Lie Algebroids and Non-Geometry in String Theory

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Abstract

The occurrence of non-geometric fluxes in T-duality orbits of ordinary flux compactifications led to constructions of target space actions for the corresponding fields. In this thesis, which is based on [1], we attempt to clarify their relation and their meaning. In particular, we will see that such actions can be described as supergravity actions on Lie algebroids: Using techniques of generalized geometry (which we will briefly review) we can associate to each $O(d,d)$ transformation a new metric and a redefined Kalb-Ramond field. These naturally live on a Lie algebroid over the compactification manifold. Using the Lie algebroid anchor, we can pull back the standard NS-NS action to this algebroid and find that for certain non-geometric $O(d,d)$ transformations, the previously known actions drop out. Furthermore, we find that these results can also be derived by using an appropriate solution of the strong constraint in double field theory. We can extend the construction to get a full supergravity action on Lie algebroids to all orders in α'. By examining the symmetries of the actions, we find that they are not globally well-defined on non-geometric backgrounds like T-folds – their description requires more general approaches.

Contents

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Chapter 1

Introduction

The path to find a consistent quantum theory of gravity is rarely undertaken with a well-defined goal in mind. Although several interesting ideas have been suggested, and many have been in fruitful development for several decades, it is still not quite clear whether any of these theories will end up describing the physical phenomena we see in the real world. Two very well-developed proposals are *non-commutative quantum field theory* and *string theory*.
The idea of non-commutative QFT is relatively simple: To cure the non-renormalizability of the Einstein-Hilbert action, one 'smears out' the space by introducing coordinates that are non-commutative, i.e.

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}(x). \tag{1.1}$$

Another approach would be a pointwise Lie algebra structure in the following sense:

$$[x^\mu, x^\nu] = c^{\mu\nu}_\rho x^\rho. \tag{1.2}$$

The construction of such geometries is much simpler if we take the dual point of view: Instead of introducing new coordinates, we introduce a new product on the ring of functions of the spacetime manifold M – we 'deform the product'. We can observe that (1.1) is equivalent to

$$f(x) \star g(x) = f(x)g(x) + \frac{i}{2}\theta^{\mu\nu}\partial_\mu f \partial_\nu g + \mathcal{O}(\theta^2). \tag{1.3}$$

Another well-known approach to quantum gravity is string theory: We start by considering a classical theory of a (closed or open) string, and write down a general sigma model action for it (the following action is a gauge-fixed version, for the proper description see [2]):

$$S = \frac{1}{2\pi\alpha'} \int_\Sigma d^2\sigma E_{\mu\nu}(X)\partial_+ X^\mu \partial_- X^\nu, \tag{1.4}$$

where the string worldsheet Σ (the two-dimensional analog of a worldline) is parametrized by σ^1 and σ^2. To write down the action in the above form, we introduced the light cone coordinates $\sigma^{\pm} = \sigma^1 \pm \sigma^2$. The coordinates X^μ are the string embedding functions, and α', the Regge slope, is a free parameter. The last ingredients are the coupling constants $E_{\mu\nu}$, which we can split into a symmetric and an antisymmetric part:

$$E_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}, \tag{1.5}$$

where $G_{\mu\nu} = G_{\nu\mu}$ is the metric, and $B_{\mu\nu} = -B_{\nu\mu}$ is the Kalb-Ramond field. One then finds that requiring the vanishing of the beta functions of these coupling constants gives rise to Einstein's field equations in linear order of α'. Thus, string theory naturally describes gravitational degrees of freedom. This approach has been developed much further, and a very rich theory has been (partially) uncovered. Among the most important points are the requirement that M be 26-, or, for the supersymmetric extension of (1.4), 10-dimensional. In an attempt to recover physical theories in four dimensions, the approach of string compactification has laid bare a very intricate mathematical structure. Furthermore, the existence of higher-dimensional solitonic objects, D-branes, made it possible to connect open string theory with non-commutative quantum field theory (see [3], we will not cover this aspect in this thesis).

We will focus on closed strings instead: Compactifying these one-dimensional objects on a circle (or, more generally, on a torus) gives rise to two discrete quantum numbers: Like in standard Kaluza-Klein compactifications, the string's center of mass momentum in the compact direction is quantized. The second quantum number is the number of times the string winds the S^1, the *winding number*. This theory permits a duality transformation: Replacing the circle radius R by $\frac{l_S^2}{R}$ (where $l_s = \sqrt{\alpha'}$ is the string length parameter) and exchanging winding and momentum quantum numbers is a symmetry of the full string partition function. For a comprehensive review of this target space duality, or *T-duality*, see [4].

This thesis covers some results of this T-duality invariance: Compactifying a superstring theory, like IIA theory, on a torus, gives rise to a lower-dimensional theory with a diverse set of scalar fields, for which one can write down a potential. But as was derived in [5], the compactification of the T-dual theory on the T-dual torus only gives an equivalent theory if some additional fields, *non-geometric fluxes*, are included. On a 6-torus, we can obtain a chain of 3 T-dualities and find several T-dual partners of the standard H-flux (where $H = dB$):

$$H_{ijk} \xrightarrow{T^i} f^i_{jk} \xrightarrow{T^j} Q^{ij}_k \xrightarrow{T^k} R^{ijk}. \tag{1.6}$$

There is some evidence that these fluxes give rise to non-commutative (cf. [6]) and even non-associative (cf. [7]) geometry. Thus, they have the potential to connect string theory with non-commutative quantum field theories. What is missing, however, is an effective target space description of these additional fields. Recently, there has been some progress in this area, with proposed actions both for the Q-flux [8–10] and for the R-flux [11, 12]. In this thesis, we will have a closer look at these approaches and we will investigate their underlying structure.
This work is organized as follows:

- The first chapter is this introduction.
- The second chapter gives an overview over some topics that are required to understand the results, but are usually not contained in a string theory textbook. We will introduce some basic notions of generalized geometry, double field theory, the mathematical theory of Lie algebroids, and we will introduce non-geometric fluxes and their descriptions.
- In the third chapter, we will review how the T-duality group, $O(d, d)$, gives rise to redefinitions of the B and G fields, and we will present some examples for that.
- In the fourth chapter, a geometry on Lie algebroids is introduced, and we will apply these results to the examples of chapter 3, by assigning a Lie algebroid to every $O(d, d)$ transformation.
- In the fifth chapter, we will derive the final result: For each of these Lie algebroids, we can write down a supergravity action naturally defined on these algebroids, and we will see how these actions are related to non-geometric fluxes.
- Finally, in the conclusions chapter, we will give a short outlook on future directions.

For an introduction to the string theory which is required to understand this thesis, there is an abundance of excellent textbooks available. Most necessary background materials, however, should be covered by [2, 13, 14].

Chapter 2

Preliminaries

Before we can start describing supergravity actions on Lie algebroids and their relation to non-geometric frames, we have to review some basic concepts.

2.1 Generalized geometry

In the following section, we will introduce the concept of generalized geometry: It is based on [15] and was further developed in [16]; this summary mostly follows the introduction of [17].
The main idea of generalized geometry is to combine the tangent and the cotangent bundle of a manifold M into the generalized tangent bundle $E = TM \oplus T^*M$. On its sections, the generalized vector fields, we can define a non-degenerate bilinear form:

$$\langle X, Y \rangle := \xi(y) + \upsilon(x) \tag{2.1}$$

for $X = (x + \xi), Y = (y + \upsilon) \in \Gamma(E)$. By using the notation

$$X = \begin{pmatrix} x \\ \xi \end{pmatrix}, \quad Y = \begin{pmatrix} y \\ \upsilon \end{pmatrix}, \tag{2.2}$$

we can write this as

$$\langle X, Y \rangle = \frac{1}{2} X^t \eta Y \tag{2.3}$$

for

$$\eta = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \tag{2.4}$$

where $\mathbb{I}$ refers to the d-dimensional identity matrix. The symmetry group which leaves this scalar product invariant is $O(d, d; \mathbb{R})$, which also is the

group that relates different Narain lattices in compactifications of bosonic strings on a d-torus. This is a hint that generalized geometry can be used to describe T-duality transformations (For a classical review of target space duality and the $O(d,d)$ group, see [4]). In this thesis, we will allow for these $O(d,d)$ transformations to depend on the spacetime coordinates, unless we explicitly mention that they are constant (as in section 2.2.)
Of particular importance is the Courant bracket that we can define on generalized vector fields. It is defined as (cf. e.g. [17]):

$$[x+\xi, y+\upsilon]_{\text{Cour}} := [x,y] + \mathcal{L}_x \upsilon - \mathcal{L}_y \xi - \tfrac{1}{2} d(\iota_x \upsilon - \iota_y \xi), \tag{2.5}$$

where $[\cdot,\cdot]$ is the Lie bracket of two vector fields, $\mathcal{L}$ is the standard Lie derivative and ι is the insertion of a vector field into a differential form.
A general element $\mathtt{h} \in O(d,d)$, i.e. $\mathtt{h}^t\,\eta\,\mathtt{h} = \eta$, will have the form

$$\mathtt{h} = \begin{pmatrix} \mathtt{a} & \mathtt{b} \\ \mathtt{c} & \mathtt{d} \end{pmatrix} \tag{2.6}$$

with

$$\mathtt{a}^t\mathtt{c} + \mathtt{c}^t\mathtt{a} = 0, \tag{2.7}$$
$$\mathtt{b}^t\mathtt{d} + \mathtt{d}^t\mathtt{b} = 0, \tag{2.8}$$
$$\mathtt{a}^t\mathtt{d} + \mathtt{c}^t\mathtt{b} = \mathbb{I}. \tag{2.9}$$

It acts on generalized vectors $X = \begin{pmatrix} x^i \\ \xi_i \end{pmatrix}$ as $X' = \mathtt{h}\,X$, so the index structure of $\mathtt{h}$ is

$$\mathtt{h}_{IJ} = \begin{pmatrix} \mathtt{a}^i{}_j & \mathtt{b}^{ij} \\ \mathtt{c}_{ij} & \mathtt{d}_i{}^j \end{pmatrix} \tag{2.10}$$

with $I, J = 1, \ldots, 2d$.
To give these transformations a physical interpretation, we introduce a second metric on E by combining the Kalb-Ramond two-form B and the metric G into the *generalized metric* $\mathcal{H}$:

$$\mathcal{H} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}. \tag{2.11}$$

A simple calculation shows that $\mathcal{H}$ is symmetric and actually an element of $O(d,d)$ itself. We can let an $O(d,d)$ transformation $\mathtt{h}$ act on $\mathcal{H}$ by conjugation, i.e. $\mathcal{H}' = \mathtt{h}^t\,\mathcal{H}\,\mathtt{h}$. This allows us to generalize diffeomorphisms and B field gauge transformations, because they are contained as subgroups:

$$\mathtt{h}_{\text{gauge}} = \begin{pmatrix} \mathbb{I} & 0 \\ -d\Lambda & \mathbb{I} \end{pmatrix},$$
$$\mathtt{h}_{\text{diff}} = \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}.$$

We can actually replace $d\Lambda$ by a general two-form, it need not be exact for $\mathtt{h}_{\text{gauge}} \in O(d,d)$; the matrix A has to be invertible. If we require these to be symmetries of the Courant bracket (2.5), however, $d\Lambda$ has to be at least closed. As we will investigate in chapter 3, these transformations act on the G and B fields as follows:

$$\mathtt{h}_{\text{gauge}}: \quad G' = G, \qquad B' = B + d\Lambda, \tag{2.12}$$
$$\mathtt{h}_{\text{diff}}: \quad G' = A^t G A, \quad B' = A^t B A. \tag{2.13}$$

We will denote the group generated by $\mathtt{h}_{\text{gauge}}$ and $\mathtt{h}_{\text{diff}}$ as the geometric subgroup $O(d,d)_{\text{geom}}$ of $O(d,d)$.
Because the dimension of $O(d,d)_{\text{geom}}$ is $\frac{3}{2}d^2 - \frac{1}{2}d$, and the dimension of $O(d,d)$ is $2d^2 - d$, there are some additional $O(d,d)$ elements which do not have such a simple interpretation. An example of these non-geometric transformations are the β-transforms, which do not leave the Courant bracket invariant:

$$\mathtt{h}_{\beta} = \begin{pmatrix} \mathbb{I} & -\beta \\ 0 & \mathbb{I} \end{pmatrix} \tag{2.14}$$

for a bivector β. Taken together, all of these transformations generate the identity component $O(d,d)_0$ of $O(d,d)$ (for details, see appendix A).
In section 4.2 we will use the structures provided by generalized geometry to find a Lie algebroid associated to each of these $O(d,d)$ elements. This will allow us to elucidate the physical interpretation of the non-geometric transformations.

2.2 Double Field Theory

In this section, we will briefly review some basic concepts of double field theory; it is based on the very accessible introduction [18]. For original papers, see [19–22], and some reviews with connection to the topic of this thesis are [23–25].
Double field theory (DFT) goes one step further than generalized geometry: Instead of just 'doubling' the tangent bundle, the whole (compact)

space is doubled. We introduce an additional set of coordinates $\tilde{x}_i$ canonically conjugate to the winding numbers w^i to obtain the doubled coordinates $X^M = (\tilde{x}_i, x^i)$, $M = 1, \ldots, 2d$. With the aid of these, we can write down an action that is manifestly $O(d, d)$ invariant. But to relate this to 'ordinary', d-dimensional physics, we need to impose another constraint to reduce the dimensions – for this, we can employ the level matching condition:

$$L - \bar{L} = 0. \tag{2.15}$$

In our situation, (i.e. for closed strings winding cycles in a compact manifold) this is equivalent to

$$N - \bar{N} = -p_i w^i. \tag{2.16}$$

For the fields in the massless sector with $N = \bar{N} = 0$, we arrive at the famous *section condition* in Fourier space

$$\partial_i \tilde{\partial}^i = 0. \tag{2.17}$$

However, the exact meaning of the above equation is not entirely clear: Sometimes, it is used as an off-shell constraint in the sense that $\partial_i \tilde{\partial}^i (A(x, \tilde{x}))$ is set to zero for all 'physical' fields and gauge parameters $A(x, \tilde{x})$ in the theory. This is also called the 'weak constraint'. Often, this is not enough, so one imposes this condition also on the products of fields. This is called the 'strong constraint':

$$\partial_i \tilde{\partial}^i \left((A(x, \tilde{x}) B(x, \tilde{x}) \right) = 0 \tag{2.18}$$

for all $A(x, \tilde{x})$, $B(x, \tilde{x})$ in the theory. For a further discussion of possible different interpretations of this condition, see [26]. In the following, we will use the strong constraint, and all terms vanishing under it will be dropped. The action on this doubled space is usually defined for the NS-NS sector, and involves the generalized metric $\mathcal{H}$, the redefined dilaton d with

$$e^{-2d} = e^{-2\phi} \sqrt{|G|} \tag{2.19}$$

and the combined derivatives $\partial_M = (\tilde{\partial}^i, \partial_i)$. It is obtained by imposing some constraints on the resulting theory: Invariance under global $O(d, d)$ transformations and generalized diffeomorphism invariance. The former transformations act on the generalized metric $\mathcal{H}_{MN}$ as in generalized geometry, while the coordinates transform in the fundamental and the dilaton d forms a singlet, i.e. for a constant $O(d, d)$ matrix

$$\mathtt{h} = \begin{pmatrix} \mathtt{a} & \mathtt{b} \\ \mathtt{c} & \mathtt{d} \end{pmatrix} \tag{2.20}$$

we have

$$\mathcal{H}' = \mathtt{h}^t\,\mathcal{H}\,\mathtt{h}, \quad d' = d, \tag{2.21}$$

$$X' = \mathtt{h}\,X \Longrightarrow \partial' = (\mathtt{h}^t)^{-1}\partial. \tag{2.22}$$

The generalized diffeomorphisms are the generalizations of ordinary diffeomorphisms and B field gauge transformations and are parametrized by an $O(d,d)$ vector ξ^M. These gauge symmetries can be expressed by introducing a generalized Lie derivative $\widehat{\mathcal{L}}_\xi$ that acts on generalized vectors A^M and one-forms B_M as follows [18]:

$$\widehat{\mathcal{L}}_\xi A^M := \xi^P\partial_P A^M - (\partial_P\xi^M - \partial^M\xi_P)A^P, \tag{2.23}$$

$$\widehat{\mathcal{L}}_\xi B_M := \xi^P\partial_P B_M + (\partial_M\xi^P - \partial^P\xi_M)B_P. \tag{2.24}$$

As always, the generalization to higher tensor fields follows from the Leibniz rule. The generalized diffeomorphisms then act as

$$\delta e^{-2d} = \partial_M\left(\xi^M e^{-2d}\right), \tag{2.25}$$

$$\delta\,\mathcal{H}^{MN} = \widehat{\mathcal{L}}_\xi\,\mathcal{H}^{MN}. \tag{2.26}$$

Finally, we can write down the action:

$$S_{\text{DFT}} = \int d^Dx d^D\tilde{x} e^{-2d}\Big[\frac{1}{8}\,\mathcal{H}^{MN}\,\partial_M\,\mathcal{H}^{KL}\,\partial_N\,\mathcal{H}_{KL} - \frac{1}{2}\,\mathcal{H}^{MN}\,\partial_N\,\mathcal{H}^{KL}\,\partial_L\,\mathcal{H}_{MK} - 2\partial_M d\,\partial_N\,\mathcal{H}^{MN} + 4\,\mathcal{H}^{MN}\,\partial_M d\,\partial_N d\Big]. \tag{2.27}$$

Unfortunately, this action does not have the most convenient form for our goal: We want to find a particular solution of the strong constraint to reduce it to the Lie algebroid actions that we want to derive. To facilitate that, we can combine B and G into the field $\mathcal{E}_{ij} = G_{ij} + B_{ij}$. Furthermore, we introduce the derivatives

$$D_i = \partial_i - \mathcal{E}_{ik}\tilde{\partial}^k, \tag{2.28}$$

$$\overline{D}_i = \partial_i + \mathcal{E}_{ki}\tilde{\partial}^k. \tag{2.29}$$

One can now express the DFT action in these fields:

$$S_{\text{DFT}} = \int d^Dx d^D\tilde{x} e^{-2d}\Big[- \frac{1}{4}g^{ik}g^{jl}D^p\mathcal{E}_{kl}D_p\mathcal{E}_{ij} + 4D^i d D_i d + \frac{1}{4}g^{kl}\left(D^j\mathcal{E}_{ik}D^i\mathcal{E}_{jl} + \overline{D}^j\mathcal{E}_{ki}\overline{D}^i\mathcal{E}_{lj}\right) + \left(D^i d\overline{D}^j\mathcal{E}_{ij} + \overline{D}^i d D^j\mathcal{E}_{ji}\right)\Big]. \tag{2.30}$$

The global $O(d,d)$ transformations (2.21) then act as

$$\begin{aligned} \mathcal{E}' &= (\mathtt{a}\mathcal{E} + \mathtt{b})(\mathtt{c}\mathcal{E} + \mathtt{d})^{-1}, \\ D' &= M^{-1}D, \quad \overline{D}' = \overline{M}^{-1}\overline{D}, \end{aligned} \tag{2.31}$$

where

$$M = (\mathtt{d} - \mathtt{c}\mathcal{E}^t)^t, \tag{2.32}$$

$$\overline{M} = (\mathtt{d} + \mathtt{c}\mathcal{E})^t. \tag{2.33}$$

This implies the following transformation behavior for the metric:

$$G = MG'M^t, \tag{2.34}$$

where G' is the transformed metric.
That $(\mathtt{c}\mathcal{E} + \mathtt{d})$ is always invertible for positive definite G is shown in appendix A. In section 5.2.1, we will use this form of the action and the above transformations to relate double field theory to supergravity actions on Lie algebroids. For now, it's tedious, but quite straightforward to show that (2.30) reduces to the standard NS-NS action when setting $\tilde{\partial}$ to zero.

2.3 Lie algebroids

A *Lie algebroid* is a vector bundle E over a manifold M[1] that has a (fiberwise) $\mathbb{R}$-linear Lie algebra structure $[\cdot,\cdot]_E : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$. In addition, it supports a $\mathcal{C}^\infty(M)$-linear Lie algebra morphism $\rho : E \to TM$ which fulfills a Leibniz rule:

$$[s, f \cdot t]_E = f \cdot [s,t]_E + \rho(s)(f) \cdot t,$$

for all $s,t \in \Gamma(E)$, $f \in C^\infty(M)$. As we will soon see, this structure allows us to extend many basic constructions in differential geometry, such as higher tensor fields and exterior derivatives. A comprehensive overview of the mathematical structure of Lie algebroids and the closely associated Lie groupoids can be found in [27–29].
The homomorphism property of ρ actually follows from the Leibniz rule and the Jacobi identity for $[\cdot,\cdot]_E$ (cf. [29]), and need not be imposed.
To make the above definition more concrete, we will look at some examples:

[1]Unless explicitly mentioned, all mathematical objects that can be smooth are assumed to be smooth in this thesis.

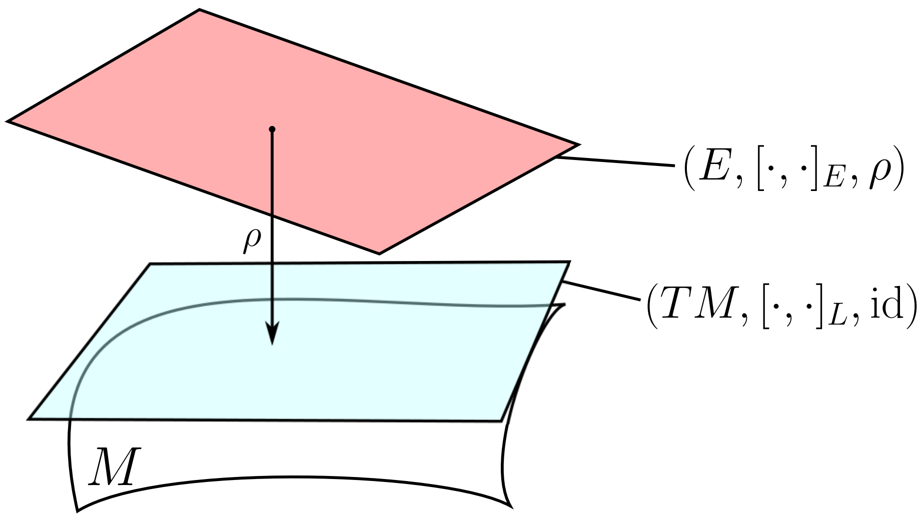

Figure 2.1: The Lie algebroids we consider have an invertible anchor ρ, so they have the same dimension as the tangent bundle. In the most simple situations, ρ can just be intepreted as a global rotation of TM; we will, however, also consider Lie algebroids without such a simple geometric interpretation.

- The most obvious example is the tangent bundle with the standard Lie bracket, $(TM, \mathrm{id}, [\cdot,\cdot])$. The proof of the Leibniz rule and the Jacobi identity are simple exercises in differential geometry. In the following, we will call this the *tangent Lie algebroid*.

- Another quite trivial example is a Lie algebra $\mathfrak{g}$ interpreted as a vector bundle over a single point p_0, with the anchor the constant map 0. If the base manifold is larger, but the anchor still vanishes at each point, each fiber of the vector bundle is a Lie algebra, and they are completely independent.

- A non-trivial example of a Lie algebroid is the *Atiyah Lie algebroid* [30], which can be used to define complex analytic connections of complex principal bundles and to study their existence. For more details on the construction, see [27].

  Let P be a (real or complex) principal G-bundle over M. Consider the following short exact sequence of vector bundles over M:

$$\frac{P \times \mathfrak{g}}{G} \overset{j}{\hookrightarrow} \frac{TP}{G} \overset{T\pi}{\twoheadrightarrow} TM. \qquad (2.35)$$

  Here, $\mathfrak{g}$ is the Lie algebra of G and the action of G on TP is the tangential map to the right multiplication by a group element. Thus, the sections of $\frac{TP}{G}$ are G-invariant vector fields on the principal bundle.

So, intuitively, the Atiyah sequence gives a decomposition of $\frac{TP}{G}$ into its Lie algebra and its TM part.

The Lie algebra structure of TP on $\frac{TP}{G}$ is well-defined, because the G-invariant vector fields in TP are closed under the Lie bracket.

The differential of the projection, $T\pi$, is the corresponding Lie algebroid anchor.

Its kernel is the adjoint bundle $\frac{P\times\mathfrak{g}}{G}$ (this is the associated bundle with respect to the adjoint action of G on $\mathfrak{g}$). The inclusion j of this bundle in $\frac{TP}{G}$ is induced by the G-equivariant

$$\begin{aligned} J : P \times \mathfrak{g} &\to TP, \\ (u, x) &\mapsto T_e(m_u)(x), \end{aligned} \tag{2.36}$$

where $m_u : G \to P, g \mapsto ug$ is the right multiplication of G on $u \in P$, and e is the unit in G.

The image of j coincides with the vertical tangent vectors $T^\pi P$ of TP, so choosing a splitting of this sequence is equivalent to choosing a connection on P. We can go a bit further by noting that $\frac{T^\pi P}{G}$ is also closed under the Lie bracket, and j is a Lie algebra morphism. Then we can also give $\frac{P\times\mathfrak{g}}{G}$ a Lie algebroid structure. All in all, we can interpret (2.35) not just as a sequence of vector bundles, but also as a short sequence of Lie algebroids. One can show that a splitting in this category is equivalent to a flat connection on P.

The triple $(\frac{TP}{G}, [\cdot,\cdot], \pi_*)$ is called the Atiyah Lie algebroid associated to P.

Despite their interesting mathematical structure, we won't be concerned with such Lie algebroids in this thesis. They still have some connection to physics, though: For example, they occur when studying Strominger's equations in heterotic string theory (see e.g. [31]) and they have connections to versions of noncommutative geometry [32] that are themselves used to define new physical theories (see e.g. [33]).

- Let $(\mathcal{M}, \beta)$ be a Poisson manifold, i.e. $[\beta,\beta]_{\mathrm{SN}} = 0$, where $[\cdot,\cdot]_{\mathrm{SN}}$ is the *Schouten-Nijenhuis bracket* of multivector fields (see e.g. [34]):

$$\begin{aligned} [X_1\wedge \ldots \wedge X_p, Y_1 \wedge \ldots \wedge Y_q]_{\mathrm{SN}} := \sum_{s,t} (-1)^{s+t} X_1 \wedge \ldots \\ \ldots \wedge \widehat{X}_s \wedge \ldots \wedge X_p \wedge [X_s, Y_t] \wedge Y_1 \wedge \ldots \wedge \widehat{Y}_t \wedge \ldots \wedge Y_q, \end{aligned} \tag{2.37}$$

where the entries with a hat are omitted.

Locally, this means that

$$\beta^{[a|d}\partial_d\beta^{|bc]} = 0.^2 \tag{2.38}$$

Then the cotangent bundle $T^*\mathcal{M}$ can be equipped with a Lie algebroid structure (cf. e.g. [29]).

The anchor, which maps covector fields to vector fields, is just $\beta^\sharp$:

$$\begin{aligned}\beta^\sharp &: T^*M \to TM,\\ \xi &= \xi_i dx^i \mapsto \beta^{ij}\xi_i\partial_j.\end{aligned}$$

The bracket is the *Koszul bracket* of one-forms:

$$\begin{aligned}[\xi,\eta]_K :&= \left(\xi_l\beta^{jl}\partial_j\eta_i - \eta_l\beta^{jl}\partial_j\xi_i + \partial_i\left(\beta^{kl}\eta_k\xi_l\right)\right)dx^i =\\ &= \iota_{\beta^\sharp(\xi)}d\eta - \iota_{\beta^\sharp(\eta)}d\xi - d\beta(\xi,\eta) =\\ &= L_{\beta^\sharp(\xi)}\eta - \iota_{\beta^\sharp(\eta)}d\xi,\end{aligned} \tag{2.39}$$

for $\xi, \eta \in \Gamma(T^*M)$.

Due to the second line of (2.39), we find that

$$[df, dg]_K = -d\{f, g\}, \tag{2.40}$$

where $f, g \in \mathcal{C}^\infty(\mathcal{M})$ and $\{\cdot,\cdot\}$ is the Poisson bracket of functions induced by β.

Note that often in the construction of the Koszul bracket, β is replaced by $-\beta$; this is just a matter of convention.

This Lie algebroid will occur later on in our examples; we will, however, also consider quasi-Poisson manifolds, where the Poisson bracket does not fulfill the Jacobi identity and $[\beta,\beta]_{\mathrm{SN}}$ will not vanish. To preserve the Lie algebroid properties, we will have to add an additional term to (2.39). This result can be interpreted as an effect of an underlying non-associative geometry.

Further constructions of Lie algebroids and related geometric concepts are developed in chapter 4.

[2] The notation $[a_1 \dots a_p]$ denotes the antisymmetrization of the indices $a_1, \dots, a_p$ with a prefactor of $\frac{1}{p!}$

2.4 Non-geometric fluxes and their actions

We'll now briefly present the theory of *non-geometric fluxes*, which was developed in [5].
In summary, these results indicate that the description of the bosonic part of string theory is incomplete: If we compactify type IIB theory on a $T^6/\mathbb{Z}_2$ orientifold, and demand that the form of the resulting physical theory is invariant under a chain of T-duality transformations, we find that we have to 'add' an additional set of fields – non-geometric fluxes.

2.4.1 Shelton-Taylor-Wecht fluxes

This exposition mostly follows [35] and originally goes back to [5].
The basic strategy is going to be as follows: We will compactify type IIB string theory on an

$$\mathcal{X} = T^6/\left(\Omega\mathbb{Z}_2(-1)^{F_L}\right) \tag{2.41}$$

background and calculate the superpotential for the scalar fields of the four-dimensional theory. Then we will compactify the dual IIA theory on a twisted torus (which we will briefly cover in 5.5, for more details on such backgrounds, see [36].) The $\mathbb{Z}_2$ acts on T^6 as $x^i \mapsto -x^i$ for $i = 1, \ldots, 6$, Ω is the worldsheet parity operator and F_L counts the left-moving fermionic modes.
We will look at the case where the T^6 is a product of three identical two-tori; this means in particular that we only have one complex structure modulus τ and one Kähler modulus U.
We want to calculate the potential for the scalar fields in the theory, which has two ingredients: The superpotential W and the Kähler potential K.
The superpotential is given by the Gukov-Vafa-Witten formula [37]

$$W = \int_{\mathcal{X}} G_3 \wedge \Omega, \tag{2.42}$$

with the holomorphic 3-form Ω and $G_3 = dC_2 - SH_3$ ($S = C_0 + ie^{-\phi}$ is the axiodilaton). The Kähler potential K is the same for the IIA and IIB compactifications we will consider, so we will ignore it.
But the superpotential is different:
For the IIB theory, it has the following expansion (cf. [35]):

$$W = a_0 - 3a_1\tau + 3a_2\tau^2 - a_3\tau^3 + S(-b_0 + 3b_1\tau - 3b_2\tau^2 + b_3\tau^3), \tag{2.43}$$

where the coefficients a_i, b_i are integrated $F_{(3)}$ and H fluxes around different cycles of the tori; as this is just a rough overview of the topic, we will not give

their explicit form. These fluxes are not independent – they are constrained by the Bianchi identity of the self-dual $\tilde{F}_5$ field strength.
For the IIA compactification on a twisted torus with geometric flux f^k_{ij} (which is itself constrained), we can also calculate the superpotential:

$$W = \alpha_0 - 3\alpha_1\tau + 3\alpha_2\tau^2 - \alpha_3\tau^3 + S(-\beta_0 + 3\beta_1\tau) + 3U(\gamma_0 + (\gamma_1 + \gamma_2 + \gamma_3)\tau), \tag{2.44}$$

where the coefficients $\alpha_i, \beta_i, \gamma_i$ are again given by integrated fluxes – this time, they correspond to the R-R fluxes, the H flux and the geometric flux of the IIA theory.
There are again some constraints these coefficients have to satisfy, but the main 'problem' is clear: The two superpotentials clearly have different expansions and cannot possibly match up. For some of these coefficients, however, we can see a pattern. For example, the well-known mapping of Ramond-Ramond fluxes under T-duality can be recovered by comparing coefficients of the superpotential. So if we want to take T-duality seriously, then there must be some way in which we can extend the field content of string theory such that the whole superpotential is T-duality invariant. And indeed, there is a unique way to do so, which is shown in the following sequence of maps, which represents which fluxes are mapped to each other under T-dualities:

$$H_{ijk} \xrightarrow{T^i} f^i_{jk} \xrightarrow{T^j} Q^{ij}_k \xrightarrow{T^k} R^{ijk}. \tag{2.45}$$

So we argue that a generic string compactification contains two new sets of fields: The Q and the R flux – these are called *non-geometric fluxes.* These are not completely independent of each other, because the Bianchi identities and relations between the geometric fields also imply relations for non-geometric fluxes.
The exact nature of these (possibly) new degrees of freedom, and their ten-dimensional origin is still a bit unclear, which has led to many new ideas and developments in recent years.
In the following, we review two different proposals for low-energy effective target space actions for non-geometric fluxes, we'll call them the R action [11,12] and the Q action [8–10]. Both roughly have the form of the standard NS-NS action and arise under field redefinitions of the G and B fields (which resemble the application of the Buscher rules for T-duality), but are derived in different ways.

2.4.2 R action

This action was constructed in [11,12] with the goal to find an action that is manifestly invariant under the ordinary diffeomorphisms and β transforma-

tions of generalized geometry.
To be able to accomplish this, we introduce a bivector field

$$\hat{\beta} = B^{-1} \tag{2.46}$$

and a metric on the cotangent bundle,

$$\hat{g} = -\hat{\beta} G \hat{\beta}. \tag{2.47}$$

So the main idea is to replace the geometric objects living on the tangent bundle by dual objects on T^*M. We can also introduce a new derivative operator with upper indices as

$$D^a f = \hat{\beta}^{am} \partial_m. \tag{2.48}$$

We define the R-flux as

$$\hat{R}^{abc} = 3\hat{\beta}^{[a|m} \partial_m \hat{\beta}^{|bc]}. \tag{2.49}$$

Then the proposed action is given by

$$S_R = -\frac{1}{2\kappa^2} \int d^n x \sqrt{|\hat{g}|} \left| \hat{\beta}^{-1} \right| \left(\hat{R} - \frac{1}{12} \hat{R}^{abc} \hat{R}_{abc} + 4 \hat{g}_{ab} D^a \phi D^b \phi \right). \tag{2.50}$$

This theory can be interpreted in a Lie algebroid setting, for more details, see chapter 4.

2.4.3 Q action

The simplest (but not the original [8]) way to derive this action employs double field theory and is described in [9] as follows: We take the standard DFT action, do a T-duality in all directions and solve the strong constraint by setting $\tilde{\partial} = 0$. Explicitly, this is realized by the $O(d,d)$ transformation

$$\mathtt{h} = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}. \tag{2.51}$$

According to (2.31), $\mathtt{h}$ acts on $\mathcal{E} = G + B$ as follows:

$$\mathcal{E}'(x, \tilde{x}) = \mathcal{E}^{-1}(\tilde{x}, x). \tag{2.52}$$

We now introduce

$$\tilde{\mathcal{E}}^{ij} = \tilde{g}^{ij} + \beta^{ij}, \tag{2.53}$$

such that $\tilde{\mathcal{E}}^{-1} = \mathcal{E}$, for a bivector field β^{ij} and a metric $\tilde{g}_{ij}$ (so $\tilde{g}^{ij}$ is its inverse). Our goal is to rewrite the DFT action such that these fields are described explicitly. But this is straightforward, because (2.51) is an $O(d,d)$ element, and the double field theory action is invariant under $O(d,d)$ transformations. According to (2.31), the action of $\mathtt{h}$ on the derivatives D and $\overline{D}$ implies that we can write the DFT action (2.30) with:

$$\widetilde{D}^i = \tilde{\partial}^i - \tilde{\mathcal{E}}^{ik}\partial_k, \tag{2.54}$$

$$\widetilde{\overline{D}} = \tilde{\partial}^i + \tilde{\mathcal{E}}^{ki}\partial_k. \tag{2.55}$$

If we then take this action and set $\tilde{\partial} = 0$, we arrive, up to total derivative terms, at

$$\begin{aligned} S_Q = -\frac{1}{2\kappa^2}\int d^n x\sqrt{|\tilde{g}|}e^{-2\tilde{\phi}}\Big(&\mathcal{R}(\tilde{g}) + 4(\partial\tilde{\phi})^2 - \frac{1}{12}R^{ijk}R_{ijk} + 4\tilde{g}_{ij}\beta^{ik}\beta^{jl}\partial_k d\partial_l d \\ &- 2\partial_k d\partial_l\left(\tilde{g}_{ij}\beta^{ik}\beta^{jl}\right) - \frac{1}{4}\tilde{g}_{ik}\tilde{g}_{jl}\tilde{g}^{rs}\mathcal{Q}_r{}^{kl}\mathcal{Q}_s{}^{ij} + \frac{1}{2}\tilde{g}_{pq}\mathcal{Q}_k{}^{lp}\mathcal{Q}_l{}^{kq} \\ &+ \tilde{g}_{jl}\tilde{g}_{pq}\beta^{jm}\left(\mathcal{Q}_k{}^{lp}\partial_m\tilde{g}^{kq} + \partial_k\tilde{g}^{lp}\mathcal{Q}_m{}^{kq}\right) \\ &- \frac{1}{4}\tilde{g}_{ik}\tilde{g}_{jl}\tilde{g}_{pq}\left(\beta^{pr}\beta^{qs}\partial_r\tilde{g}^{kl}\partial_s\tilde{g}^{ij} - 2\beta^{ir}\beta^{js}\partial_r\tilde{g}^{lp}\partial_s\tilde{g}^{kq}\right)\Big), \end{aligned} \tag{2.56}$$

where

$$\mathcal{Q}_l{}^{kq} = \partial_l\beta^{kq} \tag{2.57}$$

and

$$e^{-2\tilde{\phi}} := \frac{e^{-2d}}{\sqrt{|\tilde{g}|}}. \tag{2.58}$$

Chapter 3

O(d,d) field redefinitions & non-geometric frames

In this chapter, we will review some geometric concepts that are associated to non-geometric fluxes and we will explicitly work out how $O(d,d)$ transformations can be defined as redefinitions of the G and B field. These results will then be applied to arrive at the field redefinitions that describe the Q and R actions.

3.1 Field redefinitions

Recall from section 2.1 that an element $\mathtt{h} \in O(d,d)$ (i.e. $\mathtt{h}^t \, \eta \, \mathtt{h} = \eta$) has the following structure:

$$\mathtt{h}_{IJ} = \begin{pmatrix} \mathtt{a}^i{}_j & \mathtt{b}^{ij} \\ \mathtt{c}_{ij} & \mathtt{d}_i{}^j \end{pmatrix}, \tag{3.1}$$

where

$$\begin{aligned} \mathtt{a}^t\mathtt{c} + \mathtt{c}^t\mathtt{a} &= 0 = \mathtt{b}^t\mathtt{d} + \mathtt{d}^t\mathtt{b}, \\ \mathtt{a}^t\mathtt{d} + \mathtt{c}^t\mathtt{b} &= \mathbb{I}, \end{aligned} \tag{3.2}$$

which is equivalent to

$$\begin{aligned} \mathtt{a}\mathtt{b}^t + \mathtt{b}\mathtt{a}^t &= 0 = \mathtt{c}\mathtt{d}^t + \mathtt{d}\mathtt{c}^t, \\ \mathtt{a}\mathtt{d}^t + \mathtt{b}\mathtt{c}^t &= \mathbb{I}. \end{aligned} \tag{3.3}$$

We have already argued that the identity component of $O(d,d)$ is generated by the groups in table 3.1. (For more details and the remaining generators, T-dualities, see appendix A). We have also seen how these transformations act on the generalized metric $\mathcal{H}$:

$$\mathcal{H}' = \mathtt{h}^t \, \mathcal{H} \, \mathtt{h}, \tag{3.4}$$

Table 3.1: Generators of $O(d,d)_0$

Diffeomorphisms	$\mathtt{h}_A = \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}$
Gauge transformations	$\mathtt{h}_{\mathtt{B}} = \begin{pmatrix} \mathbb{I} & 0 \\ -\mathtt{B} & \mathbb{I} \end{pmatrix}$
β transformations	$\mathtt{h}_\beta = \begin{pmatrix} \mathbb{I} & -\beta \\ 0 & \mathbb{I} \end{pmatrix}$

where $\mathcal{H}$ was given as

$$\mathcal{H} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}. \tag{3.5}$$

This action is transitive: Given any $\mathcal{H}$ and $\mathcal{H}'$ that have the structure above (for B antisymmetric and G symmetric and positive definite), we can always find an $O(d,d)$ element $\mathtt{h}$ that relates them in this way. But the action is not injective: For each $\mathcal{H}$, there is an $O(d) \times O(d)$ subgroup that leaves $\mathcal{H}$ invariant (this can easily be seen by counting dimensions). These stabilizers can even be determined explicitly (cf. [1]):

$$\mathtt{h}_{O_1} = \begin{pmatrix} O_1 & 0 \\ BO_1 - (O_1^t)^{-1} B & (O_1^t)^{-1} \end{pmatrix}, \tag{3.6}$$

$$\mathtt{h}_{O_2} = \begin{pmatrix} -G^{-1} (O_2^t)^{-1} B & G^{-1} (O_2^t)^{-1} \\ GO_2 - BG^{-1} (O_2^t)^{-1} B & BG^{-1} (O_2^t)^{-1} \end{pmatrix}, \tag{3.7}$$

where O_1, O_2 are chosen such that

$$O_i^t G O_i = G, \tag{3.8}$$

for $i = 1, 2$.
One could now ask the question whether $O(d,d)$ transformations always transform generalized metrics into new generalized metrics in the following sense: If

$$\mathcal{H}'(G,B) = \mathtt{h}^t\, \mathcal{H}(G,B)\, \mathtt{h}, \tag{3.9}$$

can we find a new metric $\hat{g}$ and a new Kalb-Ramond field $\widehat{B}$ such that

$$\mathcal{H}'(G,B) = \mathcal{H}(\hat{g}, \widehat{B}), \tag{3.10}$$

and how can we express these new fields $\hat{g}(G,B)$, $\widehat{B}(G,B)$?
Finding out how the metric transforms is simple: We just need to look at the bottom right component of (3.4), G^{-1}, and we read off that

$$\hat{g}^{-1} = \left[\mathtt{d} + (G-B)\mathtt{b}\right]^t G^{-1} \left[\mathtt{d} + (G-B)\mathtt{b}\right]. \tag{3.11}$$

That the above expression is well-defined is non-trivial to prove: If we introduce

$$\gamma = \mathtt{d} + (G-B)\mathtt{b}, \tag{3.12}$$

we would like to define

$$\hat{g} = \gamma^{-1} G \left(\gamma^t\right)^{-1}. \tag{3.13}$$

Although it's pretty clear that this leaves $\hat{g}$ positive definite and symmetric, the redefinition only makes sense if γ is invertible. We will prove this in appendix A, but the proof depends on the positivity of G, and for backgrounds with an indefinite signature, we can actually find counterexamples.[1]
Note that this problem already arises in double field theory (consider equation (2.31), where the same issue can occur).
We can use the result for the metric to find the transformed B field. Consider the upper right component of $\mathcal{H}$, BG^{-1}; it transforms as

$$\widehat{B}\hat{g}^{-1} = (\mathtt{c} + (G-B)\mathtt{a})^t G^{-1} (\mathtt{d} + (G-B)\mathtt{b}) - \mathbb{I}. \tag{3.14}$$

By using equation (3.13), we can read off that

$$\widehat{B} = \gamma^{-1}(\gamma\delta^t - G)\left(\gamma^t\right)^{-1}, \tag{3.15}$$

where we defined

$$\delta = \mathtt{c} + (G-B)\mathtt{a}. \tag{3.16}$$

Showing that $\widehat{B}$ is antisymmetric can either be done explicitly by using the relations (3.3), or by noting that $\mathcal{H}' = \mathtt{h}^t\,\mathcal{H}\,\mathtt{h}$ is still symmetric.
To summarize, we have

$$\hat{g} = \gamma^{-1} G \left(\gamma^t\right)^{-1}, \tag{3.17}$$
$$\widehat{B} = \gamma^{-1}(\gamma\delta^t - G)\left(\gamma^t\right)^{-1}. \tag{3.18}$$

We can now check which effects the transformations of table 3.1 have on the B and G field:

[1] Take, for example, $G = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. and $\mathtt{h} = \eta = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$. Then $\gamma = G - B = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$, which is not invertible.

- **Diffeomorphisms**: Let

$$\mathtt{h}_A = \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}. \tag{3.19}$$

Because $\mathtt{b} = 0$, we have

$$\gamma_A = (A^t)^{-1}, \tag{3.20}$$
$$\delta_A = (G - B)A. \tag{3.21}$$

Then the transformed metric is simply

$$\hat{g} = \gamma_A^{-1} G \left(\gamma_A^{-1}\right)^t = A^t G A. \tag{3.22}$$

For the Kalb-Ramond field, we find

$$\begin{aligned} \widehat{B} &= \gamma_A^{-1}(\gamma_A \delta_A^t - G)\left(\gamma_A^t\right)^{-1} = A^t \left(\left(A^t\right)^{-1} A^t (G + B) - G \right) A = \\ &= A^t B A. \end{aligned} \tag{3.23}$$

This is, of course, precisely the way we expect a diffeomorphism to act on these fields. For $A \in O(d)$, this is a volume-preserving diffeomorphism, which is a symmetry of the NS-NS action.

- **Gauge transformations**: Gauge transformations have the form

$$\mathtt{h}_\mathtt{B} = \begin{pmatrix} \mathbb{I} & 0 \\ \mathtt{B} & \mathbb{I} \end{pmatrix}. \tag{3.24}$$

Note the difference between $\mathtt{B}$ and B. To represent an actual symmetry transformation, we will take $\mathtt{B}$ to be exact, i.e. $B = -d\Lambda$. We find that

$$\gamma_\mathtt{B} = \mathbb{I}, \tag{3.25}$$
$$\delta_\mathtt{B} = (G - B) + \mathtt{B}. \tag{3.26}$$

Thus, the metric is unchanged:

$$\hat{g} = G, \tag{3.27}$$

and

$$\widehat{B} = B - \mathtt{B}, \tag{3.28}$$

which, for $\mathtt{B} = -d\Lambda$, gives the expected form of a gauge transformation

$$\widehat{B} = B + d\Lambda. \tag{3.29}$$

- **β-transformations**: For

$$\mathtt{h}_\beta = \begin{pmatrix} \mathbb{I} & -\beta \\ 0 & \mathbb{I} \end{pmatrix} \tag{3.30}$$

we have

$$\gamma_\beta = \mathbb{I} - (G-B)\beta, \tag{3.31}$$
$$\delta_\beta = G - B. \tag{3.32}$$

For this case, unfortunately, $\hat{g}$ and $\widehat{B}$ don't have a simple form:

$$\hat{g} = (\mathbb{I} - (G-B)\beta)^{-1} G (\mathbb{I} + \beta(G+B)) \tag{3.33}$$

and

$$\widehat{B} = (\mathbb{I} - (G-B)\beta)^{-1}(B - (G-B)\beta(G+B))(\mathbb{I} + \beta(G+B))^{-1}. \tag{3.34}$$

Unlike the previous examples, this transformation is not a symmetry of the NS-NS action. It will, however, appear in the transformation to non-geometric frames, and can be interpreted to give rise to non-geometric effects.

- **T-dualities** The remaining $O(d,d)$ generators (which are not in table 3.1) are T-dualities:

$$\mathtt{h}_\pm = \left(\begin{array}{cccc|cccc} 0 & & & & \pm 1 & & & \\ & 1 & & & & 0 & & \\ & & \ddots & & & & \ddots & \\ & & & 1 & & & & 0 \\ \hline \pm 1 & & & & 0 & & & \\ & 0 & & & & 1 & & \\ & & \ddots & & & & \ddots & \\ & & & 0 & & & & 1 \end{array}\right). \tag{3.35}$$

These are $O(d,d)$ elements that do not live in the identity component of the group (see appendix A). They give rise to

$$\gamma_\pm = \mathbb{I} - E_1 \pm (G-B)E_1, \tag{3.36}$$
$$\delta_\pm = \pm E_1 + (G-B)(\mathbb{I} - E_1), \tag{3.37}$$

where $E_1 = \mathrm{diag}(1,0,\ldots,0)$. In the appendix, we did not need to explicitly calculate the inverse of $\gamma_\pm$, but noting that $E_1(G-B)E_1 = G_{11}E_1$, it's trivial to check that

$$\gamma_\pm^{-1} = (\mathbb{I} - E_1) + \frac{1}{G_{11}}(\pm E_1 - (\mathbb{I} - E_1)(G-B)E_1). \tag{3.38}$$

The transformations of the G and B field are then given by the well-known Buscher rules [38] (if we choose the + sign in (3.35)):

$$\begin{aligned} \hat{g}_{11} &= \frac{1}{G_{11}}, & \widehat{B}_{11} &= 0, \\ \hat{g}_{1j} &= \mp\frac{B_{1j}}{G_{11}} = \hat{g}_{j1}, & \widehat{B}_{1j} &= \mp\frac{G_{1j}}{G_{11}} = -\widehat{B}_{j1}, \\ \hat{g}_{ij} &= G_{ij} + \frac{B_{1i}B_{1j} - G_{1i}G_{1j}}{G_{11}}, & \widehat{B}_{ij} &= B_{ij} + \frac{B_{1i}G_{1j} - B_{1j}G_{1i}}{G_{11}}, \end{aligned} \tag{3.39}$$

where $i, j \neq 1$. Note that we do not transform the dilaton ϕ, the transformation factor in the string frame will be introduced into our action via a change of the measure. For the details, see the construction in chapter 5.

So these transformations correspond to a T-duality in the 1-direction (of course, a T-duality in the i-direction can be performed analogously).

Note that

$$\mathtt{h} = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \tag{3.40}$$

is a T-duality in all d directions.
This transformation relates gauge and β transforms:

$$\begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}^t \begin{pmatrix} \mathbb{I} & \beta \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} = \begin{pmatrix} \mathbb{I} & 0 \\ \beta & \mathbb{I} \end{pmatrix}. \tag{3.41}$$

To find the inverse of the field redefinitions, we just form γ and δ of the transformed fields and apply the rules above with the inverse $O(d,d)$ transformation $\mathtt{h}^{-1}$. Because $\mathtt{h}^t\,\eta\,\mathtt{h} = \eta$, and $\eta^2 = \mathbb{I}$, we find that

$$\mathtt{h}^{-1} = \eta\,\mathtt{h}^t\,\eta = \begin{pmatrix} \mathtt{d}^t & \mathtt{b}^t \\ \mathtt{c}^t & \mathtt{a}^t \end{pmatrix}. \tag{3.42}$$

So if we call these new maps $\hat{\gamma}$ and $\hat{\delta}$, we have

$$\hat{\gamma} = \mathtt{a}^t + (\hat{g} - \widehat{B})\mathtt{b}^t, \tag{3.43}$$

$$\hat{\delta} = \mathtt{c}^t + (\hat{g} - \widehat{B})\mathtt{d}^t, \tag{3.44}$$

and, as we can infer from (3.13), $\hat{\gamma} = \gamma^{-1}$.

3.2 R frame

We will now examine a frame that is reached via non-geometric β-transforms. It was introduced in [11, 12] without an $O(d,d)$ interpretation, but we can see that it fits into our framework.
In this construction, we will assume that the Kalb-Ramond field is invertible. We then introduce the bivector field

$$\hat{\beta} = B^{-1}. \tag{3.45}$$

Consider the transformation

$$\mathtt{h}_R = \begin{pmatrix} \mathbb{I} & 0 \\ 2B & \mathbb{I} \end{pmatrix} \begin{pmatrix} \mathbb{I} & -\hat{\beta} \\ 0 & \mathbb{I} \end{pmatrix} = \begin{pmatrix} \mathbb{I} & -\hat{\beta} \\ 2B & -\mathbb{I} \end{pmatrix}. \tag{3.46}$$

$\mathtt{h}_R$ is the composition of a gauge and a β-transformation. Note that this means that the transformation parameters depend explicitly on the background fields. We can calculate the effects of the transformation on these fields as before. First we find that

$$\gamma_R = (\mathbb{I} + BG^{-1})^{-1} \tag{3.47}$$

and

$$\delta_R = G + B. \tag{3.48}$$

We could then calculate the transformed fields $\tilde{g}$ and $\tilde{B}$ to get

$$\tilde{g} = -BG^{-1}B, \tag{3.49}$$

and

$$\tilde{B} = B. \tag{3.50}$$

But these fields do not give the description we are looking for; we want to describe the theory using tensors acting on covector fields, i.e. $\hat{g}, \hat{\beta} \in \Gamma(TM \otimes TM)$. So we redefine

$$\hat{g} = \tilde{g}^{-1} = -B^{-1}GB^{-1} \tag{3.51}$$

and

$$\hat{\beta} = B^{-1}. \tag{3.52}$$

We can then write the redefined generalized metric $\mathcal{H}'$ in the new fields as:

$$\mathcal{H}' = \begin{pmatrix} \hat{g}^{-1} - \hat{\beta}^{-1}\hat{g}\hat{\beta}^{-1} & \hat{\beta}^{-1}\hat{g} \\ -\hat{g}\hat{\beta}^{-1} & \hat{g} \end{pmatrix}. \tag{3.53}$$

So all the entries of $\mathcal{H}'$ look like the entries of $\mathcal{H}$, with the identifications

$$G \mapsto \hat{g}^{-1}, \tag{3.54}$$

$$B \mapsto \hat{\beta}^{-1}. \tag{3.55}$$

We will equip the R frame with a Lie algebroid structure in section 4.2.2; this will allow us to reproduce the action (2.50).

3.3 Q frame

The $O(d,d)$ matrix that was used to obtain the action for the Q frame from DFT in [9] was

$$\mathtt{h} = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}. \tag{3.56}$$

Unfortunately, in the framework of generalized geometry, for an $O(d,d)$ transformation

$$\mathtt{h} = \begin{pmatrix} \mathtt{a} & \mathtt{b} \\ \mathtt{c} & \mathtt{d} \end{pmatrix}, \tag{3.57}$$

we interpret the components as maps

$$\begin{aligned} \mathtt{a} &: TM \ \ \to TM, \\ \mathtt{b} &: T^*M \to TM, \\ \mathtt{c} &: TM \ \ \to T^*M, \\ \mathtt{d} &: T^*M \to T^*M. \end{aligned}$$

So we would need to interpret $\mathbb{I}$ as a map between the tangent and the cotangent bundle of M; such a canonical isomorphism exists, but it is usually given by the metric on M. In case of a torus, this would of course coincide, but, in general, we want to find another transformation that will give the same result.
Let's consider

$$\mathtt{h}_Q = \begin{pmatrix} 0 & (G - BG^{-1}B)^{-1} \\ G - BG^{-1}B & 0 \end{pmatrix}. \tag{3.58}$$

For this transformation, we find

$$\gamma_Q = (\mathbb{I} + BG^{-1})^{-1} \tag{3.59}$$

and

$$\delta_Q = G - BG^{-1}B = \delta_Q^t. \tag{3.60}$$

So the transformed background fields are

$$\tilde{g} = (\mathbb{I} + BG^{-1})G(\mathbb{I} - G^{-1}B) \tag{3.61}$$

and

$$\begin{aligned}\widehat{B} &= (\mathbb{I} + BG^{-1})\Big(\overbrace{(\mathbb{I} + BG^{-1})^{-1}(G - BG^{-1}B)}^{(G-B)} - G\Big)(\mathbb{I} - G^{-1}B) = \\ &= (\mathbb{I} + BG^{-1})(-B)(\mathbb{I} - G^{-1}B).\end{aligned} \tag{3.62}$$

We'll again introduce a bivector field β, as

$$\beta = \tilde{g}^{-1}\widehat{B}\tilde{g}^{-1}. \tag{3.63}$$

We can also work out the inverse of these transformations:

$$G = (\tilde{g}^{-1} - \beta)^{-1}\tilde{g}^{-1}(\tilde{g}^{-1} + \beta)^{-1}, \tag{3.64}$$

$$B = (\tilde{g}^{-1} - \beta)^{-1}(-\beta)(\tilde{g}^{-1} + \beta)^{-1}. \tag{3.65}$$

So this is the field redefinition that we have already encountered in section 2.4.3 and in [8–10], because adding both equations gives us

$$G + B = \left(\tilde{g}^{-1} + \beta\right)^{-1}. \tag{3.66}$$

By considering equation (3.61), we can rewrite the transformation $\mathtt{h}_Q$ as

$$\mathtt{h}_Q = \begin{pmatrix} 0 & \tilde{g}^{-1} \\ \tilde{g} & 0 \end{pmatrix}. \tag{3.67}$$

So in the case of a torus compactification (which we will briefly consider in section 5.5), where $\tilde{g} = \mathbb{I}$, we recover the transformation

$$\mathtt{h} = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}. \tag{3.68}$$

In the context of double field theory, where $O(d,d)$ acts on (doubled) coordinates, we could have used this transformation in the first place (as was done in [9]). In fact, the DFT action (2.30) is only invariant under constant $O(d,d)$ transformations.

Chapter 4

Lie algebroid geometry

In this chapter, we will show how concepts from ordinary differential geometry 'on TM' can be transferred to Lie algebroids. Afterwards, we will investigate these structures for the examples relevant to our work.

4.1 Riemannian geometry on Lie algebroids

Lie algebroids can be used to build physical theories due to one important aspect: We can easily develop a differential geometry on them.
Basically, we just need to replace each TM in most definitions of a textbook on differential geometry by the Lie algebroid E. To see how this actually works, we will develop all the techniques we need in this section. For a further exposition of this material, see [27, 28, 39].
In the following, let M be a Riemannian manifold and E a Lie algebroid over M with anchor ρ.
One main reason to study sections of TM (vector fields) is their roles as derivative operators on functions $f \in \mathcal{C}^\infty(M)$. Using the anchor map, we can give the same interpretation to a section $s \in \Gamma(E)$ by defining

$$D_s f := \rho(s) f. \tag{4.1}$$

Obviously, this reduces to the ordinary partial derivative on the tangent bundle.
As for any other vector bundle, we can introduce local frame fields ϵ_α on any chart of the Lie algebroid E and use them as a vector space basis at each fiber. The duals of ϵ_α will be denoted by ϵ^α (so $\epsilon^\alpha \epsilon_\beta = \delta^\alpha_\beta$). This allows us to write down local formulas in index notation, just as we are used to it from standard differential geometry – in the following, Greek indices will be

used on the Lie algebroid and Latin indices for objects 'living' on the tangent bundle.
To extend (4.1) to actions of Lie-algebroid-valued vector fields on themselves, we define *connections* on Lie algebroids as maps $\nabla : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ that satisfy

1. $\nabla_{fs+gt} u = f \nabla_s u + g \nabla_t u$,
2. $\nabla_s (t + u) = \nabla_s t + \nabla_s u$,
3. $\nabla_s (ft) = f \nabla_s t + t \cdot \rho(s) f$

for all $s, t, u \in \Gamma(E)$, $f, g \in \mathcal{C}^\infty(M)$.
Using the Leibniz rule, we can extend ∇ to higher tensor fields, i.e. sections of $\Gamma(E^{\otimes r} \otimes E^{*\otimes s})$.
In index notation, a connection is defined by its Christoffel symbols:

$$\Gamma^\alpha_{\beta\gamma} \epsilon_\alpha := \nabla_{\epsilon_\beta} \epsilon_\gamma \tag{4.2}$$

and we will denote the derivative (4.1) in the direction of a basis vector field as

$$D_\alpha f := D_{\epsilon_\alpha} f = \rho(\epsilon_\alpha) f. \tag{4.3}$$

So, the connection ∇_{ϵ_β} applied to a section $s = s^\alpha \epsilon_\alpha \in \Gamma(E)$ gives

$$\nabla_{\epsilon_\beta} (s^\alpha \epsilon_\alpha) = s^\alpha \nabla_{\epsilon_\beta} \epsilon_\alpha + \epsilon_\alpha \rho(\epsilon_\beta) s^\alpha = \epsilon_\alpha \left(D_\beta s^\alpha + \Gamma^\alpha_{\beta\gamma} s^\gamma \right), \tag{4.4}$$

which is the well-known formula from Riemannian geometry if we replace D by ∂.
We also want to find the action of the covariant derivative on sections of E^*. Let $s^* = s_\alpha \epsilon^\alpha \in \Gamma(E^*)$. Then for $t = t^\alpha \epsilon_\alpha \in \Gamma(E)$, $s^*(t)$ is a function on M. So, by the product rule, for a section $u \in \Gamma(E)$, we have

$$\rho(u)(s^*(t)) = (\nabla_u s^*)(t) + s^*(\nabla_u t), \tag{4.5}$$

and if we set $u = \epsilon_\alpha$ and $t = \epsilon_\beta$, we get in components:

$$D_\alpha s_\beta = (\nabla_\alpha s^*)_\beta + s_\gamma \Gamma^\gamma_{\alpha\beta}, \tag{4.6}$$

or

$$\nabla_\alpha (s_\beta \epsilon^\beta) = \epsilon^\beta \left(D_\alpha s_\beta - \Gamma^\gamma_{\alpha\beta} s_\gamma \right). \tag{4.7}$$

Thus, our definitions are designed to give exactly the same formula for the covariant derivative as in normal differential geometry – we get a $-\Gamma$ for each lower index and a $+\Gamma$ for each upper index of a tensor.
To write down a supergravity action on a Lie algebroid, we will obviously have to define a metric on it:

Definition 4.1. Let E be a vector bundle over a manifold M. A *metric* on E is a section $g \in \Gamma(E^* \otimes E^*)$ that induces a positive definite inner product on every fiber.

In our case, M will be a Riemannian manifold with metric G and the anchor ρ will be invertible. This allows us to pull back the metric from the tangent bundle to the Lie algebroid as follows:

$$g(s,t) = G(\rho(s), \rho(t)) \quad \forall s,t \in \Gamma(E). \tag{4.8}$$

Note that g is positive definite because ρ is invertible. We will extend this construction to connections and their curvature tensors in the next chapter. A very important tensor field associated to a connection is its curvature tensor, which is defined as in the case of the Levi-Civita connection on a Riemannian manifold:

$$R(s,t)u = [\nabla_s, \nabla_t]u - \nabla_{[s,t]_E} u, \tag{4.9}$$

where $[\cdot,\cdot]_E$ is the Lie algebroid bracket on E. A connection with vanishing curvature is called a *flat connection*.
We can also introduce the torsion tensor of the connection ∇:

$$T(s,t) = \nabla_s t - \nabla_t s - [s,t]_E. \tag{4.10}$$

To show that these maps are actually tensor fields, we need to show that they are $\mathcal{C}^\infty(M)$-linear:

$$\begin{aligned} R(fs, gt)(hu) &= fghR(s,t), \\ T(fs, gt) &= fgT(s,t), \end{aligned}$$

for any $f, g, h \in \mathcal{C}^\infty(M)$ and $s, t, u \in \Gamma(E)$. To illustrate how all our definitions ensure this result, we will prove the tensor property for the torsion:

$$\begin{aligned} T(fs, gt) &= \nabla_{fs}(gt) - \nabla_{gt}(fs) - [fs, gt]_E = \\ &= t \cdot \rho(fs)(g) + fg\nabla_s t - gs \cdot \rho(t)(f) - fg\nabla_t s - t \cdot \rho(fs)(g) + g[t, fs]_E = \\ &= fg\nabla_s t - gs \cdot \rho(t)(f) - fg\nabla_t s - fg[s,t]_E + gs \cdot \rho(t)(f) = \\ &= fg \cdot T(s,t). \end{aligned}$$

If we have a metric g on the Lie algebroid, then we can require a connection to be compatible with that metric:

$$\nabla_s(g(t,u)) = g(\nabla_s t, u) + g(t, \nabla_s u), \tag{4.11}$$

for all $s, t, u \in \Gamma(E)$. (Note that $\nabla_s(g(t,u)) = \rho(s)(g(t,u))$.)
The Levi-Civita connection $\widehat{\nabla}$ on E is defined as the unique metric-compatible connection with vanishing torsion. The uniqueness and existence can be derived by considering the expression

$$\begin{aligned}
&\rho(s)g(t,u) + \rho(t)g(s,u) - \rho(u)g(s,t) = \\
&= g(\widehat{\nabla}_s u - \widehat{\nabla}_u s, t) + g(\widehat{\nabla}_t u - \widehat{\nabla}_u t, s) + g(\widehat{\nabla}_t s - \widehat{\nabla}_s t, u) + 2g(\widehat{\nabla}_s t, u) = \\
&= g([s,u]_E, t) + g([t,u]_E, s) + g([t,s]_E, u) + 2g(\widehat{\nabla}_s t, u),
\end{aligned}$$

or, by rearranging,

$$\begin{aligned}
g(\widehat{\nabla}_s t, u) = \frac{1}{2}\big(&\rho(s)g(t,u) + \rho(t)g(s,u) - \rho(u)g(s,t) + \\
&+ g([s,t]_E, u) - g([s,u]_E, t) - g([t,u]_E, s)\big),
\end{aligned} \tag{4.12}$$

which is called the *Koszul formula*. Because the covariant derivative doesn't occur on the right-hand-side, this expression can be used to define the Levi-Civita connection on any metric Lie algebroid, and it's a simple exercise to show that it satisfies all the required properties.
Note that although $\widehat{\nabla}$ is torsion-free, its Christoffel symbols will, in general, not be symmetric in its lower indices – unless $[\epsilon_\alpha, \epsilon_\beta]_E = 0$ for the basis ϵ_α we are working in.
We already considered higher tensor fields as sections of $E^{\otimes r} \otimes E^{*\otimes s}$, and we can define differential k-forms as sections of $\bigwedge^k E^*$. On these, we can introduce a nilpotent exterior derivative via the following formula:

$$\begin{aligned}
d_E\omega\,(s_1, \ldots, s_k, s_{k+1}) = &\sum_i (-1)^i \rho(s_i)\left(\omega\,(s_1, \ldots, \hat{s}_i, \ldots, s_{k+1})\right) + \\
&\sum_{i<j} (-1)^{i+j} \omega\left([s_i, s_j]_E, s_1, \ldots, \hat{s}_i, \ldots, \hat{s}_j, \ldots, s_{k+1}\right)
\end{aligned} \tag{4.13}$$

where $\omega \in \Gamma(\bigwedge^k E^*)$, $s_1, \ldots, s_{k+1} \in \Gamma(E)$. This defines the *Chevalley-Eilenberg algebra* of E.
Using the exterior derivative d_E, we can define cocycles and coboundaries in the usual way:

$$Z_E^k := \{\omega \in \Gamma(\Lambda^k E^*) | \omega \in \ker(d_E)\}, \tag{4.14}$$

$$B_E^k := \{\omega \in \Gamma(\Lambda^k E^*) | \omega \in \mathrm{im}(d_E)\}. \tag{4.15}$$

Then we define the k-th cohomology group as

$$H^k(E) := \frac{Z_E^k}{B_E^k}. \tag{4.16}$$

For the Lie algebroids we consider, i.e. those with an invertible anchor ρ, our definitions imply that

$$H^k_{\mathrm{dR}}(M) \cong H^k(E), \tag{4.17}$$

where $H^k_{\mathrm{dR}}(M)$ is the de Rham cohomology of M. To see why this holds, consider a k-form ω in $\Gamma(\Lambda^k E^*)$. Then we can pull back ω to $\Gamma(\Lambda^k T^*M)$ using exterior products of

$$\left(\rho^{-1}\right)^t : E^* \to T^*M. \tag{4.18}$$

For this map we find that

$$\begin{aligned}
&\left(\left(\Lambda^{k+1}\left(\rho^{-1}\right)^t\right)(d_E\omega)\right)(X_1,\dots,X_{k+1}) = d_E\omega\left(\rho^{-1}(X_1),\dots,\rho^{-1}(X_{k+1})\right) \\
&= \sum_i (-1)^i \rho(\rho^{-1}(X_i))\left(\omega\left(\rho^{-1}(X_1),\dots,\widehat{\rho^{-1}(X_i)},\dots,\rho^{-1}(X_{k+1})\right)\right) + \\
&\quad + \sum_{i<j} (-1)^{i+j} \omega\Big([\rho^{-1}(X_i),\rho^{-1}(X_j)]_E, \rho^{-1}(X_1),\dots,\widehat{\rho^{-1}(X_i)},\dots \\
&\qquad \dots,\widehat{\rho^{-1}(X_j)},\dots,\rho^{-1}(X_{k+1})\Big) = \\
&= d\left(\left(\Lambda^k\left(\rho^{-1}\right)^t\right)\omega\right)(X_1,\dots X_{k+1}),
\end{aligned} \tag{4.19}$$

where we used that ρ^{-1} is an isomorphism of Lie algebras.
So if ω is closed on E, then $\left(\Lambda^k\left(\rho^{-1}\right)^t\right)\omega$ is closed on TM, and if ω is exact on E, then $\left(\Lambda^k\left(\rho^{-1}\right)^t\right)\omega$ is exact on E. As the map is invertible, the cohomologies are isomorphic.

4.2 Examples

We will now start to connect the theory of non-geometric frames with the constructions above. For each $O(d,d)$ field redefinition, we can associate a Lie algebroid that, as a vector bundle, is a subbundle of $TM \oplus T^*M$, the generalized tangent bundle. For our two main examples, which will be used to describe the Q action and the R action, we will simply use TM and T^*M and equip each of them with a fiberwise Lie algebra structure and a compatible anchor.
But let's look at some simpler Lie algebroids first:
In the case of the tangent Lie algebroid, i.e. the tangent bundle with the standard Lie bracket between two vector fields, all the constructions of the last section give the well-known results of Riemannian geometry.

If the Lie algebroid E is just a Lie algebra $\mathfrak{g}$ (i.e. if M is a point), then the k-forms on E with the differential (4.19) reduce to the Chevalley-Eilenberg algebra of $\mathfrak{g}$.
For the case of the Poisson Lie algebroid, the nilpotent differential is an operator on $\Gamma(\Lambda^k TM)$, the multivector fields of M. The exterior derivative d_E is then just given by

$$d_E \alpha = [\beta, \alpha]_{\mathrm{SN}}, \tag{4.20}$$

where α is a multivector field, $[\cdot,\cdot]_{\mathrm{SN}}$ is the Schouten-Nijenhuis bracket and β is the Poisson bivector with

$$[\beta, \beta]_{\mathrm{SN}} = 0. \tag{4.21}$$

4.2.1 TM

The idea of the following construction is rather simple: We want to equip TM with a Lie algebroid structure that is different from $(TM, [\cdot,\cdot]_L, \mathrm{id})$ and allows us to describe non-geometric frames. We want to find an anchor map ρ from TM to itself that induces the transformation behavior (4.8) for the metric. So if G is the metric on M, then

$$v^t \rho^t G \rho v = \rho(v)^t G \rho(v) = v^t \gamma^{-1} G \left(\gamma^{-1}\right)^t v \quad \forall v \in \Gamma(TM), \tag{4.22}$$

where $\gamma = \mathtt{d} + (G - B)\mathtt{b}$. Now we can simply read off that

$$\rho = \left(\gamma^{-1}\right)^t. \tag{4.23}$$

Because it can be somewhat confusing, we should remind ourselves of the index structure of all the maps we will use in this section. For convenience, and because it will help later on, we will use different indices for the tangent bundle with the normal Lie bracket and the tangent bundle with the different Lie algebroid structure. The conventions will be as follows: Latin indices will be used on $(TM, [\cdot,\cdot]_L, \mathrm{id})$ and Greek indices on $E = (TM, [\cdot,\cdot]_E, \rho)$. Then the anchor, its inverses and transposes have the following index form:

$$\begin{array}{rll} \rho : E & \to TM & \rho^a{}_\alpha \\ \rho^t : T^*M & \to E^* & \rho_\alpha{}^a \\ \rho^{-1} : TM & \to E & \left(\rho^{-1}\right)^\alpha{}_a \\ \left(\rho^{-1}\right)^t : E^* & \to T^*M & \left(\rho^{-1}\right)_a{}^\alpha \end{array}$$

For the partial derivative on the Lie algebroid (4.1), this means that

$$D_\alpha f := D_{e_\alpha} f = \rho(e_\alpha) f = \rho^c{}_\alpha e_c f = \left(\rho^t\right)_\alpha{}^c e_c f. \tag{4.24}$$

We also want to introduce the bracket $[\cdot,\cdot]_E$ such that

$$\rho\big([s,t]_E\big) = [\rho(s),\rho(t)]_L \quad \forall s,t\in\Gamma(E). \tag{4.25}$$

Because ρ is invertible, this determines $[\cdot,\cdot]_E$ uniquely. Thus, the following result will actually be valid for all the Lie algebroids we consider: As soon as the anchor ρ is identified, the bracket is already fixed.
If $s = s^\alpha e_\alpha$, $t = t^\alpha e_\alpha$ for an (in general) non-holonomic basis e_α, we find that

$$\begin{aligned}
[s,t]_E &= \rho^{-1}\big([\rho(s),\rho(t)]_L\big)\\
&= \rho^{-1}\left(\left(\rho^b{}_\beta\, s^\beta \partial_b(\rho^a{}_\gamma t^\gamma) - \rho^b{}_\beta\, t^\beta \partial(\rho^a{}_\gamma s^\gamma) + s^\beta t^\delta \rho^b{}_\beta \rho^c{}_\delta [e_b,e_c]^a\right) e_a\right)\\
&= \Big[s^\beta D_\beta t^\alpha - t^\beta D_\beta s^\alpha + s^\gamma t^\delta \underbrace{\left(\rho^{-1}\right)^\alpha{}_a \left(D_\gamma \rho^a{}_\delta - D_\delta \rho^a{}_\gamma + \rho^b{}_\gamma \rho^c{}_\delta [e_b,e_c]^a\right)}_{:=F^\alpha_{\gamma\delta}}\Big] e_\alpha,
\end{aligned}$$

where we defined the *structure constants* $F^\gamma_{\alpha\beta}$ of the Lie algebroid.
So, to summarize,

$$[s,t]_E := \big(s^\beta D_\beta t^\alpha - t^\beta D_\beta s^\alpha + s^\gamma t^\delta F^\alpha_{\gamma\delta}\big) e_\alpha. \tag{4.26}$$

The Jacobi identity follows from the linearity of the anchor and the Jacobi identity for $[\cdot,\cdot]_L$; for the Leibniz rule we observe

$$\begin{aligned}
[s, f\cdot t]_E &= \rho^{-1}\left([\rho(s),\rho(ft)]_L\right) = \rho^{-1}\left([\rho(s), f\rho(t)]_L\right)\\
&= \rho^{-1}\big(f\cdot[\rho(s),\rho(t)]_L + \rho(t)\cdot\rho(s)f\big)\\
&= f\cdot[s,t]_E + t\cdot\rho(s)f.
\end{aligned}$$

Q frame

An application of the construction above is the field redefinition corresponding to the Q frame [8–10] of section 3.3. The $O(d,d)$ transformation we used was

$$\mathtt{h}_Q = \begin{pmatrix} 0 & (G-BG^{-1}B)^{-1}\\ G-BG^{-1}B & 0\end{pmatrix}, \tag{4.27}$$

so we have

$$\begin{aligned}
\gamma_Q &= (G-B)(G-BG^{-1}B)^{-1} = (\mathbb{I}-BG^{-1})(\mathbb{I}-BG^{-1}BG^{-1})^{-1}\\
&= (\mathbb{I}-BG^{-1})((\mathbb{I}+BG^{-1})(\mathbb{I}-BG^{-1}))^{-1}\\
&= (\mathbb{I}+BG^{-1})^{-1}.
\end{aligned} \tag{4.28}$$

Thus, to set up the Lie algebroid $E_Q = (TM,[\cdot,\cdot]_Q,\rho_Q)$, we will use the anchor

$$\rho_Q = (\gamma_Q^t)^{-1} = \mathbb{I} - G^{-1}B = \mathbb{I} + \beta\tilde{g}. \tag{4.29}$$

To write down our formulas, we will use the coordinate basis e_a on both E_Q and $(TM, [\cdot,\cdot]_L, \mathrm{id})$, for which $[e_a, e_b] = 0$.
We can then use definition (4.26) to write down the corresponding Lie bracket $[\cdot,\cdot]_Q$:

$$\begin{aligned}[s,t]_Q := \Big(& s^b \left(\partial_b - \tilde{g}_{bc}\beta^{cd}\partial_d\right) t^a - t^b \left(\partial_b - \tilde{g}_{bc}\beta^{cd}\partial_d\right) s^a + \\ & + 2s^c t^d \left(\mathbb{I} + \beta\tilde{g}\right)^{-1\,a}{}_b \left(\partial_{[c|} - \tilde{g}_{[c|e}\beta^{ef}\partial_f\right) \beta^{bg}\tilde{g}_{g|d]}\Big) e_a,\end{aligned} \tag{4.30}$$

where the partial derivative is

$$D_a f = \partial_a f + B_{ab}G^{bc}\partial_c = \partial_a f - \tilde{g}_{ab}\beta^{bc}\partial_c. \tag{4.31}$$

We can already see that $Q_m{}^{nk} = \partial_m \beta^{nk}$ is contained in the structure constants of the algebra, but it's not easy to give a physical interpretation of the bracket.

4.2.2 T^*M

We also want to apply our findings to the setting of [11, 12], the R frame. There, all the fields are naturally defined on the cotangent bundle, instead of the tangent bundle. Thus, we also want to equip T^*M with a Lie algebroid structure E associated to an $O(d,d)$ field redefinition.
This time, however, it's not quite clear what our anchor is supposed to be, but we can impose the following condition: On TM, we required the $O(d,d)$ transform of the standard metric to be its pullback to the Lie algebroid; for T^*M, a metric, i.e. a positive definite bilinear form for one-forms, would be G^{-1}. Thus, we will require that the Lie algebroid metric g is given as the $O(d,d)$ transformed G^{-1}:

$$g = \rho^t G \rho = \left(\gamma^{-1} G \left(\gamma^{-1}\right)^t\right)^{-1} = \gamma^t G^{-1}\gamma = \left(G^{-1}\gamma\right)^t G G^{-1}\gamma, \tag{4.32}$$

so

$$\rho = G^{-1}\gamma. \tag{4.33}$$

Again, we should be aware of the index structure of the involved maps:

$$\begin{aligned} \rho &: E \to TM & \rho^{a\alpha} \\ \rho^t &: T^*M \to E^* & \rho^{\alpha a} \\ \rho^{-1} &: TM \to E & \left(\rho^{-1}\right)_{\alpha a} \\ \left(\rho^{-1}\right)^t &: E^* \to T^*M & \left(\rho^{-1}\right)_{a\alpha} \end{aligned}$$

Just as before, the bracket is already determined because ρ is invertible, and we can basically just copy the formula:

$$[s,t]_{T^*M} = \rho^{-1}\left([\rho(s),\rho(t)]_L\right) = \tag{4.34}$$
$$= \left[s_\beta D^\beta t_\alpha - t_\beta D^\beta s_\alpha + s_\gamma t_\delta \overbrace{\left(\rho^{-1}\right)_{a\alpha}\left(D^\gamma \rho^{a\delta} - D^\delta \rho^{a\gamma} + \rho^{b\gamma}\rho^{c\delta}[e_b,e_c]^a\right)}^{:=\mathcal{G}_\alpha^{\gamma\delta}} \right] e^\alpha,$$

where $s = s_\alpha e^\alpha, t = t_\alpha e^\alpha \in \Gamma(T^*M)$ and $\mathcal{G}_\alpha^{\gamma\delta}$ are again called the structure constants.
The partial derivative also has the same formula as before:

$$D^\alpha f := D_{e^\alpha} f = \rho^{a\alpha}\partial_a = \left(\rho^t\right)^{\alpha a}\partial_a. \tag{4.35}$$

We now described two ways to find Lie algebroids associated to non-geometric frames, and we should find out whether they provide the same geometric descriptions. Fortunately, the two constructions are related in the following way:

$$\rho_T = \rho_{T^*}\cdot \widehat{G}, \tag{4.36}$$

where $\widehat{G}$ is the $O(d,d)$ transformed metric $\widehat{G} = \gamma^{-1}G\left(\gamma^{-1}\right)^t$. This also implies that the brackets are related:

$$\begin{aligned}[s,t]_{TM} &= \rho_T^{-1}\left([\rho_T(s),\rho_T(t)]_L\right) = \widehat{G}^{-1}\rho_{T^*}\left(\left[\rho_{T^*}(\widehat{G}s),\rho_{T^*}(\widehat{G}t)\right]_L\right) = \\ &= \widehat{G}^{-1}\left([\widehat{G}s,\widehat{G}t]_{T^*M}\right).\end{aligned} \tag{4.37}$$

So as long as we use the transformed metric $\widehat{G}$ to relate our quantities, all geometric objects of both constructions can be identified.

R frame

The main objects of the R frame [11, 12] of section 3.2 live on the cotangent bundle, so the T^*M Lie algebroid provides the natural setting to describe it. We can, however, also use the equivalent TM algebroid.
So in this section we will consider three Lie algebroids: The standard tangent bundle $(TM,[\cdot,\cdot]_L,\mathrm{id})$, the tangent bundle with the Lie algebroid structure 4.2.1, $(TM,[\cdot,\cdot]_{TM},\rho_T)$, and the cotangent bundle 4.2.2, $(T^*M,[\cdot,\cdot]_{T^*M},\rho_{T^*})$.
The $O(d,d)$ matrix realizing the transformation to our frame is

$$\mathtt{h}_R = \begin{pmatrix} \mathbb{I} & -\hat{\beta} \\ 2B & -\mathbb{I}\end{pmatrix}, \tag{4.38}$$

where $\hat{\beta} = B^{-1}$. This gives

$$\gamma_R = -\mathbb{I} - (G - B)\,\hat{\beta} = -G\hat{\beta}. \tag{4.39}$$

Thus, the anchors are

$$\rho_T = G^{-1}\hat{\beta}^{-1} = G^{-1}B, \tag{4.40}$$

$$\rho_{T^*} = -\hat{\beta}. \tag{4.41}$$

They induce the partial derivatives

$$D_a f = \rho_T(e_a) f = -\hat{\beta}^{-1}_{ac}\, G^{cb}\partial_b f,$$
$$D^a f = \rho_{T^*}(e^a) f = \hat{\beta}^{ab}\partial_b f.$$

Note that $e^a \in \Gamma(T^*M)$ is the dual basis to $e_a \in \Gamma(TM)$, and we'll assume $[e_a, e_b]_L = 0$.
The brackets are

$$[u, v]_{TM} = \left[u^b D_b v^a - v^b D_b u^a + 2u^b v^c G_{de}\hat{\beta}^{ea}\hat{\beta}^{-1}_{[b|f}G^{fg}\partial_g \left(G^{dh}\hat{\beta}^{-1}_{h|c]}\right)\right] e_a$$

and

$$[s, t]_{T^*M} = \left[s_b\hat{\beta}^{bc}\partial_c t_a - t_b\hat{\beta}^{bc}\partial_c s_a + 2s_c t_d \left(\hat{\beta}^{-1}\right)_{ea}\hat{\beta}^{[c|f}\partial_f\hat{\beta}^{|d]e}\right] e^a.$$

Playing around with the terms of $[s, t]_{T^*M}$, we find that that the bracket coincides with the H-twisted Koszul bracket (up to a sign):

$$[s, t]_{T^*M} = L_{-\hat{\beta}^\sharp(s)}t - \iota_{-\hat{\beta}^\sharp(t)}dt + \iota_{\hat{\beta}^\sharp(s)}\iota_{\hat{\beta}^\sharp(t)}H. \tag{4.42}$$

For more details, see [12].

Remark: Non-holonomic bases & Non-geometry

We still have to consider an important question: Can these non-geometric effects really not be explained by geometry? Or, more precisely: Can we realize these non-geometric Lie algebroids by simply choosing a non-holonomic basis of the tangent bundle? So we want to find a basis $\hat{e}_a \in \Gamma(TM)$ such that

$$[\hat{e}_a, \hat{e}_b] = F^c_{ab}\hat{e}_c. \tag{4.43}$$

But a simple calculation shows that this can be realized by

$$\hat{e}_c = \rho^d{}_c e_d. \tag{4.44}$$

Equivalently, we could replace a generic $O(d,d)$ transformation

$$\mathtt{h} = \begin{pmatrix} \mathtt{a} & \mathtt{b} \\ \mathtt{c} & \mathtt{d} \end{pmatrix} \tag{4.45}$$

by a diffeomorphism

$$\mathtt{h}_A = \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix} \tag{4.46}$$

with

$$A = (\mathtt{d}^t + \mathtt{b}^t(G+B))^{-1}, \tag{4.47}$$

and because the anchor would not change, the geometry would be the same. But the difference to a 'normal' diffeomorphism is that it explicitly depends on the metric and the B-field; thus, we need a generalized geometry to describe these frames.

Chapter 5

Supergravity on Lie algebroids

In this chapter, we will finally work out a supergravity action on Lie algebroids that are isomorphic to the tangent bundle. Our constructions will not depend on any specific algebroid, like a Dirac subbundle of $TM \oplus T^*M$; they can be used in more general settings.
We will start with a general Lie algebroid $(E, [\cdot,\cdot]_E, \rho)$ over a Riemannian manifold (M, G), where ρ is invertible and $G \in \Gamma(T^*M \otimes_{sym} T^*M)$ is a positive definite metric on M.
Then we have already seen in (4.8) how we can pull back the metric G from the tangent bundle to a metric $\hat{g} \in \Gamma(E^* \otimes_{sym} E^*)$ on the Lie algebroid:

$$\hat{g}(s,t) = G(\rho(s), \rho(t)) \quad \forall s,t \in \Gamma(E). \tag{5.1}$$

Locally, we can write this formula as

$$\hat{g}_{\alpha\beta} = {\rho^a}_\alpha {\rho^b}_\beta G_{ab}. \tag{5.2}$$

Note that the ρ-factors on the right hand side are transposed.
For this metric, we can determine the Levi-Civita connection $\widehat{\nabla}$ by the Koszul formula (4.12). By using the anchor properties, the definition of $\hat{g}$, and the non-degeneracy of the metric, we find that

$$\widehat{\nabla}_s t = \rho^{-1}\left(\nabla_{\rho(s)}\rho(t)\right) \quad \forall s,t \in \Gamma(E), \tag{5.3}$$

where ∇ is the Levi Civita connection with respect to G. To find the corresponding formula for one-forms, note that for $s,t \in \Gamma(E)$ and $\tau \in \Gamma(E^*)$, we have

$$D_t(\tau(s)) = \left(\widehat{\nabla}_t \tau\right)(s) + \tau\left(\widehat{\nabla}_t s\right), \tag{5.4}$$

and

$$\begin{aligned} D_t(\tau(s)) = D_t\left[\left((\rho^{-1})^t(\tau)\right)(\rho(s))\right] = \rho(t)\left[\left((\rho^{-1})^t(\tau)\right)(\rho(s))\right] = \\ = \left(\nabla_{\rho(t)}(\rho^{-1})^t(\tau)\right)(\rho(s)) + (\rho^{-1})^t(\tau)\left(\nabla_{\rho(t)}\rho(s)\right) = \\ = \rho^t\left(\nabla_{\rho(t)}(\rho^{-1})^t(\tau)\right)(s) + \tau\left(\widehat{\nabla}_t s\right). \end{aligned} \tag{5.5}$$

Combining these equations gives

$$\left(\widehat{\nabla}_t \tau\right)(s) = \rho^t\left(\nabla_{\rho(t)}(\rho^{-1})^t(\tau)\right)(s) \tag{5.6}$$

for all $s, t \in \Gamma(E)$ and $\tau \in \Gamma(E^*)$.
We can determine the Christoffel symbols $\widehat{\Gamma}^\alpha_{\beta\gamma}$ of $\widehat{\nabla}$ by using (5.3):

$$\widehat{\Gamma}^\alpha_{\beta\gamma} = (\rho^{-1})^\alpha{}_c\, \rho^a{}_\beta \rho^b{}_\gamma \Gamma^c_{ab} + (\rho^{-1})^\alpha{}_b\, \rho^a{}_\beta \partial_a \rho^b{}_\gamma. \tag{5.7}$$

Because the curvature and torsion tensors are defined via the connection and the bracket, we can easily see that

$$\widehat{R}(s,t)u = \rho^{-1}\left(R(\rho(s),\rho(t))\rho(u)\right), \tag{5.8}$$

$$\widehat{T}(s,t) = \rho^{-1}\left(T(\rho(s),\rho(t))\right), \tag{5.9}$$

where $s, t, u \in \Gamma(E)$, $\widehat{R}$ and $\widehat{T}$ are the torsion and the curvature of $\widehat{\nabla}$ and R and T are torsion and curvature of ∇. Thus, $\widehat{T}$ vanishes (but remember that this doesn't imply that the Christoffel symbols $\widehat{\Gamma}^\alpha_{\beta\gamma}$ are symmetric in (β,γ) – this is, in general, not the case).
Because $\widehat{R}$ is $\mathcal{C}^\infty(M)$-linear, we find the local relation

$$\widehat{R}^\alpha{}_{\beta\gamma\delta} = (\rho^{-1})^\alpha{}_a \rho^b{}_\beta \rho^c{}_\gamma \rho^d{}_\delta R^a{}_{bcd}. \tag{5.10}$$

Notice that there seems to be a pattern that fits the following, more general picture: Given an (r,s)-tensor-field $A^{a_1\ldots a_r}{}_{b_1\ldots b_s}$ on $(\bigotimes^r TM) \otimes (\bigotimes^s T^*M)$, we can define an (r,s)-tensor-field on $(\bigotimes^r E) \otimes (\bigotimes^s E^*)$ as follows:

$$\widehat{A}^{\alpha_1\ldots\alpha_r}{}_{\beta_1\ldots\beta_s} = \left(\rho^{-1}\right)^{\alpha_1}{}_{a_1} \cdots \left(\rho^{-1}\right)^{\alpha_r}{}_{a_r} \rho^{b_1}{}_{\beta_1} \cdots \rho^{b_s}{}_{\beta_s} A^{a_1\ldots a_r}{}_{b_1\ldots b_s}. \tag{5.11}$$

We will call such a field $\widehat{A}$ a ρ-tensor.
As we have already seen, the covariant derivatives on the tangent bundle and on the Lie algebroid are compatible, so the above map also works for derivatives of fields, in the same way that we observed for the exterior derivative.
For such a ρ-tensor, it's a simple exercise to show that all full contractions of (r,s)-tensors on E are equal to the contractions of the corresponding (r,s)-tensors on TM. An example for this is the Ricci scalar:

$$\widehat{R} = \hat{g}^{\beta\gamma} \widehat{R}^\alpha{}_{\beta\alpha\gamma} = G^{bc} R^a{}_{bac} = R. \tag{5.12}$$

5.1 Gauge fields

As we discussed in chapter 3, the transformation of the B field is a bit more complicated than just applying the anchor. To write down the Kalb-Ramond field B on TM as a pushforward of a two-form on E, we have to define this two-form in the first place. Because $B \in \Gamma(T^*M \bigwedge T^*M)$, we want to write

$$B = \left((\rho^{-1})^t \wedge (\rho^{-1})^t\right) \mathfrak{b} \tag{5.13}$$

for a $\mathfrak{b} \in \Gamma(E^* \bigwedge E^*)$. For example, in the TM frame, we have $\rho = (\gamma^{-1})^t = \hat{\gamma}^t$, and

$$B = \hat{\gamma}^{-1}(\hat{\gamma}\hat{\delta}^t - \hat{g})(\hat{\gamma}^{-1})^t. \tag{5.14}$$

So in this case, we would have

$$\mathfrak{b} = \hat{\gamma}\hat{\delta}^t - \hat{g}. \tag{5.15}$$

We want to investigate the behavior of the Lie algebroid Kalb-Ramond field $\mathfrak{b}$ under gauge transformations $B \mapsto B + d\xi$ for a $\xi \in \Gamma(T^*M)$. Naively, we would say that

$$\left(\Lambda^2(\rho^{-1})^t\right)(d\xi) = d_E\left((\rho^{-1})^t\xi\right) \tag{5.16}$$

implies that the gauge symmetry corresponds to

$$\mathfrak{b} \mapsto \mathfrak{b} + d_E\xi, \tag{5.17}$$

for a $\xi \in \Gamma(T^*M)$, but unfortunately, in the situations we will consider, the anchor ρ itself depends on the B field as well; this will add a correction term Δ_ξ to the gauge transformations of every field that's pulled back to the Lie algebroid. We will avoid these complications by only considering gauge independent objects; and if we define the 3-form

$$\Theta = d_E\mathfrak{b}, \tag{5.18}$$

we find that

$$\Theta_{\alpha\beta\gamma} = \rho^a{}_\alpha\rho^b{}_\beta\rho^c{}_\gamma H_{abc}, \tag{5.19}$$

and

$$\Theta^2 = \Theta_{\alpha\beta\gamma}\Theta_{\delta\epsilon\zeta}\,\hat{g}^{\alpha\delta}\hat{g}^{\beta\epsilon}\hat{g}^{\gamma\zeta} = H^2 \tag{5.20}$$

is invariant under gauge transformations of the B field.

5.2 Constructing the action

We have now defined all the ingredients we need to write down the supergravity action in Lie algebroid frames. Consider the standard NS-NS action on TM:

$$S^{\text{NS-NS}} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{G}e^{-2\phi}\left(R - \frac{1}{12}H_{abc}H^{abc} + 4\partial_a\phi\partial^a\phi\right). \tag{5.21}$$

We have actually already seen how all of these terms can be transferred to the Lie algebroid:

$$R \mapsto \widehat{R}, \tag{5.22}$$
$$H_{abc}H^{abc} \mapsto \Theta_{\alpha\beta\gamma}\Theta^{\alpha\beta\gamma}, \tag{5.23}$$
$$\partial_a\phi\partial^a\phi \mapsto D_\alpha\phi D^\alpha\phi. \tag{5.24}$$

Finally, for the determinant of the metric we get

$$\det G = \det\hat{g}(\det\rho^{-1})^2. \tag{5.25}$$

So we have to make the replacement

$$\sqrt{G} \mapsto \sqrt{\hat{g}}\,|\rho|^{-1}. \tag{5.26}$$

Then finally, we found that $S^{\text{NS-NS}}$ is completely equivalent to the following action on Lie algebroids:

$$\hat{S}^{\text{NS-NS}} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{\hat{g}}\,|\rho|^{-1}\,e^{-2\phi}\left(\widehat{R} - \frac{1}{12}\Theta_{\alpha\beta\gamma}\Theta^{\alpha\beta\gamma} + 4D_\alpha\phi D^\alpha\phi\right). \tag{5.27}$$

Let's find out what this means for the Q and the R frame:

Q-frame action

As one can imagine when reading the papers [8–10], expanding all the terms in the action (2.56) leads to very messy expressions, that are not necessarily intuitive. One should note that the actions in these different papers are equivalent up to total boundary terms, whereas the DFT derivation of [9] or our Lie algebroid result gives an action that is exactly equal to the standard NS-NS action.
Remember that the anchor is given by

$$\rho_Q = \mathbb{I} + \beta\tilde{g}, \tag{5.28}$$

so we get for the measure

$$\sqrt{G} = \frac{\sqrt{|\tilde{g}|}}{|\mathbb{I} + \beta\tilde{g}|} = \frac{1}{\sqrt{|\tilde{g}|}} \cdot \frac{1}{|\tilde{g}^{-1} + \beta|} \tag{5.29}$$

Note that in the convention normally used when describing this action, the dilaton ϕ is redefined to 'swallow' this measure factor.
We can also express the three-form Θ by using the fact that it is a ρ-tensor, and find that

$$\Theta_{abc} = 3\tilde{g}_{ad}\tilde{g}_{be}\tilde{g}_{cf}\beta^{g[d|}\partial_g\beta^{|ef]} + \mathcal{O}(\partial\tilde{g}) + \mathcal{O}(\partial\beta). \tag{5.30}$$

R-frame action

We have already seen that we can recover the partial derivative of [11, 12] in this frame. Because $\rho_R = -\hat{\beta}$, we can also see that the measure factor $\det\beta^{-1}$ is included in the action. The Ricci scalar $\widehat{R}$ is constructed exactly like in [11, 12], and for the three-form we have

$$\Theta^{abc} = 3\hat{\beta}^{d[a|}\partial_d\hat{\beta}^{|bc]}. \tag{5.31}$$

Thus, we recovered the action (2.50).

5.2.1 DFT based construction

We can derive (5.27) for the case of **constant $\mathbf{O(d, d)}$ transformations**

$$\mathtt{h} = \begin{pmatrix} \mathtt{a} & \mathtt{b} \\ \mathtt{c} & \mathtt{d} \end{pmatrix} \tag{5.32}$$

by using double field theory techniques. The key is to use both the $O(d, d)$ invariance of the action and a special solution of the strong constraint (2.18). Our argument will somewhat be in reverse: we start with the DFT action (2.30) in the transformed fields $\hat{g}$ and $\widehat{B}$, and reduce it to d dimensions by giving an explicit solution to the strong constraint (2.18). Then we will apply an $O(d, d)$ transformation to these fields and their corresponding coordinates. This leaves the action invariant, but maps the transformed fields $\hat{g}$ and $\widehat{B}$ to the original metric G and Kalb-Ramond field B; furthermore, it acts on the derivatives in such a way that the chosen solution to the strong constraint will just be $\tilde{\partial} = 0$ in the new frame, which reduces the action to the NS-NS action.

So let's consider S_{DFT} in the frame where $\widehat{\mathcal{E}} = \hat{g} + \widehat{B}$. Then we will solve the strong constraint by making the ansatz

$$\hat{\tilde{\partial}}^i = (\mathtt{b}^t)^{ij}\partial_j, \tag{5.33}$$

$$\hat{\partial}_i = (\mathtt{a}^t)_i{}^j\partial_j. \tag{5.34}$$

Then

$$\hat{\tilde{\partial}}^i\hat{\partial}_i(A\cdot B) = \partial_i A\partial_j B(\mathtt{a}\cdot\mathtt{b}^t + \mathtt{b}\cdot\mathtt{a}^t)^{ij} = 0 \tag{5.35}$$

for $\mathtt{h} \in O(d,d)$.

As we stated before, we will now perform an $O(d,d)$ transformation on the whole frame. The transformation we will choose is

$$\mathtt{h}' = \left(\mathtt{h}^t\right)^{-1} = \begin{pmatrix} \mathtt{d} & \mathtt{c} \\ \mathtt{b} & \mathtt{a} \end{pmatrix}. \tag{5.36}$$

Then the transformation rules of double field theory, (2.21), imply that the new winding derivative is given by

$$\tilde{\partial}' = \mathtt{a}\cdot\hat{\tilde{\partial}} + \mathtt{b}\cdot\hat{\partial}, \tag{5.37}$$

which vanishes for the previously chosen solution for the strong constraint. Furthermore, the metric transforms via (2.34) with $M = (\mathtt{a} - \mathtt{b}\widehat{\mathcal{E}}^t)^t$:

$$\hat{g} = (\mathtt{a}^t + (\hat{g} - \widehat{B})\mathtt{b}^t)G(\mathtt{a}^t + (\hat{g} - \widehat{B})\mathtt{b}^t)^t, \tag{5.38}$$

which is, due to (3.43), exactly the transformation that transforms us back to the standard (G,B) frame; the dilaton d stays invariant, which implies the correct transformation of the measure.

Thus, DFT can be used to show the equivalence of the Lie algebroid and the standard NS-NS actions for constant $O(d,d)$ transformations.

5.3 R-R fields and fermions

In this section, we will extend the results above to define a full low-energy effective superstring action on a Lie algebroid.

As we have to allow for flat spinor indices, we will slightly amend our notation: tangent bundle indices will be denoted by greek indices starting with μ, Lie algebroid indices will be Greek indices starting with α and flat Lorentz frame indices will be Latin indices starting with a.

As an example, we will consider type IIA theory without higher-order α' corrections (which wouldn't pose a problem either). This is a ten-dimensional

theory containing the bosonic fields G, B and ϕ in the NS-NS sector, the Ramond-Ramond 1- and 3-forms C_μ and $C_{\mu_1\mu_2\mu_3}$ and the fermionic dilatino λ and the gravitino ψ_μ.
The type IIA supergravity action (in the Einstein frame) is given by (cf. e.g. [40]):

$$S_{\text{IIA}} = S^{\text{NS-NS}} + S^{\text{R-R}} + S^{\text{CS}} + S^{\text{f}} + \mathcal{O}(\alpha'^2), \tag{5.39}$$

where

$$S^{\text{NS-NS}} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{G}\left(R - \frac{1}{3}e^{-\phi}H_{\mu\nu\rho}H^{\mu\nu\rho} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi\right) \tag{5.40}$$

is the action for the NS-NS fields,

$$S^{\text{R-R}} = -\frac{1}{2\kappa^2}\int d^{10}x\sqrt{G}\left(e^{\frac{3\phi}{2}}\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{12}e^{\frac{\phi}{2}}F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma}\right) \tag{5.41}$$

is the action for the R-R field strengths,

$$S^{\text{CS}} = \frac{1}{2\kappa^2}\int d^{10}x\frac{1}{288}\epsilon^{\mu_1\ldots\mu_{10}}B_{\mu_1\mu_2}\partial_{[\mu_3}C_{\mu_4\mu_5\mu_6]}\partial_{[\mu_7}C_{\mu_8\mu_9\mu_{10}]} \tag{5.42}$$

is the Chern-Simons term and

$$\begin{aligned}
S^{\text{f}} =& \frac{2}{\kappa^2}\int d^{10}x\sqrt{G}\Big[-\frac{1}{2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}\left(\nabla_\nu - \frac{i}{4}\omega_\nu{}^{ab}\gamma_{ab}\right)\psi_\rho - \\
&- \frac{1}{2}\bar{\lambda}\gamma^\mu\left(\partial_\mu - \frac{i}{4}\omega_\mu{}^{ab}\gamma_{ab}\right)\lambda + \frac{\sqrt{2}}{4}\bar{\lambda}\gamma^{11}\gamma^\mu\gamma^\nu\psi_\mu\partial_\nu\sigma + \\
&+ \frac{1}{96}e^{\frac{\phi}{4}}\left(-\bar{\psi}_{\mu_1}\gamma^{\mu_1\ldots\mu_6}\psi_{\mu_2} - 12\bar{\psi}^{\mu_3}\gamma^{\mu_4\mu_5}\psi^{\mu_6} + \frac{1}{\sqrt{2}}\bar{\lambda}\gamma^{11}\gamma^{\mu_3\ldots\mu_6}\lambda\right)F_{\mu_3\ldots\mu_6} \\
&- \frac{1}{24}e^{-\frac{\phi}{2}}\left(\bar{\psi}_{\mu_1}\gamma^{11}\gamma^{\mu_1\ldots\mu_5}\psi_{\mu_2} - 6\bar{\psi}^{\mu_3}\gamma^{11}\gamma^{\mu_4}\psi^{\mu_5} - \sqrt{2}\bar{\lambda}\gamma^\nu\gamma^{\mu_3\mu_4\mu_5}\psi_\nu\right)F_{\mu_3\mu_4\mu_5} \\
&- \frac{1}{16}e^{\frac{3\phi}{4}}\Big(\bar{\psi}_{\mu_1}\gamma^{11}\gamma^{\mu_1\ldots\mu_4}\psi_{\mu_2} + 2\bar{\psi}^{\mu_3}\gamma^{11}\psi^{\mu_4} + \frac{3}{\sqrt{2}}\bar{\lambda}\gamma^{\mu_1}\gamma^{\mu_3\mu_4}\psi_{\mu_1} \\
&- \frac{5}{4}\bar{\lambda}\gamma^{11}\gamma^{\mu_3\mu_4}\lambda\Big)F_{\mu_3\mu_4} + \text{four-fermion interactions}\Big]
\end{aligned} \tag{5.43}$$

is the action describing the fermionic fields.
The R-R field strengths $F_{\mu\nu}$ and $F_{\mu\nu\rho\sigma}$ are given by[1]

$$F_{\mu\nu} = 2\nabla_{[\mu}C_{\nu]}, \tag{5.44}$$

$$F_{\mu\nu\rho\sigma} = 4\left(\nabla_{[\mu}C_{\nu\rho\sigma]} + C_{[\mu}H_{\nu\rho\sigma]}\right). \tag{5.45}$$

[1]These fields are usually defined with a partial derivative – but because we antisymmetrize, and the standard Christoffel symbols are symmetric, this doesn't make a difference. But on the Lie algebroid, $\widehat{\Gamma}^\alpha_{\beta\gamma}$ is not symmetric anymore, and we will have to make these replacements whenever they are necessary.

These fields are invariant under two gauge symmetries:

$$\delta_{\Lambda_{(0)}} C_\mu = \partial_\mu \Lambda, \qquad \delta_{\Lambda_{(0)}} C_{\mu\nu\rho} = -\Lambda H_{\mu\nu\rho}, \tag{5.46}$$

$$\delta_{\Lambda_{(2)}} C_{\mu\nu\rho} = \nabla_{[\mu} \Lambda_{\nu\rho]}, \tag{5.47}$$

for any two-forms $\Lambda_{(2)}$ and functions $\Lambda_{(0)}$.
By pulling back the R-R fields $C_{(1)}$ and $C_{(3)}$ to the Lie algebroid, we can define their field strengths in exactly the same way:

$$\widehat{F}_{\alpha\beta} = 2\widehat{\nabla}_{[\alpha}\widehat{C}_{\beta]}, \tag{5.48}$$

$$\widehat{F}_{\alpha\beta\gamma\delta} = 4\left(\widehat{\nabla}_{[\alpha}\widehat{C}_{\beta\gamma\delta]} + \widehat{C}_{[\alpha}\Theta_{\beta\gamma\delta]}\right). \tag{5.49}$$

The gauge symmetries are preserved under this transformation:

$$\delta_{\Lambda_{(0)}} \widehat{C}_\alpha = D_\alpha \Lambda, \qquad \delta_{\Lambda_{(0)}} \widehat{C}_{\alpha\beta\gamma} = -\Lambda \Theta_{\alpha\beta\gamma}, \tag{5.50}$$

$$\delta_{\Lambda_{(2)}} \widehat{C}_{\alpha\beta\gamma} = \widehat{\nabla}_{[\alpha}\Lambda_{\beta\gamma]} \tag{5.51}$$

For the invariance under $\delta_{\Lambda_{(0)}}$, observe that $\widehat{\nabla}_{[\alpha}\Theta_{\beta\gamma\delta]} = 0$.
The last remaining ingredients are the spin connection ${\omega_\mu}^{ab}$ and the covariant gamma matrices γ^μ. To obtain these, we will define the frame fields e^a_μ by

$$e^a_\mu e^b_\nu G^{\mu\nu} = \delta^{ab}, \tag{5.52}$$

and their inverses will be denoted by e^μ_a.
Then the spin connection is given by

$${\omega_\mu}^a{}_b = e^a_\nu e^\rho_b \Gamma^\nu_{\rho\mu} + e^a_\nu \partial_\mu e^\nu_b. \tag{5.53}$$

The gamma matrices γ_μ are defined as

$$\gamma_\mu = e^a_\mu \gamma_a, \tag{5.54}$$

where γ_a are the classical gamma matrices on flat space.
All of these objects can be consistently defined as ρ-tensors on the Lie algebroid E. We can introduce the vielbeins e^a_α by

$$\hat{e}^a_\alpha \hat{e}^b_\beta \hat{g}^{\alpha\beta} = \delta^{\alpha\beta}. \tag{5.55}$$

Comparing (5.52) with (5.55), and using $\hat{g}_{\alpha\beta} = \rho^\mu{}_\alpha \rho^\nu{}_\beta G_{\mu\nu}$, we find that

$$\hat{e}^a_\alpha = \rho^\mu{}_\alpha e^a_\mu. \tag{5.56}$$

Thus, the vielbein is a ρ-tensor.

Then we can write down the spin connection on the Lie algebroid as

$$\widehat{\omega}_\alpha{}^a{}_b = \hat{e}^a_\beta \hat{e}^\gamma_b \widehat{\Gamma}^\beta_{\gamma\alpha} + \hat{e}^a_\beta D_\alpha \hat{e}^\beta_b. \tag{5.57}$$

Using the formula (5.7) for the Lie algebroid Christoffel symbols (which were not ρ-tensors) gives the result

$$\widehat{\omega}_\alpha{}^a{}_b = \rho^\mu{}_\alpha \omega_\mu{}^a{}_b. \tag{5.58}$$

Finally, for the gamma matrices $\hat{\gamma}$ we have

$$\hat{\gamma}_\alpha = \hat{e}^a_\alpha \gamma_a = \rho^\mu{}_\alpha e^a_\mu \gamma_a = \rho^\mu{}_\alpha \gamma_\mu. \tag{5.59}$$

Note that the epsilon-tensor in the Chern-Simons part of the action, which is defined by contracting the epsilon symbol with the vielbeins, transforms as

$$\hat{\epsilon}^{\alpha_1 \ldots \alpha_{10}} = (\rho^{-1})^{\alpha_1}{}_{\mu_1} \cdots (\rho^{-1})^{\alpha_{10}}{}_{\mu_{10}} \epsilon^{\mu_1 \ldots \mu_{10}}. \tag{5.60}$$

Finally, for the gravitino we define

$$\hat{\psi}_\alpha = \rho^\mu{}_\alpha \psi_\mu. \tag{5.61}$$

With the above procedures, we can pull back the full action of type IIA supergravity to the Lie algebroid.
We've already seen how this works for $S^{\text{NS-NS}}$, and for the Einstein frame we get:

$$\hat{S}^{\text{NS-NS}} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{\hat{g}}\, |\rho|^{-1} \left(\widehat{R} - \frac{1}{3} e^{-\phi} \Theta_{\alpha\beta\gamma} \Theta^{\alpha\beta\gamma} - \frac{1}{2} D_\alpha \phi D^\alpha \phi \right). \tag{5.62}$$

For the Ramond-Ramond part of the action, we have

$$\hat{S}^{\text{R-R}} = -\frac{1}{2\kappa^2} \int d^{10}x \sqrt{\hat{g}}\, |\rho|^{-1} \left(e^{\frac{3\phi}{2}} \frac{1}{4} \widehat{F}_{\alpha\beta} \widehat{F}^{\alpha\beta} + \frac{1}{12} e^{\frac{\phi}{2}} \widehat{F}_{\alpha\beta\gamma\delta} \widehat{F}^{\alpha\beta\gamma\delta} \right). \tag{5.63}$$

For the Chern-Simons part, we get

$$\hat{S}^{\text{CS}} = \frac{1}{2\kappa^2} \int d^{10}x \frac{1}{288} \hat{\epsilon}^{\alpha_1 \ldots \alpha_{10}} \mathfrak{b}_{\alpha_1\alpha_2} \widehat{\nabla}_{[\alpha_3} \widehat{C}_{\alpha_4\alpha_5\alpha_6]} \widehat{\nabla}_{[\alpha_7} \widehat{C}_{\alpha_8\alpha_9\alpha_{10}]}. \tag{5.64}$$

So finally, for the fermionic part of the action we find

$$
\begin{aligned}
\hat{S}^{\mathrm{f}} =& \frac{2}{\kappa^2}\int d^{10}x\sqrt{\hat{g}}\,|\rho|^{-1}\Big[-\frac{1}{2}\bar{\hat{\psi}}_\alpha\hat{\gamma}^{\alpha\beta\gamma}\left(\widehat{\nabla}_\beta-\frac{i}{4}\widehat{\omega}_\beta{}^{ab}\gamma_{ab}\right)\hat{\psi}_\gamma- \\
&-\frac{1}{2}\bar{\lambda}\hat{\gamma}^\alpha\left(D_\alpha-\frac{i}{4}\widehat{\omega}_\alpha{}^{ab}\gamma_{ab}\right)\lambda+\frac{\sqrt{2}}{4}\bar{\lambda}\hat{\gamma}^{11}\hat{\gamma}^\alpha\hat{\gamma}^\beta\hat{\psi}_\alpha D_\beta\phi+ \\
&+\frac{1}{96}e^{\frac{\phi}{4}}\left(-\bar{\hat{\psi}}_{\alpha_1}\hat{\gamma}^{\alpha_1\ldots\alpha_6}\hat{\psi}_{\alpha_2}-12\bar{\hat{\psi}}^{\alpha_3}\hat{\gamma}^{\alpha_4\alpha_5}\hat{\psi}^{\alpha_6}+\frac{1}{\sqrt{2}}\bar{\lambda}\hat{\gamma}^{11}\hat{\gamma}^{\alpha_3\ldots\alpha_6}\lambda\right)\widehat{F}_{\alpha_3\ldots\alpha_6} \\
&-\frac{1}{24}e^{-\frac{\phi}{2}}\left(\bar{\hat{\psi}}_{\alpha_1}\hat{\gamma}^{11}\hat{\gamma}^{\alpha_1\ldots\alpha_5}\hat{\psi}_{\alpha_2}-6\bar{\hat{\psi}}^{\mu_3}\gamma^{11}\hat{\gamma}^{\alpha_4}\hat{\psi}^{\alpha_5}-\sqrt{2}\bar{\lambda}\hat{\gamma}^\beta\gamma^{\alpha_3\alpha_4\alpha_5}\hat{\psi}_\beta\right)\widehat{F}_{\alpha_3\alpha_4\alpha_5} \\
&-\frac{1}{16}e^{\frac{3\phi}{4}}\Big(\bar{\hat{\psi}}_{\alpha_1}\hat{\gamma}^{11}\hat{\gamma}^{\alpha_1\ldots\alpha_4}\hat{\psi}_{\alpha_2}+2\bar{\hat{\psi}}^{\alpha_3}\hat{\gamma}^{11}\hat{\psi}^{\alpha_4}+\frac{3}{\sqrt{2}}\bar{\lambda}\hat{\gamma}^{\alpha_1}\hat{\gamma}^{\alpha_3\alpha_4}\hat{\psi}_{\alpha_1} \\
&-\frac{5}{4}\bar{\lambda}\hat{\gamma}^{11}\hat{\gamma}^{\alpha_3\alpha_4}\lambda\Big)\widehat{F}_{\alpha_3\alpha_4}+\text{four-fermion interactions}\Big].
\end{aligned}
\tag{5.65}
$$

In summary, the Lie algebroid action is given by

$$
\hat{S}_{\mathrm{IIA}}=\hat{S}^{\text{NS-NS}}+\hat{S}^{\text{R-R}}+\hat{S}^{\mathrm{CS}}+\hat{S}^{\mathrm{f}}+\mathcal{O}(\alpha'^2) \tag{5.66}
$$

and is completely equivalent to S_{IIA}.

5.4 Equations of motion

The equations of motion for the standard NS-NS action in the string frame can be derived quite easily. Requiring the action to be stationary under field variations gives

$$
\begin{aligned}
0 &= R_{\mu\nu}+2\nabla_\mu\nabla_\nu\phi-\frac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma}, \\
0 &= -\frac{1}{2}\nabla^\mu H_{\mu\nu\rho}+\nabla^\mu\phi H_{\mu\nu\rho}, \\
0 &= -\frac{1}{2}g^{\mu\nu}\nabla_\mu\nabla_\nu\phi+g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi-\frac{1}{24}H_{\mu\nu\rho}H^{\mu\nu\rho}.
\end{aligned}
\tag{5.67}
$$

To arrive at the equations of motion for $\hat{S}_{\text{NS-NS}}$, we can either apply our field redefinitions to the equations above, or we can re-derive them from the new action. In both cases, we get

$$\begin{aligned}
0 &= \widehat{R}_{\alpha\beta} + 2\widehat{\nabla}_\alpha \widehat{\nabla}_\beta \phi - \frac{1}{4}\Theta_{\alpha\gamma\delta}\Theta_\beta{}^{\gamma\delta}, \\
0 &= -\frac{1}{2}\widehat{\nabla}^\gamma \Theta_{\gamma\alpha\beta} + \widehat{\nabla}^\gamma \phi \Theta_{\gamma\alpha\beta}, \\
0 &= -\frac{1}{2}\hat{g}^{\alpha\beta}\widehat{\nabla}_\alpha \widehat{\nabla}_\beta \phi + \hat{g}^{\alpha\beta}\widehat{\nabla}_\alpha \phi \widehat{\nabla}_\beta \phi - \frac{1}{24}\Theta_{\alpha\beta\gamma}\Theta^{\alpha\beta\gamma}.
\end{aligned} \tag{5.68}$$

This coincides with the results of [12] for the R frame.

5.5 Symmetries & non-geometric backgrounds

In this section, we have to face the question whether the action (5.66) can accurately describe non-geometric backgrounds. These backgrounds are solutions to the equations of motion (5.68), and we will briefly present the standard way of obtaining them by considering a torus compactification (cf. e.g. [36]).

Let $M = T^3$ and

$$G = dx^2 + dy^2 + dz^2, \tag{5.69}$$

$$B = Nz \cdot dx \wedge dy, \tag{5.70}$$

$$\phi = \text{const.} \tag{5.71}$$

Then $H = N dx \wedge dy \wedge dz$. The radii of the torus are set to $\frac{1}{2\pi}$, so

$$(x, y, z) \sim (x + 1, y, z) \sim (x, y + 1, z) \sim (x, y, z + 1). \tag{5.72}$$

Thus, while the B-field has a monodromy, its values at the 'boundary' are related by a gauge transformation, and the H flux is still single-valued. Note that this is only a solution of (5.67) up to linear order in the H-flux, but this shall suffice for our considerations.

Because G and B contain two isometric directions, x and y, we can perform T-dualities for these components by applying the Buscher rules (3.39). By applying one T-duality, we end up with a *twisted torus*:

$$G = (dx - Nz \cdot dy)^2 + dy^2 + dz^2, \tag{5.73}$$

$$B = 0. \tag{5.74}$$

We define the vielbeins

$$\sigma^x = dx - Nz \cdot dy, \tag{5.75}$$

$$\sigma^y = dy, \tag{5.76}$$

$$\sigma^z = dz. \tag{5.77}$$

Then we can define the *geometric flux* $f^{\mu}_{\nu\rho}$ as

$$d\sigma^{\mu} = f^{\mu}_{\nu\rho}\sigma^{\nu} \wedge \sigma^{\rho}. \tag{5.78}$$

In our case, we have a geometric flux $f^{x}_{yz} = N$. This flux is called geometric because we can still interpret it geometrically; we just need to amend the identification rules (5.72). For more information about such backgrounds, see [36].

We still have a remaining isometry direction, so let's see which background we end up with after a T-duality in the y direction:

$$G = \frac{1}{1+(Nz)^2}(dx^2 + dy^2) + dz^2, \tag{5.79}$$

$$B = \frac{-Nz}{1+(Nz)^2} \cdot dx \wedge dy. \tag{5.80}$$

This *non-geometric background* is obviously a bit more problematic: The B-field cannot be patched up by a gauge transformation; the fields are only well-defined in a non-geometric frame. Indeed, if we transform to the Q frame of section 3.3, we find that the background is described by

$$\tilde{g} = dx^2 + dy^2 + dz^2, \tag{5.81}$$

$$\widehat{B} = -Nz \cdot dx \wedge dy. \tag{5.82}$$

The bivector (3.63) is given by

$$\beta = -Nz\partial_x \wedge \partial_y. \tag{5.83}$$

So the Q-flux $\partial\beta$ is given by

$$\mathcal{Q}_z{}^{xy} = -N. \tag{5.84}$$

Thus, to make the Q-flux background consistent, we would need a β transformation to patch up the fields, which is the way in which non-geometry 'transcends' geometry.

For a further exposition of such non-geometric string backgrounds, see [41].

Note that another (forbidden) T-duality in the z direction leads us to a locally ill-defined background with a non-vanishing R flux.

We now want to investigate which low-energy effective action can describe such backgrounds in the following, abstract sense:

Suppose we are considering a physical theory on a manifold M. (See figure 5.1) Then M is completely described by its patches, like U and V, and its transition functions. So to describe the theory on M, we will give local actions

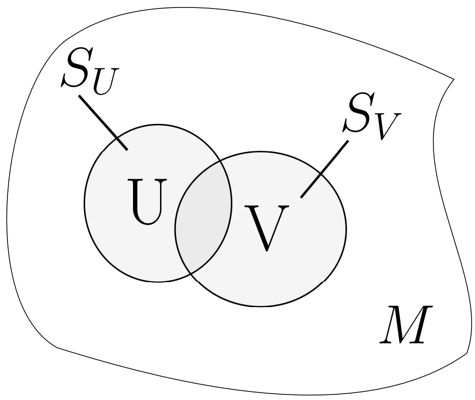

Figure 5.1: The manifold M contains the overlapping patches U and V, and, for consistency, the actions S_U on U and S_V on V have to be equivalent on $U \cap V$.

on each patch, like S_U and S_V in the figure; because we want the action to be globally well-defined, we need to make sure that S_U is completely equivalent to S_V on the overlap $U \cap V$.
Now let's consider our theories on Lie algebroids (5.66) (we will focus on the NS-NS part of the action).
The field content is the dilaton ϕ and the generalized metric $\mathcal{H}$. If we are given a solution $\mathcal{H}$ to the equations of motion, then we can describe $\mathcal{H}$ locally on U by $\mathcal{H}_U$ and on V by $\mathcal{H}_V$. Then, in general, $\mathcal{H}_U$ and $\mathcal{H}_V$ will not coincide on $U \cap V$. But as long as the transformation that's required to 'patch up' the solution is a symmetry of the action, it is still well-defined.
So in our case of interest, we will have

$$\mathcal{H}_U = \mathtt{g}^t \, \mathcal{H}_V \, \mathtt{g} \tag{5.85}$$

for a β-transformation $\mathtt{g}$. Now we transform the whole situation to a new frame, i.e. we redefine the $\mathcal{H}$ field to get

$$\mathcal{H}'_U = \mathtt{h}^t \, \mathcal{H}_U \, \mathtt{h}, \tag{5.86}$$
$$\mathcal{H}'_V = \mathtt{h}^t \, \mathcal{H}_V \, \mathtt{h} \, . \tag{5.87}$$

So the relation between the transformed generalized metrics (or the transition function between the patches) is

$$\mathcal{H}'_U = (\mathtt{h}^{-1} \, \mathtt{g} \, \mathtt{h})^t \, \mathcal{H}'_V (\mathtt{h}^{-1} \, \mathtt{g} \, \mathtt{h}). \tag{5.88}$$

So the group of symmetries of the action transforms via conjugation by the transformation $\mathtt{h} \in O(d, d)$.

So we can see that a non-geometric background like the Q flux background, where

$$\widehat{B} = -Nz \cdot dx \wedge dy \tag{5.89}$$

can not be patched up by elements in the symmetry group of the Q action, because it doesn't include shifts in $\widehat{B}$. (Alternatively, we could also see that the constant shifts are not d_E-exact.)
So the non-geometric actions can describe non-geometric backgrounds on each patch, but they are not globally well-defined.

Chapter 6

Conclusions & Outlook

In this thesis, we have worked out how actions for non-geometric fluxes can be obtained by employing ideas of generalized geometry. We determined which $O(d,d)$ field redefinitions give rise to non-geometric fluxes, and we found a Lie algebroid corresponding to each of these non-geometric frames. On these algebroids, we could write down a supergravity action that is exactly equivalent to the standard supergravity action on the tangent bundle.
We also found that supergravity on Lie algebroids, as it was presented in this thesis, only seems to provide a local description of non-geometric fluxes. The problem can most easily be understood by taking the DFT approach: Although all solutions of double field theory can, as long as they satisfy the weak constraint, locally be described by only d coordinates, we need to allow for some winding dependence to connect different patches of the compactification manifold. This is not an issue for geometric backgrounds, as the necessary transformations to do so, diffeomorphisms and B-field gauge transformations, are not winding-dependent. But for non-geometric backgrounds, choosing a global solution to the strong constraint and reducing the action seems to inevitably cause a loss of information.
Some more recent studies of non-geometry 'go beyond' generalized geometry and use the framework of double field theory to investigate effects of non-associativity and non-commutativity. In [42], non-associative deformations were analyzed, and it was found that on-shell associativity was preserved due to the section condition. Furthermore, the authors of [43] present explicit non-geometric solutions in double field theory, which elucidates the role that winding coordinates play for such backgrounds. A more recent connection between Lie (bi-)algebroids and double field theory is presented in [44].

Appendix A

Invertibility of the anchor

In this appendix we will show that the map $\gamma = \mathtt{d} + (G - B)\mathtt{b}$, and thus the anchor ρ, is invertible for arbitrary coordinate-dependent $O(d, d)$ transformations

$$\mathtt{h} = \begin{pmatrix} \mathtt{a} & \mathtt{b} \\ \mathtt{c} & \mathtt{d} \end{pmatrix}.$$

To prove this result, we will use that any such $\mathtt{h}$ can be written as a finite products of a small class of $O(d, d)$ transformations, because:

Lemma A.1. *The identity component $O(d, d; \mathbb{R})_0$ of $O(d, d; \mathbb{R})$ is generated by the diffeomorphisms, B-field gauge transformations and β-transformations of table 3.1.*

Proof. The group generated by these matrices is a closed Lie subgroup of $O(d, d; \mathbb{R})$, and its dimension is $d^2 + \frac{d(d-1)}{2} + \frac{d(d-1)}{2}$, which is $2d^2 - d = \frac{2d(2d-1)}{2}$, so already the dimension of the whole group. As all smoothly embedded codimension 0 submanifolds of a smooth manifold are open subsets, this is an open neighbourhood of the identity. Because the identity component of a Lie group is generated by any such open neighbourhood, the result follows. □

As all $O(p, q; \mathbb{R})$ groups with $p, q > 0$, $O(d, d; \mathbb{R})$ is not connected, its group of components is $O(d, d; \mathbb{R})/O(d, d; \mathbb{R})_0 = \pi_0(O(d, d; \mathbb{R})) = \mathbb{Z}_2 \times \mathbb{Z}_2$. To find the generators of that group, it's convenient to do a basis transformation by

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{I} & \mathbb{I} \\ -\mathbb{I} & \mathbb{I} \end{pmatrix}. \tag{A.1}$$

In this basis, the $O(d, d)$ metric η transforms to

$$\eta' = P \eta P^t = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}.$$

Thus, an $O(d,d)$ transformation in this basis has to fulfill

$$\begin{pmatrix} A^t & C^t \\ B^t & D^t \end{pmatrix} \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}. \tag{A.2}$$

This implies that

$$A^t A = \mathbb{I} + C^t C, \tag{A.3}$$
$$D^t D = \mathbb{I} + B^t B. \tag{A.4}$$

Now assume that v is an eigenvector of A with unit norm and the (complex) eigenvalue λ. Then we can sandwich (A.3) between v and v and we get

$$|\lambda|^2 = 1 + \|Cv\|^2. \tag{A.5}$$

Because the determinant of A is the product of its eigenvalues, we find that $(\det A)^2 \geq 1$, and similarly $(\det D)^2 \geq 1$. This makes it very easy to identify the four components of $O(d,d)$ in this basis: Their upper-left and lower-right submatrices have positive or negative determinants, respectively. As generators of the component group, we can now choose

$$H_- = \left(\begin{array}{cccc|ccc} -1 & & & & & & \\ & 1 & & & & 0 & \\ & & \ddots & & & & \\ & & & 1 & & & \\ \hline & & & & 1 & & \\ & 0 & & & & \ddots & \\ & & & & & & 1 \end{array}\right), \tag{A.6}$$

$$H_+ = \left(\begin{array}{ccc|cccc} 1 & & & & & & \\ & \ddots & & & 0 & & \\ & & 1 & & & & \\ \hline & & & -1 & & & \\ & & & & 1 & & \\ & 0 & & & & \ddots & \\ & & & & & & 1 \end{array}\right). \tag{A.7}$$

In the standard basis, these transformations are

$$\mathtt{h}_{\pm} = \left(\begin{array}{cccc|cccc} 0 & & & & \pm 1 & & & \\ & 1 & & & & 0 & & \\ & & \ddots & & & & \ddots & \\ & & & 1 & & & & 0 \\ \hline \pm 1 & & & & 0 & & & \\ & 0 & & & & 1 & & \\ & & \ddots & & & & \ddots & \\ & & & 0 & & & & 1 \end{array}\right). \tag{A.8}$$

So, to summarize, a general $O(d,d)$ transformation has the following form:

$$\mathtt{h} = (\mathtt{h}_{+})^{\eta_{+}}(\mathtt{h}_{-})^{\eta_{-}} \prod_{i=1}^{n} \mathtt{h}_{B_i}\, \mathtt{h}_{A_i}\, \mathtt{h}_{\beta_i}, \tag{A.9}$$

with $\eta_{\pm} \in \{0,1\}$.
To show that γ for an arbitrary $O(d,d)$ transform is invertible, it thus suffices to show that it's invertible for each of these generators – we can just apply the field redefinitions successively.
The map γ corresponding to each of these generators can be read off from the following table:

Transformation	γ
$\mathtt{h}_{\pm}$	$\mathbb{I} - E_1 \pm (G-B)E_1$
$\mathtt{h}_{B}$	$\mathbb{I}$
$\mathtt{h}_{\beta}$	$\mathbb{I} - (G-B)\beta$
$\mathtt{h}_{A}$	$(A^t)^{-1}$

where $E_1 = \mathrm{diag}(1,0,\dots,0)$.
For γ_B and γ_A, it's clear that they are invertible.
For $\mathtt{h}_\beta$, we first note that $(G-B)$ is invertible because its kernel is trivial: If there were a nonzero vector v such that $(G-B)v = 0$, then

$$0 = \langle v, (G-B)v\rangle = \langle v, Gv\rangle - \underbrace{\langle v, Bv\rangle}_{=0} > 0,$$

which is a contradiction.
Then we can write γ_β as

$$\gamma_\beta = (G-B)[(G-B)^{-1} - \beta].$$

For the second factor, we can do the analogous analysis as before: Assume $[(G-B)^{-1}-\beta]v=0$; because $(G-B)$ is invertible, we can find a vector w such that $v=(G-B)w$. Then

$$0=\langle v,[(G-B)^{-1}-\beta]v\rangle=\langle v,(G-B)^{-1}v\rangle-\underbrace{\langle v,\beta v\rangle}_{=0}=\langle (G-B)w,w\rangle>0.$$

Finally, $\gamma_\pm$ is invertible as well: Again, assume that $v\in\ker(\gamma_\pm)$. Then

$$0=(\mathbb{I}-E_1)v\pm(G-B)E_1v.$$

Thus, if $E_1v=0$, then $v=0$. So if $E_1v\neq 0$, then

$$0=\langle E_1v,(\mathbb{I}-E_1)v\pm(G-B)E_1v\rangle=\underbrace{\langle E_1v,(\mathbb{I}-E_1)v\rangle}_{=0}\pm\langle E_1v,(G-B)E_1v\rangle\neq 0,$$

which is a contradiction.

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Hiermit erkläre ich, Christian Schmid, die vorliegende Masterarbeit selbstständig angefertigt und alle verwendeten Quellen nach bestem Wissen und Gewissen deutlich gekennzeichnet zu haben.

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