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Composite and Background Fields in Non-Abelian Gauge Models

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Abstract: A joint introduction of composite and background fields into non-Abelian quantum gauge theories is suggested based on the symmetries of the generating functional of Green's functions, with the systematic analysis focused on quantum Yang–Mills theories, including the properties of the generating functional of vertex Green's functions (effective action). For the effective action in such theories, gauge dependence is found in terms of a nilpotent operator with composite and background fields, and on-shell independence from gauge fixing is established. The basic concept of a joint introduction of composite and background fields into non-Abelian gauge theories is extended to the Volovich–Katanaev model of two-dimensional gravity with dynamical torsion, as well as to the Gribov–Zwanziger theory.

Keywords: composite fields; background fields; Yang–Mills theory; Volovich–Katanaev model; Gribov–Zwanziger model; background effective action; gauge-dependence

1. Introduction

Composite [1,2] and background [3–5] fields are widely used in quantum gauge theories. The attention to composite fields (see [2] for an overview) stems from the fact that the effective action for composite fields suggested in [1] has been applied to quantum field models such as [6–8], including the Early Universe, Inflationary Universe, Standard Model and SUSY theories [9–13]. The study of BRST-invariant renormalizability in Yang–Mills theories, which includes $N = 1$ SUSY formulations [14–16], the functional renormalization group [17–21] and the Gribov horizon [22–24], appears to be promising within the concept of soft BRST symmetry breaking [25–28] and the local composite operator technique [29,30] as applied to arbitrary backgrounds [31]. In its turn, the background field method [3–5] presents the quantization of Yang–Mills theories using background gauges [5,32,33] in such a way that provides an invariance of the effective action under the gauge transformations of background fields and reproduces physical results with essential simplifications in the Feynman diagrams, thereby providing insight into diverse quantum properties of gauge theories [34–43]; for recent developments, see [44–48].

The present article is devoted to quantum non-Abelian gauge models with composite and background fields, whose consistent analysis isfocused on Yang–Mills theories quantized using the Faddeev–Popov method [49] in a combined presence of composite and background fields. A joint treatment of Yang–Mills fields A_μ with composite and background ones requires a systematic consideration of these ingredients as interrelated. We forward the symmetry principle as such a systematic concept. In fact, suppose that a generating functional $Z(J, L)$ of Green's functions with composite fields is given, depending on sources J_A for the quantum fields ϕ^A, as well as on sources L_m for the composite fields $\sigma^m(\phi)$. A question then naturally arises of how one can introduce some background fields B_μ in a way that leads to an extended functional $Z(B,J,L)$ reflecting the possible symmetries of $Z(J,L)$. Let us next suppose that a generating functional $Z(B,J)$ of Green's functions, also with certain symmetries, is given in the background field method, and then the question is how some composite fields $\sigma^m(\phi,B)$ with sources L_m can be introduced for the resulting $Z(B,J,L)$ to inherit the given symmetries. It turns out that these two approaches are equivalent in the following sense.

According to the first approach, a generating functional $Z(J,L)$ is given,

$$Z(J,L) = \int d\phi \exp\left\{\frac{i}{\hbar}\left[S_{\rm FP}(\phi) + J_A\phi^A + L_m\sigma^m(\phi)\right]\right\}, \tag{1}$$

corresponding to the Faddeev–Popov action $S_{\rm FP}(\phi)$ of a Yang–Mills theory with composite fields $\sigma^m(\phi)$. A background field B_μ can then be introduced by *localizing* the inherent global symmetry of $Z(J,L)$ under $SU(N)$ transformations (rotations for J_A and tensor transformations for L_m) in such a way that $Z(B,J,L)$ defined as

$$Z(B,J,L) = Z(J,L)|_{\partial_\mu \to D_\mu(B)} \tag{2}$$

is invariant under local $SU(N)$ transformations of the sources J_A, L_m, accompanied by gauge transformations of the field B_μ with an associated covariant derivative $D_\mu(B)$, where the precise meaning of $\partial_\mu \to D_\mu(B)$ in (2) is given by (19) of Section 2. The original $S_{\rm FP}(\phi)$ becomes thereby modified to the Faddeev–Popov action of the background field method, $S_{\rm FP}(\phi,B)$, which is related to $S_{\rm FP}(\phi)$ by so-called *background* and *quantum* transformations of this method (see Section 2.2).

According to the second approach, a generating functional $Z(B,J)$ is constructed using the background field method for Yang–Mills theories,

$$Z(B,J) = \int d\phi \exp\left\{\frac{i}{\hbar}\left[S_{\rm FP}(\phi,B) + J_A\phi^A\right]\right\}, \tag{3}$$

which implies

$$S_{\rm FP}(\phi,B) = S_{\rm FP}(\phi)|_{\partial_\mu \to D_\mu(B)} \,.$$

Some composite fields $\sigma^m(\phi,B)$ with sources L_m can then be introduced on condition that the resulting generating functional

$$Z(B,J,L) = \int d\phi \exp\left\{\frac{i}{\hbar}\left[S_{\rm FP}(\phi,B) + J_A\phi^A + L_m\sigma^m(\phi,B)\right]\right\} \tag{4}$$

should inherit the symmetry of $Z(B,J)$ under local $SU(N)$ rotations of the sources J_A accompanied by gauge transformations of the background field B_μ with the covariant derivative $D_\mu(B)$. This symmetry requirement for $Z(B,J,L)$ is satisfied by a local $SU(N)$ tensor transformation law imposed on $\sigma^m(\phi,B)$ and is provided by B_μ entering the composite fields $\sigma^m(\phi,B)$ by means of the covariant derivative $D_\mu(B)$, which implies

$$\sigma^m(\phi,B) = \sigma^m(\phi)|_{\partial_\mu \to D_\mu(B)} \tag{5}$$

for certain $\sigma^m(\phi)$, and thereby we return to the first approach. In the main part of the article, we implement the first approach as a starting point of our systematic analysis, assuming the composite fields to be *local*, whereas in the remaining part we show how the first and second approaches can be extended beyond the given assumptions by considering the Volovich–Katanaev model of two-dimensional gravity with dynamical torsion [50] and the Gribov–Zwanziger theory [23,24].

Two-dimensional models of gravity [50–61] and supergravity [62–65] are of interest in view of their close relation to string and superstring theory. Simple two-dimensional models also provide a deeper insight into classical and quantum properties of gravity in higher dimensions, while in some cases these models are exactly solvable at the classical level. One of the two-dimensional gravity models that has been widely discussed at the classical [66–70] and quantum [71–75] levels is the model [50] suggested in the context of bosonic string theory with dynamical torsion [76] in order to address some problems of string theory. Thus, it has been shown [76], using the path integral approach, that a string with dynamical torsion has no critical dimension. The model [50] presents the most general theory of two-dimensional R^2-gravity with independent dynamical torsion leading to second-order equations of motion for the zweibein and Lorentz connection. The model also contains solutions with constant curvature and zero torsion, thereby incorporating some other two-dimensional gravity models [51,52,61] whose actions, as compared to that of [50], do not allow a purely geometric interpretation. Being quantized in the background field method, the model yields a gauge-invariant background effective action [75].

In quantum Yang–Mills theories using differential (e.g., Landau or Feynman) gauges, the non-Abelian nature of the gauge group features the Gribov ambiguity [22] implying a residual gauge-invariance due to Gribov copies, which are removed by means of a Gribov horizon [22] implemented in the Gribov–Zwanziger model [23,24] being a quantum Yang–Mills theory in Landau gauge and including an additive horizon functional in terms of a non-local composite field [77]. The issue of bringing the horizon functional to other gauges has been settled due to the concept of finite field-dependent BRST transformations [26,28,78–84], which allows one to present the horizon functional using different gauges in a way consistent with the gauge-independence of the path integral, based on the Gribov–Zwanziger recipe [23,24] and starting from a BRST-invariant Yang–Mills quantum action in Landau gauge. The horizon functional in covariant R_ξ gauges has been given by [26,28,82,85] (see also [86,87] for a BRST-invariant horizon) and later extended to the Standard Model in [88,89].

The article is organized as follows. In Section 2, a generating functional of Green's functions with composite and background fields in Yang–Mills theories is introduced, as well as a generating functional of vertex Green's functions (effective action). Analyzing the dependence of the generating functionals of Green's functions upon a choice of gauge-fixing, we find a gauge variation of the effective action in terms of a nilpotent operator depending on the composite fields and determine the conditions of on-shell gauge-independence. Besides, the effective action $\Gamma_{\text{eff}}(B,\Sigma)$, with a background field B_μ and a set of auxiliary tensor fields Σ^m associated with σ^m, is found to exhibit a local symmetry under the gauge transformations of B_μ combined with the local $SU(N)$ transformations of Σ^m. In Section 3, we examine the Volovich–Katanaev model [50] quantized according to the background field method in [75]. As an extension of our second approach (3)–(5) beyond the Yang–Mills case, the quantized two-dimensional gravity [75] is modified by the presence of local composite fields, and the corresponding background effective action is found to be gauge-invariant in a way similar to the Yang–Mills case. In Section 4, we modify the Gribov–Zwanziger model [23,24] by introducing a background field, which extends our first approach (1), (2) beyond the Yang–Mills case with local composite fields and thereby yields a gauge-invariant background effective action. Section 5 is devoted to concluding remarks.

We use DeWitt's condensed notation [90]. The Grassmann parity and ghost number of a quantity F are denoted by $\epsilon(F)$, $\mathrm{gh}(F)$, respectively. The supercommutator $[F,G\}$ of any quantities F,G with definite Grassmann parities is given by $[F\,,G\} = FG - (-1)^{\epsilon(F)\epsilon(G)}GF$. Unless specified by an arrow, derivatives with respect to fields and sources are regarded as left-hand ones.

2. Generating Functionals and Their Properties

Let us examine a generating functional $Z(J,L)$ corresponding to the Faddeev–Popov action $S_{\text{FP}}(\phi)$ of a Yang–Mills theory with local composite fields,

$$Z(J,L) = \int d\phi \exp\left\{\frac{i}{\hbar}\left[S_{\text{FP}}(\phi) + J_A\phi^A + L_m\sigma^m(\phi)\right]\right\}, \tag{6}$$

where L_m are sources to the composite fields $\sigma^m(\phi)$,

$$\sigma^m(\phi) = \sum_{n=2} \frac{1}{n!} \Lambda^m_{A_1 \ldots A_n} \phi^{A_n} \ldots \phi^{A_1} , \tag{7}$$

and J_A are sources to the fields $\phi^A = (A^i, b^\alpha, \bar{c}^\alpha, c^\alpha)$ composed by gauge fields A^i, (anti)ghost fields $\bar{c}^\alpha$, c^α, and Nakanishi–Lautrup fields b^α, with the following distribution of Grassmann parity and ghost number:

$$\epsilon(\phi^A) = (0,0,1,1)\,,\ \ \mathrm{gh}(\phi^A) = (0,0,-1,1)\,,\ \ \epsilon(J_A, L_m) = \epsilon(\phi^A, \sigma^m),\ \ \mathrm{gh}(J_A, L_m) = -\mathrm{gh}(\phi^A, \sigma^m).$$

The Faddeev–Popov action $S_{\mathrm{FP}}(\phi)$,

$$S_{\mathrm{FP}}(\phi) = S_0(A) + \Psi(\phi)\overleftarrow{s}\,,$$

is given in terms of a gauge-invariant classical action $S_0(A)$, invariant, $\delta_\xi S_0(A) = 0$, under infinitesimal gauge transformations $\delta_\xi A^i = R^i_\alpha(A)\,\xi^\alpha$ with a closed algebra of gauge generators $R^i_\alpha(A)$,

$$R^i_{\alpha,j}(A)\,R^j_\beta(A) - R^i_{\beta,j}(A)\,R^j_\alpha(A) = F^\gamma_{\alpha\beta} R^i_\gamma(A)\,,\ \ F^\gamma_{\alpha\beta} = \mathrm{const},\ \ R^i_{\alpha,j} \equiv R^i_\alpha \frac{\overleftarrow{\delta}}{\delta A^j}\,,$$

and a nilpotent Slavnov variation $\overleftarrow{s}$ applied to a gauge Fermion $\Psi(\phi)$, $\epsilon(\Psi) = 1$,

$$S_{\mathrm{FP}}(\phi) = S_0(A) + \Psi(\phi)\overleftarrow{s}\,,\ \ \Psi(\phi) = \bar{c}^\alpha \chi_\alpha(\phi)\,,\ \overleftarrow{s}^{\,2} = 0, \tag{8}$$

where

$$\phi^A \overleftarrow{s} = \left(R^i_\alpha(A)\,c^\alpha, 0, b^\alpha, 1/2 F^\alpha_{\beta\gamma} c^\gamma c^\beta \right).$$

For the explicit field content

$$\begin{aligned} i &= (x,p,\mu),\ \ \alpha = (x,p)\,,\ \ \mu = 0,\ldots,D-1,\ \ p = 1,\ldots,N^2-1, \\ \phi^A &= \left(A^{p|\mu}, b^p, \bar{c}^p, c^p \right),\ \ (A^\mu, b, \bar{c}, c) \equiv T^p \left(A^{p|\mu}, b^p, \bar{c}^p, c^p \right),\ \ [T^p, T^q] = f^{pqr} T^r, \end{aligned}$$

the field variations $\phi^A \overleftarrow{s}$ have the form

$$(A_\mu, b, \bar{c}, c)\overleftarrow{s} = \left([D_\mu(A), c]\,, 0, b, g/2\,[c,c]_+ \right),\ \left(A^p_\mu, b^p, \bar{c}^p, c^p \right)\overleftarrow{s} = \left(D^{pq}_\mu(A)\,c^q, 0, b^p, g/2 f^{pqr} c^q c^r \right),$$

where

$$D_\mu(A) \equiv \partial_\mu + g A_\mu\,,\ \ D^{pq}_\mu(A) = \delta^{pq}\partial_\mu + g f^{prq} A^r_\mu\,,$$

The classical action $S_0(A)$ has the form (in the adjoint representation with Hermitian T^p),

$$\begin{aligned} S_0(A) &= \frac{1}{2g^2} \int d^D x\, \mathrm{Tr}\left(F_{\mu\nu} F^{\mu\nu} \right) = \frac{1}{4} \int d^D x\, F^p_{\mu\nu} F^{p|\mu\nu},\ \ \mathrm{Tr}\left(T^p T^q \right) = \frac{1}{2}\delta^{pq}, \\ F_{\mu\nu} &\equiv \left[D_\mu(A), D_\nu(A) \right],\ \ F^p_{\mu\nu} = \partial_\mu A^p_\nu - \partial_\nu A^p_\mu + g f^{prs} A^r_\mu A^s_\nu\,, \end{aligned} \tag{9}$$

and the gauge Fermion $\Psi(\phi) = \bar{c}^\alpha \chi_\alpha(\phi)$ with some gauge-fixing functions $\chi_\alpha(\phi) = \chi^p(\phi(x))$ reads

$$\Psi(\phi) = \int d^D x\, \bar{c}^p \chi^p(\phi) = 2 \int d^D x\, \mathrm{Tr}\left[\bar{c}\chi(\phi) \right],\ \ \chi(\phi) = T^p \chi^p(\phi)\,. \tag{10}$$

The Faddeev–Popov action $S_{\mathrm{FP}}(\phi)$ in (8), (9) is invariant under two kinds of global transformations: BRST transformations [91–93], $\delta_\lambda \phi^A = \phi^A \overleftarrow{s} \lambda$, with an anticommuting parameter λ, $\epsilon(\lambda) = 1$, and $SU(N)$ rotations (finite $\phi^A \overset{U}{\to} \phi'^A$ and infinitesimal $\delta_\varsigma \phi^A$) with even parameters ς^p,

$$\begin{aligned} \left(A_\mu, b, \bar{c}, c\right) \overset{U}{\to} \left(A_\mu, b, \bar{c}, c\right)' &= U\left(A_\mu, b, \bar{c}, c\right) U^{-1}, \quad U = \exp\left(-gT^p\varsigma^p\right), \quad \varsigma^p = \mathrm{const}, \\ \delta_\varsigma \left(A^p_\mu, b^p, \bar{c}^p, c^p\right) &= gf^{prq}\left(A^r_\mu, b^r, \bar{c}^r, c^r\right)\varsigma^q, \end{aligned}$$

or, in a tensor form, via the adjoint representation with a matrix $M^{pq}(\varsigma)$,

$$\left(A^p_\mu, b^p, \bar{c}^p, c^p\right)' = M^{pq}(\varsigma)\left(A^q_\mu, b^q, \bar{c}^q, c^q\right), \quad M^{pq}(\varsigma) = \delta^{pq} + gf^{pqr}\varsigma^r + O\left(\varsigma^2\right).$$

The classical action $S_0(A)$ in $S_{\mathrm{FP}}(\phi) = S_0(A) + \Psi(\phi)\overleftarrow{s}$ is invariant under $A_\mu \overset{U}{\to} A'_\mu$ as a particular case ($\xi^p(x) = \mathrm{const}$) of invariance under the finite form $A_\mu \overset{V}{\to} A'_\mu$ of gauge transformations

$$A'_\mu = VA_\mu V^{-1} + g^{-1}V\left(\partial_\mu V^{-1}\right), \quad D_\mu(A') = VD_\mu(A)V^{-1}, \quad V = \exp\left(-gT^p\xi^p\right), \quad \xi^p = \xi^p(x), \tag{11}$$

while the invariance of $\Psi(\phi)\overleftarrow{s}$ under $\phi^A \overset{U}{\to} \phi'^A$ reflects the explicit form of $\overleftarrow{s}$ and the fact that the gauge functions $\chi^p(\phi)$ are local and constructed from the fields ϕ^A, structure constants f^{pqr} and derivatives ∂_μ, for instance, in Landau and Feynman gauges,

$$\chi^p_{\mathrm{L}}(\phi) = \partial^\mu A^p_\mu, \quad \chi^p_{\mathrm{F}}(\phi) = b^p + \partial^\mu A^p_\mu, \tag{12}$$

so that, in particular, $\chi^p(\phi)$ transform as $SU(N)$ vectors, with $\Psi(\phi)$ being invariant under $\phi^A \overset{U}{\to} \phi'^A$,

$$\chi(\phi') = U\chi(\phi)U^{-1}, \quad \delta\chi^p(\phi) = gf^{prq}\chi^r(\phi)\varsigma^q$$

Due to the same reason, local composite fields $\sigma^m(\phi)$ constructed from the fields ϕ^A, structure constants f^{pqr} and derivatives ∂_μ,

$$\sigma^m(\phi) = \sigma^{p_1\cdots p_k|\mu_1\cdots\mu_l}(\phi(x)), \quad m = (x, p_1\cdots p_k, \mu_1\cdots\mu_l),$$

in the path integral (6) for $Z(J,L)$ transform under $\phi^A \overset{U}{\to} \phi'^A$ as tensors with respect to the indices $p_1, \ldots, p_k$,

$$\begin{aligned} \sigma'^{\,p_1\cdots p_k|\mu_1\cdots\mu_l} &= M^{p_1q_1}\ldots M^{p_kq_k}\sigma^{q_1\cdots q_k|\mu_1\cdots\mu_l}, \\ \delta_\varsigma \sigma^{p_1\cdots p_k|\mu_1\cdots\mu_l} &= g\sum_{r_s\in\{r_1,\ldots,r_k\}} f^{p_sr_sq}\sigma^{p_1\cdots r_s\cdots p_k|\mu_1\cdots\mu_l}\varsigma^q \equiv gf^{\{p\}\hat{r}q}\sigma^{p_1\cdots\hat{r}\cdots p_k|\mu_1\cdots\mu_l}\varsigma^q, \end{aligned} \tag{13}$$

which generalizes the vector transformation. As a result, the exponential in the path integral (6) for $Z(J,L)$ is invariant under $\phi^A \overset{U}{\to} \phi'^A$, along with some global transformations of the sources J_A, L_m,

$$(J_A, L_m) \overset{U}{\to} (J_A, L_m)', \quad J_A = (J^p_{(A)\mu}, J^p_{(b)}, J^p_{(\bar{c})}, J^p_{(c)}), \quad L_m = L^{p_1\cdots p_k}_{\mu_1\cdots\mu_l}, \tag{14}$$

in a tensor and infinitesimal form,

$$\begin{aligned} &(J^{p|\mu}_{(A)}, J^p_{(b)}, J^p_{(\bar{c})}, J^p_{(c)})' = M^{pq}(J^{q|\mu}_{(A)}, J^q_{(b)}, J^q_{(\bar{c})}, J^q_{(c)}), \qquad L'^{\,p_1\cdots p_k}_{\mu_1\cdots\mu_l} = M^{p_1q_1}\cdots M^{p_kq_k}L^{q_1\cdots q_k}_{\mu_1\cdots\mu_l}, \\ &\delta_\varsigma(J^{p|\mu}_{(A)}, J^p_{(b)}, J^p_{(\bar{c})}, J^p_{(c)}) = gf^{prq}(J^{r|\mu}_{(A)}, J^r_{(b)}, J^r_{(\bar{c})}, J^r_{(c)})\varsigma^q, \quad \delta_\varsigma L^{p_1\cdots p_k}_{\mu_1\cdots\mu_l} = gf^{\{p\}\hat{r}q}L^{p_1\cdots\hat{r}\cdots p_k}_{\mu_1\cdots\mu_l}\varsigma^q, \end{aligned} \tag{15}$$

which entails the invariance of the source term $J_A\phi^A + L_m\sigma^m(\phi)$.

Let us introduce an additional field $B_\mu = B_\mu^p T^p$ having a gauge transformation as in (11),

$$B_\mu \xrightarrow{V} B'_\mu = V B_\mu V^{-1} + g^{-1} V \left(\partial_\mu V^{-1}\right), \tag{16}$$

with the inherent property

$$D_\mu \left(B'\right) = V D_\mu \left(B\right) V^{-1}, \quad D_\mu \left(B\right) \equiv \partial_\mu + g B_\mu , \tag{17}$$

and subject the exponential in the path integral (6) to the modification

$$\exp\left\{\frac{i}{\hbar}\left[S_{\mathrm{FP}}\left(\phi\right) + J\phi + L\sigma\left(\phi\right)\right]\right\}\Big|_{\partial_\mu \to D_\mu(B)} \equiv \exp\left\{\frac{i}{\hbar}\left[S_{\mathrm{FP}}\left(\phi, B\right) + J\phi + L\sigma\left(\phi, B\right)\right]\right\}, \tag{18}$$

where the replacement $\partial_\mu \to D_\mu\left(B\right)$ reads as

$$\partial_\mu \to D_\mu\left(B\right): \quad \left[\partial_\mu, \bullet\right] \to \left[D_\mu\left(B\right), \bullet\right] \Rightarrow \left[D_\mu\left(A\right), \bullet\right] \to \left[D_\mu\left(A+B\right), \bullet\right], \tag{19}$$

so that

$$F_{\mu\nu}\left(A\right) \to \left[D_\mu\left(A+B\right), D_\nu\left(A+B\right)\right] = F_{\mu\nu}\left(A+B\right). \tag{20}$$

Due to the transformation property (17) of the derivative $D_\mu\left(B\right)$, the generating functional $Z\left(B, J, L\right)$ modified by the field B_μ according to (18), (19),

$$Z\left(B, J, L\right) = \int d\phi \exp\left\{\frac{i}{\hbar}\left[S_{\mathrm{FP}}\left(\phi, B\right) + J_A \phi^A + L_m \sigma^m\left(\phi, B\right)\right]\right\}, \tag{21}$$

is invariant under a set of local transformations,

$$\begin{aligned} \delta_\xi B_\mu^p &= D_\mu^{pq}\left(B\right)\xi^q, \\ \delta_\xi\left(J_{(A)}^{p|\mu}, J_{(b)}^p, J_{(\bar{c})}^p, J_{(c)}^p\right) &= g f^{prq}\left(J_{(A)}^{r|\mu}, J_{(b)}^r, J_{(\bar{c})}^r, J_{(c)}^r\right)\xi^q, \\ \delta_\xi L_{\mu_1\cdots\mu_l}^{p_1\cdots p_k} &= g f^{\{p\}\hat{r}q} L_{\mu_1\cdots\mu_l}^{p_1\cdots\hat{r}\cdots p_k}\xi^q, \end{aligned} \tag{22}$$

given by the gauge transformations (16) of the field B_μ combined with a localized form $U\left(\varsigma\right) \to V\left(\xi\right)$ of the transformations (14), (15) for the sources J_A, L_m with infinitesimal parameters ξ^p,

$$\left(B_\mu, J_A, L_m\right) \xrightarrow{V} \left(B_\mu, J_A, L_m\right)'.$$

The invariance property $Z\left(B', J', L'\right) = Z\left(B, J, L\right)$ can be established by applying to the transformed path integral $Z\left(B', J', L'\right)$ a compensating change of the integration variables:

$$\delta_\xi\left(A_\mu^p, b^p, \bar{c}^p, c^p\right) = g f^{prq}\left(A_\mu^r, b^r, \bar{c}^r, c^r\right)\xi^q, \quad \left(A_\mu, b, \bar{c}, c\right) \xrightarrow{V} \left(A_\mu, b, \bar{c}, c\right)',$$

whose Jacobian equals to unity in view of the complete antisymmetry of the structure constants. The invariance of $Z\left(B, J, L\right)$ can be recast in the form

$$\begin{aligned} \int d^D x \, \Bigg\{ & \left[D_\mu^{pq}\left(B\right)\xi^q\right]\frac{\overrightarrow{\delta}}{\delta B_\mu^p} + g\xi^q f^{\{p\}\hat{r}q} L_{\mu_1\cdots\mu_l}^{p_1\cdots\hat{r}\cdots p_k}\frac{\overrightarrow{\delta}}{\delta L_{\mu_1\cdots\mu_l}^{p_1\cdots p_k}} \\ & + g\xi^q f^{prq}\left(J_{(A)}^{r|\mu}\frac{\overrightarrow{\delta}}{\delta J_{(A)}^{p|\mu}} + J_{(b)}^r\frac{\overrightarrow{\delta}}{\delta J_{(b)}^p} + J_{(\bar{c})}^r\frac{\overrightarrow{\delta}}{\delta J_{(\bar{c})}^p} + J_{(c)}^r\frac{\overrightarrow{\delta}}{\delta J_{(c)}^p}\right)\Bigg\} Z\left(B, J, L\right) = 0. \end{aligned} \tag{23}$$

2.1. Effective Action and Gauge Dependence

Let us present a generating functional of vertex Green's functions and examine its gauge-dependence properties. To do so, we first introduce an extended generating functional $Z\left(B,J,L,\phi^{*}\right)$,

$$Z\left(B,J,L,\phi^{*}\right)=\int d\phi\exp\left\{\frac{i}{\hbar}\left[S_{\mathrm{ext}}\left(\phi,\phi^{*},B\right)+J_{A}\phi^{A}+L_{m}\sigma^{m}\left(\phi,B\right)\right]\right\}, \tag{24}$$

with an extended quantum action $S_{\mathrm{ext}}\left(\phi,\phi^{*},B\right)$ defined as

$$S_{\mathrm{ext}}\left(\phi,\phi^{*},B\right)=S_{\mathrm{FP}}\left(\phi,B\right)+\phi_{A}^{*}(\phi^{A}\overleftarrow{s}_{\mathrm{q}}),\quad \overleftarrow{s}_{\mathrm{q}}=\overleftarrow{s}|_{\partial_{\mu}\to D_{\mu}(B)}\ ,\quad \overleftarrow{s}_{\mathrm{q}}\overleftarrow{s}_{\mathrm{q}}=0,$$

where ϕ_{A}^{*} is a set of antifields introduced as sources to the variations $\phi^{A}\overleftarrow{s}_{\mathrm{q}}$. Given the properties $S_{\mathrm{FP}}\left(\phi\right)\overleftarrow{s}=0$ and $\overleftarrow{s}_{\mathrm{q}}\overleftarrow{s}_{\mathrm{q}}=0$, the extended quantum action $S_{\mathrm{ext}}\left(\phi,\phi^{*},B\right)$ satisfies the identity

$$S_{\mathrm{ext}}\left(\phi,\phi^{*},B\right)\overleftarrow{s}_{\mathrm{q}}=S_{\mathrm{FP}}\left(\phi,B\right)\overleftarrow{s}_{\mathrm{q}}=S_{\mathrm{FP}}\left(\phi\right)\overleftarrow{s}|_{\partial_{\mu}\to D_{\mu}(B)}=0,$$

having an equivalent form, implied by the antisymmetry of the structure constants:

$$\Delta\exp\left(i/\hbar\right)S_{\mathrm{ext}}=0,\quad \Delta\equiv\left(-1\right)^{\epsilon\left(\phi^{A}\right)}\frac{\overrightarrow{\delta}}{\delta\phi^{A}}\frac{\overrightarrow{\delta}}{\delta\phi_{A}^{*}},\quad \Delta^{2}=0. \tag{25}$$

In view of (24), (25), the variation $\delta_{\Psi}Z\left(B,J,L,\phi^{*}\right)$ related to a change $\delta\Psi\left(\phi,B\right)$ has the form

$$\begin{aligned}\delta_{\Psi}Z\left(B,J,L,\phi^{*}\right) &= \int d\phi\ \exp\left\{\frac{i}{\hbar}\left[J_{A}\phi^{A}+L_{m}\sigma^{m}\left(\phi,B\right)\right]\right\}\delta_{\Psi}\exp\left[\frac{i}{\hbar}S_{\mathrm{ext}}\left(\phi,\phi^{*},B\right)\right], \\ \delta_{\Psi}\exp\left(\frac{i}{\hbar}S_{\mathrm{ext}}\right) &= \frac{i}{\hbar}\delta\Psi\overleftarrow{s}_{\mathrm{q}}\exp\left(\frac{i}{\hbar}S_{\mathrm{ext}}\right)=-\Delta\left[\delta\Psi\exp\left(\frac{i}{\hbar}S_{\mathrm{ext}}\right)\right],\quad \delta\Psi\overleftarrow{s}_{\mathrm{q}}=\delta\Psi_{,A}(\phi^{A}\overleftarrow{s}_{\mathrm{q}}),\end{aligned} \tag{26}$$

which can be represented as

$$\delta_{\Psi}Z=\delta\Psi_{,A}\left(\frac{\hbar}{i}\frac{\overrightarrow{\delta}}{\delta J},B\right)\frac{\overrightarrow{\delta}}{\delta\phi_{A}^{*}}Z=\frac{i}{\hbar}\hat{\omega}\delta\Psi\left(\frac{\hbar}{i}\frac{\overrightarrow{\delta}}{\delta J},B\right)Z, \tag{27}$$

where the second equality is a result of integration by parts in (26), and $\hat{\omega}$ is a nilpotent operator (which is also found in the Ward identity $\hat{\omega}Z=0$), namely,

$$\hat{\omega}=\left[J_{A}+L_{m}\sigma_{,A}^{m}\left(\frac{\hbar}{i}\frac{\overrightarrow{\delta}}{\delta J},B\right)\right]\frac{\overrightarrow{\delta}}{\delta\phi_{A}^{*}},\quad \hat{\omega}^{2}=0. \tag{28}$$

In terms of a generating functional $\Gamma\left(B,\phi,\Sigma,\phi^{*}\right)$ of vertex Green's functions with composite fields (on a background B_{μ}) given by a double Legendre transformation [94],

$$\Gamma\left(B,\phi,\Sigma,\phi^{*}\right)=W\left(B,J,L,\phi^{*}\right)-J_{A}\phi^{A}-L_{m}\left[\sigma^{m}\left(\phi,B\right)+\Sigma^{m}\right]\ ,\quad W=\left(\hbar/i\right)\ln Z, \tag{29}$$

where

$$\phi^{A}=\frac{\overrightarrow{\delta}W}{\delta J_{A}},\ \Sigma^{m}=\frac{\overrightarrow{\delta}W}{\delta L_{m}}-\sigma^{m}\left(\frac{\overrightarrow{\delta}W}{\delta J},B\right),\ -J_{A}=\Gamma\frac{\overleftarrow{\delta}}{\delta\phi^{A}}+L_{m}\sigma_{,A}^{m}(\phi,B),\ -L_{m}=\Gamma\frac{\overleftarrow{\delta}}{\delta\Sigma^{m}}.$$

The variation $\delta_\Psi \Gamma\left(B, \phi, \Sigma, \phi^*\right)$ induced by (27), with account taken of $\delta_\Psi \Gamma = \delta_\Psi W$, reads

$$
\begin{aligned}
\delta_\Psi \Gamma \;=\; & \langle\langle\delta\Psi\rangle\rangle \frac{\overleftarrow{\delta}}{\delta\phi^A}\frac{\overrightarrow{\delta}\Gamma}{\delta\phi^*_A} - \langle\langle\delta\Psi\rangle\rangle \frac{\overleftarrow{\delta}}{\delta\phi^*_A}\frac{\overrightarrow{\delta}\Gamma}{\delta\phi^A} + \left[\sigma^m_{,A}(\hat{\phi},B) - \sigma^m_{,A}(\phi,B)\right]\frac{\overrightarrow{\delta}\Gamma}{\delta\phi^*_A}\frac{\overrightarrow{\delta}}{\delta\Sigma^m}\langle\langle\delta\Psi\rangle\rangle \\
& -\frac{i}{\hbar}\left(\left[\Gamma\frac{\overleftarrow{\delta}}{\delta\Sigma^m}\left(\sigma^m_{,C}(\hat{\phi},B)\frac{\overrightarrow{\delta}\Gamma}{\delta\phi^*_C}\right), \Phi^{\mathsf{a}}\right\}\frac{\overrightarrow{\delta}}{\delta\Phi^{\mathsf{a}}} + \left[\Gamma\frac{\overleftarrow{\delta}}{\delta\Sigma^m}\left(\sigma^m_{,C}(\hat{\phi},B)\frac{\overrightarrow{\delta}\Gamma}{\delta\phi^*_C}\right), \sigma^n(\phi,B)\right\}\frac{\overrightarrow{\delta}}{\delta\Sigma^n}\right)\langle\langle\delta\Psi\rangle\rangle \\
& +\frac{i}{\hbar}(-1)^{\epsilon(\sigma^n)+\epsilon(\phi^D)}\sigma^n_{,D}(\phi,B)\left[\Gamma\frac{\overleftarrow{\delta}}{\delta\Sigma^m}\left(\sigma^m_{,C}(\hat{\phi},B)\frac{\overrightarrow{\delta}\Gamma}{\delta\phi^*_C}\right), \phi^D\right\}\frac{\overrightarrow{\delta}}{\delta\Sigma^n}\langle\langle\delta\Psi\rangle\rangle \\
& +(-1)^{\epsilon(\sigma^m)+\epsilon(\phi^D)\epsilon(\phi^A)}\left[\sigma^m_{,D}(\phi,B), \Gamma\frac{\overleftarrow{\delta}}{\delta\Sigma^n}\sigma^n_{,A}(\hat{\phi},B)\right\}\left(G''^{-1}\right)^{A\mathsf{a}}\left(\frac{\overrightarrow{\delta}}{\delta\Phi^{\mathsf{a}}}\frac{\overrightarrow{\delta}\Gamma}{\delta\phi^*_D}\right)\frac{\overrightarrow{\delta}\langle\langle\delta\Psi\rangle\rangle}{\delta\Sigma^m}, \qquad (30)\\
\delta_\Psi \Gamma \equiv\; & \hat{\omega}_\Gamma\langle\langle\delta\Psi\rangle\rangle, \quad \langle\langle\delta\Psi\rangle\rangle \equiv \delta\Psi(\hat{\phi},B), \qquad (31)
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{\phi}^A &= \phi^A + i\hbar\left(G''^{-1}\right)^{A\mathsf{a}}\frac{\overrightarrow{\delta}}{\delta\Phi^{\mathsf{a}}}, \quad \Phi^{\mathsf{a}} = \left(\phi^A, \Sigma^m\right), \\
G''_{\mathsf{ab}} &= \frac{\overrightarrow{\delta}}{\delta\Phi^{\mathsf{a}}}F_{\mathsf{b}}, \quad F_{\mathsf{a}} = \left(\Gamma\frac{\overleftarrow{\delta}}{\delta\phi^A} - \Gamma\frac{\overleftarrow{\delta}}{\delta\Sigma^n}\sigma^n_{,A}(\phi,B),\ \Gamma\frac{\overleftarrow{\delta}}{\delta\Sigma^m}\right).
\end{aligned}
$$

In (30), (31), $\hat{\omega}_\Gamma$ is a Legendre transform of $\hat{\omega}_W \equiv e^{-i/\hbar W}\hat{\omega}e^{i/\hbar W}$, with $\hat{\omega}$ given by (28), and therefore $\hat{\omega}_\Gamma$ inherits the nilpotency of $\hat{\omega}$, namely, $\hat{\omega}^2_\Gamma = \hat{\omega}^2_W = 0$. From (30) it follows, according to [95,96], that the generating functional $\Gamma\left(B, \phi, \Sigma, \phi^*\right)$ is gauge-independent, $\delta_\Psi \Gamma = 0$, on the extremals

$$
\frac{\delta\Gamma}{\delta\phi^A} = \frac{\delta\Gamma}{\delta\Sigma^m} = 0, \qquad (32)
$$

so that the effective action $\Gamma_{\text{eff}} = \Gamma_{\text{eff}}\left(B, \Sigma\right)$ with composite and background fields defined as

$$
\Gamma_{\text{eff}}\left(B, \Sigma\right) = \Gamma\left(B, \phi, \Sigma, \phi^*\right)\big|_{\phi=\phi^*=0} \qquad (33)
$$

is gauge-independent, $\delta_\Psi \Gamma_{\text{eff}} = 0$, on the extremals (32) restricted to the hypersurface $\phi^A = \phi^*_A = 0$ of vanishing quantum fields ϕ^A and antifields ϕ^*_A. This result on gauge dependence, in fact, holds true for general gauge theories, whose gauge algebra may be open, and whose gauge generators may be reducible, given an appropriate field-antifield structure. Besides, since the restricted generating functional $Z\left(B, J, L\right)$, as we set $\phi^* = 0$ in (24), satisfies the identity (23), which is also valid for the related functional $W\left(B, J, L\right)$ as we substitute $Z = \exp\left[(i/\hbar)W\right]$, the effective action $\Gamma_{\text{eff}}\left(B, \Sigma\right)$ defined by (29), (33) obeys an equality resulting from a Legendre transform of (23) written in terms of $W\left(B, J, L\right) = W\left(B, J, L, \phi^*\right)|_{\phi^*=0}$ and then reduced to $\phi^A = 0$,

$$
\int d^D x\, \left\{\left[D^{pq}_\mu\left(B\right)\xi^q\right]\frac{\overrightarrow{\delta}}{\delta B^p_\mu} + g f^{\{p\}\hat{r}q}\Sigma^{p_1\cdots\hat{r}\cdots p_k}_{\mu_1\cdots\mu_l}\xi^q\frac{\overrightarrow{\delta}}{\delta\Sigma^{p_1\cdots p_k}_{\mu_1\cdots\mu_l}}\right\}\Gamma_{\text{eff}}\left(B, \Sigma\right) = 0, \qquad (34)
$$

as a consequence of the following identity implied by the notation $f^{\{p\}\hat{r}q}$ in (13), due to the antisymmetry of the structure constants:

$$
f^{\{p\}\hat{r}q}\Sigma^{p_1\cdots p_k}_{\mu_1\cdots\mu_l}\frac{\overrightarrow{\delta}}{\delta\Sigma^{p_1\cdots\hat{r}\cdots p_k}_{\mu_1\cdots\mu_l}} = -f^{\{p\}\hat{r}q}\Sigma^{p_1\cdots\hat{r}\cdots p_k}_{\mu_1\cdots\mu_l}\frac{\overrightarrow{\delta}}{\delta\Sigma^{p_1\cdots p_k}_{\mu_1\cdots\mu_l}}. \qquad (35)
$$

The effective action is thereby invariant, $\delta_\xi \Gamma_{\text{eff}} = 0$, under the local transformations

$$
\delta_\xi B^p_\mu = D^{pq}_\mu\left(B\right)\xi^q, \quad \delta_\xi \Sigma^{p_1\cdots p_k}_{\mu_1\cdots\mu_l} = g f^{\{p\}\hat{r}q}\Sigma^{p_1\cdots\hat{r}\cdots p_k}_{\mu_1\cdots\mu_l}\xi^q, \qquad (36)
$$

which consist of the initial gauge transformations for the background field B^p_μ and of the local $SU(N)$ transformations for the fields $\Sigma^{p_1\cdots p_i}_{\mu_1\cdots\mu_j}$. Note that we assume the existence of a "deep" gauge-invariant regularization preserving the Ward identities (see, e.g., [14]), and expect the corresponding renormalized generating functionals to obey the same properties as the unrenormalized ones.

2.2. Background Field Interpretation

In order to interpret the generating functional $Z(B,J,L)$ in (21), note that the modified Faddeev–Popov action $S_{\mathrm{FP}}(\phi,B)$ defined as (18)–(20) is invariant under the finite local transformations

$$\left(A_\mu,b,\bar{c},c\right)\xrightarrow{V}V\left(A_\mu,b,\bar{c},c\right)V^{-1}\,,\quad B_\mu\xrightarrow{V}VB_\mu V^{-1}+g^{-1}V\partial_\mu V^{-1} \tag{37}$$

and acquires the form

$$S_{\mathrm{FP}}(\phi,B)=S_0(A+B)+\Psi(\phi,B)\overleftarrow{s}_{\mathrm{q}}\,, \tag{38}$$

where

$$\begin{aligned} S_0(A+B) &= S_0(A)|_{[\partial_\mu,\bullet]\to[D_\mu(B),\bullet]}=S_0(A)|_{D_\mu(A)\to D_\mu(A+B)}\,,\\ \Psi(\phi,B) &= \Psi(\phi)|_{[\partial_\mu,\bullet]\to[D_\mu(B),\bullet]}\,,\quad \overleftarrow{s}_{\mathrm{q}}=\overleftarrow{s}|_{D_\mu(A)\to D_\mu(A+B)}\,, \end{aligned} \tag{39}$$

and the variation $\overleftarrow{s}_{\mathrm{q}}$ reads explicitly

$$\begin{aligned} \left(A_\mu,b,\bar{c},c\right)\overleftarrow{s}_{\mathrm{q}} &= \left(\left[D_\mu(A+B),c\right],0,b,g/2\left[c,c\right]_+\right)\,,\\ \left(A^p_\mu,b^p,\bar{c}^p,c^p\right)\overleftarrow{s}_{\mathrm{q}} &= \left(D^{pq}_\mu(A+B)\,c^q,0,b^p,g/2f^{pqr}c^qc^r\right)\,. \end{aligned} \tag{40}$$

Thus, the Landau and Feynman gauges (12) are modified to the respective background gauges

$$\chi^p_{\mathrm{L}}(\phi,B)=D^{pq}_\mu(B)\,A^{q|\mu}\,,\quad \chi^p_{\mathrm{F}}(\phi,B)=b^p+D^{pq}_\mu(B)\,A^{q|\mu}. \tag{41}$$

In the background field method, a quantum action $S_{\mathrm{FP}}(\phi,B)$ given by (38), (40) is known as the Faddeev–Popov action with a background field B_μ. The quantum action $S_{\mathrm{FP}}(\phi,B)$ is invariant under global transformations of ϕ^A, with a nilpotent generator $\overleftarrow{s}_{\mathrm{q}}$ and an anticommuting parameter λ:

$$\delta_\lambda S_{\mathrm{FP}}(\phi,B)=0,\quad \delta\phi^A=\phi^A\overleftarrow{s}_{\mathrm{q}}\lambda,\quad \epsilon(\lambda)=1. \tag{42}$$

Infinitesimally, the local transformations (37) for the fields A_μ, B_μ are known as *background* transformations, $\delta_{\mathrm{b}}(A_\mu,B_\mu)$, and the transformations of A_μ, B_μ corresponding to the modified Slavnov variation $\overleftarrow{s}_{\mathrm{q}}$ in (38), (40), (42) are known as *quantum* transformations, $\delta_{\mathrm{q}}(A_\mu,B_\mu)$,

$$\begin{aligned} &\delta_{\mathrm{b}}A_\mu=g\left[A_\mu,T^p\xi^p\right]\,, && \delta_{\mathrm{b}}B_\mu=\left[D_\mu(B)\,,T^p\xi^p\right]\,,\\ &\delta_{\mathrm{q}}A_\mu=\left[D_\mu(A+B)\,,T^p\xi^p\right]\,, && \delta_{\mathrm{q}}B_\mu=0, \end{aligned}$$

while the classical action $S_0(A+B)$ is invariant under both types of such transformations. In this regard, the family of background gauges $\chi^p(\phi,B)=\tilde{\chi}^p(A,B)+(\alpha/2)\,b^p$, parameterized by $\alpha\neq0$ and defined as (39),

$$\chi^p(A,B)=\chi^p(A)|_{[\partial_\mu,\bullet]\to[D_\mu(B),\bullet]}\,,$$

with the Nakanishi–Lautrup fields b^p integrated out of (21) by the shift $b^p \to b^p + \alpha^{-1}\tilde{\chi}$ at the vanishing sources, $J = L = 0$, transforms the vacuum functional $Z(B)$ to the representation (for the convenience of Section 3, we denote $A \equiv Q$),

$$\begin{aligned} Z(B) &= \int dQ\, d\bar{c}\, dc\, \exp\left\{\frac{i}{\hbar}\left[S_0(Q+B) + S_{\mathrm{gf}}(Q,B) + S_{\mathrm{gh}}(Q,B;\bar{c},c)\right]\right\}, \\ S_{\mathrm{gf}}(Q,B) &= -\frac{1}{2\alpha}\int d^Dx\, \tilde{\chi}^p\tilde{\chi}^p, \quad S_{\mathrm{gh}}(Q,B) = \int d^Dx\, \bar{c}^p\, \delta_{\mathrm{q}}\tilde{\chi}^p|_{\xi\to c}\,, \end{aligned} \tag{43}$$

where the gauge-fixing term $S_{\mathrm{gf}} = S_{\mathrm{gf}}(Q,B)$ is invariant, $\delta_{\mathrm{b}} S_{\mathrm{gf}} = 0$ under the background transformations, due to $\delta_{\mathrm{b}}\tilde{\chi}^p = gf^{prq}\tilde{\chi}^r\xi^q$, which can be employed as a definition for the quantum action in background gauges $\tilde{\chi}^p(Q,B)$ depending on the quantum and background fields, with the related background and quantum transformations (see also [5]),

$$\begin{array}{ll} \delta_{\mathrm{b}} B^p_\mu = D^{pq}_\mu(B)\,\xi^q, & \delta_{\mathrm{b}} Q^p_\mu = gf^{prq}Q^r_\mu\xi^q, \\ \delta_{\mathrm{q}} B^p_\mu = 0, & \delta_{\mathrm{q}} Q^p_\mu = D^{pq}_\mu(Q+B)\,\xi^q. \end{array} \tag{44}$$

The quantum action and the integrand of $Z(B)$ in (43) are invariant under the residual local transformations (37),

$$\left(Q_\mu,\bar{c},c\right) \xrightarrow{V} V\left(Q_\mu,\bar{c},c\right)V^{-1}, \quad B_\mu \xrightarrow{V} VB_\mu V^{-1} + g^{-1}V\partial_\mu V^{-1}, \tag{45}$$

corresponding, at the infinitesimal level $\delta_\xi\left(B_\mu, Q_\mu, \bar{c}, c\right)$, to the background transformations δ_{b} along with some compensating local transformations of the ghost fields:

$$\delta_\xi\left(B^p_\mu,\, Q^p_\mu,\, \bar{c}^p,\, c^p\right) = \left(\delta_{\mathrm{b}} B^p_\mu,\, \delta_{\mathrm{b}} Q^p_\mu,\, gf^{prq}\bar{c}^r\xi^q,\, gf^{prq}c^r\xi^q\right).$$

In view of the above, we interpret $Z(B,J,L)$ in (6), (18), (19), (21) as a generating functional of Green's functions for Yang–Mills theories with composite fields in the background field method, or as a generating functional of Green's functions with composite and background fields in such theories. As has been shown, this interpretation provides for such theories the existence of a corresponding gauge-invariant (36) effective action, being gauge-independent on the extremals (32) for the extended generating functional of vertex functions (29) when these extremals are restricted to the hypersurface of vanishing antifields and quantum fields.

3. Volovich–Katanaev Model

Let us examine the model of two-dimensional gravity with dynamical torsion described in terms of a zweibein e^i_μ and a Lorentz connection ω_μ by the action [50]

$$S_0(e,\omega) = \int d^2x\, e\left(\frac{1}{16\alpha}R_{\mu\nu}{}^{ij}R^{\mu\nu}{}_{ij} - \frac{1}{8\beta}T_{\mu\nu}{}^iT^{\mu\nu}{}_i - \gamma\right), \tag{46}$$

where α, β, γ are constant parameters, and the following notation is used:

$$\begin{aligned} e &= \det e^i_\mu, \\ R_{\mu\nu}{}^{ij} &= \varepsilon^{ij}R_{\mu\nu}, \quad R_{\mu\nu} = \partial_\mu\omega_\nu - (\mu \leftrightarrow \nu), \\ T_{\mu\nu}{}^i &= \partial_\mu e^i_\nu + \varepsilon^{ij}\omega_\mu e_{\nu j} - (\mu \leftrightarrow \nu). \end{aligned} \tag{47}$$

Here, the indices of quantities transforming under the local Lorentz group are denoted by Latin characters, $i, j, k\ldots$ $(i = 0,1)$, with ε^{ij} being a constant antisymmetric second-rank pseudo-tensor subject to the normalization $\varepsilon^{01} = 1$. The indices of quantities transforming as (pseudo-)tensors under the general coordinate transformations are labelled by Greek characters, $\lambda, \mu, \nu\ldots$ $(\lambda = 0,1)$. The Latin indices are raised and lowered by the Minkowski metric η_{ij} $(+,-)$, and the Greek indices, by the metric tensor $g_{\mu\nu} = \eta_{ij}e^i_\mu e^j_\nu$. The action (46) is invariant under the local Lorentz transformations, $e^i_\mu \to e'^i_\mu$, $\omega_\mu \to \omega'_\mu$,

$$\begin{aligned} e'^i_\mu &= (\Lambda e_\mu)^i, \\ (\Omega'_\mu)^i_j &= (\Lambda\Omega_\mu\Lambda^{-1})^i_j + (\Lambda\partial_\mu\Lambda^{-1})^i_j, \quad (\Omega_\mu)^i_j \equiv \varepsilon^{ik}\eta_{kj}\omega_\mu, \end{aligned} \tag{48}$$

or, infinitesimally, with a parameter ζ,

$$\delta_\zeta e^i_\mu = \varepsilon^{ij}e_{\mu j}\zeta, \quad \delta_\zeta\omega_\mu = -\partial_\mu\zeta, \tag{49}$$

as well as under the general coordinate transformations, $x \to x' = x'(x)$,

$$\begin{aligned} e^i_\mu &\to e'^i_\mu(x') = \frac{\partial x^\lambda}{\partial x'^\mu}e^i_\lambda(x), \\ \omega_\mu &\to \omega'_\mu(x') = \frac{\partial x^\lambda}{\partial x'^\mu}\omega_\lambda(x), \end{aligned} \tag{50}$$

implying the infinitesimal field variations, with some parameters ξ^μ,

$$\delta_\xi e^i_\mu = e^i_\nu\partial_\mu\xi^\nu + (\partial_\nu e^i_\mu)\xi^\nu, \quad \delta_\xi\omega_\mu = \omega_\nu\partial_\mu\xi^\nu + (\partial_\nu\omega_\mu)\xi^\nu. \tag{51}$$

The given model can be quantized using the Faddeev–Popov method, since the gauge transformations (49), (51) form a closed algebra:

$$\begin{aligned} [\delta_{\zeta(1)},\, \delta_{\zeta(2)}] &= 0, \\ [\delta_{\xi(1)},\, \delta_{\xi(2)}] &= \delta_{\xi(1,2)}, \\ [\delta_\zeta,\, \delta_\xi] &= \delta_{\zeta'}, \end{aligned} \tag{52}$$

where

$$\xi^\mu{}_{(1,2)} = \xi^\nu{}_{(1)}\partial_\nu\xi^\mu{}_{(2)} - (\partial_\nu\xi^\mu{}_{(1)})\xi^\nu{}_{(2)}, \quad \zeta' = (\partial_\mu\zeta)\xi^\mu.$$

In [75], a quantum theory for the gauge model (46), (49), (51), (52) has been presented according to the background field method, by using an ansatz for the vacuum functional which corresponds to $Z\,(B)$ of (43) for a Yang–Mills theory with the Nakanishi–Lautrup fields removes using a background gauge, see also [5]. Thus, the initial classical fields are ascribed the sets of quantum Q and background B fields, to be denoted by $Q = (q^i_\mu, q_\mu)$ and $B = (e^i_\mu, \omega_\mu)$, with the expressions for $g_{\mu\nu}$, as well as e, $(\Omega_\mu)^i_j$ in (47), (48), being related only to the background fields. One also associates the gauge transformations (49), (51) with two forms of infinitesimal transformations, background δ_{b} and quantum δ_{q}, being constructed by analogy with (44), so that the action $S_0(Q+B)$ in (46) remains invariant under both of these forms of transformations,

$$\begin{aligned} &\delta_{\mathrm{b}}e^i_\mu = \varepsilon^{ij}e_{\mu j}\zeta + e^i_\nu\partial_\mu\xi^\nu + (\partial_\nu e^i_\mu)\xi^\nu, \qquad \delta_{\mathrm{b}}q^i_\mu = \varepsilon^{ij}q_{\mu j}\zeta + q^i_\nu\partial_\mu\xi^\nu + (\partial_\nu q^a_\mu)\xi^\nu, \\ &\delta_{\mathrm{b}}\omega_\mu = -\partial_\mu\zeta + \omega_\nu\partial_\mu\xi^\nu + (\partial_\nu\omega_\mu)\xi^\nu, \quad \delta_{\mathrm{b}}q_\mu = q_\nu\partial_\mu\xi^\nu + (\partial_\nu q_\mu)\xi^\nu, \end{aligned} \tag{53}$$

$$\begin{aligned} &\delta_{\mathrm{q}}e^i_\mu = 0, \quad \delta_{\mathrm{q}}q^i_\mu = \varepsilon^{ij}(e_{\mu j}+q_{\mu j})\zeta + (e^i_\nu + q^i_\nu)\partial_\mu\xi^\nu + (\partial_\nu e^i_\mu + \partial_\nu q^i_\mu)\xi^\nu, \\ &\delta_{\mathrm{q}}\omega_\mu = 0, \quad \delta_{\mathrm{q}}q_\mu = -\partial_\mu\zeta + (\omega_\nu + q_\nu)\partial_\mu\xi^\nu + (\partial_\nu\omega_\mu + \partial_\nu q_\mu)\xi^\nu. \end{aligned} \tag{54}$$

One next introduces the Faddeev–Popov ghosts $(\bar{c}, c)$, $(\bar{c}^\mu, c^\mu)$ according to the respective number of gauge parameters ζ, ξ^μ in (49), (51),

$$\epsilon(\bar{c},\bar{c}^\mu,c,c^\mu) = 1, \;\; \mathrm{gh}\,(c,c^\mu) = -\mathrm{gh}(\bar{c},\bar{c}^\mu) = 1,$$

and considers an analogue [5] of the generating functional of Green's functions, $(\bar{c}, \bar{c}^{\mu}, c, c^{\mu}) = (\overline{C}, C)$,

$$Z(B,J) = \int dQ\, d\overline{C}\, dC\ \exp\left\{\frac{i}{\hbar}\left[S_0(Q+B) + S_{\rm gf}(Q,B) + S_{\rm gh}(Q,B;\overline{C},C) + JQ\right]\right\}, \tag{55}$$

where $J = (J_i^{\mu}, J^{\mu})$ are sources to the quantum fields $Q = (q_{\mu}^{i}, q_{\mu})$, and the functional $S_{\rm gf} = S_{\rm gf}(Q,B)$ is constructed using some background gauge functions χ, χ_{μ} (for the respective gauge parameters ζ, ξ^{μ}) as one demands its invariance, $\delta_{\rm b} S_{\rm gf} = 0$, under the background transformations, whereas the ghost term $S_{\rm gh} = S_{\rm gh}(Q,B;\overline{C},C)$ is then determined by

$$S_{\rm gh} = \int d^2x\ \left(\bar{c}\delta_{\rm q}\chi + \bar{c}^{\mu}\delta_{\rm q}\chi_{\mu}\right)\big|_{(\zeta,\xi^{\mu})\to(c,c^{\mu})}\,. \tag{56}$$

Let us choose the gauge functions $\chi = \chi(Q,B)$ and $\chi_{\mu} = \chi_{\mu}(Q,B)$ to be linear in the quantum fields $Q = (q_{\mu}^{i}, q_{\mu})$ as in [75],

$$\chi = e g^{\mu\nu}\nabla_{\mu} q_{\nu}\ ,\quad \chi_{\mu} = e g^{\lambda\nu} e_{\mu i}\nabla_{\lambda} q_{\nu}^{i}\ ,\quad g^{\mu\lambda} g_{\lambda\nu} = \delta^{\mu}_{\nu}, \tag{57}$$

where $e = \det e_{\mu}^{i}$, $g_{\mu\nu} = \eta_{ij} e_{\mu}^{i} e_{\nu}^{j}$, and ∇_{μ} is a covariant derivative acting on an arbitrary (psedo-)tensor field $T^{\nu_1\ldots\nu_l\ j_1\ldots j_n}_{\mu_1\ldots\mu_k i_1\ldots i_m}$, in terms of the connection $(\Omega_{\mu})^{i}_{j} = \varepsilon^{ik}\eta_{kj}\omega_{\mu}$ and the Christoffel symbols $\Gamma^{\lambda}_{\mu\nu}$

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}\left(\partial_{\nu} g_{\mu\sigma} + \partial_{\mu} g_{\nu\sigma} - \partial_{\sigma} g_{\mu\nu}\right), \tag{58}$$

by the rule

$$\begin{aligned} \nabla_{\mu} T^{\nu_1\ldots\nu_l\ j_1\ldots j_n}_{\mu_1\ldots\mu_k i_1\ldots i_m} &= \partial_{\mu} T^{\nu_1\ldots\nu_l\ j_1\ldots j_n}_{\mu_1\ldots\mu_k i_1\ldots i_m} - \Gamma^{\hat{\lambda}}_{\mu\{\mu\}} T^{\nu_1\ldots\nu_l\qquad j_1\ldots j_n}_{\mu_1\ldots\hat{\lambda}\ldots\mu_k i_1\ldots i_m} + \Gamma^{\{\nu\}}_{\mu\hat{\lambda}} T^{\nu_1\ldots\hat{\lambda}\ldots\nu_l j_1\ldots j_n}_{\mu_1\ldots\mu_k\qquad i_1\ldots i_m} \\ &\quad + (\Omega_{\mu})^{\{j\}}_{\hat{p}} T^{\nu_1\ldots\nu_l\ j_1\ldots\hat{p}\ldots j_n}_{\mu_1\ldots\mu_k i_1\ldots i_m} - (\Omega_{\mu})^{\hat{p}}_{\{i\}} T^{\nu_1\ldots\nu_l\ j_1\ldots j_n}_{\mu_1\ldots\mu_k i_1\ldots\hat{p}\ldots i_m}\,, \end{aligned} \tag{59}$$

with the shorthand notation, $p_k \in \{p_1, \ldots, p_m\}$, $\nu_k \in \{\nu_1, \ldots, \nu_n\}$,

$$\begin{aligned} F^{\hat{p}}_{\{i\}} T^{\mu_1\cdots\mu_n}_{i_1\cdots\hat{p}\cdots i_m} &= \sum_{p_k} F^{p_k}_{i_k} T^{\mu_1\cdots\mu_n}_{i_1\cdots p_k\cdots i_m}\,,\quad G^{\{\mu\}}_{\hat{\nu}} T^{\mu_1\cdots\hat{\nu}\cdots\mu_n}_{i_1\cdots i_m} = \sum_{\nu_k} G^{\mu_k}_{\nu_k} T^{\mu_1\cdots\nu_k\cdots\mu_n}_{i_1\cdots i_m}\,, \\ F^{\{i\}}_{\hat{p}} T^{i_1\cdots\hat{p}\cdots i_m}_{\mu_1\cdots\mu_n} &= \sum_{p_k} F^{i_k}_{p_k} T^{i_1\cdots p_k\cdots i_m}_{\mu_1\cdots\mu_n}\,,\quad G^{\hat{\nu}}_{\{\mu\}} T^{i_1\cdots i_m}_{\mu_1\cdots\hat{\nu}\cdots\mu_n} = \sum_{\nu_k} G^{\nu_k}_{\mu_k} T^{i_1\cdots i_m}_{\mu_1\cdots\nu_k\cdots\mu_n}\,, \end{aligned} \tag{60}$$

so that the covariant derivative ∇_{μ} features the standard properties (with F, G being arbitrary (psedo-)tensor fields)

$$\nabla_{\sigma} g_{\mu\nu} = \nabla_{\sigma} g^{\mu\nu} = 0,\quad \nabla_{\mu}(FG) = F\nabla_{\mu} G + (\nabla_{\mu} F) G. \tag{61}$$

The above objects make it possible to construct the gauge-fixing term $S_{\rm gf}$ as a functional quadratic in the background gauges χ, χ^{μ} (including some numeric parameters a, b)

$$S_{\rm gf} = \frac{1}{2}\int d^2x\, e^{-1}\left(a\chi^2 + b\chi_{\mu}\chi^{\mu}\right) \tag{62}$$

and invariant under the local Lorentz transformations

$$\begin{aligned} e'^{i}_{\mu} &= (\Lambda e_{\mu})^{i},\ \ q'^{i}_{\mu} = (\Lambda q_{\mu})^{i}, \\ (\Omega'_{\mu})^{i}_{b} &= (\Lambda\Omega_{\mu}\Lambda^{-1})^{i}_{j} + (\Lambda\partial_{\mu}\Lambda^{-1})^{i}_{j},\ \ q'_{\mu} = q_{\mu}, \end{aligned} \tag{63}$$

as well as under the general coordinate transformations, $x \to x' = x'(x)$,

$$e'^{i}_{\mu}(x') = \frac{\partial x^{\lambda}}{\partial x'^{\mu}} e^{i}_{\lambda}(x), \quad \omega'_{\mu}(x') = \frac{\partial x^{\lambda}}{\partial x'^{\mu}} \omega_{\lambda}(x),$$
$$q'^{i}_{\mu}(x') = \frac{\partial x^{\lambda}}{\partial x'^{\mu}} q^{i}_{\lambda}(x), \quad q'_{\mu}(x') = \frac{\partial x^{\lambda}}{\partial x'^{\mu}} q_{\lambda}(x). \tag{64}$$

The field transformations (63) and (64) coincide infinitesimally with the background transformations (53), which thereby implies the fulfillment of $\delta_{\mathrm{b}} S_{\mathrm{gf}} = 0$. Having this in mind, as well as the fact that the non-zero quantum transformations (54), with account taken of (59), can be represented as

$$\begin{aligned} \delta_{\mathrm{q}} q^{i}_{\mu} &= \varepsilon^{ij}(e_{\mu j} + q_{\mu j})c + (e^{i}_{\nu} + q^{i}_{\nu})\nabla_{\mu} c^{\nu} + (\nabla_{\nu} e^{i}_{\mu} + \nabla_{\nu} q^{i}_{\mu})c^{\nu} - \varepsilon^{ij}\omega_{\nu}(e_{\mu j} + q_{\mu b})c^{\nu}, \\ \delta_{\mathrm{q}} q_{\mu} &= -\nabla_{\mu} c + (\omega_{\nu} + q_{\nu})\nabla_{\mu} c^{\nu} + (\nabla_{\nu}\omega_{\mu} + \nabla_{\nu} q_{\mu})c^{\nu} \end{aligned}$$

under $\zeta \to c$, $\xi^{\mu} \to c^{\mu}$, the ghost contribution S_{gh} in (56) reads

$$\begin{aligned} S_{\mathrm{gh}} &= \int d^2x\, e\, \Big\{ -\bar{c}\nabla_{\mu}\nabla^{\mu} c + \bar{c}\nabla^{\mu}[(\nabla_{\nu}\omega_{\mu} + \nabla_{\nu} q_{\mu})c^{\nu} + (\omega_{\nu} + q_{\nu})\nabla_{\mu} c^{\nu}] \\ &\quad + \varepsilon^{ij}\bar{c}^{\mu} e_{\mu i}\nabla^{\nu}[(e_{\nu j} + q_{\nu j})(c - \omega_{\lambda} c^{\lambda})] \\ &\quad + \bar{c}^{\mu} e_{\mu i}\nabla^{\nu}[(\nabla_{\lambda} e^{i}_{\nu} + \nabla_{\lambda} q^{i}_{\nu})c^{\lambda} + (e^{i}_{\lambda} + q^{i}_{\lambda})\nabla_{\nu} c^{\lambda}] \Big\}. \end{aligned} \tag{65}$$

The quantum action in (55) represented by (46), (57), (62), (65) turns out to be invariant (which is also valid for the integrand in (55) once the sources are put to zero $J = 0$ under the assumption $\delta(x) = \partial_{\mu}\delta(x)|_{x=0} = 0$) with respect to the background transformations (53) accompanied by a set of compensating local transformations for the ghost fields [75],

$$\begin{aligned} \delta\bar{c} &= (\partial_{\mu}\bar{c})\xi^{\mu}, & \delta\bar{c}^{\mu} &= -\bar{c}^{\nu}\partial_{\nu}\xi^{\mu} + (\partial_{\nu}\bar{c}^{\mu})\xi^{\nu}, \\ \delta c &= -c^{\mu}\partial_{\mu}\zeta + (\partial_{\mu} c)\xi^{\mu}, & \delta c^{\mu} &= -c^{\nu}\partial_{\nu}\xi^{\mu} + (\partial_{\nu} c^{\mu})\xi^{\nu}. \end{aligned} \tag{66}$$

From (53) and (66), it follows that the generating functional $Z(B, J)$ in (55) is invariant [75] under the initial gauge transformations (49), (51) of the background fields $B = (e^{i}_{\mu}, \omega_{\mu})$ along with the following local transformations of the sources $J = (J^{\mu}_{i}, J^{\mu})$:

$$\delta J^{\mu}_{i} = -\varepsilon^{k}_{i} J^{\mu}_{k}\zeta - J^{\nu}_{i}\partial_{\nu}\xi^{\mu} + \partial_{\nu}(J^{\mu}_{i}\xi^{\nu}), \quad \delta J^{\mu} = -J^{\nu}\partial_{\nu}\xi^{\mu} + \partial_{\nu}(J^{\mu}\xi^{\nu}), \quad \varepsilon^{i}_{j} \equiv \varepsilon^{ik}\eta_{kj}\,, \tag{67}$$

which implies the property

$$\delta(JQ) = \int d^2x\, \partial_{\mu} F^{\mu}\,, \quad F^{\mu}(x) \equiv \left(J^{\nu}_{i} q^{i}_{\nu} + J^{\nu} q_{\nu}\right)\xi^{\mu}$$

for the source term JQ in (55) and extends an infinitesimal tensor transformation law for the sources J^{μ}_{i} and J^{μ} by adding the contributions $J^{\mu}_{i}\partial_{\nu}\xi^{\nu}$ and $J^{\mu}\partial_{\nu}\xi^{\nu}$, indeed,

$$\delta J^{\mu}_{i} = \left[-\varepsilon^{k}_{i} J^{\mu}_{k}\zeta - J^{\nu}_{i}\partial_{\nu}\xi^{\mu} + (\partial_{\nu} J^{\mu}_{i})\xi^{\nu}\right] + J^{\mu}_{i}\partial_{\nu}\xi^{\nu}, \quad \delta J^{\mu} = [-J^{\nu}\partial_{\nu}\xi^{\mu} + (\partial_{\nu} J^{\mu})\xi^{\nu}] + J^{\mu}\partial_{\nu}\xi^{\nu}.$$

Given the invariance of $Z(B, J) = \exp\{(i/h)\, W(B, J)\}$ under (49), (51), (67), one establishes the invariance [75] of a functional $\Gamma = \Gamma(B, Q)$ defined as

$$\Gamma(B, Q) = W(B, J) - JQ, \quad Q = \frac{\delta W}{\delta J}, \quad J = -\frac{\delta\Gamma}{\delta Q}, \quad Q = (q^{i}_{\mu}, q_{\mu}) \tag{68}$$

under the background transformations (53) of the fields B and Q,

$$\delta_{\mathrm{b}}\Gamma = \int d^2x \left[\frac{\delta\Gamma}{\delta B\,(x)}\delta_{\mathrm{b}}B\,(x) + \frac{\delta\Gamma}{\delta Q\,(x)}\delta_{\mathrm{b}}Q\,(x)\right] = 0,$$

which entails that the effective action $\Gamma_{\mathrm{eff}}(B)$ of the background field method,

$$\Gamma_{\mathrm{eff}}(B) = \Gamma(B,Q)|_{Q=0}\;,$$

proves to be invariant, $\delta_{(\zeta,\xi)}\Gamma_{\mathrm{eff}} = 0$, under the gauge transformations (49), (51) of the background fields $B = (e^i_\mu, \omega_\mu)$.

Composite Field Introduction

Let us extend the generating functional (55) proposed in [75] to a functional $Z(B,J,L)$, as we introduce a set of local background-dependent composite fields $\sigma^m\,(Q,B)$ with sources L_m,

$$\sigma^m\,(Q,B) = \sigma^{i_1\cdots i_k}_{\mu_1\cdots\mu_l}\,(Q\,(x)\,,B\,(x))\,,\;\; L_m = L^{\mu_1\cdots\mu_l}_{i_1\cdots i_k}\,(x)\,,\;\; m = (x, i_1, \ldots, i_k, \mu_1, \ldots, \mu_l)\,,$$

and denote the entire quantum action by $S(Q,B;\overline{C},C)$,

$$Z(B,J,L) = \int dQ\,d\overline{C}\,dC\,\exp\left\{\frac{i}{\hbar}\left[S(Q,B;\overline{C},C) + JQ + L_m\sigma^m\,(Q,B)\right]\right\}. \tag{69}$$

In this setting, we demand that the extended functional $Z(B,J,L)$ inherit the local symmetry of $Z(B,J)$ under the transformations (49), (51), (67) of the background fields $B = (e^i_\mu, \omega_\mu)$ and the sources $J = (J^\mu_i, J^\mu)$. To do so, we impose the condition that the composite fields $\sigma^{i_1\cdots i_m}_{\mu_1\cdots\mu_n}(x) = \sigma^{i_1\cdots i_m}_{\mu_1\cdots\mu_n}\,(Q(x),B(x))$ should behave as tensors with respect to the Lorentz (63) and general coordinate (64) transformations of the quantum and background fields,

$$\begin{aligned} \sigma'^{i_1\cdots i_m}_{\mu_1\cdots\mu_n}(x) &= \Lambda^{i_1}_{j_1}\cdots\Lambda^{i_m}_{j_m}\sigma^{j_1\cdots j_m}_{\mu_1\cdots\mu_n}(x)\,, \\ \sigma'^{i_1\cdots i_m}_{\mu_1\cdots\mu_n}(x') &= \frac{\partial x^{\nu_1}}{\partial x'^{\mu_1}}\cdots\frac{\partial x^{\nu_n}}{\partial x'^{\mu_n}}\sigma^{i_1\cdots i_m}_{\nu_1\cdots\nu_n}(x)\,,\quad x' = x'(x)\,. \end{aligned} \tag{70}$$

Generally, a composite field $\sigma^{i_1\cdots i_m}_{\mu_1\cdots\mu_n}\,(Q,B)$ transforming as (70) is multiplicative with respect to the quantum fields $Q = (q^i_\mu, q_\mu)$ and the background field objects e^i_μ, $g_{\mu\nu}$, $R_{\mu\nu}{}^{ij}$, $T_{\mu\nu}{}^i$, see (47). Such a field may also include some background covariant derivatives ∇_μ acting as (59), (58) and having the properties (61). Given this, any composite field limited by $\sigma^{i_1\cdots i_m}_{\mu_1\cdots\mu_n}\,(Q,0) \neq 0$ is allowed to contain the background fields B only via the covariant derivative ∇_μ, defined in terms of $\Gamma^\lambda_{\mu\nu}$, $(\Omega_\mu)^i_j$ and acting on q^i_μ, q_μ as follows:

$$\nabla_\mu q^i_\nu = \partial_\mu q^i_\nu - \Gamma^\lambda_{\mu\nu}q^i_\lambda + (\Omega_\mu)^i_j q^j_\nu\,,\quad \nabla_\mu q_\nu = \partial_\mu q_\nu - \Gamma^\lambda_{\mu\nu}q_\lambda\,.$$

The transformations (70) correspond infinitesimally to local tensor variations $\delta\sigma^{i_1\cdots i_m}_{\mu_1\cdots\mu_n}$ with parameters ζ and ξ^μ,

$$\delta\sigma^{i_1\cdots i_m}_{\mu_1\cdots\mu_n} = \varepsilon^{\{i\}}_{\hat{p}}\,\sigma^{i_1\cdots\hat{p}\cdots i_m}_{\mu_1\cdots\mu_n}\zeta + \sigma^{i_1\cdots i_m}_{\mu_1\cdots\hat{\nu}\cdots\mu_n}\partial_{\{\mu\}}\xi^{\hat{\nu}} + (\partial_\nu\sigma^{i_1\cdots i_m}_{\mu_1\cdots\mu_n})\xi^\nu. \tag{71}$$

Under these assumptions and the invariance of the vacuum functional in (69) with respect to the background transformations (53), accompanied by the compensating local transformations (66) of the ghost fields, the modified generating functional $Z(B,J,L)$ in (69) is invariant under the initial gauge transformations (49), (51) of the background fields $B = (e^i_\mu, \omega_\mu)$ along with the local transformations (67) of the sources $J = \left(J^\mu_i, J^\mu\right)$ and some local transformations of the sources $L^{\mu_1\cdots\mu_n}_{i_1\cdots i_m}$,

$$\delta L^{\mu_1\cdots\mu_n}_{i_1\cdots i_m} = -\varepsilon^{\hat{p}}_{\{i\}} L^{\mu_1\cdots\mu_n}_{i_1\cdots\hat{p}\cdots i_m}\zeta - L^{\mu_1\cdots\hat{\nu}\cdots\mu_n}_{i_1\cdots i_m}\partial_{\hat{\nu}}\xi^{\{\mu\}} + \partial_\nu(L^{\mu_1\cdots\mu_n}_{i_1\cdots i_m}\xi^\nu)\,, \tag{72}$$

which diverges from the infinitesimal tensor transformation law by the contribution $L^{\mu_1\cdots\mu_n}_{i_1\cdots i_m}\partial_\nu\xi^\nu$,

$$\delta L^{\mu_1\cdots\mu_n}_{i_1\cdots i_m} = \left[-\varepsilon^{\hat{p}}_{\{i\}} L^{\mu_1\cdots\mu_n}_{i_1\cdots\hat{p}\cdots i_m}\zeta - L^{\mu_1\cdots\hat{\nu}\cdots\mu_n}_{i_1\cdots i_m}\partial_{\hat{\nu}}\xi^{\{\mu\}} + (\partial_\nu L^{\mu_1\cdots\mu_n}_{i_1\cdots i_m})\xi^\nu\right] + L^{\mu_1\cdots\mu_n}_{i_1\cdots i_m}\partial_\nu\xi^\nu\,,$$

and provides for the source term $L_m\sigma^m$ in (69) the transformation property

$$\delta\left(L_m\sigma^m\right) = \int d^2x\,\partial_\mu G^\mu\,,\quad G^\mu \equiv L^{\nu_1\cdots\nu_n}_{i_1\cdots i_m}\sigma^{i_1\cdots i_m}_{\nu_1\cdots\nu_n}\xi^\mu\,.$$

The invariance of $Z(B, J, L)$ under (49), (51), (67), (72) can be represented as

$$\begin{aligned}&\int d^2x \Bigg\{\left[\varepsilon^{ij}e_{\mu j}\zeta + e^i_\nu(\partial_\mu\xi^\nu) + (\partial_\nu e^i_\mu)\xi^\nu\right]\frac{\delta}{\delta e^i_\mu} + \left[-\partial_\mu\zeta + \omega_\nu(\partial_\mu\xi^\nu) + (\partial_\nu\omega_\mu)\xi^\nu\right]\frac{\delta}{\delta\omega_\mu}\\
&-\left[\varepsilon^k_i J^\mu_k\zeta + J^\nu_i(\partial_\nu\xi^\mu) - \partial_\nu(J^\mu_i\xi^\nu)\right]\frac{\delta}{\delta J^\mu_i} - \left[J^\nu(\partial_\nu\xi^\mu) - \partial_\nu(J^\mu\xi^\nu)\right]\frac{\delta}{\delta J^\mu}\\
&-\left[\varepsilon^{\hat{p}}_{\{i\}} L^{\mu_1\cdots\mu_n}_{i_1\cdots\hat{p}\cdots i_m}\zeta + L^{\mu_1\cdots\hat{\nu}\cdots\mu_n}_{i_1\cdots i_m}(\partial_{\hat{\nu}}\xi^{\{\mu\}}) - \partial_\nu(L^{\mu_1\cdots\mu_n}_{i_1\cdots i_m}\xi^\nu)\right]\frac{\delta}{\delta L^{\mu_1\cdots\mu_n}_{i_1\cdots i_m}}\Bigg\} Z(B, J, L) = 0.\end{aligned} \tag{73}$$

Let us introduce a functional $\Gamma = \Gamma(B, Q, \Sigma)$ given by the double Legendre transformation

$$\Gamma\left(B, Q, \Sigma\right) = W\left(B, J, L\right) - JQ - L_m\left[\sigma^m\left(Q, B\right) + \Sigma^m\right],\;\; W = (\hbar/i)\ln Z \tag{74}$$

in terms of additional fields $\Sigma^m = \Sigma^{i_1\cdots i_m}_{\mu_1\cdots\mu_n}\left(x\right)$,

$$Q = \frac{\delta W}{\delta J},\;\; \Sigma^m = \frac{\delta W}{\delta L_m} - \sigma^m\left(\frac{\delta W}{\delta J}, B\right),\;\; -J = \frac{\delta\Gamma}{\delta Q} + L_m\frac{\delta\sigma^m}{\delta Q},\;\; -L_m = \frac{\delta\Gamma}{\delta\Sigma^m}\,.$$

The effective action $\Gamma_{\mathrm{eff}}\left(B, \Sigma\right)$ with composite and background fields,

$$\Gamma_{\mathrm{eff}}\left(B, \Sigma\right) = \Gamma\left(B, Q, \Sigma\right)|_{Q=0}\,, \tag{75}$$

then satisfies the identity

$$\begin{aligned}&\int d^2x \Bigg\{\left[\varepsilon^{ij}e_{\mu j}\zeta + e^i_\nu(\partial_\mu\xi^\nu) + (\partial_\nu e^i_\mu)\xi^\nu\right]\frac{\delta}{\delta e^i_\mu} + \left[-\partial_\mu\zeta + \omega_\nu(\partial_\mu\xi^\nu) + (\partial_\nu\omega_\mu)\xi^\nu\right]\frac{\delta}{\delta\omega_\mu}\\
&+\left[\varepsilon^{\hat{p}}_{\{i\}}\Sigma^{i_1\cdots i_m}_{\mu_1\cdots\mu_n}\zeta\frac{\delta}{\delta\Sigma^{i_1\cdots\hat{p}\cdots i_m}_{\mu_1\cdots\mu_n}} + \Sigma^{i_1\cdots i_m}_{\mu_1\cdots\mu_n}(\partial_{\hat{\nu}}\xi^{\{\mu\}})\frac{\delta}{\delta\Sigma^{i_1\cdots i_m}_{\mu_1\cdots\hat{\nu}\cdots\mu_n}} + (\partial_\nu\Sigma^{i_1\cdots i_m}_{\mu_1\cdots\mu_n})\xi^\nu\frac{\delta}{\delta\Sigma^{i_1\cdots i_m}_{\mu_1\cdots\mu_n}}\right]\Bigg\}\Gamma_{\mathrm{eff}}\left(B, \Sigma\right) = 0,\end{aligned}$$

due to integration by parts in (73), as written in terms of $\Gamma(B, Q, \Sigma)$, which is completed by setting $Q = \sigma^m\left(0, B\right) = 0$. Using the latter identity for $\Gamma_{\mathrm{eff}}\left(B, \Sigma\right)$ and the following consequences, cf. (35), of the notation (60),

$$\begin{aligned}\varepsilon^{\hat{p}}_{\{i\}}\Sigma^{i_1\cdots i_m}_{\mu_1\cdots\mu_n}\frac{\delta}{\delta\Sigma^{i_1\cdots\hat{p}\cdots i_m}_{\mu_1\cdots\mu_n}} &= \varepsilon^{\{i\}}_{\hat{p}}\Sigma^{i_1\cdots\hat{p}\cdots i_m}_{\mu_1\cdots\mu_n}\frac{\delta}{\delta\Sigma^{i_1\cdots i_m}_{\mu_1\cdots\mu_n}}\,,\\
\Sigma^{i_1\cdots i_m}_{\mu_1\cdots\mu_n}(\partial_{\hat{\nu}}\xi^{\{\mu\}})\frac{\delta}{\delta\Sigma^{i_1\cdots i_m}_{\mu_1\cdots\hat{\nu}\cdots\mu_n}} &= \Sigma^{i_1\cdots i_m}_{\mu_1\cdots\hat{\nu}\cdots\mu_n}(\partial_{\{\mu\}}\xi^{\hat{\nu}})\frac{\delta}{\delta\Sigma^{i_1\cdots i_m}_{\mu_1\cdots\mu_n}}\,,\end{aligned}$$

by virtue of $\varepsilon^{j}_{k} = \varepsilon^{ji}\eta_{ik} = \varepsilon^{k}_{j}$ ($\varepsilon^{0}_{1} = \varepsilon^{1}_{0} = -1$, $\varepsilon^{0}_{0} = \varepsilon^{1}_{1} = 0$), we find that $\Gamma_{\text{eff}}(B, \Sigma)$ is invariant, $\delta_{(\zeta,\xi)}\Gamma_{\text{eff}} = 0$, under a set of local transformations comprised by the gauge transformations (49), (51) of the background fields $B = (e^{i}_{\mu}, \omega_{\mu})$ along with the infinitesimal local tensor transformations

$$\delta_{(\zeta,\xi)}\Sigma^{i_1\cdots i_m}_{\mu_1\cdots\mu_n} = \varepsilon^{\{i\}}_{\hat{p}}\Sigma^{i_1\cdots\hat{p}\cdots i_m}_{\mu_1\cdots\mu_n}\zeta + \Sigma^{i_1\cdots i_m}_{\mu_1\cdots\hat{\nu}\cdots\mu_n}\partial_{\{\mu\}}\xi^{\hat{\nu}} + (\partial_\nu\Sigma^{i_1\cdots i_m}_{\mu_1\cdots\mu_n})\xi^{\nu} \tag{76}$$

of the additional fields $\Sigma^{i_1\cdots i_m}_{\mu_1\cdots\mu_n}$, cf. (70), (71).

4. Gribov–Zwanziger Model

Let us examine the Gribov–Zwanziger model [23,24] implementing the concept of Gribov horizon [22] in quantum Yang–Mills theories by using a non-local composite field. To this end, we consider a Euclidean form of the Faddeev–Popov action $S_{\text{FP}}(\phi)$ in Landau gauge $\chi^{p}_{\text{L}}(\phi) = \partial^{\mu}A^{p}_{\mu}$, (8), (9), (10), and examine a non-local horizon functional $H(A)$,

$$H(A) = \gamma \int d^{D}x \left[\int d^{D}y\, f^{prt} g A^{r}_{\mu}(x) \left(K^{-1}\right)^{pq}(x;y)\, f^{qst} g A^{s|\mu}(y) + D\left(N^{2}-1\right)\right], \tag{77}$$

where, in Euclidean metric $A_{\mu} = A^{\mu}$, we use the signature $\eta_{\mu\nu} = (-,+,\ldots,+)$ under the Wick rotation $x^{0} \to ix^{0}$, $A^{p|0} \to iA^{p|0}$, $S_{\text{FP}} \to iS_{\text{FP}}$ and maintain the summation convention $A_{\mu}B_{\mu} = A_{\mu}B^{\mu}$; besides, K^{-1} is the inverse,

$$\int d^{D}z \left(K^{-1}\right)^{pr}(x;z)\,(K)^{rq}(z;y) = \int d^{D}z\,(K)^{pr}(x;z)\left(K^{-1}\right)^{rq}(z;y) = \delta^{pq}\delta(x-y), \tag{78}$$

of the Faddeev–Popov operator K in terms of the gauge condition $\partial^{\mu}A^{p}_{\mu} = 0$,

$$K^{pq}(x;y) = \left(\delta^{pq}\partial^{2} + g f^{prq} A^{r}_{\mu}\partial^{\mu}\right)\delta(x-y), \quad K^{pq}(x;y) = K^{qp}(y;x), \tag{79}$$

while γ is a Gribov thermodynamic parameter [23,24], introduced in a self-consistent way by using a gap equation (horizon condition) for a Gribov–Zwanziger action $S_{\text{GZ}} = S_{\text{GZ}}(\phi)$,

$$\frac{\partial E_{\text{vac}}}{\partial\gamma} = 0, \quad \exp\left(-\hbar^{-1}E_{\text{vac}}\right) \equiv \int d\phi\, \exp\left(-\hbar^{-1}S_{\text{GZ}}\right),$$

where E_{vac} is the vacuum energy, and the action S_{GZ} reads

$$S_{\text{GZ}}(\phi) = S_{\text{FP}}(\phi) - H(A)\,. \tag{80}$$

A generating functional of Green's functions $Z_{H}(J)$ for the quantum theory under study can be given in terms of a Faddeev–Popov action shifted by a constant value, $S_{\text{FP}}(\phi) - H(0)$,

$$Z_{H}(J) = Z_{H}(J,L)|_{L=1}, \quad Z_{H}(J,L) = \int d\phi\, \exp\left\{-\hbar^{-1}\left[S_{\text{FP}}(\phi) - H(0) + J_{A}\phi^{A} + L\sigma(A)\right]\right\}, \tag{81}$$

where $L = L(x)$, $\epsilon(L) = \text{gh}(L) = 0$, is a source to the non-local composite field

$$\sigma(A)(x) = \gamma \int d^{D}y\, f^{trp} g A^{r}_{\mu}(x)\,(\tilde{K}^{-1})^{pq}(x;y)\, f^{qst} g A^{s|\mu}(y), \tag{82}$$

with $\tilde{K}^{-1}$ being the inverse,

$$\int d^{D}z\,(\tilde{K}^{-1})^{pr}(x;z)\,(\tilde{K})^{rq}(z;y) = \int d^{D}z\,(\tilde{K})^{pr}(x;z)\,(\tilde{K}^{-1})^{rq}(z;y) = \delta^{pq}\delta(x-y), \tag{83}$$

of an operator $\tilde{K}$ defined for a quantity $F^p = F^p(x)$,

$$\int d^D y\, (\tilde{K})^{pq}(x;y)\, F^q(y) \equiv \left[\partial_\mu, [D^\mu(A), F]\right]^p(x)\,, \quad A_\mu(x) = T^p A^p_\mu(x)\,, \quad F(x) = T^p F^p(x)\,, \tag{84}$$

which implies

$$\tilde{K}^{pq}(x;y) = \partial^\mu D^{pq}_\mu(A)\,\delta(x-y)\,, \tag{85}$$

and reduces to the operator K of (79) under the Landau gauge condition due to $S_{\mathrm{FP}}(\phi)$ in the path integral (81).

4.1. Formal Background Introduction

Let us now formally extend the generating functional $Z_H(J,L)$ with a non-local composite field (81)–(84) to the case of an additional background field B_μ with a covariant derivative $D_\mu(B)$ and the gauge properties (16), (17), by using the approach (2), (19), as adapted to Euclidean QFT, which implies a modification of derivatives $\partial_\mu \to D_\mu(B)$ in (81), according to

$$Z_H(B,J,L) = Z_H(J,L)|_{\partial_\mu \to D_\mu(B)} = \int d\phi \exp\left\{-\hbar^{-1}\left[S_{\mathrm{FP}}(\phi,B) - H(0) + J_A\phi^A + L\sigma(A,B)\right]\right\}, \tag{86}$$

where $S_{\mathrm{FP}}(\phi,B)$ is the Faddeev–Popov action in the Landau background gauge $\chi^p_{\mathrm{L}}(\phi,B) = 0$, see (41), and $\sigma(A,B)$ is a non-local composite field on a background,

$$\sigma(A,B)(x) = \gamma \int d^D y\, f^{trp} g A^r_\mu(x)\, (\tilde{K}^{-1}_B)^{pq}(x;y)\, f^{qst} g A^{s|\mu}(y)\,. \tag{87}$$

Here, $\tilde{K}^{-1}_B$ is a modified operator $\tilde{K}^{-1}$ as in (78), with the related inverse $\tilde{K}_B$ given by the replacement $\tilde{K} \to \tilde{K}_B$,

$$\int d^D y\, (\tilde{K})^{pq}_B(x;y)\, F^q(y) \equiv \left[D_\mu(B), [D^\mu(A+B), F]\right]^p(x)\,, \quad F(x) = F^p(x)\,T^p, \tag{88}$$

and having the explicit form

$$(\tilde{K})^{pq}_B(x;y) = D^{pr}_\mu(B)\, D^{rq|\mu}(A+B)\,\delta(x-y)\,. \tag{89}$$

In the particular case, cf. (81),

$$Z_H(B,J) = Z_H(B,J,L)|_{L=1}\,,$$

we obtain the generating functional

$$Z_H(B,J) = \int d\phi \exp\left\{-\hbar^{-1}\left[S_{\mathrm{GZ}}(\phi,B) + J_A\phi^A\right]\right\}, \quad S_{\mathrm{GZ}}(\phi,B) \equiv S_{\mathrm{FP}}(\phi,B) - H(A,B)\,, \tag{90}$$

with a non-local term $H(A,B)$ given by

$$H(A,B) = \gamma \int d^D x \left[\int d^D y\, f^{prt} g A^r_\mu(x) \left(K^{-1}_B\right)^{pq}(x;y)\, f^{qst} g A^{s|\mu}(y) + D\left(N^2-1\right)\right]. \tag{91}$$

Here, K^{-1}_B is the inverse of an operator K_B as in (83), which is identical with $\tilde{K}_B$ in (88) being expressed, due to $S_{\mathrm{FP}}(\phi,B)$ in (90), by utilizing the gauge condition $D^{pq}_\mu(B)\,A^{q|\mu} = 0$ and the properties of f^{pqr} including the Jacobi identity,

$$K_B(x;y) = \left[\partial^2 + g\left(\partial_\mu B^\mu\right) + g\left(A_\mu + 2B_\mu\right)\partial^\mu + g^2\left(A_\mu + B_\mu\right)B^\mu\right]\delta(x-y)\,, \tag{92}$$

where A_μ, B_μ are matrices with the elements $\left(A^{pq}_\mu, B^{pq}_\mu\right) = f^{prq}\left(A^r_\mu, B^r_\mu\right)$, and $K_B\,(x;y)$ is related to $\tilde{K}_B\,(x;y)$ in (89) as follows:

$$K_B\,(x;y) = \tilde{K}_B\,(x;y) - g\left[D_\mu\,(B)\,, A^\mu\right]\delta\,(x-y) = D^\mu\,(A+B)\,D_\mu\,(B)\,\delta\,(x-y)\,. \tag{93}$$

The operator K_B extends the original operator K in (79) and exhibits the properties

$$K_B|_{B=0} = K, \quad (K_B)^{pq}\,(x;y) = (K_B)^{qp}\,(y;x)\,, \tag{94}$$

which can be verified by a direct calculation:

$$\int d^Dy\ \left[(K_B)^{pq}\,(x;y) - (K_B)^{qp}\,(y;x)\right]F^q\,(y) = gf^{prq}\left[D^{rs}_\mu\,(B)\,A^{s|\mu}\right]F^q\,(x) = 0.$$

By virtue of (94), one may formally interpret $S_{\mathrm{GZ}}\,(\phi, B)$ as a Gribov–Zwanziger action on a background B_μ, with the corresponding non-local horizon term $H\,(A,B)$ in (91), (93) and the generating functional $Z_H\,(B,J)$ in (90). However, since the background introduction (2), (19) has been applied to a non-local composite field it is natural to examine the consistency of the above interpretation with the symmetries exhibited by $Z_H\,(B,J)$. To this end, it is useful to recast $Z_H\,(B,J)$ in a local form by extending the configuration space according to [97], in such a way that one introduces a set of commuting $(\bar{\varphi}^{pq}_\mu, \varphi^{pq}_\mu)$ and anticommuting $(\bar{\omega}^{pq}_\mu, \omega^{pq}_\mu)$ auxiliary fields, where $\bar{\varphi}^{pq}_\mu$ and φ^{pq}_μ are mutually complex-conjugate,

$$\epsilon\left(\bar{\varphi}^{pq}_\mu, \varphi^{pq}_\mu, \bar{\omega}^{pq}_\mu, \omega^{pq}_\mu\right) = (0,0,1,1)\,, \;\; \mathrm{gh}\left(\bar{\varphi}^{pq}_\mu, \varphi^{pq}_\mu, \bar{\omega}^{pq}_\mu, \omega^{pq}_\mu\right) = (0,0,-1,1)\,.$$

This makes it possible to construct the parametrization

$$\exp\left\{\hbar^{-1}\left[H\,(A,B) - H\,(0,B)\right]\right\} = \int d\bar{\varphi}\; d\varphi\; d\bar{\omega}\; d\omega \exp\left[-\hbar^{-1}S_\gamma\,(A,B;\bar{\varphi}\;\varphi,\bar{\omega},\omega)\right], \tag{95}$$

where

$$S_\gamma = \int d^Dx\ \left[-\bar{\varphi}^{rp}_\mu K^{pq}_B \varphi^{rq|\mu} + \bar{\omega}^{rp}_\mu K^{pq}_B \omega^{rq|\mu} + i\gamma^{1/2} g\ f^{prq} A^{r|\mu}\left(\bar{\varphi}^{pq}_\mu + \varphi^{pq}_\mu\right)\right], \tag{96}$$

as we denote

$$K^{pq}_B \varphi^{rq}_\mu\,(x) = \int d^Dy\; K^{pq}_B\,(x,y)\,\varphi^{rq}_\mu\,(y)\,, \quad K^{pq}_B \omega^{rq}_\mu\,(x) = \int d^Dy\; K^{pq}_B\,(x,y)\,\omega^{rq}_\mu\,(y)\,.$$

In the configuration space $\Phi = (\phi, \bar{\varphi}, \;\varphi, \bar{\omega}, \omega)$, the functional $Z_H\,(B,J)$ of (90) takes the form

$$Z_H\,(B,J) = \int d\Phi \exp\left\{-\hbar^{-1}\left[S_{\mathrm{GZ}}\,(\Phi,B) + J_A\phi^A\right]\right\}, \tag{97}$$

where the local action $S_{\mathrm{GZ}}\,(\Phi, B)$ of the Gribov–Zwanziger theory on a background reads (note the antisymmetry of f^{pqr})

$$\begin{aligned} S_{\mathrm{GZ}}\,(\Phi,B) \;&=\; S_{\mathrm{FP}}\,(\phi,B) - H\,(0,B) + S_K\,(\Phi,B) - i\gamma^{1/2}g\int d^Dx\ \mathrm{Tr}\ A^\mu\left(\bar{\varphi}_\mu - \varphi^{\mathrm{T}}_\mu\right), \\ S_K\,(\Phi,B) \;&\equiv\; \int d^Dx\; d^Dy\ \mathrm{Tr}\left[-\bar{\varphi}^\mu\,(x)\,K_B\,(x,y)\,\varphi^{\mathrm{T}}_\mu\,(y) + \bar{\omega}^\mu\,(x)\,K_B\,(x,y)\,\omega^{\mathrm{T}}_\mu\,(y)\right]. \end{aligned} \tag{98}$$

Using the explicit form of $K_B\,(x,y)$ in (93), the antisymmetry of f^{pqr}, and a repeated integration by parts, one can remove the delta-function $\delta\,(x-y)$ absorbed in $K_B\,(x,y)$ and present $S_K\,(\Phi, B)$ in (98) as follows:

$$S_K\,(\Phi,B) = \int d^Dx\ \mathrm{Tr}\left[-\bar{\varphi}^\mu D_\nu\,(A+B)\,D^\nu\,(B)\,\varphi^{\mathrm{T}}_\mu + \bar{\omega}^\mu D_\nu\,(A+B)\,D^\nu\,(B)\,\omega^{\mathrm{T}}_\mu\right]. \tag{99}$$

The action $S_{\mathrm{GZ}}\left(\Phi,B\right)$ is invariant under the global $SU\left(N\right)$ transformations

$$\left(A_{\mu},B_{\mu},b,\bar{c},c,\bar{\varphi}_{\mu},\varphi_{\mu}^{\mathrm{T}},\bar{\omega}_{\mu},\omega_{\mu}^{\mathrm{T}}\right)\stackrel{U}{\rightarrow}U\left(A_{\mu},B_{\mu},b,\bar{c},c,\bar{\varphi}_{\mu},\varphi_{\mu}^{\mathrm{T}},\bar{\omega}_{\mu},\omega_{\mu}^{\mathrm{T}}\right)U^{-1}. \tag{100}$$

The infinitesimal form of (100)

$$\delta_{\varsigma}F^{pq}=gf^{prs}F^{rq}\varsigma^{s}+gf^{qrs}F^{pr}\varsigma^{s},\quad F^{pq}=\left(A_{\mu},B_{\mu},b,\bar{c},c,\bar{\varphi}_{\mu},\varphi_{\mu}^{\mathrm{T}},\bar{\omega}_{\mu},\omega_{\mu}^{\mathrm{T}}\right)^{pq} \tag{101}$$

produces a unit Jacobian (due to the antisymmetry of f^{pqr}) in the integration measure of (97) and leaves the functional $Z_{H}\left(B,J\right)$ invariant under infinitesimal global $SU\left(N\right)$ transformations for the background field B_{μ} and the sources J_{A}, having the adjoint representation form

$$\delta_{\varsigma}G^{pq}=gf^{prs}G^{rq}\varsigma^{s}+gf^{qrs}G^{pr}\varsigma^{s},\quad G^{pq}=\left(B_{\mu},J_{\mu(A)},J_{(b)},J_{(\bar{c})},J_{(c)}\right)^{pq}. \tag{102}$$

This behavior of $Z_{H}\left(B,J\right)$ obviously entails an invariance of the restricted generating functional $Z_{H}\left(J\right)$ under the global $SU\left(N\right)$ transformations of the sources; however, the global $SU\left(N\right)$ invariance of $Z_{H}\left(J\right)$ fails to translate itself into a local symmetry of the background functional $Z_{H}\left(J,B\right)$. Indeed, let us apply an infinitesimal form of the local change of variables ($U\rightarrow V$)

$$\Phi\stackrel{V}{\rightarrow}\Phi',\quad B_{\mu}\stackrel{V}{\rightarrow}B'_{\mu}=VB_{\mu}V^{-1}+g^{-1}V\left(\partial_{\mu}V^{-1}\right), \tag{103}$$

with a unitary matrix $V=V\left(\xi\right)$, $\xi=\xi\left(x\right)$, to the integrand in (97), which produces a unit Jacobian and a variation $\delta_{\xi}S_{\mathrm{GZ}}=\delta_{\xi}S_{K}$ in (98), so that the presence of extra derivatives $\partial_{\mu}V^{-1}$ and $\partial^{2}V^{-1}$ in the transformed expression

$$\begin{aligned}S_{K}\left(\Phi',B'\right)&=\int d^{D}x\ \mathrm{Tr}\left[-V\bar{\varphi}^{\mu}D_{\nu}\left(A+B\right)D^{\nu}\left(B\right)\varphi_{\mu}^{\mathrm{T}}V^{-1}+V\bar{\omega}^{\mu}D_{\nu}\left(A+B\right)D^{\nu}\left(B\right)\omega_{\mu}^{\mathrm{T}}V^{-1}\right]\\&\neq\int d^{D}x\ \mathrm{Tr}\left[-V^{-1}V\bar{\varphi}^{\mu}D_{\nu}\left(A+B\right)D^{\nu}\left(B\right)\varphi_{\mu}^{\mathrm{T}}+V^{-1}V\bar{\omega}^{\mu}D_{\nu}\left(A+B\right)D^{\nu}\left(B\right)\omega_{\mu}^{\mathrm{T}}\right],\end{aligned}$$

implies $S_{K}\left(\Phi',B'\right)\neq S_{K}\left(\Phi,B\right)$, also involving a local non-invariance of the background horizon term, $H\left(A',B'\right)\neq H\left(A,B\right)$. Finally, in view of $\delta_{\xi}\left(J_{A}\phi^{A}\right)=0$, the functional $Z_{H}\left(B,J\right)$ is not invariant under the background gauge transformations of B_{μ} with the local $SU\left(N\right)$ transformations of J_{A}, since the latter do not compensate the variation $\delta_{\xi}S_{\mathrm{GZ}}\left(\Phi,B\right)\neq0$.

4.2. Modified Background Introduction and Effective Action

The local non-invariance of $Z_{H}\left(B,J\right)$ can be explained by the fact that the background field B_{μ} has been introduced directly into the non-local horizon term $H\left(A\right)$ by means of (93), while localizing the resultant background term $H\left(A,B\right)$ by the auxiliary fields $\left(\bar{\varphi}_{\mu},\varphi_{\mu}\right)$ and $\left(\bar{\omega}_{\mu},\omega_{\mu}\right)$ does not provide them with a covariant derivative as in (19), which is evident from (99). In order to remedy this issue, we examine an alternative introduction of a background field, i.e., by using a local parametrization of the original horizon term $H\left(A\right)$ at a point before a background has been switched on. To this end, let us consider the expressions (95)–(98) when restricted to $B_{\mu}=0$ and recast the functional $Z_{H}\left(J\right)$ given by (81) in the form

$$Z_{H}\left(J\right)=\int d\Phi\ \exp\left\{-\hbar^{-1}\left[S_{\mathrm{GZ}}\left(\Phi\right)+J_{A}\phi^{A}\right]\right\},\ S_{\mathrm{GZ}}\left(\Phi\right)=S_{\mathrm{GZ}}\left(\Phi,0\right),\ \Phi=\left(\phi,\bar{\varphi},\ \varphi,\bar{\omega},\omega\right).$$

For a consideration of the auxiliary fields $(\bar{\varphi}, \bar{\omega})$ and $\left(\varphi^{\mathrm{T}}, \omega^{\mathrm{T}}\right)$ on equal terms, note that the action $S_{\mathrm{GZ}}(\Phi)$ in the above integrand with the Landau gauge condition $\partial_\mu A^\mu = 0$ encoded in $\exp\left[-\hbar S_{\mathrm{FP}}(\phi)\right]$ is equivalent to an action $\mathcal{S}_{\mathrm{GZ}}(\Phi)$ which arises from replacing $K(x,y)$ by $\mathcal{K}(x,y)$ defined as

$$\mathcal{K}(x,y) \equiv \frac{1}{2}\left[K(x,y) + \tilde{K}(x,y)\right], \quad \tilde{K}(x,y) = K(x,y) + g\left[\partial_\mu, A^\mu\right]\delta(x-y). \tag{104}$$

Using integration by parts, permutations under the sign of Tr, and the Landau gauge condition in the path integral, one brings the equivalent action $\mathcal{S}_{\mathrm{GZ}}(\Phi)$ to the form

$$\begin{aligned}
\mathcal{S}_{\mathrm{GZ}}(\Phi) &= S_{\mathrm{FP}}(\phi) - H(0) + S_{\mathcal{K}}(\Phi) - i\gamma^{1/2} g \int d^D x \operatorname{Tr} A^\mu \left(\bar{\varphi}_\mu - \varphi^{\mathrm{T}}_\mu\right), \\
S_{\mathcal{K}}(\Phi) &\equiv \frac{1}{2}\int d^D x \operatorname{Tr}\left\{\left[D^\nu(A), \bar{\varphi}^\mu\right]\partial_\nu \varphi^{\mathrm{T}}_\mu + \left(\partial^\nu \bar{\varphi}^\mu\right)\left[D_\nu(A), \varphi^{\mathrm{T}}_\mu\right]\right\} - (\bar{\varphi}, \varphi \to \bar{\omega}, \omega),
\end{aligned} \tag{105}$$

which invites a natural introduction of a background, $\mathcal{S}_{\mathrm{GZ}}(\Phi) \to \mathcal{S}_{\mathrm{GZ}}(\Phi, B)$, according to (19),

$$\begin{aligned}
\mathcal{S}_{\mathrm{GZ}}(\Phi, B) &= S_{\mathrm{FP}}(\phi, B) - H(0) + S_{\mathcal{K}}(\Phi, B) - i\gamma^{1/2} g \int d^D x \operatorname{Tr} A^\mu \left(\bar{\varphi}_\mu - \varphi^{\mathrm{T}}_\mu\right), \\
S_{\mathcal{K}}(\Phi, B) &= \frac{1}{2}\int d^D x \operatorname{Tr}\left\{\left[D^\nu(A+B), \bar{\varphi}^\mu\right]\left[D_\nu(B), \varphi^{\mathrm{T}}_\mu\right] + \left[D^\nu(B), \bar{\varphi}^\mu\right]\left[D_\nu(A+B), \varphi^{\mathrm{T}}_\mu\right]\right\} \\
&\quad - (\bar{\varphi}, \varphi \to \bar{\omega}, \omega) \equiv \int d^D x \left(-\bar{\varphi}^{pq}_\mu \mathcal{K}^{pq|rs}_B \varphi^{rs|\mu} + \bar{\omega}^{pq}_\mu \mathcal{K}^{pq|rs}_B \omega^{rs|\mu}\right).
\end{aligned} \tag{106}$$

Let us consider the expression

$$\langle F, G\rangle \equiv -\frac{1}{2}\int d^D x \operatorname{Tr}\left\{\left[D_\mu(A+B), F\right]\left[D^\mu(B), G\right] + \left[D_\mu(B), F\right]\left[D^\mu(A+B), G\right]\right\}, \quad \epsilon(F) = \epsilon(G),$$

and present it in the form

$$\langle F, G\rangle = \int d^D x \, F^{pq} \mathcal{K}^{pq|rs}_B G^{rs} = \int d^D x \, d^D y \, F^{pq}(x) \, \mathcal{K}^{pq|rs}_B(x;y) \, G^{rs}(y).$$

Then, due to the (anti)symmetry of $\langle F, G\rangle$ under the replacement $F \leftrightarrow G$, we find

$$\mathcal{K}^{pq|rs}_B(x;y) = \mathcal{K}^{rs|pq}_B(y;x).$$

Given the above, we interpret $\mathcal{S}_{\mathrm{GZ}}(\Phi, B)$ in (106) as an alternative local Gribov–Zwanziger action on the background B_μ, with the corresponding modified background horizon term $\mathcal{H}(A, B)$ defined as

$$\exp\left\{\hbar^{-1}\left[\mathcal{H}(A,B) - \mathcal{H}(0,B)\right]\right\} = \int d\bar{\varphi}\, d\varphi\, d\bar{\omega}\, d\omega \exp\left[-\hbar^{-1}\mathcal{S}_\gamma(\Phi, B)\right], \quad \mathcal{H}(0,B) \equiv H(0), \tag{107}$$

where

$$\begin{aligned}
\mathcal{S}_\gamma(\Phi, B) &= S_{\mathcal{K}}(\Phi, B) - i\gamma^{1/2} g \int d^D x \operatorname{Tr} A^\mu \left(\bar{\varphi}_\mu - \varphi^{\mathrm{T}}_\mu\right) \\
&= \int d^D x \left[-\bar{\varphi}^{pq}_\mu \mathcal{K}^{pq|rs}_B \varphi^{rs|\mu} + \bar{\omega}^{pq}_\mu \mathcal{K}^{pq|rs}_B \omega^{rs|\mu} + i\gamma^{1/2} g f^{prq} A^{r|\mu}\left(\bar{\varphi}^{pq}_\mu + \varphi^{pq}_\mu\right)\right].
\end{aligned}$$

By construction, the action $\mathcal{S}_{\mathrm{GZ}}(\Phi, B)$ in (106) is invariant under the local transformations (103), which implies a unit Jacobian in the infinitesimal case and leads to an invariance of the background generating functional

$$Z_{\mathcal{H}}(B, J) = \int d\Phi \, \exp\left\{-\hbar^{-1}\left[\mathcal{S}_{\mathrm{GZ}}(\Phi, B) + J_A \Phi^A\right]\right\}, \quad Z_{\mathcal{H}}(0, J) = Z_H(J), \tag{108}$$

with respect to local transformations of the sources and the background field:

$$\delta_\xi B^p_\mu = D^{pq}_\mu (B)\, \xi^q, \quad \delta_\xi (J_{\mu(A)}, J_{(b)}, J_{(\bar{c})}, J_{(c)})^p = g f^{prq} (J_{\mu(A)}, J_{(b)}, J_{(\bar{c})}, J_{(c)})^r \xi^q. \tag{109}$$

The transformations (103) also provide a local invariance of the modified background horizon term $\mathcal{H}(A, B)$ in (107),

$$\delta_\xi \mathcal{H}(A, B) = 0, \ \delta_\xi B^p_\mu = D^{pq}_\mu (B)\, \xi^q, \ \delta_\xi A^p_\mu = g f^{prq} A^r_\mu \xi^q.$$

Introducing the generating functionals of connected $W_\mathcal{H}(B, J)$ and vertex $\Gamma_\mathcal{H}(B, \phi)$ Green's functions on a background,

$$Z_\mathcal{H} = \exp\left(-\hbar^{-1} W_\mathcal{H}\right), \ \Gamma_\mathcal{H}(B, \phi) = W_\mathcal{H}(B, J) - J_A \phi^A, \ \phi^A = \frac{\overrightarrow{\delta}}{\delta J_A} W_\mathcal{H}, \ J_A = -\Gamma_\mathcal{H} \frac{\overleftarrow{\delta}}{\delta \phi^A}, \tag{110}$$

one translates the invariance of $Z_\mathcal{H}(B, J)$ under (109) into an invariance of $\Gamma_\mathcal{H}(B, \phi)$ under the following local transformations, cf. (22), (23), (34), (36),

$$\delta_\xi B^p_\mu = D^{pq}_\mu (B)\, \xi^q, \quad \delta_\xi (A_\mu, b, \bar{c}, c)^p = g f^{prq} (A_\mu, b, \bar{c}, c)^r \xi^q, \tag{111}$$

which consist of the gauge transformations for the background field B_μ and of the local $SU(N)$ transformations for the quantum fields ϕ^A, so that the background effective action $\Gamma_{\text{eff}}(B)$ for the Gribov–Zwanziger model defined as

$$\Gamma_{\text{eff}}(B) = \Gamma_\mathcal{H}(B, \phi)|_{\phi=0} \tag{112}$$

is invariant, $\delta_\xi \Gamma_{\text{eff}} = 0$, under the gauge transformations of the background field B_μ.

5. Conclusions

In the present article, we have approached the issue of a joint introduction of composite and background fields into non-Abelian quantum gauge models on the basis of symmetries exhibited by the generating functional of Green's functions. Thus, we examine the Yang–Mills theory, the Volovich–Katanaev model of two-dimensional gravity with dynamical torsion, and the Gribov–Zwanziger theory.

Our systematic analysis of the problem focuses on quantum Yang–Mills theories and local composite fields. As our first approach, we consider a theory quantized according to Faddeev and Popov and including some composite fields (1), with the resultant generating functional $Z(J, L)$ having an inherent symmetry under global $SU(N)$ transformations of the sources J_A, L_m for the quantum ϕ^A and composite $\sigma^m(\phi)$ fields. As our second approach, we examine a theory quantized in the background field method (3), with the background generating functional $Z(B, J)$ being inherently invariant under local $SU(N)$ transformations of the sources J_A accompanied by gauge transformations of the background field B_μ with an associated covariant derivative $D_\mu(B)$. In the first approach, we introduce a background field B_μ equipped with $D_\mu(B)$, whereas in the second approach we introduce some composite fields $\sigma^m(\phi, B)$ with sources L_m. In doing so, we demand that the functional $Z(B, J, L)$ given by $Z(J, L) \xrightarrow{B} Z(B, J, L)$ and $Z(B, J) \xrightarrow{L} Z(B, J, L)$ reflect the symmetries inherent in $Z(J, L)$ and $Z(B, J)$. The resultant $Z(B, J, L)$ is thereby invariant under the local $SU(N)$ transformations of J_A, L_m combined with the gauge transformations of B_μ, whereas our first and second approaches are equivalent in the sense of (2) and (4), (5) with $\partial_\mu \to D_\mu(B)$ given by (19).

In quantum Yang–Mills theories with composite and background fields, we have introduced an extended (due to antifields ϕ^*_A) generating functional of vertex Green's functions (24), (29), including a background effective action (33), and investigated its properties. For the generating functional of vertex Green's functions (effective action), gauge dependence has been obtained in terms of a nilpotent operator with composite and background fields (30), (31), and conditions of on-shell independence from gauge-fixing have been established. Thus, the generating functional of vertex functions is found to be gauge-independent on its extremals (32), whereas the background effective action is is independent of any specific choice of gauge-fixing at the extremals (32) restricted to the hypersurface of vanishing antifields ϕ^*_A and quantum fields ϕ^A. The background effective action $\Gamma_{\text{eff}}\,(B,\Sigma)$, depending on the background field B_μ and a set of auxiliary fields Σ^m associated with the composite fields σ^m, is found to be invariant under the gauge transformations of B_μ and the local $SU(N)$ transformations of Σ^m, see (36).

Introduction of composite fields in the approach $Z\,(B,J) \xrightarrow{L} Z\,(B,J,L)$ starting from a background generating functional $Z\,(B,J)$ has been extended beyond the Yang–Mills case by considering the Volovich–Katanaev model [50] of two-dimensional gravity with dynamical torsion (in terms of a zweibein e^i_μ and a Lorentz connection ω_μ) quantized using the background field method in [75] and exhibiting a gauge-invariant background effective action. Thus, the quantum theory [75] has been modified by introducing local composite fields $\sigma^{i_1\cdots i_m}_{\mu_1\cdots\mu_n}$ with sources $L^{\mu_1\cdots\mu_n}_{i_1\cdots i_m}$, as we demand for $\sigma^{i_1\cdots i_m}_{\mu_1\cdots\mu_n}$ to behave as tensors (70) under the Lorentz (63) and general coordinate (64) transformations of background $B = (e^i_\mu,\omega_\mu)$ and quantum $Q = (q^i_\mu,q_\mu)$ fields. These transformations coincide infinitesimally with the background transformations $\delta_{\text{b}}B$ and $\delta_{\text{b}}Q$ in (53), constructed by analogy with the Yang–Mills case (44). The corresponding background effective action $\Gamma_{\text{eff}}\,(B,\Sigma)$ defined as (69), (74), (75) has been found invariant under the gauge transformations (49), (51) of background fields B combined with the local transformations (76) of additional fields $\Sigma^{i_1\cdots i_m}_{\mu_1\cdots\mu_n}$ related to $\sigma^{i_1\cdots i_m}_{\mu_1\cdots\mu_n}$. As in the Yang–Mills case, (76) is an infinitesimal form of tensor transformations, cf. (70), (71).

Introduction of background fields in the approach $Z\,(J,L) \xrightarrow{B} Z\,(B,J,L)$ starting from a generating functional $Z\,(J,L)$ has been extended beyond the case of local composite fields by considering the Gribov–Zwanziger model [23,24] implementing the Gribov horizon [22] for quantum Yang–Mills theories in Landau gauge by means of an additive horizon functional $H\,(A)$ which can be presented in terms of a non-local composite field $\sigma\,(A)$. Direct introduction of a background field B_μ into the composite field, $\sigma\,(A) \to \sigma\,(A,B)$, produces a formally consistent (94) background-modified generating functional for the Gribov–Zwanziger theory, see $Z_H\,(B,J)$ in (90) with the corresponding non-local horizon term $H\,(A,B)$ in (91), (93). However, it has been shown that $Z_H\,(B,J)$ does not inherit (in a localized form) the symmetry of the original Gribov–Zwanziger functional $Z_H\,(J)$ in (81) under global $SU\,(N)$ transformations of the sources, as $H\,(A,B)$ fails to be invariant under local $SU\,(N)$ transformations of quantum Yang–Mills field A_μ combined with gauge transformations of the background field B_μ. Further, representing the original horizon term $H\,(A)$ in a local parametrization using auxiliary fields according to [97], we bring the related functional $Z_H\,(J)$ to an equivalent (under the Landau gauge) form with a modified horizon term $\mathcal{H}\,(A)$. This local parametrization admits a natural introduction of a covariant derivative $D_\mu\,(B)$ as acting on the auxiliary fields by extension $\partial_\mu \to D_\mu\,(B)$ according to (19), which implies a background horizon term $\mathcal{H}\,(A,B)$ invariant under the local $SU\,(N)$ transformations of A_μ along with the gauge transformations of B_μ. The resulting modified background functional $Z_{\mathcal{H}}\,(B,J)$ in (108) is consistent, $Z_{\mathcal{H}}\,(B,J)|_{B=0} = Z_H\,(J)$, and invariant under the local $SU\,(N)$ transformations of the sources combined with the gauge transformations of the background field. The corresponding background effective action $\Gamma_{\text{eff}}\,(B)$ for the Gribov–Zwanziger model defined as (106), (108), (110), (112) proves to be invariant under the gauge transformations $\delta_\xi B_\mu = \left[D_\mu\,(B)\,,\xi\right]$ of the background field B_μ. This effective action seems advantageous as a starting point in approaching a renormalization analysis of the Gribov–Zwanziger model as one accounts for the horizon in the background field method.

One should note, the calculations maybe independently done in the Computer Algebra Systems, like Mathematica Singular or Maple. We checked the correctness some of the equations with use of the Mathematica.

One more application of the suggested approach may be developed for QCD-gauge theory of strong interactions with $SU(3)$ gauge group [33,88]-to describe hadron (meson and baryon. e.g., proton, neutron) particles as the composite fields, $\sigma^m(\bar{u}, u, \bar{d}, d, A^{\overline{\alpha}}_{\mu})$, of up u and down d quarks (including left-an right-handed spinors) of spin $1/2$ and $A^{\overline{\alpha}}_{\mu}$, $\overline{\alpha} = 1, ..., 8$ of spin 1 gluons fields. Here, the Grassmann parity $\epsilon(\sigma^m)$ should equal to 1 for baryon and to 0 for meson particles.

Another interesting application of the presented background field method with composite fields is a consideration of so-called Generalized Lagrange space (for metric fields) to exploit its properties of curvature, torsion and deflection in order to take into account the asymmetries and anisotropies arising from physical phenomena basically at the cosmological level.

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