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CANCELLATION OF THE CENTRIFUGAL SPACE-CHARGE FORCE†

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The transverse dynamics of high-energy electrons confined in curved geometry are examined, including the effects of space-charge-induced fields. Attention is restricted to the centrifugal-space-charge force, which is the result of noncancellation of beam-induced transverse electric and magnetic fields in the curved geometry. This force is shown to be nearly cancelled in the evaluation of the horizontal tune and chromaticity by another, often overlooked term in the equations of motion. The additional term is the consequence of oscillations of the kinetic energy, which accompany betatron oscillations in the beam-induced electric potential. In curved geometry this term is of first order in the amplitude of the radial oscillation. A highly simplified system model is employed so that physical effects appear in as clear a form as possible. We assume azimuthal and median plane symmetry, static fields, and ultrarelativistic particle velocity ($1/\gamma^2 \to 0$).

1. INTRODUCTION

At high energies it is well known that beam-induced transverse electric and magnetic fields of a long beam bunch cancel to order $1/\gamma^2$ provided the beam pipe is straight and smooth:

$$\left(\mathbf{E}+\frac{\mathbf{v}}{c}\times\mathbf{B}\right)_b \sim \mathrm{O}\left(\frac{2\lambda}{\gamma^2 a}\right)\to 0,$$

where λ is the line charge density, a is the scale beam radius, and subscript b denotes beam-induced components of fields. In curved geometry this relativistic cancellation is incomplete, with resultant order of magnitude[1]

$$\left(\mathbf{E}+\frac{\mathbf{v}}{c}\times\mathbf{B}\right)_b \sim \mathrm{O}\left(\frac{\lambda}{R_0}\right),$$

where R_0 is the radius of curvature of the design orbit. This residual force has been termed[2] the centrifugal space-charge force (CSCF) in recent work and was identified during 1985 as a serious concern for its effect on dynamics in electron storage rings. The CSCF was the subject of considerable theoretical study at that time,[1–6] with effort concentrated primarily on calculation of the force in various

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simplified geometries. Phenomena associated with CSCF include[6] a shift in design energy for given bend field, shift in horizontal tune, nonlinear resonances,[2] and a significant contribution to chromaticity.

A vital point that was overlooked in the recent development of the CSCF is the existence of a second space-charge/curvature effect in the particle dynamics. A particle undergoing betatron oscillations has simultaneous oscillations of its kinetic energy due to its motion through the beam's electric potential. In curved geometry the kinetic-energy oscillation results in a first-order dynamical term in the horizontal equation of motion, which shifts the betatron frequency. For a highly relativistic beam this additional term nearly cancels a term proportional to the gradient of the CSCF. Specifically, in addition to the expected CSCF gradient term of form

$$\frac{\partial}{\partial r}(E_r + B_z)_b \sim O\left(\frac{\lambda}{R_0 a}\right),$$

there is a term E_r/r produced by the energy oscillation that produces the cancellation

$$\left(\frac{E_r}{r} + \frac{\partial E_r}{\partial r} + \frac{\partial B_z}{\partial r}\right)_b \sim O\left(\frac{\lambda}{R_0^2}\right).$$

With this cancellation taken into account, the redisual effect is on the order of 10^{-3}–10^{-5} smaller than that of the CSCF gradient alone for typical storage-ring parameters. This cancellation is clearly critical for a proper evaluation of space-charge effects on dynamics.

It should not be assumed that the CSCF never has a significant effect on tune and chromaticity. These quantities can, in some situations, be shifted from their low-current values because the relation between total energy and equilibrium orbit is altered by the CSCF. This feature, which is also briefly treated here, differs qualitatively from the direct dynamical role of the CSCF gradient that is the main concern of this work.

Kinetic-energy oscillations were taken into account in some older work (1965–70) with application to the electron-ring approach to collective acceleration of ions.[7,8] Unfortunately, much of the literature associated with that effort appears in reports not generally available. Correct expressions for tunes of an intense stored beam were derived at that time, but the cancellation of space-charge/curvature effects at high energy was not emphasized.

The purpose of the present paper is to derive the space-charge/curvature terms for the simplest model systems and demonstrate their cancellation. Therefore, an azimuthally symmetric ring (weak external focusing only) is assumed. The particles are ultrarelativistic, so effects of order $1/\gamma^2$ are dropped. The subject has been controversial, and to avoid introducing further confusion a relatively large number of elementary steps is included in the derivations. Section 2 contains details of the model. Section 3 presents the expression for the equilibrium orbit and Section 4 contains a derivation of horizontal and vertical tunes. A detailed calculation of fields and their gradients is made in Section 5, and the cancellation of space-charge/curvature effects is demonstrated. A summary of elementary consequences of the CSCF is given in Section 6.

2. SYSTEM SPECIFICATION

To evaluate the space-charge/curvature effects in a simple, unambiguous manner, we consider a weak-focusing system with all fields constant in time and having azimuthal and median plane symmetry. A cylindrical coordinate system (r, θ, z) is employed, with electrons (charge $= -e$) circulating in the positive θ direction. The kinetic energy is assumed to be large enough so that terms proportional to $1/\gamma^2$ can be neglected $(P = \gamma mc)$. Then the equations of motion for a single electron are

$$\dot{\gamma}\dot{r} + \gamma(\ddot{r} - r\dot{\theta}^2) = -\frac{e}{m}\left(E_r + \frac{r\dot{\theta}}{c}B_z\right), \tag{1}$$

$$\dot{\gamma}\dot{z} + \gamma\ddot{z} = -\frac{e}{m}\left(E_z - \frac{r\dot{\theta}}{c}B_r\right), \tag{2}$$

$$\dot{\gamma} = -\frac{e}{mc^2}(\dot{r}E_r + \dot{z}E_z), \tag{3}$$

$$\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2 = c^2. \tag{4}$$

An externally imposed magnetic field is specified with median plane value $B_e(r)\hat{e}_z$. In addition to this field, which is treated separately, there are the beam-induced electric and magnetic fields, whose respective sources are the beam charge and current densities and which are influenced by the beam pipe geometry (shielding). Because of the assumed symmetries, the beam-induced fields can be derived from scalar and vector potentials whose nonzero components satisfy

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = -4\pi\rho, \tag{5}$$

$$\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}rA_\theta + \frac{\partial^2 A_\theta}{\partial z^2} = -\frac{4\pi}{c}J_\theta, \tag{6}$$

where the beam's charge density is ρ. Gaussian units are employed. The electric and magnetic fields induced by the beam are given by the usual relations, i.e., $E_{r_b} = -\partial\phi/\partial r$, $B_{r_b} = -\partial A_\theta/\partial z$, etc. For the present calculation we neglect the nonlinear effect on current resulting from betatron oscillations and, therefore, have

$$J_\theta = \rho c, \tag{7}$$

the modification of J_θ being second order in the mean betatron oscillation amplitudes. Both ϕ and A_θ are taken to vanish on the conducting beam-pipe boundary; that is, the fully shielded case is treated when a pipe is present. Some previous calculations have treated an unshielded charge ring; this is also treated here in Section 5.

3. EQUILIBRIUM ORBIT

The total particle energy

$$E = \gamma mc^2 - e\phi \tag{8}$$

is conserved. From Eq. (1) the circular median plane orbit associated with energy E has radius R satisfying

$$\gamma mc^2 = Re(E_r + B_z), \tag{9}$$

where the fields are the sum of external and beam-induced components. We define the CSCF with an overall minus sign to agree with the conventions of other authors:

$$F = -(E_r + B_z)_b. \tag{10}$$

Then, since

$$E_r + B_z = B_{ze} - F$$

we have the equilibrium relation

$$\gamma(R)mc^2 = Re[B_e(R) - F(R)]. \tag{11}$$

Here $B_e(R)$ and $F(R)$ are the median plane values of B_{ze} and F at R. It will be shown in Section 5 that in a good approximation, $F = -\phi/r$, which is positive. Hence for given R the value of γ is reduced somewhat from the low-current relation ($\gamma = eRB_e/mc^2$), and we have the approximate relation $E \cong eRB_e(R)$, which includes the effect of the beam-induced fields. Note that the circular radius R is not in general equal to the design orbit radius (R_0) since we are considering a beam with a spread of energies around the design energy E_0. In fact, R_0 may be set arbitrarily, although it is usually convenient to define it to be the beam center.

4. BETATRON FREQUENCIES

Every electron is characterized by an equilibrium radius $R(E)$ for a circular orbit in the median plane and may undergo small-amplitude (betatron) oscillations in r and z around that orbit. The particle variables satisfy the linearized equations of motion derived from Eqs. (1)–(4). We define

$$r = R + \delta r, \tag{12a}$$

$$\gamma = \gamma(R) + \delta\gamma, \tag{12b}$$

$$z = \delta z, \tag{12c}$$

$$B_z = B_z(R) + \frac{\partial B_z}{\partial r}(R)\,\delta r, \tag{12d}$$

$$B_r = \frac{\partial B_r}{\partial z}(R)\,\delta z, \tag{12e}$$

$$E_r = E_r(R) + \frac{\partial E_r}{\partial r}(R)\,\delta r, \tag{12f}$$

$$E_z = \frac{\partial E_z}{\partial z}(R)\,\delta z. \tag{12g}$$

The time-averaged values of the small-amplitude quantities vanish in linear (first-order) theory. Note also that $r\dot{\theta}$ is constant $(=c)$ through this order. The linearized equations of motion are

$$-\frac{c^2}{R}\delta\gamma+\gamma(R)\left(\ddot{\delta r}+\frac{c^2}{R^2}\delta r\right)=-\frac{e}{m}\left(\frac{\partial E_r}{\partial r}+\frac{\partial B_z}{\partial r}\right)_R\delta r, \tag{13}$$

$$\gamma(R)\ddot{\delta z}=-\frac{e}{m}\left(\frac{\partial E_z}{\partial z}-\frac{\partial B_r}{\partial z}\right)_R\delta z, \tag{14}$$

$$\delta\dot{\gamma}=-\frac{e}{mc^2}E_r(R)\delta\dot{r}, \tag{15}$$

$$\frac{c}{R}\delta r+R\delta\dot{\theta}=\delta(r\dot{\theta})=0. \tag{16}$$

The feature that is different from the usual in this system is the retention of the first term on the left-hand side of Eq. (13) $(=-c^2\delta\gamma/R)$. It is found to nearly cancel the portion of the right-hand side of Eq. (13), which is induced by the beam. As mentioned, some previous work includes this term, but it is overlooked in the recent analyses.

To obtain the betatron frequencies we first integrate Eq. (15) in time to obtain

$$\delta\gamma=-\frac{e}{mc^2}E_r(R)\,\delta r. \tag{17}$$

This expression is used to eliminate $\delta\gamma$ from Eq. (13), giving

$$\ddot{\delta r}=\left[-\frac{c^2}{R^2}-\frac{e}{\gamma m}\left(\frac{E_r}{r}+\frac{\partial E_r}{\partial r}+\frac{\partial B_z}{\partial r}\right)_R\right]\delta r. \tag{18}$$

Expressions for the betatron frequencies $(\omega_r,\ \omega_z)$ for a particle with equilibrium radius R follow immediately from Eqs. (14) and (18):

$$\omega_r^2=\frac{c^2}{R^2}+\frac{e}{\gamma m}\left(\frac{E_r}{r}+\frac{\partial E_r}{\partial r}+\frac{\partial B_z}{\partial r}\right)_R, \tag{19}$$

$$\omega_z^2=\frac{e}{\gamma m}\left(\frac{\partial E_z}{\partial z}-\frac{\partial B_r}{\partial z}\right)_R, \tag{20}$$

with all quantities evaluated at $(z=0,\ r=R)$. The corresponding tunes are

$$(Q_r,\ Q_z)=\frac{R}{c}(\omega_r,\ \omega_z). \tag{21}$$

It is convenient to define here a quantity $\partial G/\partial r$, analogous to F, that contains both of the beam-induced terms in Eq. (19):

$$\frac{\partial G}{\partial r}\equiv-\left(\frac{1}{r}\frac{\partial}{\partial r}rE_r+\frac{\partial B_z}{\partial r}\right)_b. \tag{22}$$

Then we have from Eqs. (19) and (21)

$$Q_r^2 = 1 + \frac{eR^2}{\gamma(R)mc^2}\left[\frac{dB_e(R)}{dR} - \frac{dG(R)}{dR}\right]. \tag{23}$$

It follows from Eq. (7) (and Maxwell's equations) that

$$\left(\frac{\partial E_z}{\partial z} - \frac{\partial B_r}{\partial z}\right)_b + \left(\frac{1}{r}\frac{\partial}{\partial r} rE_r + \frac{\partial B_z}{\partial r}\right)_b = 4\pi\left(\rho - \frac{J_\theta}{c}\right) = 0. \tag{24}$$

Using Eqs. (20)–(24) we get

$$Q_z^2 = -\frac{eR^2}{\gamma(R)mc^2}\left[\frac{dB_e(R)}{dR} - \frac{dG(R)}{dR}\right], \tag{25}$$

and the usual vacuum relation for tunes in a weak-focusing machine is recovered despite the presence of space charge:

$$Q_r^2 + Q_z^2 = 1.0. \tag{26}$$

An evaluation of Q_r that neglects the term $(-c^2\,\delta\gamma/R)$ in Eq. (13) leads to an incorrect version of Eq. (23) in which dG/dR is replaced by dF/dR.

5. CALCULATION OF FIELDS

5.1. *Case 1: Infinite Vertical Annulus*

Consider first the unphysical case of beam charge distributed between radii R_1 and R_2 but extending uniformly and without limit in the vertical (z) direction, i.e.,

$$\rho = \rho(r) \qquad R_1 < r < R_2. \tag{27}$$

The fields are also functions of r only, and the external magnetic field has the constant value B_e. It follows immediately from Eqs. (5)–(7) that

$$-\frac{\partial G}{\partial r} = \left(\frac{1}{r}\frac{\partial}{\partial r} rE_r + \frac{\partial B_z}{\partial r}\right)_b = -\frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial\phi}{\partial r} + \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} rA_\theta = 4\pi\left(\rho - \frac{J_\theta}{c}\right) = 0, \tag{28}$$

and Eq. (23) gives

$$Q_r^2 = 1.0. \tag{29}$$

The beam-induced modifications of tune cancel exactly in this special case, and chromaticity ($\xi = P\,dQ_r/dP = \gamma\,dQ_r/d\gamma$) also vanishes.

5.2. *Case 2: Unshielded Ring of Charge of Vanishing Height and Thickness*

This case has been examined by several authors;[1,2,7,8] we set

$$\rho(r, z) = \lambda\,\delta(r - R_0)\,\delta(z). \tag{30}$$

It is of interest to evaluate fields in the median plane at values of r near R_0. The scalar potential is readily evaluated using complete elliptic integrals:

$$\phi = \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}' - \mathbf{r}|} = 2\lambda \sqrt{\frac{R_0}{r}}\, kK(k), \tag{31}$$

where $K(k)$ is an elliptic integral of the first kind,[9] and

$$k = \sqrt{\frac{4rR_0}{(r+R_0)^2 + z^2}}. \tag{32}$$

Similarly, the vector potential is

$$A_\theta = \int d^3r' \frac{\rho(\mathbf{r}')\mathbf{e}_\theta \cdot \mathbf{e}'_\theta}{|\mathbf{r} - \mathbf{r}'|} \tag{33}$$

$$= \left(\frac{2}{k^2} - 1\right)\phi - 4\lambda \sqrt{\frac{R_0}{r}} \frac{E(k)}{k}, \tag{34}$$

where $E(k)$ is an elliptic integral of the second kind.[9] Defining the variable $x = r - R_0$ and making expansions in the small quantity

$$w = \frac{x}{R_0} = \frac{r - R_0}{R_0} \ll 1, \tag{35}$$

we obtain beam-induced fields in the median plane, just outside of the charge ring:

$$\phi = 2\lambda\left[-\ln\left(\frac{w}{8}\right) + \frac{w}{2} - \frac{7}{16}w^2 + \frac{w}{2}\ln\left(\frac{w}{8}\right) - \frac{5}{16}w^2\ln\left(\frac{w}{8}\right) + \cdots\right], \tag{36}$$

$$A_\theta = 2\lambda\left[-\ln\left(\frac{w}{8}\right) - 2 + \frac{3}{2}w - \frac{21}{16}w^2 + \frac{w}{2}\ln\left(\frac{w}{8}\right) - \frac{9}{16}w^2\ln\left(\frac{w}{8}\right) + \cdots\right], \tag{37}$$

$$E_{r_b} = \frac{2\lambda}{R_0}\left[\frac{1}{w} - 1 + \frac{19}{16}w - \frac{1}{2}\ln\left(\frac{w}{8}\right) + \frac{5w}{8}\ln\left(\frac{w}{8}\right) + \cdots\right], \tag{38}$$

$$\frac{\partial E_{r_b}}{\partial r} = -\frac{2\lambda}{R_0^2}\left[\frac{1}{w^2} - \frac{29}{16} + \frac{1}{2w} - \frac{5}{8}\ln\left(\frac{w}{8}\right) + \cdots\right], \tag{39}$$

$$B_{z_b} = \frac{2\lambda}{R_0}\left[-\frac{1}{w} + \frac{5}{16}w - \frac{1}{2}\ln\left(\frac{w}{8}\right) + \frac{3}{8}w\ln\left(\frac{w}{8}\right) + \cdots\right], \tag{40}$$

$$\frac{\partial B_{z_b}}{\partial r} = \frac{2\lambda}{R_0^2}\left[\frac{1}{w^2} + \frac{11}{16} - \frac{1}{2w} + \frac{3}{8}\ln\left(\frac{w}{8}\right) + \cdots\right], \tag{41}$$

$$F = \frac{2\lambda}{R_0}\left[1 + \ln\left(\frac{w}{8}\right) - \frac{3}{2}w - w\ln\left(\frac{w}{8}\right) + \cdots\right], \tag{42}$$

$$\frac{\partial F}{\partial r} = \frac{2\lambda}{R_0^2}\left[\frac{1}{w} - \frac{5}{2} - \ln\left(\frac{w}{8}\right) + \cdots\right], \tag{43}$$

$$\frac{\partial G}{\partial r} = -\frac{\lambda}{R_0^2}\left[1 + \ln\left(\frac{w}{8}\right) + \cdots\right]. \tag{44}$$

An examination of Eqs. (36)–(44) reveals several essential features of the fields. Since a beam of infinitesimal diameter was assumed, the fields E_{r_b} and B_{z_b} diverge ($\sim 2\lambda/x$), but these terms cancel (as expected) in forming the sum F. The order of magnitude of F is λ/R_0, which can be predicted from a simple dimensional analysis of the unshielded ring. The most significant feature for the present study is the difference between the expressions for $\partial G/\partial r$ and $\partial F/\partial r$, which appear, respectively, in the correct and incorrect formulas for the horizontal tune [see Eq. (23)]. The strong divergence $\partial F/\partial r \sim 2\lambda/R_0 x$ is cancelled from the expression for $\partial G/\partial r$ by the additional term E_r/r, which is the consequence of the oscillations of γ in curved geometry. The order of magnitude of $\partial G/\partial r$ is the very small quantity λ/R_0^2. Weak (logarithmic) divergences in the fields are of no real concern since they are insensitive to a cutoff applied at a finite beam edge radius ($a \ll R_0$).

5.3. *Case 3: Shielded Beam*

Two useful relations, readily derived from Eqs. (5)–(7) and Eq. (22), are

$$\frac{\partial G}{\partial r} = \frac{\partial^2}{\partial z^2}(A_\theta - \phi), \tag{45}$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2}\right)(A_\theta - \phi) + \frac{1}{r}\frac{\partial}{\partial r}(A_\theta - \phi) = = \frac{A_\theta}{r^2}. \tag{46}$$

Since the quantity $(A_\theta - \phi)$ vanishes on the conducting boundary around the beam, it is clear from Eq. (46) that A_θ/r^2 acts as a source for $(A_\theta - \phi)$ within the shielded region, and that $\partial G/\partial r$ is thereby determined. An essential point here is that A_θ/r^2 is small (or order λ/R_0^2) and that the quantity $(A_\theta - \phi)$ will therefore be of the order $\lambda b^2/R_0^2$, where b is the scale pipe radius.

A good approximation to $(A_\theta - \phi)$ is obtained by substituting the straight-pipe limiting form of A_θ in the right-hand side of Eq. (46), along with $r^{-2} \to R_0^{-2}$. Further, it is clear that we may neglect the term $r^{-1}\,\partial/\partial r(A_\theta - \phi)$, which is of the order $\lambda b/R_0^3$. Eq. (46) becomes

$$\left(\frac{\partial}{\partial x^2} + \frac{\partial_2}{\partial z^2}\right)(A_\theta - \phi) = \frac{A_\theta^s}{R_0^2}, \tag{47}$$

where $A_0^s = \phi^s$ is the straight-pipe solution, i.e.,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)(A_\theta^s \text{ or } \phi^s) = -4\pi\rho. \tag{48}$$

A simple application of Eqs. (47) and (48) is to a round beam (constant density $\rho = \lambda/\pi a^2$ inside radius a) centered in a round pipe of radius b. Defining the variable $u = \sqrt{x^2 + z^2}$ we have

$$\frac{1}{u}\frac{d}{du}u\frac{dA_\theta^s}{du} = -4\pi\rho H(a - u), \tag{49}$$

where H is the step function. The solution of Eq. (49) with $A^s_\theta(u=b)=0$ is

$$A^s_\theta = 2\lambda\left\{H(u-a)\ln\left(\frac{b}{u}\right) + H(a-u)\left[\ln\left(\frac{b}{a}\right) + \frac{a^2-u^2}{2a^2}\right]\right\}. \tag{50}$$

To find $(A_\theta - \phi)$ we must integrate Eq. (47), which because of cylindrical symmetry around the beam axis may be written

$$\frac{1}{u}\frac{d}{du}u\frac{d}{du}(A_\theta - \phi) = \frac{A^s_\theta}{R_0^2}. \tag{51}$$

A first integration yields, for $u < a$

$$u\frac{d}{du}(A_\theta - \phi) = \frac{2\lambda}{R_0^2}\left[\frac{u^2}{2}\ln\left(\frac{b}{a}\right) + \frac{u^2}{4} - \frac{u^4}{8a^2}\right]. \tag{52}$$

Then, from Eq. (45)

$$\frac{\partial G}{\partial r} = \frac{\partial^2}{\partial z^2}(A_\theta - \phi) = \left(1 + \frac{z^2}{u}\frac{d}{du}\right)\frac{1}{u}\frac{d}{du}(A_\theta - \phi)$$
$$= \frac{2\lambda}{R_0^2}\left[\frac{1}{2}\ln\left(\frac{b}{a}\right) + \frac{1}{4} - \frac{3}{8}\frac{z^2}{a^2} - \frac{1}{8}\frac{x^2}{a^2}\right]. \tag{53}$$

Note that, for this simple case, the second derivative $\partial^2 G/\partial r^2$ vanishes at the pipe center. This is a consequence of the cylindrical symmetry. However, in general $\partial^2 G/\partial r^2$ will be of the order $\lambda/R_0^2 a$.

Since the quantity $(A_\theta - \phi)$ has the negligible magnitude ($\sim \lambda b^2/R_0^2$) we obtain the approximate value of F:

$$F = -(E_r + B_z)_b = -\frac{1}{r}\frac{\partial}{\partial r}r(A_\theta - \phi) - \frac{\phi}{r} \cong -\frac{\phi}{r} \equiv -\frac{\phi^s}{r} = O\left(\frac{\lambda}{R_0}\right). \tag{54}$$

Other approximate formulas for the shielded beam fields are

$$\frac{dF}{dR} \approx -\frac{1}{R_0}\frac{d\phi^s}{dx} = O\left(\frac{\lambda}{R_0 a}\right), \tag{55}$$

$$\frac{d^2F}{dR^2} \approx -\frac{1}{R_0}\frac{d^2\phi^s}{dx^2} = O\left(\frac{\lambda}{R_0 a^2}\right). \tag{56}$$

$$\frac{dG}{dR} = O\left(\frac{\lambda}{R_0^2}\right) \ll \frac{dF}{dR}, \tag{57}$$

$$\frac{d^2G}{dR^2} = O\left(\frac{\lambda}{R_0^2 a}\right) \ll \frac{d^2F}{dR^2}. \tag{58}$$

6. CONSEQUENCES

Recall the basic formulas [Eqs. (11) for the orbit and (23) for the radial tune]. It has been shown that dG/dR is $O(\lambda/R_0^2)$ and may therefore be neglected.

Furthermore, $F \approx -\phi^s/R$, so we have the elegant formula:

$$E \approx \gamma mc^2 - e\phi^s = eRB_e. \tag{59}$$

The total energy (E), rather than the kinetic energy, relates radius (R) to the external field $B_e(R)$. As previously noted, ϕ^s is negative, so the kinetic energy is slightly lower for given RB_e than it would be in the absence of space charge.

Dropping dG/dR from the tune formula [Eq. (23)] and eliminating γ with Eq. (11) we obtain

$$Q_r^2 = 1 + \frac{R(dB_e/dR - dG/dR)}{B_e - F} \tag{60}$$

$$\cong 1 + \frac{RdB_e/dR}{B_e - F}. \tag{61}$$

Equation (60) was given by Laslett.[8] Although the incorrect term dF/dR has been displaced by the negligible dG/dR in the tune formula, F still enters through the equilibrium expression for $\gamma(R)$ and in general affects both tune and chromaticity. We consider the simple case

$$B_e(R) = B_0\left(\frac{R_0}{R}\right)^n, \tag{62}$$

where n in the constant-field index

$$n = -\frac{d \ln B_e}{d \ln R} \geq 0. \tag{63}$$

In the absence of space charge

$$Q_r^2 = 1 - n, \tag{64}$$

which is independent of energy, so chromaticity vanishes in this limit. With finite space charge

$$Q_r^2 = 1 - \frac{n}{1 - F/B_e} \approx 1 - n\left(1 - \frac{\phi^s}{RB_e}\right). \tag{65}$$

Since ϕ^s is negative, Q_r is slightly decreased from the vacuum limit (except for the case of constant B_e, where n vanishes).

For the chromaticity (ξ) we have

$$\xi = \left(\gamma\frac{dQ_r}{d\gamma}\right)_0 = \left(\frac{\gamma}{2Q_r}\frac{dQ_r^2/dR}{d\gamma/dR}\right)_0, \tag{66}$$

where all quantities and derivatives are evaluated on the design orbit (R_0). If only the linear terms in F is kept, this expression becomes

$$\xi = \left[\frac{RB_e}{2\sqrt{1-n}}\frac{d(-nF/B_e)/dR}{d(RB_e)/dR}\right]_0$$

$$= -\frac{n}{2\sqrt{1-n}^3}\left[\frac{1}{B_e}\frac{dRF}{dR} - \frac{F}{B_e^2}\frac{dRB_e}{dR}\right]_0$$

$$= -\frac{n}{2\sqrt{1-n^3}}\left[-\frac{1}{B_e}\frac{d\phi^s}{dR}+\frac{1-n}{RB_e}\phi^s\right]_0$$

$$\approx -\frac{n}{2\sqrt{1-n^3}}\left(\frac{E_r^s}{B_e}\right)_0. \tag{67}$$

This expression vanishes if the beam and pipe are symmetric across the design orbit; however, for the unsymmetrical situation ξ has a small positive value. It is tempting to define chromaticity as $\xi = (E\, dQ_r/dE)_0$ since total particle energy (E) is conserved. The resulting expression for ξ is unchanged from Eq. (67) in lowest order in F.

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