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Equivalence of the (Generalised) Hadamard and Microlocal Spectrum Condition for (Generalised) Free Fields in Curved Spacetime

Abstract We prove that the singularity structure of all n-point distributions of a state of a generalised real free scalar field in curved spacetime can be estimated if the two-point distribution is of Hadamard form. In particular this applies to the free field and the result has applications in perturbative quantum field theory, showing that the class of all Hadamard states is the state space of interest. In our proof we assume that the field is a generalised free field, i.e. that it satisfies scalar (c-number) commutation relations, but it need not satisfy an equation of motion. The same arguments also work for anti-commutation relations and for vector-valued fields. To indicate the strengths and limitations of our assumption we also prove the analogues of a theorem by Borchers and Zimmermann on the self-adjointness of field operators and of a weak form of the Jost-Schroer theorem. The original proofs of these results make use of analytic continuation arguments. In our case no analyticity is assumed, but to some extent the scalar commutation relations can take its place.

1 Introduction

The study of quantum field theories in curved spacetime is simplified considerably by the use of techniques from microlocal analysis to study the singularities of n-point distributions. Ever since Radzikowski (16) has shown that Hadamard states of the real free scalar field can be characterised by the wave front set of their two-point distributions, these techniques have been on the increase as a suitable replacement of the Fourier transform in Minkowski spacetime. This enabled Brunetti, Fredenhagen and Köhler (2) to introduce a microlocal spectrum condi-

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tion (μSC) for general real scalar fields, which is a (smoothly) generally covariant condition that generalises Wightman's spectrum condition.

The generalisation is only possible at a price: whereas the n-point distributions of a Wightman field are the boundary values of analytic functions, this is no longer so in curved spacetimes. In (21) an analytic microlocal spectrum condition (AμSC) was introduced on analytic spacetimes in order to provide an amount of analyticity analogous to the Wightman case, but the requirement that the metric be analytic in some analytic structure on the manifold, although technically advantageous, seems to be unphysically restrictive. In fact, a generic curved spacetime cannot be expected to be analytic at all, so all arguments involving analytic continuation have to be reexamined in the context of quantum field theory in curved spacetime.

In this work we will not require any analyticity, but instead we consider a real scalar field which satisfies scalar (i.e. c-number) commutation relations[1]. These fields, which include the real free scalar field, will be called generalised free fields, following the terminology for the Wightman framework in Minkowski spacetime (see e.g. (10)), although in curved spacetime not much seems to be known about them. As our main result we will prove that an estimate on the singularities of the two-point distribution ("generalised Hadamard condition") implies estimates on the singularities of all n-point distributions. In particular, all truncated n-point distributions with $n \neq 2$ will be shown to be smooth and consequently the state will satisfy the μSC. An easy application is that the class of generalised Hadamard states is closed under operations from the algebra of observables. Moreover, all Hadamard states of a free field can be extended to the extended algebra of Wick polynomials and time-ordered products as constructed by Hollands and Wald (7; 8).

After that we will investigate the strength of our assumption by proving the analogues of a result by Borchers and Zimmermann (1) on the self-adjointness of field operators and a very weak version of the Jost-Schroer theorem (4; 11; 15). In both cases the original proofs rely on analytic continuation arguments, but in our case no analyticity is assumed. The commutation relations take the place of analyticity to a certain extent, but not fully, and we have had to weaken the conclusion of the Jost-Schroer theorem accordingly.

The organisation of our paper is as follows: we first establish our notation for quantum field theory in curved spacetime in Sect. 2. There we also present the μSC, the (generalised) Hadamard condition and the truncated n-point distributions and we collect some results concerning the singularities of the two-point distribution. In Sect. 3 we introduce the commutation relations and give two equivalent characterisations of generalised free fields. Section 4 contains our main results concerning the singularity structure of higher n-point distributions and truncated n-point distributions, as well as a result on the comparison of n-point distributions of different states. In Sect. 5 we discuss the generalisations of the result by Borchers and Zimmermann and the Jost-Schroer theorem. We conclude with some easy applications and an outlook in Sect. 6. For an introduction to microlocal analysis we refer to Chap. 8 of (9).

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[1] Our results also work for anti-commutation relations and for vector-valued fields.

2 Real Scalar Quantum Fields and the Microlocal Spectrum Condition

Let $M = (\mathcal{M}, g)$ be a spacetime, i.e. $\mathcal{M}$ is a smooth, connected manifold of dimension $D \geq 2$ with the smooth Lorentzian metric g, where we use the signature convention $+ - \cdots -$. We let $V \subset TM$ denote the set of all causal tangent vectors (including
0-vectors) and we let $V^* \subset T^*M$ be its dual, i.e. the image of V under the identification of TM with T^*M via the metric. We assume that M is time-oriented, so we can define the future and past causal cones $V^{\pm} \subset TM$ and their duals, $V^{*\pm} \subset T^*M$. We use $\mathscr{Z}$ to denote the zero section of a vector bundle (it will always be clear from the context which vector bundle is meant).

A real scalar quantum field on the spacetime M can be described using the Borchers-Uhlmann algebra. Here we adopt the convention that the space $M^{\times 0}$ consists of a single point, so that $C_0^\infty(M^{\times 0}) = \mathbb{C}$.

Definition 21 *The (scalar)* **Borchers-Uhlmann** *algebra on the spacetime M is defined to be the topological *-algebra $\mathscr{U}_M := \oplus_{n=0}^\infty C_0^\infty(M^{\times n})$, where we allow only finite direct sums and where*

1. *the product is determined by the linear extension of*
   $f(x_{n+m}, \ldots, x_{n+1}) g(x_n, \ldots, x_1) := (f \otimes g)(x_{n+m}, \ldots, x_1),$
2. *the *-operation is determined by anti-linear extension of*
   $f^*(x_n, \ldots, x_1) := \overline{f(x_1, \ldots, x_n)},$
3. *as a topological space $\mathscr{U}_M$ is the strict inductive limit*

$$\mathscr{U}_M = \cup_{N=0}^\infty \oplus_{n=0}^N C_0^\infty(K_N^{\times n}),$$

   where K_N is an exhausting (and increasing) sequence of compact subsets of $\mathcal{M}$ and each $C_0^\infty(K_N^{\times n})$ is given the test-function topology (cf. (19) Theorem 2.6.4).

A **state** *on the Borchers-Uhlmann algebra is a normalised continuous positive linear map $\omega \colon \mathscr{U}_M \to \mathbb{C}$.*

The topology of $\mathscr{U}_M$ is such that $f_j = \oplus_n f_j^{(n)}$ converges to $f = \oplus_n f^{(n)}$ if and only if for all n we have $f_j^{(n)} \to f^{(n)}$ in $C_0^\infty(M^{\times n})$ and all $f_j^{(n)}$ vanish if $n \geq N$ for some $N > 0$. A state therefore consists of a sequence of n-point distributions, $\omega = \{\omega_n\}_{n=0}^\infty$, where ω_n is a distribution on $M^{\times n}$. The algebra $\mathscr{U}_M$ has the unit $I = 1 \oplus 0 \oplus 0 \ldots$ and the normalisation of the state ω means that $\omega(I) = \omega_0 = 1$. Given a state one can construct the GNS-representation π_ω on a Hilbert space $\mathscr{H}_\omega$ with a dense domain $\mathscr{D}_\omega$ that contains a vector Ω_ω such that: $\mathscr{D}_\omega = \pi_\omega(\mathscr{U}_M)\Omega_\omega$ and $\omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle$ for each $A \in \mathscr{U}_M$. The GNS-quadruple $(\pi_\omega, \mathscr{H}_\omega, \mathscr{D}_\omega, \Omega_\omega)$ is the unique quadruple with these properties, up to unitary equivalence (see (20) Theorem 8.6.2).

Instead of the n-point distributions one often considers the truncated n-point distributions of a state ω, which we will now define. For $n \geq 1$ we let $\mathscr{P}_n$ denote the set of all partitions of the set $\{1, \ldots, n\}$ into pairwise disjoint subsets, which are ordered from low to high. If r is an ordered set in the partition $P \in \mathscr{P}_n$ we write $r \in P$ and we denote the elements of r by $r(1) < \cdots < r(|r|)$, where $|r|$ is the

number of elements in r. The truncated n-point distributions ω_n^T, $n \geq 1$, of a state ω are defined implicitly in terms of the n-point distributions ω_n by:

$$\omega_n(x_n, \ldots, x_1) = \sum_{P \in \mathscr{P}_n} \prod_{r \in P} \omega_{|r|}^T(x_{r(|r|)}, \ldots, x_{r(1)}). \tag{1}$$

Note that this equation can be solved iteratively for ω_n^T order by order.

Definition 22 *A state ω is called* **quasi-free** *if and only if $\omega_n^T \equiv 0$ for all $n \neq 2$.*

We will denote by Φ the canonical injection $C_0^\infty(M) \subset \mathscr{U}_M$, which sends f to

$$\Phi(f) := 0 \oplus f \oplus 0 \oplus \ldots.$$

The map Φ is a distribution with values in $\mathscr{U}_M$ and it represents the real scalar quantum field. In the GNS-representation of a state ω the field is represented by $\Phi^\omega(f) := \pi_\omega(\Phi(f))$. For our current purposes it is convenient not to impose commutation
relations, causality or an equation of motion on the field Φ, but to let them be dictated for Φ^ω by the state. This will be done in Sect. 3.

We now give an equivalent reformulation of the μSC of Brunetti, Fredenhagen and Köhler (2), starting with the introduction of some terminology.

Definition 23 *We let $\mathscr{G}_n$ denote the set of all graphs with n vertices and finitely many edges. An* **immersion** *of a graph $G \in \mathscr{G}_n$ into the spacetime M consists of an assignment of*

1. *a point $x(i) \in M$ to each vertex v_i of G,*
2. *a piecewise smooth curve γ_r between $x(i)$ and $x(j)$ to every edge e_r of G that connects v_i and v_j,*
3. *a causal, future pointing covector field ξ_r on γ_r to each e_r, so that ξ_r is covariantly constant, $\nabla \xi_r = 0$, along γ_r.*

An immersion of a graph $G \in \mathscr{G}_n$ into the spacetime M is called **causal**, *resp.* **light-like**, *iff the curves γ_r are causal, resp. light-like.*

We say that a point $(x_n, k_n; \ldots; x_1, k_1) \in T^*M^{\times n} \setminus \mathscr{Z}$ is **instantiated** *by an immersion of a graph $G \in \mathscr{G}_n$ if and only if for each $i = 1, \ldots, n$ the immersion sends the vertex v_i to x_i and*

$$k_i = \sum_{e_r \text{ between } i \text{ and } j>i} \xi_r(x_i) - \sum_{e_r \text{ between } j<i \text{ and } i} \xi_r(x_i).$$

Recall that $\mathscr{Z}$ denotes the zero section of a vector bundle. The covector field ξ_r is to be thought of as a singularity, propagating along the curve γ_r from $x(i)$ to $x(j)$. The following sets describe the singularities that we allow the n-point distributions to have:

$$\Gamma_n := \big\{ (x_n, k_n; \ldots; x_1, k_1) \in T^*M^{\times n} \setminus \mathscr{Z} \mid \exists G \in \mathscr{G}_n \text{ and an immersion of } G$$
$$\text{into } M \text{ which instantiates the point } (x_n, k_n; \ldots; x_1, k_1) \big\}. \tag{2}$$

The sets Γ_n^c, resp. Γ_n^{ll}, are defined similarly, but using only causal, resp. light-like, immersions of graphs. In general we will write $\Gamma_n^\bullet$, where $\bullet$ denotes either no superscript, or c or ll.

Definition 24 *A state ω satisfies the* **microlocal spectrum condition** *(μSC) with smooth, resp. causal, resp. light-like immersions, iff for all $n \in \mathbb{N}$ we have $WF(\omega_n) \subset \Gamma_n^\bullet$, where $^\bullet$ denotes no superscript, resp. c, resp. ll.*

If M is an analytic spacetime then ω satisfies the **analytic microlocal spectrum condition** *($A\mu$SC) with smooth, resp. causal, resp. light-like immersions, iff for all $n \in \mathbb{N}$ we have $WF_A(\omega_n) \subset \Gamma_n^\bullet$, where $^\bullet$ denotes no superscript, resp. c, resp. ll.*

The usefulness of these restrictions on the singularities of the *n*-point distributions derives largely from the properties of the sets $\Gamma_n^\bullet$:

Proposition 25 *The sets $\Gamma_n^\bullet$, with a fixed choice for the superscript $^\bullet$, have the following properties:*

1. *each $\Gamma_n^\bullet \subset T^*M^{\times n} \setminus \mathscr{Z}$ is a convex cone,*
2. *$\Gamma_n^\bullet \cap -\Gamma_n^\bullet = \emptyset$,*
3. *for every partition $P \in \mathscr{P}_n$ with ordered subsets $r_1, \ldots, r_m$ we have $\pi((\Gamma_{|r_1|}^\bullet \cup \mathscr{Z}) \times \ldots \times (\Gamma_{|r_m|}^\bullet \cup \mathscr{Z})) \subset \Gamma_n^\bullet \cup \mathscr{Z}$, where the permutation π acts on the indices and sends $(r_1, \ldots, r_m)$ to $\{1, \ldots, n\}$.*
4. *$(x_1, k_1; \ldots; x_n, k_n) \in -\Gamma_n^\bullet$ iff $(x_n, k_n; \ldots; x_1, k_1) \in \Gamma_n^\bullet$.*

Proof We refer to (2) Lemma 4.2 for a proof of the first property. The second property follows from the first and the third property follows immediately from the definitions, using the unions of disconnected instantiating graphs (cf. (2) Prop. 4.3). The fourth property follows directly from the definitions. $\quad\square$

Lemma 26 *A state ω satisfies the μSC with smooth, resp. causal, resp. light-like immersions iff $WF(\omega_n^T) \subset \Gamma_n^\bullet$ for all $n \in \mathbb{N}$, where $^\bullet$ denotes no superscript, resp. c, resp. ll. The same result holds in the analytic case.*

Proof We prove by induction on $n \in \mathbb{N}$ that $WF(\omega_n^T) \subset \Gamma_n^\bullet$ if and only if $WF(\omega_n) \subset \Gamma_n^\bullet$. For $n = 1$ this holds because $\omega_1^T = \omega_1$. Now assume that the claim holds for $i = 1, \ldots, n-1$ for some $n \geq 2$. From Eq. (1) we see that $\omega_n - \omega_n^T$ can be expressed as a sum, whose wave front set is contained in $\Gamma_n^\bullet$ by Items 1 and 3 in Proposition 25. Using Item 1 of this proposition once more we see that $WF(\omega_n) \subset \Gamma_n^\bullet$ if and only if $WF(\omega_n^T) \subset \Gamma_n^\bullet$. The argument is purely combinatorical, so it remains true in the analytic case. $\quad\square$

A much weaker condition than the μSC is the Hadamard condition. This condition only places a restriction on the singularities of the two-point distribution so as to enable the renormalisation of the stress-energy-momentum tensor of a free scalar field (see (23)). Because our field need not be free we will consider the following immediate generalisation of the Hadamard condition:

Definition 27 *A state ω is called a* **generalised Hadamard state** *iff $WF(\omega_2) \subset \Gamma_2$.*

Note that a (generalised) Hadamard state need not be quasi-free. We will show in Sect. 3 that the generalised Hadamard condition reduces to the Hadamard condition in the case of free fields.

To complete this section we will now collect some small but useful results on the relation between the generalised Hadamard condition and the two-point

distribution. For this purpose we define the symmetric and anti-symmetric part of the two-point distribution by

$$\omega_{2\pm}(x_2,x_1) := \frac{1}{2}\left(\omega_2(x_2,x_1) \pm \omega_2(x_1,x_2)\right). \tag{3}$$

Proposition 28 *If ω is a generalised Hadamard state, then we have:*

1. $(x_2,k_2;x_1,k_1) \in WF(\omega_{2\pm})$ *iff* $(x_1,k_1;x_2,k_2) \in WF(\omega_{2\pm})$ *iff* $(x_2,-k_2;x_1,-k_1) \in WF(\omega_{2\pm})$,
2. $WF(\omega_{2+}) = WF(\omega_{2-}) \subset \Gamma_2 \cup -\Gamma_2$,
3. $WF(\omega_2) = WF(\omega_{2-}) \cap \Gamma_2$.

Proof The idea of this proof is borrowed from Proposition 6.1 in (21). The positivity of ω implies that $\overline{\omega_2}(x_2,x_1) = \omega_2(x_1,x_2)$, so from Eq. (3) we find that $\overline{\omega_{2\pm}}(x_2,x_1) = \omega_{2\pm}(x_1,x_2) = \pm\omega_{2\pm}(x_2,x_1)$, from which the first item follows. That $WF(\omega_{2\pm}) \subset \Gamma_2 \cup -\Gamma_2$ for a generalised Hadamard state is clear from the definition. Now suppose that $(x_2,k_2;x_1,k_1) \in WF(\omega_{2\pm})$. Then we can distinguish two cases, namely either $(x_2,k_2;x_1,k_1) \notin WF(\omega_2)$ or $(x_2,-k_2;x_1,-k_1) \notin WF(\omega_2)$ by Eq. (3) and Statement 2 of Proposition 25. Using $\omega_{2\pm} = \omega_2 - \omega_{2\mp}$ and the properties under the first item we find that either $(x_2,k_2;x_1,k_1) \in WF(\omega_{2\mp})$ or $(x_2,-k_2;x_1,-k_1) \in WF(\omega_{2\mp})$, as the case may be, and hence that $(x_2,k_2;x_1,k_1) \in WF(\omega_{2\mp})$. Thus $WF(\omega_{2+}) \subset WF(\omega_{2-})$ and the opposite inclusion can be proved in the same way, which proves the second item. For the last item we use again the definition $2\omega_{2-} = \omega_2 - \tilde{\omega}_2$, where $\tilde{\omega}_2(x_2,x_1) := \omega_2(x_1,x_2)$. By the assumption on ω_2 we have $WF(\omega_2) \cap WF(\tilde{\omega}_2) = \emptyset$. Hence we deduce: $WF(\omega_2) \subset WF(\omega_{2-})$, $WF(\tilde{\omega}_2) \subset WF(\omega_{2-})$ and $WF(\omega_{2-}) \subset WF(\omega_2) \cup WF(\tilde{\omega}_2)$, from which it follows that $WF(\omega_{2-}) = WF(\omega_2) \cup WF(\tilde{\omega}_2)$. Intersecting with Γ_2 then gives the result. □

The following result on the comparison of two generalised Hadamard states is well known and lies at the basis of the renormalisation of the stress-energy-momentum tensor in the free field case:

Lemma 29 *For two generalised Hadamard states ω, ω' we have that $\omega_2 - \omega'_2$ is smooth iff $\omega_{2-} - \omega'_{2-}$ is smooth.*

Proof We define $w(x_2,x_1) := (\omega_2 - \omega'_2)(x_2,x_1)$, $\tilde{w}(x_2,x_1) := w(x_1,x_2)$ and $w_{2-} := \frac{1}{2}(w - \tilde{w})$ and argue as in the proof of Proposition 28: $WF(w) \subset \Gamma_2$, $WF(\tilde{w}) \subset -\Gamma_2$, and hence $WF(w) \cap WF(\tilde{w}) = \emptyset$. It then follows from $w - \tilde{w} = 2w_{2-}$ that $WF(w_{2-}) = WF(w) \cup WF(\tilde{w})$. Now $WF(w_{2-}) = \emptyset$ if and only if $WF(w) = \emptyset$, which proves the statement. □

3 Generalised Free Fields in Curved Spacetime

In this section we define a number of physical properties that the state ω may satisfy and derive some easy results concerning them:

Definition 31 *The state ω is called **causal** iff ω descends to a state on $\mathscr{U}_M/J$, where $J \subset \mathscr{U}_M$ is the *-ideal generated by all elements of the form $f \otimes h - h \otimes f$ where the supports of $f, h \in C_0^\infty(M)$ are causally disjoint.*

*A state ω satisfies the **Klein-Gordon equation** with mass m and scalar curvature coupling ξ iff ω descends to a state on $\mathscr{U}_M/J$, where $J \subset \mathscr{U}_M$ is the *-ideal generated by all elements of the form $(\Box + m^2 + \xi R)f$, where $\Box$ is the d'Alembertian and R the scalar curvature.*

*Given a bi-distribution E on $M^{\times 2}$ we say that the state ω is a **generalised free field state with commutator** E iff ω **satisfies the commutation relations with commutator** E, i.e. iff ω descends to a state on $\mathscr{U}_M/J$, where $J \subset \mathscr{U}_M$ is the *-ideal generated by all elements of the form $f \otimes h - h \otimes f - iE(f,h)I$.*

*A generalised free field state ω on a globally hyperbolic spacetime is called a **free field state** iff it satisfies the Klein-Gordon equation with mass m and scalar curvature coupling ξ and $E = E_{m,\xi}$, the difference of the advanced and retarded fundamental solutions of the Klein-Gordon equation $(\Box + m^2 + \xi R)\phi = 0$.*

The first three properties above can be written equivalently in terms of the represented field as:

$$[\Phi^\omega(f), \Phi^\omega(h)] = 0, \quad \operatorname{supp} f \cap J(\operatorname{supp} h) = \emptyset, \tag{4}$$

$$(\Box + m^2 + \xi R)\Phi^\omega = 0, \tag{5}$$

$$[\Phi^\omega(f), \Phi^\omega(h)] = iE(f,h)I. \tag{6}$$

We have chosen to allow very general distributions E to appear in the commutation relations in order to emphasise that their precise form does not matter for our arguments. In particular, our commutation relations need not imply causality and our arguments also hold for anti-commutation relations. However, it is important that the commutator of two smeared field operators is a scalar. Note that E must be anti-symmetric, $E(f,h) = -E(h,f)$ for all $f, h \in C_0^\infty(M)$, for the commutation relations to make sense.

Instead of the free field commutator $E = E_{m,\xi}$ one can take for example $E(x_2, x_1) := \int_0^\infty \int E_{m,\xi}(x_2, x_1) f(m, \xi) d\xi dm$, where f is a compactly supported smooth function. In fact, in the Wightman framework in Minkowski spacetime one can use the Källen-Lehmann representation of the two-point distribution to prove that E must be of this form for a suitable distribution $f(m)\delta(\xi)$. (We can take $\xi = 0$ because $R \equiv 0$ in Minkowski spacetime.) Whether such a result still holds in curved spacetime is not clear, because no suitable replacement of the Källen-Lehmann is currently available. We leave these issues for a future investigation.

It is worthwhile to note the following:

Proposition 32 *If ω is a generalised Hadamard state and a generalised free field state with commutator E, then $E = -2i\omega_{2-}$, $WF(E) \subset \Gamma_2 \cup -\Gamma_2$ and $WF(\omega_2) = WF(E) \cap \Gamma_2$.*

Proof The first equality follows by applying ω to the commutation relations (6). The others follow from the last two items of Proposition 28. $\square$

Note that $WF(E_{m,\xi}) \subset \Gamma_2^{ll} \cup -\Gamma_2^{ll}$, so in this case we must have $WF(\omega_2) \subset \Gamma_2^{ll}$.

Corollary 33 *If a state ω satisfies the Klein-Gordon equation with parameters m, ξ on a globally hyperbolic spacetime M and the commutation relations with commutator $E_{m,\xi}$, then it is a generalised Hadamard state if and only if it is a Hadamard state.*

Proof The result of Radzikowski implies that a free field state on a globally hyperbolic spacetime which satisfies the commutation relations with commutator $E_{m,\xi}$ is a Hadamard state if and only if $WF(\omega_2) = WF(E_{m,\xi}) \cap \Gamma_2$. The result therefore follows from Proposition 32.   □

We see from Proposition 32 and Lemma 29 that for two generalised free field states ω and ω' with commutator functions E and E' respectively, $\omega_2 - \omega'_2$ is smooth iff $E - E'$ is smooth. In general, however, even $E_{m,\xi} - E_{m',\xi'}$ will not be smooth, even though both have the same wave front sets. Indeed, if $E_{m,\xi} - E_{m',\xi'}$ is smooth and if we define $K_{x_2} := \Box + m^2 + \xi R$ acting on the variable x_2 and similarly for K', then the following is also smooth:

$$
\begin{aligned}
K'_{x_2}(E_{m,\xi} - E_{m',\xi'})(x_2, x_1) &= K'_{x_2} E_{m,\xi}(x_2, x_1) \\
&= (K'_{x_2} - K_{x_2})E_{m,\xi}(x_2, x_1) \\
&= ((m')^2 - m^2 + \xi' R(x_2) - \xi R(x_2))E_{m,\xi}(x_2, x_1).
\end{aligned}
$$

Because $E_{m,\xi}(x_2, x_1)$ is singular whenever x_1 and x_2 can be connected by a light-like geodesic, we would then have to have $(m')^2 - m^2 + \xi' R(x_2) - \xi R(x_2) \equiv 0$. In general, however, this is not the case.

We conclude this section by proving a useful equivalent characterisation of generalised free fields in terms of the truncated n-point distributions (see note added in proof):

Proposition 34 *A state ω is a generalised free field state iff all the truncated n-point distributions ω_n^T with $n \neq 2$ are symmetric in their arguments.*

In the case where we have anti-commutation relations instead of commutation relations a similar proof shows that the truncated n-point distributions are anti-symmetric for $n \neq 2$.

Proof First assume that ω_n^T is symmetric for all $n \neq 2$. For $n \geq 2$ we then use Eq. (1) to see that for any $1 \leq i < n$,

$$
\begin{aligned}
&\omega_n(x_n, \ldots, x_1) - \omega_n(x_n, \ldots, x_i, x_{i+1}, \ldots, x_1) \\
&\quad = 2\omega_{2-}(x_{i+1}, x_i)\omega_{n-2}(x_n, \ldots, \hat{x}_{i+1}, \hat{x}_i, \ldots, x_1).
\end{aligned}
$$

Here we noted that most terms cancel out, either by the hypothesis or by the fact that i and $i + 1$ are subsequent indices. The remaining terms have been collected together using once again Eq. (1). By definition this equation means that ω satisfies the commutation relations with commutator $E = -2i\omega_{2-}$.

For the opposite direction we assume that ω satisfies the commutation relations (necessarily with $E = -2i\omega_{2-}$). We use similar arguments as above to prove by induction that ω_n^T is symmetric for $n \geq 3$ (for $n = 1, 2$ there is nothing to prove):

$$
\begin{aligned}
&\omega_3^T(x_3, x_2, x_1) - \omega_3^T(x_2, x_3, x_1) \\
&= \omega_3(x_3, x_2, x_1) - \omega_3(x_2, x_3, x_1) - (\omega_2^T(x_3, x_2) - \omega_2^T(x_2, x_3))\omega_1^T(x_1) \\
&= 2\omega_{2-}(x_3, x_2)\omega_1(x_1) - 2\omega_{2-}(x_3, x_2)\omega_1(x_1) = 0.
\end{aligned}
$$

A similar result holds for the transposition in the indices 1 and 2, from which the invariance under all permutations follows for $n = 3$. Next we consider $n > 3$ and assume that the claim holds for all $\omega_{n'}^T$ with $0 \leq n' \leq n-1$. Again it suffices to prove that $\omega_n^T(x_n, \ldots, x_1)$ is invariant under a transposition of the indices i and $i+1$ for some $1 \leq i \leq n-1$, because such transpositions generate the group of all permutations. Using the induction hypothesis we find similarly:

$$\omega_n^T(x_n, \ldots, x_1) - \omega_n^T(x_n, \ldots, x_i, x_{i+1}, \ldots, x_1)$$
$$= \omega_n(x_n, \ldots, x_1) - \omega_n(x_n, \ldots, x_i, x_{i+1}, \ldots, x_1)$$
$$- (\omega_2^T(x_{i+1}, x_i) - \omega_2^T(x_i, x_{i+1}))\omega_{n-2}(x_n, \ldots, \hat{x}_{i+1}, \hat{x}_i, \ldots, x_1) = 0.$$

This completes the proof. $\square$

The previous proposition is reminiscent of, but certainly not equivalent to, the result in (10) that a vacuum state ω of a Wightman field theory is causal if and only if the n-point distributions ω_n, extended to suitable complex domains, are symmetric in their arguments in those domains. That result, however, uses the Bargmann-Hall-Wightman theorem, whereas our result relies solely on elementary combinatorics (cf. (10) Sect. 4.4, (6; 11)).

Finally we note the following corollary of Proposition 34[2]:

Corollary 35 *A quasi-free state satisfies the commutation relations with commutator $E = -2i\omega_{2-}$.*

Proof By Definition 22 of a quasi-free state ω_n^T is symmetric for $n \neq 2$. $\square$

4 Equivalence of the Hamadard and Microlocal Spectrum Conditions

We now start our analysis of the singularities of higher n-point distributions of a generalised free field state with a result that exploits the positivity of the state.

Proposition 41 *Let ω be a generalised Hadamard state and assume that for $n \geq 1$ we have $(x_n, k_n; \ldots; x_1, k_1) \in WF(\omega_n)$. Then $(x_1, k_1) \in V^{*+} \cup \mathscr{Z}$ and $(x_n, k_n) \in V^{*-} \cup \mathscr{Z}$. In particular, $WF(\omega_1) = \emptyset$.*

Proof The positivity of ω implies $\overline{\omega_n(f_n, \ldots, f_1)} = \omega_n(\bar{f}_1, \ldots, \bar{f}_n)$, and hence the second statement follows from the first. In fact, the positivity allows us to perform the GNS-construction, which yields a representation π_ω of $\mathscr{U}_M$ on a Hilbert space $\mathscr{H}_\omega$ by closable operators and a vector $\Omega_\omega \in \mathscr{H}_\omega$ such that $\omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle$ for all $A \in \mathscr{U}_M$. We can then define the $\mathscr{H}_\omega$-valued distributions $\phi_m(f_m, \ldots, f_1) := \pi_\omega(f_m \otimes \ldots \otimes f_1)\Omega_\omega$ for all $m \in \mathbb{N}$. Using the inner product of $\mathscr{H}_\omega$ we can write:

$$\omega_n(f_n, \ldots, f_1) = \langle \phi_{n-1}(\bar{f}_2, \ldots, \bar{f}_n), \phi_1(f_1) \rangle,$$
$$\omega_2(f_2, f_1) = \langle \phi_1(\bar{f}_2), \phi_1(f_1) \rangle.$$

The calculus of Hilbert space-valued distributions (see e.g. (21) Proposition 2.2 or (18) Theorem A.1.6) now means that $(x_n, k_n; \ldots; x_1, k_1) \in WF(\omega_n)$ implies $(x_1, k_1) \in WF(\phi_1) \cup \mathscr{Z}$ and if $k_1 \neq 0$ then $(x_1, -k_1; x_1, k_1) \in WF(\omega_2)$. The conclusion follows from the assumption that $WF(\omega_2) \subset \Gamma_2$. $\square$

---

[2] We thank Prof. Rehren for pointing this out to us at an early stage.

Proposition 41 has some nice consequences in the case of generalised free fields:

Theorem 42 *Let ω be a generalised Hadamard state which is also a generalised free field state. Then $\omega_2^T - \omega_2$ and ω_n^T for all $n \neq 2$ are smooth functions.*

Proof From Proposition 41 and Eq. (1) we see that $(x_n, k_n; \ldots; x_1, k_1) \in WF(\omega_n^T)$ implies $(x_1, k_1) \in V^{*+} \cup \mathscr{Z}$ and $(x_n, k_n) \in V^{*-} \cup \mathscr{Z}$. However, because ω is a generalised free field state all truncated n-point distributions with $n \neq 2$ are symmetric by Proposition 34. This means that each (x_i, k_i) must be in $(V^{*+} \cup \mathscr{Z}) \cap (V^{*-} \cup \mathscr{Z}) = \mathscr{Z}$, i.e. $k_i = 0$. It follows that $WF(\omega_n^T) = \emptyset$ and hence ω_n^T is smooth for $n \neq 2$. The result for $n = 2$ follows from $\omega_2 - \omega_2^T = \omega_1 \otimes \omega_1$. $\square$

Corollary 43 *Let ω be a generalised Hadamard state which is also a generalised free field state. Then ω satisfies the μSC with smooth, resp. causal, resp. light-like immersions if $WF(\omega_{2-}) \subset \Gamma_2^{\bullet}$, where $^{\bullet}$ denotes no superscript, resp. c, resp. ll. More precisely, for each point in $WF(\omega_n)$ we can find an instantiating graph $G \in \mathscr{G}_n$ which is a disconnected union of graphs in $\mathscr{G}_2$ that instantiate points in $WF(\omega_2) = WF(E) \cap \Gamma_2$.*

Proof This follows immediately from Theorem 42, Eq. (1) and the properties of the cones $\Gamma_n^{\bullet}$ in Proposition 25. $\square$

The singularity structure that we derived in Theorem 42 and Corollary 43 is what one would expect of quasi-free states, because of Eq. (1) (see (2)). It is nice to see that this form persists when the state is only required to satisfy scalar commutation relations. Analogous results also hold in the analytic case, for vector-valued fields and in the case of anti-commutation relations.

(2) describes a point in $T^* M^5 \setminus \mathscr{Z}$ that is not in Γ_5^c and one wonders whether such a point can be in the wave front set of the 5-point distribution of a state. We have just proved that for generalised free fields this possibility is excluded. Moreover, our result also implies that the μSC with light-like curves includes more than just free fields and their Wick powers (2), namely generalised free fields with any suitable commutator function. (We will leave the existence of a sufficiently large class of such fields in curved spacetime for a future publication.)

An easy consequence of the analytic case of Theorem 42 is the following characterisation of generalised free field states:

Proposition 44 *Let ω be a causal state satisfying the $A\mu SC$. Then ω is a generalised free field state if and only if ω_n^T is analytic for all $n \neq 2$.*

Proof If ω is a generalised free field state the conclusion follows from the analytic version of Theorem 42. For the converse we use causality to prove by induction on n that every ω_n^T is symmetric when all arguments are space-like separated. Analytic continuation for $n \neq 2$ then proves their symmetry everywhere and we may then apply Proposition 34. $\square$

As another easy result we show that the class of generalised Hadamard states of a generalised free field is closed under operations:

Proposition 45 *Let ω be a generalised Hadamard and generalised free field state on $\mathscr{U}_M$ and let $A \in \mathscr{U}_M$ be any operator such that $\omega(A^*A) = 1$. Then the state ω^A,*

defined by $\omega^A(B) := \omega(A^*BA)$, is a generalised Hadamard and generalised free field state on $\mathscr{U}_M$.

Notice that for given A the expression $\omega(A^*BA)$ may involve arbitrary high n-point distributions, depending on the choice of B, so without an estimate on the wave front sets of higher n-point distributions this result sounds rather surprising.

Proof We may write $A = \sum_{i=1}^n f_i^{(i)} \otimes \cdots \otimes f_1^{(i)}$ for some n and $f_j^{(i)} \in C_0^\infty(M)$. The two-point distribution of ω^A is then a sum of terms of the form

$$\omega_{i+k+2}\left(\overline{f_1^{(i)}}, \ldots, \overline{f_i^{(i)}}, x_2, x_1, f_k^{(k)}, \ldots, f_1^{(k)}\right)$$

which are distributions in x_1, x_2. The wave front set of each such term can be estimated using standard arguments (see (9) Theorem 8.2.12) as a subset of

$$\{(x_2, k_2; x_1, k_1)|\ (y_1, 0; \ldots; y_i, 0; x_2, k_2; x_1, k_1; z_k, 0; \ldots; z_1, 0) \in WF(\omega_{i+k+2})\}$$

which is a subset of Γ_2. The wave front set of a sum of such terms is also contained in Γ_2 by Proposition 25 and therefore ω^A is a generalised Hadamard state. That it is a generalised free field state follows from Eq. (6). $\quad\square$

To close this section we prove the following lemma on the comparisons of the n-point distributions of two states, generalising Lemma 29.

Lemma 46 *Consider two generalised Hadamard states ω, ω', which both satisfy commutation relations with the same commutator E such that $WF(E) \neq \emptyset$. For any $n \geq 0$ we have that $\omega_{n+2} - \omega'_{n+2}$ is smooth if and only if $\omega_n \equiv \omega'_n$.*

Proof The case $n = 0$ follows from Lemma 29. For $n \geq 1$ we first suppose that $\omega_n \equiv \omega'_n$. For any index $1 \leq i < n$ we then have $(\omega_{n+2} - \omega'_{n+2})(x_{n+2}, \ldots, x_1) = (\omega_{n+2} - \omega'_{n+2})(x_{n+2}, \ldots, x_i, x_{i+1}, \ldots, x_1)$, where we swapped the indices i and $i + 1$ and the commutator terms vanish by the assumption. We can therefore permute indices ad lib. and in this way we derive $(\omega_{n+2} - \omega'_{n+2})(x_{n+2}, \ldots, x_1) = (\omega_{n+2} - \omega'_{n+2})(x_1, \ldots, x_{n+2})$. Using the assumption that both states are generalised Hadamard states and Items two and four of Proposition 25 we find that $WF(\omega_{n+2} - \omega'_{n+2}) \subset \Gamma_{n+2} \cap -\Gamma_{n+2} = \emptyset$. This proves that $\omega_{n+2} - \omega'_{n+2}$ is smooth.

For the opposite direction we assume that $\omega_{n+2} - \omega'_{n+2}$ is smooth and we let the symbol $\sim$ denote equality modulo terms w such that $WF(w) \cap T^*M \times V^{*+} \times T^*M \times \cdots \times T^*M = \emptyset$, i.e. we are interested in the direction of the covectors in the $n + 1^{\text{st}}$ slot (from the right). Using the expressions for ω_{n+2} and ω'_{n+2} in terms of truncated n-point distributions (1) we compute:

$$0 \sim \omega_{n+2} - \omega'_{n+2} \sim \omega_2 \otimes \omega_n - \omega'_2 \otimes \omega'_n$$
$$\sim \omega_2 \otimes \omega_n - \omega_2 \otimes \omega'_n = \omega_2 \otimes (\omega_n - \omega'_n),$$

where we used the result for $n = 0$ to get to the last line. If $w_n := \omega_n - \omega'_n \neq 0$ is not identically 0 then we can find test-functions $f_1, \ldots, f_n$ such that $c :=$

$w_n(f_n, \ldots, f_1) \neq 0$, which leads to a contradiction as follows. Notice that

$$
\begin{aligned}
WF(\omega_2) = WF(c \cdot \omega_2) &= WF(\omega_2 \cdot w_n(f_n, \ldots, f_1)) \\
&\subset \{(x_{n+2}, k_{n+2}; x_{n+1}, k_{n+1}) | \text{ for some } x_i \in \mathrm{supp}(f_i), \ i = 1, \ldots, n \\
&\quad (x_{n+2}, k_{n+2}; x_{n+1}, k_{n+1}; x_n, 0; \ldots; x_1, 0) \in WF(\omega_2 \otimes w_n)\},
\end{aligned}
$$

by Theorem 8.2.12 of (9). Because $\omega_2 \otimes w_n \sim 0$ and because ω_2 is a generalised Hadamard state we find that $WF(\omega_2) = \emptyset$. However, by Proposition 32 this implies that $WF(E) \cap \Gamma_2 = \emptyset$ and hence $WF(E) \cap -\Gamma_2 = \emptyset$ and $WF(E) = \emptyset$. This contradicts the assumption on E, so we must have $w_n \equiv 0$. □

The same statement still holds when the commutators E and E' of the two states differ by a smooth function.

5 Two Theorems Generalised to Curved Spacetimes

We now discuss the generalisation of two theorems from Wightman field theory to curved spacetimes, illustrating the strength and the limitations of the commutation relations in that setting. First we generalise a result due to Borchers and Zimmermann (1) concerning the self-adjointness of field operators. Then we consider the generalisation of (a weak form of) the Jost-Schroer theorem.

The result of (1) gives a sufficient condition for the symmetric operator $\Phi^\omega(f)$ with a given $f \in C_0^\infty(M, \mathbb{R})$ to be self-adjoint. To discuss its generalisation we recall the following notion:

Definition 51 *A vector ψ in a Hilbert space $\mathcal{H}$ is an* **analytic vector** *for a (possibly unbounded) linear operator T on $\mathcal{H}$ iff the series $\sum_{n=0}^\infty \frac{\|T^n\psi\|}{n!} z^n$ has a non-zero radius of convergence. (In particular we require that ψ is in the domain of each T^n.)*

Notice that for a bounded linear operator T all vectors are analytic. The following elementary lemma is adapted from (1):

Lemma 52 *For a vector ψ in the Hilbert space $\mathcal{H}$ and a symmetric linear operator T on $\mathcal{H}$ the following are equivalent:*

1. *ψ is analytic for T,*
2. *there is a constant $c > 0$ such that $\|T^n\psi\| \leq n! c^n$,*
3. *$\sum_{n=0}^\infty \frac{|\langle \psi, T^n\psi \rangle|}{n!} z^n$ has a non-zero radius of convergence,*
4. *there is a constant $c > 0$ such that $|\langle \psi, T^n\psi \rangle| \leq n! c^n$.*

Proof See loc. cit. Sect. 2. □

For a Wightman field theory in Minkowski spacetime Borchers and Zimmermann (1) used causality and the Reeh-Schlieder theorem to prove that a field operator $\Phi^\omega(f)$ is self-adjoint as soon as the vacuum vector Ω_ω is analytic. An analogous proof can be given in curved spacetime, whenever the state ω is causal and has the Reeh-Schlieder property, i.e. the GNS-vector Ω_ω is cyclic for all local algebras. The latter can be ensured e.g. by imposing the AμSC (see (17; 21)), but unfortunately it is not clear whether all analytic spacetimes admit states satisfying

the AμSC, or whether all (smooth) spacetimes have states with the Reeh-Schlieder property. We now prove that the conclusion of Borchers and Zimmermann can also be obtained without recourse to the Reeh-Schlieder theorem if we assume that the state is a generalised free field state. For the proof we adapt an idea of Nelson (14).

Theorem 53 *If ω is a generalised free field state on $\mathscr{U}_M$ with some commutator E and Ω_ω is an analytic vector for $\Phi^\omega(f)$ for some $f \in C_0^\infty(M, \mathbb{R})$, then all vectors $\pi_\omega(A)\Omega_\omega$ with $A \in \mathscr{U}_M$ are analytic vectors for $\Phi^\omega(f)$ and this operator is essentially self-adjoint.*

Proof First assume that $\psi \in \pi_\omega(\mathscr{U}_M)\Omega_\omega$ is an analytic vector for $\Phi^\omega(f)$ for given $f \in C_0^\infty(M, \mathbb{R})$. For any $h \in C_0^\infty(M)$ we will prove that $\Phi^\omega(h)\psi$ is an analytic vector for $\Phi^\omega(f)$. To see this we note that for $n \geq 1$ we have

$$\Phi(f)^n\Phi(h) = \Phi(h)\Phi(f)^n + niE(f,h)\Phi(f)^{n-1},$$

which may easily be proved by induction. Using this we compute:

$$|\langle\Phi^\omega(h)\psi, \Phi^\omega(f)^n\Phi^\omega(h)\psi\rangle| \leq |\langle\Phi^\omega(\bar{h})\Phi^\omega(h)\psi, \Phi^\omega(f)^n\psi\rangle|$$
$$+ n|E(F,h)| \cdot |\langle\Phi^\omega(h)\psi, \Phi^\omega(f)^{n-1}\psi\rangle|$$
$$\leq c\|\Phi^\omega(f)^n\psi\| + cn\|\Phi^\omega(f)^{n-1}\psi\|,$$

where the constant $c > 0$ may depend on f and h, but not on n. The assumption that ψ is analytic then implies that (see Lemma 52)

$$|\langle\Phi^\omega(h)\psi, \Phi^\omega(f)^n\Phi^\omega(h)\psi\rangle| \leq c(c')^n n! + cn(c')^{n-1}(n-1)! \leq C^n n!$$

for suitable constants $c', C > 0$. By Lemma 52 this implies that $\Phi^\omega(h)\psi$ is an analytic vector for $\Phi^\omega(f)$.

Now assume that Ω_ω is an analytic vector for $\Phi^\omega(f)$. We can then repeatedly apply the result of the previous paragraph to prove that any vector of the form $\Phi^\omega(h_m)\cdots\Phi^\omega(h_1)$
Ω_ω is an analytic vector. Because the set of analytic vectors for a given operator is a linear space, every vector in $\pi_\omega(\mathscr{U}_M)\Omega_\omega$ is analytic. This provides a dense set of analytic vectors, so we can apply Nelson's theorem ((14) Lemma 5.1) to conclude that $\Phi^\omega(f)$ is essentially self-adjoint. $\square$

The analyticity of Ω_ω can be formulated conveniently in terms of the n-point distributions by Lemma 52 and in terms of the truncated n-point distributions too (for a proof we refer to (1)):

Proposition 54 *Ω_ω is an analytic vector for $\Phi^\omega(f)$ if and only if there is a $d > 0$ such that $|\omega_n^T(f^{\otimes n})| < n!d^n$ for all $n \in \mathbb{N}$.*

The condition of the previous theorem may not always be satisfied, as we will now illustrate with the following

Example In Minkowski spacetime we will construct a translation invariant free field state $\tilde{\omega}$ which satisfies the AμSC, but whose *GNS*-vector $\Omega_{\tilde{\omega}}$ is not analytic for any non-zero smeared field operator $\Phi^{\tilde{\omega}}(f)$, $f \in C_0^\infty(M_0, \mathbb{R})$. (We will not discuss the question whether these operators are essentially self-adjoint.)

Let ω denote the Minkowski vacuum state with two-point distribution ω_2. We set $w_2(x_2, x_1) := \int e^{-ik \cdot (x_1 - x_2)} e^{-k_0^2} \delta(k^2 - m^2) dk$, which is an analytic, real-valued, symmetric and translation invariant bi-solution of the Klein-Gordon equation of positive type. Next we define the two-point distributions $\omega_2^j := e^j w_2 + \omega_2$ for each $j \in \mathbb{N}$ and we note that the anti-symmetric part is $\omega_{2-}^j = \omega_{2-}$. Each ω_2^j defines a quasi-free state ω^j on the Weyl-algebra (see (12)) and hence also on the Borchers-Uhlmann algebra, because a quasi-free state is regular (cf. Proposition 54). Each of the states ω^j is a translation invariant, Hadamard, free field state satisfying the AμSC. (Note however that they are not Lorentz-invariant, because w_2 is not Lorentz invariant.)

Now we define the state $\tilde{\omega}$ by $\tilde{\omega} := e^{-1} \sum_{j=0}^{\infty} \frac{1}{j!} \omega^j$. Note that $\tilde{\omega}(A^*A) \geq 0$ and $\tilde{\omega}_0(I) = 1$, so it is indeed a state. It follows from the properties of the ω^j that $\tilde{\omega}$ is translation invariant and that it is a free field state. To see that $\tilde{\omega}$ is continuous we note that $\tilde{\omega}_{2n-1} = 0$ for $n \in \mathbb{N}$ and that for all $n, N \in \mathbb{N}$:

$$e^{-1} \sum_{j=0}^{N} \frac{1}{j!} \omega_{2n}^j = e^{-1} \sum_{P \in \mathscr{P}_n} \sum_{j=0}^{N} \frac{1}{j!} (e^j w_2 + \omega_2)^{\otimes n} \circ \pi_P$$

$$= e^{-1} \sum_{P \in \mathscr{P}_n} \sum_{k=0}^{n} \sum_{j=0}^{N} \frac{e^{kj}}{j!} \left(w_2^{\otimes k} \otimes \omega_2^{\otimes(n-k)} + \cdots + \omega_2^{\otimes(n-k)} \otimes w_2^{\otimes(n-k)} \right) \circ \pi_P,$$

where the operation π_P denotes the permutation that corresponds to the partition P of the set $\{1, \ldots, n\}$ (see Eq. (1) and Definition 22) and the dots in the last line indicate all the different orderings of the factors w_2 and ω_2. Taking the limit we see that the sum over j converges so that

$$\tilde{\omega}_{2n} = \sum_{P \in \mathscr{P}_n} \sum_{k=0}^{n} e^{e^k - 1} \left(w_2^{\otimes k} \otimes \omega_2^{\otimes(n-k)} + \cdots + \omega_2^{\otimes(n-k)} \otimes w_2^{\otimes(n-k)} \right) \circ \pi_P, \quad (7)$$

which exhibits $\tilde{\omega}_{2n}$ as a finite sum of distributions. It also follows from Eq. (7) that $\tilde{\omega}$ satisfies the AμSC.

Finally we prove that $\Omega_{\tilde{\omega}}$ is not an analytic vector for any non-zero $\Phi^{\tilde{\omega}}(f)$ with $f \in C_0^\infty(M_0, \mathbb{R})$. Suppose that $\Omega_{\tilde{\omega}}$ is an analytic vector for a given $\Phi^{\tilde{\omega}}(f)$. By Lemma 52 there is a constant $c > 0$ such that

$$c^{2n}(2n)! \geq \tilde{\omega}_{2n}(f^{\otimes 2n}) \geq \frac{(2n)!}{2^n n!} e^{e^n - 1} w_2(f, f)^n,$$

where we used Eq. (7) and the positive type of ω_2 and w_2 for the last inequality. Using $\ln n \leq n$ we find $n! \leq n^n \leq e^{n^2}$ and hence

$$c^{2n} \geq \left(\frac{w_2(f, f)}{2} \right)^n e^{n^3/6 - n^2}.$$

If $w_2(f, f) \neq 0$ we can take logarithms on both sides and let $n \to \infty$ to find a contradiction. If $w_2(f, f) = 0$, on the other hand, we use the positivity and the support of $\hat{w}_2$ to deduce that $\omega_2(f, f) = 0$ too and hence $\tilde{\omega}_2(f, f) = 0$. This means that $\Phi^{\tilde{\omega}}(f)$ annihilates $\Omega_{\tilde{\omega}}$ and it commutes with all other smeared field operators, so that $\Phi^{\tilde{\omega}}(f) = 0$ (cf. the proof of Proposition 55 below).

Now we turn to an analogue of the Jost-Schroer theorem (see (4; 11; 15)), which provides a way to recognise free field states. In the Wightman framework this theorem says that any state whose two-point distribution is that of a free field must be a free field state[3]. (Recall that this means it satisfies the Klein-Gordon equation and the canonical commutation relations.)

As before we can prove our result by using commutation relations to replace the analyticity that is due to the spectrum condition of the Wightman axioms. Note, however, that this makes part of the result, namely the proof of the commutation relations, trivial. The following is therefore a generalisation of a very weak form of the Jost-Schroer theorem:

Proposition 55 *Let ω be a generalised free field state and assume that ω_2 is the two-point distribution of a free-field state, i.e. it satisfies the Klein-Gordon equation for some mass m and scalar curvature coupling ξ and $\omega_{2-} = \frac{i}{2}E_{m,\xi}$. Then ω is a free field state.*

(The same result also works for other linear partial differential operators.)

Proof Let K denote the Klein-Gordon operator with mass m and coupling ξ. For any $f \in C_0^\infty(M)$ we have $K\Phi(f) = \Phi(Kf)$, because the Klein-Gordon operator is formally self-adjoint. This implies that

$$|\omega_n(f_n,\ldots,f_2,Kf_1)| \leq \|\Phi^\omega(\overline{f}_2)\cdots\Phi^\omega(\overline{f}_n)\Omega_\omega\| \cdot \|\Phi^\omega(Kf_1)\Omega_\omega\| = 0,$$

because $\|\Phi^\omega(Kf_1)\Omega_\omega\|^2 = \omega_2(K\overline{f}_1, Kf_1) = 0$. Therefore every ω_n satisfies the Klein-Gordon equation in the first (rightmost) argument. One proves by induction that the same is then true for ω_n^T, using Eq. (1). For a generalised free field state we can then apply Proposition 34 and find that ω_n^T satisfies the Klein-Gordon equation in all arguments for $n \neq 2$. For $n = 2$ this is true by the assumption on ω_2. Using Eq. (1) once more shows that the ω_n satisfy the Klein-Gordon equation in all arguments, which completes the proof. $\square$

Alternatively we could drop the assumption that ω is a generalised free field state and require causality and the AμSC (or the Reeh-Schlieder property) instead. This certainly allows us to prove that ω satisfies the Klein-Gordon equation as follows:

Proposition 56 *Let ω be a causal state satisfying the AμSC. If $K_x\omega_2(x,y) = 0$ then ω satisfies the Klein-Gordon equation.*

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[3] A related result, due to Greenberg (5), says that a state must be a generalised free field state if the Källen-Lehmann representation of the two-point distribution

$$\omega_{2-}(x_2,x_1) = \int \rho_{KL}(m^2)\omega_{2-}^m(x_2,x_1)dm^2$$

in terms of the free field commutator functions of mass m, ω_{2-}^m, has a positive measure ρ_{KL} whose support satisfies certain restrictions. In the Wightman framework every ω_{2-} allows a Källen-Lehmann representation, but in curved spacetime such a tool is not available, so at present it makes no sense to consider the generalisation of this result. Moreover, our current strategy of weakening the Wightman axioms and assuming commutation relations instead would render the statement trivial.

Proof By AμSC, ω has the Reeh-Schlieder property, i.e. Ω_ω is a cyclic vector for every local algebra (21). Now $\Phi^\omega(Kf)$ annihilates Ω_ω for every $f \in C_0^\infty(M)$ and $\Phi^\omega(Kf) \cdot \pi_\omega(B)\Omega_\omega = 0$ for any B that commutes with $\Phi(f)$. By causality and the Reeh-Schlieder property we conclude that $\Phi^\omega(Kf)$ annihilates a dense set of vectors and hence $\Phi^\omega(Kf) = 0$ (because the operator is closable). $\quad\square$

Note, however, that it is not at all clear whether the state also satisfies the canonical commutation relations. The proof of (15), e.g., uses Poincaré invariance, the full strength of the spectrum condition and the uniqueness of the vacuum[4]. We will not investigate what other assumptions are necessary to recover the strong version of the Jost-Schroer theorem, but for completeness we do provide the following:

Example We construct a state satisfying the assumptions of Proposition 56 with the canonical commutator function, but which is not a generalised free field state. For this purpose we let ω^1 denote the quasi-free state on Minkowski spacetime with two-point distribution $\omega_2^1 = 2\omega_2^0$, where ω^0 is the Minkowski vacuum. We let ω^2 be the state with $\omega_n^2 = 0$ for all $n > 0$ and we note that the mixed state $\omega^3 := \frac{1}{2}(\omega^1 + \omega^2)$ serves our purpose by considering the four-point distribution:

$$\omega_4^3(x_4, x_3, x_2, x_1) - \omega_4^3(x_3, x_4, x_2, x_1) = 2i\omega_{2-}^3(x_4, x_3)\omega_2^3(x_2, x_1).$$

6 Applications and Outlook

(13) already mentions the class of Hadamard states whose truncated n-point distributions are smooth functions for all $n \neq 2$ as an interesting class. Later (7; 8) discuss perturbation theory by constructing an extended *-algebra of Wick powers and time-ordered products of a free field and find that the continuous states on this algebra are exactly the Hadamard states of this class. Our Theorem 42 shows that the condition on the truncated n-point distributions is automatically satisfied for (generalised) free fields due to the scalar commutation relations, so the class of *all* Hadamard states is the class of interest for perturbative quantum field theory. Furthermore, Corollary 43 shows that for a generalised free field any generalised Hadamard state satisfies the μSC and Proposition 45 tells us that the class of generalised Hadamard states is closed under operations, which is useful to know from a fundamental point of view. Our Theorem 42 and Corollary 43 could find further applications in perturbative quantum field theory around a generalised free field, rather than around a free field. Such an approach has been suggested in (3) as a way to gain insight in the AdS-CFT correspondence.

Concerning the strength of the assumption that a state is a generalised free field state we have discussed the generalisation of two results from the Wightman framework to curved spacetimes. We showed that in some circumstances our assumption can replace the existing arguments based on analyticity, as in Theorem 53 that generalised a result of Borchers and Zimmerman. For the Jost-Schroer theorem the situation was more delicate: a weak form of this theorem can be proved

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[4] In this connection it should also be noted that generalised free fields need not have the time-slice property, so then the commutation relations cannot be proved in curved spacetime via a spacetime-deformation argument as in (22).

in curved spacetimes by assuming that a state is a generalised free field state. However, it is not known if one can prove that a state is a (generalised) free field under suitable circumstances without assuming commutation relations in the first place.

Finally we note that the proofs we used were all elementary applications of the calculus of wave front sets of (Hilbert space-valued) distributions and the combinatorics of (truncated) n-point distributions. Both can be generalised to vector-valued fields and to anti-commutation relations in a straightforward manner (see e.g. (18) Prop. 4.2.17 for the result that a Hadamard state of the free Dirac field satisfies the μSC).

Note added in proof. Our Proposition 3.4 was already known as Lemma 5.2 of Gottschalk and Thaler, "An indefinite metric model for interacting quantum fields on globally hyperbolic spacetimes", Ann. Henri Poincare 4 (2003) 637–659. We thank Thomas-Paul Hack for bringing this to our attention.

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