - g_{15} + g_{16} - g_{18} + g_{19} + g_{20} + g_{21} + g_{22}, \\ b_2 &= g_{23}. \\ b_3 &= g_2 + g_4 + g_{19} \\ b_4 &= g_{13} \\ b_5 &= g_{15} \\ b_6 &= g_9 \\ b_7 &= g_{18} \\ b_8 &= g_4 \end{aligned} \tag{4.25}$$

4.2. Non-planar Double Box

In this section we present the results for the systematic analysis of numerators with constant leading singularities for the non-planar double box. Using the labeling of figure 4.4 the integrand in four dimensions is

$$J^{\text{np}}_{a_1,\ldots,a_9} = \frac{d^4k_1 d^4k_2}{[k_1^2]^{a_1}[(k_1+p_2)^2]^{a_2}[(k_1-p_3-p_4)^2]^{a_3}} \tag{4.26}$$

$$\times \frac{[(k_1-p_3)^2]^{-a_8}[(k_2+p_2)^2]^{-a_9}}{[(k_1-k_2)^2]^{a_4}[k_2^2]^{a_5}[(k_2-p_3)^2]^{a_6}[(k_1-k_2-p_4)^2]^{a_7}}, \tag{4.27}$$

where the first seven indices a_i are associated to the propagators of the non-planar double box if they are positive and to numerators if they are negative. The last two indices must always be zero or negative and thus can only be associated to numerators.

Different from the planar double box there are five instead of four propagators dependent on k_1, so we also have to consider numerators scaling like $(k_1^2)^2$ for $k_1 \to \infty$.

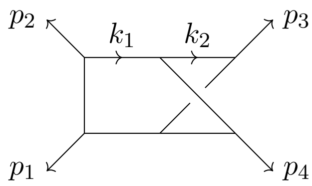


Figure 4.4.: Non-planar double box

We make the ansatz

$$J^{\mathrm{np}} = \sum_i n_i J^{\mathrm{np}}_{a_{1,i},\ldots,a_{9,i}}, \tag{4.28}$$

where $a_j \leq 1$ for $j \in \{1, ..., 7\}$, $a_j \leq 0$ for $j \in \{8, 9\}$, $a_1 + a_2 + a_3 + a_4 + a_7 + a_8 \geq 3$, and $a_4 + a_5 + a_6 + a_7 + a_9 \geq 3$ for each integrand in the sum. This defines an ansatz with 70 free parameters n_i and using the algorithm of section 3.3.2 we have to fix 34 parameters to remove double poles. Eventually we get 36 linear independent leading singularities and the same number of free parameters. So we can find 36 linear independent integrands with constant leading singularities listed in tables 4.2 and 4.3. 22 of the integrands are purely planar and are thus already known by the analysis of the planar double box. The other 14 integrands containing non-planar diagrams are represented graphically in figure 4.5. From section 3.4.3 we already know $j^{(\mathrm{np})}_{26}$, which is the non-planar double box with numerator $su(k_1 - p_3)^2$ (note, that the labeling of the propagators is different in section 3.4.3). From the symmetry of the diagram we can directly conclude that also $st(k_1 - p_4)^2$ must be a numerator of an integrand with constant leading singularities. However, $(k_1 - p_4)^2$ is not part of the expression in (4.26), so we have to write it as a linear combination of the propagator terms in (4.26) as

$$st(k_1 - p_4)^2 = [(k_1 - p_3 - p - 4)^2 - (k_1 - p_3)^2 + k_1^2 - s]st. \tag{4.29}$$

Thus we can identify $j^{(\mathrm{np})}_{31}$ as the integrand with the numerator $st(k_1 - p_4)^2$. Using equation (4.29) one can find further integrands that are related by symmetry.

Since unlike in the case of the planar double box we do not have a list of master integrals where the uniform transcendental weight property is already proven, we show it for our solution by directly deriving the differential equations. Using IBP-relations we find 12 master integrals and we can choose

$$\vec{f}^{(\mathrm{np})} = \left(f^{(\mathrm{np})}_1, f^{(\mathrm{np})}_2, f^{(\mathrm{np})}_3, f^{(\mathrm{np})}_4, f^{(\mathrm{np})}_5, f^{(\mathrm{np})}_7, f^{(\mathrm{np})}_{15}, f^{(\mathrm{np})}_{16}, f^{(\mathrm{np})}_{23}, f^{(\mathrm{np})}_{24}, f^{(\mathrm{np})}_{26}, f^{(\mathrm{np})}_{31}\right)^T \tag{4.30}$$

to be the set of our master integrals, where we used the definition

$$f^{(\mathrm{np})}_i := \frac{e^{2\epsilon\gamma_E}}{(i\pi^{D/2})^2} \int j^{(\mathrm{np})}_i. \tag{4.31}$$

Deriving the differential equation

$$\partial_x \vec{f}^{(\mathrm{np})}(x,\epsilon) = \epsilon \left(\frac{a}{x} + \frac{b}{1+x} \right) \vec{f}^{(\mathrm{np})}(x,\epsilon), \tag{4.32}$$

where $x = t/s$ we find

$$a = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 3 & 0 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & -6 & 12 & 0 & -6 & 0 & 0 & -2 & 6 & 2 & 1 & 1 \\ -6 & -6 & 0 & 0 & 6 & 0 & 0 & 2 & 6 & 0 & 0 & -2 \end{pmatrix} \tag{4.33}$$

and

$$b = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 3 & -3 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & -6 & 6 & 0 & 0 & 2 & 0 & -6 & 0 & -2 & 0 \\ -4 & 12 & -6 & -6 & 0 & 0 & -2 & 0 & -6 & -2 & 1 & 1 \end{pmatrix} \tag{4.34}$$

we see that all entries are numerical constants and thus our master integrals are all uniform transcendental weight functions.

The other integrals can just be related to these master integrals as

$$
\begin{aligned}
&f_6 = f_1, \quad f_8 = f_9 = f_{10} = f_7, \quad f_{11} = f_4, \quad f_{12} = f_5, \quad f_{13} = f_2, \quad f_{14} = f_3, \\
&f_{17} = f_{15}, \quad f_{18} = f_{16}, \quad f_{19} = f_{20} = f_3 + f_7 + \frac{f_{16}}{3}, \\
&f_{21} = 2f_2 - f_4 - f_7 - \frac{2f_{15}}{3}, \quad f_{22} = 2f_2 - f_4 - f_7 - \frac{2f_{15}}{3}, \\
&f_{25} = 2f_1 - 2f_{24}, \quad f_{27} = \frac{3f_1}{2} + \frac{f_2}{2} + \frac{3f_4}{2} - \frac{3f_3}{2} - \frac{f_5}{2} - \frac{f_{23}}{2}, \\
&f_{28} = \frac{f_1}{2} + \frac{3f_2}{2} + \frac{3f_3}{2} - \frac{3f_4}{2} - \frac{f_5}{2} - \frac{f_{23}}{2}, \quad f_{29} = \frac{3f_1}{2} + \frac{3f_2}{2} + \frac{f_4}{2} - \frac{f_3}{2} - \frac{3f_5}{2} - \frac{3f_{23}}{2}, \\
&f_{30} = -f_2 - f_3 + f_4 + f_5 + f_{23}, \quad f_{32} = -\frac{3f_1}{2} + \frac{f_3}{2} + \frac{3f_5}{2} + \frac{3f_{23}}{2} + f_{31} - \frac{3f_2}{2} - \frac{f_4}{2}, \\
&f_{33} = f_1 + f_{15} - f_{24} + \frac{f_{31}}{2} - \frac{f_{26}}{2}, \quad f_{36} = f_1 + f_2 - f_5 - f_{23} - f_{31} \\
&f_{34} = -f_1 + \frac{3f_2}{2} + \frac{f_4}{2} + \frac{5f_5}{2} - f_7 + \frac{f_{16}}{3} - \frac{9f_3}{2} - \frac{f_{23}}{2} - \frac{f_{24}}{2} - \frac{f_{15}}{3} - \frac{3f_{26}}{4} - \frac{f_{31}}{4}, \\
&f_{35} = -\frac{f_1}{4} + \frac{f_2}{2} + \frac{5f_3}{2} + f_{23} + \frac{3f_{24}}{4} + \frac{3f_{26}}{8} + \frac{f_{31}}{8} - \frac{3f_4}{2} - \frac{f_5}{2} - \frac{f_7}{2} - \frac{f_{15}}{2} - \frac{f_{16}}{6},
\end{aligned}
\tag{4.35}
$$

and since all coefficients are numerical values we have proven that all other integrals are also uniform transcendental weight functions.

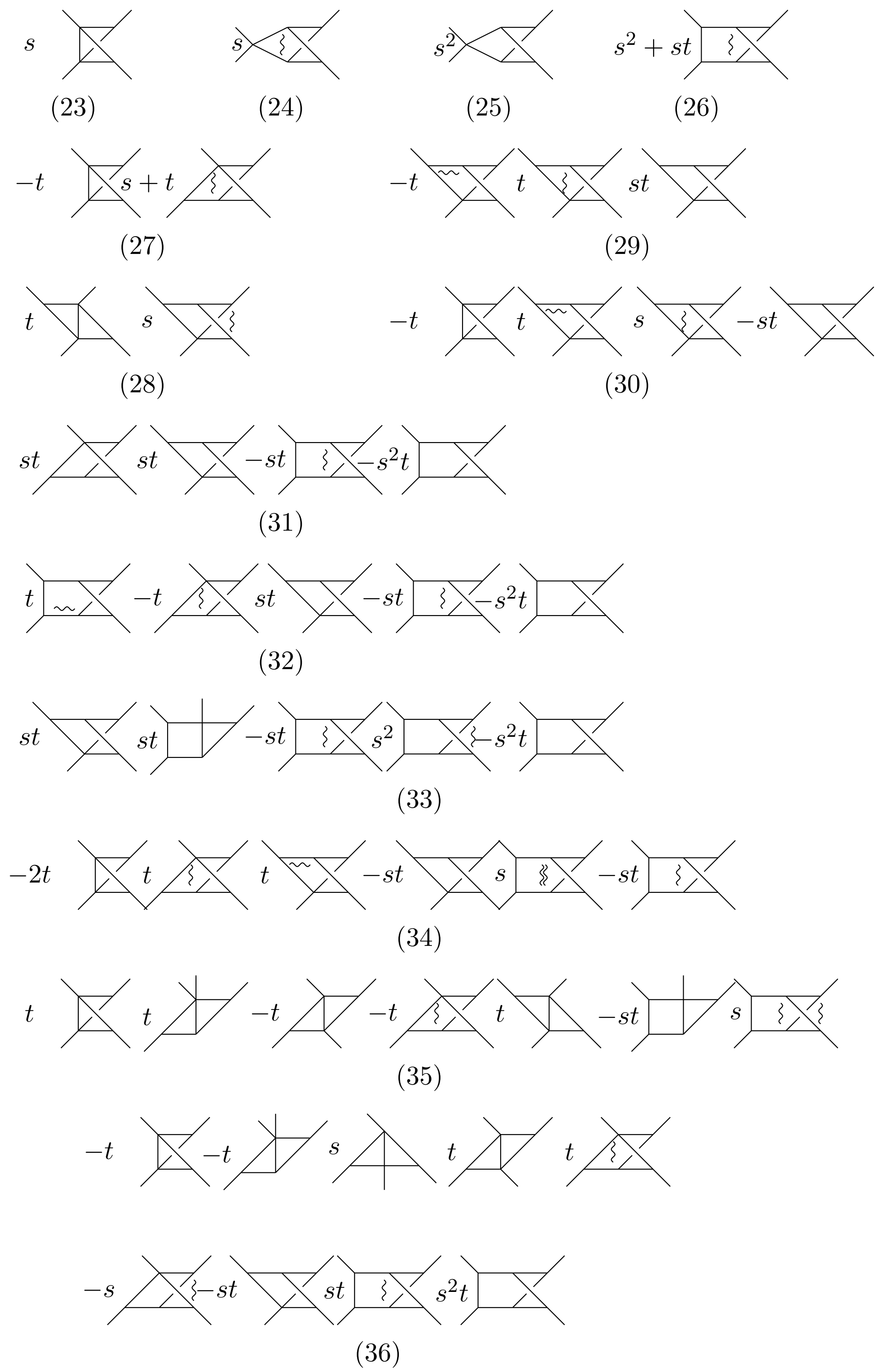


Figure 4.5.: Non-planar double box integrands with constant leading singularities. A wavy line in a loop indicates a numerator depending on the corresponding loop variable. For diagrams containing vertical external lines the order of the external momenta is changed.

$$
\begin{array}{ll}
j^{\text{np}}_1 = sJ^{\text{np}}_{0,0,1,1,1,1,1,0,0} & j^{\text{np}}_2 = (s+t)J^{\text{np}}_{0,1,1,0,1,1,1,0,0} \\
j^{\text{np}}_3 = tJ^{\text{np}}_{0,1,1,1,0,1,1,0,0} & j^{\text{np}}_4 = tJ^{\text{np}}_{0,1,1,1,1,0,1,0,0} \\
j^{\text{np}}_5 = (s+t)J^{\text{np}}_{0,1,1,1,1,1,0,0,0} & j^{\text{np}}_6 = sJ^{\text{np}}_{1,0,0,1,1,1,1,0,0} \\
j^{\text{np}}_7 = sJ^{\text{np}}_{1,0,1,0,1,1,1,0,0} & j^{\text{np}}_8 = sJ^{\text{np}}_{1,0,1,1,0,1,1,0,0} \\
j^{\text{np}}_9 = sJ^{\text{np}}_{1,0,1,1,1,0,1,0,0} & j^{\text{np}}_{10} = sJ^{\text{np}}_{1,0,1,1,1,1,0,0,0} \\
j^{\text{np}}_{11} = tJ^{\text{np}}_{1,1,0,0,1,1,1,0,0} & j^{\text{np}}_{12} = (s+t)J^{\text{np}}_{1,1,0,1,0,1,1,0,0} \\
j^{\text{np}}_{13} = (s+t)J^{\text{np}}_{1,1,0,1,1,0,1,0,0} & j^{\text{np}}_{14} = tJ^{\text{np}}_{1,1,0,1,1,1,0,0,0} \\
j^{\text{np}}_{15} = (s^2+st)J^{\text{np}}_{1,1,1,0,1,1,1,0,0} & j^{\text{np}}_{16} = stJ^{\text{np}}_{1,1,1,1,0,1,1,0,0} \\
j^{\text{np}}_{17} = (s^2+st)J^{\text{np}}_{1,1,1,1,1,0,1,0,0} & j^{\text{np}}_{18} = stJ^{\text{np}}_{1,1,1,1,1,1,0,0,0} \\
\multicolumn{2}{l}{j^{\text{np}}_{19} = sJ^{\text{np}}_{1,1,1,1,0,1,1,-1,0} + tJ^{\text{np}}_{1,1,0,1,0,1,1,0,0}} \\
\multicolumn{2}{l}{j^{\text{np}}_{20} = sJ^{\text{np}}_{1,1,1,1,1,1,0,-1,0} + tJ^{\text{np}}_{0,1,1,1,1,1,0,0,0}} \\
\multicolumn{2}{l}{j^{\text{np}}_{21} = sJ^{\text{np}}_{1,1,1,0,1,1,1,-1,0} - stJ^{\text{np}}_{1,1,1,0,1,1,1,0,0} + tJ^{\text{np}}_{0,1,1,0,1,1,1,0,0}} \\
\multicolumn{2}{l}{j^{\text{np}}_{22} = sJ^{\text{np}}_{1,1,1,1,1,0,1,-1,0} - stJ^{\text{np}}_{1,1,1,1,1,0,1,0,0} + tJ^{\text{np}}_{1,1,0,1,1,0,1,0,0}}
\end{array}
$$

Table 4.2.: Planar subdiagrams of non-planar double box with constant leading singularities

$$
\begin{array}{ll}
j^{\text{np}}_{23} = sJ^{\text{np}}_{0,1,0,1,1,1,1,0,0} & j^{\text{np}}_{24} = sJ^{\text{np}}_{1,0,1,1,1,1,1,-1,0} \\
j^{\text{np}}_{25} = s^2J^{\text{np}}_{1,0,1,1,1,1,1,0,0} & j^{\text{np}}_{26} = (s^2+st)J^{\text{np}}_{1,1,1,1,1,1,1,-1,0}
\end{array}
$$

$$
\begin{aligned}
j^{\text{np}}_{27} &= (s+t)J^{\text{np}}_{0,1,1,1,1,1,1,-1,0} - tJ^{\text{np}}_{0,1,0,1,1,1,1,0,0} \\
j^{\text{np}}_{28} &= sJ^{\text{np}}_{1,1,0,1,1,1,1,0,-1} + tJ^{\text{np}}_{1,1,0,1,0,1,1,0,0} \\
j^{\text{np}}_{29} &= stJ^{\text{np}}_{1,1,0,1,1,1,1,0,0} + tJ^{\text{np}}_{1,1,0,1,1,1,1,-1,0} - tJ^{\text{np}}_{1,1,-1,1,1,1,1,0,0} \\
j^{\text{np}}_{30} &= sJ^{\text{np}}_{1,1,0,1,1,1,1,-1,0} - stJ^{\text{np}}_{1,1,0,1,1,1,1,0,0} - tJ^{\text{np}}_{0,1,0,1,1,1,1,0,0} + tJ^{\text{np}}_{1,1,-1,1,1,1,1,0,0} \\
j^{\text{np}}_{31} &= -s^2tJ^{\text{np}}_{1,1,1,1,1,1,1,0,0} + stJ^{\text{np}}_{0,1,1,1,1,1,1,0,0} + stJ^{\text{np}}_{1,1,0,1,1,1,1,0,0} - stJ^{\text{np}}_{1,1,1,1,1,1,1,-1,0} \\
j^{\text{np}}_{32} &= -s^2tJ^{\text{np}}_{1,1,1,1,1,1,1,0,0} + stJ^{\text{np}}_{1,1,0,1,1,1,1,0,0} - stJ^{\text{np}}_{1,1,1,1,1,1,1,-1,0} \\
&\quad -tJ^{\text{np}}_{0,1,1,1,1,1,1,-1,0} + tJ^{\text{np}}_{-1,1,1,1,1,1,1,0,0} \\
j^{\text{np}}_{33} &= s^2J^{\text{np}}_{1,1,1,1,1,1,1,0,-1} - s^2tJ^{\text{np}}_{1,1,1,1,1,1,1,0,0} + stJ^{\text{np}}_{1,1,0,1,1,1,1,0,0} \\
&\quad +stJ^{\text{np}}_{1,1,1,0,1,1,1,0,0} - stJ^{\text{np}}_{1,1,1,1,1,1,1,-1,0} \\
j^{\text{np}}_{34} &= -2tJ^{\text{np}}_{0,1,0,1,1,1,1,0,0} + sJ^{\text{np}}_{1,1,1,1,1,1,1,-2,0} - stJ^{\text{np}}_{1,1,0,1,1,1,1,0,0} \\
&\quad -stJ^{\text{np}}_{1,1,1,1,1,1,1,-1,0} + tJ^{\text{np}}_{0,1,1,1,1,1,1,-1,0} + tJ^{\text{np}}_{1,1,-1,1,1,1,1,0,0} \\
j^{\text{np}}_{35} &= sJ^{\text{np}}_{1,1,1,1,1,1,1,-1,-1} - stJ^{\text{np}}_{1,1,1,0,1,1,1,0,0} + tJ^{\text{np}}_{0,1,0,1,1,1,1,0,0} + tJ^{\text{np}}_{0,1,1,0,1,1,1,0,0} \\
&\quad -tJ^{\text{np}}_{0,1,1,1,1,1,0,0,0} - tJ^{\text{np}}_{0,1,1,1,1,1,1,-1,0} + tJ^{\text{np}}_{1,1,0,1,0,1,1,0,0} \\
j^{\text{np}}_{36} &= s^2tJ^{\text{np}}_{1,1,1,1,1,1,1,0,0} + sJ^{\text{np}}_{0,1,1,1,1,0,1,0,0} - sJ^{\text{np}}_{0,1,1,1,1,1,1,0,-1} \\
&\quad -stJ^{\text{np}}_{1,1,0,1,1,1,1,0,0} + stJ^{\text{np}}_{1,1,1,1,1,1,1,-1,0} - tJ^{\text{np}}_{0,1,0,1,1,1,1,0,0} - \\
&\quad tJ^{\text{np}}_{0,1,1,0,1,1,1,0,0} + tJ^{\text{np}}_{0,1,1,1,1,1,0,0,0} + tJ^{\text{np}}_{0,1,1,1,1,1,1,-1,0}
\end{aligned}
$$

Table 4.3.: Non-planar diagrams of non-planar double box with constant leading singularities

4.3. Diagram A

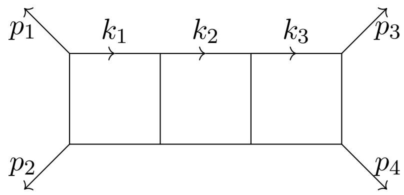


Figure 4.6.: Diagram A: Planar triple box

Diagram A is the planar triple box shown in figure 4.6 and for the labeling of the propagators and numerators we follow the notation in [20] by defining

$$J^{\mathrm{A}}_{a_1,\ldots,a_{15}} = \frac{d^Dk_1 d^Dk_2 d^Dk_3 [-(k_1-p_3)^2]^{-a_{11}}[-(k_2+p_1)^2]^{-a_{12}}}{[-k_1^2]^{a_1}[-(k_1+p_1+p_2)^2]^{a_2}[-k_2^2]^{a_3}[-(k_2+p_1+p_2)^2]^{a_4}[-k_3^2]^{a_5}}$$
$$\times \frac{[-(k_2-p_3)^2]^{-a_{13}}[-(k_3+p_1)^2]^{-a_{14}}[-(k_1-k_3)^2]^{-a_{15}}}{[-(k_3+p_1+p_2)^2]^{a_6}[-(k_1+p_1)^2]^{a_7}[-(k_1-k_2)^2]^{a_8}[-(k_2-k_3)^2]^{a_9}[-(k_3-p_3)^2]^{a_{10}}}. \tag{4.36}$$

In comparison to all diagrams we have discussed so far in this notation the external momenta p_2 and p_3 are not neighboring and consequently the definition of the Mandelstam variable $t = (p_1+p_3)^2$ is adjusted, while $s = (p_1+p_2)^2$ stays unchanged. Here the first ten indices $a_1, \ldots a_{10}$ are associated to propagators and the last 5 indices $a_{11}, \ldots, a_{15}$ are associated to numerators.

We make a general numerator ansatz with 141 free parameters n_i

$$J^{\mathrm{A}} = \sum_i n_i J^{\mathrm{A}}_{a_{1,i},\ldots,a_{15,i}}, \tag{4.37}$$

where $a_j \leq 1$ for $j \in \{1, \ldots, 10\}$, $a_j \leq 0$ for $j \in \{11, \ldots, 15\}$, $a_1+a_2+a_7+a_8+a_{11}+a_{15} \geq 3$, $a_3+a_4+a_8+a_9+a_{12}+a_{13} \geq 3$, and $a_5+a_6+a_9+a_{10}+a_{14}+a_{15} \geq 3$.

Applying the algorithm to this ansatz we get a solution with 101 linear independent leading singularities and the same number of free parameters so we can choose a set of 101 linear independent integrands with constant leading singularities. The solution is given in table 4.4 and 4.5. In figure 4.8 and 4.9 the diagrams are graphically represented and for all diagrams that are equal up to vertical and horizontal reflection only one representative is shown. Using again the graphical rules and also taking into account the integrands of the planar double box one may verify the constant leading singularity property for most of the integrands.

Similar to the planar double box we again tested if all found integrands are UT-functions after integration. So we build the integrals

$$I^{\mathrm{A}}_i := \frac{\epsilon^6 e^{3\epsilon\gamma_E}(-s)^{3\epsilon}}{(i\pi^{D/2})^3} \int J^{\mathrm{A}}_i \tag{4.38}$$

and relate them to a basis of integrals where the UT-property is already proven. Such a basis can be found in [20] and again we found that all integrals are UT-functions. The integral basis in [20] consists of 26 integrals, however, the 101 integrands we found turned out to correspond to only 25 linear independent integrals. One diagram of the basis in [20] that is linear independent to the 25 integrals is the triple bubble integral

$$\frac{\epsilon^3 e^{3\epsilon\gamma_E}(-s)^{3\epsilon}}{(i\pi^{D/2})^3} \int j^{\mathrm{A}}_{0,2,0,0,1,0,0,2,2,0,0,0,0,0,0}. \tag{4.39}$$

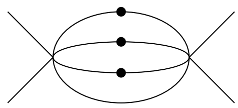


Figure 4.7.: Integral that is linear independent to all integrals defined in equation (4.38). Dots denote squared propagators.

However, if we combine the 101 integrand solutions with the integrands we get by rotating diagram A by 90 degrees, which is equivalent to exchanging the Mandelstam variables s and t after integration, we get a set of integrals that also includes the integral in (4.39).

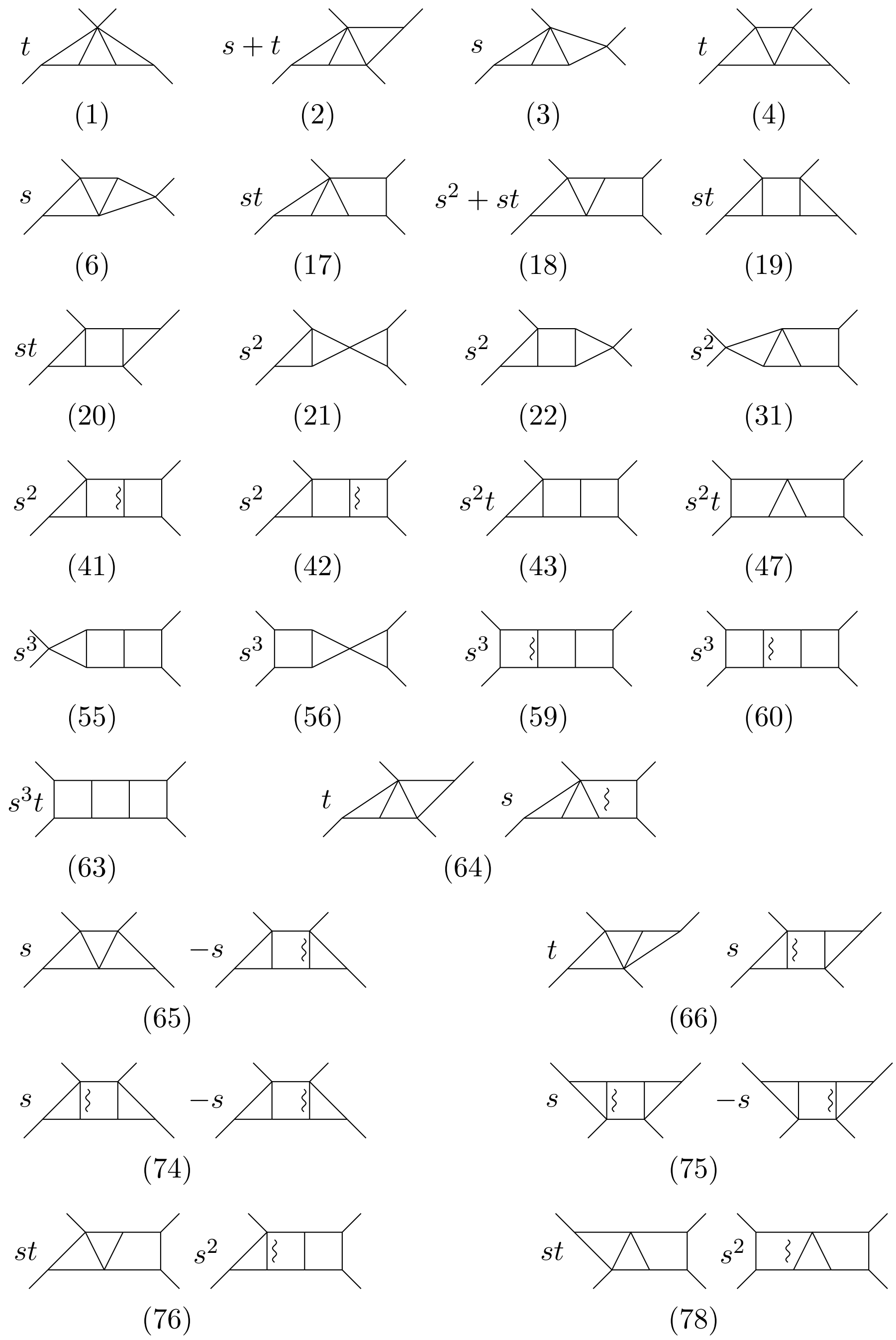


Figure 4.8.: Diagram A integrands with constant leading singularities. A wavy line in a loop indicates a numerator depending on the corresponding loop variable.

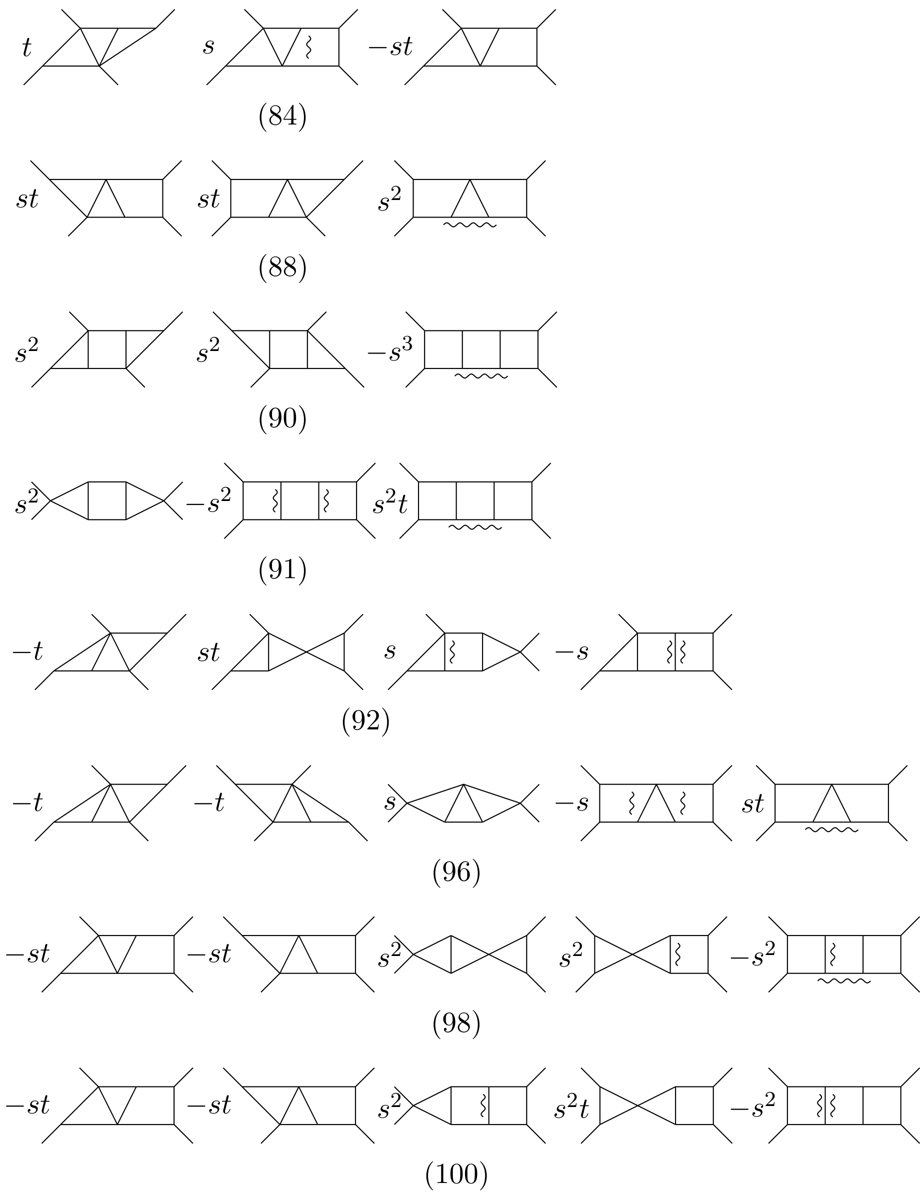


Figure 4.9.: Diagram A (continued).

$j_1^{\mathrm{A}} = tJ^{\mathrm{A}}_{0,1,0,1,0,1,1,1,1,1,0,0,0,0,0}$	$j_2^{\mathrm{A}} = (s+t)J^{\mathrm{A}}_{0,1,0,1,1,0,1,1,1,1,0,0,0,0,0}$
$j_3^{\mathrm{A}} = sJ^{\mathrm{A}}_{0,1,0,1,1,1,1,1,1,0,0,0,0,0,0}$	$j_4^{\mathrm{A}} = tJ^{\mathrm{A}}_{0,1,1,0,0,1,1,1,1,1,0,0,0,0,0}$
$j_5^{\mathrm{A}} = (s+t)J^{\mathrm{A}}_{0,1,1,0,1,0,1,1,1,1,0,0,0,0,0}$	$j_6^{\mathrm{A}} = sJ^{\mathrm{A}}_{0,1,1,0,1,1,1,1,1,0,0,0,0,0,0}$
$j_7^{\mathrm{A}} = (s+t)J^{\mathrm{A}}_{1,0,0,1,0,1,1,1,1,1,0,0,0,0,0}$	$j_8^{\mathrm{A}} = tJ^{\mathrm{A}}_{1,0,0,1,1,0,1,1,1,1,0,0,0,0,0}$
$j_9^{\mathrm{A}} = sJ^{\mathrm{A}}_{1,0,0,1,1,1,1,1,1,0,0,0,0,0,0}$	$j_{10}^{\mathrm{A}} = (s+t)J^{\mathrm{A}}_{1,0,1,0,0,1,1,1,1,1,0,0,0,0,0}$
$j_{11}^{\mathrm{A}} = tJ^{\mathrm{A}}_{1,0,1,0,1,0,1,1,1,1,0,0,0,0,0}$	$j_{12}^{\mathrm{A}} = sJ^{\mathrm{A}}_{1,0,1,0,1,1,1,1,1,0,0,0,0,0,0}$
$j_{13}^{\mathrm{A}} = sJ^{\mathrm{A}}_{1,1,0,1,0,1,0,1,1,1,0,0,0,0,0}$	$j_{14}^{\mathrm{A}} = sJ^{\mathrm{A}}_{1,1,0,1,1,0,0,1,1,1,0,0,0,0,0}$
$j_{15}^{\mathrm{A}} = sJ^{\mathrm{A}}_{1,1,1,0,0,1,0,1,1,1,0,0,0,0,0}$	$j_{16}^{\mathrm{A}} = sJ^{\mathrm{A}}_{1,1,1,0,1,0,0,1,1,1,0,0,0,0,0}$
$j_{17}^{\mathrm{A}} = stJ^{\mathrm{A}}_{0,1,0,1,1,1,1,1,1,1,0,0,0,0,0}$	$j_{18}^{\mathrm{A}} = (s^2+st)J^{\mathrm{A}}_{0,1,1,0,1,1,1,1,1,1,0,0,0,0,0}$
$j_{19}^{\mathrm{A}} = stJ^{\mathrm{A}}_{0,1,1,1,0,1,1,1,1,1,0,0,0,0,0}$	$j_{20}^{\mathrm{A}} = stJ^{\mathrm{A}}_{0,1,1,1,1,0,1,1,1,1,0,0,0,0,0}$
$j_{21}^{\mathrm{A}} = s^2J^{\mathrm{A}}_{0,1,1,1,1,1,1,1,0,1,0,0,0,0,0}$	$j_{22}^{\mathrm{A}} = s^2J^{\mathrm{A}}_{0,1,1,1,1,1,1,1,1,0,0,0,0,0,0}$
$j_{23}^{\mathrm{A}} = (s^2+st)J^{\mathrm{A}}_{1,0,0,1,1,1,1,1,1,1,0,0,0,0,0}$	$j_{24}^{\mathrm{A}} = stJ^{\mathrm{A}}_{1,0,1,0,1,1,1,1,1,1,0,0,0,0,0}$
$j_{25}^{\mathrm{A}} = stJ^{\mathrm{A}}_{1,0,1,1,0,1,1,1,1,1,0,0,0,0,0}$	$j_{26}^{\mathrm{A}} = stJ^{\mathrm{A}}_{1,0,1,1,1,0,1,1,1,1,0,0,0,0,0}$
$j_{27}^{\mathrm{A}} = s^2J^{\mathrm{A}}_{1,0,1,1,1,1,1,1,0,1,0,0,0,0,0}$	$j_{28}^{\mathrm{A}} = s^2J^{\mathrm{A}}_{1,0,1,1,1,1,1,1,1,0,0,0,0,0,0}$
$j_{29}^{\mathrm{A}} = stJ^{\mathrm{A}}_{1,1,0,1,0,1,1,1,1,1,0,0,0,0,0}$	$j_{30}^{\mathrm{A}} = (s^2+st)J^{\mathrm{A}}_{1,1,0,1,1,0,1,1,1,1,0,0,0,0,0}$
$j_{31}^{\mathrm{A}} = s^2J^{\mathrm{A}}_{1,1,0,1,1,1,0,1,1,1,0,0,0,0,0}$	$j_{32}^{\mathrm{A}} = s^2J^{\mathrm{A}}_{1,1,0,1,1,1,1,1,1,0,0,0,0,0,0}$
$j_{33}^{\mathrm{A}} = (s^2+st)J^{\mathrm{A}}_{1,1,1,0,0,1,1,1,1,1,0,0,0,0,0}$	$j_{34}^{\mathrm{A}} = stJ^{\mathrm{A}}_{1,1,1,0,1,0,1,1,1,1,0,0,0,0,0}$
$j_{35}^{\mathrm{A}} = s^2J^{\mathrm{A}}_{1,1,1,0,1,1,0,1,1,1,0,0,0,0,0}$	$j_{36}^{\mathrm{A}} = s^2J^{\mathrm{A}}_{1,1,1,0,1,1,1,1,1,0,0,0,0,0,0}$
$j_{37}^{\mathrm{A}} = s^2J^{\mathrm{A}}_{1,1,1,1,0,1,0,1,1,1,0,0,0,0,0}$	$j_{38}^{\mathrm{A}} = s^2J^{\mathrm{A}}_{1,1,1,1,0,1,1,0,1,1,0,0,0,0,0}$
$j_{39}^{\mathrm{A}} = s^2J^{\mathrm{A}}_{1,1,1,1,1,0,0,1,1,1,0,0,0,0,0}$	$j_{40}^{\mathrm{A}} = s^2J^{\mathrm{A}}_{1,1,1,1,1,0,1,0,1,1,0,0,0,0,0}$
$j_{41}^{\mathrm{A}} = s^2J^{\mathrm{A}}_{0,1,1,1,1,1,1,1,1,1,0,0,-1,0,0}$	$j_{42}^{\mathrm{A}} = s^2J^{\mathrm{A}}_{0,1,1,1,1,1,1,1,1,1,0,0,0,-1,0}$
$j_{43}^{\mathrm{A}} = s^2tJ^{\mathrm{A}}_{0,1,1,1,1,1,1,1,1,1,0,0,0,0,0}$	$j_{44}^{\mathrm{A}} = s^2J^{\mathrm{A}}_{1,0,1,1,1,1,1,1,1,1,0,0,-1,0,0}$
$j_{45}^{\mathrm{A}} = s^2J^{\mathrm{A}}_{1,0,1,1,1,1,1,1,1,1,0,0,0,-1,0}$	$j_{46}^{\mathrm{A}} = s^2tJ^{\mathrm{A}}_{1,0,1,1,1,1,1,1,1,1,0,0,0,0,0}$
$j_{47}^{\mathrm{A}} = s^2tJ^{\mathrm{A}}_{1,1,0,1,1,1,1,1,1,1,0,0,0,0,0}$	$j_{48}^{\mathrm{A}} = s^2tJ^{\mathrm{A}}_{1,1,1,0,1,1,1,1,1,1,0,0,0,0,0}$
$j_{49}^{\mathrm{A}} = s^2J^{\mathrm{A}}_{1,1,1,1,0,1,1,1,1,1,-1,0,0,0,0}$	$j_{50}^{\mathrm{A}} = s^2J^{\mathrm{A}}_{1,1,1,1,0,1,1,1,1,1,0,-1,0,0,0}$
$j_{51}^{\mathrm{A}} = s^2tJ^{\mathrm{A}}_{1,1,1,1,0,1,1,1,1,1,0,0,0,0,0}$	$j_{52}^{\mathrm{A}} = s^2J^{\mathrm{A}}_{1,1,1,1,1,0,1,1,1,1,-1,0,0,0,0}$
$j_{53}^{\mathrm{A}} = s^2J^{\mathrm{A}}_{1,1,1,1,1,0,1,1,1,1,0,-1,0,0,0}$	$j_{54}^{\mathrm{A}} = s^2tJ^{\mathrm{A}}_{1,1,1,1,1,0,1,1,1,1,0,0,0,0,0}$
$j_{55}^{\mathrm{A}} = s^3J^{\mathrm{A}}_{1,1,1,1,1,1,0,1,1,1,0,0,0,0,0}$	$j_{56}^{\mathrm{A}} = s^3J^{\mathrm{A}}_{1,1,1,1,1,1,1,0,1,1,0,0,0,0,0}$
$j_{57}^{\mathrm{A}} = s^3J^{\mathrm{A}}_{1,1,1,1,1,1,1,1,0,1,0,0,0,0,0}$	$j_{58}^{\mathrm{A}} = s^3J^{\mathrm{A}}_{1,1,1,1,1,1,1,1,1,0,0,0,0,0,0}$
$j_{59}^{\mathrm{A}} = s^3J^{\mathrm{A}}_{1,1,1,1,1,1,1,1,1,1,-1,0,0,0,0}$	$j_{60}^{\mathrm{A}} = s^3J^{\mathrm{A}}_{1,1,1,1,1,1,1,1,1,1,0,-1,0,0,0}$
$j_{61}^{\mathrm{A}} = s^3J^{\mathrm{A}}_{1,1,1,1,1,1,1,1,1,1,0,0,-1,0,0}$	$j_{62}^{\mathrm{A}} = s^3J^{\mathrm{A}}_{1,1,1,1,1,1,1,1,1,1,0,0,0,-1,0}$
$j_{63}^{\mathrm{A}} = s^3tJ^{\mathrm{A}}_{1,1,1,1,1,1,1,1,1,1,0,0,0,0,0}$	

$$j_{64}^{\mathrm{A}} = sJ^{\mathrm{A}}_{0,1,0,1,1,1,1,1,1,1,0,0,0,-1,0} + tJ^{\mathrm{A}}_{0,1,0,1,1,0,1,1,1,1,0,0,0,0,0}$$

$$j_{65}^{\mathrm{A}} = sJ^{\mathrm{A}}_{0,1,1,0,0,1,1,1,1,1,0,0,0,0,0} - sJ^{\mathrm{A}}_{0,1,1,1,0,1,1,1,1,1,0,0,-1,0,0}$$

$$j_{66}^{\mathrm{A}} = sJ^{\mathrm{A}}_{0,1,1,1,1,0,1,1,1,1,0,-1,0,0,0} + tJ^{\mathrm{A}}_{0,1,1,0,1,0,1,1,1,1,0,0,0,0,0}$$

$$j_{67}^{\mathrm{A}} = sJ^{\mathrm{A}}_{0,1,1,1,1,0,1,1,1,1,0,0,-1,0,0} + tJ^{\mathrm{A}}_{0,1,0,1,1,0,1,1,1,1,0,0,0,0,0}$$

$$j_{68}^{\mathrm{A}} = sJ^{\mathrm{A}}_{1,0,1,0,1,1,1,1,1,1,0,0,0,-1,0} + tJ^{\mathrm{A}}_{1,0,1,0,0,1,1,1,1,1,0,0,0,0,0}$$

$$j_{69}^{\mathrm{A}} = sJ^{\mathrm{A}}_{1,0,1,1,0,1,1,1,1,1,0,-1,0,0,0} + tJ^{\mathrm{A}}_{1,0,0,1,0,1,1,1,1,1,0,0,0,0,0}$$

$$j_{70}^{\mathrm{A}} = sJ^{\mathrm{A}}_{1,0,1,1,0,1,1,1,1,1,0,0,-1,0,0} + tJ^{\mathrm{A}}_{1,0,1,0,0,1,1,1,1,1,0,0,0,0,0}$$

Table 4.4.: Integrands of Diagram A with constant leading singularities

$$
\begin{aligned}
j^{\mathrm{A}}_{71} = \quad & sJ^{\mathrm{A}}_{1,0,0,1,1,0,1,1,1,1,0,0,0,0,0} - sJ^{\mathrm{A}}_{1,0,1,1,1,0,1,1,1,1,0,0,-1,0,0} \\
j^{\mathrm{A}}_{72} = \quad & sJ^{\mathrm{A}}_{1,1,0,1,0,1,1,1,1,1,-1,0,0,0,0} + tJ^{\mathrm{A}}_{1,0,0,1,0,1,1,1,1,1,0,0,0,0,0} \\
j^{\mathrm{A}}_{73} = \quad & sJ^{\mathrm{A}}_{1,1,1,0,1,0,1,1,1,1,-1,0,0,0,0} + tJ^{\mathrm{A}}_{0,1,1,0,1,0,1,1,1,1,0,0,0,0,0} \\
j^{\mathrm{A}}_{74} = \quad & -sJ^{\mathrm{A}}_{0,1,1,1,0,1,1,1,1,1,0,0,-1,0,0} + sJ^{\mathrm{A}}_{0,1,1,1,0,1,1,1,1,1,0,-1,0,0,0} \\
j^{\mathrm{A}}_{75} = \quad & -sJ^{\mathrm{A}}_{1,0,1,1,1,0,1,1,1,1,0,0,-1,0,0} + sJ^{\mathrm{A}}_{1,0,1,1,1,0,1,1,1,1,0,-1,0,0,0} \\
j^{\mathrm{A}}_{76} = \quad & stJ^{\mathrm{A}}_{0,1,1,0,1,1,1,1,1,1,0,0,0,0,0} + s^2J^{\mathrm{A}}_{0,1,1,1,1,1,1,1,1,1,0,-1,0,0,0} \\
j^{\mathrm{A}}_{77} = \quad & stJ^{\mathrm{A}}_{1,0,0,1,1,1,1,1,1,1,0,0,0,0,0} + s^2J^{\mathrm{A}}_{1,0,1,1,1,1,1,1,1,1,0,-1,0,0,0} \\
j^{\mathrm{A}}_{78} = \quad & stJ^{\mathrm{A}}_{1,0,0,1,1,1,1,1,1,1,0,0,0,0,0} + s^2J^{\mathrm{A}}_{1,1,0,1,1,1,1,1,1,1,-1,0,0,0,0} \\
j^{\mathrm{A}}_{79} = \quad & stJ^{\mathrm{A}}_{1,1,0,1,1,0,1,1,1,1,0,0,0,0,0} + s^2J^{\mathrm{A}}_{1,1,0,1,1,1,1,1,1,1,0,0,0,-1,0} \\
j^{\mathrm{A}}_{80} = \quad & stJ^{\mathrm{A}}_{0,1,1,0,1,1,1,1,1,1,0,0,0,0,0} + s^2J^{\mathrm{A}}_{1,1,1,0,1,1,1,1,1,1,-1,0,0,0,0} \\
j^{\mathrm{A}}_{81} = \quad & stJ^{\mathrm{A}}_{1,1,1,0,0,1,1,1,1,1,0,0,0,0,0} + s^2J^{\mathrm{A}}_{1,1,1,0,1,1,1,1,1,1,0,0,0,-1,0} \\
j^{\mathrm{A}}_{82} = \quad & stJ^{\mathrm{A}}_{1,1,1,0,0,1,1,1,1,1,0,0,0,0,0} + s^2J^{\mathrm{A}}_{1,1,1,1,0,1,1,1,1,1,0,0,-1,0,0} \\
j^{\mathrm{A}}_{83} = \quad & stJ^{\mathrm{A}}_{1,1,0,1,1,0,1,1,1,1,0,0,0,0,0} + s^2J^{\mathrm{A}}_{1,1,1,1,1,0,1,1,1,1,0,0,-1,0,0} \\
j^{\mathrm{A}}_{84} = \quad & sJ^{\mathrm{A}}_{0,1,1,0,1,1,1,1,1,1,0,0,0,-1,0} - stJ^{\mathrm{A}}_{0,1,1,0,1,1,1,1,1,1,0,0,0,0,0} + tJ^{\mathrm{A}}_{0,1,1,0,1,0,1,1,1,1,0,0,0,0,0} \\
j^{\mathrm{A}}_{85} = \quad & sJ^{\mathrm{A}}_{1,0,0,1,1,1,1,1,1,1,0,0,0,-1,0} - stJ^{\mathrm{A}}_{1,0,0,1,1,1,1,1,1,1,0,0,0,0,0} + tJ^{\mathrm{A}}_{1,0,0,1,0,1,1,1,1,1,0,0,0,0,0} \\
j^{\mathrm{A}}_{86} = \quad & sJ^{\mathrm{A}}_{1,1,0,1,1,0,1,1,1,1,-1,0,0,0,0} - stJ^{\mathrm{A}}_{1,1,0,1,1,0,1,1,1,1,0,0,0,0,0} + tJ^{\mathrm{A}}_{0,1,0,1,1,0,1,1,1,1,0,0,0,0,0} \\
j^{\mathrm{A}}_{87} = \quad & sJ^{\mathrm{A}}_{1,1,1,0,0,1,1,1,1,1,-1,0,0,0,0} - stJ^{\mathrm{A}}_{1,1,1,0,0,1,1,1,1,1,0,0,0,0,0} + tJ^{\mathrm{A}}_{1,0,1,0,0,1,1,1,1,1,0,0,0,0,0} \\
j^{\mathrm{A}}_{88} = \quad & stJ^{\mathrm{A}}_{1,0,0,1,1,1,1,1,1,1,0,0,0,0,0} + stJ^{\mathrm{A}}_{1,1,0,1,1,0,1,1,1,1,0,0,0,0,0} + s^2J^{\mathrm{A}}_{1,1,0,1,1,1,1,1,1,1,0,0,0,0,-1} \\
j^{\mathrm{A}}_{89} = \quad & stJ^{\mathrm{A}}_{0,1,1,0,1,1,1,1,1,1,0,0,0,0,0} + stJ^{\mathrm{A}}_{1,1,1,0,0,1,1,1,1,1,0,0,0,0,0} + s^2J^{\mathrm{A}}_{1,1,1,0,1,1,1,1,1,1,0,0,0,0,-1} \\
j^{\mathrm{A}}_{90} = \quad & s^2J^{\mathrm{A}}_{0,1,1,1,1,0,1,1,1,1,0,0,0,0,0} + s^2J^{\mathrm{A}}_{1,0,1,1,0,1,1,1,1,1,0,0,0,0,0} - s^3J^{\mathrm{A}}_{1,1,1,1,1,1,1,1,1,1,0,0,0,0,-1} \\
j^{\mathrm{A}}_{91} = \quad & s^2tJ^{\mathrm{A}}_{1,1,1,1,1,1,1,1,1,1,0,0,0,0,-1} + s^2J^{\mathrm{A}}_{1,1,1,1,1,1,0,1,1,0,0,0,0,0,0} - s^2J^{\mathrm{A}}_{1,1,1,1,1,1,1,1,1,1,-1,0,0,-1,0} \\
j^{\mathrm{A}}_{92} = \quad & sJ^{\mathrm{A}}_{0,1,1,1,1,1,1,1,1,0,0,-1,0,0,0} - sJ^{\mathrm{A}}_{0,1,1,1,1,1,1,1,1,1,0,0,-1,-1,0} \\
& +stJ^{\mathrm{A}}_{0,1,1,1,1,1,1,1,0,1,0,0,0,0,0} - tJ^{\mathrm{A}}_{0,1,0,1,1,0,1,1,1,1,0,0,0,0,0} \\
j^{\mathrm{A}}_{93} = \quad & sJ^{\mathrm{A}}_{1,0,1,1,1,1,1,1,1,0,0,-1,0,0,0} - sJ^{\mathrm{A}}_{1,0,1,1,1,1,1,1,1,1,0,0,-1,-1,0} \\
& +stJ^{\mathrm{A}}_{1,0,1,1,1,1,1,1,0,1,0,0,0,0,0} - tJ^{\mathrm{A}}_{1,0,1,0,0,1,1,1,1,1,0,0,0,0,0} \\
j^{\mathrm{A}}_{94} = \quad & sJ^{\mathrm{A}}_{1,1,1,1,0,1,0,1,1,1,0,0,-1,0,0} - sJ^{\mathrm{A}}_{1,1,1,1,0,1,1,1,1,1,-1,-1,0,0,0} \\
& +stJ^{\mathrm{A}}_{1,1,1,1,0,1,1,0,1,1,0,0,0,0,0} - tJ^{\mathrm{A}}_{1,0,0,1,0,1,1,1,1,1,0,0,0,0,0} \\
j^{\mathrm{A}}_{95} = \quad & sJ^{\mathrm{A}}_{1,1,1,1,1,0,0,1,1,1,0,0,-1,0,0} - sJ^{\mathrm{A}}_{1,1,1,1,1,0,1,1,1,1,-1,-1,0,0,0} \\
& +stJ^{\mathrm{A}}_{1,1,1,1,1,0,1,0,1,1,0,0,0,0,0} - tJ^{\mathrm{A}}_{0,1,1,0,1,0,1,1,1,1,0,0,0,0,0} \\
j^{\mathrm{A}}_{96} = \quad & sJ^{\mathrm{A}}_{1,1,0,1,1,1,0,1,1,0,0,0,0,0,0} - sJ^{\mathrm{A}}_{1,1,0,1,1,1,1,1,1,1,-1,0,0,-1,0} \\
& +stJ^{\mathrm{A}}_{1,1,0,1,1,1,1,1,1,1,0,0,0,0,-1} - tJ^{\mathrm{A}}_{0,1,0,1,1,0,1,1,1,1,0,0,0,0,0} - tJ^{\mathrm{A}}_{1,0,0,1,0,1,1,1,1,1,0,0,0,0,0} \\
j^{\mathrm{A}}_{97} = \quad & sJ^{\mathrm{A}}_{1,1,1,0,1,1,0,1,1,0,0,0,0,0,0} - sJ^{\mathrm{A}}_{1,1,1,0,1,1,1,1,1,1,-1,0,0,-1,0} + stJ^{\mathrm{A}}_{1,1,1,0,1,1,1,1,1,1,0,0,0,0,-1} \\
& -tJ^{\mathrm{A}}_{0,1,1,0,1,0,1,1,1,1,0,0,0,0,0} - tJ^{\mathrm{A}}_{1,0,1,0,0,1,1,1,1,1,0,0,0,0,0} \\
j^{\mathrm{A}}_{98} = \quad & -stJ^{\mathrm{A}}_{0,1,1,0,1,1,1,1,1,1,0,0,0,0,0} - stJ^{\mathrm{A}}_{1,0,0,1,1,1,1,1,1,1,0,0,0,0,0} + s^2J^{\mathrm{A}}_{1,1,1,1,1,1,0,1,0,1,0,0,0,0,0} \\
& +s^2J^{\mathrm{A}}_{1,1,1,1,1,1,1,0,1,1,0,0,0,-1,0} - s^2J^{\mathrm{A}}_{1,1,1,1,1,1,1,1,1,1,0,-1,0,0,-1} \\
j^{\mathrm{A}}_{99} = \quad & -stJ^{\mathrm{A}}_{1,1,0,1,1,0,1,1,1,1,0,0,0,0,0} - stJ^{\mathrm{A}}_{1,1,1,0,0,1,1,1,1,1,0,0,0,0,0} + s^2J^{\mathrm{A}}_{1,1,1,1,1,1,1,0,1,0,0,0,0,0,0} \\
& +s^2J^{\mathrm{A}}_{1,1,1,1,1,1,1,1,0,1,-1,0,0,0,0} - s^2J^{\mathrm{A}}_{1,1,1,1,1,1,1,1,1,1,0,0,-1,0,-1} \\
j^{\mathrm{A}}_{100} = \quad & -stJ^{\mathrm{A}}_{0,1,1,0,1,1,1,1,1,1,0,0,0,0,0} - stJ^{\mathrm{A}}_{1,0,0,1,1,1,1,1,1,1,0,0,0,0,0} + s^2tJ^{\mathrm{A}}_{1,1,1,1,1,1,1,0,1,1,0,0,0,0,0} \\
& +s^2J^{\mathrm{A}}_{1,1,1,1,1,1,0,1,1,1,0,0,-1,0,0} - s^2J^{\mathrm{A}}_{1,1,1,1,1,1,1,1,1,1,-1,-1,0,0,0} \\
j^{\mathrm{A}}_{101} = \quad & -stJ^{\mathrm{A}}_{1,1,0,1,1,0,1,1,1,1,0,0,0,0,0} - stJ^{\mathrm{A}}_{1,1,1,0,0,1,1,1,1,1,0,0,0,0,0} + s^2tJ^{\mathrm{A}}_{1,1,1,1,1,1,1,1,0,1,0,0,0,0,0} \\
& +s^2J^{\mathrm{A}}_{1,1,1,1,1,1,1,1,1,0,0,-1,0,0,0} - s^2J^{\mathrm{A}}_{1,1,1,1,1,1,1,1,1,1,0,0,-1,-1,0}
\end{aligned}
$$

Table 4.5.: Diagram A with constant leading singularities (continued).

4.4. Diagram E

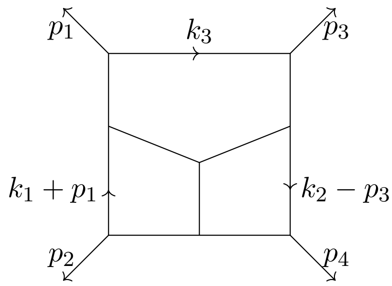


Figure 4.10.: Diagram E, "tennis court diagram"

Diagram E, sometimes referred to as the tennis court diagram, is the second planar three-loop diagram and for the momentum labeling we again follow the notation in [20] and define the integrand family as

$$J^{\mathrm{E}}_{a_1,\ldots,a_{15}} = \frac{d^D k_1 d^D k_2 d^D k_3 [-(k_3+p_1+p_2)^2]^{-a_{11}} [-(k_2+p_1)^2]^{-a_{12}}}{[-(k_1-k_3)^2]^{a_1} [-(k_1+p_1)^2]^{a_2} [-(k_1+p_1+p_2)^2]^{a_3} [-(k_2+p_1+p_2)^2]^{a_4}} \times \frac{[-(k_1-p_3)^2]^{-a_{13}} [-k_1^2]^{-a_{14}} [-k_2^2]^{-a_{15}}}{[-(k_2-p_3)^2]^{a_5} [-(k_2-k_3)^2]^{a_6} [-(k_1-k_2)^2]^{a_7} [-k_3^2]^{a_8} [-(k_3+p_1)^2]^{a_9} [-(k_3-p_3)^2]^{a_{10}}}, \tag{4.40}$$

where the first ten indices $a_1, \ldots, a_{10}$ are associated to propagators if they are positive and all indicies $a_1, \ldots, a_{15}$ are associated to numerators if they are negative.

The integrand ansatz for diagram E is bigger than the ansatz for diagram A, since diagram E has five propagators instead of four that depend on k_3. This means that numerators may also scale as $(k_3^2)^2$ for $k_3 \to \infty$ and so we define

$$J^{\mathrm{E}} = \sum_i n_i J^{\mathrm{E}}_{a_{1,i},\ldots,a_{15,i}}, \tag{4.41}$$

where $a_j \leq 1$ for $j \in \{1, \ldots, 10\}$, $a_j \leq 0$ for $j \in \{11, \ldots, 15\}$, $a_1+a_2+a_3+a_7+a_{13}+a_{14} \geq 3$, $a_4+a_5+a_6+a_6+a_{12}+a_{15} \geq 3$, and $a_1+a_6+a_8+a_9+a_{10}+a_{11} \geq 3$. This ansatz has 441 terms and using the algorithm in section 3.3.2 we get 171 linear independent leading singularities and the same number of free parameters n_i.

The list of integrands with constant leading singularities can be found in tables A.1-A.6 in the appendix and the graphical representations of the integrands 1 - 167 can be found in figures A.1 - A.11 also in the appendix.

Again we checked the UT-property of the resulting functions after integration by comparing it to the basis in [20]. So building the integrals

$$I^{\mathrm{E}}_i := \frac{\epsilon^6 e^{3\epsilon\gamma_E} (-s)^{3\epsilon}}{(i\pi^{D/2})^3} \int J^{\mathrm{E}}_i \tag{4.42}$$

we found that I_1^E-I_{170}^{E} are UT-functions while I_{171}^{E} turned out not to have uniform transcendental weight.

This seems to be a bit surprising at first after having found so many integrands with constant leading singularities all fulfilling the uniform transcendental weight property, but we will show that one could have anticipated such a result and it has to do with fact that we compute the leading singularities in $D = 4$ dimensions as we will see in the following section.

4.5. Vanishing Gram Determinants

So far we never discussed the question, whether all the integrands in equation (4.41) and similar equations for the other diagrams are linear independent. The answer to that question depends on the dimension D in which we work and while in the cases we analyzed before, the planar and non-planar double box and diagram A, the integrands were always linear independent in $D = 4$ dimensions, this is not true anymore for the integrands we used for diagram E, where there is exactly one nontrivial solution for the n_i-parameters solving $J^{(\mathrm{E})} = 0$. This is equivalent to the existence of a nontrivial numerator for the full ten propagator integral of diagram E that vanishes in four dimensions. Nontrivial linear combinations of scalar products adding up to zero in four dimensions can easily be constructed using Gram determinants that were introduced in section 2.5. Since Gram determinants like

$$G\begin{pmatrix} v_1, v_2, v_3, v_4, v_5 \\ w_1, w_2, w_3, w_4, w_5 \end{pmatrix} \tag{4.43}$$

are always zero if either $v_1, ..., v_5$ or $w_1, ..., w_5$ are linear dependent,

$$G\begin{pmatrix} k_3, k_1, p_1, p_2, p_3 \\ k_3, k_2, p_1, p_2, p_3 \end{pmatrix} \tag{4.44}$$

is an example of a term that vanishes in $D = 4$ dimensions but not for general values of D. If we use this expression as a numerator for the parent diagram E with all ten propagators, we can rewrite it as a sum of integrands all appearing in the ansatz (4.41) and since it is zero in four dimensions we can add it to all of our constant leading singularity integrands without altering the leading singularities. Using this Gram determinant in $D = 4 - 2\epsilon$ dimensions as numerator and computing the integral will, however, not lead to a uniform transcendental weight function, which is not unexpected, because even if it would have happened to be a uniform transcendental weight function we could just multiply the Gram determinant with certain terms to spoil the uniform transcendental weight property while the integrand in four dimensions would still be zero.

5. Summary and Conclusion

The main goal of this master thesis was to find bases of integrands with constant leading singularities. So based on [4], where similar integrand bases have been derived for planar diagrams of $\mathcal{N}$=4 Super Yang Mills theory, we extended the approach by applying it also to integrands of non-planar diagrams for general non-supersymmetric Yang-Mills theories like QCD.

For the computation of the leading singularities and the closely related dlog forms we used two different approaches. The first was to develop an algorithm for a MATHEMATICA program that computed iteratively residues of rational functions. The second was to use one-loop building blocks and calculate the leading singularities in a graphical method loop by loop.

We applied these methods to planar and non-planar two loop diagrams as well as planar three loop diagrams with four massless external momenta and massless propagators. For the planar and non-planar double box we found 23 respectively 36 integrands and for the two planar three-loop diagrams, denoted as diagram A and E, we found 101 and 171 integrands.

Using integration by parts identities we could analyze the integrals corresponding to the integrands. We found that the number of linear independent integrals was always enough to form complete integral bases that can serve as master integrals for the corresponding integral family. Another part of the analysis was to find finite integrals by searching for integrands with vanishing corner cuts.

One central part of the integral analysis was to test the uniform transcendental weight property of the integrals and we found a positive result for all integrals but one exception in diagram E. The exception can be explained that by computing the leading singularities in $D = 4$ dimensions the vanishing of certain Gram-determinants leads to ambiguities in our solutions. These ambiguities appear only for integrand families containing numerators of a certain degree and diagram E was the first integrand family where this was the case. Extending the analysis of leading singularities to higher dimensions like $D = 6$ or a dimension parametrized as $D = 4 - 2\epsilon$ can remove such ambiguities

Possible extensions of our analysis are the application to further diagrams such as the non-planar three-loop diagrams and the analysis of diagrams with more general kinematics like five-point two-loop diagrams. By working out more one-loop building blocks one can try to extend the graphical method in such a way we can apply it to all integrands that we found with the algorithmic approach. So far we have not proven that the set of integrands we found for each diagram is complete in the sense that there are no further integrands of that family with constant leading singularities. For this purpose the analysis should be extended, by increasing the integrand ansatz for each diagram up to a certain power limit in the loop momenta and to proof that all numerators above this limit lead to integrands that can not be written as dlog forms.

A. Appendix

$j_1^E = -tJ^E_{0,1,1,0,1,1,1,0,1,1,0,0,0,0,0}$	$j_2^E = (-s-t)J^E_{0,1,1,0,1,1,1,1,0,1,0,0,0,0,0}$
$j_3^E = sJ^E_{0,1,1,0,1,1,1,1,1,0,0,0,0,0,0}$	$j_4^E = -tJ^E_{0,1,1,1,0,1,1,0,1,1,0,0,0,0,0}$
$j_5^E = (-s-t)J^E_{0,1,1,1,0,1,1,1,0,1,0,0,0,0,0}$	$j_6^E = sJ^E_{0,1,1,1,0,1,1,1,1,0,0,0,0,0,0}$
$j_7^E = -tJ^E_{1,0,1,0,1,1,1,0,1,1,0,0,0,0,0}$	$j_8^E = sJ^E_{1,0,1,0,1,1,1,1,0,1,0,0,0,0,0}$
$j_9^E = (-s-t)J^E_{1,0,1,0,1,1,1,1,1,0,0,0,0,0,0}$	$j_{10}^E = -tJ^E_{1,0,1,1,1,0,1,0,1,1,0,0,0,0,0}$
$j_{11}^E = sJ^E_{1,0,1,1,1,0,1,1,0,1,0,0,0,0,0}$	$j_{12}^E = (-s-t)J^E_{1,0,1,1,1,0,1,1,1,0,0,0,0,0,0}$
$j_{13}^E = -tJ^E_{1,0,1,1,1,1,1,0,1,0,0,0,0,0,0}$	$j_{14}^E = sJ^E_{1,0,1,1,1,1,1,1,0,0,0,0,0,0,0}$
$j_{15}^E = -tJ^E_{1,1,0,0,1,1,1,1,0,1,0,0,0,0,0}$	$j_{16}^E = -tJ^E_{1,1,0,0,1,1,1,1,1,0,0,0,0,0,0}$
$j_{17}^E = -tJ^E_{1,1,0,1,0,1,1,0,1,1,0,0,0,0,0}$	$j_{18}^E = (-s-t)J^E_{1,1,0,1,0,1,1,1,0,1,0,0,0,0,0}$
$j_{19}^E = sJ^E_{1,1,0,1,0,1,1,1,1,0,0,0,0,0,0}$	$j_{20}^E = -tJ^E_{1,1,0,1,1,0,1,0,1,1,0,0,0,0,0}$
$j_{21}^E = sJ^E_{1,1,0,1,1,0,1,1,0,1,0,0,0,0,0}$	$j_{22}^E = (-s-t)J^E_{1,1,0,1,1,0,1,1,1,0,0,0,0,0,0}$
$j_{23}^E = -tJ^E_{1,1,0,1,1,1,1,0,0,1,0,0,0,0,0}$	$j_{24}^E = (-s-t)J^E_{1,1,0,1,1,1,1,1,0,0,0,0,0,0,0}$
$j_{25}^E = -tJ^E_{1,1,1,0,1,1,1,0,1,0,0,0,0,0,0}$	$j_{26}^E = (-s-t)J^E_{1,1,1,0,1,1,1,1,0,0,0,0,0,0,0}$
$j_{27}^E = -tJ^E_{1,1,1,1,0,1,1,0,0,1,0,0,0,0,0}$	$j_{28}^E = sJ^E_{1,1,1,1,0,1,1,1,0,0,0,0,0,0,0}$
$j_{29}^E = -tJ^E_{1,1,1,1,1,1,0,0,0,1,0,0,0,0,0}$	$j_{30}^E = -tJ^E_{1,1,1,1,1,1,0,0,1,0,0,0,0,0,0}$
$j_{31}^E = t(s+t)J^E_{0,1,1,0,1,1,1,1,1,1,0,0,0,0,0}$	$j_{32}^E = -stJ^E_{0,1,1,1,0,1,1,1,1,1,0,0,0,0,0}$
$j_{33}^E = t^2J^E_{0,1,1,1,1,0,1,1,1,1,0,0,0,0,0}$	$j_{34}^E = -tJ^E_{0,1,1,1,1,1,1,0,1,1,0,-1,0,0,0}$
$j_{35}^E = t^2J^E_{0,1,1,1,1,1,1,0,1,1,0,0,0,0,0}$	$j_{36}^E = -stJ^E_{0,1,1,1,1,1,1,1,0,1,0,0,0,0,0}$
$j_{37}^E = -stJ^E_{0,1,1,1,1,1,1,1,1,0,0,0,0,0,0}$	$j_{38}^E = -stJ^E_{1,0,1,0,1,1,1,1,1,1,0,0,0,0,0}$
$j_{39}^E = -stJ^E_{1,0,1,1,1,0,1,1,1,1,0,0,0,0,0}$	$j_{40}^E = -tJ^E_{1,0,1,1,1,1,1,0,1,1,-1,0,0,0,0}$
$j_{41}^E = sJ^E_{1,0,1,1,1,1,1,1,0,1,-1,0,0,0,0}$	$j_{42}^E = (-s-t)J^E_{1,0,1,1,1,1,1,1,1,0,-1,0,0,0,0}$
$j_{43}^E = -stJ^E_{1,0,1,1,1,1,1,1,1,0,0,0,0,0,0}$	$j_{44}^E = t^2J^E_{1,1,0,0,1,1,1,1,1,1,0,0,0,0,0}$
$j_{45}^E = -stJ^E_{1,1,0,1,0,1,1,1,1,1,0,0,0,0,0}$	$j_{46}^E = t(s+t)J^E_{1,1,0,1,1,0,1,1,1,1,0,0,0,0,0}$
$j_{47}^E = -tJ^E_{1,1,0,1,1,1,1,0,1,1,0,-1,0,0,0}$	$j_{48}^E = t^2J^E_{1,1,0,1,1,1,1,0,1,1,0,0,0,0,0}$
$j_{49}^E = (-s-t)J^E_{1,1,0,1,1,1,1,1,1,0,0,-1,0,0,0}$	$j_{50}^E = -stJ^E_{1,1,0,1,1,1,1,1,1,0,0,0,0,0,0}$
$j_{51}^E = -tJ^E_{1,1,1,0,1,1,1,0,1,1,0,0,-1,0,0}$	$j_{52}^E = t^2J^E_{1,1,1,0,1,1,1,0,1,1,0,0,0,0,0}$
$j_{53}^E = (-s-t)J^E_{1,1,1,0,1,1,1,1,0,1,0,0,-1,0,0}$	$j_{54}^E = -stJ^E_{1,1,1,0,1,1,1,1,0,1,0,0,0,0,0}$
$j_{55}^E = -tJ^E_{1,1,1,1,0,1,1,0,1,1,-1,0,0,0,0}$	$j_{56}^E = (-s-t)J^E_{1,1,1,1,0,1,1,1,0,1,-1,0,0,0,0}$
$j_{57}^E = -stJ^E_{1,1,1,1,0,1,1,1,0,1,0,0,0,0,0}$	$j_{58}^E = sJ^E_{1,1,1,1,0,1,1,1,1,0,-1,0,0,0,0}$
$j_{59}^E = -tJ^E_{1,1,1,1,1,0,1,0,1,1,0,0,-1,0,0}$	$j_{60}^E = t^2J^E_{1,1,1,1,1,0,1,0,1,1,0,0,0,0,0}$
$j_{61}^E = -stJ^E_{1,1,1,1,1,0,1,1,0,1,0,0,0,0,0}$	$j_{62}^E = -stJ^E_{1,1,1,1,1,0,1,1,1,0,0,0,0,0,0}$
$j_{63}^E = -tJ^E_{1,1,1,1,1,1,0,0,1,1,-1,0,0,0,0}$	$j_{64}^E = -stJ^E_{1,1,1,1,1,1,0,1,0,1,0,0,0,0,0}$
$j_{65}^E = -stJ^E_{1,1,1,1,1,1,0,1,1,0,0,0,0,0,0}$	$j_{66}^E = -tJ^E_{1,1,1,1,1,1,1,0,0,1,0,0,-1,0,0}$
$j_{67}^E = -tJ^E_{1,1,1,1,1,1,1,0,1,0,0,-1,0,0,0}$	$j_{68}^E = t^2J^E_{0,1,1,1,1,1,1,1,1,1,-1,0,0,0,0}$
$j_{69}^E = t^2J^E_{0,1,1,1,1,1,1,1,1,1,0,0,0,0,-1}$	$j_{70}^E = st^2J^E_{0,1,1,1,1,1,1,1,1,1,0,0,0,0,0}$

Table A.1.: Diagram E: Integrands with constant leading singularities

$$j^E_{71} = -stJ^E_{1,0,1,1,1,1,1,1,1,1,-1,0,0,0,0} \qquad j^E_{72} = t^2J^E_{1,1,0,1,1,1,1,1,1,1,-1,0,0,0,0}$$
$$j^E_{73} = t^2J^E_{1,1,1,0,1,1,1,1,1,1,-1,0,0,0,0} \qquad j^E_{74} = -stJ^E_{1,1,1,1,0,1,1,1,1,1,-1,0,0,0,0}$$
$$j^E_{75} = t^2J^E_{1,1,1,1,1,0,1,1,1,1,-1,0,0,0,0} \qquad j^E_{76} = t^2J^E_{1,1,1,1,1,0,1,1,1,1,0,0,0,-1,0}$$
$$j^E_{77} = st^2J^E_{1,1,1,1,1,0,1,1,1,1,0,0,0,0,0} \qquad j^E_{78} = -stJ^E_{1,1,1,1,1,1,0,1,1,1,-1,0,0,0,0}$$
$$j^E_{79} = t^2J^E_{1,1,1,1,1,1,1,0,1,1,-1,0,0,0,0} \qquad j^E_{80} = (-s-t)J^E_{1,1,1,1,1,1,1,1,0,1,-1,0,-1,0,0}$$
$$j^E_{81} = -stJ^E_{1,1,1,1,1,1,1,1,0,1,-1,0,0,0,0} \qquad j^E_{82} = -stJ^E_{1,1,1,1,1,1,1,1,0,1,0,0,-1,0,0}$$
$$j^E_{83} = (-s-t)J^E_{1,1,1,1,1,1,1,1,1,0,-1,-1,0,0,0} \qquad j^E_{84} = -stJ^E_{1,1,1,1,1,1,1,1,1,0,-1,0,0,0,0}$$
$$j^E_{85} = -stJ^E_{1,1,1,1,1,1,1,1,1,0,0,-1,0,0,0} \qquad j^E_{86} = t^2J^E_{1,1,1,1,1,1,1,1,1,1,-2,0,0,0,0}$$
$$j^E_{87} = -stJ^E_{1,1,1,1,1,1,1,1,1,1,-1,-1,0,0,0} \qquad j^E_{88} = -stJ^E_{1,1,1,1,1,1,1,1,1,1,-1,0,-1,0,0}$$
$$j^E_{89} = t^2J^E_{1,1,1,1,1,1,1,1,1,1,-1,0,0,-1,0} \qquad j^E_{90} = st^2J^E_{1,1,1,1,1,1,1,1,1,1,-1,0,0,0,0}$$
$$j^E_{91} = (-s-t)J^E_{0,1,1,0,1,1,1,1,1,0,0,0,0,0,0} + (s+t)J^E_{0,1,1,1,1,1,1,1,1,0,0,-1,0,0,0}$$
$$j^E_{92} = sJ^E_{1,0,1,0,1,1,1,1,1,0,0,0,0,0,0} + tJ^E_{1,1,1,0,1,1,1,1,1,0,0,0,0,-1,0}$$
$$j^E_{93} = (-s-t)J^E_{1,1,0,1,1,0,1,1,0,1,0,0,0,0,0} + (s+t)J^E_{1,1,1,1,1,0,1,1,0,1,0,0,-1,0,0}$$
$$j^E_{94} = sJ^E_{1,0,1,1,1,0,1,1,1,0,0,0,0,0,0} + tJ^E_{1,1,1,1,1,0,1,1,1,0,0,0,0,-1,0}$$
$$j^E_{95} = (-s-t)J^E_{1,1,1,1,1,1,0,1,0,0,0,0,0,0,0} + (s+t)J^E_{1,1,1,1,1,1,0,1,1,0,-1,0,0,0,0}$$
$$j^E_{96} = (-s-t)J^E_{1,1,1,1,1,1,0,1,1,0,-1,0,0,0,0} + sJ^E_{1,1,1,1,1,1,0,1,0,0,0,0,0,0,0}$$
$$j^E_{97} = tJ^E_{0,1,1,1,1,1,1,1,1,0,0,0,0,0,-1} - tJ^E_{0,1,1,1,1,1,1,1,1,0,0,-1,0,0,0}$$
$$j^E_{98} = -tJ^E_{1,0,1,0,1,1,1,1,1,1,-1,0,0,0,0} + tJ^E_{1,1,1,0,1,1,1,1,1,0,0,0,0,-1,0}$$
$$j^E_{99} = -tJ^E_{1,0,1,1,1,0,1,1,1,1,-1,0,0,0,0} + tJ^E_{1,1,1,1,1,0,1,1,1,0,0,0,0,-1,0}$$
$$j^E_{100} = -sJ^E_{1,1,1,1,1,1,0,1,1,0,-1,0,0,0,0} + sJ^E_{1,1,1,1,1,1,0,1,1,-1,0,0,0,0,0}$$
$$j^E_{101} = (-s-t)J^E_{1,1,1,1,1,1,0,1,0,1,-1,0,0,0,0} + (s+t)J^E_{1,1,1,1,1,1,0,1,1,0,-1,0,0,0,0}$$
$$j^E_{102} = -tJ^E_{1,1,1,0,1,1,1,0,0,1,0,0,0,0,0} + tJ^E_{1,1,1,1,1,1,1,0,1,1,-1,-1,0,0,0}$$
$$j^E_{103} = -tJ^E_{1,1,0,1,1,1,1,0,1,0,0,0,0,0,0} + tJ^E_{1,1,1,1,1,1,1,0,1,1,-1,0,-1,0,0}$$
$$j^E_{104} = tJ^E_{1,1,1,1,1,1,1,1,0,0,0,0,0,0,-1} - tJ^E_{1,1,1,1,1,1,1,1,0,0,0,0,0,-1,0}$$
$$j^E_{105} = -stJ^E_{1,1,1,1,1,1,1,1,0,0,0,0,0,0,0} - tJ^E_{1,1,1,1,1,1,1,1,0,0,0,0,0,0,-1}$$
$$j^E_{106} = sJ^E_{1,1,0,1,1,1,1,1,-1,1,0,0,0,0,0} - st^2J^E_{1,1,0,1,1,1,1,1,1,1,0,0,0,0,0}$$
$$j^E_{107} = sJ^E_{1,1,0,1,1,1,1,1,0,1,0,-1,0,0,0} - st^2J^E_{1,1,0,1,1,1,1,1,1,1,0,0,0,0,0}$$
$$j^E_{108} = -st^2J^E_{1,1,0,1,1,1,1,1,1,1,0,0,0,0,0} - stJ^E_{1,1,0,1,1,1,1,1,0,1,0,0,0,0,0}$$
$$j^E_{109} = -t(2s+t)J^E_{0,1,1,0,1,1,1,1,1,0,0,0,0,0,0} - t^2J^E_{1,1,1,0,1,1,1,1,1,0,0,0,0,-1,0}$$
$$j^E_{110} = sJ^E_{1,1,1,0,1,1,1,1,1,-1,0,0,0,0,0} - st^2J^E_{1,1,1,0,1,1,1,1,1,1,0,0,0,0,0}$$
$$j^E_{111} = sJ^E_{1,1,1,0,1,1,1,1,1,0,0,0,-1,0,0} - st^2J^E_{1,1,1,0,1,1,1,1,1,1,0,0,0,0,0}$$
$$j^E_{112} = -st^2J^E_{1,1,1,0,1,1,1,1,1,1,0,0,0,0,0} - stJ^E_{1,1,1,0,1,1,1,1,1,0,0,0,0,0,0}$$
$$j^E_{113} = stJ^E_{1,1,1,1,1,0,1,1,1,1,0,0,-1,0,0} - t(2s+t)J^E_{1,1,0,1,1,0,1,1,1,1,0,0,0,0,0}$$
$$j^E_{114} = sJ^E_{1,1,1,0,1,1,1,1,-1,1,0,0,0,0,0} - sJ^E_{1,1,1,1,1,1,1,1,0,1,-1,-1,0,0,0}$$
$$j^E_{115} = -tJ^E_{1,1,1,1,0,1,1,1,0,1,0,0,0,-1,0} + tJ^E_{1,1,1,1,1,1,1,1,0,1,0,0,-1,0,-1}$$
$$j^E_{116} = sJ^E_{1,1,0,1,1,1,1,1,1,-1,0,0,0,0,0} - sJ^E_{1,1,1,1,1,1,1,1,1,0,-1,0,-1,0,0}$$
$$j^E_{117} = -tJ^E_{1,0,1,1,1,1,1,1,1,0,0,0,0,0,-1} + tJ^E_{1,1,1,1,1,1,1,1,1,0,0,-1,0,-1,0}$$
$$j^E_{118} = -stJ^E_{0,1,1,1,1,1,1,1,1,1,0,-1,0,0,0} - t^2J^E_{1,1,1,0,1,1,1,1,1,1,0,0,0,-1,0}$$
$$j^E_{119} = stJ^E_{1,1,1,1,1,0,1,1,1,1,0,0,-1,0,0} + t^2J^E_{1,1,0,1,1,1,1,1,1,1,0,0,0,0,-1}$$
$$j^E_{120} = sJ^E_{1,1,1,1,1,1,1,1,-1,1,0,0,-1,0,0} + tJ^E_{1,1,1,1,1,1,1,1,0,1,-1,0,0,-1,0}$$
$$j^E_{121} = sJ^E_{1,1,1,1,1,1,1,1,1,-1,0,-1,0,0,0} + tJ^E_{1,1,1,1,1,1,1,1,1,0,-1,0,0,0,-1}$$

Table A.2.: Diagram E: Integrands with constant leading singularities (continued)

$$j^E_{122} = -tJ^E_{1,0,1,1,0,1,1,0,1,1,0,0,0,0,0} + tJ^E_{1,1,1,1,1,1,1,0,1,1,0,-1,-1,0,0} + t^2J^E_{1,1,1,1,1,1,0,0,1,1,0,0,0,0,0}$$
$$j^E_{123} = (-s-2t)J^E_{1,1,1,1,1,1,0,1,1,0,-1,0,0,0,0} + sJ^E_{1,1,1,1,1,1,0,1,-1,1,0,0,0,0,0} + tJ^E_{1,1,1,1,1,1,0,1,1,1,-2,0,0,0,0}$$
$$j^E_{124} = (-s-2t)J^E_{1,1,1,1,1,1,0,1,1,0,-1,0,0,0,0} + sJ^E_{1,1,1,1,1,1,0,1,0,1,-1,0,0,0,0} + tJ^E_{1,1,1,1,1,1,0,1,1,1,-2,0,0,0,0}$$
$$j^E_{125} = sJ^E_{1,1,0,1,1,1,1,1,0,0,0,0,0,0,0} - sJ^E_{1,1,1,1,1,1,1,1,0,1,-1,0,-1,0,0} - tJ^E_{1,1,1,1,1,1,1,1,0,1,-1,0,0,-1,0}$$
$$j^E_{126} = sJ^E_{1,1,1,0,1,1,1,1,0,0,0,0,0,0,0} - sJ^E_{1,1,1,1,1,1,1,1,1,0,-1,-1,0,0,0} - tJ^E_{1,1,1,1,1,1,1,1,1,0,-1,0,0,0,-1}$$
$$j^E_{127} = sJ^E_{1,1,1,0,1,1,1,1,0,1,0,0,-1,0,0} + tJ^E_{1,1,1,0,1,1,1,1,1,1,-1,0,-1,0,0} - t^2J^E_{1,1,1,0,1,1,1,1,1,1,0,0,0,-1,0}$$
$$j^E_{128} = -tJ^E_{1,1,1,0,1,1,1,1,0,1,0,0,0,-1,0} + tJ^E_{1,1,1,0,1,1,1,1,1,1,-1,0,-1,0,0} - t^2J^E_{1,1,1,0,1,1,1,1,1,1,0,0,0,-1,0}$$
$$j^E_{129} = -sJ^E_{1,1,1,1,1,1,1,1,0,1,-1,0,-1,0,0} - tJ^E_{1,1,0,1,1,1,1,1,0,1,-1,0,0,0,0} - tJ^E_{1,1,1,1,1,1,1,1,0,1,-1,0,0,-1,0}$$
$$j^E_{130} = sJ^E_{1,1,1,1,1,1,1,1,0,0,0,0,-1,0,0} - sJ^E_{1,1,1,1,1,1,1,1,0,1,-1,0,-1,0,0} - tJ^E_{1,1,1,1,1,1,1,1,0,1,-1,0,0,-1,0}$$
$$j^E_{131} = -sJ^E_{1,1,1,1,1,1,1,1,1,0,-1,-1,0,0,0} - tJ^E_{1,1,1,0,1,1,1,1,1,0,-1,0,0,0,0} - tJ^E_{1,1,1,1,1,1,1,1,1,0,-1,0,0,0,-1}$$
$$j^E_{132} = sJ^E_{1,1,1,1,1,1,1,1,0,0,0,-1,0,0,0} - sJ^E_{1,1,1,1,1,1,1,1,1,0,-1,-1,0,0,0} - tJ^E_{1,1,1,1,1,1,1,1,1,0,-1,0,0,0,-1}$$
$$j^E_{133} = -st^2J^E_{1,1,1,0,1,1,1,1,1,1,0,0,0,0,0} - stJ^E_{1,1,1,0,1,1,1,1,1,1,0,0,-1,0,0} - t^2J^E_{1,1,1,0,1,1,1,1,1,1,0,0,0,-1,0}$$
$$j^E_{134} = -st^2J^E_{1,1,0,1,1,1,1,1,1,1,0,0,0,0,0} - stJ^E_{1,1,0,1,1,1,1,1,1,1,0,-1,0,0,0} + stJ^E_{1,1,1,1,1,0,1,1,1,1,0,0,-1,0,0}$$
$$j^E_{135} = -tJ^E_{1,1,1,0,1,1,1,1,0,1,-1,0,0,0,0} - tJ^E_{1,1,1,1,1,1,1,1,1,0,-1,0,0,0,-1} + tJ^E_{1,1,1,1,1,1,1,1,1,1,-2,-1,0,0,0}$$
$$j^E_{136} = -tJ^E_{1,1,0,1,1,1,1,1,1,0,-1,0,0,0,0} - tJ^E_{1,1,1,1,1,1,1,1,0,1,-1,0,0,-1,0} + tJ^E_{1,1,1,1,1,1,1,1,1,1,-2,0,-1,0,0}$$
$$\begin{aligned} j^E_{137} = \quad & (s+t)J^E_{0,1,1,1,1,1,1,1,1,0,0,-1,0,0,0} - tJ^E_{0,1,1,0,1,1,1,1,1,1,-1,0,0,0,0} \\ & -tJ^E_{0,1,1,1,1,1,1,1,1,0,0,0,0,0,-1} + tJ^E_{0,1,1,1,1,1,1,1,1,1,-1,-1,0,0,0} \end{aligned}$$
$$\begin{aligned} j^E_{138} = \quad & -tJ^E_{1,1,0,0,1,1,1,0,1,1,0,0,0,0,0} - tJ^E_{1,1,1,0,1,1,1,1,1,0,0,0,0,-1,0} \\ & +tJ^E_{1,1,1,0,1,1,1,1,1,1,-1,0,0,-1,0} - tJ^E_{1,1,1,0,1,1,1,1,1,1,-1,0,-1,0,0} \end{aligned}$$
$$\begin{aligned} j^E_{139} = \quad & sJ^E_{1,1,1,1,1,0,1,1,0,1,0,0,-1,0,0} - tJ^E_{1,1,1,1,1,0,1,1,1,0,0,0,0,-1,0} \\ & +tJ^E_{1,1,1,1,1,0,1,1,1,1,-1,0,0,-1,0} + t(s+t)J^E_{0,1,1,1,1,0,1,1,1,1,0,0,0,0,0} \end{aligned}$$
$$\begin{aligned} j^E_{140} = \quad & -tJ^E_{1,1,0,1,1,1,1,1,1,1,-1,-1,0,0,0} + tJ^E_{1,1,1,0,1,1,1,1,1,0,0,0,0,-1,0} \\ & -tJ^E_{1,1,1,1,0,1,1,1,1,1,-1,0,0,-1,0} + tJ^E_{1,1,1,1,1,1,1,1,1,1,-1,0,-1,0,-1} \end{aligned}$$
$$\begin{aligned} j^E_{141} = \quad & sJ^E_{0,1,1,1,0,1,1,1,0,1,0,0,0,0,0} + (s+t)J^E_{0,1,1,1,1,1,1,1,1,0,0,-1,0,0,0} - tJ^E_{0,1,1,1,1,1,1,1,1,0,0,0,0,0,-1} \\ & +tJ^E_{0,1,1,1,1,1,1,1,1,1,-1,0,0,0,-1} + t(s+t)J^E_{0,1,1,1,1,0,1,1,1,1,0,0,0,0,0} \end{aligned}$$
$$\begin{aligned} j^E_{142} = \quad & (s+t)J^E_{0,1,1,1,1,1,1,1,1,0,0,-1,0,0,0} - tJ^E_{0,1,1,1,0,1,1,1,1,1,-1,0,0,0,0} - tJ^E_{0,1,1,1,1,1,1,1,1,0,0,0,0,0,-1} \\ & +tJ^E_{0,1,1,1,1,1,1,1,1,1,-1,0,0,0,-1} + t(s+t)J^E_{0,1,1,1,1,0,1,1,1,1,0,0,0,0,0} \end{aligned}$$
$$\begin{aligned} j^E_{143} = \quad & (s+t)J^E_{0,1,1,1,1,1,1,1,1,0,0,-1,0,0,0} - tJ^E_{0,1,1,1,1,1,1,1,0,1,0,0,0,0,-1} - tJ^E_{0,1,1,1,1,1,1,1,1,0,0,0,0,0,-1} \\ & +tJ^E_{0,1,1,1,1,1,1,1,1,1,-1,0,0,0,-1} + t(s+t)J^E_{0,1,1,1,1,0,1,1,1,1,0,0,0,0,0} \end{aligned}$$
$$\begin{aligned} j^E_{144} = \quad & sJ^E_{0,1,1,0,1,1,1,1,0,1,0,0,0,0,0} + (s+t)J^E_{0,1,1,1,1,1,1,1,1,0,0,-1,0,0,0} - tJ^E_{0,1,1,1,1,1,1,1,1,0,0,0,0,0,-1} \\ & +tJ^E_{0,1,1,1,1,1,1,1,1,1,-1,-1,0,0,0} - t^2J^E_{1,1,1,0,1,1,1,1,1,1,0,0,0,-1,0} \end{aligned}$$
$$\begin{aligned} j^E_{145} = \quad & (s+t)J^E_{1,1,1,1,1,0,1,1,0,1,0,0,-1,0,0} - tJ^E_{1,1,1,1,1,0,1,1,0,1,0,0,0,-1,0} - tJ^E_{1,1,1,1,1,0,1,1,1,0,0,0,0,-1,0} \\ & +tJ^E_{1,1,1,1,1,0,1,1,1,1,-1,0,0,-1,0} + t(s+t)J^E_{0,1,1,1,1,0,1,1,1,1,0,0,0,0,0} \end{aligned}$$
$$\begin{aligned} j^E_{146} = \quad & sJ^E_{0,1,1,1,1,1,1,1,0,1,0,-1,0,0,0} + (s+t)J^E_{0,1,1,1,1,1,1,1,1,0,0,-1,0,0,0} - tJ^E_{0,1,1,1,1,1,1,1,1,0,0,0,0,0,-1} \\ & +tJ^E_{0,1,1,1,1,1,1,1,1,1,-1,-1,0,0,0} - t^2J^E_{1,1,1,0,1,1,1,1,1,1,0,0,0,-1,0} \end{aligned}$$
$$\begin{aligned} j^E_{147} = \quad & (-s-t)tJ^E_{0,1,1,1,1,0,1,1,1,1,0,0,0,0,0} - tJ^E_{1,1,0,1,1,0,1,1,1,1,-1,0,0,0,0} + tJ^E_{1,1,1,1,1,0,1,1,1,0,0,0,0,-1,0} \\ & -tJ^E_{1,1,1,1,1,0,1,1,1,1,-1,0,0,-1,0} + tJ^E_{1,1,1,1,1,0,1,1,1,1,-1,0,-1,0,0} \end{aligned}$$
$$\begin{aligned} j^E_{148} = \quad & sJ^E_{1,0,1,1,1,1,1,1,1,-1,0,0,0,0,0} - sJ^E_{1,1,1,1,1,1,1,1,0,1,-1,0,-1,0,0} - tJ^E_{1,1,1,1,1,1,1,0,1,1,-1,0,-1,0,0} \\ & -tJ^E_{1,1,1,1,1,1,1,1,0,1,-1,0,0,-1,0} + tJ^E_{1,1,1,1,1,1,1,1,1,0,-1,0,0,-1,0} \end{aligned}$$

Table A.3.: Diagram E: Integrands with constant leading singularities (continued).

$$
\begin{aligned}
j^E_{149} = {} & sJ^E_{1,0,1,1,1,1,1,1,0,-1,0,0,0,0} - sJ^E_{1,1,1,1,1,1,1,1,0,1,-1,0,-1,0,0} - tJ^E_{1,1,1,1,1,1,1,0,1,1,-1,0,-1,0,0} \\
& -tJ^E_{1,1,1,1,1,1,1,1,0,1,-1,0,0,-1,0} + tJ^E_{1,1,1,1,1,1,1,1,1,0,-1,0,0,-1,0} \\
j^E_{150} = {} & sJ^E_{1,1,1,1,0,1,1,1,-1,1,0,0,0,0,0} - sJ^E_{1,1,1,1,1,1,1,1,1,0,-1,-1,0,0,0} - tJ^E_{1,1,1,1,1,1,1,0,1,1,-1,-1,0,0,0} \\
& +tJ^E_{1,1,1,1,1,1,1,1,0,1,-1,0,0,0,-1} - tJ^E_{1,1,1,1,1,1,1,1,1,0,-1,0,0,0,-1} \\
j^E_{151} = {} & sJ^E_{1,1,1,1,0,1,1,1,0,1,-1,0,0,0,0} - sJ^E_{1,1,1,1,1,1,1,1,1,0,-1,-1,0,0,0} - tJ^E_{1,1,1,1,1,1,1,0,1,1,-1,-1,0,0,0} \\
& +tJ^E_{1,1,1,1,1,1,1,1,0,1,-1,0,0,0,-1} - tJ^E_{1,1,1,1,1,1,1,1,1,0,-1,0,0,0,-1} \\
j^E_{152} = {} & sJ^E_{1,0,1,1,0,1,1,1,1,0,0,0,0,0,0} - sJ^E_{1,1,1,1,1,1,1,1,1,0,0,-1,-1,0,0} - stJ^E_{1,1,1,1,1,0,1,1,1,1,0,0,-1,0,0} \\
& +tJ^E_{1,1,1,1,0,1,1,1,1,1,-1,0,0,-1,0} - tJ^E_{1,1,1,1,1,1,1,1,1,1,-1,0,-1,0,-1} \\
j^E_{153} = {} & sJ^E_{1,1,0,1,1,1,1,1,1,0,0,-1,0,0,0} + stJ^E_{1,1,1,1,1,0,1,1,1,1,0,0,-1,0,0} + tJ^E_{1,1,1,0,1,1,1,1,1,0,0,0,0,-1,0} \\
& -tJ^E_{1,1,1,1,0,1,1,1,1,1,-1,0,0,-1,0} + tJ^E_{1,1,1,1,1,1,1,1,1,1,-1,0,-1,0,-1} \\
j^E_{154} = {} & stJ^E_{1,1,1,1,1,0,1,1,1,1,0,0,-1,0,0} - tJ^E_{1,1,0,1,1,1,1,1,1,0,0,0,0,0,-1} + tJ^E_{1,1,1,0,1,1,1,1,1,0,0,0,0,-1,0} \\
& -tJ^E_{1,1,1,1,0,1,1,1,1,1,-1,0,0,-1,0} + tJ^E_{1,1,1,1,1,1,1,1,1,1,-1,0,-1,0,-1} \\
j^E_{155} = {} & -sJ^E_{1,1,1,1,1,1,1,1,0,1,-1,0,-1,0,0} - tJ^E_{1,0,1,1,1,1,1,1,1,1,-2,0,0,0,0} - tJ^E_{1,1,1,1,1,1,1,0,1,1,-1,0,-1,0,0} \\
& -tJ^E_{1,1,1,1,1,1,1,1,0,1,-1,0,0,-1,0} + tJ^E_{1,1,1,1,1,1,1,1,1,0,-1,0,0,-1,0} \\
j^E_{156} = {} & -sJ^E_{1,1,1,1,1,1,1,1,1,0,-1,-1,0,0,0} - tJ^E_{1,1,1,1,0,1,1,1,1,1,-2,0,0,0,0} - tJ^E_{1,1,1,1,1,1,1,0,1,1,-1,-1,0,0,0} \\
& +tJ^E_{1,1,1,1,1,1,1,1,0,1,-1,0,0,0,-1} - tJ^E_{1,1,1,1,1,1,1,1,1,0,-1,0,0,0,-1} \\
j^E_{157} = {} & st^2J^E_{1,1,1,1,1,1,0,1,1,1,0,0,0,0,0} - stJ^E_{1,0,1,1,0,1,1,1,1,1,0,0,0,0,0} - stJ^E_{1,1,1,1,1,0,1,1,1,1,0,0,-1,0,0} \\
& +stJ^E_{1,1,1,1,1,1,1,1,1,1,0,-1,-1,0,0} + t^2J^E_{1,1,1,0,1,1,1,1,1,1,0,0,0,-1,0} \\
j^E_{158} = {} & (-s-t)tJ^E_{0,1,1,1,1,0,1,1,1,1,0,0,0,0,0} + sJ^E_{1,1,0,1,1,0,1,1,1,0,0,0,0,0,0} + stJ^E_{1,1,1,1,1,0,1,1,1,1,0,0,-1,0,0} \\
& +tJ^E_{1,1,1,1,1,0,1,1,1,0,0,0,0,-1,0} - tJ^E_{1,1,1,1,1,0,1,1,1,1,-1,0,0,-1,0} + tJ^E_{1,1,1,1,1,0,1,1,1,1,-1,0,-1,0,0} \\
j^E_{159} = {} & (-s-t)tJ^E_{0,1,1,1,1,0,1,1,1,1,0,0,0,0,0} + sJ^E_{1,1,1,1,1,0,1,1,1,0,0,0,-1,0,0} + stJ^E_{1,1,1,1,1,0,1,1,1,1,0,0,-1,0,0} \\
& +tJ^E_{1,1,1,1,1,0,1,1,1,0,0,0,0,-1,0} - tJ^E_{1,1,1,1,1,0,1,1,1,1,-1,0,0,-1,0} + tJ^E_{1,1,1,1,1,0,1,1,1,1,-1,0,-1,0,0} \\
j^E_{160} = {} & sJ^E_{1,1,0,1,0,1,1,1,0,1,0,0,0,0,0} + tJ^E_{1,1,0,1,1,1,1,1,1,1,-1,0,0,0,-1} - tJ^E_{1,1,1,0,1,1,1,1,1,1,-1,0,0,-1,0} \\
& +tJ^E_{1,1,1,0,1,1,1,1,1,1,-1,0,-1,0,0} + tJ^E_{1,1,1,1,0,1,1,1,1,1,-1,0,0,-1,0} - tJ^E_{1,1,1,1,1,1,1,1,1,1,-1,0,-1,0,-1} \\
j^E_{161} = {} & -tJ^E_{1,1,0,1,1,1,1,0,1,1,-1,0,0,0,0} + tJ^E_{1,1,1,1,1,1,1,0,1,1,-1,0,-1,0,0} - tJ^E_{1,1,1,1,1,1,1,1,0,1,-1,0,0,-1,0} \\
& -tJ^E_{1,1,1,1,1,1,1,1,1,0,-1,0,0,-1,0} + tJ^E_{1,1,1,1,1,1,1,1,1,1,-2,0,0,-1,0} + t(s+t)J^E_{0,1,1,1,1,1,1,1,1,1,-1,0,0,0,0} \\
j^E_{162} = {} & -(-s-t)tJ^E_{1,1,1,1,1,0,1,1,1,1,-1,0,0,0,0} - tJ^E_{1,1,1,0,1,1,1,0,1,1,-1,0,0,0,0} + tJ^E_{1,1,1,1,1,1,1,0,1,1,-1,-1,0,0,0} \\
& -tJ^E_{1,1,1,1,1,1,1,1,0,1,-1,0,0,0,-1} - tJ^E_{1,1,1,1,1,1,1,1,1,0,-1,0,0,0,-1} + tJ^E_{1,1,1,1,1,1,1,1,1,1,-2,0,0,0,-1} \\
j^E_{163} = {} & -tJ^E_{1,1,0,1,0,1,1,1,1,1,-1,0,0,0,0} + tJ^E_{1,1,0,1,1,1,1,1,1,1,-1,0,0,0,-1} - tJ^E_{1,1,1,0,1,1,1,1,1,1,-1,0,0,-1,0} \\
& +tJ^E_{1,1,1,0,1,1,1,1,1,1,-1,0,-1,0,0} + tJ^E_{1,1,1,1,0,1,1,1,1,1,-1,0,0,-1,0} - tJ^E_{1,1,1,1,1,1,1,1,1,1,-1,0,-1,0,-1} \\
j^E_{164} = {} & -tJ^E_{1,1,0,1,1,1,1,1,0,1,0,0,0,0,-1} + tJ^E_{1,1,0,1,1,1,1,1,1,1,-1,0,0,0,-1} - tJ^E_{1,1,1,0,1,1,1,1,1,1,-1,0,0,-1,0} \\
& +tJ^E_{1,1,1,0,1,1,1,1,1,1,-1,0,-1,0,0} + tJ^E_{1,1,1,1,0,1,1,1,1,1,-1,0,0,-1,0} - tJ^E_{1,1,1,1,1,1,1,1,1,1,-1,0,-1,0,-1} \\
j^E_{165} = {} & -tJ^E_{1,0,1,1,1,1,1,1,1,1,-1,0,0,0,-1} + tJ^E_{1,1,0,1,1,1,1,1,1,1,-1,0,0,0,-1} - tJ^E_{1,1,1,0,1,1,1,1,1,1,-1,0,0,-1,0} \\
& +tJ^E_{1,1,1,1,0,1,1,1,1,1,-1,0,0,-1,0} - tJ^E_{1,1,1,1,1,1,1,1,1,1,-1,0,-1,0,-1} + tJ^E_{1,1,1,1,1,1,1,1,1,1,-1,-1,0,-1,0} \\
j^E_{166} = {} & sJ^E_{1,0,1,1,0,1,1,1,0,1,0,0,0,0,0} - sJ^E_{1,1,1,1,1,1,1,1,0,1,0,-1,-1,0,0} + tJ^E_{1,1,0,1,1,1,1,1,1,1,-1,0,0,0,-1} \\
& -tJ^E_{1,1,1,0,1,1,1,1,1,1,-1,0,0,-1,0} + tJ^E_{1,1,1,1,0,1,1,1,1,1,-1,0,0,-1,0} \\
& -tJ^E_{1,1,1,1,1,1,1,1,1,1,-1,0,-1,0,-1} + t^2J^E_{1,1,1,0,1,1,1,1,1,1,0,0,0,-1,0} \\
j^E_{167} = {} & 2tJ^E_{1,1,1,1,0,1,1,1,1,1,-1,0,0,-1,0} - 2tJ^E_{1,1,1,1,1,1,1,1,1,1,-1,0,-1,0,-1} + (s+t)tJ^E_{1,1,1,1,1,1,0,1,1,1,-1,0,0,0,0} \\
& -tJ^E_{1,0,1,1,0,1,1,1,1,1,-1,0,0,0,0} + tJ^E_{1,1,0,1,1,1,1,1,1,1,-1,0,0,0,-1} \\
& -tJ^E_{1,1,1,0,1,1,1,1,1,1,-1,0,0,-1,0} + tJ^E_{1,1,1,1,1,1,1,1,1,1,-1,-1,-1,0,0}
\end{aligned}
$$

Table A.4.: Diagram E: Integrands with constant leading singularities (continued).

$$
\begin{aligned}
j^{E}_{168} = \quad & -J^{E}_{1,0,1,0,1,1,1,1,1,-1,0,0,0,0,0} - J^{E}_{1,0,1,1,0,1,1,0,1,0,0,0,0,0,0} - J^{E}_{1,0,1,1,1,1,1,1,1,0,-1,0,0,0,-1} \\
& -J^{E}_{1,1,0,1,0,1,1,1,0,0,0,0,0,0,0} - J^{E}_{1,1,0,1,1,1,1,0,1,0,0,-1,0,0,0} - J^{E}_{1,1,0,1,1,1,1,1,1,-1,0,0,0,0,-1} \\
& -J^{E}_{1,1,1,0,1,1,1,0,1,0,0,0,-1,0,0} - J^{E}_{1,1,1,0,1,1,1,1,0,0,0,0,0,-1,0} - J^{E}_{1,1,1,1,0,1,1,1,1,0,-1,0,0,-1,0} \\
& -J^{E}_{1,1,1,1,1,1,1,1,0,0,0,0,-1,0,-1} - J^{E}_{1,1,1,1,1,1,1,1,1,0,-1,-1,-1,0,0} - J^{E}_{1,1,1,1,1,1,1,1,1,-1,0,-1,0,-1,0} \\
& +J^{E}_{1,0,1,0,1,1,1,0,1,0,0,0,0,0,0} + J^{E}_{1,0,1,1,0,1,1,1,1,0,-1,0,0,0,0} + J^{E}_{1,0,1,1,1,1,1,1,1,-1,0,0,0,0,-1} \\
& +J^{E}_{1,1,0,1,0,1,1,0,1,0,0,0,0,0,0} + J^{E}_{1,1,0,1,1,1,1,1,0,0,0,0,0,0,-1} + J^{E}_{1,1,0,1,1,1,1,1,1,-1,0,-1,0,0,0} \\
& +J^{E}_{1,1,1,0,1,1,1,1,0,0,0,0,-1,0,0} + J^{E}_{1,1,1,0,1,1,1,1,1,-1,0,0,0,-1,0} + J^{E}_{1,1,1,1,0,1,1,1,0,0,0,0,0,-1,0} \\
& +J^{E}_{1,1,1,1,1,1,1,0,1,0,0,-1,-1,0,0} + J^{E}_{1,1,1,1,1,1,1,1,1,0,-1,-1,0,-1,0} + J^{E}_{1,1,1,1,1,1,1,1,1,0,-1,0,-1,0,-1}
\end{aligned}
$$

$$
\begin{aligned}
j^{E}_{169} = \quad & -J^{E}_{1,0,1,0,1,1,1,1,1,0,-1,0,0,0,0} - J^{E}_{1,0,1,1,0,1,1,0,1,1,-1,0,0,0,0} - J^{E} - -_{1,0,1,1,1,1,1,1,1,1,-2,0,0,0,-1} \\
& -J^{E}_{1,1,0,1,0,1,1,1,0,1,-1,0,0,0,0} - J^{E}_{1,1,0,1,1,1,1,0,1,1,-1,-1,0,0,0} - J^{E}_{1,1,0,1,1,1,1,1,1,0,-1,0,0,0,-1} \\
& -J^{E}_{1,1,1,0,1,1,1,0,1,1,-1,0,-1,0,0} - J^{E}_{1,1,1,0,1,1,1,1,0,1,-1,0,0,-1,0} - J^{E}_{1,1,1,1,0,1,1,1,1,1,-2,0,0,-1,0} \\
& -J^{E}_{1,1,1,1,1,1,1,1,0,1,-1,0,-1,0,-1} - J^{E}_{1,1,1,1,1,1,1,1,1,0,-1,-1,0,-1,0} - J^{E}_{1,1,1,1,1,1,1,1,1,1,-2,-1,-1,0,0} \\
& +(s+t)[-J^{E}_{0,1,1,1,0,1,1,1,1,1,-1,0,0,0,0} + J^{E}_{0,1,1,1,1,1,1,1,1,1,-1,-1,0,0,0} + J^{E}_{1,0,1,1,0,1,1,1,1,1,-1,0,0,0,0}] \\
& +(s+t)[-J^{E}_{1,0,1,1,1,0,1,1,1,1,-1,0,0,0,0} + J^{E}_{1,1,1,1,1,0,1,1,1,1,-1,0,-1,0,0} - J^{E}_{1,1,1,1,1,1,1,1,1,1,-1,-1,-1,0,0}] \\
& -t(s+t)J^{E}_{1,1,1,1,1,1,0,1,1,1,-1,0,0,0,0} + J^{E}_{1,0,1,0,1,1,1,0,1,1,-1,0,0,0,0} + J^{E}_{1,0,1,1,0,1,1,1,1,1,-2,0,0,0,0} \\
& +J^{E}_{1,0,1,1,1,1,1,1,1,0,-1,0,0,0,-1} + J^{E}_{1,1,0,1,0,1,1,0,1,1,-1,0,0,0,0} + J^{E}_{1,1,0,1,1,1,1,1,0,1,-1,0,0,0,-1} \\
& +J^{E}_{1,1,0,1,1,1,1,1,1,0,-1,-1,0,0,0} + J^{E}_{1,1,1,0,1,1,1,1,0,1,-1,0,-1,0,0} + J^{E}_{1,1,1,0,1,1,1,1,1,0,-1,0,0,-1,0} \\
& +J^{E}_{1,1,1,1,0,1,1,1,0,1,-1,0,0,-1,0} + J^{E}_{1,1,1,1,1,1,1,0,1,1,-1,-1,-1,0,0} + J^{E}_{1,1,1,1,1,1,1,1,1,1,-2,-1,0,-1,0} \\
& +J^{E}_{1,1,1,1,1,1,1,1,1,1,-2,0,-1,0,-1}
\end{aligned}
$$

$$
\begin{aligned}
j^{E}_{170} = \quad & -J^{E}_{1,0,1,0,1,1,1,1,0,0,0,0,0,0,0} - J^{E}_{1,0,1,1,0,1,1,0,0,1,0,0,0,0,0} - J^{E}_{1,0,1,1,1,1,1,1,0,1,-1,0,0,0,-1} \\
& -J^{E}_{1,1,0,1,0,1,1,1,-1,1,0,0,0,0,0} - J^{E}_{1,1,0,1,1,1,1,0,0,1,0,-1,0,0,0} - J^{E}_{1,1,0,1,1,1,1,1,0,0,0,0,0,0,-1} \\
& -J^{E}_{1,1,1,0,1,1,1,0,0,1,0,0,-1,0,0} - J^{E}_{1,1,1,0,1,1,1,1,-1,1,0,0,0,-1,0} - J^{E}_{1,1,1,1,0,1,1,1,0,1,-1,0,0,-1,0} \\
& -J^{E}_{1,1,1,1,1,1,1,1,0,0,0,-1,0,-1,0} - J^{E}_{1,1,1,1,1,1,1,1,0,1,-1,-1,-1,0,0} - J^{E}_{1,1,1,1,1,1,1,1,-1,1,0,0,-1,0,-1} \\
& +J^{E}_{1,0,1,0,1,1,1,0,0,1,0,0,0,0,0} + J^{E}_{1,0,1,1,0,1,1,1,0,1,-1,0,0,0,0} + J^{E}_{1,0,1,1,1,1,1,1,0,0,0,0,0,0,-1} \\
& +J^{E}_{1,1,0,1,0,1,1,0,0,1,0,0,0,0,0} + J^{E}_{1,1,0,1,1,1,1,1,-1,1,0,0,0,0,-1} + J^{E}_{1,1,0,1,1,1,1,1,0,0,0,-1,0,0,0} \\
& +J^{E}_{1,1,1,0,1,1,1,1,-1,1,0,0,-1,0,0} + J^{E}_{1,1,1,0,1,1,1,1,0,0,0,0,0,-1,0} + J^{E}_{1,1,1,1,0,1,1,1,-1,1,0,0,0,-1,0} \\
& +J^{E}_{1,1,1,1,1,1,1,0,0,1,0,-1,-1,0,0} + J^{E}_{1,1,1,1,1,1,1,1,0,1,-1,-1,0,-1,0} + J^{E}_{1,1,1,1,1,1,1,1,0,1,-1,0,-1,0,-1}
\end{aligned}
$$

$$
\begin{aligned}
j^{E}_{171} = \quad & (st+t^2)[-J^{E}_{0,1,1,0,1,1,1,0,1,1,0,0,0,0,0} + J^{E}_{0,1,1,0,1,1,1,1,1,1,-1,0,0,0,0} - J^{E}_{0,1,1,1,1,1,1,1,1,1,-1,0,0,0,-1}] \\
& +t^2 J^{E}_{0,1,1,1,1,1,1,0,1,1,0,0,0,0,-1} + (s^2+st)[J^{E}_{0,1,1,1,0,1,1,1,1,1,-1,0,0,0,0} - J^{E}_{0,1,1,1,1,1,1,1,1,1,-1,-1,0,0,0}] \\
& +(s^2 t+st^2)J^{E}_{0,1,1,1,1,1,1,1,1,1,-1,0,0,0,0} + tJ^{E}_{1,0,1,0,1,1,1,-1,1,1,0,0,0,0,0} + 2sJ^{E}_{1,0,1,0,1,1,1,0,0,1,0,0,0,0,0} \\
& +(-s-t)J^{E}_{1,0,1,0,1,1,1,0,1,0,0,0,0,0,0} + (-s-t)J^{E}_{1,0,1,0,1,1,1,0,1,1,-1,0,0,0,0} - 2sJ^{E}_{1,0,1,0,1,1,1,1,0,0,0,0,0,0,0} \\
& +sJ^{E}_{1,0,1,0,1,1,1,1,1,-1,0,0,0,0,0} + (s+t)J^{E}_{1,0,1,0,1,1,1,1,1,0,-1,0,0,0,0} - tJ^{E}_{1,0,1,1,0,1,1,-1,1,1,0,0,0,0,0} \\
& -2sJ^{E}_{1,0,1,1,0,1,1,0,0,1,0,0,0,0,0} - sJ^{E}_{1,0,1,1,0,1,1,0,1,0,0,0,0,0,0} + (3s+2t)J^{E}_{1,0,1,1,0,1,1,0,1,1,-1,0,0,0,0} \\
& +2sJ^{E}_{1,0,1,1,0,1,1,1,0,0,0,0,0,0,0} - sJ^{E}_{1,0,1,1,0,1,1,1,1,0,-1,0,0,0,0} + (-s-t)J^{E}_{1,0,1,1,0,1,1,1,1,1,-2,0,0,0,0} \\
& +(s^2+st)[-J^{E}_{1,0,1,1,0,1,1,1,1,1,-1,0,0,0,0} + J^{E}_{1,0,1,1,1,0,1,1,1,1,-1,0,0,0,0} + 2sJ^{E}_{1,0,1,1,1,1,1,0,1,0,0,-1,0,0,0} \\
& +tJ^{E}_{1,0,1,1,1,1,1,0,1,0,0,0,0,0,-1} - 2sJ^{E}_{1,0,1,1,1,1,1,0,1,1,-1,-1,0,0,0} - tJ^{E}_{1,0,1,1,1,1,1,0,1,1,-1,0,0,0,-1} \\
& -(2s+t)J^{E}_{1,0,1,1,1,1,1,1,1,0,-1,0,0,0,-1} + sJ^{E}_{1,0,1,1,1,1,1,1,1,-1,0,0,0,0,-1} + 2sJ^{E}_{1,0,1,1,1,1,1,1,1,0,-1,-1,0,0,0} \\
& -2sJ^{E}_{1,0,1,1,1,1,1,1,1,-1,0,-1,0,0,0} + 2s^2 J^{E}_{1,0,1,1,1,1,1,1,0,0,-1,0,0,0,0} + (s+t)J^{E}_{1,0,1,1,1,1,1,1,1,1,-2,0,0,0,-1}
\end{aligned}
$$

+...(see next page)

Table A.5.: Diagram E: Integrands with constant leading singularities (continued).

...(see previous page)

$$
\begin{aligned}
&-2tJ^E_{1,1,0,0,1,1,1,-1,1,1,0,0,0,0,0} + 2tJ^E_{1,1,0,0,1,1,1,0,0,1,0,0,0,0,0} + 2tJ^E_{1,1,0,0,1,1,1,0,1,0,0,0,0,0,0} \\
&+(2st+2t^2)J^E_{1,1,0,0,1,1,1,0,1,1,0,0,0,0,0} - 2tJ^E_{1,1,0,0,1,1,1,1,0,0,0,0,0,0,0} + tJ^E_{1,1,0,1,0,1,1,-1,1,1,0,0,0,0,0} \\
&-tJ^E_{1,1,0,1,0,1,1,0,0,1,0,0,0,0,0} + sJ^E_{1,1,0,1,0,1,1,0,1,0,0,0,0,0,0} + (-s-t)J^E_{1,1,0,1,0,1,1,0,1,1,-1,0,0,0,0} \\
&-sJ^E_{1,1,0,1,0,1,1,1,0,0,0,0,0,0,0} + (s+t)J^E_{1,1,0,1,0,1,1,1,0,1,-1,0,0,0,0} + (-st-t^2)J^E_{1,1,0,1,1,0,1,0,1,1,0,0,0,0,0} \\
&+(st+t^2)J^E_{1,1,0,1,1,0,1,1,1,1,-1,0,0,0,0} + tJ^E_{1,1,0,1,1,1,1,-1,1,1,0,-1,0,0,0} + t^2J^E_{1,1,0,1,1,1,1,-1,1,1,0,0,0,0,0} \\
&-tJ^E_{1,1,0,1,1,1,1,0,0,1,0,0,0,0,-1} + (-s-t)J^E_{1,1,0,1,1,1,1,0,1,0,0,-1,0,0,0} - tJ^E_{1,1,0,1,1,1,1,0,1,0,0,0,0,0,-1} \\
&+(s-t)J^E_{1,1,0,1,1,1,1,0,1,1,-1,-1,0,0,0} + 2tJ^E_{1,1,0,1,1,1,1,0,1,1,-1,0,0,0,-1} + (-st-t^2)J^E_{1,1,0,1,1,1,1,0,1,1,-1,0,0,0,0} \\
&-t^2J^E_{1,1,0,1,1,1,1,0,1,1,0,0,0,0,-1} + (s+2t)J^E_{1,1,0,1,1,1,1,1,0,0,0,0,0,0,-1} + (-s-t)J^E_{1,1,0,1,1,1,1,1,0,1,-1,0,0,0,-1} \\
&+sJ^E_{1,1,0,1,1,1,1,1,1,-1,0,-1,0,0,0} - sJ^E_{1,1,0,1,1,1,1,1,1,-1,0,0,0,0,-1} + (-s+t)J^E_{1,1,0,1,1,1,1,1,1,0,-1,-1,0,0,0} \\
&+(s-t)J^E_{1,1,0,1,1,1,1,1,1,0,-1,0,0,0,-1} + (-st-t^2)J^E_{1,1,0,1,1,1,1,1,1,1,-1,0,0,0,-1} + tJ^E_{1,1,1,0,1,1,1,-1,1,1,0,0,-1,0,0} \\
&+t^2J^E_{1,1,1,0,1,1,1,-1,1,1,0,0,0,0,0} + (-2s-t)J^E_{1,1,1,0,1,1,1,0,0,1,0,0,-1,0,0} - tJ^E_{1,1,1,0,1,1,1,0,0,1,0,0,0,-1,0} \\
&+sJ^E_{1,1,1,0,1,1,1,0,1,0,0,0,-1,0,0} - tJ^E_{1,1,1,0,1,1,1,0,1,0,0,0,0,-1,0} + (s-t)J^E_{1,1,1,0,1,1,1,0,1,1,-1,0,-1,0,0} \\
&+2tJ^E_{1,1,1,0,1,1,1,0,1,1,-1,0,0,-1,0} + (-st-t^2)J^E_{1,1,1,0,1,1,1,0,1,1,-1,0,0,0,0} - t^2J^E_{1,1,1,0,1,1,1,0,1,1,0,0,0,-1,0} \\
&+2sJ^E_{1,1,1,0,1,1,1,1,-1,1,0,0,-1,0,0} - 2sJ^E_{1,1,1,0,1,1,1,1,-1,1,0,0,0,-1,0} - sJ^E_{1,1,1,0,1,1,1,1,0,0,0,0,-1,0,0} \\
&+(3s+2t)J^E_{1,1,1,0,1,1,1,1,0,0,0,0,0,-1,0} + (-s+t)[J^E_{1,1,1,0,1,1,1,1,0,1,-1,0,-1,0,0} - J^E_{1,1,1,0,1,1,1,1,0,1,-1,0,0,-1,0}] \\
&-sJ^E_{1,1,1,0,1,1,1,1,1,-1,0,0,0,-1,0} - (s+t)J^E_{1,1,1,0,1,1,1,1,1,0,-1,0,0,-1,0} - (st+t^2)J^E_{1,1,1,0,1,1,1,1,1,1,-1,0,0,-1,0} \\
&+2sJ^E_{1,1,1,1,0,1,1,0,0,1,0,0,-1,0,0} + tJ^E_{1,1,1,1,0,1,1,0,0,1,0,0,0,-1,0} - 2sJ^E_{1,1,1,1,0,1,1,0,1,1,-1,0,-1,0,0} \\
&-tJ^E_{1,1,1,1,0,1,1,0,1,1,-1,0,0,-1,0} - 2sJ^E_{1,1,1,1,0,1,1,1,-1,1,0,0,-1,0,0} + 2sJ^E_{1,1,1,1,0,1,1,1,-1,1,0,0,0,-1,0} \\
&-sJ^E_{1,1,1,1,0,1,1,1,0,0,0,0,0,-1,0} + 2sJ^E_{1,1,1,1,0,1,1,1,0,1,-1,0,-1,0,0} + (-3s-t)J^E_{1,1,1,1,0,1,1,1,0,1,-1,0,0,-1,0} \\
&+2s^2J^E_{1,1,1,1,0,1,1,1,0,1,0,0,-1,0,0} + sJ^E_{1,1,1,1,0,1,1,1,1,0,-1,0,0,-1,0} + (s+t)J^E_{1,1,1,1,0,1,1,1,1,1,-2,0,0,-1,0} \\
&+t^2J^E_{1,1,1,1,1,0,1,0,1,1,0,0,0,-1,0} - (s^2+st)J^E_{1,1,1,1,1,0,1,1,1,1,-1,0,-1,0,0} - (st+t^2)J^E_{1,1,1,1,1,0,1,1,1,1,-1,0,0,-1,0} \\
&+(s^2t+st^2)J^E_{1,1,1,1,1,0,1,1,1,1,-1,0,0,0,0} - t^2J^E_{1,1,1,1,1,1,0,-1,1,1,0,0,0,0,0} + (2st+2t^2)J^E_{1,1,1,1,1,1,0,0,1,1,-1,0,0,0,0} \\
&-4st^2J^E_{1,1,1,1,1,1,0,0,1,1,0,0,0,0,0} - (st+t^2)J^E_{1,1,1,1,1,1,0,1,1,1,-2,0,0,0,0} - 3st(s+t)J^E_{1,1,1,1,1,1,0,1,1,1,-1,0,0,0,0} \\
&-2s^2t^2J^E_{1,1,1,1,1,1,0,1,1,1,0,0,0,0,0} - tJ^E_{1,1,1,1,1,1,1,-1,1,1,0,-1,-1,0,0} + tJ^E_{1,1,1,1,1,1,1,0,0,1,0,0,-1,0,-1} \\
&-sJ^E_{1,1,1,1,1,1,1,0,1,0,0,-1,-1,0,0} + tJ^E_{1,1,1,1,1,1,1,0,1,0,0,-1,0,-1,0} + (s+2t)J^E_{1,1,1,1,1,1,1,0,1,1,-1,-1,-1,0,0} \\
&-tJ^E_{1,1,1,1,1,1,1,0,1,1,-1,-1,0,-1,0} - tJ^E_{1,1,1,1,1,1,1,0,1,1,-1,0,-1,0,-1} - t^2J^E_{1,1,1,1,1,1,1,0,1,1,-1,0,0,-1,0} \\
&-t^2J^E_{1,1,1,1,1,1,1,0,1,1,-1,0,0,0,-1} - 4stJ^E_{1,1,1,1,1,1,1,0,1,1,0,-1,-1,0,0} + 2sJ^E_{1,1,1,1,1,1,1,1,0,0,0,-1,-1,0,0} \\
&-2sJ^E_{1,1,1,1,1,1,1,1,0,0,0,-1,0,-1,0} - sJ^E_{1,1,1,1,1,1,1,1,0,0,0,0,-1,0,-1} - 2tJ^E_{1,1,1,1,1,1,1,1,0,0,0,0,0,-1,-1} \\
&-2stJ^E_{1,1,1,1,1,1,1,1,0,0,0,0,0,0,-1} - 2sJ^E_{1,1,1,1,1,1,1,1,0,1,-1,-1,-1,0,0} + 2sJ^E_{1,1,1,1,1,1,1,1,0,1,-1,-1,0,-1,0} \\
&+(s-t)J^E_{1,1,1,1,1,1,1,1,0,1,-1,0,-1,0,-1} + 2tJ^E_{1,1,1,1,1,1,1,1,0,1,-1,0,0,-1,-1} - 2s^2J^E_{1,1,1,1,1,1,1,1,0,1,0,-1,-1,0,0} \\
&+2stJ^E_{1,1,1,1,1,1,1,1,0,1,0,0,-1,0,-1} + sJ^E_{1,1,1,1,1,1,1,1,1,-1,0,-1,0,-1,0} - sJ^E_{1,1,1,1,1,1,1,1,1,0,-1,-1,-1,0,0} \\
&-tJ^E_{1,1,1,1,1,1,1,1,1,0,-1,-1,0,-1,0} + sJ^E_{1,1,1,1,1,1,1,1,1,0,-1,0,-1,0,-1} + 2tJ^E_{1,1,1,1,1,1,1,1,1,0,-1,0,0,-1,-1} \\
&-2s^2J^E_{1,1,1,1,1,1,1,1,1,0,0,-1,-1,0,0} + 2stJ^E_{1,1,1,1,1,1,1,1,1,0,0,-1,0,-1,0} + (s-t)J^E_{1,1,1,1,1,1,1,1,1,1,-2,-1,-1,0,0} \\
&+(-s+t)[J^E_{1,1,1,1,1,1,1,1,1,1,-2,-1,0,-1,0} + J^E_{1,1,1,1,1,1,1,1,1,1,-2,0,-1,0,-1}] - 2tJ^E_{1,1,1,1,1,1,1,1,1,1,-2,0,0,-1,-1} \\
&+(st+t^2)J^E_{1,1,1,1,1,1,1,1,1,1,-2,0,0,-1,0} + (st+t^2)J^E_{1,1,1,1,1,1,1,1,1,1,-2,0,0,0,-1} \\
&+2stJ^E_{1,1,1,1,1,1,1,1,1,1,-1,-1,0,-1,0} + 2stJ^E_{1,1,1,1,1,1,1,1,1,1,-1,0,-1,0,-1} + 2t^2J^E_{1,1,1,1,1,1,1,1,1,1,-1,0,0,-1,-1} \\
&-2s^2tJ^E_{1,1,1,1,1,1,1,1,1,1,0,-1,-1,0,0} + (s^2-3st)J^E_{1,1,1,1,1,1,1,1,1,1,-1,-1,-1,0,0}
\end{aligned}
$$

Table A.6.: Diagram E: Integrands with constant leading singularities (continued).

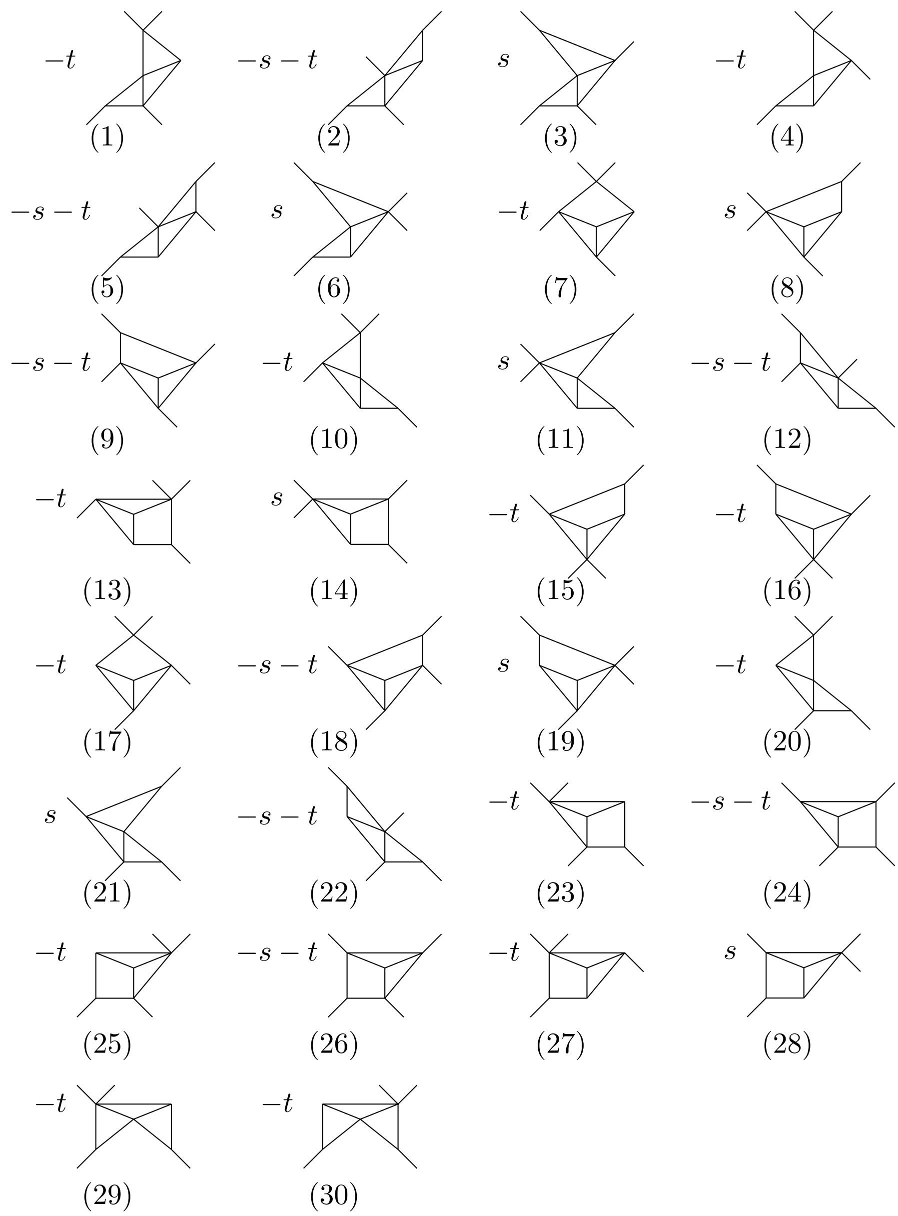


Figure A.1.: Diagram E integrands with constant leading singularities. A wavy line in a loop indicates a numerator depending on the corresponding loop variable.

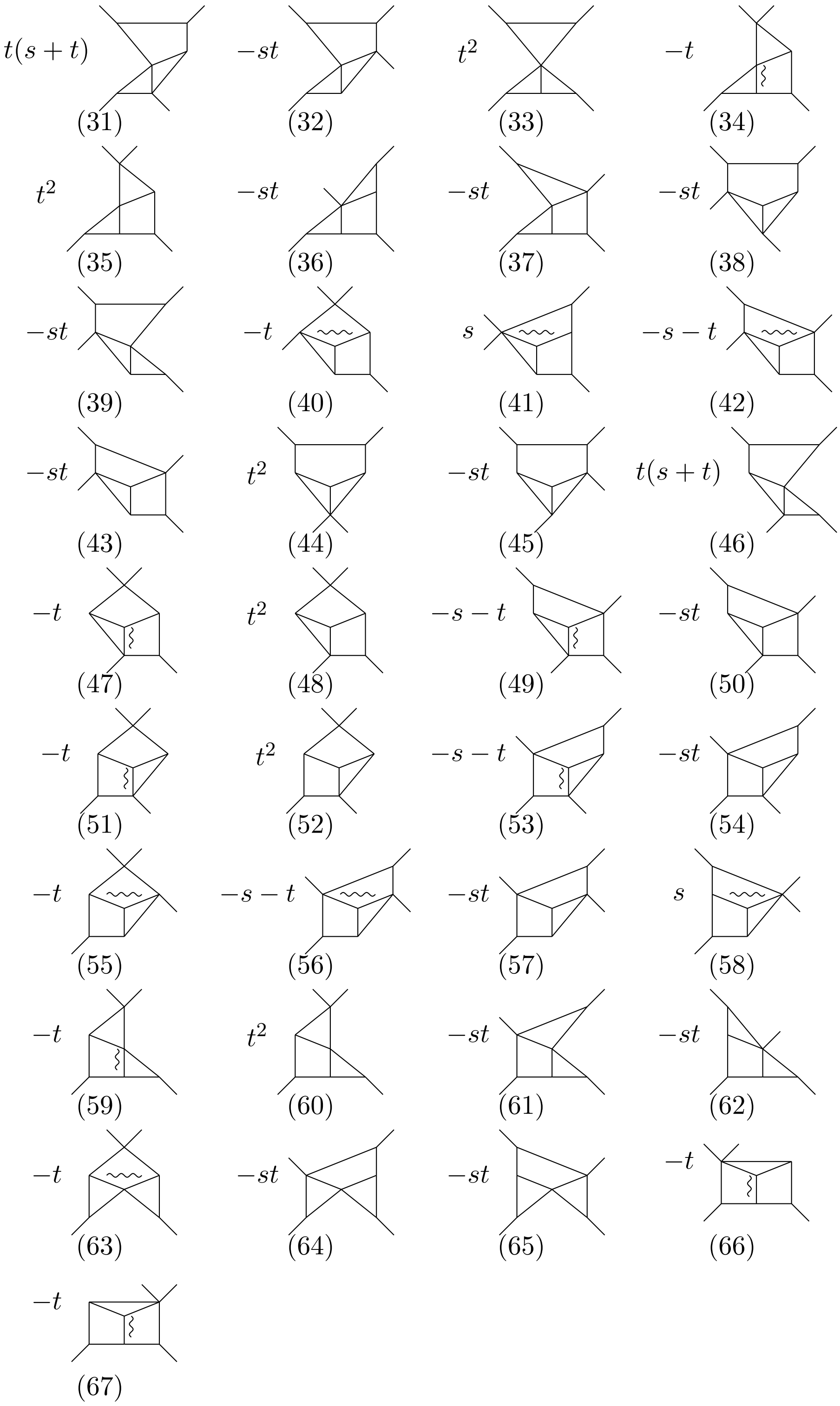


Figure A.2.: Diagram E (continued).

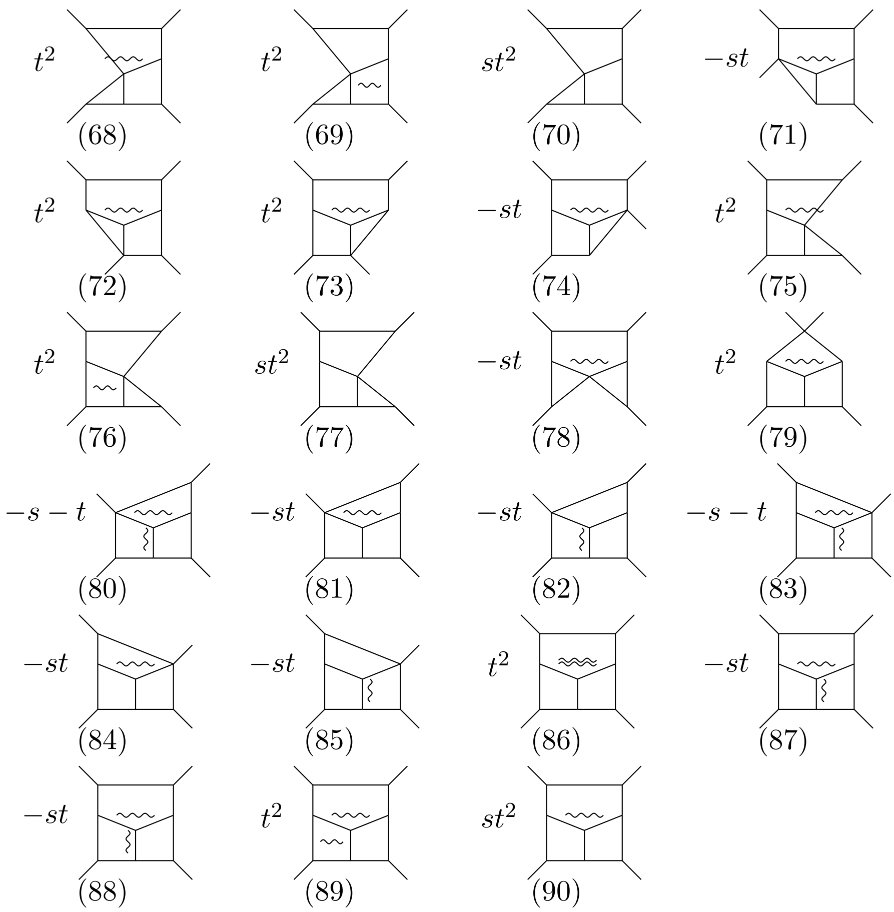


Figure A.3.: Diagram E (continued).

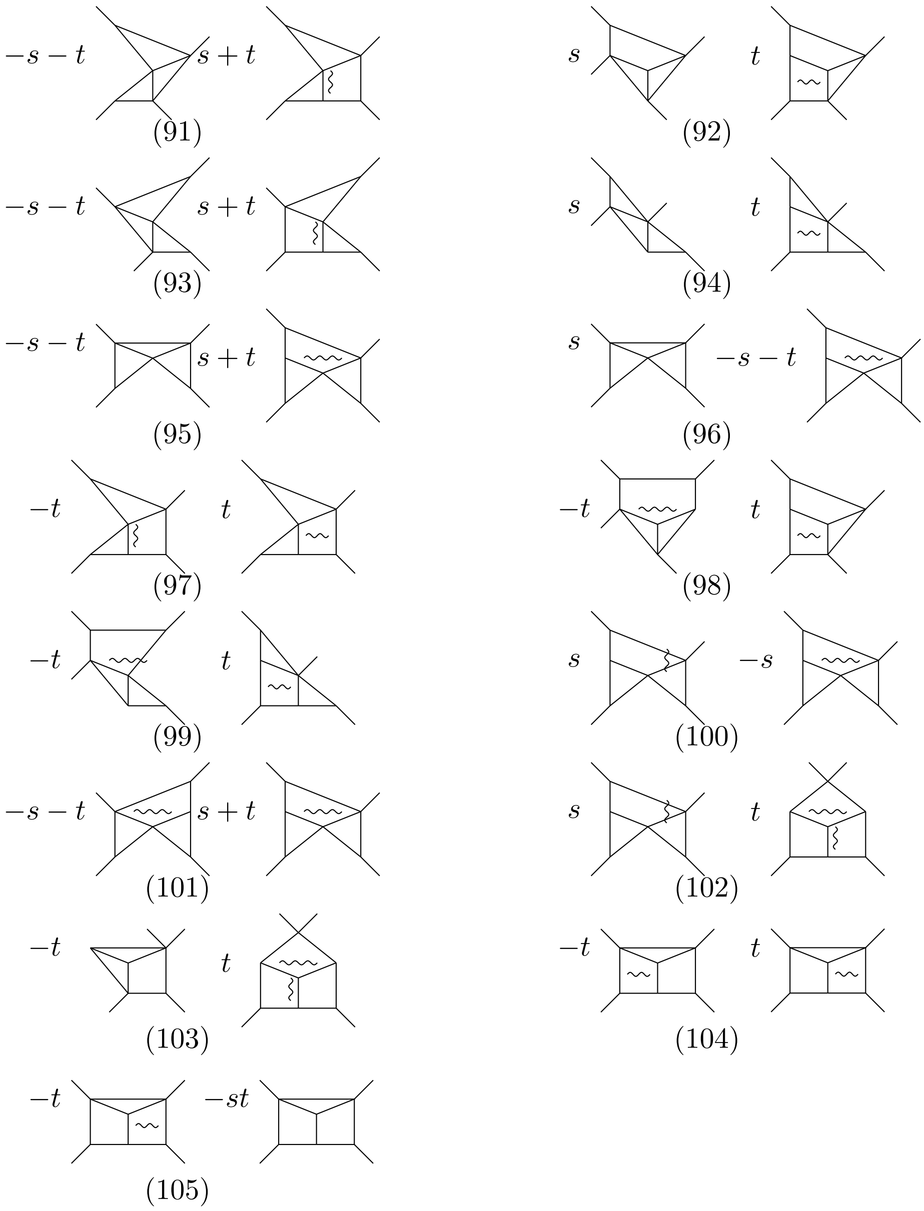


Figure A.4.: Diagram E (continued).

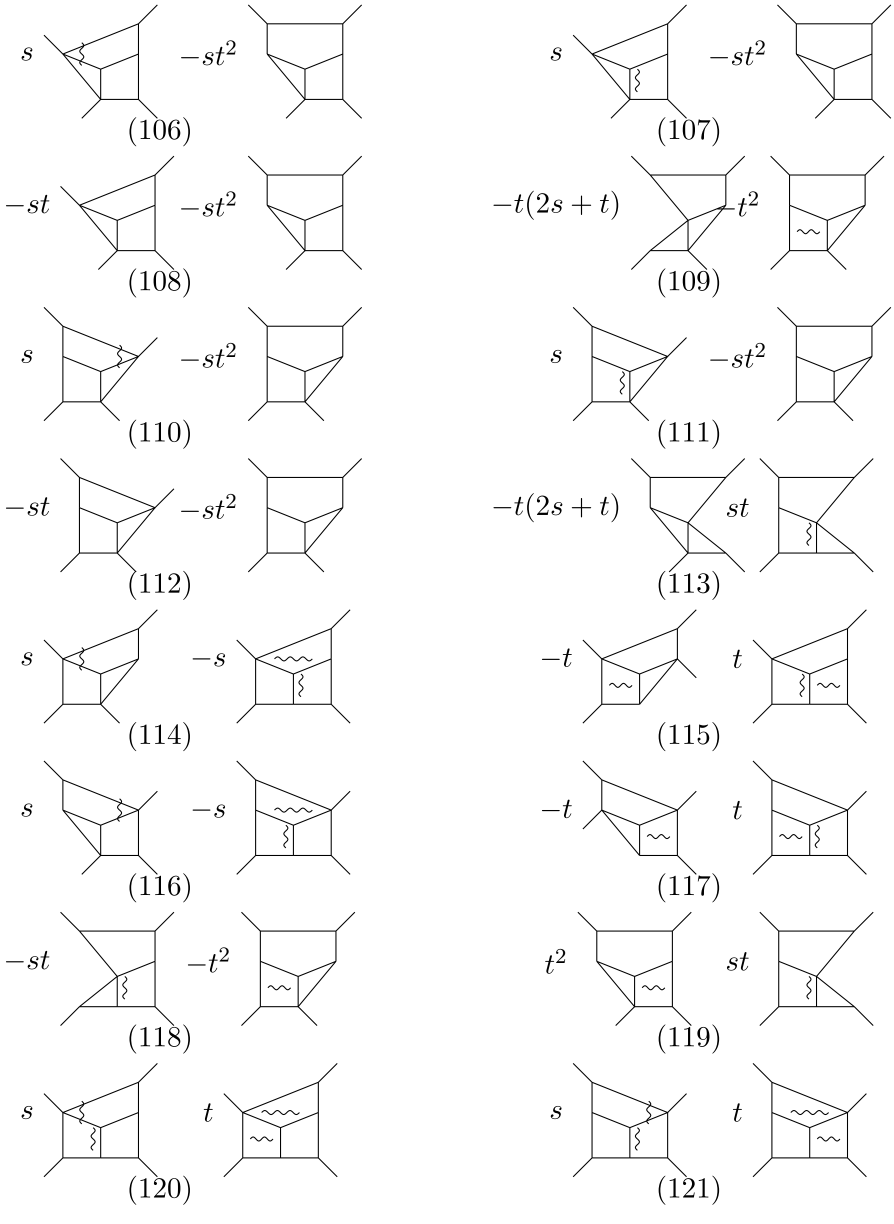


Figure A.5.: Diagram E (continued).

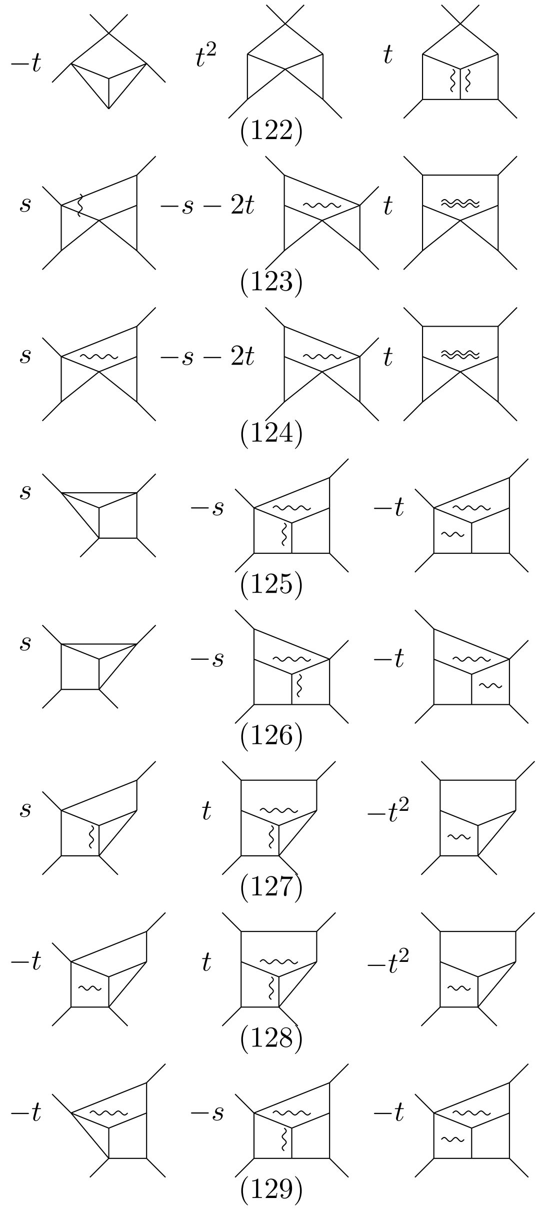


Figure A.6.: Diagram E (continued).

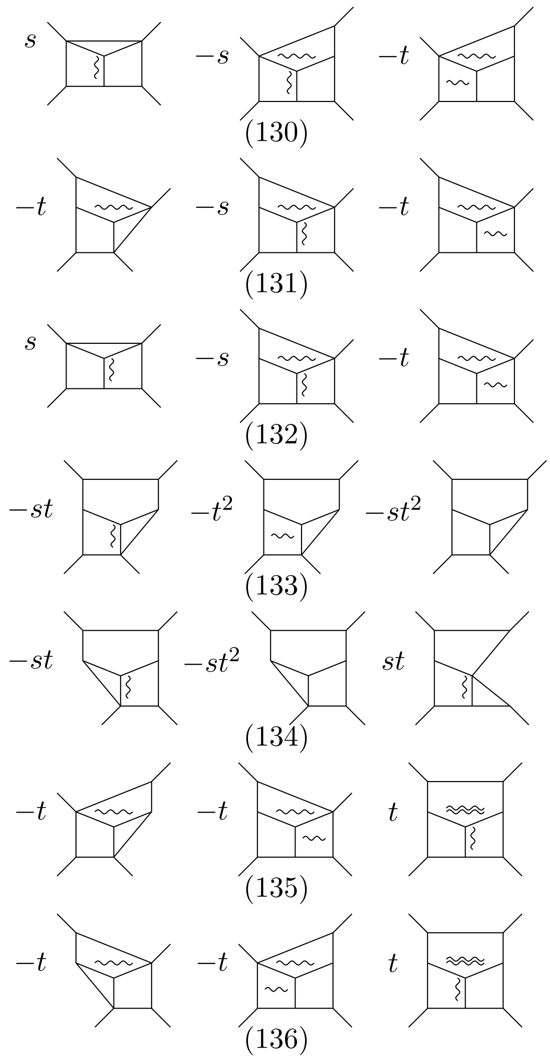


Figure A.7.: Diagram E (continued).

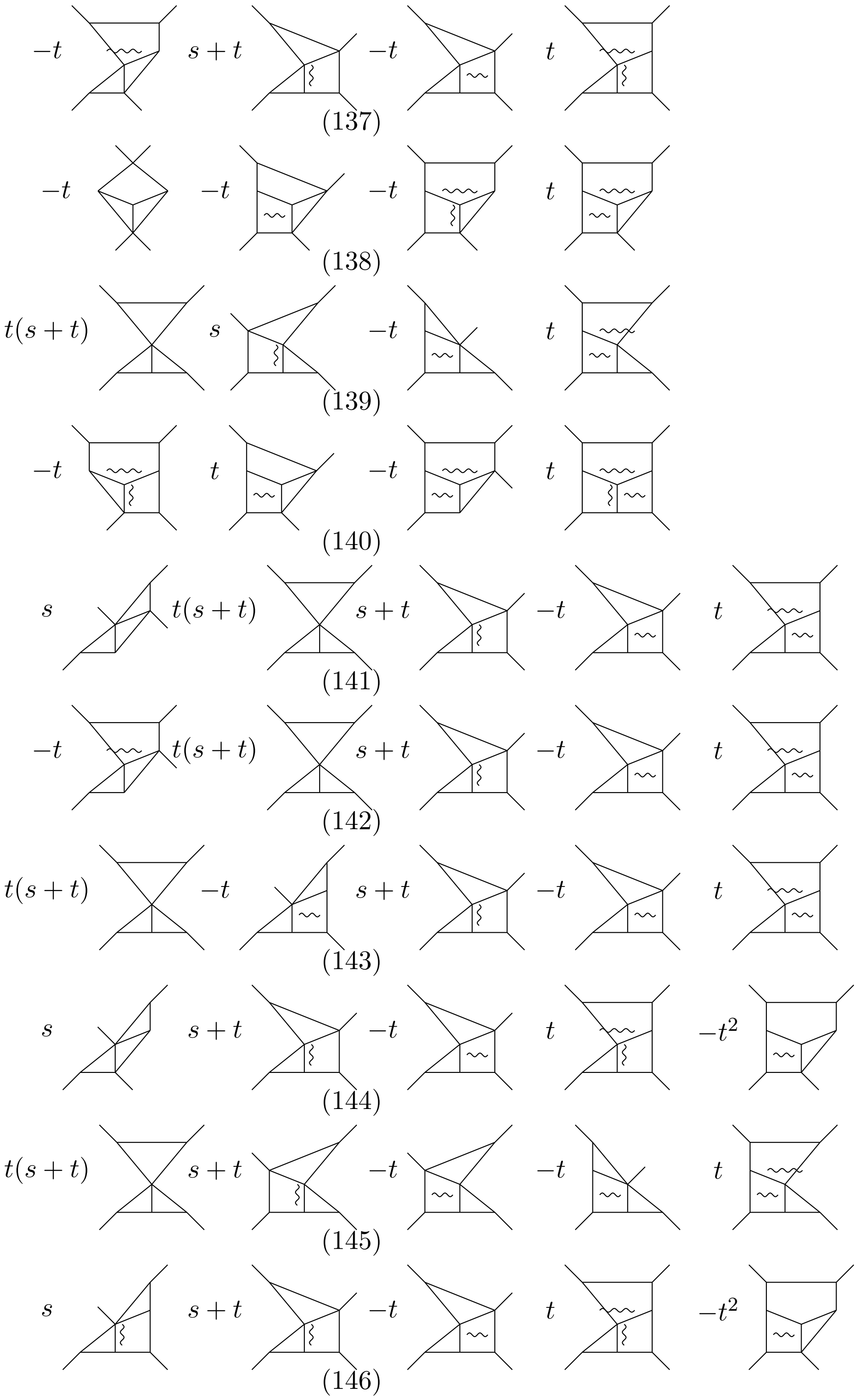


Figure A.8.: Diagram E (continued).

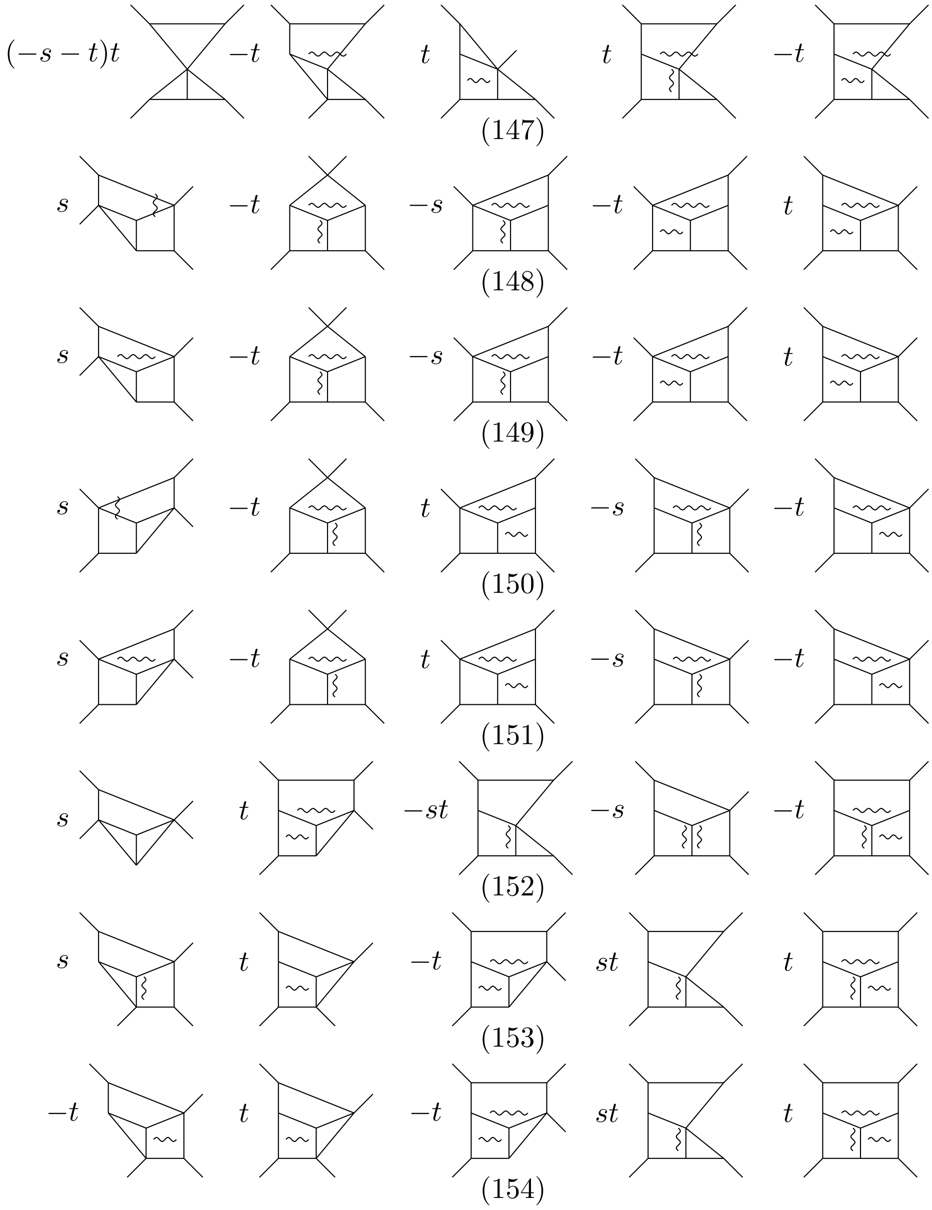


Figure A.9.: Diagram E (continued).

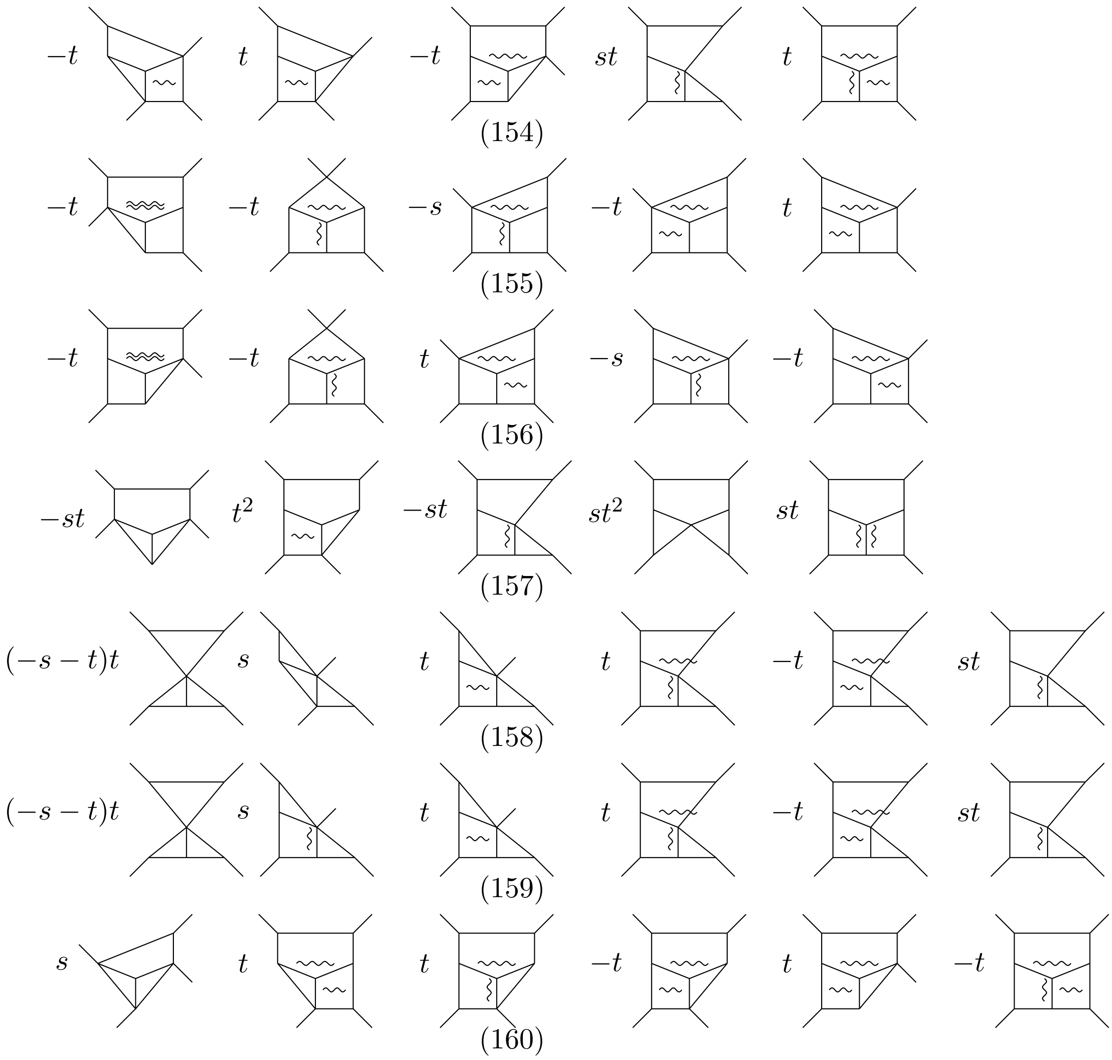


Figure A.10.: Diagram E (continued).

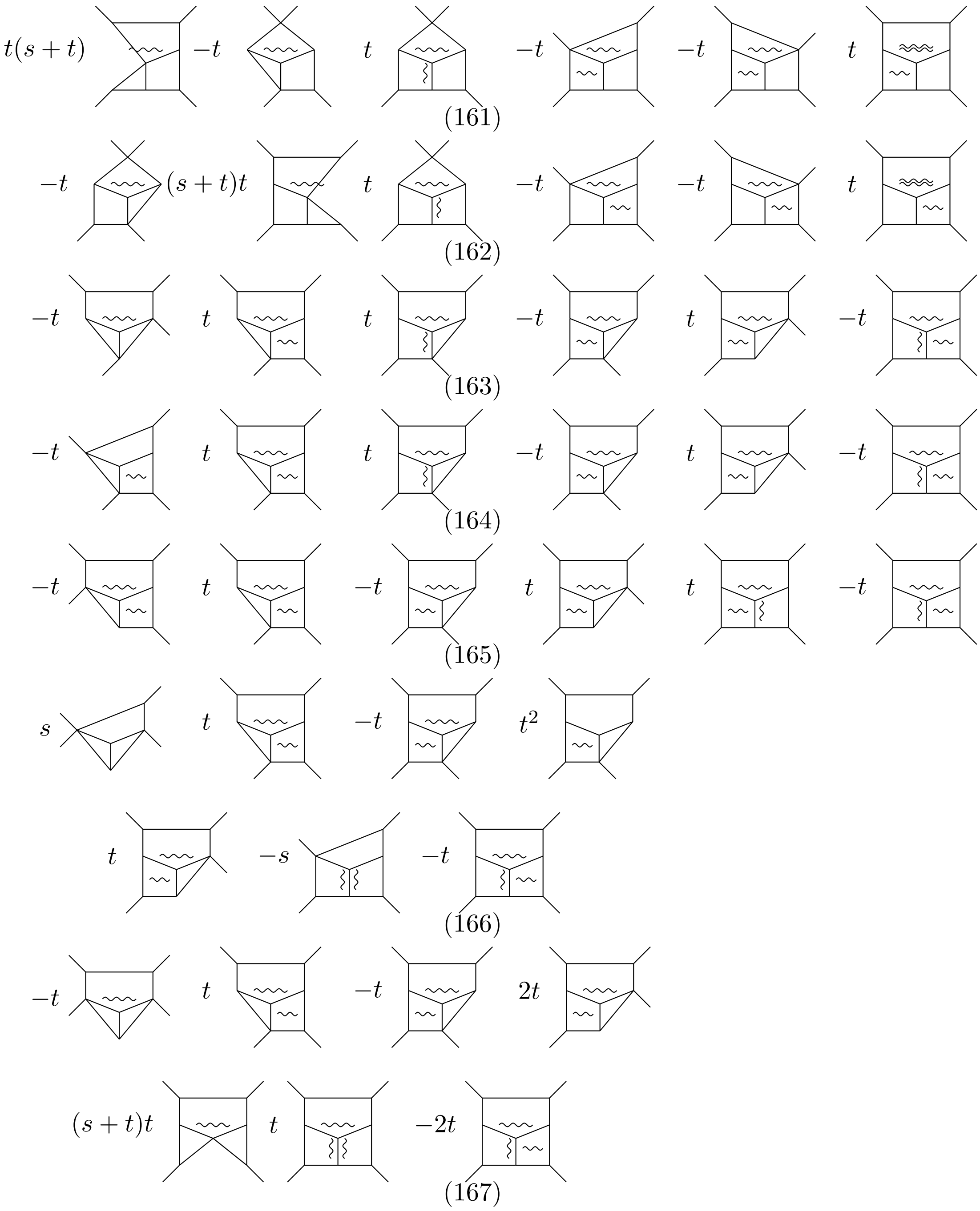


Figure A.11.: Diagram E (continued).

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