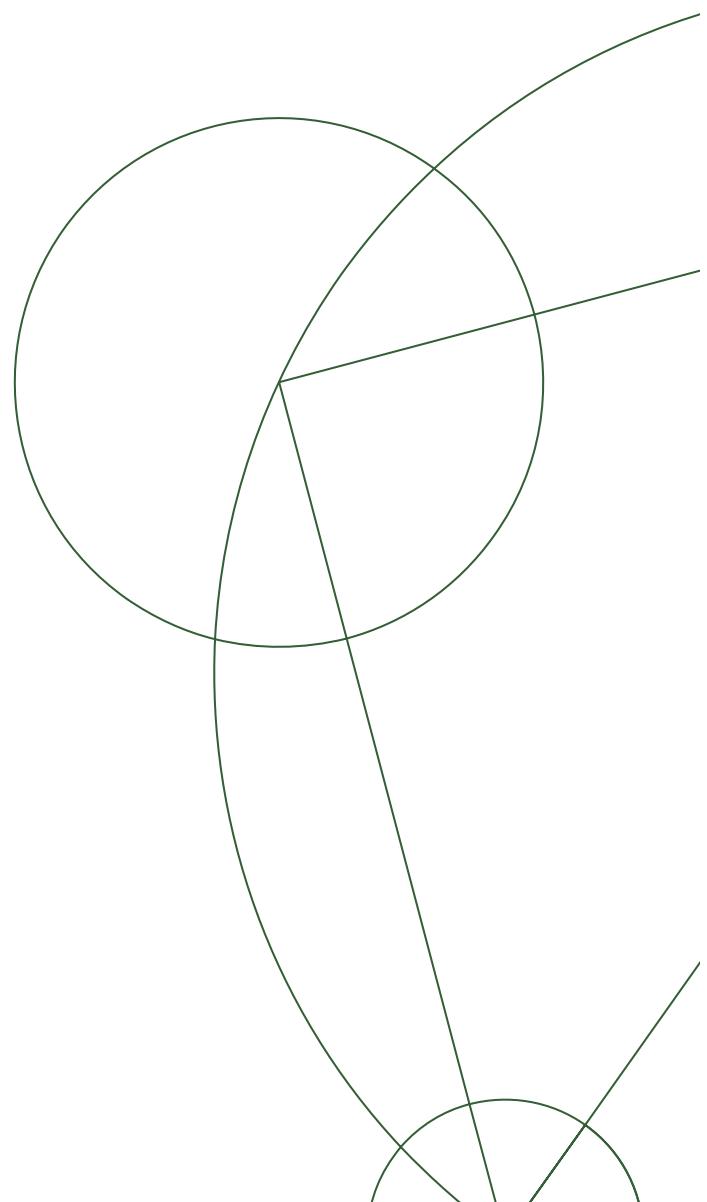




PhD thesis

Paweł Piotr Caputa

Non-planar corrections to the dilatation operator
of AdS/CFT



Academic advisor: Charlotte Fløe Kristjansen

Submitted:

Abstract

In this thesis we study aspects of non-planar corrections to the dilatation operator of the AdS/CFT correspondence.

The material is presented in two parts. In the first one we review main theoretical tools that were used in our research. These include basics of integrability and detailed derivation of the one loop dilatation generator of the $\mathcal{N} = 4$ SYM theory. The second part contains our main calculations and results that are based on the two articles [21, 22]. First, in a clear and pedagogical introductions we guide the reader through our projects and then present them in their published form.

Our key results are:

- derivation and analysis of the full one-loop dilatation generator of the $\mathcal{N} = 4$ SYM with orthogonal and symplectic gauge groups
- revealing a new class of non-planar corrections to the dilatation operator that does not lead to splitting or joining the spin-chain
- derivation of the analytic form for these $1/N$ corrections in a basis of BMN operators
- performing standard tests for integrability of the new corrections (with negative outcome)
- derivation of the full, two-loop dilatation generator of the ABJ theory and its diagonalization in a basis of short operators
- observation that $1/N$ corrections in the ABJ theory break parity that was unexpectedly present in the planar spectrum.

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During my studies I also visited Jagiellonian University, Nagoya University and Brown University. I would like to thank Andrzej Wereszczynski, Shinji Hirano and Anastasia Volovich for their wonderful hospitality and sharing their experience and ideas on physics with me.

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Last but not least, I thank my family for their support and dedicate this work to them.

*Pawel Caputa
Copenhagen, 09/09/2011*

Publications

Articles reviewed in this thesis

- “On the spectral problem of N=4 SYM with orthogonal or symplectic gauge group”
with C. Kristjansen and K. Zoubos
Published in : JHEP **1010**, 082 (2010)
arXiv:1005.2611 [hep-th]
- “Non-planar ABJ Theory and Parity”
with C. Kristjansen and K. Zoubos
Published in : Phys. Lett. **B677**, 197-202 (2009)
arXiv:0903.3354 [hep-th]

Additional projects

- “Observations on Open and Closed String Scattering Amplitudes at High Energies”,
with S. Hirano
To appear in JHEP
arXiv:1108.2381 [hep-th]

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Chapter 1

Introduction

The AdS/CFT correspondence is a conjecture about the equivalence of strings propagating in D-dimensional, negatively curved, Anti-de Sitter (AdS) space-time and certain conformal field theories (CFT) on the D-1 dimensional boundary of *AdS*. It manifestly incorporates the idea of holography[2] into modern approaches to field theory and gravity. The first example of such a duality was proposed by Maldacena in 1997 [1]. It identifies a gauge theory without gravity, maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills in four dimensions and a theory of quantum gravity, the type IIB superstring theory on $AdS_5 \times S^5$. The model attracted enormous attention within the high energy community and reshaped our way of thinking about both ingredients of the conjecture. Up to date the proposal has passed so many non-trivial tests that hardly any high energy theoretical physicists doubt that it is correct. One of the most spectacular checks of AdS/CFT, the solution of the planar spectral problem, has been possible due to integrability present in this limit. This is the starting point of our discussion so let us elaborate more on this point.

The dictionary between the two sides of the duality relates eigenvalues of a dilatation generator [23] in a basis of composite operators that are traces of the $\mathcal{N} = 4$ SYM fields and string energies in IIB theory. The verification that for “dual” observables these quantum numbers indeed match is called the spectral problem of AdS/CFT.

In general, the dilatation generator has a very complicated mixing problem. In other words its action mixes states with different number of traces and the number of states at every order in perturbation theory grows very fast. This way the task of diagonalizing the operator becomes more and more in-

tractable. Fortunately, the operator has a well defined 't Hooft expansion in $1/N$. This allows, as a first approximation, to consider the mixing problem when the number of colors is very large $N \rightarrow \infty$. "Magically", in this regime the dilatation generator is equivalent to a Hamiltonian of an integrable spin-chain [18]. This discovery started a new field of integrability in gauge and string theory and after a huge amount of effort led to, a very convincing, conjecture solution of the spectral problem in the large N limit[28]. We will neither discuss the path to this result nor the details of the solution itself. Readers interested in these developments and many more aspects of integrability in gauge and string theory are referred to a recent set of excellent reviews on the subject [74].

After solving the planar theory a natural step is to investigate what happens when N is finite. This subject is still largely unexplored. The main obstacle there is the breakdown of the spin-chain picture that was so successful at the planar level. More precisely, non-planar corrections split and join spin-chains and the standard integrability machinery of the Bethe ansatz seems inapplicable. There exists a conventional wisdom that integrable structure is lost beyond the planar limit. It is based on the fact that certain degeneracies between planar parity pairs are lifted when $1/N$ corrections are included in the spectrum (see [23] and later chapters of this thesis). This should naturally not stop us from understanding the structure of the full theory. Even though, as most of the realistic particle physics theories, it might not be all integrable but still due to the large amount of symmetry is probably the simplest possible model for understanding $1/N$ physics.

Some activity beyond the planar limit has been initiated on the level of the three point functions [93, 95, 94] which can be regarded as building blocks for non-planar objects. Namely, they are correlation functions of 3 traces so when we take two of the 3 points coincident they resemble the two point function of a single and double trace operators.

Also, very recently, some traces of integrability beyond the planar limit were noticed in a basis of Schur polynomials [96, 97, 98] in which the mixing problem of the non-planar dilatation generator was equivalent to a system of two integrable hamiltonians. This might be a promising new direction and framework for understanding the full dilatation operator and the lack or presence of integrability.

In any case, before we will reach the final conclusion about the possibility of using integrability at any N and λ , all methods of addressing the problem are on an equal footing.

Together with C. Kristjansen and K. Zoubos we embarked on the analysis of $1/N$ corrections with conventional techniques of integrability [21]. In order to do that we considered the dilatation generator in $\mathcal{N} = 4$ SYM with $U(N)$, $SO(N)$ and $Sp(n)$ gauge groups. All of these theories are equivalent at the planar level, however their non-planar corrections are different. For orthogonal and symplectic gauge groups we uncovered a new class of $1/N$ corrections which does not mix operators with different number of traces. In the spin chain nomenclature they do not split or join the spin-chain.

This allowed us to derive an analytic formula for energies of these corrections in a basis of two-impurity operators [5] which is a strong prediction for energies of semiclassical strings on $AdS_5 \times \mathbb{RP}^5$ (background dual to the field theory with orthogonal and symplectic gauge groups). Moreover, for the first time, we were able search for signs of integrability in a standard way. That means, by trying to construct a modified Bethe Ansatz that would solve their diagonalization problem. In addition, based on experience with planar integrability, we tried to construct $1/N$ corrections to higher commuting charges, hoping that they would commute at every order in N . Unfortunately this construction was unsuccessful.

Even though all our tests for possible integrability of the new corrections were negative, comparing our them with semiclassical string solutions will be an important and non-trivial test of the AdS/CFT correspondence beyond the planar limit.

In our second project, we studied the interplay between integrability and parity. The parity operator acts on a single trace operator by inverting the order of fields inside the trace [75]

$$\hat{P} \text{Tr} (X_1 X_2 \dots X_n) = \text{Tr} (X_n \dots X_2 X_1), \quad (1.1)$$

and on multiple trace operators acts on each of the single trace components. \hat{P} commutes with the planar hamiltonian (dilatation operator D_0)

$$[\hat{P}, \hat{D}_0] = 0, \quad (1.2)$$

so planar eigenstates can be brought into a form with definite parity.

A manifestation of the integrable structure is the presence of a certain degeneracies in the spectrum between states with opposite parity[23]. This is explained by the existence of a tower of higher conserved charges¹ that lead

¹The second charge is usually the Hamiltonian

to a complete solubility of the planar theory. For example the third charge Q_3 maps one state from the parity pair into the other and vice versa. $1/N$ corrections lift these degeneracies, which is usually interpreted as a loss of the charges. In [22] we investigated these phenomena in another example of the gauge/gravity duality, the ABJ theory [77]. Even though by construction the model was manifestly parity breaking, surprisingly, the planar spectrum was still parity invariant (see e.g.[100]).

We decided to study the non-planar corrections in this model to gain more understanding about this puzzle. Following [83], we derived the full dilatation operator at two loops and diagonalized it in a basis of short states (6 14 fields inside the trace) using computer algebra. Our observation was that, as expected, non-planar corrections broke parity and the planar invariance seemed to be a coincidence. Moreover, similarly to $AdS_5 \times S^5$, $1/N$ corrections lifted degeneracies in the spectrum.

In short, this is the content of the thesis. A more pedagogical introduction and details of the projects can be found in the main text.

Outline

The material is presented in two parts. Part I is mostly introductory and is meant to be pedagogical.

After a brief introduction to the AdS/CFT duality we review basics of integrability that are necessary to understand the context and the logic behind our reasoning in the research part of the thesis. This includes Coordinate Bethe Ansatz, Asymptotic Bethe Ansatz, construction of conserved charges with boost operator and the parity operator. Then we present a detailed derivation of the one-loop dilatation generator for $\mathcal{N} = 4$ SYM with unitary, special-unitary, orthogonal and symplectic gauge groups. We then study its structure in the context of mixing of composite operators. We identify appropriate substructures and classify them into planar and non-planar, or leading and subleading in the number of colors N respectively. Part II contains the summary of the results obtained in the projects and both articles in their published versions. Finally additional material about orthogonal and symplectic contractions, Chan-Paton factors and perturbation theory is collected in three appendices.

Part I

Introductory Material

Chapter 2

A brief introduction into the AdS/CFT correspondence.

This chapter is a very short introduction to the correspondence between string theory in Anti-de Sitter (AdS) space and conformal field theory (CFT), AdS/CFT for short. The duality was first conjectured by Maldacena more than a decade ago [1]. We assume that the readers are already familiar with AdS/CFT and here, we will only state basic facts that will be relevant for our discussion in later parts of the thesis.

An extensive and pedagogical introduction to the correspondence can be found in [16].

2.1 The conjecture

By now we have a large amount of evidence that, at the microscopic level, information about gravity in d -dimensions can be encoded in a certain quantum field theory in $d - 1$ dimensions [2]. This observation is known as the holographic principle (see review [3]).

One of its most concrete examples was proposed by Maldacena in 1997 [1]. It is a conjecture about equivalence between type IIB superstring theory on $AdS_5 \times S^5$ background and the four dimensional $\mathcal{N} = 4$ maximally supersymmetric Yang-Mills theory without gravity.

This very non-trivial proposal comes from two ways of looking at the same physical system, namely closed strings propagating in the background of N $D3$ -branes in flat ten dimensions (Fig.2.1).

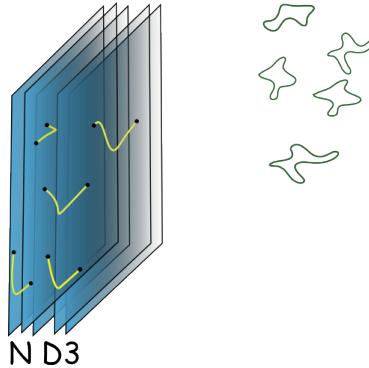


Figure 2.1: *Closed strings propagating in the background of N D3 branes.*

On one hand, D3 branes can be regarded as sources, and effectively replaced by the solution of type IIB supergravity. The solution consists of a metric $g_{\mu\nu}$, a dilaton field ϕ and a R-R five form field strength F_5 . They are explicitly given by

$$ds^2 = H^{-1/2}(r) (-dt^2 + (dx_i)^2) + H^{1/2} (dr^2 + r^2 d\Omega_5^2),$$

$$e^\phi = g_s, \quad F_{tj_1j_2j_3r} = \epsilon_{j_1j_2j_3} H^{-2}(r) \frac{Q}{r^5}, \quad (2.1)$$

where g_s is the string coupling constant and Q the charge of the solution. In the metric, the first bracket describes coordinates “on” D3 branes (parallel to the branes), whereas the second bracket, coordinates perpendicular to the branes. Coordinate r is the radial distance from the branes. Finally the prefactor

$$H(r) = 1 + \left(\frac{R}{r} \right)^4. \quad (2.2)$$

can be physically interpreted as a scale factor between the energy E_r , of the D3 brane system measured from some constant distance r , and the energy seen from infinity E_∞ , such that

$$E_\infty = H^{-\frac{1}{4}}(r) E_r. \quad (2.3)$$

Now, according to the observer at ∞ , low energy limit is equivalent to “fo-

cusing” on D3 branes ($r \rightarrow 0$ so $H \sim \frac{R^4}{r^4}$). In this limit, the metric becomes¹

$$ds^2 = R^2 \left(\frac{\eta_{\mu\nu} dx^\mu dx^\nu}{r^2} + \frac{dr^2}{r^2} \right) + R^2 d\Omega_5^2. \quad (2.4)$$

It describes a product manifold $AdS_5 \times S^5$, both with radius R . This way, we can interpret the system upon consideration as closed super-strings (described by e.g. Green-Schwarz action) propagating in $AdS_5 \times S^5$ supergravity background.

On the other hand, when we focus on branes, they can be described by the effective theory of end-points of the open strings that stretch between them. It is known that for N incident $D3$ -branes this theory is the $\mathcal{N} = 4$ maximally supersymmetric Yang-Mills theory in Minkowski spacetime (for short $\mathcal{N} = 4$ SYM). $\mathcal{N} = 4$ SYM is a theory of “gluons” A_μ , six scalars ϕ_i and fermions ψ . It is given explicitly by

$$S = \frac{2}{g_{YM}^2} \int d^4x Tr \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu \phi_i D_\mu \phi_i - \frac{1}{4} [\phi_i, \phi_j] [\phi_i, \phi_j] + \frac{1}{2} \bar{\psi} \Gamma_\mu D_\mu \psi - \frac{i}{2} \bar{\psi} \Gamma_i [\phi_i, \psi] \right), \quad (2.5)$$

where the field strength is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i [A_\mu, A_\nu], \quad (2.6)$$

and the covariant derivative acts on scalars and fermions as

$$D_\mu \phi_i = \partial_\mu \phi_i - i [A_\mu, \phi_i], \quad (2.7)$$

$$D_\mu \psi = \partial_\mu \psi - i [A_\mu, \psi]. \quad (2.8)$$

Parameter g_{YM} is the coupling constant of the gauge theory.

All the fields $Y = \{A_\mu, \phi_i, \psi\}$ are $N \times N$ matrices

$$Y_{ij} = Y^a(x) T_{ij}^a, \quad (2.9)$$

in the adjoint representation of the unitary, special unitary, orthogonal or symplectic Lie algebra (see B for details). Since the fields are matrices, we

¹after change of variables $r \rightarrow R^2/r$

can introduce the standard 't Hooft expansion [4] (generalized to other gauge groups in [32]) and the effective coupling constant of the theory becomes the 't Hooft coupling

$$\lambda = g_{YM}^2 N, \quad (2.10)$$

where N is the size of the matrix.

By looking at these two descriptions of the same physical system we could roughly argue that type IIB superstring theory on $AdS_5 \times S^5$ background and the four dimensional $\mathcal{N} = 4$ should be identical and parameters of both theories satisfy²

$$g_{YM}^2 = g_s, \quad R^4 = 4\pi g_s N \alpha'^2 = 4\pi \lambda \alpha'^2. \quad (2.11)$$

Usually one can find these relations with the string length related to α' by

$$\alpha' = l_s^2. \quad (2.12)$$

Manifestation of holography in the conjecture can be seen in the principal formula of the correspondence[33, 34], which states that gauge invariant operators \mathcal{O} on the boundary of AdS_5 , serve as sources for supergravity fields ϕ (with boundary condition ϕ_0)

$$\langle e^{\int d^4x \phi_0(\vec{x}) \mathcal{O}(\vec{x})} \rangle_{CFT} = \mathcal{Z}_{string} [\phi(\vec{x}, z)|_{z=0} = \phi_0(\vec{x})]. \quad (2.13)$$

In other words, each field in Anti-de Sitter space can be mapped 1-1 to an operator in the field theory. The mass m of the field and the classical scaling dimension Δ of the gauge invariant operator are related by

$$\Delta = 2 + \sqrt{4 + R^2 m^2}. \quad (2.14)$$

This is the famous Maldacena's AdS/CFT conjecture.

2.2 Limits

Looking closer at the relation between parameters(2.11), we can see that when the coupling λ is small

$$g_{YM}^2 N = \lambda = \frac{R^4}{4\pi l_s^4} \ll 1, \quad (2.15)$$

²The first relation comes from DBI action and second, from identifying the ADM mass of the $AdS_5 \times S^5$ spacetime with the tension of N $D3$ -branes.

we can trust the perturbative expansion in the gauge theory, whereas on the gravity side we have the radius of AdS comparable with the string length l_s , hence difficult to deal with quantum gravity. On the other hand, when λ becomes large, and we lose control on the gauge theory, classical supergravity for large R (in l_s units) is a good description. Therefore, we see that these two disconnected regimes make AdS/CFT very difficult to prove, since it would require a non-perturbative solution of either the $\mathcal{N}=4$ SYM or the string theory in $AdS_5 \times S^5$ background with R-R flux. Fortunately, we can take certain limits which slightly “weaken” the conjecture but allow for explicit checks. They are usually two limits that one may take

1. The 't Hooft Limit

On the gauge side we fix the coupling $\lambda \equiv g_{YM}^2 N$, while $N \rightarrow \infty$. In the 't Hooft limit only planar diagrams contribute to the perturbative series. On the gravity side, since $g_s = \lambda/N$, we end up with non-interacting strings $g_s \rightarrow 0$ in curved space with constant radius R . This way we weaken the AdS/CFT to a duality between large N , planar $\mathcal{N}=4$ SYM and non-interacting strings on $AdS_5 \times S^5$.

2. The Large λ Limit

Taking the 't Hooft coupling $\lambda \rightarrow \infty$, physically, makes the string tension $T \sim \lambda$ very large, such that all the string massive modes become extremely heavy. They decouple from low energies. An effective theory is then approximated by type IIB supergravity on $AdS_5 \times S^5$.

In this thesis we will be interested in the first limit, the 't Hooft limit on the gauge theory side, and non-planar corrections (next to leading order in N) to it.

2.3 Conformal symmetry and observables

Both parts of the conjecture have the same global symmetry, the superconformal group $PSU(2, 2/4)$ (see [6]). It is then appropriate to classify observables using representations of this symmetry. In all our discussions we will specialize to the bosonic part, namely $SO(2, 4) \times SO(6)$. $SO(2, 4)$ is the conformal group in four dimensions and the isometry group of the AdS_5 spacetime. Similarly $SO(6)$ is the global symmetry of the five-sphere S_5 and the R-symmetry of the $\mathcal{N} = 4$ SYM.

A natural choice of labels for bosonic observables in any of the two systems are then the six Cartan generators of the symmetry algebra. Physically, they are the three angular momenta on the sphere (J_1, J_2, J_3) together with two spins (S_1, S_2) and the energy E corresponding to rotations and time translation symmetries of the AdS_5 . On the gauge theory side the only difference is that instead of the energy we have dilatations D .

Set of observables on the gauge theory side consists of gauge invariant operators built of the fields of $\mathcal{N} = 4$ SYM and their covariant derivatives. More precisely they are linear combinations of traces

$$O(x) = C_{i_1 \dots i_k i_{k+1} \dots i_n} \dots \text{Tr} (\mathcal{D} Y^{i_1} \dots Y^{i_k}) \text{Tr} (Y^{i_{k+1}} \dots Y^{i_n}) \dots \quad (2.16)$$

By analyzing the action of the superconformal algebra on the operators we can distinguish a special class of operators called chiral primary (or BPS). For example in the sector of scalar fields they have traceless symmetric tensors $C_{i\dots}$. Their conformal dimension does not receive any quantum corrections (they are protected) hence it is equal to the classical dimension Δ_0 . In order to construct non-BPS operators we insert different fields into traceless symmetric combinations. They lead to "anomalous" corrections to the classical dimension.

On the string theory side we consider strings propagating in $AdS_5 \times S^5$. Unfortunately up to date it has been impossible to quantize superstrings on this background³, so one usually takes the limit where strings are classical. Then the task is to find classical string solutions to the equations of motion and compute their energies. To be more precise, string's energy in global AdS coordinates is the eigenvalue of the zero components of the momentum and special conformal transformation operators

$$\frac{1}{2} (P_0 + K_0), \quad (2.17)$$

and, according to the conjecture, it should be "dual" to i times the eigenvalues of the dilatation operator D of the corresponding operator.

In the two cases angular momenta and spins of the operators/solutions are just integer constants but the energy and dilatations depend non-trivially

³It is possible thought if one takes PP-wave limit of the geometry[5]. This limit is called the BMN limit.

on parameters of both theories (e.g. receive quantum corrections). Therefore, verifying that energies are indeed equal to dilatations is in general a non-trivial task. This problem is called the "Spectral Problem of AdS/CFT".

2.4 Anomalous dimension and Dilatation generator

In any conformal field theory two point functions of gauge invariant operators are completely fixed by the symmetry. They have the form

$$\langle O(x)\bar{O}(y) \rangle \sim \frac{1}{|x-y|^{2\Delta}}. \quad (2.18)$$

If now the classical conformal dimension of the field (determined from the Lagrangian) receives a small "quantum correction"

$$\Delta = \Delta_0 + \gamma, \quad (2.19)$$

the correlator becomes

$$\langle O(x)\bar{O}(y) \rangle \sim \frac{1}{|x-y|^{2\Delta_0}} (1 - \gamma \ln \Lambda^2 |x-y|^2), \quad (2.20)$$

where Λ is a cutoff scale.

If, on the other hand, we analyze the two point correlator from perturbation theory (see 4), we can see that γ can be effectively obtained as an eigenvalue of a local operator acting on the trace. For example, at one loop, in the scalar sector we have

$$\check{D}^1 O(x) = \gamma^1 O(x), \quad (2.21)$$

where

$$D^1 = \frac{\lambda}{8\pi^2} \sum_{l=1}^L \left(1 - P_{l,l+1} + \frac{1}{2} K_{l,l+1} \right), \quad (2.22)$$

where P exchange flavor indices on sites l and $l+1$ and K contracts the indices. Action of the generator will often mix different operators O_i

$$D O^i = \sum_j c_{ij} O_j, \quad (2.23)$$

and in order to find the set of anomalous dimensions γ_i we have to solve the mixing problem. This will be explained in later chapters together with the solution given by Bethe Ansatz.

In general the dilatation generator has a well defined expansion in the 't Hooft coupling λ and the number of colors N . Formally we can write

$$\check{D} = \sum_{l=1} \sum_{k=0} D^{l,k}(\lambda^l, \frac{1}{N^k}), \quad (2.24)$$

where at every order in λ diagrams can be grouped into leading in N , usually called planar due to the fact that they can be drawn on the plane, and subleading in N , with the topology of the higher genus Riemann surfaces. In the limit of $N \rightarrow \infty$ planar diagrams will dominate the answer and equivalently we will talk about the planar dilatation generator. Especially in this thesis we will try to analyze non-planar contributions too. Then we will treat them as small perturbations to the planar generator and the mixing problem will be solved by means of the usual quantum mechanical perturbation theory (see C).

Chapter 3

Basic techniques from integrability

This chapter is a summary on spin chains and some integrability technology. On the example of the Heisenberg’s XXX hamiltonian we review the problem of diagonalization by Bethe Ansatz and discuss details of the asymptotic Bethe equations. Then we demonstrate how to construct the tower of conserved charges using boost operator and define the concept of a spin chain parity. Finally we provide a dictionary between the operators in AdS/CFT and spin-chains. This chapter is meant to be pedagogical and explicit, so that later, the reader can easily follow derivations strongly based on these techniques. Readers that want to know more about integrability than just the minimum presented here are referred to [7],[8] and [74].

Classical and Quantum Integrability

Before starting, we have to specify “what we talk about when we talk about integrability”. Roughly speaking, a classical system with N -degrees of freedom, described by a hamiltonian H , is integrable (Liouville integrable), if there exist N conserved charges Q_i with zero Poisson bracket (they are in involution)

$$\{Q_i, Q_j\} = 0, \tag{3.1}$$

and H is one of the charges. Each of Q ’s yields a conservation law that can be solved (integrated) to fix all the independent degrees of freedom.

A quantum theory defined by hamiltonian-operator H , will be called inte-

grable if there exist N conserved charges-operators Q_i that commute

$$[Q_i, Q_j] = 0, \quad (3.2)$$

and the hamiltonian is one of them. Therefore all the charges can be diagonalized simultaneously and the set of complete eigenstates with corresponding eigenvalues can be determined. This way we completely fix the N degrees of freedom of the quantum system.

Equivalently, by quantum integrable model we will mean a model that can be "solved" by some sort of Bethe ansatz (described below).

3.1 Coordinate Bethe Ansatz

The simplest example of a discrete quantum integrable model is a system of spins on a periodic chain with the nearest-neighbor interactions given by the famous Heisenberg's $XXX_{1/2}$ hamiltonian. It is the seminal example for application of the Bethe ansatz so we analyze it in great detail. As we will learn later this hamiltonian plays a crucial role in the problem of finding planar and non-planar anomalous dimensions of the composite operators in AdS/CFT . Let us then proceed slowly with defining the setup and considering some explicit examples.

3.1.1 Heisenberg's hamiltonian

Consider a collection of ordered points (sites) on a circle that are labelled by an integer n from 1 to L such that $n = n + L$ (Fig. 3.1). L plays the role of the volume of the space and is usually referred to as a fundamental domain. With each site we associate a vector space $V = \mathbb{C}^2$ with a basis that consists

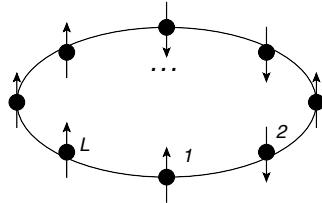


Figure 3.1: Periodic spin-chain with L -sites

of a spin up $|\uparrow\rangle$ and a spin down $|\downarrow\rangle$ states represented as two-component vectors

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (3.3)$$

Therefore an arbitrary state of the chain with length L , is an object in the 2^L dimensional Hilbert space that is obtained by tensoring L V

$$\mathcal{H} = \prod_{n=1}^L \otimes V_n = V_1 \otimes \dots \otimes V_L. \quad (3.4)$$

To describe spin interactions recall first that the algebra of spins is governed by Pauli matrices

$$\sigma_n^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_n^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_n^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.5)$$

that satisfy the $su(2)$ lie commutation relations

$$[\sigma_n^i, \sigma_m^j] = 2i\epsilon^{ijk}\sigma_n^k\delta_{mn}. \quad (3.6)$$

The creation and annihilation operators at site n are¹

$$\sigma_n^\pm = \frac{1}{2}(\sigma_n^1 \pm i\sigma_n^2), \quad (3.7)$$

and they rise or lower the spin at the site respectively

$$\sigma^+ |\downarrow\rangle = |\uparrow\rangle, \quad \sigma^- |\uparrow\rangle = |\downarrow\rangle. \quad (3.8)$$

We choose to call the spin up an excitation.

The simplest dynamical situation on a spin-chain is when only the nearest neighbors interact with each other. As it was first demonstrated by Heisenberg, the hamiltonian of such a system is given by

$$\begin{aligned} H_0 &= \sum_{n=1}^L \left(\mathbf{1} - \hat{P}_{n,n+1} \right) = \frac{1}{2} \sum_{n=1}^L \mathbf{1} - \bar{\sigma}_n \otimes \bar{\sigma}_{n+1} = \\ &= \frac{1}{2} \sum_{n=1}^L \mathbf{1} - (\sigma_n^1 \otimes \sigma_{n+1}^1 + \sigma_n^2 \otimes \sigma_{n+1}^2 + \sigma_n^3 \otimes \sigma_{n+1}^3). \end{aligned} \quad (3.9)$$

¹They are given explicitly by

$$\sigma_n^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_n^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

$P_{n,n+1}$ is the permutation operator acting on sites n and $n+1$ and $\bar{\sigma}_n \otimes \bar{\sigma}_{n+1}$ is the tensor product of the Pauli matrices at n and $n+1$.

In order to get accustomed to the notation let us derive (3.9). For simplicity take the length of the chain $L = 2$. First, remind ourselves the tensor product of two arbitrary square matrices

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \otimes \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{pmatrix}. \quad (3.10)$$

As a particular example, the product of two-component column vectors is simply given by

$$x \otimes y = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1y_1 \\ x_1y_2 \\ x_2y_1 \\ x_2y_2 \end{pmatrix}. \quad (3.11)$$

The only non-trivial object that we have in the first representation of H_0 is the permutation operator $P_{i,j}$. By definition, it interchanges states between the sites that it acts on

$$\hat{P}(x \otimes y) = y \otimes x. \quad (3.12)$$

Writing tensor products in components, the above equation is equivalent to

$$\hat{P} \begin{pmatrix} x_1y_1 \\ x_1y_2 \\ x_2y_1 \\ x_2y_2 \end{pmatrix} = \begin{pmatrix} y_1x_1 \\ y_1x_2 \\ y_2x_1 \\ y_2x_2 \end{pmatrix}, \quad (3.13)$$

so we can easily write down the matrix of the permutation operator as a 4×4 matrix

$$\hat{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (3.14)$$

Therefore the matrix representation of the $(1 - P)$ part of the Hamiltonian in a basis of length $L = 2$ is

$$(1 - P) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.15)$$

Generalization to chains with more sites is straightforward. Permutation operator on sites i and j of the state

$$x_1 \otimes \dots \otimes x_i \otimes \dots \otimes x_j \otimes \dots \otimes x_L, \quad (3.16)$$

will be the $2^L \times 2^L$ matrix that permutes only states i and j and acts as the identity on the remaining sites.

For the second representation of the hamiltonian let us evaluate the tensor products of Pauli matrices (3.5). They are

$$\begin{aligned} \sigma^1 \otimes \sigma^1 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\ \sigma^2 \otimes \sigma^2 &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \\ \sigma^3 \otimes \sigma^3 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \end{aligned} \quad (3.17)$$

so the scalar product in (3.9) in the length two basis is

$$\bar{\sigma} \otimes \bar{\sigma} = \sigma^1 \otimes \sigma^1 + \sigma^2 \otimes \sigma^2 + \sigma^3 \otimes \sigma^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (3.18)$$

It is then easy to see that the two representations of H_0 are equivalent

$$1 - P = \frac{1}{2} (1 - \bar{\sigma} \otimes \bar{\sigma}). \quad (3.19)$$

Let us now move to the eigenvalue problem of the Heisenberg's hamiltonian.

3.1.2 Bethe Ansatz

Once we understand the structure of the hamiltonian the problem that we want to solve is its diagonalization. By this we mean, for a given fundamental domain L and the number of excitations M^2 (spins up or down), determine the set of eigenvalues E_M^L and eigenstates $|\psi_M^L\rangle$ of (3.9). This is expressed by the quantum mechanical eigenequation

$$H_0 |\psi_M^L\rangle = E_M^L |\psi_M^L\rangle. \quad (3.20)$$

The solution to this problem was first given by Hans Bethe in 1931[9] and we present it below.

The most general eigenstate of H_0 of length L with M -excitations can be written as the following superposition

$$|\psi_M^L\rangle = \sum_{1 \leq n_1 \leq \dots \leq n_M}^L \psi(n_1, \dots, n_M) |n_1, \dots, n_M\rangle, \quad (3.21)$$

where ψ is a periodic wave function

$$\psi(n_2, \dots, n_M, n_1 + L) = \psi(n_1, n_2, \dots, n_M). \quad (3.22)$$

Integers n_i from 1 to M denote the position of the excitation (e.g. $|5\rangle$ represents a magnon at site 5 and $|1, 3\rangle$, one magnon at site 1 and the other at 3 etc.), so we sum over all possible insertions of the excitations that respect positions of the other magnons on the chain. The celebrated wave function was proposed by Bethe and it is given by

$$\psi(n_1, \dots, n_M) = \sum_{\pi \in S_M} A_\pi \exp \left(i \sum_{j=1}^M p_{\pi(j)} n_j \right). \quad (3.23)$$

The sum is over all $M!$ permutations π from S_M that correspond to the distributions of M magnons on the chain. p 's are so called "pseudo-momenta" of the excitations and A 's are "constants" that can only depend on the pseudo-momenta but not on the position of the insertion. p_i 's and A_π 's are to be determined from the eigenvalue equation (C.4) and the periodicity condition (3.22). To see how it works in practice we will now analyze examples with zero, one and two magnons on the periodic chain of an arbitrary length L .

²Excitations are usually called magnons or impurities in the condensed matter literature

3.1.3 M=0

It is easy to see that the state with all spins down (or all spins up)

$$|0\rangle = |\downarrow \dots \downarrow\rangle, \quad (3.24)$$

is the eigenstate with zero eigenvalue, or the lowest energy ($E_M^L = 0$). It is usually chosen as the ferromagnetic vacuum state of the Heisenberg's hamiltonian³.

3.1.4 M=1

Eigenstate with a single magnon is simply

$$|\psi_1^L\rangle = A \sum_{n_1=1}^L e^{ip_1 n_1} |n_1\rangle. \quad (3.25)$$

We will now use the eigenvalue equation and the periodicity condition to find the energy and the set of eigenstates. In other words we have to find the set of possible momenta p_1 in the Bethe ansatz (in this example the overall constant A is irrelevant).

When we act on a state with one magnon inserted at $|n_1\rangle$, most of the contributions to the $1 - P$ hamiltonian are zero. Namely, the permutation operator is equivalent to the identity when acts on sites with the same value of the spin. There are only two contributions that are non trivially subtracted from the identity

$$\hat{P}_{n_1-1, n_1} |n_1\rangle = |n_1 - 1\rangle, \quad \hat{P}_{n_1, n_1+1} |n_1\rangle = |n_1 + 1\rangle. \quad (3.26)$$

This way the eigenvalue equation becomes⁴

$$H_0 |\psi_1^L\rangle = L |\psi_1^L\rangle - (L - 2) |\psi_1^L\rangle - \sum_{n_1=1}^L e^{ip_1 n_1} |n_1 - 1\rangle - \sum_{n_1=1}^L e^{ip_1 n_1} |n_1 + 1\rangle. \quad (3.27)$$

³Often in the condensed matter literature one can find a anti-ferromagnetic vacuum with half spins up and half spins down alternating. Such a state is sometimes also called the “Neel state”.

⁴in this simple case A drops out

The first term is just the identity part of the hamiltonian, the second is the sum of $L - 2$ terms when P acts on sites with the same spin and finally the last two contributions are from the action of P in the nearest neighborhood of the excitation. Since in both of the last terms we sum over the entire fundamental domain, we can shift n_1 to $n_1 + 1$ and $n_1 - 1$ respectively. This yields the eigenequation

$$H_0 |\psi_1^L\rangle = (2 - e^{ip_1} - e^{-ip_1}) |\psi_1^L\rangle = E_1^L |\psi_1^L\rangle, \quad (3.28)$$

with the energy of the single excitation on the chain of length L equal to

$$E_1^L = 2 - e^{ip_1} - e^{-ip_1} = 2(1 - \cos p) = 4 \sin^2 \frac{p_1}{2}. \quad (3.29)$$

Notice that we don't have any explicit L dependence, yet. This naturally comes from the periodicity condition (3.22). For a single magnon it reads

$$e^{ip_1(n_1+L)} = e^{ip_1 n_1}. \quad (3.30)$$

This implies the set of $\frac{L!}{(L-1)!1!} = L$ allowed momenta, quantized in the units of the fundamental domain L

$$p_1 = \frac{2\pi k}{L}, \quad k = 0, \dots, L-1. \quad (3.31)$$

This is a general feature of the Bethe ansatz that the periodicity condition is a sort of quantization condition for the pseudo-momenta.

Since neither of the two relevant equations required the presence of A , for convenience we can set it to one. For more magnons it will not be the case and these constants will be very important.

Summarizing, given a periodic chain of length L with only one excitation the Bethe Ansatz solution is

$$\begin{aligned} E_1^L &= 4 \sin^2 \frac{\pi k}{L}, \\ |\psi_1^L\rangle &= \sum_{n_1=1}^L e^{i \frac{2\pi k}{L} n_1} |n_1\rangle, \\ k &= 0, \dots, L-1. \end{aligned} \quad (3.32)$$

The $M = 1$ example is very simple, nevertheless in integrable models, the energy of the chain with more magnons is just a sum of the one magnon contributions (of course with the appropriate pseudo-momenta). In *AdS/CFT* $M = 1$ example is trivial due to the additional zero momentum condition that comes from cyclicity of the chain (trace operator).

3.1.5 M=2

The example with two excitations is already more non-trivial. In the *AdS/CFT* correspondence this example can be directly applied to the problem of diagonalization of the planar (one-loop) dilatation operator in the basis of so called BMN states. Let us then write the Bethe ansatz for two impurities and analyze the corresponding eigenequation.

At $M = 2$ we have to deal with a scattering matrix which describes the situation when two magnons cross each other. Therefore the Bethe wave function for two magnons consists of two terms⁵

$$\psi(n_1, n_2) = e^{ip_1 n_1 + ip_2 n_2} + S(p_2, p_1) e^{ip_2 n_1 + ip_1 n_2}, \quad (3.33)$$

where $S(p_2, p_1)$ is the "scattering matrix" (or simply S-matrix) that captures the effect of interchanging of the two magnons. Notice that the term with the S-matrix in front has momenta p_1 and p_2 swapped in the exponent.

When solving the eigenequation, we have to distinguish two cases: the first, when magnons are "well separated" ($n_2 \geq n_1 + 1$), and second when they are on the neighboring sites ($n_2 = n_1 + 1$). The two corresponding equations that we get are

- For $n_2 \geq n_1 + 1$, after acting with H_0 and shifting n_1 and n_2 appropriately we get

$$\begin{aligned} E_2^L \psi(n_1, n_2) = & 4\psi(n_1, n_2) - \psi(n_1 - 1, n_2) - \psi(n_1 + 1, n_2) \\ & - \psi(n_1, n_2 - 1) - \psi(n_1, n_2 + 1). \end{aligned} \quad (3.34)$$

- For the two magnons next to each other $n_2 = n_1 + 1$ we get

$$E_2^L \psi(n_1, n_1 + 1) = 2\psi(n_1, n_1 + 1) - \psi(n_1 - 1, n_1 + 1) - \psi(n_1, n_1 + 2). \quad (3.35)$$

From the well-separated case we can determine the energy. Namely, inserting the ansatz for the wave function (3.33) yields

$$\left(E_2^L - 4 \sin^2 \frac{p_1}{2} - 4 \sin^2 \frac{p_2}{2} \right) \psi(n_1, n_2) = 0. \quad (3.36)$$

⁵according to (3.23) we should have two constants in the wave function. We can just normalize by the first one and keep the second. As we will see it will be equal to the S-matrix.

The energy of the two-magnon system is just the sum of the individual magnon contributions

$$E_2^L(p_1, p_2) = 4 \sin^2 \frac{p_1}{2} + 4 \sin^2 \frac{p_2}{2} = E_1^L(p_1) + E_1^L(p_2). \quad (3.37)$$

This additivity of the energy is one of the consequences of the fact that the hamiltonian of the system is integrable.

The situation when two magnons are on adjacent sites allows us to fix the S-matrix. Inserting ansatz (3.33) and energy (3.37) into condition (3.35) can be solved to

$$S(p_2, p_1) = -\frac{e^{i(p_1+p_2)} - 2e^{ip_2} + 1}{e^{i(p_1+p_2)} - 2e^{ip_1} + 1}. \quad (3.38)$$

An important property of this S-matrix is that reflection of its arguments is equivalent to the inversion

$$S(p_1, p_2)S(p_2, p_1) = 1. \quad (3.39)$$

We will use this condition frequently.

Periodicity implies that for $M = 2$ the wave function satisfies

$$\psi(n_2, n_1 + L) = \psi(n_1, n_2). \quad (3.40)$$

Inserting Bethe ansatz (3.33) gives

$$e^{ip_1 n_2 + ip_2 n_1} e^{ip_2 L} + S(p_2, p_1) e^{ip_1 L} e^{ip_2 n_2 + ip_1 n_1} = e^{ip_1 n_1 + ip_2 n_2} + S(p_2, p_1) e^{ip_2 n_1 + ip_1 n_2}. \quad (3.41)$$

Now comparing the coefficients in front of the appropriate exponents leaves us with two equations

$$e^{ip_1 L} = \frac{1}{S(p_2, p_1)} = S(p_1, p_2), \quad e^{ip_2 L} = S(p_2, p_1), \quad (3.42)$$

where in the second equality we used (3.39). They are known as the Bethe equations. One can see them as quantization conditions for the pseudomomentum of each of the magnons in the presence of the other.

Solving Bethe equations is in general a difficult problem, however some simplifications appear when we assume more conditions on the pseudomomenta. For example, if they are real the S-matrix can be represented as a pure phase

$$S(p_2, p_1) = e^{i\theta(p_2, p_1)}. \quad (3.43)$$

Then taking the logarithm of (3.42) yields

$$p_1 = \frac{2\pi m_1}{L} + \frac{\theta(p_1, p_2)}{L}, \quad p_2 = \frac{2\pi m_2}{L} + \frac{\theta(p_2, p_1)}{L}, \quad (3.44)$$

and the problem boils down to finding of all the possible pairs (m_1, m_2) (Bethe quantum numbers) that satisfy (3.42). We do not discuss the properties of the solutions here but an interested reader is referred to an extensive review with many details and further references therein [17].

3.1.6 General M

The above procedure can be repeated for an arbitrary number of excitations (magnons). A very non-trivial fact, and a consequence of integrability, is that the energy is always the sum of the energies of the single excitations

$$E_M^L = \sum_{k=1}^M 4 \sin^2 \frac{p_k}{2}. \quad (3.45)$$

In addition the the S-matrix in the multi magnon wave function is a product of the two body scattering matrices. This way the system of M Bethe equations can be written as

$$e^{ip_k L} = \prod_{j=1, j \neq k}^M S(p_k, p_j). \quad (3.46)$$

This particular property of the scattering on a spin-chain is called "factorized scattering" and it can be regarded as a definition of integrability of the hamiltonian that governs the system.

Before we move to the discussion about asymptotic Bethe ansatz let us give an equivalent, but a very useful, representation of the Bethe equations.

3.1.7 Rapidity variables

Bethe equations give a set of quantization conditions for the momenta of the excitations on the chain. In practice, for more than two excitations, the equations are quite complicated to solve analytically and one has to use numerical analysis. Nevertheless, there is a very useful set of variables that puts Bethe equations into more elegant and usually easier-to-work-with form. These

variables are called rapidities u and are defined for every pseudomomentum p_k as

$$u_k = \frac{1}{2} \cot \frac{p_k}{2}. \quad (3.47)$$

The first advantage of this parametrization can be already seen on the level of the S-matrix. When expressed on the "rapidity plane", the two magnon scattering matrix has the following form

$$S(u_1, u_2) = \frac{u_1 - u_2 + i}{u_1 - u_2 - i}. \quad (3.48)$$

Similarly, the exponent of p_k (that appears in Bethe equations) is given by

$$e^{ip_k} = \frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}, \quad (3.49)$$

so finally for the spin-chain of length L with M , (3.46) becomes

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} \right)^L = \prod_{j \neq k}^M \frac{u_k - u_j + i}{u_k - u_j - i}. \quad (3.50)$$

One can also check that the energy of a single magnon is

$$E(u_k) = \frac{i}{u_k + \frac{i}{2}} - \frac{i}{u_k - \frac{i}{2}} = \frac{1}{u_k^2 + \frac{1}{4}}, \quad (3.51)$$

hence, the total energy of M magnons is simply

$$E = \sum_k^M \frac{1}{u_k^2 + \frac{1}{4}}. \quad (3.52)$$

For AdS/CFT chains we have one additional constraint. It comes from the fact that single trace operators are by definition symmetric under cyclic permutations which means in particular that the translation operator

$$e^P \equiv \exp \left(\sum_{k=1}^M p_k \right), \quad (3.53)$$

where p_k are magnons momenta, should leave the trace invariant. Therefore we require

$$e^P = 1, \quad (3.54)$$

which implies that the total momentum should be zero

$$P = \sum_{k=1}^M p_k = 0. \quad (3.55)$$

In terms of rapidities it reads

$$e^{iP} = \prod_{k=1}^M \frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} = 1. \quad (3.56)$$

In the literature the set of rapidities is usually referred to as Bethe roots. We will use these names interchangeably.

3.1.8 M=2 on the rapidity plane

It is an instructive exercise to solve Bethe equations on the rapidity plane (for $M = 2$). If we assume that the total momentum is conserved, from (3.56) we have that

$$\frac{u_1 + i/2}{u_1 - i/2} = \frac{u_2 - i/2}{u_2 + i/2}. \quad (3.57)$$

This equation is solved by real roots $u_2 = -u_1 = u$. Inserting them back into Bethe equations (3.50), and dividing by the right hand side we have

$$\left(\frac{u + i/2}{u - i/2} \right)^{J+1} = 1, \quad (3.58)$$

where $J = L - 2$ is the number of sites with the vacuum spin value. The solution can be found in a straightforward way and it is given by the cotangent function

$$u = \frac{1}{2} \cot \frac{\pi n}{J+1}, \quad n = 0, \dots, J+1. \quad (3.59)$$

From the definition of the roots we can read the associated pseudomomenta

$$p = \frac{2\pi n}{J+1}, \quad n = 0, \dots, J+1. \quad (3.60)$$

Then magnon's energy (3.52) is the sum of two identical contributions

$$E = \sum_{k=2}^2 \frac{1}{u_k^2 + \frac{1}{4}} = \frac{2}{u^2 + \frac{1}{4}} = 4 \sin^2 \frac{2\pi n}{J+1}. \quad (3.61)$$

Finally we can notice that $S(u, -u) = 1$ so Bethe states (3.33) take a very elegant form

$$\begin{aligned}
|\psi_k\rangle &= \sum_{1 \leq n_1 \leq n_2 \leq L} \left(e^{i \frac{2\pi k}{J+1} (n_2 - n_1)} + e^{-i \frac{2\pi k}{J+1} (n_2 - n_1)} \right) |n_1, n_2\rangle \\
&= \sum_{n_1=1}^{J+1} \sum_{p=0}^{J+1-n_1} \left(e^{i \frac{2\pi k}{J+1} (p+1)} + e^{-i \frac{2\pi k}{J+1} (p+1)} \right) |n_1, n_1 + p + 1\rangle \\
&= \sum_{n_1=1}^{J+1} \sum_{p=0}^{J+1-n_1} 2 \cos \frac{2\pi k (p+1)}{J+1} |n_1, n_1 + p + 1\rangle. \tag{3.62}
\end{aligned}$$

We will use these states for further analysis of the flip operator in orthogonal and symplectic AdS/CFT dualities.

3.1.9 Asymptotic Bethe Ansatz

Very often we have to deal with a situation when hamiltonian H_0 of the system is perturbed by one or more operators⁶

$$H = H_0 + gH_1 + g^2H_2 + \dots \tag{3.63}$$

If these corrections "respect" the integrable structure, one can still use a modification of the Bethe ansatz, the Asymptotic Bethe Ansatz⁷ (ABA) [10], to new energies and states that diagonalize these perturbations. We describe this procedure for a single correction added to H_0 . Further terms, similarly to the perturbation theory in the ordinary quantum mechanics, can be computed iteratively using the standard algorithm (see App.C for a short brush up on quantum mechanical perturbation theory).

The idea is that the presence of the perturbation will cause a shift to the Bethe roots

$$u_k = u_k^0 + g u_k^1. \tag{3.64}$$

⁶In principle they should be "smaller" corrections to the leading, dominant behavior.

⁷Following the condensed matter literature we used here the term Asymptotic Bethe Ansatz but probably more appropriate name would be the "Perturbative Bethe Ansatz". In AdS/CFT integrability literature term Asymptotic Bethe Ansatz is usually reserved for $psu(2, 2/4)$ Bethe equations for the spin-chain of $\mathcal{N} = 4$ SYM with infinite length [107]. From our construction it should be clear that we are only concerned with diagonalizing a hamiltonian defined by perturbative series in some expansion parameter.

Then we can modify Bethe equations (3.50) in an appropriate way, expand in g and require that they are satisfied order by order in powers of g .

There are few ways that one can modify Bethe ansatz. We can for example write the "constants" in (3.23) as a series in g , add a phase factor to the S-Matrix [11] or modify Bethe equations with some function of the roots that is also a series in g . Here we concentrate on this last method.

Let us introduce the modifying function $x(u_k)$

$$x(u_k) = u_k (1 - g f(u_k)), \quad (3.65)$$

with roots u_k given by (3.64), and postulate that Bethe equation is satisfied

$$\left(\frac{x(u_k + \frac{i}{2})}{x(u_k - \frac{i}{2})} \right)^L = \prod_{j \neq k}^M \frac{u_k - u_j + i}{u_k - u_j - i}. \quad (3.66)$$

If we insert (3.64) and (3.65), at order g^0 we get the standard Bethe equations

$$\left(\frac{u_k^0 + \frac{i}{2}}{u_k^0 - \frac{i}{2}} \right)^L = \prod_{j \neq k}^M \frac{u_k^0 - u_j^0 + i}{u_k^0 - u_j^0 - i}, \quad (3.67)$$

which are satisfied by definition of u_k^0 .

At order g^1 we can divide each side by the zeroth order contribution and get the first consistency condition for functions f

$$L \left(f(u_k^0 + \frac{i}{2}) - f(u_k^0 - \frac{i}{2}) + \frac{i u_k^1}{(u_k^0)^2 + \frac{1}{4}} \right) = 2i \sum_{j \neq k}^M \frac{u_k^1 - u_j^1}{(u_k^0 - u_j^0)^2 + 1}, \quad (3.68)$$

or more transparently

$$f(u_k^0 + \frac{i}{2}) - f(u_k^0 - \frac{i}{2}) = -\frac{i u_k^1}{(u_k^0)^2 + \frac{1}{4}} + \frac{2i}{L} \sum_{j \neq k}^M \frac{u_k^1 - u_j^1}{(u_k^0 - u_j^0)^2 + 1}. \quad (3.69)$$

In addition we have the momentum conservation condition (3.56). At g^0 it is simply expressed in terms of the leading roots u_k^0

$$\prod_{k=1}^M \frac{u_k^0 + \frac{i}{2}}{u_k^0 - \frac{i}{2}} = 1. \quad (3.70)$$

At first order in g we get another condition for f 's

$$\sum_{k=1}^M \left(f(u_k^0 + \frac{i}{2}) - f(u_k^0 - \frac{i}{2}) \right) = \sum_{k=1}^M \frac{-iu_k^1}{(u_k^0)^2 + \frac{1}{4}}. \quad (3.71)$$

We will also need the formula for the new energy from the ABA. In general if we add a correction to the energy that also depends on the modified Bethe roots $E_k^1(u_k)$, then the expansion in g is

$$\begin{aligned} E_k &= E_k^0(u_k^0 + g u_k^1) + g E_k^1(u_k^0 + g u_k^1) \\ &= E_k^0(u_k^0) + g (E_k^1(u_k^0) + u_k^1 (E_k^0(u_k^0))') + O(g^2). \end{aligned} \quad (3.72)$$

This could also be incorporated into the ABA. Namely, by following the procedure of modification of the Bethe equations and substituting $x(u_k^\pm)$ into formula (3.51)

$$E_k = \frac{i}{x(u_k^0 + g u_k^1 + \frac{i}{2})} - \frac{i}{x(u_k^0 + g u_k^1 - \frac{i}{2})}. \quad (3.73)$$

Then at the first order we get standard formula in terms of u_k^0 whereas at $O(g)$ the correction is

$$\frac{\delta E_k}{E_k^0(u_k^0)} = i(u_k^0 - \frac{i}{2})f(u_k^0 + \frac{i}{2}) - i(u_k^0 + \frac{i}{2})f(u_k^0 - \frac{i}{2}) - \frac{2u_k^0 u_k^1}{(u_k^0)^2 + \frac{1}{4}}. \quad (3.74)$$

3.1.10 ABA for M=2

Here we examine in details how ABA works for two magnons. First, we do not assume any specific form of the perturbation hamiltonian and give a general result for $M = 2$. Then, to make the reader even more confident with the techniques, we analyze a known example of correction from the AdS/CFT duality.

For $M = 2$, at the leading order everything stays the same as it should, so we have $u_2^0 = -u_1^0 = u$. However at $O(g)$ both, the momentum conservation and Bethe equations give a set of three equations that are at the same time consistency conditions for f and equations for u^1 and u^2 . The momentum conservation gives

$$f(u + \frac{i}{2}) - f(u - \frac{i}{2}) + f(-(u - \frac{i}{2})) - f(-(u + \frac{i}{2})) = -\frac{i(u_2^1 + u_1^1)}{u^2 + \frac{1}{4}}, \quad (3.75)$$

whereas from the Bethe equations we get

$$\begin{aligned} f(-(u - \frac{i}{2})) - f(-(u + \frac{i}{2})) &= \frac{i}{2L} \frac{(1 - 2L)u_1^1 - u_2^1}{u^2 + \frac{1}{4}}, \\ f(u + \frac{i}{2}) - f(u - \frac{i}{2}) &= \frac{i}{2L} \frac{(1 - 2L)u_2^1 - u_1^1}{u^2 + \frac{1}{4}}. \end{aligned} \quad (3.76)$$

Notice that in principle we have several ways to fulfil the conservation of momentum. However the most "natural" is to assume $u_2 = -u_1$ and function f to be either even or odd.

Now consider an explicit form of the modification given by Zhukovsky map

$$x(u) = \frac{u}{2} + \frac{u}{2} \sqrt{1 - \frac{2g}{u^2}} = u \left(1 - g \frac{1}{2u^2} - g^2 \frac{1}{4u^4} + O(g^3) \right). \quad (3.77)$$

We are only interested in the first order correction so the relevant function for us is

$$f(u) = \frac{1}{2u^2}. \quad (3.78)$$

An important fact is that it satisfies

$$f(u + \frac{i}{2}) - f(u - \frac{i}{2}) = -\frac{iu}{(u^2 + \frac{1}{4})^2}. \quad (3.79)$$

Notice also that this function is even so the conservation of the momenta implies that $u_2^1 = -u_1^1 = u^1$. Then the two Bethe equations reduce to

$$f(u + \frac{i}{2}) - f(u - \frac{i}{2}) = \frac{i(1 - L)}{L} \frac{u^1}{u^2 + \frac{1}{4}}, \quad (3.80)$$

and can be solved by

$$u^1 = \frac{J+2}{J+1} \frac{u}{u^2 + \frac{1}{4}}. \quad (3.81)$$

Then the correction to the energy (3.74) is

$$\delta E = -16 \sin^4 \left(\frac{n\pi}{j+1} \right) - 64 \frac{1}{J+1} \cos^2 \left(\frac{n\pi}{j+1} \right) \sin^4 \left(\frac{n\pi}{j+1} \right). \quad (3.82)$$

We will perform a similar analysis for the flip part of the $SO(N)$ and $Sp(N)$ dilatation operators in later sections.

3.2 Boost Operator and Conserved Charges

The existence of a complete set of commuting integrals of motion is the key feature of the integrability of the Heisenberg's hamiltonian. In [14] Luscher showed that charges (like the hamiltonian) are local operators that act on the chain and can be put into the form

$$Q_n = \sum_{i_1, \dots, i_{n-1}} G_{n-1}^T(i_1, \dots, i_{n-1}), \quad (3.83)$$

where $\{i_1, \dots, i_{n-1}\}$ is an ordered subset of the chain's sites, and G^T is a translationally covariant and symmetric function that obeys the property of locality

$$G_n^T(i_1, \dots, i_n) = 0, \quad |i_n - i_1| \geq n. \quad (3.84)$$

Formally, charges are related to the logarithmic derivatives with respect to spectral parameter λ of the so-called transfer matrix $T(\lambda)$ ⁸

$$Q_n = 2i \frac{d^{n-1}}{d\lambda^{n-1}} \ln T^{-1}(\lambda_0) T(\lambda)|_{\lambda=\lambda_0}, \quad (3.85)$$

where $\lambda_0 = i/2$. Nevertheless, it is very difficult to extract their explicit form this way, mostly because the size of the transfer matrix grows exponentially with the length of the chain. Fortunately there exists an alternative method of constructing the charges with a boost operator. For more extensive review and further references see [13]. The boost operator is defined as the first moment of the hamiltonian

$$\hat{B} = \frac{1}{2i} \sum_{j=1}^L j \sigma_j \cdot \sigma_{j+1}. \quad (3.86)$$

Its commutator with the transfer matrix is equal to the derivative

$$[\hat{B}, T(\lambda)] = \frac{\partial}{\partial \lambda} T(\lambda), \quad (3.87)$$

therefore, up to some constant terms, boost operator generates conserved charges recursively

$$[\hat{B}, \hat{Q}_n] = \hat{Q}_{n+1}. \quad (3.88)$$

⁸For more details on formal aspects of Bethe Ansatz see [12].

We are not going to prove these formula but instead we construct the first two conserved charges of the Heisenberg's hamiltonian⁹.
The second conserved charge Q_2 is usually related to the hamiltonian as

$$Q_2 = aH_0 + c, \quad (3.89)$$

hence for construction of the higher charges we will only need the part of H_0 with Pauli matrices. Namely, take

$$Q_2 = \sum_i \sigma_i^a \sigma_{i+1}^a, \quad (3.90)$$

where we sum over $a = 1, 2, 3$.

Let us then construct Q_3 using boost operator. We will need three useful identities for commutators

$$[A, BC] = [A, B]C + B[A, C], \quad [AB, C] = A[B, C] + [A, C]B, \quad (3.91)$$

and

$$[AB, CD] = A[B, C]D + [A, C]B D + C A[B, D] + C[A, D]B. \quad (3.92)$$

By definition (3.88) , Q_3 is just the commutator of the boost operator with Q_2

$$Q_3 = [\hat{B}, Q_2] = \frac{1}{2i} \sum_{i,j} j [\sigma_j^a \sigma_{j+1}^a, \sigma_i^b \sigma_{i+1}^b]. \quad (3.93)$$

We can use the above identities and the $su(2)$ algebra of spins

$$[\sigma_i^a, \sigma_j^b] = 2i \delta_{ij} \epsilon^{abc} \sigma_j^c, \quad (3.94)$$

to write

$$\begin{aligned} Q_3 &= \sum_{i,j} j \epsilon^{abc} (\delta_{j+1,i} \sigma_j^a \sigma_i^c \sigma_{i+1}^b + \delta_{j,i} \sigma_i^c \sigma_{j+1}^a \sigma_{i+1}^b \\ &\quad + \delta_{j+1,i+1} \sigma_i^b \sigma_j^a \sigma_{i+1}^c + \delta_{j,i+1} \sigma_i^b \sigma_{i+1}^c \sigma_{j+1}^a) \\ &= \sum_i \epsilon^{abc} [(i-1) \sigma_{i-1}^a \sigma_i^c \sigma_{i+1}^b + i \sigma_i^c \sigma_{i+1}^a \sigma_{i+1}^b \\ &\quad + i \sigma_i^b \sigma_i^a \sigma_{i+1}^c + (i+1) \sigma_i^b \sigma_{i+1}^c \sigma_{i+2}^a]. \end{aligned} \quad (3.95)$$

⁹Readers interested in the details of proof are again referred to [13]

Notice that because of the antisymmetry of ϵ^{abc} , terms with two σ 's acting on the same site vanish. If we shift by 1 the sum over the first term and then recombine both nonzero terms again, we get

$$\begin{aligned} Q_3 &= \sum_i i \epsilon^{abc} \sigma_i^a \sigma_{i+1}^c \sigma_{i+2}^b + (i+1) \epsilon^{abc} \sigma_i^b \sigma_{i+1}^c \sigma_{i+2}^a \\ &= \sum_i (i \epsilon^{acb} + (i+1) \epsilon^{cab}) \sigma_i^a \sigma_{i+1}^b \sigma_{i+2}^c = \sum_i \epsilon^{abc} \sigma_i^a \sigma_{i+1}^b \sigma_{i+2}^c, \end{aligned} \quad (3.96)$$

where in the second line we first relabeled the indices (a, b, c) that are being summed over and then used the definition of the anti-symmetric Levi-Civita symbol ϵ . Finally using the definition of the cross product

$$(A \times B)^i = \epsilon^{ijk} A^j B^k, \quad (3.97)$$

we can write the first conserved charge

$$Q_3 = \sum_i \epsilon^{abc} \sigma_i^a \sigma_{i+1}^b \sigma_{i+2}^c = \sum_i (\sigma_i \times \sigma_{i+1}) \cdot \sigma_{i+2}. \quad (3.98)$$

It indeed has a manifestly local form that is invariant under shifts along the chain and acts on the three nearest sites at the time.

The next charge Q_4 can be obtained in the same straightforward (but a bit tedious) method. By definition

$$Q_4 = [\hat{B}, Q_3] = \frac{1}{2i} \sum_{j,i} j \epsilon^{abc} [\sigma_j^\alpha \sigma_{j+1}^\alpha, \sigma_i^a \sigma_{i+1}^b \sigma_{i+2}^c] \quad (3.99)$$

Iterative application of our commutator identities gives

$$\begin{aligned} [AB, CDE] &= A[B, C]DE + [A, C]BDE + CA[B, D]E \\ &\quad + CAD[BE] + C[A, D]EB + CD[A, E]B. \end{aligned} \quad (3.100)$$

Then we have

$$\begin{aligned} \frac{1}{2i} \sum_{i,j} j \epsilon^{abc} & \left(\sigma_j^\alpha [\sigma_{j+1}^\alpha, \sigma_i^a] \sigma_{i+1}^b \sigma_{i+2}^c + [\sigma_j^\alpha, \sigma_i^a] \sigma_{j+1}^\alpha \sigma_{i+1}^b \sigma_{i+2}^c \right. \\ & \left. + \sigma_i^a \sigma_j^\alpha [\sigma_{j+1}^\alpha, \sigma_{i+1}^b] \sigma_{i+2}^c + \sigma_i^a \sigma_j^\alpha \sigma_{i+1}^b [\sigma_{j+1}^\alpha, \sigma_{i+2}^c] \right. \\ & \left. + \sigma_i^a [\sigma_j^\alpha, \sigma_{i+1}^b] \sigma_{i+2}^c \sigma_{j+1}^\alpha + \sigma_i^a \sigma_{i+1}^b [\sigma_j^\alpha, \sigma_{i+2}^c] \sigma_{j+1}^\alpha \right). \end{aligned} \quad (3.101)$$

Now, using the commutation relations for Pauli matrices yields

$$\begin{aligned} & \sum_{i,j} j \epsilon^{abc} (\epsilon^{\alpha ad} \delta_{j+1,i} \sigma_j^\alpha \sigma_i^d \sigma_{i+1}^b \sigma_{i+2}^c + \epsilon^{\alpha ad} \delta_{j,i} \sigma_i^d \sigma_{j+1}^\alpha \sigma_{i+1}^b \sigma_{i+2}^c \\ & + \epsilon^{\alpha bd} \delta_{j+1,i+1} \sigma_i^a \sigma_j^\alpha \sigma_{i+1}^d \sigma_{i+2}^c + \epsilon^{\alpha cd} \delta_{j+1,i+2} \sigma_i^a \sigma_j^\alpha \sigma_{i+1}^b \sigma_{i+2}^d \\ & + \epsilon^{\alpha bd} \delta_{j,i+1} \sigma_i^a \sigma_{i+1}^d \sigma_{i+2}^c \sigma_{j+1}^\alpha + \epsilon^{\alpha cd} \delta_{j,i+2} \sigma_i^a \sigma_{i+1}^b \sigma_{i+2}^d \sigma_{j+1}^\alpha). \end{aligned} \quad (3.102)$$

Performing the sum over j and shifting appropriate terms we finally get

$$Q_4 = 2 \sum_i (\sigma_i \times \sigma_{i+1}) \times \sigma_{i+2} \cdot \sigma_{i+3} + \sigma_i \cdot \sigma_{i+2} - 4Q_2. \quad (3.103)$$

Again it is a local operator acting on four nearest neighbors on the chain. Just for the reference we write Q_4 in terms of the permutation operators. Using the identity

$$\epsilon^{abc} \epsilon^{ade} = \delta^{bd} \delta^{ce} - \delta^{be} \delta^{cd}, \quad (3.104)$$

and then

$$\sigma_i \cdot \sigma_j = 2 P_{i,j} - \mathbf{1}_{i,j}, \quad (3.105)$$

we can write it as

$$\begin{aligned} Q_4 = & -8 \sum_{i=1} (1_{i,i+1} - P_{i,i+1}) - 4 \sum_i P_{i,i+2} + 4 \sum_i P_{i,i+3} \\ & - 8 \sum_i (P_{i,i+3} P_{i+1,i+2} - P_{i,i+2} P_{i+1,i+3}). \end{aligned} \quad (3.106)$$

This is the familiar form that can be compared e.g. with [15].

3.3 Parity operator.

Parity operator can be defined formally [75] by its action on a spin at site X_n within a chain of length L as

$$\Pi X_n \Pi^{-1} = X_{L-n+1}. \quad (3.107)$$

Equivalently its action on the tensor product state is given by the product of permutation operators

$$\Pi = P_{1,L} P_{2,L-1} \dots P_{\frac{L}{2}, \frac{L+2}{2}}, \quad L - \text{even}, \quad (3.108)$$

$$\Pi = P_{1,L} P_{2,L-1} \dots P_{\frac{L-1}{2}, \frac{L+3}{2}}, \quad L - \text{odd}. \quad (3.109)$$

So the tensor product state after the parity transforms to

$$\begin{aligned} & \Pi X_1 \otimes X_2 \otimes \dots \otimes X_{L-1} \otimes X_L \Pi^{-1} \\ &= X_L \otimes X_{L-1} \otimes \dots \otimes X_2 \otimes X_1. \end{aligned} \tag{3.110}$$

It is a unitary operator

$$\Pi \Pi^\dagger = 1, \quad \Pi = \Pi^{-1} = \Pi^\dagger, \tag{3.111}$$

that commutes with the hamiltonian and flips the sign of the momenta. This definition will require only minor modifications in the AdS/CFT context the operator itself will play an important role in searches of non-planar integrability.

Chapter 4

One-loop Dilatation Generator of $\mathcal{N} = 4$ SYM

Here we include the details of the dilatation generator of $\mathcal{N} = 4$ SYM at one loop in $SO(6)$ and $SU(2)$ scalar sectors. First we review their derivations for scalar fields in the adjoint representation of unitary, special-unitary, orthogonal and symplectic group. Then we specialize to the $SU(2)$ subsector and analyze planar and non-planar contributions that are obtained by acting with the operator according to the rules for each of the gauge groups. At the planar level, orthogonal and symplectic contributions are equal to a half of the (special) unitary and are captured by the well known Heisenberg's hamiltonian. At the non planar level, in addition to the "standard" cut and join action, we discover a new interesting class of operators that do not change the number of traces. We close the chapter with a summary table on all the possible one-loop contributions and their action on single and double trace states. Broader discussion of these issues can be found in [19], [18], [23].

4.1 One-loop $SO(6)$ dilatation generator.

In this section we review the derivation the dilatation operator in $\mathcal{N} = 4$ SYM to one-loop in the scalar sector. Building blocks will be the six scalar fields of the theory ϕ_i , $i = 1, \dots, 6$, a subset usually referred to as the $SO(6)$ sector.

The derivation is based on the method of an effective vertex described in [23], and the main idea behind it is as follows. As we discussed before, the

anomalous dimension of a composite operator can be deduced from it's two point correlation function

$$\langle O_1(x)O_2(0) \rangle = \langle \text{Tr}(\phi_1 \dots \phi_L)(x) \text{Tr}(\phi_1 \dots \phi_L)(0) \rangle, \quad (4.1)$$

that we evaluate using perturbation theory. At each level of perturbative expansion we have a certain number of Feynman diagrams that are representation of the contractions between a given number of fields from the operator at 0, as well as at x . For example at tree level only single sites are contracted, at one-loop, at a time, we can contract at most a pair of fields from the operator etc. This way, at each level, we can always contract certain number of fields (with accordance to the diagram) such that what is left is an effective operator that can only be contracted with a certain number (depending on the level) of external scalars. Schematically we have

$$\langle O_1(x)O_2(0) \rangle = \langle O_1 | \check{D}_0 | O_2 \rangle + \lambda \langle O_1 | \check{D}_1 | O_2 \rangle + \dots \quad (4.2)$$

This operator(s) is called an effective vertex.

For example the tree level contribution is just the classical scaling dimension of the operator which is its length times the scaling dimension of the field that it is built of. Scaling dimension of the scalar field is one $[\phi_i] = 1$ so the tree level operator should just measure the length of the state

$$\langle O_1 | \check{D}_0 | O_2 \rangle = L. \quad (4.3)$$

It is then not too difficult to deduce that

$$D_0 = \left(\frac{8\pi^2}{g_{YM}^2} \right)^2 \sum \text{Tr} \phi_i^- \phi_i^+ \quad (4.4)$$

where ϕ^+ is contracted with scalars at 0 and ϕ^- at x respectively. The sum runs over all sites of the state.

The one loop example is more involved. In $\mathcal{N} = 4$ SYM there are three diagrams (Fig.4.1) that contribute to the two point function at this level: scalar four point vertex, gluon exchange between two scalar lines and the self energy correction to the scalar propagator, that at one loop consists of the gluon intermediate state and a fermion loop. Then, in order to derive the effective vertex, we can proceed with a general algorithm. First, we look at the Lagrangian of the theory and identify terms in the potential that give rise to these interactions. They are:

- four-scalar vertex

$$U_{4s} = -\frac{1}{2g_{YM}^2} \text{Tr} [\phi_i, \phi_j] [\phi_i, \phi_j], \quad (4.5)$$

- gluon-scalars vertex

$$U_{sg} = -\frac{2i}{g_{YM}^2} \text{Tr} \partial_\mu \phi_i [A_\mu, \phi_i], \quad (4.6)$$

- scalar-fermions vertex

$$U_{s2f} = -\frac{i}{g_{YM}^2} \text{Tr} \bar{\psi} \Gamma_i [\phi_i, \psi]. \quad (4.7)$$

Next, we use them to write each of the four terms corresponding to the Feynman diagrams. Then contract gluons, fermions and some scalars (in the intermediate gluon diagram) so that what is left can only be contracted with the external fields inside the operators (scalars). Finally, in front of each term we put the "coupling" constant that is the value of the Feynman integral of the diagram. Below we go through these steps explicitly so the reader can learn the general procedure.

4.1.1 Scalar vertex

Let us start with the interaction vertex of four scalar fields. This is the simplest diagram because we do not need to contract any fields inside it. In other words, the only interesting thing is the non-trivial combinatorics that the vertex

$$U_{4s} = -\frac{1}{2g_{YM}^2} \text{Tr} [\phi_i, \phi_j] [\phi_i, \phi_j]. \quad (4.8)$$

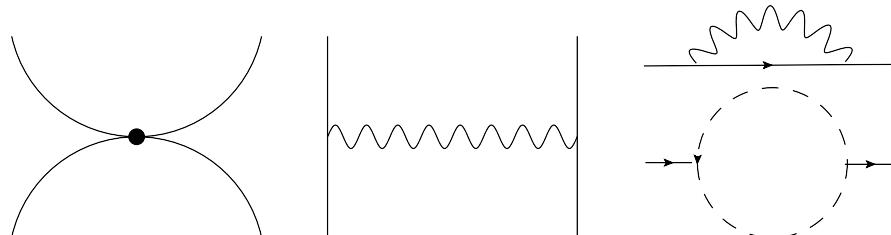


Figure 4.1: One-loop diagrams.

leads to. Since all the fields have to be contracted with either the operator at x (O^+) or 0 (O^-), we split scalars to $\phi_i = \phi_i^+ + \phi_i^- = i^+ + i^-$, and remember that the only nonvanishing contractions are¹

$$\langle i^-, j^+ \rangle = \delta^{ij}. \quad (4.9)$$

Plugging it into (4.8) and leaving only terms with two pluses and two minuses gives six traces

$$\begin{aligned} \text{Tr} [i^+, j^+] [i^-, j^-], \quad \text{Tr} [i^-, j^-] [i^+, j^+], \quad \text{Tr} [i^+, j^-] [i^+, j^-], \\ \text{Tr} [i^-, j^+] [i^-, j^+], \quad \text{Tr} [i^-, j^+] [i^+, j^-], \quad \text{Tr} [i^+, j^-] [i^-, j^+]. \end{aligned} \quad (4.10)$$

The first and the last pairs consist of elements that are cyclic permutations of each other. Similarly, the third and fourth terms are equal because of the summation over i and j (we can freely relabel $i \leftrightarrow j$). This yields

$$U_{4s} = -\frac{1}{g_{YM}^2} (Tr [i^+, j^-] [i^+, j^-] + Tr [i^+, j^-] [i^-, j^+] + Tr [i^+, j^+] [i^-, j^-]). \quad (4.11)$$

It is convenient to further rewrite the second term using the Jacobi identity. Namely, notice that

$$\text{Tr} [i^+, j^-] [i^-, j^+] = \text{Tr} (i^+ j^- - j^- i^+) [i^-, j^+] = \text{Tr} i^+ [j^-, [i^-, j^+]], \quad (4.12)$$

so using the Jacobi identity for the commutator

$$[j^-, [i^-, j^+]] = -[i^-, [j^+, j^-]] - [j^+, [j^-, i^-]], \quad (4.13)$$

one can insert it back to get

$$\text{Tr} [i^+ [j^-, [i^-, j^+]]] = \text{Tr} [i^+, j^+] [i^-, j^-] - \text{Tr} [i^+, i^-] [j^+, j^-]. \quad (4.14)$$

Adding all together we can write U_{4s} as a sum of three terms

$$U_{4s} = \frac{2}{g_{YM}^2} : (V_D + V_F + V_K) :, \quad (4.15)$$

¹we neglect the space-time dependence that can be easily restored at each stage of the computation

where

$$\begin{aligned} V_D &= \frac{1}{2} \text{Tr} [i^+, i^-] [j^+, j^-], \\ V_F &= -\text{Tr} [i^+, j^+] [i^-, j^-], \\ V_K &= -\frac{1}{2} \text{Tr} [i^+, j^-] [i^+, j^-]. \end{aligned} \quad (4.16)$$

Normal ordering symbol $::$ means that we only contract fields with external operators (and not inside the effective vertex).

Next, the coupling constant, the one loop Feynman integral in dimensional regularisation² (divided by the factor of L tree level propagators), is given by

$$\left(\frac{g_{YM}^2}{8\pi^2 x^2} \right)^{L-2-L} \left(\frac{g_{YM}^2}{8\pi^2} \right)^4 \int \frac{dz}{(x-z)^4 z^4} = \frac{g_{YM}^2 \Lambda}{32\pi^2}, \quad (4.17)$$

where

$$\Lambda = \log x^{-2} - \left(\frac{1}{\epsilon} + \gamma + \log \pi + 2 \right). \quad (4.18)$$

Finally, the contribution to the effective vertex from the four-point scalar interaction is

$$V_{4s} = \frac{g_{YM}^2 \Lambda}{16\pi^2} (: V_D : + : V_F : + : V_K :). \quad (4.19)$$

Note that since we did not perform any contractions, this part of the effective vertex will remain invariant under the change of the gauge group. In other words, we will always have three terms (4.16) coming from the four scalar interaction but they will be built from matrices in the representation of the unitary, orthogonal or symplectic Lie algebra, depending on which of the gauge groups we consider.

With this warm up we can proceed to the other two diagrams.

4.1.2 Gluon exchange

The diagram for gluon exchange comes from expanding the exponent of the action to the order that we have two gluon-scalars (4.6) vertices

$$\frac{4}{g_{YM}^4} \text{Tr} \partial_\mu \phi_i [A_\mu, \phi_i] \text{Tr} \partial_\nu \phi_j [A_\nu, \phi_j]. \quad (4.20)$$

²in $D = 4 - 2\epsilon$

Even though the spacetime dependence will be taken into account by the Feynman integral, we keep derivatives to distinguish scalar fields within each term.

The only contraction that is needed is the one between gluons. The result will contain four scalars that can be appended to an arbitrary nearest-neighbors fields inside any operator. Before contracting gluons it is convenient to rewrite the vertices as

$$\begin{aligned}\text{Tr } \partial_\mu \phi_i [A_\mu, \phi_i] &= \text{Tr } \partial_\mu \phi_i (A_\mu \phi_i - \phi_i A_\mu) \\ \text{Tr } A_\mu (\phi_i \partial_\mu \phi_i - \partial_\mu \phi_i \phi_i) &= \text{Tr } A_\mu [\phi_i, \partial_\mu \phi_i].\end{aligned}\quad (4.21)$$

While contracting the gluon fields, we can see that the result is exactly the same for all the gauge groups! Let us see this explicitly.

With $U(N)$ contraction (A.4) we simply have

$$\frac{4}{g_{YM}^4} \text{Tr} [\phi_i, \partial_\mu \phi_i] [\phi_j, \partial_\mu \phi_j]. \quad (4.22)$$

For $SU(N)$ we get an extra $1/N$ term that has two traces of a single commutator. This vanishes due to the cyclicity of the trace.

For $SO(N)$ contraction (A.17), the result is

$$\frac{1}{2} (\text{Tr} [\phi_i, \partial_\mu \phi_i] [\phi_j, \partial_\mu \phi_j] - \text{Tr} [\phi_i, \partial_\mu \phi_i] ([\phi_j, \partial_\mu \phi_j])^T). \quad (4.23)$$

After using $([A, B])^T = -[A, B]$ terms add up again to (4.22).

For $Sp(N)$ contraction (A.24), one gets

$$\frac{1}{2} (\text{Tr} [\phi_i, \partial_\mu \phi_i] [\phi_j, \partial_\mu \phi_j] + \text{Tr} [\phi_i, \partial_\mu \phi_i] J([\phi_j, \partial_\mu \phi_j])^T J), \quad (4.24)$$

which after substituting

$$([\phi_j, \partial_\mu \phi_j])^T = J \partial_\mu \phi_i J^2 \phi_i J - J \phi_i J^2 \partial_\mu \phi_i J = J [\phi_i, \partial_\mu \phi_i] J, \quad (4.25)$$

combines with the first term into (4.22).

Finally inserting $\phi_i = i^+ + i^-$, the combinatorics from the gluon exchange is

$$\frac{8}{g_{YM}^4} \text{Tr} [i^+, i^-] [j^+, j^-]. \quad (4.26)$$

Combining it with the Feynman integral coefficient (that can be found in e.g. [43]) we have the part of the effective vertex that comes from gluon exchange

$$U_{ge} = \frac{g_{YM}^2 (\Lambda + 2)}{32\pi^2} (\text{Tr} [i^+, i^-] [j^+, j^-]). \quad (4.27)$$

4.1.3 Self-Energy

The last contribution comes from the self energy correction to the scalar propagator. It contains two diagrams, one with the gluon propagator and one with the fermionic loop. We consider them separately.

Gluon propagator

The gluon propagator can be obtained from (4.26) by contracting an additional pair of scalars. For the unitary and special unitary ($1/N$ terms appear with opposite signs) contractions the result is

$$\frac{32}{g_{YM}^4} (N \text{Tr } i^+ i^- - \text{Tr } i^+ \text{Tr } i^-), \quad (4.28)$$

where for traceless $SU(N)$ generators the second term disappears.

For orthogonal group we have

$$\begin{aligned} \frac{16}{g_{YM}^4} & ((N-1) \text{Tr } i^+ i^- - \text{Tr } i^+ \text{Tr } i^- + \text{Tr } i^+ (i^-)^T) \\ & = \frac{16}{g_{YM}^4} (N-2) \text{Tr } i^+ i^-, \end{aligned} \quad (4.29)$$

where again terms with traces over single $SO(N)$ fields vanish due to anti-symmetry. Finally the symplectic group contractions yield

$$\begin{aligned} \frac{16}{g_{YM}^4} & ((N+1) \text{Tr } i^+ i^- - \text{Tr } i^+ \text{Tr } i^- + \text{Tr } i^+ J (i^-)^T J) \\ & = \frac{16}{g_{YM}^4} (N+2) \text{Tr } i^+ i^-. \end{aligned} \quad (4.30)$$

Fermion loop

Similar combinatorial contribution comes from fermions. If we first rewrite the two terms that come from the expansion of the exponent of the $\mathcal{N} = 4$ action, as

$$\frac{1}{g_{YM}^4} \text{Tr } \bar{\psi} [\phi_i, \psi] \text{Tr } \psi [\bar{\psi}, \phi_i], \quad (4.31)$$

it is clear that after contracting $\bar{\psi}$ and ψ , the structure that emerges is identical to (4.26). Therefore this step is the same for all gauge groups

$$-\frac{2}{g_{YM}^4} \text{Tr } [\bar{\psi}, \phi_i] [\psi, \phi_i]. \quad (4.32)$$

Now contracting the remaining fermions produces the same traces as we got from the gluon propagator. Taking into account Feynman integrals (see [20]) we have the scalar self energy contributions to the effective vertex

$$U_{se}^{U(N)} = \frac{g_{YM}^2(\Lambda+1)}{8\pi^2} (N \operatorname{Tr} i^+ i^- - \operatorname{Tr} i^+ \operatorname{Tr} i^-), \quad (4.33)$$

$$U_{se}^{SU(N)} = \frac{g_{YM}^2(\Lambda+1)}{8\pi^2} (N \operatorname{Tr} i^+ i^-), \quad (4.34)$$

$$U_{se}^{SO(N)} = \frac{g_{YM}^2(\Lambda+1)}{16\pi^2} (N-2) \operatorname{Tr} i^+ i^-, \quad (4.35)$$

$$U_{se}^{Sp(N)} = \frac{g_{YM}^2(\Lambda+1)}{16\pi^2} (N+2) \operatorname{Tr} i^+ i^-. \quad (4.36)$$

4.1.4 Cancellation of D-terms

One loop effective vertex is the sum of the contributions that we derived above. Nevertheless, when acting on an arbitrary single trace operator in the scalar sector, we can demonstrate that for all the gauge groups D -term from the four point scalars cancels against the gluon exchange and the self energy. The argument goes as follows.

- $U(N)$

The sum of D -term, gluon exchange and scalar self energy can be written as

$$\frac{g_{YM}^2(\Lambda+1)}{8\pi^2} \left(\frac{1}{2} : \operatorname{Tr} [i^+, i^-] [j^+, j^-] : + N : \operatorname{Tr} i^+ i^- : - \operatorname{Tr} i^+ \operatorname{Tr} i^- \right). \quad (4.37)$$

From Wick's theorem

$$\phi_1 \dots \phi_n =: \phi_1 \dots \phi_n : + \sum_{\text{contractions}} \phi_1 \dots \phi_n : \quad (4.38)$$

we can change the normal ordered terms into non-normally ordered minus (normally ordered) contractions. This way the double commutator becomes

$$\begin{aligned} & \frac{1}{2} \operatorname{Tr} [i^+, i^-] [j^+, j^-] - 2 : (N \operatorname{Tr} i^- i^+ - \operatorname{Tr} i^- \operatorname{Tr} i^+) : \\ &= \frac{1}{2} \operatorname{Tr} [i^+, i^-] [j^+, j^-] - 2 (N \operatorname{Tr} i^- i^+ - \operatorname{Tr} i^- \operatorname{Tr} i^+) \\ & \quad + 6N(N^2 - 1). \end{aligned} \quad (4.39)$$

On the other hand the two terms that come from self energy are

$$N \text{Tr } i^- i^+ - \text{Tr } i^- \text{Tr } i^+ - 6N(N^2 - 1), \quad (4.40)$$

so the sum of the three terms is finally

$$\frac{1}{2} \text{Tr} [i^+, i^-] [j^+, j^-] - N \text{Tr } i^- i^+ + \text{Tr } i^- \text{Tr } i^+. \quad (4.41)$$

Important thing to notice now is that when we apply this part to an arbitrary operator we also have to include self-contractions. Now, it is easy to see that when we contract i^+ with j^- inside the vertex, this precisely cancels the two remaining terms. It is then sufficient to show that when one apply the double commutator on a single trace, the result gives zero. First write it as

$$\text{Tr} (i^+ i^- - i^- i^+) [j^+, j^-] = \text{Tr } i^+ [i^-, [j^+, j^-]]. \quad (4.42)$$

Then acting with i^+ on a state we have

$$\begin{aligned} \text{Tr } i^+ [i^-, [j^+, j^-]] \text{Tr } i_1^- i_2^- \dots i_L^- &= \text{Tr} [i_1^-, [j^+, j^-]] i_2^- \dots i_L^- \\ &+ \text{Tr} i_1^- [i_2^-, [j^+, j^-]] \dots i_L^- + \dots + \text{Tr} i_1^- \dots [i_L^-, [j^+, j^-]] = 0, \end{aligned} \quad (4.43)$$

where we get site k inside the commutator due to the δ^{ik} from the contraction. Then once we expand the commutators it is clear that each of the terms cancels one on the right and one on the left (the one on the left in the first term cancels the one on the right of L). Hence there is no contribution from D -terms, self energy and the gluon exchange.

- $SU(N), SO(N), Sp(N)$

The cancellation can be shown analogously for the other three groups. Always the D -term and the gluon exchange (that are both invariant under the change of the gauge group), when written without normal order, cancel the self energy part with one of the internal contractions. Similarly we can see that acting on an arbitrary single trace will lead to null result as well (for $SO(N)$ and $Sp(N)$ we will get extra terms with transpositions but, up to a minus sign depending on the length, they are equivalent to these from the $U(N)$ contraction and therefore cancel mutually).

Finally we have the one loop dilatation operator of $\mathcal{N} = 4$ SYM in the $SO(6)$ sector

$$D_1 = -\frac{g_{YM}^2 \Lambda}{16\pi^2} \left(: \text{Tr} [i^+, j^+] [i^-, j^-] : + \frac{1}{2} : \text{Tr} [i^+, j^-] [i^-, j^+] : \right) \quad (4.44)$$

4.1.5 $SU(2)$ subsector

There are several subsectors of the theory in which we focus only on a subgroup of fields of $\mathcal{N} = 4$ SYM. In everything what follows, we restrict to the $SU(2)$ subsector that consists of two of the three complex scalars, say

$$Z = \frac{1}{\sqrt{2}} (\phi_5 + i \phi_6), \quad W = \frac{1}{\sqrt{2}} (\phi_3 + i \phi_4). \quad (4.45)$$

For these fields the K -term vanishes and our dilatation operator (4.44) (for all the gauge groups) reduces to

$$\hat{D} = -\frac{g_{YM}^2}{8\pi^2} : \text{Tr} [Z, W] [\check{Z}, \check{W}] : \quad (4.46)$$

where fields act on states as matrix derivatives

$$\check{Z}_i^j = \frac{\delta}{\delta Z_j^i}, \quad \check{W}_i^j = \frac{\delta}{\delta W_j^i}. \quad (4.47)$$

Normal ordering symbol $::$ means that \check{Z} and \check{W} act only on fields inside the state and not within the effective vertex.

This will be our main tool to analyze planar and non-planar interactions and search for integrability at one loop.

4.2 Structure of the dilatation operator

In this section we analyze the structure of the one loop dilatation operator (4.46) in the $\mathcal{N} = 4$ SYM with $U(N)$, $SU(N)$, $SO(N)$ and $Sp(N)$ gauge groups. The analysis can be performed in a very systematic way by applying it to a generic single and double trace states. While acting with (4.46), we use the contraction rules derived in Appendix A. In general we distinguish two contributions. A leading in color N called "planar" and a subleading in $1/N$ called "non-planar". Planar contributions come from the action on the

nearest-neighbor sites of the state/chain, while the non-planar from appending the dilatation operator to all the other non-adjacent sites. Furthermore, the non-planar contributions can be grouped into those that increase the number of traces within a state that they act on (H_+), those that decrease the number of traces within the state (H_-), and finally those that do not change the number of traces, but still act in a highly non-local way (H_{flip}). All the details and derivations are presented below. The results are summarized in table Tab.(4.3).

4.2.1 $U(N)$

We start with fields in the adjoint representation of the unitary $U(N)$ group. They are $N \times N$ matrices

$$A_{ij}(x) = A^a(x)T_{ij}^a, \quad a = 1, \dots, N^2. \quad (4.48)$$

where T^a are the N^2 generators of the $U(N)$ in the adjoint representation

$$(T^a)^{bc} = if^{abc}. \quad (4.49)$$

We drop the space-time dependence (since it can be simply restored) and from now on when writing (Z,W,A,B,C,...etc.) we refer to matrices.

Before we start the derivations, as a warm up, let us consider the action of one part of (4.46), say $Tr(ZW\check{Z}\check{W})$, on $Tr(AZBWC)$

$$Tr(ZW\check{Z}\check{W}) Tr(AZBWC). \quad (4.50)$$

First we write both terms in components

$$Tr(ZW\check{Z}\check{W}) Tr(AZBWC) = Z_\alpha^\beta W_\beta^\gamma \check{Z}_\gamma^\kappa \check{W}_\kappa^\alpha A_\mu^\nu Z_\nu^\rho B_\rho^\phi W_\phi^\sigma C_\sigma^\mu, \quad (4.51)$$

then using the $U(N)$ contraction rules

$$\check{Z}_\gamma^\kappa Z_\nu^\rho = \delta_\nu^\kappa \delta_\gamma^\rho, \quad \check{W}_\kappa^\alpha W_\phi^\sigma = \delta_\phi^\alpha \delta_\kappa^\sigma, \quad (4.52)$$

we have

$$Z_\alpha^\beta W_\beta^\gamma \delta_\nu^\kappa \delta_\gamma^\rho \delta_\phi^\sigma A_\mu^\nu B_\rho^\phi C_\sigma^\mu = Z_\phi^\beta W_\beta^\rho A_\mu^\kappa B_\rho^\phi C_\kappa^\mu. \quad (4.53)$$

This again can be written in terms of traces so the result is

$$Tr(ZWB) Tr(AC). \quad (4.54)$$

This is the standard algorithm that one can apply when acting with the dilatation operator on any state with any gauge group contraction rule. In order to see the general structure of the dilatation operator, it is then sufficient to look how it acts on single and double trace states. We define them as

$$\begin{aligned} O^1 &= \text{Tr}(ZBWC) = A_\mu^\nu Z_\nu^\rho B_\rho^\phi W_\phi^\sigma C_\sigma^\mu \\ O^2 &= \text{Tr}(AZB) \text{Tr}(CWD) = A_\alpha^\beta Z_\beta^\kappa B_\kappa^\alpha C_\mu^\nu W_\nu^\rho D_\rho^\mu. \end{aligned} \quad (4.55)$$

Z and W are the single fields/matrices that will be contracted with \check{Z} and \check{W} from the dilatation operator. On the other hand, A , B , C and D are any words of the $U(N)$ fields of an arbitrary length.

We start with the action on O^1 . It does not matter which field we append first but we present the derivation by always first applying \check{Z} and then \check{W} . The result is

$$\begin{aligned} \check{D}^{U(N)} O^1 &= -\frac{g_{YM}^2}{8\pi^2} \{ \text{Tr}(\check{W}[Z, W]BWCA) - \text{Tr}([Z, W]\check{W}BWCA) \} \\ &= -\frac{g_{YM}^2}{8\pi^2} \{ \text{Tr}([Z, W]B) \text{Tr}(AC) - \text{Tr}(B) \text{Tr}(A[Z, W]C) \} \\ &= \frac{g_{YM}^2}{8\pi^2} \{ \text{Tr}(B) \text{Tr}(A[Z, W]C) + \text{Tr}([W, Z]B) \text{Tr}(AC) \} \\ &= \frac{g_{YM}^2}{8\pi^2} \{ \text{Tr}(B) \text{Tr}(A[Z, W]C) + \text{Tr}([ZBW]) \text{Tr}(AC) \}. \end{aligned} \quad (4.56)$$

The first line is just the contraction of \check{Z} and the second, contraction of \check{W} . Remember that both of them are only contracted with fields from the state. In the third line we swap the commutator and bring the overall minus sign inside the brackets. Finally in the last line we introduced a new notation

$$[ZBW] \equiv ZBW - WBZ, \quad (4.57)$$

which will prove to be very convenient in finding general patterns for dilatation operators.

Similarly, using the same set of contraction rules, the action on the double

trace operators³ is

$$\begin{aligned}
\check{D}^{U(N)} O^2 &= -\frac{g_{YM}^2}{8\pi^2} \{ \text{Tr}(\check{W}[Z, W]A) \text{Tr}(WB) - \text{Tr}([Z, W]\check{W}A) \text{Tr}(WB) \} \\
&= -\frac{g_{YM}^2}{8\pi^2} \{ \text{Tr}([Z, W]AB) - \text{Tr}(A[Z, W]B) \} \\
&= \frac{g_{YM}^2}{8\pi^2} \{ \text{Tr}(A[Z, W]B) + \text{Tr}([W, Z]AB) \} \\
&= \frac{g_{YM}^2}{8\pi^2} \{ \text{Tr}(A[Z, W]B) + \text{Tr}([ZABW]) \}. \tag{4.58}
\end{aligned}$$

Again the first two lines are contractions of \check{Z} and \check{W} respectively, the third is a swap of the commutator and the overall minus and the last line is just the third line rewritten in the new notation defined in (4.57).

Now we are ready to classify different contributions to the dilatation operator. Let us start with the nearest neighbors. This so-called planar or the leading N part can be obtained from formula (4.56) by setting $B_j^i = \delta_j^i$. This way the trace over B brings a factor of N and the second term with the commutator vanishes due to the cyclicity of the trace. So we have

$$\text{Tr}(B) = \text{Tr}(\delta) = \delta_i^i = N, \quad \text{Tr}([W\delta Z]) = 0, \tag{4.59}$$

and the planar contribution is

$$\check{D}^{U(N)} \text{Tr}(AZWC) \equiv \check{H}_0^{U(N)} \text{Tr}(AZWC) = \frac{g^2 N}{8\pi^2} \text{Tr}(A[Z, W]C). \tag{4.60}$$

It is not too hard to notice that H_0 acts on two sites Z and W as

$$\check{H}_0 \text{Tr}(AZWB) = \frac{\lambda}{8\pi^2} (1 - P) \text{Tr}(AZWB), \tag{4.61}$$

where 1 is just the identity and P is the permutation operator that interchanges the position of Z and W . Notice that we have used the 't Hooft coupling

$$g_{YM}^2 N = \lambda. \tag{4.62}$$

We can picture Z as for example a spin up \uparrow and W as a spin down \downarrow . Then a single trace state can be viewed as a periodic chain of spins. Furthermore,

³here and for $SU(N)$ we use $O^2 = \text{Tr}(ZA) \text{Tr}(WB)$ but for orthogonal and symplectic contractions it was more transparent to use definitions (4.55).

H_0 is plays a role of the gauge theory Heisenberg's hamiltonian governing the nearest-neighbor interactions in the system. This was in fact one of most important discoveries[18] that triggered a new field of planar integrability in gauge and string theories.

Let us now move to the non-planar contributions. From (4.56), if $B \neq \delta$, we can identify two terms with an action of the following operator

$$H_+ = \frac{1}{N} H_{ij}^0 \check{S}_{ij}. \quad (4.63)$$

When appended to Z and W at sites i and j , it first splits the chain with a "split" (or "cut") operator \check{S}_{ij} as

$$\check{S} \text{Tr}(AZBWC) = \text{Tr}(B) \text{Tr}(AZWC) + \text{Tr}(AC) \text{Tr}(ZBW), \quad (4.64)$$

and then acts on Z and W by $\frac{\lambda}{8\pi^2}(1 - P)$ what we denote by H^0 . ⁴ Notice that because we absorbed a factor of N into λ we have $1/N$ in front. Another way to put it is that H_+ is a subleading contribution in the number of colors with respect to H_0 .

Often in the literature the action of the split (or cut) operator is presented on a spin chain like state⁵ built of fields ϕ_i at site i that can have one of the two values, Z (\uparrow) or W (\downarrow). In this convention S acts as

$$\begin{aligned} & \check{S}_{i,j} \text{Tr}(\phi_1 \dots \phi_{i-1} \phi_i \phi_{i+1} \dots \phi_{j-1} \phi_j \phi_{j+1} \dots \phi_L) \\ &= \text{Tr}(\phi_{i+1} \dots \phi_{j-1}) \text{Tr}(\phi_1 \dots \phi_{i-1} \phi_i \phi_j \phi_{j+1} \dots \phi_L) \\ &+ \text{Tr}(\phi_1 \dots \phi_{i-1} \phi_{j+1} \dots \phi_L) \text{Tr}(\phi_i \phi_{i+1} \dots \phi_{j-1} \phi_j). \end{aligned} \quad (4.65)$$

From formula (4.76), we can identify yet another $1/N$ contribution

$$H_- = \frac{1}{N} H_{i,j}^0 \check{J}_{ij}. \quad (4.66)$$

When appended into sites i and j inside two different traces, it first unites the traces with a "join" operator

$$\check{J} \text{Tr}(AZB) \text{Tr}(CWD) = \text{Tr}(BAZWDC) + \text{Tr}(ZBADCW), \quad (4.67)$$

⁴We will use the symbol H_0 only for planar (nearest neighbors) part of the dilatation operator and H_{ij}^0 for $\frac{\lambda}{8\pi^2}(1 - P)$ on arbitrary, non-adjacent sites.

⁵Do not confuse it with the $SO(6)$ sector!

and then acts again with $(1 - P)$ similarly to the H_+ case. Again, on chain like states, the "join" operator is given by

$$\begin{aligned} \check{J}_{i,j} Tr(\phi_1 \dots \phi_{i-1} \phi_i \phi_{i+1} \dots \phi_p) Tr(\phi_{p+1} \dots \phi_{j-1} \phi_j \phi_{j+1} \dots \phi_L) \\ = Tr(\phi_{i+1} \dots \phi_p \phi_1 \dots \phi_i \phi_j \dots \phi_L \phi_{p+1} \dots \phi_{j-1}) \\ + Tr(\phi_i \dots \phi_p \phi_1 \dots \phi_{i-1} \phi_{j+1} \dots \phi_L \phi_{p+1} \dots \phi_j) \end{aligned} \quad (4.68)$$

In the discussion above we skipped an important fact that when the dilatation operator is "applied" on a state, we sum over all possible contractions. Hence we should write all the operators as sums over appropriate sites that we append them into. This way the nearest neighbors part, like the Heisenberg's hamiltonian, is a sum over all L (the length of the state) sites

$$H_0 O^{1,L} = \sum_{i=1}^L H_{i,i+1}^0 O^{1,L}. \quad (4.69)$$

The splitting part is a sum over $1 \leq i < j \leq L$ within a single trace

$$H_+ O^{1,L} = \sum_{1 \leq i < j \leq L}^L H_{i,j}^0 \check{S}_{ij} O^{1,L}, \quad (4.70)$$

and the joining contribution is a sum over all p sites inside the first and $L-p$ inside the second trace⁶

$$H_- O^{p,L} = \sum_{i=1}^p \sum_{j=p+1}^L H_{i,j}^0 \check{J}_{ij} O^{p,L}. \quad (4.71)$$

Summarizing, the most general structure of the $U(N)$ dilatation operator is

$$\check{D}^{U(N)} = H_0 + \frac{1}{N} H_- + \frac{1}{N} H_+ \quad (4.72)$$

where H_0 acts on the nearest neighbors within the same trace, H_- acts on the states with more than one trace and reduces the number of traces by one and finally H_+ splits the trace increasing the number of traces by one.

⁶Of course we can act with split or join operators on multi-trace operators so the summation will have to be adjusted in an obvious way.

4.2.2 $SU(N)$

After the detailed explanation of the $U(N)$ gauge group example we can completely analogously consider the dilatation operator and states built of the matrices in the adjoint representation of $SU(N)$. Since

$$U(N) = SU(N) \times U(1), \quad (4.73)$$

$SU(N)$ differs from $U(N)$ only by subtraction of the singlet. This is reflected in the completeness relations

$$(T^a)_\alpha^\beta (T^a)_\nu^\mu = \delta_\nu^\beta \delta_\alpha^\mu - \frac{1}{N} \delta_\alpha^\beta \delta_\nu^\mu, \quad (4.74)$$

that lead to the set of $SU(N)$ contraction rules (see A.2). We use those to derive the action of the dilatation operator on single and double trace states. On single trace states the $SU(N)$ dilatation operator acts in the following way

$$\begin{aligned} \check{D}^{SU(N)} O^1 &= -\frac{g_{YM}^2}{8\pi^2} \left\{ Tr(\check{W}[Z,W]BWCA) - Tr([Z,W]\check{W}BWCA) \right. \\ &\quad \left. - \frac{1}{N} Tr(\check{W}[Z,W]) Tr(BWCA) + \frac{1}{N} Tr([Z,W]\check{W}) Tr(BWCA) \right\} \\ &= -\frac{g_{YM}^2}{8\pi^2} \left\{ Tr([Z,W]B) Tr(AC) - Tr(B) Tr(A[Z,W]C) \right. \\ &\quad \left. - \frac{1}{N} Tr([Z,W]BCA) + \frac{1}{N} Tr(BCA[Z,W]) \right\} \\ &= \frac{g_{YM}^2}{8\pi^2} \{ Tr(B) Tr(A[Z,W]C) + Tr([W,Z]B) Tr(AC) \} \\ &= \frac{g_{YM}^2}{8\pi^2} \{ Tr(B) Tr(A[Z,W]C) + Tr([ZBW]) Tr(AC) \}. \end{aligned} \quad (4.75)$$

The first two lines come from the $SU(N)$ contraction of \check{Z} . We can clearly see that $1/N$ terms cancel with each other. The third and fourth lines are the results of appending \check{W} . Again, $1/N$ terms cancel and the result has the same form as for $U(N)$. This is not a surprize due to a well known fact that the auxiliary $U(1)$ "photon" in $U(N) = SU(N) \times U(1)$ does not couple directly to gluons.

The same situation happens on double trace states

$$\begin{aligned}
\check{D}^{SU(N)} O^2 &= -\frac{g_{YM}^2}{8\pi^2} \left\{ Tr(\check{W}[Z,W]A) Tr(WB) - Tr([Z,W]\check{W}A) Tr(WB) \right. \\
&\quad \left. - \frac{1}{N} Tr(\check{W}[Z,W]) Tr(A) Tr(WB) + \frac{1}{N} Tr([Z,W]\check{W}) Tr(A) Tr(WB) \right\} \\
&= -\frac{g_{YM}^2}{8\pi^2} \left\{ Tr([Z,W]AB) - \frac{1}{N} Tr([Z,W]A) Tr(B) - Tr(A[Z,W]B) \right. \\
&\quad \left. + \frac{1}{N} Tr(A[Z,W]) Tr(B) \right\} = \frac{g_{YM}^2}{8\pi^2} \{ Tr(A[Z,W]B) + Tr([W,Z]AB) \} \\
&= \frac{g_{YM}^2}{8\pi^2} \{ Tr(A[Z,W]B) + Tr([ZABW]) \},
\end{aligned} \tag{4.76}$$

where we first contracted \check{Z} , then \check{W} and finally we rewrote the expression into the same form as $U(N)$.

The conclusion is then that the one loop dilatation operator is the same for $U(N)$ and $SU(N)$ gauge groups

$$\check{D}^{SU(N)} = \check{D}^{U(N)} = H_0 + \frac{1}{N} H_+ + \frac{1}{N} H_- \tag{4.77}$$

This statement should be true to all loops.

4.2.3 $SO(N)$

Now we move to the orthogonal matrices. As one could expect, the structure of the planar contribution will be the same as for the unitary and special unitary matrices. Nevertheless, at the non-planar level, we uncover two new contributions. One that joins two traces with a simultaneous transposition and the other, more promising, that conserves the number of traces. Let us first work out the details in the systematic way.

All the fields are now $N \times N$ antisymmetric matrices

$$Z = Z_{ij} = -Z_{ji} = -Z^T. \tag{4.78}$$

Generators in the adjoint representation of $SO(N)$ satisfy the completeness relations

$$(T^a)_{\alpha\beta} (T^a)_{\nu\mu} = \frac{1}{2} (\delta_{\beta\nu} \delta_{\alpha\mu} - \delta_{\alpha\nu} \delta_{\beta\mu}), \tag{4.79}$$

that lead to the set of contraction rules derived in A.3. The two crucial ones are

$$Tr [\check{X} OXY] = \frac{1}{2} (Tr [O] Tr [Y] - Tr [OY^T]) \quad (4.80)$$

$$Tr [\check{X} O] Tr [XY] = \frac{1}{2} (Tr [OY] - Tr [OY^T]). \quad (4.81)$$

Similarly to the previous cases we can use them to identify the action of the $SO(N)$ operator on single and double trace states. States are now⁷

$$\begin{aligned} O^1 &= Tr (AZBWC) = A_{\mu\nu} Z_{\nu\rho} B_{\rho\phi} W_{\phi\sigma} C_{\sigma\mu} \\ O^2 &= Tr (AZB) Tr (CWD) = A_{\alpha\beta} Z_{\beta\kappa} B_{\kappa\alpha} C_{\mu\nu} W_{\nu\rho} D_{\rho\mu}. \end{aligned} \quad (4.82)$$

Since the contractions are bit more involved than before and it is not hard to lose a minus sign, we proceed with a step-by-step action of the $SO(N)$ dilatation operator on a single trace state. After contracting \check{Z} , we get

$$\begin{aligned} \check{D}^{SO(N)} O^1 &= -\frac{g_{YM}^2}{8\pi^2} \frac{1}{2} \{ Tr (\check{W}[Z, W]BWCA) - Tr ([Z, W]\check{W}BWCA) \\ &\quad - Tr (\check{W}[Z, W](BWCA)^T) + Tr ([Z, W]\check{W}(BWCA)^T) \}. \end{aligned} \quad (4.83)$$

To rewrite the last two terms in a more convenient form, we can just use the obvious property of the trace, namely $Tr (A) = Tr (A^T)$, and the anti-symmetry of the $SO(N)$ matrices. In detail, we write the first of them as

$$\begin{aligned} Tr (\check{W}[Z, W](BWCA)^T) &= Tr ((W^T Z^T \check{W}^T - Z^T W^T \check{W}^T) BWCA) \\ &= Tr (Z W \check{W} BWCA) - Tr (W Z \check{W} BWCA) = Tr ([Z, W]\check{W}BWCA) \end{aligned} \quad (4.84)$$

and similarly the second

$$\begin{aligned} Tr ([Z, W]\check{W}(BWCA)^T) &= Tr ((\check{W}^T W^T Z^T - \check{W}^T Z^T W^T) BWCA) \\ &= Tr (\check{W} Z W \check{W} BWCA) - Tr (\check{W} W Z \check{W} BWCA) = Tr (\check{W}[Z, W]BWCA). \end{aligned} \quad (4.85)$$

⁷We postpone the $SO(N)$ projection of states to later chapters and here only care about the dilatation operator itself.

They nicely combine with the first two terms and cancel the factor of $1/2$ in front. This way (4.83) becomes

$$\begin{aligned}
& -\frac{g_{YM}^2}{8\pi^2} \{ Tr(\check{W}[Z,W]BWCA) - Tr([Z,W]\check{W}BWCA) \} \\
& = -\frac{g_{YM}^2}{8\pi^2} \frac{1}{2} \left\{ Tr([Z,W]B) Tr(CA) - Tr([Z,W]B(CA)^T) \right. \\
& \quad \left. - Tr(B) Tr(CA[Z,W]) + Tr(B(CA[Z,W])^T) \right\}. \quad (4.86)
\end{aligned}$$

After the second equality sign, follows the result of contracting \check{W} . We can already recognize the familiar, first and the third terms. Leave them for a moment and rewrite the other two. Using the symmetry of the trace under transpositions, we can swap the $(^T)$ into B in the fourth term. Also, in accordance to what we did show in (4.84) or (4.85), the second term is

$$Tr\left(([Z,W]B)^T CA\right) = -Tr\left(B^T[Z,W]CA\right). \quad (4.87)$$

Summing up, we have

$$\begin{aligned}
& -\frac{g_{YM}^2}{8\pi^2} \frac{1}{2} \left\{ -Tr([W,Z]B) Tr(AC) - Tr(B) Tr(A[Z,W]C) \right. \\
& \quad \left. + Tr(A[Z,W]B^T C) + Tr(AB^T[Z,W]C) \right\}, \quad (4.88)
\end{aligned}$$

or into a more clear and elegant form

$$\begin{aligned}
\check{D}^{SO(N)} O^1 = & \frac{g_{YM}^2}{8\pi^2} \frac{1}{2} \left\{ Tr(B) Tr(A[Z,W]C) + Tr([ZBW]) Tr(AC) \right. \\
& \quad \left. - Tr(AB^T[Z,W]C) - Tr(A[Z,W]B^T C) \right\}. \quad (4.89)
\end{aligned}$$

Before analyzing it as we did for the $U(N)$ and $SU(N)$, let us derive the action on double trace operators. After the first contraction the result is

$$\begin{aligned}
\check{D}^{SO(N)} Tr(AZB) Tr(CWD) = & -\frac{g_{YM}^2}{8\pi^2} \frac{1}{2} \left\{ Tr(\check{W}[Z,W]BA) Tr(CWD) \right. \\
& - Tr(\check{W}[Z,W](BA)^T) Tr(CWD) - Tr([Z,W]\check{W}BA) Tr(CWD) \\
& \quad \left. + Tr([Z,W]\check{W}(BA)^T) Tr(CWD) \right\}. \quad (4.90)
\end{aligned}$$

With the experience from previous derivations we already know that the above traces with transpositions can be rewritten as

$$\begin{aligned} \text{Tr}(\check{W}[Z, W](BA)^T) &= \text{Tr}([Z, W]\check{W}BA), \\ \text{Tr}([Z, W]\check{W}(BA)^T) &= \text{Tr}(\check{W}[Z, W]BA), \end{aligned} \quad (4.91)$$

and the factor of $1/2$ is cancelled again to produce

$$\begin{aligned} &-\frac{g_{YM}^2}{8\pi^2}(\text{Tr}(\check{W}[Z, W]BA) - \text{Tr}([Z, W]\check{W}BA))\text{Tr}(CWD) \\ &= -\frac{g_{YM}^2}{8\pi^2}\frac{1}{2}\{\text{Tr}([Z, W]BADC) - \text{Tr}([Z, W]BA(DC)^T) \\ &\quad - \text{Tr}(BA[Z, W]DC) + \text{Tr}(BA[Z, W](DC)^T)\}. \end{aligned} \quad (4.92)$$

The last two lines are obviously obtained after contracting \check{W} . Now if we rewrite the second term as

$$\begin{aligned} \text{Tr}(([Z, W]BA)^TDC) &= \text{Tr}\left((ZWBA - WZBA)^TDC\right) \\ &= \text{Tr}\left((BA)^TWZ - (BA)^TZW\right)DC = -\text{Tr}((BA)^T[Z, W]DC), \end{aligned} \quad (4.93)$$

the result of acting with (4.46) on double trace states can be brought into this elegant form

$$\begin{aligned} \check{D}^{SO(N)}O^2 &= \frac{g_{YM}^2}{8\pi^2}\frac{1}{2}\{\text{Tr}(BA[Z, W]DC) + \text{Tr}([ZBADCW]) \\ &\quad - \text{Tr}((BA)^T[Z, W]DC) - \text{Tr}(BA[Z, W](DC)^T)\}. \end{aligned} \quad (4.94)$$

Now, planar contribution to the anomalous dimension of the $SO(N)$ states is obtained from (4.89) by setting $B_{ij} = \delta_{ij}$. As one gets $\text{Tr}(B) = N$ and other three terms in (4.89) vanish, the result that emerges is

$$H_0^{SO(N)}O^1 = \frac{1}{2}\frac{g_{YM}^2N}{8\pi^2}\text{Tr}(A[Z, W]C) = \frac{1}{2}H_0^{U(N)}O^1. \quad (4.95)$$

As far as only gauge theories are under consideration, this factor of $1/2$ is just a consequence of the completeness relations. Later, we will get some hints from the dual string theory that requires a "mirror" string in order for a classical solution to satisfy the orientifold projection. This might lead to a factor of two, but it still requires better understanding.

Another familiar structure is the $SO(N)$ split operator H_+^{SO} that can be determined directly from (4.89), with the condition that $B \neq \delta$. It is simply equal to the half of the $U(N)$ counterpart

$$H_+^{SO(N)} O^1 = \frac{1}{2} H_+^{U(N)} O^1. \quad (4.96)$$

Now a new structure. From (4.94) we see that the joining contribution for $SO(N)$, $H_-^{SO(N)}$, consists of two terms

$$H_-^{SO(N)} O^2 = \frac{1}{2} H_-^{U(N)} O^2 + H_-^{SO;f} O^2. \quad (4.97)$$

One is just a half of the unitary counterpart, $H_-^{U(N)}$, but the extra piece, $H_-^{SO;f}$, joins traces with an additional transposition. More precisely it acts as

$$H_-^{SO;f} O^2 = -\frac{1}{2} \frac{\lambda}{8\pi^2} (Tr ((BA)^T [Z, W] DC) + Tr (BA [Z, W] (DC)^T)). \quad (4.98)$$

Or on two sites, i and j , inside two different traces, it acts as a product of a "join-with-flip" operator $J^{SO;f}$

$$J^{SO;f} Tr (AZB) Tr (CWD) = Tr ((BA)^T ZWDC) + Tr (BAZW (DC)^T). \quad (4.99)$$

and the usual H^0

$$H_-^{SO;f} = -\frac{1}{2} H_{ij}^0 J_{ij}^{SO;f}. \quad (4.100)$$

This is a novel contribution that was discovered in [21].

Just for the future reference we write the action of $J^{SO;f}$ on a spin chain like operators

$$\begin{aligned} \check{J}_{i,j}^{SO} Tr (\phi_1 \dots \phi_{i-1} \phi_i \phi_{i+1} \dots \phi_p) Tr (\phi_{p+1} \dots \phi_{j-1} \phi_j \phi_{j+1} \dots \phi_L) \\ = Tr ((\phi_{i+1} \dots \phi_p \phi_1 \dots \phi_{i-1})^T \phi_i \phi_j \dots \phi_L \phi_{p+1} \dots \phi_{j-1}) \\ + Tr (\phi_{i+1} \dots \phi_p \phi_1 \dots \phi_i \phi_j (\phi_{j+1} \dots \phi_L \phi_{p+1} \dots \phi_{j-1})^T). \end{aligned} \quad (4.101)$$

Last but not least, we have another new contribution, $H_{flip}^{SO(N)}$. It does not change the number of traces inside the state that it acts on

$$H_{flip}^{SO(N)} Tr (AZBWC) = -\frac{1}{2} \frac{\lambda}{8\pi^2} (Tr (AB^T [Z, W] C) + Tr (A [Z, W] B^T C)). \quad (4.102)$$

We can identify a substructure that consists of a flip operator F^{SO} that cuts out the part of the operator between the sites, transposes it and gules back before and after the two sites

$$F^{SO} \text{Tr} (AZBWC) = \text{Tr} (AB^T ZWC) + \text{Tr} (AZWB^TC). \quad (4.103)$$

Then, "as usually" for our non-planar hamiltonians, it is followed by $H_{i,j}^0$

$$\left(H_{\text{flip}}^{SO(N)} \right)_{ij} = -\frac{1}{2} H_{ij}^0 F_{ij}^{SO}. \quad (4.104)$$

Also for the future reference we write F^{SO} acting on a spin chain like operator

$$\begin{aligned} \hat{F}_{i,j}^{SO} \text{Tr} (\phi_1 \dots \phi_{i-1} \phi_i \phi_{i+1} \dots \phi_{j-1} \phi_j \phi_{j+1} \dots \phi_L) \\ = \text{Tr} (\phi_1 \dots \phi_{i-1} (\phi_{i+1} \dots \phi_{j-1})^T \phi_i \phi_j \phi_{j+1} \dots \phi_L) \\ + \text{Tr} (\phi_1 \dots \phi_{i-1} \phi_i \phi_j (\phi_{i+1} \dots \phi_{j-1})^T \phi_{j+1} \dots \phi_L). \end{aligned} \quad (4.105)$$

Since the flip operator does not break the chain, we can use it to test integrability at the non-planar level by standard tools! Note that this has not been possible with $1/N$ corrections known up to date. Their action always involved interaction of two chains which did not allow for a known Bethe-like ansatz solution. More on this issues and tests of the possible integrable structure of H_{flip} will be discussed in later chapters.

Summarizing, the $SO(N)$ dilatation operator can be written as

$$\check{D}^{SO(N)} = H_0^{SO(N)} + \frac{1}{N} H_+^{SO(N)} + \frac{1}{N} H_-^{SO(N)} + \frac{1}{N} H_{\text{flip}}^{SO(N)}, \quad (4.106)$$

where the planar (nearest-neighbors) part is a half of the $U(N)$ contribution, joining and joining-with-flip parts act on two traces, the split part cuts a single chain into two pieces and finally the flip contribution is a non-local interaction within a single trace.

4.2.4 $Sp(N)$

We will see that the results for the symplectic gauge group are closely related to those of $SO(N)$. All the derivations are carefully presented below with a stress on the differences with respect to the orthogonal group.

The rules for appending the operator into a state built of symplectic matrices are based on the completeness relations

$$(T^a)_{\alpha\beta} (T^a)_{\nu\mu} = \frac{1}{2} (\delta_{\beta\nu} \delta_{\alpha\mu} - J_{\alpha\nu} J_{\beta\mu}), \quad (4.107)$$

where J is an antisymmetric, traceless matrix that squares to minus one

$$J^T = -J, \quad J^2 = -1. \quad (4.108)$$

In appendix A.4 we derived the two crucial contraction rules

$$Tr (\check{X} O X Y) = \frac{1}{2} (Tr (O) Tr (Y) - Tr (O J Y^T J)) \quad (4.109)$$

$$Tr (\check{X} O) Tr (X Y) = \frac{1}{2} (Tr (O Y) + Tr (O J Y^T J)). \quad (4.110)$$

Notice that the first one can be obtained from the $SO(N)$ counterpart (A.16) just by changing $Y^T \rightarrow J Y^T J$. However, if we want to get the second rule from (A.17), we have to change $Y^T \rightarrow -J Y^T J$. This relative minus sign will reappear below in the part of the dilatation operator that joins two traces with a simultaneous flip.

Let us now go through the action on a single trace state O^1 . Contracting \check{Z} yields

$$\begin{aligned} \check{D}^{Sp(N)} O^1 = & -\frac{g_{YM}^2}{8\pi^2} \frac{1}{2} \{ Tr (\check{W} [Z, W] B W C A) - Tr ([Z, W] \check{W} B W C A) \\ & + Tr (\check{W} [Z, W] J (B W C A)^T J) - Tr ([Z, W] \check{W} J (B W C A)^T J) \}. \end{aligned} \quad (4.111)$$

Due to (4.108), the two terms with transpositions can be rewritten as

$$\begin{aligned} Tr (\check{W} [Z, W] J (B W C A)^T J) &= Tr (J^T W^T Z^T \check{W}^T J^T B W C A) \\ -Tr (J^T Z^T W^T \check{W}^T J^T B W C A) &= Tr (J J W J J Z J J \check{W} J J B W C A) \\ -Tr (J J Z J J W J J \check{W} J J B W C A) &= -Tr ([Z, W] \check{W} B W C A) \end{aligned} \quad (4.112)$$

and similarly

$$\begin{aligned} Tr ([Z, W] \check{W} J (B W C A)^T J) &= Tr (J^T \check{W}^T W^T Z^T J^T B W C A) \\ -Tr (J^T \check{W}^T Z^T W^T J^T B W C A) &= Tr (J J \check{W} J J Z J J W J J B W C A) \\ -Tr (J J \check{W} J J Z J J W J J B W C A) &= -Tr (\check{W} [Z, W] B W C A). \end{aligned} \quad (4.113)$$

Therefore, as in the previous cases, they nicely combine with the other two terms to cancel the factor of $1/2$ in front

$$\begin{aligned} & -\frac{g_{YM}^2}{8\pi^2} \{ \text{Tr}(\check{W}[Z,W]BWC A) - \text{Tr}([Z,W]\check{W}BWC A) \} \\ & = -\frac{g_{YM}^2}{8\pi^2} \frac{1}{2} \{ \text{Tr}([Z,W]B) \text{Tr}(AC) - \text{Tr}(B) \text{Tr}(A[Z,W]C) \\ & \quad - \text{Tr}([Z,W]BJ(CA)^T J) + \text{Tr}(BJ(CA[Z,W])^T J) \}. \end{aligned} \quad (4.114)$$

The second equality sign is followed by the result of contracting \check{W} . Again we can juggle the transposition in order to obtain a formula structurally familiar from the orthogonal case. First rewrite

$$\begin{aligned} \text{Tr}([Z,W]BJ(CA)^T J) & = \text{Tr}((J[Z,W]BJ)^T CA) \\ & = \text{Tr}((JB^T W^T Z^T J - JB^T Z^T W^T J) CA) \\ & = \text{Tr}((JB^T JWJJZJJ - JB^T JZJJWJJ) CA) \\ & = -\text{Tr}(JB^T J[Z,W]CA), \end{aligned} \quad (4.115)$$

where in the second line we dropped the transpositions from J 's because they are always paired and the minus signs cancel. Then, we shuffle

$$\begin{aligned} \text{Tr}(BJ(CA[Z,W])^T J) & = \text{Tr}(JB^T JCA[Z,W]) \\ & = \text{Tr}(A[Z,W]JB^T JC). \end{aligned} \quad (4.116)$$

Finally the action of the $Sp(N)$ dilatation operator on a single trace operator is

$$\begin{aligned} \check{D}^{Sp(N)} O^1 & = \frac{g_{YM}^2}{8\pi^2} \frac{1}{2} \{ \text{Tr}(B) \text{Tr}(A[Z,W]C) + \text{Tr}([ZBW]) \text{Tr}(AC) \\ & \quad - \text{Tr}(AJB^T J[Z,W]C) - \text{Tr}(A[Z,W]JB^T JC) \}. \end{aligned} \quad (4.117)$$

This can be linked with the orthogonal result by a map $B^T \leftrightarrow JB^T J$. The same way, on an arbitrary double trace state, the dilatation operator acts as

$$\begin{aligned} \check{D}^{Sp(N)} O^2 & = -\frac{g_{YM}^2}{8\pi^2} \frac{1}{2} \{ \text{Tr}(\check{W}[Z,W]BA) \text{Tr}(CWD) \\ & \quad + \text{Tr}(\check{W}[Z,W]J(BA)^T J) \text{Tr}(CWD) - \text{Tr}([Z,W]\check{W}BA) \text{Tr}(CWD) \\ & \quad - \text{Tr}([Z,W]\check{W}J(BA)^T J) \text{Tr}(CWD) \}. \end{aligned} \quad (4.118)$$

As before, the transposition can be shifted from BA into J 's and single symplectic matrices. We rewrite

$$Tr(\check{W}[Z, W]J(BA)^T J) = -Tr([Z, W]\check{W}BA), \quad (4.119)$$

and

$$Tr([Z, W]\check{W}J(BA)^T J) = -Tr(\check{W}[Z, W]BA), \quad (4.120)$$

so that $1/2$ cancels again and, after the first contraction, the result is

$$-\frac{g_{YM}^2}{8\pi^2} (Tr(\check{W}[Z, W]BA) - Tr([Z, W]\check{W}BA)) Tr(CWD). \quad (4.121)$$

At this point, a crucial sign difference with $SO(N)$ pops up. Namely, in order to contract \check{W} we use rule (4.110), with the relative minus sign with respect to the orthogonal (A.17). Then when contracting within each term we do not change the sign in front of the term with transpositions. What we mean is

$$\begin{aligned} & \frac{g_{YM}^2}{8\pi^2} \frac{1}{2} \{ Tr(BA[Z, W]DC) + Tr(\lfloor ZBADCW \rfloor) \\ & - Tr([Z, W]BAJ(DC)^T J) + Tr(BA[Z, W]J(DC)^T J) \}. \end{aligned} \quad (4.122)$$

The last term already has the required form and we only rewrite the third term so that it becomes

$$\begin{aligned} & Tr([Z, W]BAJ(DC)^T J) = Tr((J[Z, W]BAJ)^T DC) = \\ & Tr(J(BA)^T (W^T Z^T - Z^T W^T) JDC) = Tr(J(BA)^T JWJJZJJDC) \\ & - Tr(J(BA)^T JZJJWJJDC) = -Tr(J(BA)^T J[Z, W]DC). \end{aligned} \quad (4.123)$$

The action of \check{D} on O^2 is then

$$\begin{aligned} \check{D}^{Sp(N)} O^{2,p} = & \frac{g_{YM}^2}{8\pi^2} \frac{1}{2} \{ Tr(BA[Z, W]DC) + Tr(\lfloor ZBADCW \rfloor) \\ & + Tr(J(BA)^T J[Z, W]DC) + Tr(BA[Z, W]J(DC)^T J) \}. \end{aligned} \quad (4.124)$$

With (4.117) and (4.124) at hand we can analyze the general structure of the $Sp(N)$ dilatation operator.

Planar part is the same as for $SO(N)$, equal to the half of the $U(N)$ contribution

$$H_0^{Sp(N)} = \frac{1}{2} H_0^{U(N)}. \quad (4.125)$$

Similarly, operator H_+ , that increases the number of traces, is equal to the half of its $U(N)$ counterpart

$$H_+^{Sp(N)} = \frac{1}{2} H_+^{U(N)}. \quad (4.126)$$

Also H_- is a sum of a half of the $U(N)$ "join" operator and a "join-with-flip"

$$H_-^{Sp(N)} = \frac{1}{2} H_-^{U(N)} + H_-^{Sp;f}. \quad (4.127)$$

The new $Sp(N)$ correction, acting on two sites within two different traces, joins them by first transposing the part from the first trace and inserting J before and after and then does the same with the part in the second trace

$$H_-^{Sp;f} O^{2,p} = \frac{\lambda}{8\pi^2} \frac{1}{2} (Tr (J(BA)^T J[Z, W] DC) + Tr (BA[Z, W] J(DC)^T J)). \quad (4.128)$$

Note that the there is a sign difference with $SO(N)$, so the map would have to be $J(X)^T J \leftrightarrow -(X)^T$.

Finally the $Sp(N)$ flip acts within a single trace as

$$H_{flip}^{Sp(N)} O^1 = -\frac{\lambda}{8\pi^2} \frac{1}{2} (Tr (AJB^T J[Z, W] C) - Tr (A[Z, W] JB^T JC)). \quad (4.129)$$

Both, "flip" and "join-with-flip" $Sp(N)$ operators have the generic substructures of the non-planar contributions. The flip operator acting on sites i and j is just given by

$$\left(H_{flip}^{Sp(N)} \right)_{ij} = -\frac{1}{2} H_{ij}^0 F_{ij}^{Sp}, \quad (4.130)$$

where F^{Sp} cuts out the part of the operator between the sites that it acts on, transposes it and gules back in between two J 's before and after the two sites

$$F^{Sp} Tr (AZBWC) = Tr (AJB^T JZWC) + Tr (AZWJB^T JC). \quad (4.131)$$

Similarly the join-with-flip on i and j is

$$\left(H_-^{Sp;f} \right)_{i,j} = \frac{1}{2} H_{ij}^0 J_{ij}^{Sp;f}, \quad (4.132)$$

where

$$J^{Sp,f} \text{Tr} (AZB) \text{Tr} (CWD) = \text{Tr} (J(BA)^T JZWDC) + \text{Tr} (BAZWJ(DC)^T J). \quad (4.133)$$

To see their action in more detail we write it on a spin-chain like operators

$$\begin{aligned} & \hat{F}_{i,j}^{Sp} \text{Tr} (\phi_1 \dots \phi_{i-1} \phi_i \phi_{i+1} \dots \phi_{j-1} \phi_j \phi_{j+1} \dots \phi_L) \\ &= \text{Tr} (\phi_1 \dots \phi_{i-1} J(\phi_{i+1} \dots \phi_{j-1})^T J \phi_i \phi_j \phi_{j+1} \dots \phi_L) \\ &+ \text{Tr} (\phi_1 \dots \phi_{i-1} \phi_i \phi_j J(\phi_{i+1} \dots \phi_{j-1})^T J \phi_{j+1} \dots \phi_L), \end{aligned} \quad (4.134)$$

and

$$\begin{aligned} & \hat{J}_{i,j}^{Sp} \text{Tr} (\phi_1 \dots \phi_{i-1} \phi_i \phi_{i+1} \dots \phi_p) \text{Tr} (\phi_{p+1} \dots \phi_{j-1} \phi_j \phi_{j+1} \dots \phi_L) \\ &= \text{Tr} (J(\phi_{i+1} \dots \phi_p \phi_1 \dots \phi_{i-1})^T J \phi_i \phi_j \dots \phi_L \phi_{p+1} \dots \phi_{j-1}) \\ &+ \text{Tr} (\phi_{i+1} \dots \phi_p \phi_1 \dots \phi_i \phi_j J(\phi_{j+1} \dots \phi_L \phi_{p+1} \dots \phi_{j-1})^T J). \end{aligned} \quad (4.135)$$

Summarizing the symplectic gauge group dilatation operator can be written as

$$\check{D}^{Sp(N)} = H_0^{Sp(N)} + \frac{1}{N} H_+^{Sp(N)} + \frac{1}{N} H_-^{Sp(N)} + \frac{1}{N} H_{flip}^{Sp(N)}. \quad (4.136)$$

4.3 Summary

Here we once again give a summary of the one-loop formulas for the dilatation operator. We assume that a single trace states have length L and double trace

\check{D}	$U(N)$	$SU(N)$	$SO(N)$	$Sp(N)$
<i>Planar</i>	H_0	H_0	$\frac{1}{2}H_0$	$\frac{1}{2}H_0$
<i>Split</i>	H_+	H_+	$\frac{1}{2}H_+$	$\frac{1}{2}H_+$
<i>Join</i>	H_-	H_-	$\frac{1}{2}H_-$	$\frac{1}{2}H_-$
<i>Flip</i>	\times	\times	H_{flip}^{SO}	H_{flip}^{Sp}
<i>Join + Flip</i>	\times	\times	$H_-^{SO;f}$	$H_-^{Sp;f}$

Table 4.1: Planar and Non-planar contributions to the one-loop dilatation operator

states, p fields on the first trace and $L-p$ on the second. Planar contributions are obtained from the nearest neighbor Heisenberg hamiltonian

$$H_0 = \frac{\lambda}{8\pi^2} \sum_i^L (1_{i,i+1} - P_{i,i+1}), \quad (4.137)$$

and $i = L + 1 = 1$.

Cutting operator H_+ that acts within the same trace is given by

$$H_+ = \frac{\lambda}{8\pi^2} \sum_{1 \leq i < j \leq L} (1_{ij} - P_{ij}) S_{ij} \equiv \sum_{1 \leq i < j \leq L} H_{ij}^0 S_{ij}. \quad (4.138)$$

Join operator acts within two traces by

$$H_- = \frac{\lambda}{8\pi^2} \sum_{i=1}^p \sum_{j=p+1}^L (1_{ij} - P_{ij}) J_{ij} \equiv \sum_{i=1}^p \sum_{j=p+1}^L H_{ij}^0 J_{ij}. \quad (4.139)$$

Join with flip operators for orthogonal and symplectic gauge groups are

$$H_-^{SO;f} = -\frac{1}{2} \frac{\lambda}{8\pi^2} \sum_{i=1}^p \sum_{j=p+1}^L (1_{ij} - P_{ij}) J_{ij}^{SO;f} = -\frac{1}{2} \sum_{i=1}^p \sum_{j=p+1}^L H_{ij}^0 J_{ij}^{SO;f} \quad (4.140)$$

$$H_-^{Sp;f} = \frac{1}{2} \frac{\lambda}{8\pi^2} \sum_{i=1}^p \sum_{j=p+1}^L (1_{ij} - P_{ij}) J_{ij}^{Sp;f} = \frac{1}{2} \sum_{i=1}^p \sum_{j=p+1}^L H_{ij}^0 J_{ij}^{Sp;f} \quad (4.141)$$

Orthogonal and symplectic flip operators are

$$H_{flip}^{SO(N)} = \frac{1}{2} \frac{\lambda}{8\pi^2} \sum_{1 \leq i < j \leq L}^L (1_{ij} - P_{ij}) F_{ij}^{SO} \equiv \frac{1}{2} \sum_{1 \leq i < j \leq L}^L H_{ij}^0 F_{ij}^{SO} \quad (4.142)$$

$$H_{flip}^{Sp(N)} = \frac{1}{2} \frac{\lambda}{8\pi^2} \sum_{1 \leq i < j \leq L}^L (1_{ij} - P_{ij}) F_{ij}^{Sp} \equiv \frac{1}{2} \sum_{1 \leq i < j \leq L}^L H_{ij}^0 F_{ij}^{Sp}. \quad (4.143)$$

The one-loop non-planar corrections: cut, join, join+flip and flip act on

$$\begin{aligned} O^1 &= \text{Tr}(AZBWC), \\ O^2 &= \text{Tr}(AZB) \text{Tr}(CWD), \end{aligned} \quad (4.144)$$

as

$$\check{S}O^1 = \text{Tr}(B) \text{Tr}(AZWC) + \text{Tr}(AC) \text{Tr}(ZBW) \quad (4.145)$$

$$\check{J}O^2 = \text{Tr}(BAZWDC) + \text{Tr}(ZBADCW) \quad (4.146)$$

$$\check{J}^{SO;f} O^2 = \text{Tr}((BA)^T ZWDC) + \text{Tr}(BAZW(DC)^T) \quad (4.147)$$

$$\check{J}^{Sp;f} O^2 = \text{Tr}(J(BA)^T JZWDC) + \text{Tr}(BAZWJ(DC)^T J) \quad (4.148)$$

$$\check{F}^{SO} O^1 = \text{Tr}(AB^T ZWC) + \text{Tr}(AZWB^T C) \quad (4.149)$$

$$\check{F}^{Sp} O^1 = \text{Tr}(AJB^T JZW) + \text{Tr}(AZWJB^T JC). \quad (4.150)$$

Part II

Main Results

Chapter 5

Spectral problem for Orthogonal and Symplectic groups

5.1 Summary

By now, we have a solid understanding of the spectral problem of AdS_5/CFT_4 at the planar limit. It is believed (and has been checked on several examples) that in this limit, a set of equations called Y-system [28] allows one to obtain the anomalous dimension of an arbitrary single trace operator at any value of the 't Hooft coupling λ . The key to this solution is the integrable structure of the planar $\mathcal{N} = 4$ SYM.

Once non-planar ($\geq \frac{1}{N}$) corrections to the dilatation generator are taken into account, the situation complicates. First of all is not fully understood if integrability can help to solve this problem as well. The paradigm seems to be that non-planar corrections “break integrability”. It comes from an observation about explicit diagonalization of the full dilatation generator in a basis of short operators. There, one can notice that planar parity pairs (two states with the same planar energy but opposite parity) are lifted by non-planar corrections. As pointed out in [23] these states with opposite parity can be mapped to each other with the third conserved charge Q_3 . Therefore, their absence in the full spectrum is usually considered as a hint that with $1/N$ corrections included higher charges are absent, so is integrability. However in a series of papers [96, 97, 98] authors showed that there exists

special basis in which some non-planar corrections can be mapped into integrable model.

The second important fact about non-planar corrections is their non-local action on the spin-chain. More precisely, subleading corrections to the dilatation generator in $\mathcal{N}=4$ SYM with $U(N)$ gauge group, lead to mixing of operators with different number of traces. In the spin-chain language they split or join chains. These are not common phenomena in one-dimensional exactly solvable systems so we cannot use any existing technology like in the case of planar limit described by Heisenberg's hamiltonian.

These two arguments motivated us to consider $\mathcal{N}=4$ SYM with orthogonal and symplectic gauge groups. Their construction, similarly to $U(N)$, is based on Chan-Paton factors of open strings that end on N D3 branes (see B). In order to share our excitement we encourage the reader to look at 't Hooft double line diagrams on Fig. 5.1. The first and third are standard planar and non-planar diagram in 't Hooft expansion for fields in the adjoint representation of the $U(N)$. On the contrary, the middle diagram is obtained by contracting fields in the adjoint representation of the orthogonal or symplectic group (see A). First, notice that for diagram in the middle, with the propagator with lines that cross each other, it is impossible to consistently draw arrows (orient the diagram) on each line that point in opposite directions. These type of diagrams is called non-orientable and correspond non-orientable string worldsheets.

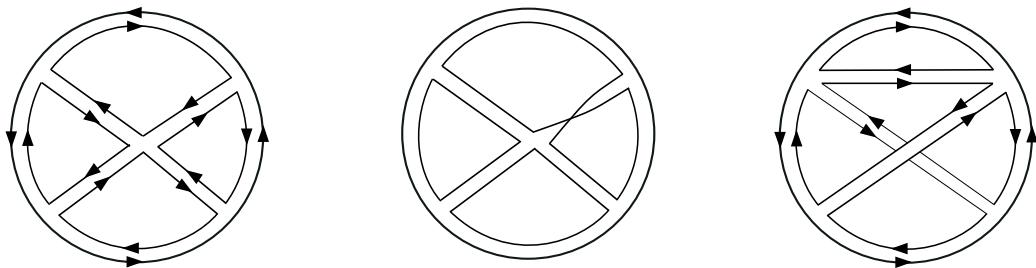


Figure 5.1: Planar and non-planar diagrams. The most right and most left are in $U(N)$ gauge theory whereas the middle comes from $SO(N)$ (or $Sp(N)$) contractions.

Now let us associate with each diagram a weight

$$N^L(g_{YM}^2)^{P-V}, \quad (5.1)$$

where L stands for the number of closed loops, V for the number of vertices and P for the number of propagators. For the $U(N)$ diagrams we have

$$(a) \quad N^5(g_{YM}^2)^{8-5} = N^2\lambda^3, \quad (5.2)$$

$$(c) \quad N^3(g_{YM}^2)^{7-4} = \lambda^3, \quad (5.3)$$

whereas for the middle diagram

$$(b) \quad N^4(g_{YM}^2)^{8-5} = N\lambda^3. \quad (5.4)$$

As we can see, at order λ^3 in $U(N)$ theory, we have the leading planar contribution (a) the first subleading correction (b) at order N^{-2} . A novelty for orthogonal and symplectic gauge groups is the correction of order N^{-1} . Naturally when we take $N \rightarrow \infty$ the topology of the leading diagrams is the same for all gauge groups.

It is not obvious from this simple example, but on the level of the dilatation operator these new corrections do not lead to splitting or joining of traces. Hence there is a hope and a possibility to search for integrability in a usual manner (e.g. find some generalized Bethe Ansatz that would diagonalize these corrections).

Last but not least, matrices in the adjoint representation of $SO(N)$ (or $Sp(N)$) are antisymmetric and for a given length L of the gauge invariant operator O_L , we can only have states of one parity. This can be seen from the fundamental property of trace, that for every matrix A

$$\text{Tr } A^T = \text{Tr } A. \quad (5.5)$$

Then e.g. for a single trace operator¹ composed of antisymmetric matrices $X_i^T = -X_i$, we have

$$O_L = \text{Tr } (X_1 \dots X_L)^T = (-1)^L \text{Tr } (X_L \dots X_1) \equiv (-1)^L \check{P} O_L, \quad (5.6)$$

so for L even we can only have operators with positive parity whereas for L odd, only the negative parity is allowed.

¹the same argument holds for multi trace operators

In other words, the argument with planar parity pairs loses its meaning but we still know that planar theory is integrable. Considering orthogonal and symplectic groups gives us then more general perspective on what are the crucial features of an integrable theory and a possibility to understand how (if) integrability breaks at the non-planar level.

With these motivations we embarked on analyzing the “flip” $1/N$ corrections to the planar dilatation in $\mathcal{N}=4$ SYM with orthogonal and symplectic gauge groups.

First, based on the experience with constructing conserved charges by boost operator we tried to guess the correction to Q_4^2 , such that it would commute with the hamiltonian at each order in $1/N$. More precisely, we assumed that if integrability is present at $1/N$, the corrected operators

$$\hat{H} = \hat{H}_0 + \frac{1}{N} \hat{H}_{flip}, \quad \hat{Q}_4 = \hat{Q}_4^{(0)} + \frac{1}{N} \hat{Q}_4^{(1)}, \quad (5.7)$$

will also commute at $1/N$. This gave the following constraint

$$[\hat{H}_0, \hat{Q}_4^{(1)}] + [\hat{H}_{flip}, \hat{Q}_4^{(0)}] = 0. \quad (5.8)$$

We then tried several (semi-constructive) guesses, however none of them satisfied this relation³. It remains unclear if our failure was due to breaking of integrability or our lack of a more systematic approach to the construction Q_4^1 .

Then we have managed to find an analytic formula for the correction to the energy of BMN states from H_{flip} . As reader remembers these are the states with two magnons that diagonalize planar XXX hamiltonian. They can be written in a compact form as

$$|n\rangle \equiv \mathcal{O}_n^J = \frac{1}{J+1} \sum_{p=0}^J \cos\left(\frac{\pi n(2p+1)}{J+1}\right) \mathcal{O}_p^J, \quad 0 \leq n \leq \frac{J}{2}. \quad (5.9)$$

²Notice that since we only have states of one parity for a given length, Q_3 loses its meaning.

³we used computer algebra to check this constraint.

The first correction to the energy is then given by

$$\begin{aligned}
E_1^n &= \langle n | \hat{H}_{flip} | n \rangle \\
&= -\frac{2}{J+1} \tan^2 \left(\frac{\pi n}{J+1} \right) - \frac{1}{J+1} \sin^2 \left(\frac{\pi n}{J+1} \right) \left(J - \frac{1}{\cos \left(\frac{2\pi n}{J+1} \right)} \right).
\end{aligned} \tag{5.10}$$

Our prediction is that, on the string theory side, this formula should correspond to the energy of two excitations propagating on the string's worldsheet with a crosscap. So far this has not been confirmed by an explicit computation and we leave it for a future project.

With (5.10) at hand we tried to find a modification of Bethe Ansatz in the spirit of perturbative ABA in $1/N$ that would reproduce our correction. As we discussed, there are several ways to modify the ansatz and we tried each of them. Unfortunately all the modifications lead to contradictory conditions. Below we present the article in the published version. It contains more details about the key steps that we just summarized and some additional comments on the string theory side of the AdS/CFT with orthogonal and symplectic gauge groups.

On the spectral problem of $\mathcal{N} = 4$ SYM with orthogonal or symplectic gauge group

Pawel Caputa^a, Charlotte Kristjansen^b and Konstantinos Zoubos^b

^aThe Niels Bohr International Academy,
The Niels Bohr Institute, Copenhagen University
Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

^b The Niels Bohr Institute, Copenhagen University
Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

`caputa@nbi.dk, kristjan@nbi.dk, kzoubos@nbi.dk`

Abstract

We study the spectral problem of $\mathcal{N} = 4$ SYM with gauge group $SO(N)$ and $Sp(N)$. At the planar level, the difference to the case of gauge group $SU(N)$ is only due to certain states being projected out, however at the non-planar level novel effects appear: While $\frac{1}{N}$ -corrections in the $SU(N)$ case are always associated with splitting and joining of spin chains, this is not so for $SO(N)$ and $Sp(N)$. Here the leading $\frac{1}{N}$ -corrections, which are due to non-orientable Feynman diagrams in the field theory, originate from a term in the dilatation operator which acts inside a single spin chain. This makes it possible to test for integrability of the leading $\frac{1}{N}$ -corrections by standard (Bethe ansatz) means and we carry out various such tests. For orthogonal and symplectic gauge group the dual string theory lives on the orientifold $AdS_5 \times \mathbb{R}P^5$. We discuss various issues related to semi-classical strings on this background.

5.2 Introduction

Whereas the planar spectral problem of $\mathcal{N} = 4$ SYM seems to be close to resolution [18, 23, 107, 25, 26, 27, 28, 29, 30], much less has been achieved in the non-planar case. Non-planar corrections, when studied perturbatively in $\frac{1}{N}$, lead to a breakdown of the spin chain picture which was the key to the progress at the planar level. More precisely, $\frac{1}{N}$ -corrections to the dilatation generator lead to interactions which split and join spin chains [43]. This enormously enlarges the Hilbert space of states and, furthermore, implies that excitations on different chains can interact, rendering the standard tools of integrable spin chains inapplicable and leaving little hope for the existence of a Bethe ansatz in the usual sense.⁴

In order to gain further insight into $\frac{1}{N}$ -corrections we will study $\mathcal{N} = 4$ SYM with gauge groups $SO(N)$ and $Sp(N)$. At the planar level, the only essential difference of these theories from the traditionally studied $SU(N)$ case is that certain states are projected out. However, at the non-planar level new effects arise. Namely, for orthogonal and symplectic gauge group the leading non-planar corrections originate from non-orientable Feynman diagrams with a single cross-cap [32]. At the level of the dilatation generator these leading non-planar corrections are described by an operator which acts entirely inside a *single* spin chain. This implies that restricting oneself to the leading $\frac{1}{N}$ -corrections one does not face the problems mentioned above. The Hilbert space of states remains the same as on the planar level and all interactions take place inside a single spin chain. Thus the existence of a usual Bethe ansatz is not a priori excluded and one may test for integrability using standard methods.

In the AdS/CFT correspondence, changing the gauge group on the field theory side translates into a modification of the background geometry on the string theory side. For orthogonal and symplectic gauge groups the relevant geometry becomes that of the orientifold $AdS_5 \times \mathbb{R}P^5$ where the case of $Sp(N)$ differs from that of $SO(N)$ by the presence of an additional B -field [36]. In the case of $\mathcal{N} = 4$ SYM with gauge group $SU(N)$ the leading non-planar effects on the string theory side have their origin in string diagrams of genus one but in the case of orthogonal and symplectic gauge groups the leading non-planar corrections should be associated with non-orientable string worldsheets with a single cross-cap. At least naively, it

⁴The situation is the same in the three-dimensional ABJM and ABJ theories [83, 22].

seems easier to deal with cross-caps than higher genus surfaces so our study might open new avenues for comparison of gauge and string theory beyond the planar limit.

Our main focus will be on the gauge theory side where we will study in depth the one-loop dilatation generator. We start in section 5.3 by explaining the reduction of the space of states compared to the theory with gauge group $SU(N)$ and subsequently write down the one-loop dilatation generator including all non-planar corrections. In section 5.5 we determine analytically the leading $\frac{1}{N}$ -correction to the anomalous dimension of two-excitation states, thereby providing a prediction for the dual string theory. After that, in section 5.6, we search for integrability in the non-planar spectrum in various ways. We look for unexpected degeneracies and for conserved charges. In addition, we put forward various possible modifications of the planar Bethe equations which would produce the correct $\frac{1}{N}$ -correction for two-excitation states and test numerically if these equations also work for higher numbers of excitations. Unfortunately, the outcome of these tests is negative. In section 5.7, we discuss the dual string theory picture and, in particular, mention a number of interesting open problems. Finally, section 6.7 contains our conclusion.

5.3 $\mathcal{N} = 4$ SYM with gauge group $SO(N)$

In this section we will study non-planar effects in the spectrum of $\mathcal{N} = 4$ SYM with gauge group $SO(N)$. Before doing so, it is useful to briefly recall how this theory arises as a suitable projection of the $SU(N)$ theory. As is well known, in string theory the latter is constructed by taking the low-energy limit of a stack of N D3-branes in ten-dimensional Minkowski space. The group $SU(N)$ arises because the matrices λ_j^i encoding the Chan-Paton factors of the open strings stretching between the D3-branes are hermitian.

In order to obtain an orthogonal gauge group, one performs an orientifold projection which, on bosonic states, amounts to relating the Chan-Paton matrices to their transpose matrices as [55]

$$\lambda = -\eta^{-1} \lambda^T \eta \quad (5.11)$$

where η is a symmetric matrix which can simply be taken to be unity. The Chan-Paton matrices are thus restricted to be antisymmetric $N \times N$ matrices, which generate the adjoint representation of the group $SO(N)$. As

explained in [36], in order to ensure that this procedure does not break $\mathcal{N} = 4$ supersymmetry one has to combine it with a spacetime identification of the six transverse to the brane coordinates X^i as $X^i \rightarrow -X^i$. This procedure leaves us with $\mathcal{N} = 4$ SYM with gauge group $SO(N)$.

We will restrict ourselves to considering the $SU(2)$ sub-sector of the theory, consisting of multi-trace operators built from two complex fields, say ϕ and Z , i.e. operators of the form

$$\mathcal{O} = \text{Tr}(Z \dots Z \phi \dots \phi Z \dots) \text{Tr}(Z \dots Z \phi \dots \phi Z \dots) \dots \quad (5.12)$$

The adjoint fields Z and ϕ , being elements of the algebra of $SO(N)$, fulfill

$$\phi^T = -\phi, \quad Z^T = -Z. \quad (5.13)$$

The dilatation generator of the $SU(2)$ sub-sector at one and two-loops can formally be written in the same way as for the $SU(N)$ case [23]. At one loop order it reads⁵

$$\hat{D} = -\frac{g_{\text{YM}}^2}{8\pi^2} \text{Tr}[\phi, Z][\check{\phi}, \check{Z}] \equiv \frac{g_{\text{YM}}^2}{8\pi^2} \hat{H}. \quad (5.14)$$

Here \check{Z} is an operator which acts on a field Z by contraction of $SO(N)$ indices, i.e.

$$\check{Z}_{\alpha\beta} Z_{\gamma\epsilon} = \frac{1}{2}(\delta_{\alpha\epsilon}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\epsilon}), \quad (5.15)$$

and similarly for $\check{\phi}$.

In the analysis of $\mathcal{N} = 4$ SYM with gauge group $SU(N)$ the concept of parity played a central role. In a spin chain context, parity is the operation which inverts the order of operators inside a given trace, i.e. [75]

$$\hat{P} \text{Tr}(X_{i_1} X_{i_2} \dots X_{i_L}) = \text{Tr}(X_{i_L} X_{i_{L-1}} \dots X_{i_1}). \quad (5.16)$$

Parity commutes with \hat{H} which means that eigenstates of \hat{H} can be chosen to be states with definite parity. (The same is the case for ABJM theory, whereas for ABJ theory parity is broken at the non-planar level [83, 22].) In general, for $\mathcal{N} = 4$ SYM with gauge group $SU(N)$, for a given length L the spectrum will then contain operators of positive as well as negative

⁵We chose to keep the normalization of generators $\text{Tr}T^a T^b = \delta^{ab}$ when passing from $SU(N)$ to $SO(N)$.

parity. However, since the group generators for gauge group $SO(N)$ are antisymmetric, a state is related to its parity conjugate in the following way:

$$\hat{P}\text{Tr}(X_{i_1}X_{i_2}\dots X_{i_L}) = (-1)^L \text{Tr}(X_{i_1}X_{i_2}\dots X_{i_L}). \quad (5.17)$$

In other words, parity has been gauged. We thus see that, compared to the case of $SU(N)$, the $SO(N)$ theory has a lot fewer states: For even length only positive parity states survive whereas for odd length only negative parity states survive. When acting on operators of the type (6.10), the one-loop dilatation generator \hat{H} can be usefully decomposed as

$$\hat{H} = N \hat{H}_0 + \hat{H}_+ + \hat{H}_- + \hat{H}_{flip}. \quad (5.18)$$

Here \hat{H}_0 is the planar part which, up to a factor of two, is the same as for $SU(N)$, i.e.⁶

$$\hat{H}_0^{SO(N)} \equiv \hat{H}_0 = \frac{1}{2} \sum_{i=1}^L (1 - P_{i,i+1}) = \frac{1}{2} \hat{H}_0^{SU(N)}. \quad (5.19)$$

In particular, this means that the information about the planar anomalous dimensions in the case of gauge group $SO(N)$ is encoded in the same Heisenberg spin chain Bethe equations as for $SU(N)$. However, due to the fact that certain states are projected out, some of the other information encoded in these equations becomes redundant.

For single trace operators consisting of M fields of type ϕ and $(L - M)$ fields of type Z , where $M \leq L/2$, the Bethe equations are expressed in terms of M rapidities $\{u_k\}_{k=1}^M$ and read

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} \right)^L = \prod_{j=1, j \neq k}^M \frac{u_k - u_j + i}{u_k - u_j - i}. \quad (5.20)$$

The rapidity u is related to the momentum p via

$$u = \frac{1}{2} \cot \left(\frac{p}{2} \right), \quad (5.21)$$

and the eigenvalues of \hat{H}_0 are given by

$$E_0 = \frac{1}{2} \sum_{k=1}^M \frac{1}{u_k^2 + \frac{1}{4}} = 2 \sum_{k=1}^M \sin^2 \left(\frac{p_k}{2} \right). \quad (5.22)$$

⁶The relative factor of $\frac{1}{2}$ in the hamiltonian arises because of our normalisation of the gauge group generators.

The momenta have to satisfy the condition

$$\sum_k p_k = 0, \quad (5.23)$$

which reflects the cyclicity of the trace. The Bethe equations, the cyclicity constraint and the expression for the energy are all invariant under $u_k \rightarrow -u_k$. This implies that for any solution, $\{u_k\}$, either $\{-u_k\} = \{u_k\}$ or $\{-u_k\}$ is a partner solution of the same energy. Following [46, 47], we will refer to the first type of solutions as unpaired solutions and the second type as paired. In $SU(N)$ terminology, the two solutions in a pair are each other's parity conjugates. The values of the higher conserved charges for the two states are identical for even charges and differ by a sign for odd charges. Unpaired states have vanishing odd charges. Considering gauge group $SO(N)$ instead of $SU(N)$, the two states in a pair get identified via eqn. (5.17) and the odd charges lose their meaning. An unpaired state survives the projection if it has parity $(-1)^L$ where L is its length. The reduction procedure is hence clear on the level of solutions. It would be neat, however, if it could be formulated at the level of the Bethe equations.⁷

At the non-planar level the dilatation operator contains the three terms \hat{H}_+ , \hat{H}_- and \hat{H}_{flip} . The operators \hat{H}_+ and \hat{H}_- respectively increase and decrease the trace number by one and have analogues in the case of $SU(N)$. The operator \hat{H}_{flip} is trace conserving and does not have any analogue in the case of $SU(N)$. In the language of string theory the operators \hat{H}_+ and \hat{H}_- correspond to string splitting and joining whereas \hat{H}_{flip} corresponds to the insertion of a cross-cap on the string worldsheet. It is well-known that for gauge theories with orthogonal or symplectic gauge group the topological expansion includes Feynman diagrams which correspond to non-orientable surfaces, i.e. surfaces with cross-caps [32]. Each occurrence of a cross-cap is associated with a factor of $\frac{1}{N}$ whereas a handle as usual gives rise to a factor of $\frac{1}{N^2}$, see Fig. 5.2. Acting with \hat{H}_{flip} on a single trace operator gives

⁷One can show that the surviving unpaired states always have L and M even [56]. For these states, one can hence directly see that the Bethe equations will take a form like

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} \right)^{L-1} = \prod_{j=1, j \neq k}^{M/2} \frac{u_k - u_j + i}{u_k - u_j - i} \frac{u_k + u_j + i}{u_k + u_j - i}$$

which is similar to the (not completely unrelated) case of open strings [58, 59, 63, 60].

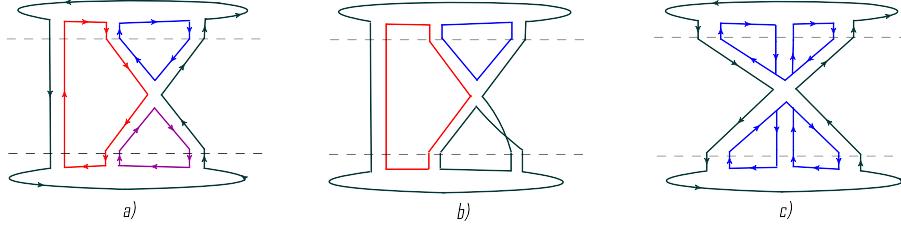


Figure 5.2: A planar Feynman diagram (a), a non-orientable diagram with a single cross-cap (b) and a diagram of genus one (c).

a contribution for each pair of fields of type ϕ, Z that the operator contains. This contribution is most conveniently described in the following way

$$\hat{H}_{flip} \text{Tr}(\phi X Z Y) = \frac{1}{2} \text{Tr}(X^T Y [Z, \phi]) + \frac{1}{2} \text{Tr}(Y X^T [Z, \phi]). \quad (5.24)$$

Here X and Y are arbitrary operators, and it is understood that the \check{Z} and $\check{\phi}$ in \hat{H}_{flip} are contracted with the explicitly written Z and ϕ in $\text{Tr}(\phi X Z Y)$. The operator \hat{H}_{flip} hence cuts out a piece of the operator and reinserts it with the opposite orientation. Since this piece can be of arbitrary length, we see that all sites in the chain are involved in the interaction. So, although \hat{H}_{flip} takes single-trace operators to single-trace operators, and can thus be interpreted as a spin-chain interaction, in contrast with the planar part of the dilatation operator its action on the spin chain is highly non-local.

Up to a factor of 2, the operator \hat{H}_+ takes the same form for $SU(N)$ and $SO(N)$ whereas the operator \hat{H}_- has extra terms for $SO(N)$. More precisely

$$\hat{H}_+^{SO(N)} = \frac{1}{2} \hat{H}_+^{SU(N)} \quad (5.25)$$

$$\begin{aligned} \hat{H}_-^{SO(N)} \text{Tr}(\phi X) \text{Tr}(ZY) &= \frac{1}{2} \hat{H}_-^{SU(N)} \text{Tr}(\phi X) \text{Tr}(ZY) \\ &\quad + \frac{1}{2} \text{Tr}(X^T Y [\phi, Z]) + \frac{1}{2} \text{Tr}(Y X^T [Z, \phi]), \end{aligned} \quad (5.26)$$

where the notation is as above and where $\hat{H}_\pm^{SU(N)}$ can be found in [43]. The extra terms in $\hat{H}_-^{SO(N)}$ are natural since for non-orientable surfaces there are two possible ways of gluing objects together. We notice that in a basis of planar eigenstates the perturbations \hat{H}_+ and \hat{H}_- are always off-diagonal. Only

\hat{H}_{flip} can have diagonal matrix elements in such a basis. Treating the energy corrections perturbatively in $\frac{1}{N}$, \hat{H}_+ and \hat{H}_- will thus generically give corrections to the energy of order $\frac{1}{N^2}$ whereas \hat{H}_{flip} can give corrections already at order $\frac{1}{N}$. The expansion of the anomalous dimensions hence generically takes the form

$$\mathcal{E} = \frac{g_{\text{YM}}^2 N}{8\pi^2} \left(E_0 + \frac{1}{N} E_1 + \frac{1}{N^2} E_2 + \mathcal{O}\left(\frac{1}{N^3}\right) \right), \quad (5.27)$$

where the contribution E_1 is mainly due to \hat{H}_{flip} . It should be noticed, however, that if there are degeneracies in the planar spectrum, energy corrections induced by \hat{H}_+ and \hat{H}_- can also be of order $\frac{1}{N}$. This phenomenon does not occur for strong coupling where the closed string perturbation theory taking into account string splitting and joining always gives rise to an expansion in $\frac{1}{N^2}$. The $\frac{1}{N}$ corrections to the energies induced by \hat{H}_+ and \hat{H}_- are hence expected to vanish for strong coupling (and only arise here due to an order of limits issue). Assuming this to be true we can thus study corrections to the string energy induced by cross-caps by considering only the corrections coming from \hat{H}_{flip} .

5.4 $\mathcal{N} = 4$ SYM with gauge group $Sp(N)$.

We now consider the case of $\mathcal{N} = 4$ SYM with gauge group $Sp(N)$, the group of $N \times N$ symplectic matrices. The construction of this theory in terms of an orientifold projection is also well known [55]: The projection in this case relates the Chan-Paton matrices of open-string states as

$$\lambda = -J^{-1} \lambda^T J \quad (5.28)$$

where J is an antisymmetric matrix satisfying $J^2 = -1_{N \times N}$, which can be taken to be (N is even):

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{N \times N}. \quad (5.29)$$

The Chan-Paton matrices in this case turn out to be symmetric, and generate the adjoint representation of $Sp(N)$. Combining this with the identification $X^i \rightarrow -X^i$ of the $\mathcal{N} = 4$ SYM scalars leads to $\mathcal{N} = 4$ SYM theory with gauge group $Sp(N)$ [36].

In $Sp(N)$, indices are raised and lowered with the matrix J , and adjoint fields with both indices down are symmetric. Thus an adjoint field $Z_\beta^\alpha = J^{\alpha\gamma} Z_{\gamma\beta}$ behaves in the following way under transposition

$$Z^T = JZJ. \quad (5.30)$$

This in particular implies that a single trace operator is again related to its parity conjugate as given in eqn. (5.17) and parity is gauged in the same way as before. Furthermore, for gauge group $Sp(N)$ the one-loop dilatation generator of $\mathcal{N} = 4$ SYM can again formally be expressed in exactly the same form as for $SU(N)$, cf. eqn. (5.14). Only the contraction rules are different. More precisely one has

$$\check{Z}_{\alpha\beta} Z_{\gamma\epsilon} = \frac{1}{2}(\delta_{\alpha\epsilon}\delta_{\beta\gamma} - J_{\alpha\gamma}J_{\beta\epsilon}). \quad (5.31)$$

Again one finds that the Hamiltonian can be written in the form given in (5.18). The action of $\hat{H}_{flip}^{Sp(N)}$ can be presented in the following way

$$\hat{H}_{flip}^{Sp(N)} \text{Tr}(\phi XZY) = \frac{1}{2} \text{Tr}(JX^T JY[Z, \phi]) + \frac{1}{2} \text{Tr}(YJX^T J[Z, \phi]). \quad (5.32)$$

We notice that the result differs from that of $SO(N)$ by X^T being replaced by $JX^T J$. This difference amounts to a shift of sign as we have for an operator X of length L

$$SO(N) : \quad X^T = (-1)^L \hat{P}X, \quad (5.33)$$

$$Sp(N) : \quad JX^T J = (-1)^{L+1} \hat{P}X, \quad (5.34)$$

where \hat{P} is the parity operator. This is in full accordance with the general result that $SO(N)$ can be understood as $Sp(-N)$ [31, 35]. Notice that this sign difference need not explicitly manifest itself in the off-diagonal terms \hat{H}_+ and \hat{H}_- since these will generically give rise to energy corrections of order $\frac{1}{N^2}$. For $Sp(N)$ we again find that the operator \hat{H}_+ differs from that of $SU(N)$ only by a factor of $\frac{1}{2}$ whereas the operator \hat{H}_- has extra terms compared to the corresponding operator for $SU(N)$. More precisely

$$\hat{H}_+^{Sp(N)} = \frac{1}{2} \hat{H}_+^{SU(N)} \quad (5.35)$$

$$\begin{aligned} \hat{H}_-^{Sp(N)} \text{Tr}(\phi X) \text{Tr}(ZY) &= \frac{1}{2} \hat{H}_-^{SU(N)} \text{Tr}(\phi X) \text{Tr}(ZY) \\ &\quad + \frac{1}{2} \text{Tr}(JX^T JY[\phi, Z]) + \frac{1}{2} \text{Tr}(YJX^T J[Z, \phi]). \end{aligned} \quad (5.36)$$

The difference between the extra terms for $Sp(N)$ and $SO(N)$ is that X^T is replaced by JX^TJ , cf. eqn (5.26), which as before amounts to a change of sign.

5.5 Analysis of BMN operators

BMN operators are operators consisting of a background of Z fields and a finite number of excitations in the form of ϕ -fields. We will restrict ourselves to discussing the simplest operators of this type, i.e. those having two excitations. Two-excitation BMN operators always have positive parity and therefore in the case of gauge group $SO(N)$ exist only for even length. At the planar level a basis for the two-excitation states can be chosen as

$$O_p^J = \text{Tr}(\phi Z^p \phi Z^{J-p}), \quad 0 \leq p \leq J. \quad (5.37)$$

In terms of these the eigenstates of \hat{H}_0 read

$$|n\rangle \equiv \mathcal{O}_n^J = \frac{1}{J+1} \sum_{p=0}^J \cos\left(\frac{\pi n(2p+1)}{J+1}\right) \mathcal{O}_p^J, \quad 0 \leq n \leq \frac{J}{2}, \quad (5.38)$$

and the corresponding eigenvalues are

$$E_0^n = 4 \sin^2\left(\frac{\pi n}{J+1}\right). \quad (5.39)$$

The inverse transformation giving \mathcal{O}_p^J in terms of $|n\rangle$ takes the form

$$\mathcal{O}_p^J = |0\rangle + 2 \sum_{n=1}^{J/2} \cos\left(\frac{\pi n(2p+1)}{J+1}\right) |n\rangle. \quad (5.40)$$

The energy correction induced by the perturbation \hat{H}_{flip} is simply given by the expression from first order quantum mechanical perturbation theory, i.e.

$$E_1^n = \langle n | \hat{H}_{flip} | n \rangle. \quad (5.41)$$

In order to determine this quantity we first evaluate $\hat{H}_{flip} \mathcal{O}_p^J$ where J is assumed to be even. We find (after some manipulations)

$$\begin{aligned} \hat{H}_{flip} \mathcal{O}_p^J &= -\frac{1}{4}(1 - (-1)^p) \{2\mathcal{O}_p^J - \mathcal{O}_{p-1}^J - \mathcal{O}_{p+1}^J\} \\ &\quad -\frac{1}{2}(-1)^p \left\{ \mathcal{O}_0^J + \mathcal{O}_J^J + 2 \sum_{k=1}^{J-1} (-1)^k \mathcal{O}_k^J \right\}. \end{aligned} \quad (5.42)$$

Having this expression, it is straightforward to determine the general matrix element of \hat{H}_{flip} as all sums involved are geometric sums. The result reads

$$\begin{aligned} \langle m | \hat{H}_{flip} | n \rangle = & \\ & -\frac{1}{J+1} \sin^2 \left(\frac{\pi m}{J+1} \right) \left\{ \delta_{n,m}(J+1) - \frac{1}{\cos \left(\frac{\pi(n-m)}{J+1} \right)} - \frac{1}{\cos \left(\frac{\pi(n+m)}{J+1} \right)} \right\} \\ & -\frac{2}{J+1} \frac{\sin^2 \left(\frac{\pi m}{J+1} \right)}{\cos \left(\frac{\pi n}{J+1} \right) \cos \left(\frac{\pi m}{J+1} \right)}. \end{aligned} \quad (5.43)$$

We notice that \hat{H}_{flip} is not hermitian but this phenomenon is well-known [37, 43]: The operator \hat{H}_{flip} is related to its hermitian conjugate by a similarity transformation. For $n = m$ the expression (5.43) reduces to

$$\begin{aligned} E_1^n &= \langle n | \hat{H}_{flip} | n \rangle \\ &= -\frac{2}{J+1} \tan^2 \left(\frac{\pi n}{J+1} \right) - \frac{1}{J+1} \sin^2 \left(\frac{\pi n}{J+1} \right) \left(J - \frac{1}{\cos \left(\frac{2\pi n}{J+1} \right)} \right). \end{aligned} \quad (5.44)$$

This should correspond to the energy correction to a closed string state resulting from the insertion of a cross-cap on its worldsheet. Defining $\lambda' = g_{\text{YM}}^2 N/J^2$ and $g_2 = J^2/N$, the anomalous dimensions of BMN operators were originally believed to have a double expansion in λ' and g_2 in the limit $\lambda, J, N \rightarrow \infty$ with λ', g_2 fixed [5, 49, 50]. This double expansion worked for BMN operators in $\mathcal{N} = 4$ SYM with gauge group $SU(N)$ for the first few terms in λ' and g_2 and led to some success in reproducing the first non-planar correction on the gauge theory side from LCSFT, for a review see [51]. Later it was understood that planar BMN scaling breaks down at four loop order in the gauge theory [52, 25, 53]. Furthermore, on the string theory side a BMN expansion would involve half-integer powers of λ' starting at one-loop order [54]. Here the first few terms of the expansion in powers of λ' and g_2 for the anomalous dimension in (5.45) read

$$\mathcal{E}^n = \frac{\lambda'}{2} \left(n^2 - g_2 \frac{n^2}{4J^2} \right), \quad (5.45)$$

meaning that the first non-planar contribution would not survive the above mentioned limit. Still it would be interesting to analyse the cross-cap scenario in the pp-wave geometry by some version of LCSFT.

5.6 Search for integrability at finite N

For gauge group $SU(N)$ an important concept in the search for integrability was the occurrence of so-called planar parity pairs, i.e. pairs of operators which at the planar level had the same anomalous dimension but opposite parity. The existence of such parity pairs could be traced back to the existence of an extra conserved charge commuting with the Hamiltonian but anti-commuting with parity [23]. When splitting and joining of traces were taken into account the degeneracy between the operators in a parity pair disappeared and this was taken as an indication that integrability was lost beyond the planar level [23]. The situation was the same for ABJM theory [83]. In the case of gauge group $SO(N)$ where parity is gauged one obviously does not even have planar parity pairs. Thus one has to invent other means to test for integrability.

One option is to look for other types of degeneracies in the spectrum which could survive the non-planar corrections. One such type of degeneracy is that between anomalous dimensions of certain single- and multi-trace operators, for instance between BMN operators with different number of traces, i.e. operators of the type

$$\mathcal{O}_n^{J_0;J_1,\dots,J_k} \equiv \mathcal{O}_n^{J_0} \text{Tr}(Z^{J_1})\text{Tr}(Z^{J_2})\dots\text{Tr}(Z^{J_k}), \quad (5.46)$$

with anomalous dimension

$$E_{0;n}^{J_0;J_1,\dots,J_k} = 4 \sin^2\left(\frac{\pi n}{J_0 + 1}\right). \quad (5.47)$$

These degeneracies between BMN states with different numbers of traces were what rendered the non-planar problem of $\mathcal{N} = 4$ SYM with gauge group $SU(N)$ intractable. The degeneracies are less pronounced in the case of gauge group $SO(N)$ due to the gauging of the parity symmetry. The first case of planar degenerate BMN states in the $SO(N)$ case is the degeneracy between the states \mathcal{O}_3^8 and $\mathcal{O}_1^{2;4}$. The second case is the degeneracy between the operators \mathcal{O}_5^{14} and $\mathcal{O}_3^{8;6}$. Using the full Hamiltonian we can easily check if the first non-planar correction which is of order $\frac{1}{N}$ lifts the degeneracy in these two cases and it turns out that it does. There is thus no hint of non-planar integrability from this analysis.

Another option to test for integrability is to directly try to construct conserved charges commuting with the Hamiltonian. In the higher loop analysis of $\mathcal{N} = 4$ SYM it was found that such conserved charges could be constructed

order by order in the coupling constant, λ [23]. More generally one can generate perturbatively integrable long range spin chains with $GL(K)$ symmetry starting from chains with nearest neighbour interactions [38, 60]. The construction can be elegantly described in terms of a master symmetry [39] or a boost operator [40] and leads to a large family of long range perturbatively integrable spin chains [61, 62]. These techniques do unfortunately not immediately apply to our case as they require that the spin chain length exceeds the range of the interaction. Nevertheless, we will discuss the possibility of constructing higher conserved charges perturbatively in $\frac{1}{N}$. For spin chains with local interactions integrability follows as soon as a single additional charge commuting with the Hamiltonian can be found [41, 42]. Again, this does not necessarily apply to our type of spin chain.

Since, as discussed earlier, the odd charges lose their meaning in our setting, where parity is gauged, at planar level the next higher conserved charge after the hamiltonian $\hat{H} = \hat{Q}_2$ is the even charge \hat{Q}_4 . If we expand to first order in $1/N$,

$$\hat{H} = \hat{H}_0 + \frac{1}{N} \hat{H}_{flip}, \quad \hat{Q}_4 = \hat{Q}_4^{(0)} + \frac{1}{N} \hat{Q}_4^{(1)}, \quad (5.48)$$

our task is to determine a suitable $\hat{Q}_4^{(1)}$ such that

$$[\hat{H}_0, \hat{Q}_4^{(1)}] + [\hat{H}_{flip}, \hat{Q}_4^{(0)}] = 0. \quad (5.49)$$

Since \hat{H}_{flip} only acts within a single trace, we can assume the same about $\hat{Q}_4^{(1)}$. At the planar level, the higher charges can be constructed iteratively starting from the Hamiltonian by means of the boost operator \hat{B} [13], i.e.

$$[\hat{B}, \hat{Q}_n^{(0)}] = \hat{Q}_{n+1}^{(0)}, \quad (5.50)$$

where \hat{B} is a moment of the Hamiltonian:

$$\hat{B} = \frac{1}{2i} \sum_{j=1}^L j \sigma_j \cdot \sigma_{j+1}, \quad (5.51)$$

with the σ 's being the Pauli matrices.

Ignoring constants and terms commuting with $\hat{H}^{(0)}$, this gives⁸

$$\hat{Q}_4^{(0)} = \sum_{i=1}^L (-8 [P_{i,i+3}P_{i+1,i+2} - P_{i,i+2}P_{i+1,i+3}] + 4P_{i,i+3} - 4P_{i,i+2}) . \quad (5.52)$$

⁸This matches the expression for $\hat{Q}_4^{(0)}$ given in [15], up to the terms mentioned.

Lacking a constructive way of extending this expression beyond the planar level, we have tried to guess a possible form by first rewriting all the permutation operators in terms of nearest-neighbour ones:

$$\begin{aligned} P_{i,i+3} &= P_{i+2,i+3}P_{i+1,i+2}P_{i,i+1}P_{i+1,i+2}P_{i+2,i+3} \quad \text{and} \\ P_{i,i+2} &= P_{i+1,i+2}P_{i,i+1}P_{i+1,i+2} \end{aligned} \quad (5.53)$$

and then using the relation $P_{i,i+1} = I_{i,i+1} - 2H_{i,i+1}^{(0)}$ (cf. eqn. (5.19)) to rewrite $\hat{Q}_4^{(0)}$ in terms of the planar Hamiltonian. Having done this (with the caveat that the rewritings in (5.53) are not unique), it is then natural to introduce a dependence on \hat{H}^{flip} by perturbing as:

$$H_{i,i+1}^{(0)} \rightarrow H_{i,i+1}^{(0)} + \frac{1}{N}H_i^{flip}, \quad (5.54)$$

where we have decomposed \hat{H}^{flip} as

$$\hat{H}_{flip} = \sum_{i=1}^L H_i^{flip}. \quad (5.55)$$

More precisely, we define H_i^{flip} by

$$H_i^{flip} = \sum_{j=1}^L H_{ij}^{flip}, \quad (5.56)$$

with H_{ij}^{flip} acting on sites i and j of a *periodic* chain of length L as, (cf. eqn. (5.24))⁹

$$\begin{aligned} H_{ij}^{flip} &(\mathcal{M}_{L-j+1,L-j+i-1} \otimes a_i \otimes \mathcal{N}_{1,j-i-1} \otimes b_j \otimes \mathcal{M}_{1,L-j}) \\ &= -\frac{1}{2} ((\mathcal{N}^T \otimes \mathcal{M})_{L-i,L-2} \otimes [a_i, b_j]_{\otimes} \otimes (\mathcal{N}^T \otimes \mathcal{M})_{1,L-i-1}) \\ &\quad -\frac{1}{2} ((\mathcal{M} \otimes \mathcal{N}^T)_{L-i,L-2} \otimes [a_i, b_j]_{\otimes} \otimes (\mathcal{M} \otimes \mathcal{N}^T)_{1,L-i-1}). \end{aligned} \quad (5.57)$$

⁹Note that there is an ambiguity in the location of the index i on the chain after the action of H_i^{flip} , which we have fixed by cyclically shifting the resulting chain by a suitable number of sites, such that the first term of the commutator $[a_i, b_j]$ always ends up at position i . Keeping track of i is important when deforming the higher charges, since in a typical term H_i^{flip} will be preceded or followed by e.g. $H_{i,i+1}^{(0)}$ or $H_{i+1,i+2}^{(0)}$ and the sum over i is performed only at the end.

Here we have defined $\mathcal{M}_{k,l} = m_k \otimes m_{k+1} \cdots m_{l-1} \otimes m_l$ and similarly for \mathcal{N} .

The expression for $\hat{Q}_4^{(1)}$ obtained by inserting (5.54) into (5.52) is too long to be reproduced here, but with the help of computer algebra we can check whether (5.49) is satisfied. This turns out not to be the case for our naive guess for $\hat{Q}_4^{(1)}$. Given the amount of ambiguity involved in obtaining $\hat{Q}_4^{(1)}$, this is perhaps not surprising, and outlines the need for a more systematic approach.

A third way to look for integrability is to see if the first few non-planar corrections can be reproduced from a perturbative Bethe ansatz as was the case in the higher loop analysis of [23, 47]. The most obvious way to check this is to simply try and derive a set of Bethe equations, for instance using the coordinate space approach. This direct approach is, however, not straightforward. First, it is not clear how to implement the gauging of parity in a convenient way in this language. Secondly, it is obvious that our spin chain does not have an asymptotic regime since, as soon as we go beyond the planar limit, all sites of the chain interact with each other. Therefore, we will take a more naive approach.

Let us recall the perturbative Bethe equation for $\mathcal{N} = 4$ SYM with gauge group $SU(N)$. For operators of length L containing M ϕ -fields and $(L - M)$ Z -fields (with $M \leq L/2$) it reads

$$\left(\frac{x(u_k + \frac{i}{2})}{x(u_k - \frac{i}{2})} \right)^L = \prod_{j \neq k}^M \frac{u_k - u_j + i}{u_k - u_j - i}, \quad (5.58)$$

where

$$x(u) = \frac{1}{2}u + \frac{1}{2}\sqrt{u^2 - 2g^2} \equiv u(1 - g^2 f(u)), \quad (5.59)$$

and where $g^2 = \frac{g_{\text{YM}}^2 N}{8\pi^2}$. Here u is related to the momentum p via

$$e^{ip} = \frac{x^+(u)}{x^-(u)}, \quad (5.60)$$

with

$$x^\pm(u) = x(u \pm \frac{i}{2}). \quad (5.61)$$

For later convenience we notice that purely algebraic arguments pertaining to the symmetry properties of the full $\mathcal{N} = 4$ SYM (and not just its $SU(2)$ -sector) imply that one needs [44]

$$x^+ + \frac{g^2}{2x^+} - x^- - \frac{g^2}{2x^-} = i, \quad (5.62)$$

which is of course fulfilled by the function $x(u)$ given above. Furthermore, we have the cyclicity constraint (5.23) and the energy is given as

$$E = \sum_k \frac{1}{g^2} \left(\sqrt{1 + 8g^2 \sin^2(\frac{p_k}{2})} - 1 \right). \quad (5.63)$$

For BMN states with two excitations we have $M = 2$, $L = J + 2$. Following [47] and expanding the Bethe root $u \equiv u_1 = -u_2$ as

$$u = u_0 + g^2 \delta u, \quad (5.64)$$

we find from the Bethe equation to order g^2

$$\delta u = \frac{u_0}{u_0^2 + \frac{1}{4}} \left(\frac{J+2}{J+1} \right), \quad (5.65)$$

and consequently, with $E = E_0 + g^2 \delta E$,

$$\delta E_{SU(N)} = -16 \sin^4 \left(\frac{n\pi}{J+1} \right) - 64 \frac{1}{J+1} \cos^2 \left(\frac{n\pi}{J+1} \right) \sin^4 \left(\frac{n\pi}{J+1} \right), \quad (5.66)$$

where the first term comes from the correction to the dispersion relation and the second one from the correction of the momenta. Let us rewrite the first $\frac{1}{N}$ -correction to the BMN states of the $SO(N)$ gauge theory in a similar way

$$\begin{aligned} \delta E_{SO(N)} &= -\sin^2 \left(\frac{n\pi}{J+1} \right) \\ &\quad - \frac{1}{J+1} \left\{ 2 \tan^2 \left(\frac{\pi n}{J+1} \right) - \frac{1}{2} \tan^2 \left(\frac{2\pi n}{J+1} \right) \cos \left(\frac{2\pi n}{J+1} \right) \right\}. \end{aligned} \quad (5.67)$$

From this expression it is clear that if this were to arise from a Bethe system the first term would have to originate from a correction of the dispersion relation and the second one from a correction of the rapidities, i.e. a correction of the Bethe equations. The needed correction of the rapidities would be

$$\delta u = -\frac{1}{J+1} \frac{4u_0^2 + 1}{64u_0^3(4u_0^2 - 1)}. \quad (5.68)$$

There are of course many possible ways to deform the Bethe equations so that we would get the rapidity corrections for two-excitation states appearing in (5.68). Given a plausible deformation one can test if it gives the

correct answer for the energy of states with more excitations which we can of course again compute using quantum mechanical perturbation theory. Let us illustrate this with a simple example. Parametrising the function $x(u)$ as

$$x(u) = u(1 - \frac{1}{N}f(u)), \quad (5.69)$$

we find that in order to correctly reproduce the $\frac{1}{N}$ -correction to the energies of the two-excitation states the function $f(u)$ needs to fulfill the following equation

$$f_-(u) \equiv f(u + \frac{i}{2}) - f(u - \frac{i}{2}) = -i \frac{1}{16u^3(4u^2 - 1)}. \quad (5.70)$$

This implies that $f(u)$ can neither be written as a Taylor expansion nor as a Laurent expansion in u . Notice, however, that to solve the modified Bethe equations perturbatively we would only need to know $f_-(u)$. We have checked whether the Bethe equations with the expression for the $x(u)$ given in eqn. (5.69) and the dispersion relation corrected by the first term in eqn. (5.67) correctly reproduce the energy of states with four excitations and length eight, cf. Appendix 5.8. We found that the simple modification of the Bethe ansatz described above does not lead to the correct non-planar correction to the energy of any of these states. Now, one may ask whether the algebraic arguments which led to (5.62) and (5.63) are valid for the non-planar case as well. It follows from the analysis of reference [44] that the dispersion relation can indeed be modified to include a correction which would lead to the first term in the relation (5.67). However, the relation (5.62) to leading order in λ simply becomes $x^+(u) - x^-(u) = i$ which leads to the following constraint on the function $f(u)$

$$f(u + \frac{i}{2}) + f(u - \frac{i}{2}) = 2iu \left[f(u + \frac{i}{2}) - f(u - \frac{i}{2}) \right]. \quad (5.71)$$

This constraint is unfortunately incompatible with the relation (5.70). Thus the naive proposal for the modification of the Bethe ansatz would anyway not have a chance to work for the full $\mathcal{N} = 4$ SYM theory.

Obviously, there are many other possible ways to deform the Bethe ansatz. In particular, there is the possibility of including a phase factor [48]. This would, in the simplest possible approach, mean modifying the Bethe ansatz

to

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L = \prod_{j \neq k}^M \frac{u_k - u_j + i}{u_k - u_j - i} \left(1 + \frac{i}{N} h(u_k - u_j)\right). \quad (5.72)$$

Here we have for simplicity assumed that the phase factor depends only on the difference of rapidities and that the modification of the Bethe equations is due to the appearance of a phase factor alone. Demanding again the modification of rapidities to be given by (5.68) we find for the function $h(u)$

$$h(u) = \frac{1}{2u^3(u^2 - 1)}. \quad (5.73)$$

Note the non-trivial fact that $h(u)$ is real for real u and that $h(u)$ does not depend on the length of the spin chain. We have checked if the modified Bethe equation (5.72) correctly reproduce the energy correction for length eight and four excitations. Unfortunately, this is not the case. Needless to say that the tests performed here do not exclude the existence of a modified Bethe ansatz.

5.7 Comments on the string theory side

As discussed in the previous sections, the spectral problem of $SO(N)$ and $Sp(N)$ $\mathcal{N} = 4$ SYM theory exhibits several interesting differences compared to the $SU(N)$ case. In this section we make some preliminary observations on how these differences manifest themselves on the string theory side.

In sections 2 and 3 we sketched how the $\mathcal{N} = 4$ SYM theory with orthogonal or symplectic gauge group can be obtained by performing an orientifold operation on a stack of D3-branes. Taking the near-horizon limit we find that the AdS/CFT dual gravity background should be given by an orientifold of $\text{AdS}_5 \times \text{S}^5$ [36]. Embedding the sphere in \mathbb{R}^6 as

$$\sum_{i=1}^6 (X^i)^2 = 1, \quad (5.74)$$

this orientifold is a combination of the \mathbb{Z}_2 action $X^i \rightarrow -X^i$ and the worldsheet orientation reversal $\sigma \rightarrow 2\pi - \sigma$. Note that the \mathbb{Z}_2 acts without fixed points on S^5 and thus there is no orientifold plane. Consequently, there is no need for additional branes to cancel the orientifold plane charge, and thus no

open string sector. Therefore, this setting still corresponds to an $\mathcal{N} = 4$ theory.¹⁰ The dual geometry is now $\text{AdS}_5 \times \mathbb{R}\text{P}^5$, and the difference between the $SO(N)$ and $Sp(N)$ projections lies in the presence of an additional B-field.

As discussed in [64], in the strict planar (free string) limit all correlation function calculations in the orientifolded theory can be reduced, up to trivial rescalings, to those in the oriented one. We thus do not expect our picture of planar integrability to be modified in a major way. Of course, any spinning string solutions on S^5 not invariant under the orientifold procedure will be projected out.

Therefore, in the planar limit the differences to the S^5 case are relatively minor and arise only because some spinning string solutions on S^5 are not invariant under the orientifold transformation and are projected out. This corresponds to the fact, discussed in section 2, that certain gauge theory operators are projected out, depending on their length and parity. Unfortunately, since the semi-classical string solutions have large length, the distinction between odd and even length is not as apparent as on the gauge theory side. It would be interesting to do a thorough analysis of spinning strings on $\text{AdS}_5 \times \mathbb{R}\text{P}^5$ along the lines of [65, 66, 67] and we hope to return to this problem in the future.

For the moment, however, we will confine ourselves to the straightforward observation that, by analogy with other contexts involving orientifolds, one can obtain invariant solutions by extending known ones with the addition of mirror strings. Let us demonstrate this for the $SU(2)$ sector, in which classical string solutions can be described in terms of their profile on an S^2 inside S^5 . This S^2 is defined by $\sum_{i=1}^3 (x^i)^2 = 1$, where we have written the coordinates of S^5 as $X_1 \pm iX_4 = x^1 \exp(\pm i\phi_1)$, etc. Then the orientifold projection can be taken to act on the coordinates of this S^2 as $x^i \rightarrow -x^i$, resulting in the real projective space $\mathbb{R}\text{P}^2$. Now, given any string solution with a profile $x^i(\sigma)$ for $0 \leq \sigma < 2\pi$ on S^2 , we can construct a “doubled” solution on $\mathbb{R}\text{P}^2$ by taking the profile to be $x^i(\sigma)$ for $0 \leq \sigma < \pi$ and $-x^i(\sigma)$ for $\pi \leq \sigma < 2\pi$. See Fig. 5.3 for a drawing of such a solution on $\mathbb{R}\text{P}^2$. Note that, despite appearances, the string in the figure is a closed string, since antipodal points are identified on $\mathbb{R}\text{P}^2$. The energy of such strings is always quadratic in $x^i(\sigma)$, so it will be exactly the same as the solution on S^2 .¹¹

¹⁰Orientifolds of $\mathcal{N} = 4$ SYM with fixed planes, which lead to $\mathcal{N} = 2$ conformal theories with additional flavours, have been considered in an integrability context in [57, 58, 59].

¹¹For the purpose of comparing with weak coupling results, it might thus be more appropriate to use a different normalisation of the $SU(N)$ and $SO(N)$ generators in the gauge

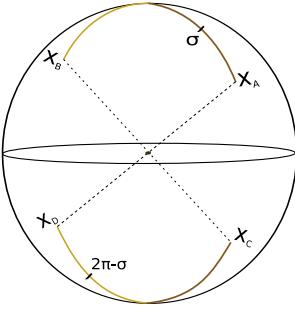


Figure 5.3: A closed string solution on \mathbb{RP}^2 which is invariant under the orientifold. The configuration $X(\sigma = 0) = x_A$, $X(\sigma = \pi) = x_B = x_C \sim -x_B$, $X(\sigma = 2\pi) = x_D$ is invariant under $X^i \rightarrow -X^i$ and $\sigma \rightarrow 2\pi - \sigma$.

Arguing in this way, it seems that any solution which in the original $AdS_5 \times S^5$ geometry is confined to a half S^2 (the fundamental domain of \mathbb{RP}^2) inside the S^5 , can be extended to a solution in $AdS_5 \times \mathbb{RP}^5$ by superimposing it with its mirror under the transformation $X_i \rightarrow -X_i$ and $\sigma \rightarrow 2\pi - \sigma$. This includes for instance the giant magnon solution [45] and the folded spinning string solution [65].¹²

Things become more interesting when considering $\frac{1}{N}$ -corrections, which correspond to turning on string interactions. Recall that the analogue of a spin chain splitting-and-joining operation is a process where a string decays into two strings, which later recombine, creating a worldsheet of genus one. Such processes are not well understood, even in the pp-wave geometry, the main obstacle coming from the necessity of summing over the infinite number of intermediate states (see [51] for a discussion). A simple model for splitting and joining of semi-classical strings in $AdS_5 \times S^5$ was presented in [72]. However, as discussed (in a simplified model) in [73], semi-classical splitting-and-joining does not seem to capture all of the relevant physics.

In our $SO(N)$ case, apart from the splitting-and-joining terms \hat{H}_+ and \hat{H}_- , the dilatation operator contains an additional term which we have denoted by \hat{H}_{flip} . What is the analogue of this term on the string side? It

theory, or alternatively rescale the length of the string before and after the orientifold.

¹²Giant magnon solutions on \mathbb{RP}^2 have previously appeared in the context of the $AdS_4 \times \mathbb{CP}^3$ dual of ABJM theory, where the \mathbb{RP}^2 in that context arises as a suitable subspace of \mathbb{CP}^3 [68, 69, 70, 71]. The main difference in our case is that, since we are dealing with an orientifold, we additionally need to implement the worldsheet identification $\sigma \rightarrow 2\pi - \sigma$.

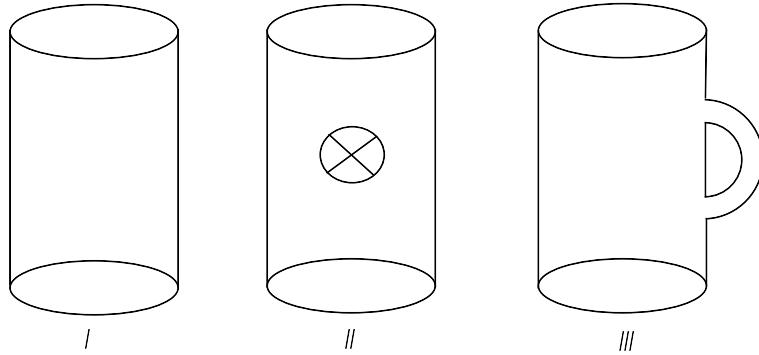


Figure 5.4: Two-point string amplitudes. (I) The (planar) cylinder amplitude. (II) A cylinder with a cross-cap, contributing at order $\frac{1}{N}$. (III) A cylinder with a handle, contributing at order $\frac{1}{N^2}$.

will clearly be related to the fact that, due to the orientifold operation, one should now also consider non-orientable string worldsheets, or in other words worldsheets with cross-caps. Recall the weighting of a worldsheet with b boundaries (each with N Chan-Paton factors), c cross-caps and g handles:

$$(Ng_s)^b g_s^c g_s^{2g-2} = \lambda^{2g-2+b+c} N^{-c-2g+2}, \quad (5.75)$$

where on the right-hand side we have rewritten the result in terms of gauge theory quantities, where the 't Hooft coupling is $\lambda = g_{YM}^2 N = g_s N$. We see that a cross-cap weights the amplitude by a factor of $\frac{1}{N}$ compared to the oriented amplitude, while a handle by a factor of $\frac{1}{N^2}$. See Fig. 5.4. The cross-cap contribution thus, as expected, appears at the same order as the leading contribution from \hat{H}_{flip} on the gauge theory side and it is natural to identify the two. Intuitively, it is also clear that \hat{H}_{flip} is associated with cross-caps since the operator acts by cutting out a piece of an operator and gluing it back in with the opposite orientation. Since it does not require summation over all intermediate states, the cross-cap calculation on the string theory side could be expected to be simpler than the genus-one case.

It would be very interesting to perform such a non-oriented string calculation and compare with the gauge theory side. Especially using a pp-wave geometry one might be able to compare with our gauge theory results for BMN operators, cf. section 5.5.

5.8 Conclusion

We have studied a number of features which distinguish the spectral problem of $\mathcal{N} = 4$ SYM with gauge group $SO(N)$ or $Sp(N)$ from that of $\mathcal{N} = 4$ SYM with gauge group $SU(N)$. Of particular interest to us was the difference in the leading non-planar corrections. For orthogonal and symplectic gauge groups the leading non-planar corrections define a novel type of spin chain interaction of highly non-local nature which cuts out a piece of the chain and re-inserts it with the opposite orientation. Unlike the case of gauge group $SU(N)$, the leading non-planar corrections a priori could fit into the standard framework of integrability. However, the resulting spin chain did not show any signs of integrability when studied by usual methods. In particular, our attempts to describe the diagonalization problem for \hat{H}_{flip} by means of a Bethe ansatz were unsuccessful. However, given that the spin chain described by this Hamiltonian seems to lack an asymptotic regime (since all sites of the chain are involved in the interaction) it could still be that integrability, if present, simply cannot be formulated in terms of a Bethe ansatz.

Just as $\mathcal{N} = 4$ SYM with orthogonal or symplectic gauge group is much less studied than its $SU(N)$ cousin, the same holds for the dual string theories. Here we briefly discussed some issues related to studying the spectrum of type IIB string theory on the $AdS_5 \times \mathbb{R}P^5$ background. We mentioned some features of spinning string solutions and discussed how the leading non-planar corrections to anomalous dimensions on the gauge theory side should originate from non-oriented string worldsheets with a single cross-cap. By considering such worldsheets, one might hope to reproduce the leading non-planar corrections for two-excitation states that we found from the gauge theory side. More generally, as cross-caps might be easier to handle than higher genus surfaces, this might open new possibilities for comparing gauge and string theories beyond the planar limit.

A Numerical tests of Bethe equations.

We specify here the details of the numerical tests we performed. We focused on the (single trace) states of length eight with four excitations. There are three such highest weight states. At one loop order at the planar level they can be described in terms of the corresponding roots of the Bethe equations given in (5.20). The three sets of roots $\{u_i^1\}$, $\{u_i^2\}$ and $\{u_i^3\}$, $i \in \{1, 2, 3, 4\}$

read¹³

$$\{u_i^1\} = \{\pm 0.525, \pm 0.129\}, \quad (5.76)$$

$$\{u_i^2\} = \{\pm 0.0413, \pm 1.026i\}, \quad (5.77)$$

$$\{u_i^3\} = \{\pm 0.463 \pm 0.502i\}, \quad (5.78)$$

and the corresponding planar one-loop energies, E_0^j , $j = 1, 2, 3$ are the roots of the polynomial

$$-x^3 + 10x^2 - 29x + 200 = 0. \quad (5.79)$$

By direct diagonalization of $H_0 + \frac{1}{N} \hat{H}_{flip}$ we find the $\frac{1}{N}$ -corrections to the energies, E_1^i to be¹⁴

$$E_1^1 = 1.618, \quad E_1^2 = -6.75, \quad E_1^3 = -19.85. \quad (5.80)$$

On the other hand solving the Bethe ansatz (5.58) with $x(u)$ given by (5.69) and (5.70) we find the following $\frac{1}{N}$ -correction to the rapidities

$$\delta u_1^i = \{\pm 0.0255 \pm 0.000893i\}, \quad (5.81)$$

$$\delta u_2^i = \{\pm 47.6, \pm 138.4i\}, \quad (5.82)$$

$$\delta u_3^i = \{\pm 3.65, \pm 10.74\}, \quad (5.83)$$

which leads to the following $\frac{1}{N}$ -correction to the energies

$$E_1^1 = -0.43, \quad E_1^2 = -504, \quad E_1^3 = -26.6. \quad (5.84)$$

These values clearly differ from the exact ones given in eqn. (5.80).

Using instead the deformed Bethe ansatz given by (5.72) and (5.73) the $\frac{1}{N}$ -correction to the Bethe roots are

$$\{\delta u_i^1\} = \{\pm 1.146 \pm 0.0327i\}, \quad (5.85)$$

$$\{\delta u_i^2\} = \{\pm 5.96, \pm 17.29i\}, \quad (5.86)$$

$$\{\delta u_i^3\} = \{\pm 0.799, \pm 1.045\}, \quad (5.87)$$

and the energy corrections, E_1^i become

$$E_1^1 = -2.07, \quad E_1^2 = -63.6, \quad E_1^3 = -8.25. \quad (5.88)$$

These values also fail to agree with the exact ones given in eqn. (5.80).

¹³These roots as well as others can be found in references [46, 47].

¹⁴We remark that the operators considered here do not exhibit degeneracy with any multi-trace states and thus there are no further corrections to their energies of order $\frac{1}{N}$.

Chapter 6

Non-planar ABJ theory and parity

6.1 Summary

The most famous and concrete example of gauge/gravity duality, the Maldacena's conjecture about $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM, drastically changed our way of thinking about both, gravity and gauge theories. Nevertheless, it is just an example of the duality, so one, natural, next step is to understand which of its properties are universal and shared with other holographic systems and which are not.

In 2008, new class of AdS_4/CFT_3 dualities was constructed by Aharony, Bergman, Jafferis and Maldacena [76]. The proposal came from studying N coincident M2-branes on the $\mathbb{C}^4/\mathbb{Z}_k$ orbifold. According to authors of [76], such system is described by 3 dimensional $\mathcal{N} = 6$ supersymmetric Chern-Simons-matter theory with gauge group $U(N)_k \times U(N)_{-k}$. The theory was abbreviated to the ABJM model after the names of the authors (for pedagogical review of the ABJM model see [99]).

Actually, as shown by [77], one can consider generalized ABJM model with gauge group $SU(M)_k \times SU(N)_{-k}$ that on the M theory side corresponds $\min(M, N)$ M2 branes moving on $\mathbb{C}^4/\mathbb{Z}_k$ and $|M - N|$ fractional branes localized at the singularity. This example is known as the ABJ model. Since it is straightforward to extract all the results for *ABJM* theory from ABJ, below we summarize the former one.

The 3D gauge theory has three parameters, Chern-Simons level k and size

of the rectangular matrices $M \times N$. k plays a role of the coupling constant, in a sense that all interactions are suppressed as $1/k$. Therefore, by referring to weak coupling regime we will mean the limit of large k .

The number of parameters makes the duality somehow richer than the Maldacena's model. Namely, we can consider different relative scaling of N , M and k that lead to different gravity duals. The most interesting case from the perspective of integrability is when $N^{1/5} \ll k$ and $M^{1/5} \ll k$. M theory is then well approximated by weakly coupled type IIA string theory on $AdS_4 \times \mathbb{CP}^3$ with additional NS B-field B_2 .

Action for the gauge theory was first written down explicitly in [84]. Its global symmetry (and the isometry group of the $AdS_4 \times \mathbb{CP}^3$) is the orthosymplectic supergroup $OSp(6/4)$ that contains as bosonic subgroups $SU(4)$ R-symmetry and $SO(2, 3)$ conformal group in 3 dimensions¹. The theory has $\mathcal{N} = 6$ supersymmetry.

Observables of the gauge theory are gauge invariant operators constructed out of gauge fields A_m and \hat{A}_m ($m = 0, 1, 2$), four complex scalars Y^I , four Majorana fermions ψ_I ($I = 1, 2, 3, 4$), and their complex conjugates. Matter fields (scalars and fermions) are $M \times N$ matrices in the bi-fundamental representation of the gauge group.

Formally it is possible to introduce 't Hooft large N (M) expansion. This is done by introducing two 't Hooft parameters

$$\lambda = \frac{4\pi N}{k}, \quad \hat{\lambda} = \frac{4\pi M}{k}. \quad (6.1)$$

and taking double 't Hooft limit

$$N, M \rightarrow \infty, \quad k \rightarrow \infty, \quad \lambda, \hat{\lambda} \text{ fixed.} \quad (6.2)$$

According to the AdS_4/CFT_3 dictionary, string coupling is related to k via

$$g_s \sim \left(\frac{N}{k^5} \right)^{1/4} = \frac{\lambda^{5/4}}{N}, \quad (6.3)$$

hence in the planar limit strings do not interact. As before, non-planar contributions come from splitting and joining of strings.

As one could expect from the $\mathcal{N} = 4$ SYM experience, the spin-chain is the $OSp(6/4)$ chain that represents single trace operators. A novelty is

¹spacetime rotations and dilatations

though due to the bi-fundamental $M \times N$ fields. They lead to a staggered spin-chain that can be seen as two chains intertwined. Also, since fields on even sites are different from those on odd sites, length of the chain must be even.

The two loop planar dilatation generator for this model was derived in [80] and asymptotic Bethe Ansatz proposed [80, 103].

One can consider a simple sector to work with, the $SU(2) \times SU(2)$ (the analog of the $SU(2)$ from AdS_5/CFT_4), where only two pairs of scalar fields are taken into account. For example we can choose the vacuum of the chain to be

$$\text{Tr} \left(Y^1 Y_4^\dagger Y^1 Y_4^\dagger \dots \right) \quad (6.4)$$

where Y^1 is the vacuum on one chain and Y^4 on the other. Then excitations are Y^2 and Y_3^\dagger on the first and second chain respectively. The full two loop² dilatation operator in this sector was derived in [83, 22]. It has the following structure

$$D = \lambda \hat{\lambda} \left\{ D_0 + \frac{1}{\mathcal{M}} (D_+ + D_-) + \frac{1}{\mathcal{M}^2} (D_{00} + D_{++} + D_{--}) \right\}, \quad (6.5)$$

where D_+ and D_{++} increase the number of traces by one and two respectively and D_- and D_{--} decrease the number of traces by one and two. Moreover, D_0 does not change the number of traces and D_{00} first adds one trace and subsequently removes one or vice versa. Finally $\frac{1}{\mathcal{M}}$ stands for $\frac{1}{N}$ or $\frac{1}{M}$ and $\frac{1}{\mathcal{M}^2}$ stands for $\frac{1}{N^2}$, $\frac{1}{M^2}$ or $\frac{1}{MN}$. It was our main tool in the article presented in the next section so see there for more details and its explicit form in terms of the scalar fields.

Our research was largely motivated by aspects of parity and their relations to integrable structure. In AdS_4/CFT_3 these issues are particularly subtle. By construction, the ABJ theory breaks parity. On the string side the difference between M and N leads to additional θ angle on the world-sheet that in general breaks parity. On the gauge theory, since the spin-chain consists of two alternating chains, one might expect that this parity breaking will appear as a different dispersion relation for magnons on different chains. Nevertheless, explicit perturbative computations (up to four loops)

²there are two chains so the first correction to the dilatation generator comes at two loops. Chains decouple in perturbation theory up to 6 loops

[100, 101, 102] showed no signs of parity breaking (all the formulas are invariant under the exchange of λ and $\hat{\lambda}$). These gives us a very interesting puzzle and a testing ground for relations between parity and integrable structure.

In the project we decided to address these questions on the gauge theory side using the full two loop dilatation operator that we derived. More precisely we diagonalized it in a basis of short operators with excitations on one and two chains at the same time. For states with excitations on one chain only there was no mixing between states with different parities, hence parity was still present. However, once we considered excitations on both chains, mixing corrections at non-planar level appeared. As we expected, their form was proportional to $M - N$, hence for the parity invariant ABJM model ($M = N$) they disappeared.

In both models, ABJM and ABJ, nonplanar corrections lifted degeneracies between planar parity pairs which seems to be a universal phenomena among non-planar corrections.

Below we present the article in it's published version. Readers interested in details of derivations of the full two loop dilatation generator of ABJM theory are referred to [83].

Non-planar ABJ Theory and Parity

Pawel Caputa, Charlotte Kristjansen and Konstantinos Zoubos

The Niels Bohr Institute, Copenhagen University
Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark.

`caputa@nbi.dk, kristjan@nbi.dk, kzoubos@nbi.dk`

Abstract

While the ABJ Chern–Simons–matter theory and its string theory dual manifestly lack parity invariance, no sign of parity violation has so far been observed on the weak coupling spin chain side. In particular, the planar two-loop dilatation generator of ABJ theory is parity invariant. In this letter we derive the non-planar part of the two-loop dilatation generator of ABJ theory in its $SU(2) \times SU(2)$ sub-sector. Applying the dilatation generator to short operators, we explicitly demonstrate that, for operators carrying excitations on both spin chains, the non-planar part breaks parity invariance. For operators with only one type of excitation, however, parity remains conserved at the non-planar level. We furthermore observe that, as for ABJM theory, the degeneracy between planar parity pairs is lifted when non-planar corrections are taken into account.

6.2 Summary of the project

While the ABJ Chern–Simons–matter theory and its string theory dual manifestly lack parity invariance, no sign of parity violation has so far been observed on the weak coupling spin chain side. In particular, the planar two-loop dilatation generator of ABJ theory is parity invariant. In this letter we derive the non-planar part of the two-loop dilatation generator of ABJ theory in its $SU(2) \times SU(2)$ sub-sector. Applying the dilatation generator to short operators, we explicitly demonstrate that, for operators carrying excitations on both spin chains, the non-planar part breaks parity invariance. For operators with only one type of excitation, however, parity remains conserved at the non-planar level. We furthermore observe that, as for ABJM theory, the degeneracy between planar parity pairs is lifted when non-planar corrections are taken into account.

6.3 Introduction

The concept of spin chain parity [75] played a crucial role in the discovery of higher loop integrability of the planar spectral problem of $\mathcal{N} = 4$ SYM [23]. For a spin chain state the parity operation simply inverts the order of spins at the sites of the chain. In the field theory language the operation correspondingly inverts the order of fields inside a single trace operator or equivalently complex conjugates the gauge group generators. $\mathcal{N} = 4$ SYM theory is parity invariant. In particular, the theory’s dilatation generator commutes with parity. Integrability of the planar spectral problem at one loop order, discovered first in [18], implies the existence of a tower of higher conserved charges. The first of these, while commuting with the dilatation generator, anti-commutes with parity. As a consequence one finds in the planar spectrum pairs of operators with opposite parity but the same conformal dimension, denoted as planar parity pairs. The fact that these planar parity pairs survived higher loop corrections constituted the seed for the unveiling of higher loop integrability [23, 47]. When non-planar corrections were taken into account, parity was still a good quantum number but the degeneracies between planar parity pairs disappeared [23]. While not disproving integrability this shows that the standard construction of conserved charges does not work any more.

The discovery of a novel AdS_4/CFT_3 correspondence [76, 77] has pro-

vided us with the possibility of studying the effects of parity violation in a supersymmetric gauge theory and its dual string theory. A supersymmetric $\mathcal{N} = 6$ Chern–Simons–matter theory with gauge group $SU(M)_k \times \overline{SU(N)}_{-k}$, where k denotes the Chern–Simons level, has been found to be dual to type IIA string theory on $AdS_4 \times CP^3$ with a background NS B –field B_2 having non-trivial holonomy on $CP^1 \subset CP^3$. More precisely³

$$\frac{1}{2\pi} \int_{CP^1 \subset CP^3} B_2 = \frac{M - N}{k}. \quad (6.6)$$

This B –field holonomy causes breaking of world-sheet parity for $M \neq N$ and results in a string background which breaks target-space parity [77]. Correspondingly, the dual field theory does not respect three-dimensional parity invariance. For $M = N$ the Chern–Simons–matter theory is known as ABJM theory whereas the general version is denoted as ABJ theory. Our aim is to investigate how the parity breaking on the field theory side manifests itself in the spin chain language. The first steps in this direction were taken in [78, 79] where the two-loop planar dilatation generator of ABJ theory was derived, respectively in an $SU(4)$ sub-sector and for the full set of fields. However, rather surprisingly, in these studies no effects of parity violation were seen. In fact the planar two-loop dilatation generator of ABJ theory differs from that of ABJM theory [80, 81, 82] only by an overall pre-factor. This raises the question of whether the parity symmetry of the spin chain has a deeper significance, or is simply an accidental symmetry of the two-loop planar approximation. In the present letter we will derive the two-loop non-planar dilatation generator of ABJ theory in a $SU(2) \times SU(2) \subset SU(4)$ sub-sector and explicitly demonstrate parity-breaking effects.

We start by, in section 6.4, briefly describing ABJ theory and subsequently proceed to derive its full (planar plus non-planar) two-loop dilatation generator in the $SU(2) \times SU(2)$ sector in section 6.5. As the derivation follows closely that of ABJM theory [83] we shall be very brief. In section 6.6 we explicitly apply the dilatation generator to a series of short operators and determine their spectrum. In particular, we show that the non-planar part of the dilatation generator does *not* conserve parity. In addition, we observe a lifting of all planar degeneracies. Finally, section 6.7 contains our conclusion.

³Here we have assumed that $M \geq N$. Quantum consistency of the theory requires in addition that $M - N \leq k$ [77].

6.4 ABJ theory

Our notation will follow that of references [84, 81]. ABJ theory [77] (see also [85] for a discussion at the classical level) is a three-dimensional $\mathcal{N} = 6$ superconformal Chern–Simons–matter theory with gauge group $U(M)_k \times \overline{U(N)}_{-k}$ and R -symmetry group $SU(4)$. The parameter k denotes the Chern–Simons level. The fields of ABJ theory consist of gauge fields A_m and \bar{A}_m , complex scalars Y^I and Majorana spinors Ψ_I , $I \in \{1, \dots, 4\}$. The two gauge fields A_m and \bar{A}_m belong to the adjoint representation of $U(M)$ and $\overline{U(N)}$ respectively. For $N = M$, ABJ theory reduces to ABJM theory. The scalars Y^I and the spinors Ψ_I are bi-fundamental and transform in the $M \times \overline{N}$ representation of the gauge group and in the fundamental and anti-fundamental representation of $SU(4)$ respectively. For our purposes it proves convenient to write the scalars and spinors explicitly in terms of their $SU(2)$ component fields, i.e. [84]

$$\begin{aligned} Y^I &= \{Z^A, W^{\dagger A}\}, & Y_I^\dagger &= \{Z_A^\dagger, W_A\}, \\ \Psi_I &= \{\epsilon_{AB} \xi^B e^{i\pi/4}, \epsilon_{AB} \omega^{\dagger B} e^{-i\pi/4}, \}, \\ \Psi^{I\dagger} &= \{-\epsilon^{AB} \xi_B^\dagger e^{-i\pi/4}, -\epsilon^{AB} \omega_B e^{i\pi/4}\}, \end{aligned}$$

where now $A, B \in \{1, 2\}$. Expressed in terms of these fields the action reads

$$\begin{aligned} S = \int d^3x \left[& \frac{k}{4\pi} \epsilon^{mnp} \text{Tr}(A_m \partial_n A_p + \frac{2i}{3} A_m A_n A_p) - \frac{k}{4\pi} \epsilon^{mnp} \text{Tr}(\bar{A}_m \partial_n \bar{A}_p + \frac{2i}{3} \bar{A}_m \bar{A}_n \bar{A}_p) \right. \\ & - \text{Tr}(\mathcal{D}_m Z)^\dagger \mathcal{D}^m Z - \text{Tr}(\mathcal{D}_m W)^\dagger \mathcal{D}^m W + i \text{Tr} \xi^\dagger \mathcal{D} \xi + i \text{Tr} \omega^\dagger \mathcal{D} \omega \\ & \left. - V_D^{ferm} - V_D^{bos} - V_F^{ferm} - V_F^{bos} \right]. \end{aligned}$$

Here the covariant derivatives are defined as

$$\mathcal{D}_m Z^A = \partial_m Z^A + i A_m Z^A - i Z^A \bar{A}_m, \quad \mathcal{D}_m W_A = \partial_m W_A + i \bar{A}_m W_A - i W_A A_m, \quad (6.7)$$

and similarly for $\mathcal{D}_m \xi^B$ and $\mathcal{D}_m \omega_B$. The decomposition of the scalars and fermions into their $SU(2)$ components has allowed us to split the bosonic as well as the fermionic potential into D –terms and F –terms. An explicit form of these can be found in [84]. The theory has two 't Hooft parameters

$$\lambda = \frac{4\pi N}{k}, \quad \hat{\lambda} = \frac{4\pi M}{k}, \quad (6.8)$$

and one can consider the double 't Hooft limit

$$N, M \rightarrow \infty, \quad k \rightarrow \infty, \quad \lambda, \hat{\lambda} \text{ fixed.} \quad (6.9)$$

Furthermore, the theory has a multiple expansion in $\lambda, \hat{\lambda}, \frac{1}{N}$ and $\frac{1}{M}$. The action of three-dimensional parity flips the levels of the Chern–Simons terms, which produces a different theory if $M \neq N$. Thus the ABJ model is not parity invariant.

In this letter we will be interested in studying non-planar corrections (i.e. $\frac{1}{N}$ and $\frac{1}{M}$ corrections) for anomalous dimensions at the leading two-loop level. We shall restrict ourselves to considering scalar operators belonging to a $SU(2) \times SU(2)$ sub-sector i.e. operators of the following type

$$\mathcal{O} = \text{Tr} (Z^{A_1} W_{B_1} \dots Z^{A_L} W_{B_L}), \quad (6.10)$$

where $A_i, B_i \in \{1, 2\}$, and their multi-trace generalizations. A central object in our analysis will be the parity operation which acts on an operator by inverting the order of the fields inside each of its traces, i.e.⁴

$$P : \text{Tr} (Z^{A_1} W_{B_1} \dots Z^{A_L} W_{B_L}) \longrightarrow \text{Tr} (W_{B_L} Z^{A_L} \dots W_{B_1} Z^{A_1}). \quad (6.11)$$

Strictly speaking the parity operation (which would be a true symmetry in ABJM theory) involves in addition a complex conjugation of the fields [78] but as complex conjugating the fields inside an operator does not change its anomalous dimension the present definition suffices for our purposes.

6.5 The derivation of the full dilatation generator

The derivation of the full two-loop dilatation generator of ABJ theory is slightly lengthy but follows closely the one for ABJM theory [83]. The contractions one has to do are the same as before, only now one has to carefully keep track of whether a given contraction gives a factor of N or a factor of M . The Feynman diagrams which contribute at two-loop order consist of the ones depicted in figure 1 plus 14 self-energy diagrams. All diagrams of course come in planar as well as non-planar versions. In order to handle most

⁴We notice that it is not possible to define in a natural and simple way a parity operation which acts only on Z or W fields.

easily the combinatorics of planar as well as non-planar diagrams it is again convenient to make use of the method of effective vertices [86]. An effective vertex is a space-time independent vertex which, when contracted with a given operator of the type (6.10) gives the combinatorial factor associated with a particular Feynman integral times the value of the integral. If things work as in $\mathcal{N} = 4$ SYM and as in ABJM theory [83] the contribution from the bosonic D -terms should cancel against contributions from gluon exchange, fermion exchange and self-interactions to all orders in the genus expansion and this is indeed what happens. To prove this we first calculate the effective vertices corresponding to the four diagrams in figure 1. We notice, however, that for operators belonging to the $SU(2) \times SU(2)$ sector there are no contributions from Fig. 1d. Adding the contributions from the bosonic potential, gluon exchange and fermion exchange we find

$$(V^{bos})^{eff} + (V^{ferm})^{eff} + (V^{gluon})^{eff} = (V_F^{bos})^{eff} + V + const : \left\{ \text{Tr} \left(Z_C^\dagger Z^C \right) + \text{Tr} \left(W_C W^{\dagger C} \right) \right\} :, \quad (6.12)$$

where

$$const = -\frac{1}{8}(\lambda^2 + \hat{\lambda}^2) - \frac{1}{2}\lambda\hat{\lambda} + \frac{5}{24}\frac{\lambda^2}{N^2} + \frac{5}{24}\frac{\hat{\lambda}^2}{M^2} + \frac{1}{3}\frac{\lambda}{N}\frac{\hat{\lambda}}{M}, \quad (6.13)$$

and where $: :$ means that self-contractions should be excluded. The quantity V is a vertex which can be shown to give a vanishing contribution when applied to any operator in the $SU(2) \times SU(2)$ sector. Furthermore, the last term in eqn. (6.12) has exactly the form expected for self-energies and one can show that it precisely cancels the contribution from these. To do so one has to check the cancellation of both the planar and the non-planar part of the constant appearing in eqn. (6.13). The planar part of the analysis can be carried out with the aid of reference [78]. The non-planar part, however, requires a careful analysis of the non-planar versions of the 14 self-energy diagrams.

Collecting everything, we thus verify that the full two-loop dilatation generator is indeed given only by the F -terms in the bosonic potential, i.e.

$$D = (V_F^{bos})^{eff} = -\frac{\lambda}{N}\frac{\hat{\lambda}}{M} : \text{Tr} \left[W^{\dagger A} Z_B^\dagger W^{\dagger C} W_A Z^B W_C - W^{\dagger A} Z_B^\dagger W^{\dagger C} W_C Z^B W_A \right. \\ \left. + Z_A^\dagger W^{\dagger B} Z_C^\dagger Z^A W_B Z^C - Z_A^\dagger W^{\dagger B} Z_C^\dagger Z^C W_B Z^A \right] :. \quad (6.14)$$

It is easy to see that the dilatation generator vanishes when acting on an operator consisting of only two of the four fields from the $SU(2) \times SU(2)$ sector. Accordingly we will denote two of the fields, say Z_1 and W_1 , as background fields and Z_2 and W_2 as excitations. It is likewise easy to see that operators with only one type of excitation, say W_2 's, form a closed set under dilatations. For operators with only W_2 -excitations the dilatation generator consists of four terms whereas in the case with two different types of excitations it has 16 terms. In both cases D is easily seen to reduce to the one of [80, 81] in the planar limit

$$D_{\text{planar}} \equiv \lambda \hat{\lambda} D_0 = \lambda \hat{\lambda} \sum_{k=1}^{2L} (1 - P_{k,k+2}), \quad (6.15)$$

where $P_{k,k+2}$ denotes the permutation between sites k and $k+2$ and $2L$ denotes the total number of fields inside an operator. It differs from the planar dilatation generator of ABJM theory only by having the pre-factor $\lambda \hat{\lambda}$ instead of λ^2 . As explained in [80, 81] this is the Hamiltonian of two alternating $SU(2)$ Heisenberg spin chains, coupled via a momentum condition. As mentioned earlier, integrability implies that there exists a tower of charges which all commute and which commute with the Hamiltonian. In particular, there exists one such charge Q_3 which anti-commutes with parity. In addition, the planar dilatation generator itself commutes with parity, i.e.

$$[D_{\text{planar}}, Q_3] = [D_{\text{planar}}, P] = \{Q_3, P\} = 0. \quad (6.16)$$

As a consequence, the spectrum of the planar theory has degenerate parity pairs, i.e. pairs of operators with identical anomalous dimension but opposite parity. In reference [83] it was shown that for ABJM theory at the non-planar level the two-loop dilatation generator still commutes with parity but the degeneracies between parity pairs are lifted. This hinted towards the absence of higher conserved charges, at least in a standard form. Below we will analyse the situation for ABJ theory and find that again the planar degeneracies disappear but in addition the non-planar two-loop dilatation generator does *not* any longer commute with parity.

When acting with the dilatation generator on a given operator we have to perform three contractions as dictated by the three hermitian conjugate fields. It is easy to see that by acting with the dilatation generator one can change the number of traces in a given operator by at most two. More

precisely, the two-loop dilatation generator has the expansion

$$D = \lambda \hat{\lambda} \left\{ D_0 + \frac{1}{\mathcal{M}} (D_+ + D_-) + \frac{1}{\mathcal{M}^2} (D_{00} + D_{++} + D_{--}) \right\}. \quad (6.17)$$

Here D_+ and D_{++} increase the number of traces by one and two respectively and D_- and D_{--} decrease the number of traces by one and two. Finally, D_0 does not change the number of traces and D_{00} first adds one trace and subsequently removes one or vice versa. The quantity $\frac{1}{\mathcal{M}}$ stands for $\frac{1}{N}$ or $\frac{1}{M}$ and $\frac{1}{\mathcal{M}^2}$ stands for $\frac{1}{N^2}$, $\frac{1}{M^2}$ or $\frac{1}{MN}$.

Even for short operators it is in practice hard to diagonalise the full dilatation generator exactly. But one can relatively easily diagonalise the planar dilatation generator, either by brute force or by means of the Bethe equations. Subsequently the non-planar terms can be treated as perturbations and the energy corrections found approximately using quantum mechanical perturbation theory [43]. Notice that while energy corrections are generically of order $\frac{1}{\mathcal{M}^2}$, degeneracies in the planar spectrum will lead to energy corrections of order $\frac{1}{\mathcal{M}}$. (For details see [83].)

6.6 Short Operators

In this section we determine non-planar corrections to the anomalous dimensions of a number of short operators. This is done by explicitly computing and diagonalising the planar mixing matrix (aided by **GPL Maxima** as well as **Mathematica**) and subsequently determining the non-planar corrections by quantum mechanical perturbation theory.

6.6.1 Operators with excitations on the same chain

In this sector, the simplest set of operators for which one observes degenerate parity pairs as well as non-trivial mixing between operators with one, two and three traces consists of operators of length 14 with three excitations. There are in total 17 such non-protected operators. Among the non-protected operators there are only eight which are not descendants. Their explicit form can be found in reference [83]. The planar anomalous dimensions (in units of $\lambda \hat{\lambda}$), trace structure and parity for these eight operators, denoted as $\mathcal{O}_1, \dots, \mathcal{O}_8$, are

Eigenvector	Eigenvalue	Trace structure	Parity
\mathcal{O}_1	5	(14)	—
\mathcal{O}_2	6	(2)(12)	—
\mathcal{O}_3	5	(14)	+
\mathcal{O}_4	$5 + \sqrt{5}$	(2)(12)	+
\mathcal{O}_5	$5 - \sqrt{5}$	(2)(12)	+
\mathcal{O}_6	4	(4)(10)	+
\mathcal{O}_7	4	(2)(2)(10)	+
\mathcal{O}_8	6	(2)(4)(8)	+

We have one pair of degenerate single trace operators with opposite parity, namely the operators \mathcal{O}_1 and \mathcal{O}_3 .⁵

Expressing the dilatation generator in the basis above and taking into account all non-planar corrections we get (in units of $\lambda\hat{\lambda}$)⁶

$$\left(\begin{array}{cc|cccccc} 5 + \frac{15}{MN} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{N} + \frac{3}{M} & 6 + \frac{24}{MN} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 5 + \frac{35}{MN} & 0 & 0 & -\frac{4}{N} - \frac{4}{M} & -\frac{4}{MN} & -\frac{2}{MN} \\ 0 & 0 & -\frac{\sqrt{5}/2 - \sqrt{5}/2}{MN} & \sqrt{5} + 5 + \frac{(5\sqrt{5} + 35)}{MN} & \frac{3\sqrt{5} - 5}{MN} & \frac{1}{MN} & 0 & \frac{1}{M} + \frac{1}{N} \\ 0 & 0 & -\frac{\sqrt{5}/2 - \sqrt{5}/2}{MN} & -\frac{5 + 3\sqrt{5}}{MN} & 5 - \sqrt{5} - \frac{5\sqrt{5} - 35}{MN} & -\frac{1}{MN} & 0 & -\frac{1}{M} - \frac{1}{N} \\ 0 & 0 & -\frac{10}{N} - \frac{10}{M} & \frac{4\sqrt{5} + 20}{MN} & -\frac{20 - 4\sqrt{5}}{MN} & 4 + \frac{28}{MN} & 0 & 0 \\ 0 & 0 & -\frac{10}{MN} & \frac{2\sqrt{5} + 10}{N} + \frac{2\sqrt{5} + 10}{M} & \frac{2\sqrt{5} - 10}{N} + \frac{2\sqrt{5} - 10}{M} & 0 & 4 + \frac{32}{MN} & -\frac{2}{MN} \\ 0 & 0 & -\frac{10}{MN} & \frac{12\sqrt{5} + 20}{N} + \frac{12\sqrt{5} + 20}{M} & \frac{12\sqrt{5} - 20}{N} + \frac{12\sqrt{5} - 20}{M} & \frac{4}{N} + \frac{4}{M} & -\frac{8}{MN} & 6 + \frac{40}{MN} \end{array} \right).$$

This mixing matrix of course reduces to that of ABJM theory for $N = M$ as it should, cf. [83]. We notice that for this type of operators the positive and negative parity states still decouple, i.e. parity is preserved. The states \mathcal{O}_1 and \mathcal{O}_2 are exact eigenstates of the full dilatation generator with non-planar corrections equal to

$$\delta E_1 = \frac{15}{NM}, \quad \delta E_2 = \frac{24}{NM}. \quad (6.18)$$

For the remaining operators we observe that all matrix elements between degenerate states vanish. Thus the leading non-planar corrections to the

⁵We also observe a degeneracy between the negative parity double trace state \mathcal{O}_2 and the positive parity triple trace state \mathcal{O}_8 as well as a degeneracy between the double trace state \mathcal{O}_6 and the triple trace state \mathcal{O}_7 both of positive parity. However, states with different numbers of traces cannot be connected via the conserved charge Q_3 .

⁶Notice that by construction the mixing matrix is not hermitian but related to its hermitian conjugate by a similarity transformation [37, 43].

anomalous dimensions can be found using second order non-degenerate perturbation theory. The results read

$$\begin{aligned}
\delta E_3 &= \frac{40}{M^2} + \frac{40}{N^2} + \frac{115}{MN}, \quad \delta E_4 = 4(5 + 2\sqrt{5}) \left(\frac{1}{N^2} + \frac{1}{M^2} \right) + \frac{3(25 + 7\sqrt{5})}{MN}, \\
\delta E_5 &= 4(5 - 2\sqrt{5}) \left(\frac{1}{N^2} + \frac{1}{M^2} \right) + \frac{3(25 - 7\sqrt{5})}{MN}, \quad \delta E_6 = -\frac{40}{N^2} - \frac{40}{M^2} - \frac{52}{MN}, \\
\delta E_7 &= \frac{32}{MN}, \quad \delta E_8 = -40 \left(\frac{1}{N^2} + \frac{1}{M^2} \right) - \frac{40}{MN}
\end{aligned} \tag{6.19}$$

We observe that all degeneracies found at the planar level get lifted when non-planar corrections are taken into account, for all values of M and N . This in particular holds for the degeneracies between the members of the planar parity pair $(\mathcal{O}_1, \mathcal{O}_3)$. We have considered a number of different types of states with only one type of excitation and have found that the same pattern persists in all cases. In fact, one can explicitly show that the matrix elements between n and $(n+1)$ -trace states of the normal ordered operator in eqn. (6.14), (i.e. D without its pre-factor) can only depend on M and N through the combination $M + N$. Thus one cannot have parity breaking.

6.6.2 Operators with excitations on both chains

The simplest multiplet of operators which have non-planar energy corrections are operators of length six with two excitations. There are in total three such non-protected highest weight states. These read

$$\begin{aligned}
\mathcal{O}_1 &= \text{Tr}(Z_1 W_1 Z_1 W_2 Z_2 W_1) + \text{Tr}(Z_1 W_1 Z_1 W_1 Z_2 W_2) - 2\text{Tr}(Z_1 W_1 Z_2 W_1 Z_1 W_2), \\
\mathcal{O}_2 &= \text{Tr}(Z_1 W_1 Z_1 W_2 Z_2 W_1) - \text{Tr}(Z_1 W_1 Z_1 W_1 Z_2 W_2), \\
\mathcal{O}_3 &= \text{Tr}(Z_1 W_1) \text{Tr}(Z_1 W_1 Z_2 W_2) - \text{Tr}(Z_1 W_1) \text{Tr}(Z_1 W_2 Z_2 W_1).
\end{aligned} \tag{6.20}$$

Their associated planar anomalous dimension (in units of $\lambda\hat{\lambda}$), parity and trace structure are

Eigenvector	Eigenvalue	Trace Structure	Parity
\mathcal{O}_1	6	(6)	+
\mathcal{O}_2	6	(6)	-
\mathcal{O}_3	8	(2)(4)	-

Already in this simple case we have one pair of degenerate states with opposite parity, namely \mathcal{O}_1 and \mathcal{O}_2 . Expressing the dilatation generator in this basis and taking into account all non-planar corrections we get (in units of $\lambda\hat{\lambda}$)

$$\left(\begin{array}{c|cc} 6 & 0 & \frac{1}{M} - \frac{1}{N} \\ \hline 0 & 6 - \frac{12}{MN} & -\frac{3}{M} - \frac{3}{N} \\ \frac{6}{M} - \frac{6}{N} & -\frac{6}{M} - \frac{6}{N} & 8 - \frac{8}{MN} \end{array} \right).$$

We observe that in this case the dilatation generator does mix states with different parity. In other words, the non-planar dilatation generator does *not* commute with P . Calculating the energies by second order quantum mechanical perturbation theory we find

$$\delta E_1 = -\frac{3}{N^2} - \frac{3}{M^2} + \frac{6}{MN}, \quad \delta E_2 = -\frac{9}{M^2} - \frac{9}{N^2} - \frac{30}{MN}, \quad \delta E_3 = \frac{4}{M^2} + \frac{4}{N^2} + \frac{4}{MN}. \quad (6.21)$$

In particular, we see that the planar degeneracy is lifted.

Let us analyse a slightly larger multiplet of operators with two excitations of different types that exhibit some more of the above mentioned non-trivial features of the topological expansion: Operators of length eight with one excitation of each type. There are in total 7 such non-protected operators. Their explicit form can be found in reference [83] and the planar anomalous dimensions (in units of $\lambda\hat{\lambda}$), trace structure and parity of these operators, denoted as $\mathcal{O}_1, \dots, \mathcal{O}_7$, are

Eigenvector	Eigenvalue	Trace Structure	Parity
\mathcal{O}_1	8	(8)	—
\mathcal{O}_2	4	(8)	—
\mathcal{O}_3	8	(4)(4)	—
\mathcal{O}_4	6	(2)(6)	—
\mathcal{O}_5	8	(2)(2)(4)	—
\mathcal{O}_6	4	(8)	+
\mathcal{O}_7	6	(2)(6)	+

Notice that we have two pairs of degenerate operators with opposite parity, namely the single trace operators \mathcal{O}_2 and \mathcal{O}_6 and the double trace operators \mathcal{O}_4 and \mathcal{O}_7 .⁷

⁷The double trace operators \mathcal{O}_4 and \mathcal{O}_7 can be related via Q_3 when letting Q_3 act only on the longer of the two constituent traces of the operators.

Expressing the dilatation generator in the basis given above and taking into account all non-planar corrections we get (in units of $\lambda\hat{\lambda}$)

$$\left(\begin{array}{cc|cc} \begin{array}{ccccc} 8 & \frac{8}{MN} & \frac{8}{N} + \frac{8}{M} & \frac{2}{N} + \frac{2}{M} & -\frac{8}{MN} \\ \frac{8}{MN} & 4 - \frac{12}{MN} & 0 & -\frac{1}{N} - \frac{1}{M} & -\frac{4}{MN} \\ \frac{8}{N} + \frac{8}{M} & \frac{4}{N} - \frac{4}{M} & 8 & 0 & 0 \\ 0 & -\frac{8}{N} - \frac{8}{M} & -\frac{8}{MN} & 6 - \frac{8}{MN} & \frac{6}{N} - \frac{6}{M} \\ 0 & \frac{8}{MN} & 0 & \frac{6}{N} - \frac{6}{M} & 8 - \frac{8}{MN} \end{array} & \begin{array}{cc} 0 & \frac{2}{M} - \frac{2}{N} \\ 0 & \frac{1}{N} - \frac{1}{M} \\ \frac{4}{M} - \frac{4}{N} & 0 \\ \frac{4}{M} - \frac{4}{N} & 0 \\ 0 & \frac{6}{N} - \frac{6}{M} \end{array} \\ \hline \begin{array}{ccccc} 0 & 0 & 0 & \frac{1}{M} - \frac{1}{N} & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{N} - \frac{2}{M} \end{array} & \begin{array}{cc} 4 + \frac{4}{MN} & \frac{1}{N} + \frac{1}{M} \\ \frac{4}{N} + \frac{4}{M} & 6 + \frac{8}{MN} \end{array} \end{array} \right).$$

This mixing matrix of course reduces to that of ABJM theory for $N = M$ as it should, cf. [83]. We observe again that the dilatation generator does mix states with different parity. To find the corrections to the eigenvalues we use perturbation theory as described in section 6.5. First, we notice that most matrix elements between degenerate states vanish. The only exception are the matrix elements between the states \mathcal{O}_1 and \mathcal{O}_3 . To find the non-planar correction to the energy of these states we diagonalise the Hamiltonian in the corresponding subspace and find

$$\delta E_{1,3} = \mp \left(\frac{8}{N} + \frac{8}{M} \right). \quad (6.22)$$

For the remaining operators the leading non-planar corrections to the energy can be found using second order non-degenerate perturbation theory. The results read

$$\begin{aligned} \delta E_2 &= -\frac{20}{NM} - \frac{4}{N^2} - \frac{4}{M^2}, & \delta E_4 &= -\frac{40}{NM} - \frac{12}{N^2} - \frac{12}{M^2}, \\ \delta E_5 &= \frac{16}{NM} + \frac{24}{N^2} + \frac{24}{M^2}, & \delta E_6 &= \frac{4}{MN} - \frac{4}{N^2} - \frac{4}{M^2}, & \delta E_7 &= \frac{24}{MN} - \frac{4}{N^2} - \frac{4}{M^2}. \end{aligned}$$

We again notice that all degeneracies observed at the planar level get lifted when non-planar corrections are taken into account, for all values of M and N . This in particular holds for the degeneracies between the members of the two parity pairs. We have examined a number of operators with excitations of two different types and found that the same pattern persists in all cases. A closer scrutiny of the action of the dilatation generator reveals that the asymmetry between M and N originates from the situation where the

operator separates two neighbouring excitations, a situation which one does not encounter when the two excitations are on the same chain. Let us note that the characteristic polynomial of the anomalous dimension matrices will always be even in $M - N$. This implies that the eigenvalues will generically be even under the interchange of M and N (as is the case above). A possible exception might arise in cases where nonzero matrix elements appear between planar degenerate states which have opposite parity and differ in trace number by one (notice that the requirement of different trace structure prevents this complication from arising for planar parity pairs). Although mixing of the above type does occur, we did not observe any asymmetry in the eigenvalues for the explicit cases we examined.

6.7 Conclusion

We have derived and analysed the non-planar corrections to the two-loop dilatation generator of ABJ theory in the $SU(2) \times SU(2)$ sub-sector. Our analysis shows that these corrections mix states with positive and negative parity, i.e.

$$[D_{non-planar}^{ABJ}, P] \neq 0. \quad (6.23)$$

More precisely, the value of the commutator is proportional to $M - N$. This is in contrast to earlier studies of the *planar* two-loop dilatation generator which did not reveal any sign of parity breaking [78, 79]. Furthermore, whereas the planar dilatation generator could be proved to be integrable, we do not see any indication of this being the case for the non-planar one, since none of the planar degeneracies between parity pairs survive the inclusion of non-planar corrections. It is an interesting question whether the planar dilatation generator remains integrable and parity invariant when higher loop corrections are taken into account. In this connection it is worth mentioning that parity breaking does not prevent integrability [78, 79]. At planar level, one could try to address the question of parity breaking at higher-loop order from the string theory side by calculating a transition amplitude between two string states of different parity living in an instanton background of the ABJ theory dual. We note that an interesting effect of parity breaking in the non-interacting string theory has been observed in [87].

One could also try to match the results of the present calculation to the behaviour of the dual string theory by calculating the semi-classical amplitude for non-parity-conserving splitting of a one-string state into a two-string

state in the spirit of [72, 73]. Of course, this calculation would at best allow us to obtain qualitative agreement between non-planar gauge theory and interacting string theory. How to achieve quantitative agreement remains a challenge.

Chapter 7

Conclusions

In order not to overlap too much with conclusions from the articles presented before we only highlight our key results and outline some suggestions for future investigation.

The central object in our studies was the one-loop dilatation generator in relatively unexplored $\mathcal{N} = 4$ SYM with orthogonal and symplectic gauge groups. As we showed, its planar limit is just a subset of the $U(N)$ cousin but the Hilbert space of states that it acts on is largely truncated due to parity constraints. At the non-planar level, in addition to known splitting and joining corrections, we revealed a new class of $1/N$ contributions that preserve the number of traces, or in other words, do not break the spin-chain. We found analytic expression for the corresponding energies in the basis of BMN states. This is our main prediction for dual string theory on $AdS_5 \times \mathbb{RP}^5$ [36].

A natural guess is that our new non-planar corrections will be linked to string worldsheets with crosscaps. Intuitively computations on the string theory should be more tractable than on worldsheets with topology of the torus that are dual to the known $1/N^2$ terms. The comparison has not yet been achieved but we hope to report on it in the future.

As far as integrability is concerned, we only tried to search for signs of it by standard methods, namely by constructing a modification of the Bethe Ansatz or higher conserved charges. The outcome of our tests was negative however it is not enough to conclude that integrability is indeed broken beyond the planar limit.

It is very likely that our naive analysis of the perturbative series in $1/N$ is not the most efficient one and the problem requires fundamental rethinking.

Conceptually, we are just taking into account a particular subclass of Feynman ('t Hooft) diagrams and these, as we know well from many examples in amplitude computations, only exhibit certain, beautiful, properties after being summed or rearranged in an appropriate way. Therefore it might be that the theory is integrable but we will only see it when we sum up all the non-planar corrections.

As we can see, even though the problem of non-planar corrections seems to be well defined and easy to work with on the gauge theory side, it is hard to see any clear integrable structure. Based on previous experience, it might be that the best way to look at the problem will be from the string theory perspective. At first, this would require better understanding of the *AdS* sigma model on higher genus Riemann surfaces. There has not been much progress in that direction but it is definitely an interesting field for future exploration.

There are several possible follow-ups on our projects. The most concrete is understanding the details of the string theory dual of $SO(N)$ and $Sp(N)$ theories and finding string configurations that would match the energy correction that we derived. This would be a non-trivial test of the gauge/gravity duality at the non-planar level.

There is definitely more room for understanding the relation between parity and integrable structure. This could be done by deriving and analyzing the full higher loop dilatation operator in *ABJM* models. Also along these lines, it would be interesting to investigate parity in the spectrum of the dilatation operator in the $\mathcal{N} = 2$ theories recently studied in [104].

Since it is quite clear that the standard language that we used to address the non-planar questions is not the most fruitful, a new framework for diagonalization of the full dilatation operator is needed. A way to proceed could be to use the basis of Schur polynomials and express all the observables in terms of the symmetric group theory data along the lines of [96, 105, 106]. This would hopefully allow for a more constructive approach to the problem and give a hint if the non-planar theory is integrable or not.

It is also worth mentioning that interesting $1/N$ contributions appear in scattering amplitudes too. As we have learned by studying them at their planar limit, they exhibit many surprising identities that are not obviously linked with integrability of the underlying theory. It will be very interesting to investigate how many of these relations hold beyond the planar approximation. It will hopefully shed some light on the non-planar structure of the dilatation operator as well.

Appendix A

Contraction rules

In this appendix we derive the contraction rules for unitary $U(N)$, special unitary $SU(N)$, special orthogonal $SO(N)$ and symplectic $Sp(N)$ gauge groups. We are only interested in the combinatorial part so the spacetime dependence is always omitted. We choose to normalize all the generators with the same constant

$$\text{Tr}T^aT^b = a\delta^{ab}. \quad (\text{A.1})$$

As a consequence the action is the same for all gauge groups and we don't need to worry about different constants in Feynman rules.

The dimensions of the adjoint representation for our gauge groups are

dim(Adj)			
$U(N)$	$SU(N)$	$SO(N)$	$Sp(N)$
N^2	$N^2 - 1$	$\frac{1}{2}N(N - 1)$	$\frac{1}{2}N(N + 1)$

They can be deduced from the completeness relations

Completeness relations	
$U(N)$	$(T^a)_{\alpha}^{\beta} (T^a)_{\nu}^{\mu} = a\delta_{\nu}^{\beta}\delta_{\alpha}^{\mu}$
$SU(N)$	$(T^a)_{\alpha}^{\beta} (T^a)_{\nu}^{\mu} = a(\delta_{\nu}^{\beta}\delta_{\alpha}^{\mu} - \frac{1}{N}\delta_{\alpha}^{\beta}\delta_{\nu}^{\mu})$
$SO(N)$	$(T^a)_{\alpha\beta} (T^a)_{\nu\mu} = \frac{a}{2}(\delta_{\beta\nu}\delta_{\alpha\mu} - \delta_{\alpha\nu}\delta_{\beta\mu})$
$Sp(N)$	$(T^a)_{\alpha\beta} (T^a)_{\nu\mu} = \frac{a}{2}(\delta_{\beta\nu}\delta_{\alpha\mu} - J_{\alpha\nu}J_{\beta\mu})$

where for the symplectic group we have $J^2 = -1$. It is then easy to check that for $\beta = \nu$ and $\alpha = \mu$ in the completeness relations give the appropriate dimensionalities.

Now we are ready to derive all the relevant contractions that are extensively

used while applying the dilatation operator to any state. A contracted pair will be denoted by \check{X} and X and all the other letters will denote words of fields of an arbitrary length. Also for simplicity we set $a = 1$.

A.1 $U(N)$

The most common contractions are

$$Tr [\check{X} X O Y] = N Tr [O Y] \quad (A.2)$$

$$Tr [\check{X} O X Y] = Tr [O] Tr [Y] \quad (A.3)$$

$$Tr [\check{X} O] Tr [X Y] = Tr [O Y] \quad (A.4)$$

$$(A.5)$$

Derivations

$$Tr [\check{X} X O Y] = \check{X}_\beta^\alpha X_\alpha^\gamma O_\gamma^\mu Y_\mu^\beta = \delta_\alpha^\alpha \delta_\beta^\gamma O_\gamma^\mu Y_\mu^\beta = N Tr [O Y] \quad (A.6)$$

$$Tr [\check{X} O X Y] = \check{X}_\beta^\alpha O_\alpha^\gamma X_\gamma^\mu Y_\mu^\beta = \delta_\gamma^\alpha \delta_\beta^\mu O_\alpha^\gamma Y_\mu^\beta = Tr [O] Tr [Y] \quad (A.7)$$

$$Tr [\check{X} O] Tr [X Y] = \check{X}_\beta^\alpha O_\alpha^\beta X_\gamma^\mu Y_\mu^\gamma = \delta_\gamma^\alpha \delta_\beta^\mu O_\alpha^\beta Y_\mu^\gamma = Tr [O Y] \quad (A.8)$$

A.2 $SU(N)$

The same contraction rules for special unitary matrices are

$$Tr [\check{X} X O Y] = N Tr [O Y] - \frac{1}{N} Tr [O Y] \quad (A.9)$$

$$Tr [\check{X} O X Y] = Tr [O] Tr [Y] - \frac{1}{N} Tr [O Y] \quad (A.10)$$

$$Tr [\check{X} O] Tr [X Y] = Tr [O Y] - \frac{1}{N} Tr [O] Tr [Y] \quad (A.11)$$

Derivations

$$\begin{aligned} Tr [\check{X} X O Y] &= \check{X}_\beta^\alpha X_\alpha^\gamma O_\gamma^\mu Y_\mu^\beta = \delta_\alpha^\alpha \delta_\beta^\gamma O_\gamma^\mu Y_\mu^\beta - \frac{1}{N} \delta_\beta^\alpha \delta_\alpha^\gamma O_\gamma^\mu Y_\mu^\beta \\ &= N Tr [O Y] - \frac{1}{N} Tr [O Y] \end{aligned} \quad (A.12)$$

$$\begin{aligned}
Tr [\check{X} O X Y] &= \check{X}_\beta^\alpha O_\alpha^\gamma X_\gamma^\mu Y_\mu^\beta = \delta_\gamma^\alpha \delta_\beta^\mu O_\alpha^\gamma Y_\mu^\beta - \frac{1}{N} \delta_\beta^\alpha \delta_\gamma^\mu O_\alpha^\gamma Y_\mu^\beta \\
&= Tr [O] Tr [Y] - \frac{1}{N} Tr [O Y] \quad (\text{A.13})
\end{aligned}$$

$$\begin{aligned}
Tr [\check{X} O] Tr [X Y] &= \check{X}_\beta^\alpha O_\alpha^\beta X_\gamma^\mu Y_\mu^\gamma = \delta_\gamma^\alpha \delta_\beta^\mu O_\alpha^\beta Y_\mu^\gamma - \frac{1}{N} \delta_\beta^\alpha \delta_\gamma^\mu O_\alpha^\beta Y_\mu^\gamma \\
&= Tr [O Y] - \frac{1}{N} Tr [O] Tr [Y] \quad (\text{A.14})
\end{aligned}$$

A.3 $SO(N)$

In the orthogonal case we have

$$Tr [\check{X} X O Y] = \frac{1}{2} (N Tr [O Y] - Tr [O Y]) \quad (\text{A.15})$$

$$Tr [\check{X} O X Y] = \frac{1}{2} (Tr [O] Tr [Y] - Tr [O Y^T]) \quad (\text{A.16})$$

$$Tr [\check{X} O] Tr [X Y] = \frac{1}{2} (Tr [O Y] - Tr [O Y^T]) \quad (\text{A.17})$$

$$(\text{A.18})$$

Derivations

$$\begin{aligned}
Tr [\check{X} X O Y] &= \check{X}_{\alpha\beta} X_{\beta\gamma} O_{\gamma\delta} Y_{\delta\alpha} = \frac{1}{2} (N \delta_{\alpha\gamma} - \delta_{\alpha\gamma}) O_{\gamma\delta} Y_{\delta\alpha} \\
&= \frac{1}{2} (N Tr [O Y] - Tr [O Y]) \quad (\text{A.19})
\end{aligned}$$

$$\begin{aligned}
Tr [\check{X} O X Y] &= \check{X}_{\alpha\beta} O_{\beta\gamma} X_{\gamma\delta} Y_{\delta\alpha} = \frac{1}{2} (\delta_{\alpha\delta} \delta_{\beta\gamma} - \delta_{\alpha\gamma} \delta_{\beta\delta}) O_{\beta\gamma} Y_{\delta\alpha} \\
&= \frac{1}{2} (O_{\beta\beta} Y_{\delta\delta} - O_{\beta\gamma} Y_{\beta\gamma}) = \frac{1}{2} (Tr [O] Tr [Y] - Tr [O Y^T]) \quad (\text{A.20})
\end{aligned}$$

$$\begin{aligned}
Tr [\check{X} O] Tr [X Y] &= \check{X}_{\alpha\beta} O_{\beta\alpha} X_{\gamma\delta} Y_{\delta\gamma} = \frac{1}{2} (\delta_{\alpha\delta} \delta_{\beta\gamma} - \delta_{\alpha\gamma} \delta_{\beta\delta}) O_{\beta\alpha} Y_{\delta\gamma} \\
&= \frac{1}{2} (O_{\beta\alpha} Y_{\alpha\beta} - O_{\beta\alpha} Y_{\beta\alpha}) = \frac{1}{2} (Tr [O Y] - Tr [O Y^T]) \quad (\text{A.21})
\end{aligned}$$

A.4 $Sp(N)$

And finally for symplectic matrices

$$Tr [\check{X} X O Y] = \frac{1}{2} (N Tr [O Y] + Tr [O Y]) \quad (A.22)$$

$$Tr [\check{X} O X Y] = \frac{1}{2} (Tr [O] Tr [Y] - Tr [O J Y^T J]) \quad (A.23)$$

$$Tr [\check{X} O] Tr [X Y] = \frac{1}{2} (Tr [O Y] + Tr [O J Y^T J]), \quad (A.24)$$

(A.25)

where J is usually chosen to be

$$J = \begin{pmatrix} 0 & \mathbf{1}_k \\ -\mathbf{1}_k & 0 \end{pmatrix}, \quad (A.26)$$

where $n = 2k$. It satisfies

$$J^2 = J_{\alpha\beta} J_{\beta\gamma} \equiv -\delta_{\alpha\gamma} = -1, \quad J^{-1} = J^T = -J. \quad (A.27)$$

Derivations

$$\begin{aligned} Tr [\check{X} X O Y] &= \check{X}_{\alpha\beta} X_{\beta\gamma} O_{\gamma\delta} Y_{\delta\alpha} = \frac{1}{2} (N \delta_{\alpha\gamma} - J_{\alpha\beta} J_{\beta\gamma}) O_{\gamma\delta} Y_{\delta\alpha} \\ &= \frac{1}{2} (N Tr [O Y] + Tr [O Y]) = \frac{N+1}{2} Tr [O Y]. \end{aligned} \quad (A.28)$$

$$\begin{aligned} Tr [\check{X} O X Y] &= \check{X}_{\alpha\beta} O_{\beta\gamma} X_{\gamma\delta} Y_{\delta\alpha} = \frac{1}{2} (\delta_{\alpha\delta} \delta_{\beta\gamma} - J_{\alpha\gamma} J_{\beta\delta}) O_{\beta\gamma} Y_{\delta\alpha} \\ &\quad \frac{1}{2} (O_{\beta\beta} Y_{\delta\delta} - J_{\alpha\gamma} O_{\gamma\beta}^T J_{\beta\delta} Y_{\delta\alpha}) = \frac{1}{2} (Tr [O] Tr [Y] - Tr [O J Y^T J]) \end{aligned} \quad (A.29)$$

$$\begin{aligned} Tr [\check{X} O] Tr [X Y] &= \check{X}_{\alpha\beta} O_{\beta\alpha} X_{\gamma\delta} Y_{\delta\gamma} = \frac{1}{2} (\delta_{\alpha\delta} \delta_{\beta\gamma} - J_{\alpha\gamma} J_{\beta\delta}) O_{\beta\alpha} Y_{\delta\gamma} \\ &= \frac{1}{2} (Tr [O Y] + Tr [O J Y^T J]) \end{aligned} \quad (A.30)$$

Appendix B

Chan-Paton factors and gauge theories

One of the main projects that this thesis is based on was to investigate the gauge/gravity duality for orthogonal $SO(n)$ and symplectic $Sp(n)$ gauge groups. Here I review some basic knowledge about Chan-Paton factors and the way they lead to a gauge theory with unitary (oriented strings), orthogonal or symplectic gauge groups (non-oriented strings). It is sufficient to consider the example of the open bosonic string since, as we will see, Chan-Paton factors are the same for open superstring theories. More details can be found in e.g. [89] or other standard books on string theory.

B.0.1 Oriented strings

Originally Chan-Paton factors were introduced in the old models of strings for strong interactions. Quark and antiquark were connected with a flux tube (a string) and since quarks had to carry the $SU(3)$ flavour quantum numbers the end points were labeled by q_i , $i = 1, 2, 3$. In the context of the contemporary open string theories these quantum numbers are naturally be generalized to an arbitrary integer n (see Fig.B.0.1). This way string states in addition to the oscillator number N and the momentum k are label by two integers, from the left and the right endpoint of the string, $i, j = 1, \dots, n$

$$|N, k, ij\rangle, \quad (B.1)$$

hence at each level we have $n \times n$ states. Then in a natural way a set of n^2

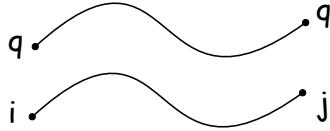


Figure B.1: Chan-Paton factors. Top: Old string for strong interactions with quark and antiquark at the endpoints connected with the flux tube. Bottom: General open sting with chages $I, J = 1, \dots, N$.

Hermitain matrices λ_{ij}^a forms a complete basis for the endpoints. Namely we can write any string state at level N with k as

$$|N, k, a\rangle = \sum_{i,j=1}^n \lambda_{ij}^a |N, k, ij\rangle. \quad (\text{B.2})$$

Matrices λ_{ij}^a are normalized to

$$\text{Tr}(\lambda^a \lambda^b) = \delta^{ab}, \quad (\text{B.3})$$

and furnish the representation of the unitary group $U(n)$. They are usually refered to as the Chan-Paton factors.

Notice that by definition the factors does not interfere with either the worldsheet or spacetime coordinates. Therefore neither the conformal or the Poincare symmetries change with these new degrees of freedom.

By analyzing the scattering amplitudes of the masless open string states we can see that are efectively described by the Yang-Mills theory with $U(n)$ gauge group. Let us see this in more details. First of all when we study the disck amplitudes of states with Chan-Paton factors the extra factor that emerges for an appropriate ordering is

$$\text{Tr}(\lambda^{a_1} \lambda^{a_2} \lambda^{a_3} \lambda^{a_4}). \quad (\text{B.4})$$

This exhibits a global $U(n)$ symmetry under transformations

$$\lambda^a \rightarrow U \lambda^a U^\dagger. \quad (\text{B.5})$$

From studying the scattering amplitudes of three and four masless gauge bosons we realize that this global worldsheet symmetry is also a local symmetry of the effective theory dercribing interactions. Namely, the interaction of the massles vector string states is described in terms of the action

$$S = -\frac{1}{4g_o'^2} \int \text{Tr}(F_{\mu\nu} F^{\mu\nu}) \quad (\text{B.6})$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i [A_\mu, A_\nu], \quad (\text{B.7})$$

and the vector fields are $n \times n$ matrices $A_\mu = A_\mu^a \lambda^a$ in the adjoint representation of the $U(n)$ gauge group. This way the low energy effective action for open strings is just given by the $U(n)$ Yang-Mills theory (B.6).

B.0.2 Non-oriented strings

The spectrum of non oriented strings, as the name indicates, should be symmetric under the change of orientation of the worldsheet. By this we mean changing parameter $\sigma \rightarrow \pi - \sigma$ for open strings and $\sigma \rightarrow 2\pi - \sigma$ for the closed. This can be expressed more precisely by introducing the worldsheet-parity operator Ω . On the open string states without Chan-Paton factors its action is

$$\Omega |N, k\rangle = \omega_N |N, k\rangle \quad (\text{B.8})$$

where

$$\omega_N = (-1)^{1+\alpha' m^2}. \quad (\text{B.9})$$

since the spectrum of unoriented strings should have $\omega_N = 1$, only states with odd masses would be allowed. Nevertheless when we introduce Chan-Paton factors the worldsheet parity acts on the states as

$$\Omega |N, k, ij\rangle = \omega_N |N, k, ji\rangle. \quad (\text{B.10})$$

If we now pick a basis for λ to be either symmetric ($s^a = 1$) or anti-symmetric ($s^a = -1$), then

$$\Omega |N, k, a\rangle = \omega_N s^a |N, k, a\rangle, \quad (\text{B.11})$$

end the eigenvalue of the worldsheet parity operator is now equal to $\omega_N s^a$. This gives us two possibilities for the states of the unoriented open strings with Chan-Paton factors, namely we can have states with odd masses and antisymmetric λ or even masses and symmetric λ .

Our main interest is on the massless gauge bosons hence we consider antisymmetric λ 's. The scattering amplitudes are simply captured by the Yang-Mills action however the fields are now $A_\mu = A_\mu^a \lambda^a$, whith $a = 1, \dots, \frac{1}{2}n(n-1)$, and transform in the adjoint representation of the special orthogonal group $SO(n)$.

One more group that we can obtain from Chan-Paton factors is the symplectic $Sp(n)$. In this case we first consider more general symmetry group of

the oriented strings which consists of the worldsheet parity action combined with a $U(n)$ rotation. It is usually denoted as Ω_γ and on a general open string state it acts as

$$\Omega_\gamma |N, k, ij\rangle = \omega_N \gamma_{jj'} |N, k, j'i'\rangle \gamma_{i'i}^{-1}. \quad (\text{B.12})$$

Analogously to the previous cases we can require that the states of unoriented theory are with the eigenvalue $\omega_\gamma = 1$. In addition we require that acting twice with Ω_γ should bring us to the same state. This is expressed by the condition

$$\Omega_\gamma^2 |N, k, ij\rangle = \omega_N^2 [\gamma (\gamma^T)]_{ii'} |N, k, i'j'\rangle (\gamma^{-1} \gamma^T)_{j'j}, \quad (\text{B.13})$$

so we have

$$\gamma^T = \pm \gamma. \quad (\text{B.14})$$

Hence we must have either symmetric or antisymmetric γ . For the symmetric case it is always possible to choose a basis that γ is equal to unity. This brings us back to the $SO(n)$ case. However the antisymmetric γ lead to a more interesting modification. Namely one can choose a basis in which

$$J = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix}. \quad (\text{B.15})$$

Then choosing a basis for Chan-Paton matrices such that

$$J(\lambda^a)^T J = s^a \lambda^a, \quad s = \pm 1, \quad (\text{B.16})$$

the states of unoriented open string theory with odd mass have $s^a = 1$, whereas with even mass $s^a = -1$. This way we get massless gauge bosons described by the Yang-Mills theory with matrices

$$J(\lambda^a)^T J = -\lambda^a, \quad (\text{B.17})$$

that transform in the adjoint representation of $Sp(2n)$ group.

The discussion is completely analogous for type IIB superstrings on $AdS_5 \times S^5$ and their relation to the $\mathcal{N} = 4$ SYM with fields in the adjoint representation of the unitary, orthogonal or symplectic gauge groups. There will be additional requirements in order for SUSY to be preserved but we will discuss them while analyzing specific examples.

Appendix C

Perturbation Theory in QM

Even though the perturbation theory is a part of the undergraduate course on quantum mechanics, we briefly review nondegenerate and degenerate, time-independent perturbation theory in this appendix. We closely follow [88] where one can find more details.

C.1 Nondegenerate perturbation theory

Let us begin with the nondegenerate case when there is no more than one state with the same energy.

Assume that we are given a Hamiltonian H_0 and a complete set of orthonormal eigenstates ϕ_n

$$\langle \phi_n | \phi_m \rangle = \delta_{nm}, \quad (C.1)$$

with corresponding eigenvalues E_n^0 , such that the eigen-equation holds

$$H_0 \phi_n = E_n^0 \phi_n. \quad (C.2)$$

This will be our starting point to which we will want to add a small perturbation. We introduce an expansion parameter $\lambda \ll 1$ and consider a system described by

$$H = H_0 + \lambda H_1 \quad (C.3)$$

where H_1 is some hermitean operator. The problem that we would like to solve is finding a complete set of states ψ_n with E_n so that

$$(H_0 + \lambda H_1) \psi_n = E_n \psi_n. \quad (C.4)$$

The way we will proceed is that we will expand the eigenstates and eigenvalues of the new Hamiltonian in powers of λ and then plugging into (C.4), we will solve equations that come with appropriate powers of λ ($\lambda^0, \lambda^1 \dots$ etc.). Naturally we require that for $\lambda \rightarrow 0$

$$E_n \rightarrow E_n^0, \quad \psi_n \rightarrow \phi_n. \quad (\text{C.5})$$

First, since ϕ_n form a complete set, we can expand ψ_n in this basis as

$$\psi_n = N(\lambda) \left\{ \phi_n + \sum_{k \neq n} C_{nk}(\lambda) \phi_k \right\}, \quad (\text{C.6})$$

where

$$C_{nk}(\lambda) = \lambda C_{nk}^{(1)} + \lambda^2 C_{nk}^{(2)} + \dots, \quad (\text{C.7})$$

and in order to satisfy (C.5)

$$N(0) = 1, \quad C_{nk}(0) = 0. \quad (\text{C.8})$$

Then we expand the eigenvalues

$$E_n = E_n^0 + \lambda E_n^1 + \lambda^2 E_n^2 + \dots, \quad (\text{C.9})$$

and plug everything into (C.4)

$$\begin{aligned} & (H_0 + \lambda H_1) \left\{ \phi_n + \lambda \sum_{k \neq n} C_{nk}^1 \phi_k + \lambda^2 \sum_{k \neq n} C_{nk}^2 \phi_k + \dots \right\} \\ &= (E_n^0 + \lambda E_n^1 + \lambda^2 E_n^2 + \dots) \left\{ \phi_n + \lambda \sum_{k \neq n} C_{nk}^1 \phi_k + \lambda^2 \sum_{k \neq n} C_{nk}^2 \phi_k + \dots \right\} \end{aligned} \quad (\text{C.10})$$

At λ^0 we simply have the eigen equation for H_0 (C.2). Then at λ^1

$$H_0 \sum_{k \neq n} C_{nk}^1 \phi_k + H_1 \phi_n = E_n^0 \sum_{k \neq n} C_{nk}^1 \phi_k + E_n^1 \phi_n, \quad (\text{C.11})$$

which with the use of (C.2) can be written as

$$E_n^1 \phi_n = H_1 \phi_n + \sum_{k \neq n} (E_k^0 - E_n^0) C_{nk}^1 \phi_k. \quad (\text{C.12})$$

By taking the scalar product with ϕ_n (and using orthonormality) we determine the first correction to the energy

$$E_n^1 = \langle \phi_n | H_1 | \phi_n \rangle. \quad (\text{C.13})$$

If instead we take a scalar product with ϕ_l , $l \neq n$ then

$$\langle \phi_l | H_1 | \phi_n \rangle = C_{nl}^1 (E_n^0 - E_l^0), \quad (\text{C.14})$$

and we can determine the correction to the eigenstate

$$C_{nl}^1 = \frac{\langle \phi_l | H_1 | \phi_n \rangle}{E_n^0 - E_l^0}. \quad (\text{C.15})$$

This can be continued to higher powers in λ but for our purposes the above analysis is sufficient.

Summarizing, the corrected wave function and the new energy are

$$\psi_n = \phi_n + \lambda \sum_{k \neq n} \frac{\langle \phi_k | H_1 | \phi_n \rangle}{E_n^0 - E_k^0} \phi_k + O(\lambda^2), \quad (\text{C.16})$$

and

$$E_n = E_n^0 + \lambda \langle \phi_n | H_1 | \phi_n \rangle. \quad (\text{C.17})$$

Let us now see how the analysis is modified when degenerations in the spectrum are present.

C.2 Degenerate perturbation theory

Perturbation theory with degenerate states works very similarly to the non-degenerate case but we have to remember that there might be a subset of eigenstates ϕ_n^i of the Hamiltonian H_0 that has the same energy E_n^0 , namely

$$H_0 \phi_n^i = E_n^0 \phi_n^i. \quad (\text{C.18})$$

In this case we first solve the problem in the non-degenerate subset according to the previous section, and then deal with degenerate states separately. This is done as follows.

Again we assume that the eigenstates are normalized as

$$\langle \phi_n^i | \phi_m^j \rangle = \delta_{nm} \delta^{ij}. \quad (\text{C.19})$$

Moreover we write ψ_n in terms of linear combinations of the degenerate states

$$\psi_n = N(\lambda) \left\{ \sum_i \alpha_i \phi_n^i + \lambda \sum_{k \neq n} C_{nk}^1 \sum_i \beta_i \phi_k^i + \dots \right\} \quad (\text{C.20})$$

and after plugging everything into (C.4), the task is to find the corrections to the energy and eigenstates, as well as α_i and β_i . To the first order in λ we get

$$H_0 \sum_{k \neq n} C_{nk}^1 \sum_i \beta_i \phi_k^i + H_1 \sum_i \alpha_i \phi_n^i = E_n^1 \sum_i \alpha_i \phi_n^i + E_n^0 \sum_{k \neq n} C_{nk}^1 \sum_i \beta_i \phi_k^i. \quad (\text{C.21})$$

By taking the scalar product with ϕ_n^j we get a set of equations

$$E_n^1 \alpha_j = \sum_i \alpha_i \langle \phi_n^j | H_1 | \phi_n^i \rangle. \quad (\text{C.22})$$

In addition one often have to assume that

$$\sum_i |\alpha_i|^2 = 1. \quad (\text{C.23})$$

On the other hand, taking the scalar product with ϕ_l^j , $l \neq n$ leads to

$$C_{nl}^1 \beta_j = \sum_i \alpha_i \frac{\langle \phi_l^j | H_1 | \phi_n^i \rangle}{E_n^0 - E_l^0}. \quad (\text{C.24})$$

Depending on the level of degeneracy we have to solve the above sets of equations to diagonalize the Hamiltonian in the degenerate perturbation theory.

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