Characterising linear growth rate in $f(R)$ gravity

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Abstract

We investigate the linear growth rate of cosmological matter density perturbations in a viable $f(R)$ model both numerically and analytically. We find that the growth rate in the scalar-tensor regime can be characterised by a simple analytic formula. We also investigate a prospect of constraining the Compton wavelength scale of the $f(R)$ model with a future weak lensing survey.

1 Introduction

Cosmological observations of distant Ia supernovae discovered that our universe is undergoing an accelerated expansion period, which is supported by other observations of the cosmic microwave background anisotropies and the large scale structure of galaxies. These observations are explained by the cosmological model with the cosmological constant Λ. Modification of the gravity theory is an alternative approach.

The key to distinguish between modified gravity and dynamical dark energy is the growth of cosmological perturbations. The growth history of cosmological perturbations can be tested with the large scale structure in the universe.

In the present paper, we investigate the growth history of matter density perturbations in $f(R)$ models.

2 A brief review of $f(R)$ model

We briefly review $f(R)$ model, which is defined by the action,

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}(R + f(R)) + \int d^4x\sqrt{-g}L^{(\mathrm{m})}, \tag{1}$$

where G is the gravitational constant, and $L^{(\mathrm{m})}$ is the matter Lagrangian density. We consider the viable models, proposed in the literatures [1, 2]. The viable models have an asymptotic formula at the late time universe ($R \gg R_c$), which can be written as $f(R) = -\lambda R_c[1-(R_c/R)^{2n}]$, where R_c is a positive constant whose value is the same order as that of the present Ricci scalar, and λ is a non-dimensional constant. Because the term λR_c plays a role of the cosmological constant, we may write $\lambda R_c = 6(1-\Omega_0)H_0^2$, where H_0 is the Hubble constant and Ω_0 is the matter density parameter. Note that we assume the spatially flat universe.

It is well known that $f_R = df(R)/dR$ plays a roll of a new degree of freedom, which behaves like a scalar field with the mass $m^2 = 1/(3f_{RR})$ where we defined $f_{RR} = d^2f(R)/dR^2$. (We have assumed $|f_R| \ll 1$ and $Rf_{RR} \ll 1$ for the viable model.)

We focus on the evolution of matter density perturbations in the $f(R)$ model, whose Fourier coefficients obey (e.g., [2])

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi\left[1 + \frac{1}{3}\frac{k^2/a^2}{k^2/a^2 + 1/(3f_{RR})}\right]G\rho\delta = 0, \tag{2}$$

where the dot denotes the differentiation with respect to the cosmic time, $H = \dot{a}/a$ is the Hubble parameter, ρ is the matter mean density, and $G_{\mathrm{eff}} = [...]G$ is the effective gravitational constant, where

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k is the wave number, and a is the scale factor normalised to unity at present epoch. As is noted in the above, the physical meaning of $m^2 = 1/(3f_{RR})$ is the square of the mass of the new degree of freedom which modifies the gravity force. We have the general relativity regime, $G_{\rm eff} = G$, for $k/a \ll m$, and the scalar-tensor regime, $G_{\rm eff} = 4G/3$, for $k/a \gg m$, respectively. Thus, the evolution of matter density perturbations depends on the wavenumber k, whose behaviour is determined by the mass $m^2 = 1/(3f_{RR})$.

For the Einstein de Sitter universe, the exact solution of Eq. (2) is found in the literature [3]. However, we consider the low density universe, where the solution of Eq. (2) is described in a different form in comparison with that of [3]. Using the formulas $\lambda R_c = 6(1-\Omega_0)H_0^2$ and $R = 3H_0^2\left[\Omega_0/a^3 + 4(1-\Omega_0)\right]$, we have

$$\frac{1}{3f_{RR}} = \frac{\Omega_0 H_0^2}{4n(2n+1)}\left(\frac{\lambda}{2}\right)^{2n}\left(\frac{\Omega_0}{1-\Omega_0}\right)^{2n+1}\left(\frac{1}{a^3}+\frac{4(1-\Omega_0)}{\Omega_0}\right)^{2n+2}. \quad (3)$$

Denoting the wavenumber corresponding to the Compton wavelength $1/m$ at the present epoch by k_C. Equation (3) is rewritten as

$$\frac{1}{3f_{RR}} = k_C^2\left(\frac{\Omega_0 a^{-3} + 4(1-\Omega_0)}{\Omega_0 + 4(1-\Omega_0)}\right)^{2n+2}. \quad (4)$$

We denote the growth factor by $D_1(a,k)$, which is the solution of Eq. (2) normalised so as to be $D_1(a,k) \simeq a$ at $a \ll 1$. The growth rate is defined by $f(a,k) = d\log D_1(a,k)/d\log a$. Using the growth rate $f(a,k)$, Eq. (2) is rephrased as

$$\frac{df}{d\ln a} + f^2 + \left(2 + \frac{\dot{H}}{H^2}\right)f = \frac{3}{2}\frac{G_{\rm eff}}{G}\Omega_m(a), \quad (5)$$

where $\Omega_m(a)$ is defined by $\Omega_m(a) = H_0^2\Omega_0 a^{-3}/H^2$. Equation (5) is useful to find an approximate solution, as we see in the next section.

3 Growth of density perturbations in $f(R)$ model

In this section, we investigate the evolution of matter density perturbations in the $f(R)$ model.

3.1 Scalar-tensor regime

In the scalar tensor regime, $k/a \gg m$, in which the wavelength is shorter than the Compton wavelength, the effective gravitational constant becomes $G_{\rm eff} = 4G/3$. In this case, we find that Eq. (5) has the solution expressed in the form

$$f(a,k) = f_0 \Omega_m(a)^{\widetilde{\gamma}(a)}, \quad (6)$$

where f_0 obeys $f_0^2 + f_0/2 = 2$, therefore $f_0 = (-1+\sqrt{33})/4$, and $\widetilde{\gamma}(a) = \sum_{\ell=0}\zeta_\ell(1-\Omega_m(a))^\ell$, where ζ_ℓ is the expansion coefficients. This can be generalised to the case when $G_{\rm eff}/G(=\xi)$ is a constant value, in which the solution of Eq. (5) has the same formula as that of (6) but with $f_0 = (-1+\sqrt{1+24\xi})/4$ and

$$\begin{aligned}\widetilde{\gamma}(a) = &\frac{-41+24\xi+\sqrt{1+24\xi}}{-70+48\xi} + \frac{1}{8(-143+24\xi)(-35+24\xi)^2}\\ &\times\Big[(-41+24\xi+\sqrt{1+24\xi})(431+\sqrt{1+24\xi}+24\xi(-13+\sqrt{1+24\xi})\\ &-36(2+\sqrt{1+24\xi})+6(-2+3\sqrt{1+24\xi}))\Big](1-\Omega_m(a)) + \mathcal{O}\left((1-\Omega_m(a))^2\right). \end{aligned} \quad (7)$$

Here we assume that $G_{\rm eff}/G(=\xi)$ is constant, but we utilise this formula by replacing ξ with $G_{\rm eff}$ of Eq. (2).

Following the previous works , the growth index $\gamma(a,k)$ is introduced by $f(a,k) = \Omega_m(a)^{\gamma(a,k)}$, which is related with $\widetilde{\gamma}(a,k)$ by $\gamma(a,k) = \frac{\ln f_0}{\ln \Omega_m(a)} + \widetilde{\gamma}(a,k)$.

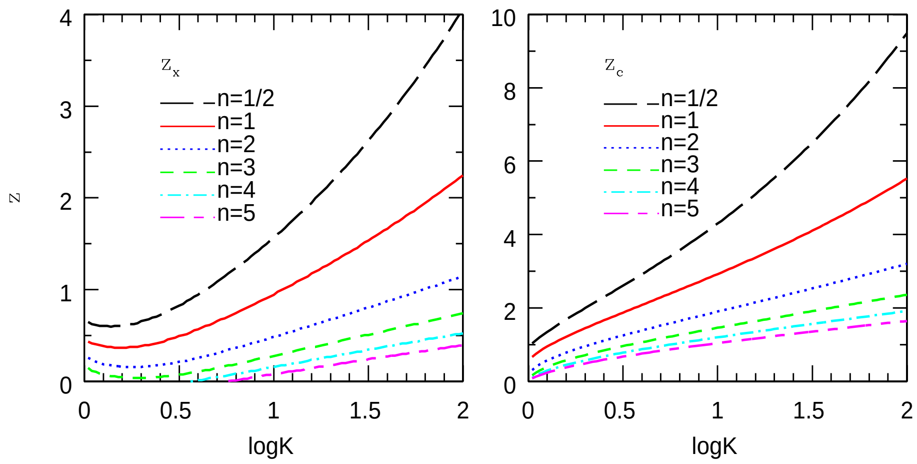

Figure 1: Left: Redshift z_x when the difference of the growth rate becomes $f^{(\mathrm{appr})} - f^{(\mathrm{exac})} = 0.03$, as a function of $K(= k/k_C)$. Right: Transition redshift z_c as a function of $K(= k/k_C)$.

Let us discuss the valid region of the approximate solution. The left panel of Fig. 1 plots the redshift z_x as a function of $K(= k/k_C)$, where z_x is defined by the redshift when the difference of the growth rate becomes $f^{(\mathrm{appr})} - f^{(\mathrm{exac})} = 0.03$. Here $f^{(\mathrm{exac})}$ is the exact solution obtained by solving Eq. (5) numerically, while $f^{(\mathrm{appr})}$ is the approximate solution. Thus, the approximate solution of the growth rate approaches the exact solution after the redshift z_x, which depends on k/k_C as well as n.

The above behaviour is related with the transition redshift z_c, when the scalar-tensor regime starts, which we defined by $k(1 + z_c) = m$, i.e.,

$$k^2(1+z_c)^2 = k_C^2\left(\frac{\Omega_0(1+z_c)^3+4(1-\Omega_0)}{\Omega_0+4(1-\Omega_0)}\right)^{2n+2}. \tag{8}$$

The right panel of Fig. 1 plots z_c as function of $K(= k/k_C)$. Figure 1 shows $z_x < z_c$. Thus the approximate formula approaches the exact solution after the scalar-tensor regime starts. For the model with larger value of n, the Compton scale evolves rapidly. Then, the transition redshift z_c becomes small as n becomes large. For the smaller value of $K(= k/k_C)$, the transition redshift z_c becomes smaller. This is the reason why z_x is smaller, as n is larger or k/k_C is smaller. Therefore, for the case when n is large and k/k_C is smaller, the redshift when the approximate formula starts to work becomes later. For the case $n \lesssim 2$, the late-time behaviour of the growth rate can be approximated by the approximate formula as long as $K \gtrsim 1$.

4 Constraint on $f(R)$ model from weak lensing survey

Cosmological constraints on the $f(R)$ model have been investigated in Refs. [4, 5]. The weak lensing statistics is useful to obtain a constraint on the growth history of cosmological density perturbations observationally. We now consider a prospect of constraining the $f(R)$ model with a future large survey of the weak lensing. To this end, we adopt the Fisher matrix analysis, which is frequently used for estimating minimal attainable constraint on the model parameters. Here we focus on the constraint on the Compton wavenumber parameter k_C defined by Eq. (4). In this analysis, we obtained the growth rate and the growth factor by numerically solving Eq. (5).

In the present paper, the modified gravity of the $f(R)$ model is supposed to be characterised by n and k_C(or λ). We perform the Fisher matrix analysis with the 9 parameters, n, λ(or k_C), w_0, w_a, Ω_0, Ω_b, h, A, and n_s, where Ω_b is the baryon density parameter, n_s is the initial spectral index, A is the amplitude of power spectrum. w_0 and w_a characterise the background expansion history and the distance-redshift relation. And we assumed the ΛCDM model as the background expansion of the universe in the previous section.

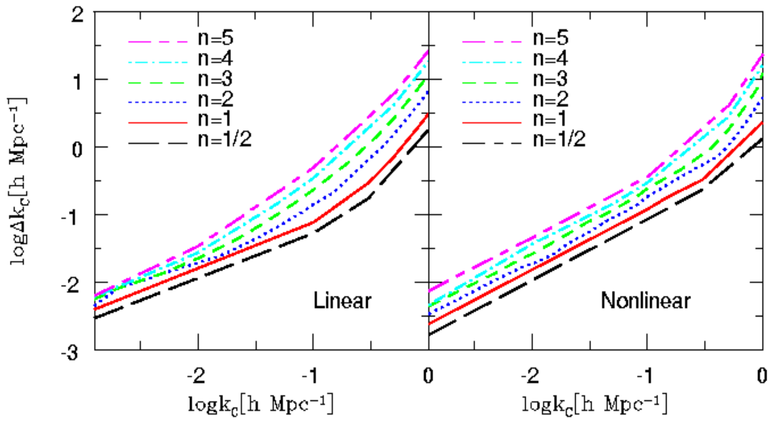

Figure 2: The 1-sigma error on k_C as a function of the target value of k_C, where the other parameters are marginalised over. The left (right) panels use the linear (nonlinear) modeling for $P_{\rm mass}(k,z)$ of the range of $10 \leq l \leq 10^3$ $(10 \leq l \ \leq 3 \times 10^3)$.

In the Fisher matrix analysis, we assume the galaxy sample of a survey with the number density $N_g = 35$ per arcmin.^{2}, the mean redshift $z_m = 0.9$, and the total survey area, $\Delta A = 2 \times 10^4$ square degrees. We also assumed the tomography with 4 redshift bins. Figure 2 is the result of the Fisher matrix analysis of the 9 parameters, n, k_C, w_0, w_a, Ω_0, Ω_b, h, A, and n_s. Figure 2 shows the 1-sigma error on k_C as a function of the target value of k_C, where the other parameters are marginalized the Fisher matrix over. The left panels are the linear theory, while the right panels are the nonlinear model. The error of k_C is the same order of k_C for the cases $n = 1/2$ and 1, but the error becomes larger as n becomes larger.

5 Summary and conclusions

In the present paper, we have investigated the linear growth rate of cosmological matter density perturbations in the viable $f(R)$ model both numerically and analytically. We found that the growth rate in the scalar-tensor regime can be characterised by a simple analytic formula (6). This is useful to understand the characteristic behaviour of the growth index in the scalar-tensor regime. We also investigate a prospect of constraining the Compton wavelength scale of the $f(R)$ model with a future weak lensing survey. This result shows that a constraint on k_C of the same order of k_C will be obtained for the model $n = 1$ and $n = 1/2$, though the constraint is weaker as n is larger. For $k_C \gtrsim 1h\mathrm{Mpc}^{-1}$, the constraint is very weak. This is because the weak lensing statistics is not very sensitive to the density perturbations on the smaller scales. A more detailed explanation is presented in Ref. [6]

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