Aspects of Phase-Space Noncommutative Quantum Mechanics

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2014/2015

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Todas as correções determinadas pelo júri, e só essas, foram efetuadas.

O Presidente do Júri,

Porto, ______/______/_________

MSc

"It doesn't make any difference how beautiful your guess is, it doesn't make any difference how smart you are, who made the guess, or what his name is. If it disagrees with experiment, it's wrong. That's all there is to it."

Richard Feynman

Acknowledgements

On the path that led to the conclusion of the present thesis, there were several people involved, in various aspects, without whom it would not be completed. For this, and in first place, I want to thank my mother, Maria, my father, Vitor, my brother, Luís, and my grandmother, Celeste. For the support, love and care in these last five years (and all the years before). Also, a very sincere word of appreciation to Catarina. Your support was invaluable and without it this path might have ended sooner. I cannot thank you enough.

My sincerest thanks to my supervisor, Professor Orfeu Bertolami, for all the help and availability during the current year and for the possibility to work with him. It was a very good experience. Also, I want to thank Catarina Bastos, for her help and insight provided during the development of this work.

For all the knowledge in physics and mathematics, for the experience provided and for stimulating the development of critical thinking, I extend my gratitude to all of you who taught me for the past five years.

And last, but not least, I want to thank all of those who shared the classroom with me, who would ensure great discussions after classes and also a special thanks to the ones who kept me playing. A particular word of appreciation to Vitor Bessa, Tiago Martins, Artur Sousa, João Silva, Marvin Fernandes, João Marques, Simão Teixeira and Rúben Alves.

Resumo

Neste trabalho são abordadas várias questões de importância no contexto da Mecânica Quântica Não Comutativa. O principal objectivo deste trabalho foi examinar se algumas das simetrias presentes na Mecânica Quântica também estão presentes na versão não comutativa no espaço de fase. Em particular, são considerados os problemas relacionados com a invariância de gauge do campo electromagnético e o Princípio de Equivalência Fraco no contexto do poço gravitacional quântico. Também é considerada a questão da simetria de Lorentz e a respectiva relação de dispersão. São impostas restrições aos parâmetros não comutativos relevantes de modo a manter as simetrias de gauge e de Lorentz. Em oposição, verifica-se que o Princípio de Equivalência Fraco se mantém na versão não comutativa, sendo apenas possível observar uma violação deste princípio se a isotropia das relações de comutação for quebrada.

Esta tese é baseada no trabalho desenvolvido na Ref. [1].

Abstract

In this work several key issues in the context of Noncommutative Quantum Mechanics (NCQM) are addressed. The main focus is on finding whether symmetries present in Quantum Mechanics still hold in the phase-space noncommutative version. In particular, the issues related with gauge invariance of the electromagnetic field and the Weak Equivalence Principle (WEP) in the context of the gravitational quantum well (GQW) are considered. The question of the Lorentz symmetry and the associated dispersion relation is also considered. Constraints are set on the relevant noncommutative parameters so that gauge invariance and Lorentz invariance holds. In opposition, the WEP is verified to hold in the noncommutative set up, and it is only possible to observe a violation whether the isotropy of the noncommutation relations is broken.

This thesis has its basis on the work developed in Ref. [1].

Contents

Chapter 1

Introduction

Noncommutative Quantum Mechanics (NCQM) is a theory whose basis is a deformation of the Heisenberg-Weyl (HW) algebra for the position and momentum operators into a noncommutative (NC) algebra. The HW algebra is given by the known commutation relations,

$$[\hat{x}_i, \hat{x}_j] = 0, \qquad [\hat{p}_i, \hat{p}_j] = 0, \qquad [\hat{x}_i, \hat{p}_j] = \mathrm{i}\hbar\delta_{ij}, \tag{1.1}$$

which are replaced by the NC algebra,

$$[\hat{q}_i, \hat{q}_j] = \mathrm{i}\theta_{ij}, \qquad [\hat{\pi}_i, \hat{\pi}_j] = \mathrm{i}\eta_{ij}, \qquad [\hat{q}_i, \hat{\pi}_j] = \mathrm{i}\hbar\delta_{ij}, \tag{1.2}$$

where θ_{ij} and η_{ij} are anti-symmetric real matrices. This algebra is the basic structure of the NCQM theory. It is not a theory that differs fundamentally from quantum mechanics in the mathematical framework it uses. In a certain sense, both are the same: they both share the same basic set of postulates for interpreting calculations and identifying mathematical objects with real world experimental results. If quantum mechanics is regarded with its basic postulates and a set of commutation relations, then NCQM is not very different from quantum mechanics. So why does it deserve special attention?

What sets NCQM apart from quantum mechanics is a different concept of space (time only gets into the equation in noncommutative field theory). In this theory, space directions and momentum directions are no longer independent, but rather are correlated with each other. These correlations are put into mathematical terms by the NC algebra, Eq. (1.2). This new concept of space and momentum incorporated into the framework of quantum mechanics gives rise to noncommutative quantum mechanics. The consequences of this change are not obvious and so a study of this theory must be pursued. If this deformed algebra is the one that describes our universe, evidence should show up in experimental results; therefore analytical or computational results are needed for this theory for comparison, either to set bounds on the parameters or to completely rule out the theory.

Historically, the idea of deformations of the HW commutation relations is not recent, especially in what concerns quantum field theory [2]. One of the first consequences of noncommutativity is the quantization of space. This is due to the impossibility of measuring two directions of space simultaneously with arbitrary precision. In fact, that was the original motivation in Ref. [2] to introduce noncommutativity in quantum field theory. More recently, interest in NCQM has been mainly due to string theory in which the entire dynamics of the strings can be described by a gauge theory in a noncommutative space [3]. Furthermore, the use of noncommutative geometry in the toroidal compactification of Matrix theory has given a boost to the study of NCQM [4]. Since quantum mechanics is the low-energy and the finite number of particles limit of more fundamental theories, such as string theory, noncommutativity might appear as a small effect at the quantum mechanical level. Motivated by such results, and by the idea of a new theory, which is easily verified or disproved experimentally, in recent years there has been a significant amount of research in NCQM with very promising results. These include: the study of charged particle in a magnetic field (Landau problem) [5]; similar problem with a harmonic oscillator potential [6, 7]; a particle in a general central potential [8]; the Gravitational Quantum Well (GQW) [9]; and applications to cosmology have been considered in Ref. [10]. Further work can be found in Refs. [11–14]. Along with these results, there has also been a development of the mathematical theory underlying NCQM, namely, its connection with NC field theories (e.g. Ref. [15]), and those endowed with extensions of the Galilei group [16], and an alternative formulation of NCQM based on the Weyl-Wigner formulation of quantum mechanics [17, 18]. This formulation has proven to be quite useful in the treatment of more complex problems, such as, for instance, the treatment of uncertainty relations in NCQM and also a noncommutative version of Ozawa's uncertainty relations [19].

The following work will focus on the study of key symmetries, namely the ones present in quantum mechanics, and it will be examined they hold in NCQM. The layout of this thesis is as follows: Chapter 2, summarizes the Weyl-Wigner formulation of quantum mechanics, as well as its noncommutative version. Chapter 3, examines gauge invariance in NCQM; in Chapter 4 the (Weak) Equivalence Principle (WEP) is studied in the context of the GQW, either for isotropic or for anisotropic noncommutativity. Finally, in Chapter 5 the question of Lorentz symmetry and the associated dispersion relation is addressed.

Chapter 2

Mathematical Background

2.1 Phase-Space Formulation of quantum mechanics

Such as classical mechanics has various equivalent formulations, e.g. Newtonian, Lagrangian, Hamiltonian, also quantum mechanics is endowed with a broad spectrum of formulations [20]. In spite of this fact, matrix, wave function and second quantization formulations are still the most used ones, and (almost) the only ones presented in most quantum mechanics courses. Nevertheless, other formulations are quite as useful and, in the following work, a particularly relevant one is phase-space formulation. In this chapter we shall illustrate how this formulation is completely equivalent to the most well known ones. While there are several representations, in the following we shall explore the Wigner-Weyl representation of this formulation. Also, the complete proofs of the results stated are out of the scope of this work. For a detailed treatment see, for instance, Refs. [21–23].

Historical motivations apart, the central object in this formulation is the Wigner function (WF), defined as

$$f(x,p) = \frac{1}{(2\pi\hbar)^n} \int \left\langle x - \frac{y}{2} \right| \hat{\rho} \left| x + \frac{y}{2} \right\rangle \mathrm{e}^{-\frac{\mathrm{i}p\cdot y}{\hbar}} \, \mathrm{d}^{\mathrm{n}} y, \tag{2.1}$$

where we consider a $2n$ dimensional phase-space and $\hat{\rho}$ is the density matrix. The integral is from $-\infty$ to $+\infty$ in all variables; this will also be the case for all integrals in this chapter, unless stated otherwise. For a one dimensional problem and considering a pure state, ψ, the WF reduces to

$$f(x,p) = \frac{1}{2\pi\hbar}\int \psi^*\left(x-\frac{y}{2}\right)\psi\left(x+\frac{y}{2}\right)\mathrm{e}^{-\frac{\mathrm{i}py}{\hbar}}\,\mathrm{d}y. \tag{2.2}$$

The WF of a problem is then related to the wave function of the same problem. However, no input wave function is needed in order to compute the WF as they are determined by a suitable differential equation in phase space. A note on the WF must be taken into account: the WF can also be expressed in terms of momentum eigenstates, having a similar expression to Eq. (2.1), only with x and p roles inverted [24]. This shows the remarkable symmetry between these variables in this formulation. Also, computing the x or p partial integration of $f(x,p)$ leads to the probability distribution for position, $|\phi(p)|^2$, and momentum, $|\psi(x)|^2$, respectively. Wigner functions are not, however, probability distributions, but instead quasi-probability distributions. They are real, normalized in phase-space, but they do not satisfy one of the axioms of probability theory: Wigner functions may have (and indeed have) negative values and so are not positive in all phase-space (see e.g. Ref. [24]). Nonetheless, this is not a problem, since the regions in which the values fall below zero are smaller than $\hbar$ and are thus shielded from us by the uncertainty principle [25].

Another important tool to study is the Wigner-Weyl (W-W) transform introduced by Weyl (see Ref. [26]). For historical reasons, the mapping from phase-space functions to operators is known as Weyl transform, while the one from operators into functions is named Wigner transform. This map, although not vital to the phase-space formalism itself, is what allows to map the operators in Hilbert space, $\boldsymbol{L}^2(\mathbb{R}^n)$, to position and momentum functions in phase-space, $\mathbb{R}^{2n}$, in a well defined way. Given an operator in Hilbert space, the W-W transform is given by:

$$W[\hat{G}] := g(x,p) = \frac{1}{(2\pi\hbar)^n}\int \left\langle x-\frac{y}{2}\right|\hat{G}\left|x+\frac{y}{2}\right\rangle \mathrm{e}^{-\frac{\mathrm{i}p\cdot y}{\hbar}}\,\mathrm{d}^{\mathrm{n}}y, \tag{2.3}$$

for which the function $g(x,p)$ is real if the operator $\hat{G}$ is self-adjoint. Also, it is a one-to-one map, and so it admits an inverse, given by,

$$\hat{G}(\hat{X},\hat{P}) = \frac{1}{(2\pi)^2}\int g(x,p)\mathrm{e}^{\mathrm{i}a(\hat{P}-p)+\mathrm{i}b(\hat{X}-x)}\,\mathrm{d}a\mathrm{d}b\mathrm{d}x\mathrm{d}p. \tag{2.4}$$

Given this correspondence, the WF can be regarded as the Wigner transform of the density matrix, $\hat{\rho}$. The usefulness of the WF in this formulation may now start to be seen: using the definition Eq. (2.4) it is easy to prove that, for a given operator $\hat{G}$ with $g(x,p)$ as Wigner transform, the expectation value is given by,

$$\langle \hat{G} \rangle = \int f(x,p) g(x,p)\, \mathrm{d}x \mathrm{d}p, \tag{2.5}$$

in which the WF acts as the weight probability distribution. Regarding the defenition, Eq. (2.3), it must be noted that the input operator must be Weyl ordered. Generally, when there are no ordering ambiguities in quantization, the Wigner transforms are obtained simply by replacing $\hat{p} \to p$ and $\hat{x} \to x$, which is a quite simple rule. If such ambiguities are present, Weyl ordering must be imposed, in order to obtain the correct result [23]. The last mathematical object to be introduced in this section is related to operator product and its Wigner transforms. Given two operators, $\hat{G}$ and $\hat{K}$, and their Wigner transforms, $g(x,p)$ and $k(x,p)$, respectively, the transform of the operator $\hat{G}\hat{K}$ is then,

$$W[\hat{G}\hat{K}] = W[\hat{G}] \star W[\hat{K}] = g(x,p) \star k(x,p), \tag{2.6}$$

where the $\star$-product is defined as [23],

$$\begin{aligned} g(x,p) \star k(x,p) := \frac{1}{(\pi\hbar)^{2n}} \int & \mathrm{d}p' \mathrm{d}p'' \mathrm{d}x' \mathrm{d}x'' g(x',p') k(x',p') \times \\ & \times \mathrm{e}^{-2\mathrm{i}/\hbar (p(x'-x'')+p'(x''-x)+p''(x-x'))}, \end{aligned} \tag{2.7}$$

or, in a more compact and useful manner,

$$g(x,p) \star k(x,p) := g(x,p) \mathrm{e}^{(\mathrm{i}\hbar/2)\overleftarrow{\partial_x}\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\overrightarrow{\partial_x}} k(x,p). \tag{2.8}$$

This product of functions, although being first introduced in Ref. [27] by Groenewold, is commonly named after Moyal, who, along with Groenewold, contributed to this formulation. It must be noted that, as x and p are c-numbers they commute, and so do all phase-space functions. All the quantum effects of this formulation are captured by the $\star$-product.

So far it has been established a direct connection between operator quantum mechanics and PSQM. Yet, if the latter is to be regarded as an independent formulation of quantum mechanics, it must not require any input from other formulations. Besides, it needs a way of dealing with the dynamical evolution of the theory. In fact, the role that the Schrödinger's equation plays in the usual formulation of QM is replaced by the Moyal equation [29] in PSQM,

$$\frac{\partial f}{\partial t} = \frac{H \star f - f \star H}{\mathrm{i}\hbar} := \frac{1}{\mathrm{i}\hbar}\{H, f\}_\star, \tag{2.9}$$

where $H = W[\hat{H}]$ is the Hamiltonian of the system and the last step is the definition of the Moyal brackets, which can be regarded as a quantum correction to the Poisson brackets [28, 29]. Usually, such as in other formulations of quantum mechanics, time dependent WF are determined considering the solutions for stationary problems, and so, stationary WF obey the equation (this is an important result, see Ref. [23] for a proof),

$$H(x, p) \star f(x, p) = E_n f(x, p). \tag{2.10}$$

The outlined formulation albeit being the tip of a complete approach to the subject will suffice for the understanding of the following work. However, for a complete approach, including the uncertainty principle, properties of the WF or perturbation theory, see e. g. the references pointed at the beginning of Section 2.1.

2.2 Wigner-Weyl formulation of NCQM

Noncommutative quantum mechanics is tightly connected to quantum mechanics through the Darboux map, D, also known as Seiberg-Witten (SW) map: a non-canonical linear transformation between the two sets of operators, $\{\hat{x}_i, \hat{p}_i\}$ (commutative) and $\{\hat{q}_i, \hat{\pi}_i\}$ (noncommutative). This sets a correspondence between each noncommutative operator and its commutative counterparts, allowing us to write any function of noncommutative variables in terms of commutative ones. The latter can then be treated using the usual tools of quantum mechanics. Thus we write

$$\begin{aligned} \hat{q}_i &= \hat{q}_i(\hat{x}_j, \hat{p}_j) \\ \hat{\pi}_i &= \hat{\pi}_i(\hat{x}_j, \hat{p}_j). \end{aligned} \tag{2.11}$$

Being connected in such a simple way, it is obvious that it's possible to construct a phase-space formulation of NCQM. The prime piece of the this formulation is the Wigner-Weyl transform and so a noncommutative version must be defined. Following the work in Refs. [17, 18], this function is defined using the Darboux map and the Weyl-Wigner transform as,

$$\hat{H}(\hat{q},\hat{\pi}) \xrightarrow{D} \hat{H}'(\hat{x},\hat{p}) \xrightarrow{W[H']} H'(x,p) \xrightarrow{D^{-1}} H(q,\pi), \tag{2.12}$$

hence we can write,

$$W^{NC} = D^{-1} \circ W \circ D. \tag{2.13}$$

This defines in a clear way the required transform and, although the Darboux map is not uniquely defined, it can be shown that, for linear Darboux maps, as the one used in NCQM, the resulting transform is independent from the Darboux map used [17]. Furthermore, as both D and the Wigner transform, W, are one-to-one and possess an inverse transformation, also the W^{NC} is one-to-one and admits inverse, as required to construct a phase-space formulation of NCQM. Using this definition of the Wigner-Weyl transform, it is easy to compute the noncommutative Wigner function (NCWF), which is then given by [17],

$$f^{NC}(q,\pi) = \frac{1}{det(\mathbf{\Omega})^{1/2}(2\pi\hbar)^d} W^{NC}[\hat{\rho}], \tag{2.14}$$

where d stands for a d-dimensional space and Ω is the simpletic matrix,

$$\mathbf{\Omega} = \begin{bmatrix} \frac{1}{\hbar}\mathbf{\Theta} & I_{d\times d} \\ I_{d\times d} & \frac{1}{\hbar}\boldsymbol{N} \end{bmatrix}, \tag{2.15}$$

and $\mathbf{\Theta}$ and $\boldsymbol{N}$ are the matrices with elements θ_{ij} and η_{ij} respectively. In the same line of reasoning as that of Section 2.1, the NCWF can now be used to compute measurable values in the same way as the commutative WF. Besides, as has been argued, all quantum mechanical information lies in the $\star$-product defined in the previous section, including the effects of $x-p$ noncommutativity. It is reasonable to expect that the same occurs for phase-space NCQM, that is, that the new (non)commutation relations are related to a new definition of a $\star$-product for phase-space functions. In fact, it has been proven that the choice of the

NC Wigner transform together with the restriction that $W^{NC}[\hat{A}(\hat{q},\hat{\pi})\hat{B}(\hat{q},\hat{\pi})] = W^{NC}[\hat{A}(\hat{q},\hat{\pi})] \star W^{NC}[\hat{B}(\hat{q},\hat{\pi})]$ is sufficient to determine the form of the new $\star$-product [17]. Hence, the new star-product is given by,

$$\star := \star_\hbar \star_\theta \star_\eta, \tag{2.16}$$

where $\star_\hbar$ is the same $\star$-product introduced in usual quantum mechanics due to $x-p$ noncommutativity and,

$$\begin{aligned} \star_\theta &:= \mathrm{e}^{(\mathrm{i}/2)(\overleftarrow{\partial_{x_i}})\theta_{ij}(\overrightarrow{\partial_{x_j}})} \\ \star_\eta &:= \mathrm{e}^{(\mathrm{i}/2)(\overleftarrow{\partial_{p_i}})\eta_{ij}(\overrightarrow{\partial_{p_j}})}. \end{aligned} \tag{2.17}$$

Therefore, if θ and η are set to zero, we obtain phase-space quantum mechanics, outlined in Section 2.1. These two terms are responsible for the noncommutativity of position and momenta and are a consequence of the commutation relations introduced in matrix NCQM. Having defined these tools, it is proven in Ref. [17] that the dynamics of NCWF is imposed by,

$$\frac{\partial f^{NC}(q,\pi)}{\partial t} = \frac{1}{\mathrm{i}\hbar}\{H(q,\pi), f^{NC}(q,\pi)\}_\star, \tag{2.18}$$

in which the deformed Moyal brackets are defined using the new $\star$-product. Thus, NCQM is formulated in terms of phase-space functions, in a completely parallel way to that of quantum mechanics. This alternative to matrix NCQM will be useful, in the next section, in the treatment of problems in which the Hamiltonian contains a potential term whose expression or properties are unknown.

This section is, once again, a shortened version of the Weyl-Wigner formulation of NCQM. For a much more detailed treatment of this matter, the reader is referred Refs. [17, 18], where all of the proofs are presented, as well as a more mathematically accurate description of the subject.

2.3 Star-product in matrix NCQM

Although phase-space formulation of NCQM is a useful tool to solve many problems, it is often simpler to use matrix formulation. However, if we consider a Hamiltonian with a potential depending on $\hat{\boldsymbol{q}}$ whose expression is not known (e.g.

$\hat{V}(\hat{\boldsymbol{q}})$), using the Darboux map to write that Hamiltonian in terms of commutative operators $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{p}}$ leads to an unknown potential, but now depending on both operators, e.g. $\hat{V}(\hat{\boldsymbol{x}},\hat{\boldsymbol{p}})$, which is not useful for solving the initial problem. This issue is rather straightforward to solve, but it is included in this work for the sake of completeness.

Consider the noncommutative Hamiltonian,

$$\hat{H} = \frac{\hat{\boldsymbol{\pi}}\cdot\hat{\boldsymbol{\pi}}}{2m} + \hat{V}(\hat{q}). \tag{2.19}$$

Using the equivalence with phase-space NCQM, if there are not ordering issues in the potential $\hat{V}(\hat{q})$, the stationary equation for the NCWF is then,

$$\begin{aligned}
&\left(\frac{\boldsymbol{\pi}\cdot\boldsymbol{\pi}}{2m} + V(q)\right)\star f^{NC}(q,\pi) = E_n f^{NC}(q,\pi)\\
\Leftrightarrow &\left(\frac{\boldsymbol{\pi}\cdot\boldsymbol{\pi}}{2m} + V(x)\right)\star_\hbar \star_\theta \star_\eta f^{NC}(x,\pi) = E_n f^{NC}(x,\pi)\\
\Leftrightarrow &\left[\left(\frac{\boldsymbol{\pi}\cdot\boldsymbol{\pi}}{2m} + V(x)\right)\star_\theta\right]\star_\hbar \star_\eta f^{NC}(x,\pi) = E_n f^{NC}(x,\pi)\\
\Leftrightarrow &\left[\left(\frac{\boldsymbol{\pi}\cdot\boldsymbol{\pi}}{2m} + V(x)\right)\mathrm{e}^{(\mathrm{i}/2)(\overleftarrow{\partial_{x_i}})\theta_{ij}(\overrightarrow{\partial_{x_j}})}\right]\star_\hbar \star_\eta f^{NC}(x,\pi) = E_n f^{NC}(x,\pi),
\end{aligned} \tag{2.20}$$

where in the first step Eq. (2.16) was used; as all noncommutative aspects are enclosed in the $\star$-product, commutative and noncommutative phase-space variables are the same, and that was used too in the first step. The obtained equation is equivalent to a Hamiltonian given by,

$$H'(x,p) = \left[\left(\frac{\boldsymbol{\pi}\cdot\boldsymbol{\pi}}{2m} + V(x)\right)\mathrm{e}^{(\mathrm{i}/2)(\overleftarrow{\partial_{x_i}})\theta_{ij}(\overrightarrow{\partial_{x_j}})}\right] \tag{2.21}$$

in a theory in which the parameter θ vanishes, so that in the definition of the Wigner transform there is no presence of such parameter. Using now the Weyl transform (in the theory of vanishing θ) in order to obtain the operator corresponding to Eq. (2.21), then,

$$\hat{H}'(\hat{x},\hat{\boldsymbol{\pi}}) = \left[\left(\frac{\hat{\boldsymbol{\pi}}\cdot\hat{\boldsymbol{\pi}}}{2m} + \hat{V}(\hat{x})\right)\mathrm{e}^{(\mathrm{i}/2)(\overleftarrow{\partial_{x_i}})\theta_{ij}(\overrightarrow{\partial_{x_j}})}\right], \tag{2.22}$$

where, for computing the Weyl transform, the defining series expansion of the exponential operator was considered. Hence, the Hamiltonian, Eq.(2.22), in a theory where only the momentum is noncommutative is equivalent to the initial Hamiltonian, Eq. (2.19), for the full noncommutative theory: position noncommutativity was removed from the theory and incorporated into the Hamiltonian. This solves the initial problem as in this new setup the argument of the potential is the usual commutative position, which can be treated as in quantum mechanics. It must be noted that the change in variables in Eq. (2.20) was only made so that the operators match the commutation relations initially defined in Chapter 1. Having these considerations into account, in treating NCQM problems, one can write,

$$\hat{H}(\hat{x}, \hat{\pi}) \star_\theta \Psi(x) = E_n \Psi(x) \tag{2.23}$$

and then use the Darboux map to alter the momentum operators their commutative counterparts. This is one of the methods employed in the literature and it shall be used in the next chapter to treat gauge invariance in NCQM.

Throughout the following Chapters, whenever need, the Darboux transformation to be used is as follows [17]:

$$\hat{q}_i = \hat{x}_i - \frac{\theta_{ij}}{2\hbar}\hat{p}_j, \qquad \hat{\pi}_i = \hat{p}_i + \frac{\eta_{ij}}{2\hbar}\hat{x}_j. \tag{2.24}$$

Chapter 3

Gauge Invariance

In order to study effects arising from NCQM we shall consider some physical systems of interest and investigate the implications of the NC deformation. The first example to consider is that of a particle with mass m and charge q in a magnetic field, with the Hamiltonian given by

$$\hat{H} = \frac{1}{2m}\left[\hat{\boldsymbol{\pi}} - q\boldsymbol{A}(\boldsymbol{q})\right]^2 . \tag{3.1}$$

In order to study this system we use the Moyal $\star$-product for the product of terms and then use the Darboux transformation, Eq. (2.24), to write the non-commuting Hamiltonian in terms of the commuting variables, $\hat{x}$ and $\hat{p}$, as outlined in Section 2.3. Thus, considering,

$$\hat{H}(\hat{q},\hat{\pi})\Psi(q) = \hat{H}(\hat{x},\hat{\pi}) \star_\theta \Psi(x) = \hat{H}(\hat{x},\hat{\pi})\mathrm{e}^{(\mathrm{i}/2)(\overleftarrow{\partial_{x_i}})\theta_{ij}(\overrightarrow{\partial_{x_j}})}\Psi(x), \tag{3.2}$$

at first order in the parameter θ,

$$\begin{aligned}&\left[\hat{H}(\hat{x},\hat{\pi}) + \frac{\mathrm{i}\theta_{ab}}{2}\partial_a \hat{H}(\hat{x},\hat{\pi})\partial_b\right]\Psi(x) = \\ &= \left[\frac{1}{2m}\left(\hat{\boldsymbol{\pi}}^2 - 2q\hat{\boldsymbol{\pi}}\cdot\boldsymbol{A}(\boldsymbol{q}) + q^2A^2(q)\right) + \frac{\mathrm{i}\theta_{ab}}{2}\partial_a\left(q^2A^2(x) - 2q\boldsymbol{A}(x)\cdot\hat{\pi}\right)\partial_b\right]\Psi(x)\end{aligned} \tag{3.3}$$

If we now consider that $\theta_{ab} = \theta\epsilon_{ab}$, where ϵ_{ab} is the rank 2 antisymmetric symbol (where $a, b = x, y, z$), the effective noncommutative Hamiltonian, at first order in θ, becomes:

$$\hat{H} = \frac{1}{2m}\left(\hat{\boldsymbol{\pi}}^2 - 2q\hat{\boldsymbol{\pi}}\cdot\boldsymbol{A}(\boldsymbol{q}) + q^2A^2(q)\right) + \frac{\mathrm{i}}{4m}\left[\nabla\left(q^2A^2(x) - 2q\boldsymbol{A}(x)\cdot\hat{\pi}\right)\times\nabla\right]\cdot\boldsymbol{\theta} \tag{3.4}$$

where $\boldsymbol{\theta} = \theta(1,-1,1)$. We now make use of the Darboux transformation, Eq. (2.24), in the momentum operator (which is now the only noncommutative operator in the Hamiltonian) to obtain:

$$\begin{aligned}\hat{H} = \frac{1}{2m}\left[(\hat{\boldsymbol{p}} - q\boldsymbol{A}(\boldsymbol{x}))^2 - \frac{1}{\hbar}(\hat{\boldsymbol{x}}\times\hat{\boldsymbol{p}})\cdot\boldsymbol{\eta} - \frac{q}{\hbar}(\hat{\boldsymbol{x}}\times\boldsymbol{A}(\boldsymbol{x}))\cdot\boldsymbol{\eta} + \frac{1}{4\hbar^2}\eta^2\epsilon_{ij}\epsilon_{ik}\hat{x}_j\hat{x}_k\right]\\ - \frac{1}{4m\hbar}\left[\nabla\left(q^2A^2(\boldsymbol{x}) - 2q\boldsymbol{A}(\boldsymbol{x})\cdot\hat{\boldsymbol{p}} - \frac{q}{\hbar}(\hat{\boldsymbol{x}}\times\boldsymbol{A}(\boldsymbol{x}))\cdot\boldsymbol{\eta}\right)\times\hat{\boldsymbol{p}}\right]\cdot\boldsymbol{\theta},\end{aligned} \tag{3.5}$$

where, as in the case of θ, $\boldsymbol{\eta} = \eta(1,-1,1)$. Presumably, θ and η, as treated here, are, as in the case of $\hbar$, new constants of Nature. We aim to see how a gauge transformation modifies the Hamiltonian and study the condition under which the NC Hamiltonian is gauge invariant. Gauge invariance must be imposed, otherwise a gauge change would lead to a modification of the system energy for the same physical configuration. For this purpose, we consider a gauge transformation to the vector potential $\boldsymbol{A}\to\boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla}\alpha$, where α is a scalar function of position. Consider now the first set of terms in the Hamiltonian, Eq. (3.5). Under the stated transformation, we get:

$$\begin{aligned}\frac{1}{2m}\left[(\hat{\boldsymbol{p}} - q\boldsymbol{A}(\boldsymbol{x}) - q\boldsymbol{\nabla}\alpha)^2 - \frac{1}{\hbar}(\hat{\boldsymbol{x}}\times\hat{\boldsymbol{p}})\cdot\boldsymbol{\eta} -\right.\\ \left.-\frac{q}{\hbar}(\hat{\boldsymbol{x}}\times\boldsymbol{A}(\boldsymbol{x}))\cdot\boldsymbol{\eta} - \frac{q}{\hbar}(\hat{\boldsymbol{x}}\times\boldsymbol{\nabla}\alpha)\cdot\boldsymbol{\eta} + \frac{1}{4\hbar^2}\eta^2\epsilon_{ij}\epsilon_{ik}\hat{x}_j\hat{x}_k\right].\end{aligned} \tag{3.6}$$

Changing the wave function on which the Hamiltonian acts, to $\Psi = \mathrm{e}^{\mathrm{i}q\alpha/\hbar}\Psi'$, ensures that the first set of extra terms in Eq. (3.5) coming from the gauge transformation will be cancelled and so we may conclude that this set of therms is not problematic. However, this is not true for the second set of terms which is transformed to,

$$\begin{aligned}\left[\nabla\left(q^2(A(\boldsymbol{x}) + \boldsymbol{\nabla}\alpha)^2 - 2q\boldsymbol{A}(\boldsymbol{x})\cdot\hat{\boldsymbol{p}} - 2q\boldsymbol{\nabla}\alpha\cdot\hat{\boldsymbol{p}} -\right.\right.\\ \left.\left.-\frac{q}{\hbar}(\hat{\boldsymbol{x}}\times\boldsymbol{A}(\boldsymbol{x}))\cdot\boldsymbol{\eta} - \frac{q}{\hbar}(\hat{\boldsymbol{x}}\times\boldsymbol{\nabla}\alpha)\cdot\boldsymbol{\eta}\right)\times\hat{\boldsymbol{p}}\right]\cdot\boldsymbol{\theta}.\end{aligned} \tag{3.7}$$

If we now consider the wave function transformation, $\Psi = e^{iq\alpha/\hbar}\Psi'$, we verify that the gauge transformation is not cancelled. This is due to the momentum operator outside the divergence in the last term of Eq. (3.5) acting on the exponential, which leads to nonlinear terms in α. Thus, the phase transformation that absorbs the gauge transformation terms in the first part of the Hamiltonian, Eq. (3.5), does not do so for the second set of terms. This comes from the fact that, in the first term, the change in $\boldsymbol{A}$ can be seen as a change in $\hat{\boldsymbol{p}}$, and a constant change in momenta can always be absorbed by a phase change. The same does not occur for the change in the second term, making it impossible to accommodate it into a change in phase. Therefore, in order to make the Hamiltonian gauge invariant, this term must vanish. To accomplish this for any $\boldsymbol{A}$, θ must vanish. This result is consistent to an explicit computation in the context of the Hamiltonian of fermionic fields [30].

Chapter 4

Gravitational Quantum Well and the Equivalence Principle in NCQM

A very interesting system to directly connect gravity to quantum mechanics is the gravitational quantum well [31–33]. As we shall see, this connection can be used to constrain quantum measurements of gravity phenomena and to test the Equivalence Principle (see also Refs. [34, 35]). It is easy to show that this principle holds for usual quantum mechanics, in the sense that a gravitational field is equivalent to an accelerated reference frame. We shall see that this also holds in the context of NCQM for isotropic noncommutativity parameters. In the following we shall study the noncommutative GQW [9] and its connection to accelerated frames of reference.

4.1 Fock space formulation of NC Gravitational Quantum Well

Let us consider the GQW in the context of NCQM. To start with we review some aspects of the usual GQW in standard quantum mechanics. The Hamiltonian is given by:

$$\hat{H} = \frac{1}{2m}\hat{\boldsymbol{p}}^2 + mg\hat{x}_i. \tag{4.1}$$

for a particle with mass, m, in a gravitational field with acceleration, g, in the x_i direction.

With the Fock space treatment in mind we define creation and annihilation operators for this Hamiltonian:

$$\hat{b}_i = \left(\frac{m^2}{\hbar^2 g}\right)^{\frac{1}{3}} \left[\hat{x}_i + \frac{i}{2}\left(\frac{g^2\hbar}{m^4}\right)^{\frac{1}{3}} \hat{p}_i\right], \tag{4.2}$$

$$\hat{b}_i^\dagger = \left(\frac{m^2}{\hbar^2 g}\right)^{\frac{1}{3}} \left[\hat{x}_i - \frac{i}{2}\left(\frac{g^2\hbar}{m^4}\right)^{\frac{1}{3}} \hat{p}_i\right], \tag{4.3}$$

where the definition concerns for the ith direction and is the same for any direction of the gravitational potential. The normalization factors are chosen so that the operators $\hat{b}_i$ and $\hat{b}_i^\dagger$ are dimensionless. We now consider the Hamiltonian, Eq. (4.1), in two dimensions, namely x and y, with the gravitational potential in the x direction. Using the above definitions for the x direction, since the particle is free in the y direction, the Hamiltonian can then be rewritten as

$$\hat{H} = K_1\left(\hat{\Gamma}_x + \hat{\Gamma}_y\right) + K_2\left(\hat{b}_x^\dagger + \hat{b}_x\right), \tag{4.4}$$

where

$$\hat{\Gamma}_i = \hat{b}_i^\dagger\hat{b}_i + \hat{b}_i\hat{b}_i^\dagger - \hat{b}_i^\dagger\hat{b}_i^\dagger - \hat{b}_i\hat{b}_i, \tag{4.5}$$

$$K_1 = \frac{1}{16}\left(\frac{\hbar^3 m^2}{g}\right)^{2/3}, \tag{4.6}$$

$$K_2 = \frac{mg}{2}\left(\frac{\hbar^2 g}{m^2}\right)^{1/3}. \tag{4.7}$$

Given the form of the Hamiltonian, it is evident that it is not diagonal in this representation, so it is not particularly useful for calculations of eigenstates and eigenvalues. This is expected from the usual solution to this problem, in which the energies involve the zeros of the Airy function, $Ai(x)$. We now examine the noncommutative Hamiltonian [9],

$$\hat{H}^{NC} = \frac{1}{2m}\left[\hat{p}_x^2 + \hat{p}_y^2\right] + mg\hat{x} + \frac{\eta}{2m\hbar}(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x) + \frac{\eta^2}{8m\hbar^2}\left(\hat{x}^2 + \hat{y}^2\right); \tag{4.8}$$

which is the equation of a particle under the influence of a gravitational field plus a fictitious "magnetic field", $\overrightarrow{B_{NC}} = -(\eta/q\hbar)\overrightarrow{e_z}$, plus a harmonic restoring force. All terms containing θ are contant and so can be absorbed by introducing a phase in the wave function. Although we are only treating the Hamiltonian, not the Schrödinger equation, we will drop them as they will not have any measurable effect on the energy of the particle. Through the definitions, Eqs. (4.2) and (4.3), it can be rewritten it, up to first order in θ and η, as:

$$\hat{H}^{NC} = K_1\left(\hat{\Gamma}_x + \hat{\Gamma}_y\right) + K_2\left(\hat{b}_x^\dagger + \hat{b}_x\right) + \frac{i\eta}{4m\hbar^{\frac{2}{3}}}\left(\hat{b}_y^\dagger\hat{b}_x - \hat{b}_x^\dagger\hat{b}_y\right). \tag{4.9}$$

It should be pointed out that this treatment considers only first order terms in either η or θ, although the latter does not show up in the Hamiltonian as its effect can be absorbed by a phase factor of the wave function, therefore not affecting any measured quantities, as stated before. Noting the similarities between both commutative and noncommutative Hamiltonians, we might ask whether there is a transformation that can turn one into the other. That might be an interesting finding as, then, noncommutativity, at least for this system, could be regarded as a modification to the commutative case, and noncommutative eigenfunctions could be constructed from commutative ones, which are well known. Furthermore, it would make noncommutativity the result of a transformation of variables, and not a fundamental property of the system under study. In order to pursue this analysis, we must introduce an operator transformation in which the new operators, $\hat{a}_i$ and $\hat{a}_i^\dagger$ for $i = x, y$, obey the same commutation relations as the original operators. Thus we define,

$$\hat{b}_i := \sum_{j=1}^{2} u_{ij}\hat{a}_j + s_{ij}\hat{a}_j^\dagger, \tag{4.10}$$

$$\hat{b}_i^\dagger := \sum_{j=1}^{2} u_{ij}^*\hat{a}_j^\dagger + s_{ij}^*\hat{a}_j, \tag{4.11}$$

where we impose the commutation relations

$$\left[\hat{a}_i, \hat{a}_j^\dagger\right] = \delta_{ij}, \tag{4.12}$$

and all the other commutation relations vanish. These conditions introduce a set of constraints on the parameters u_{ij} and s_{ij}, namely:

$$
\begin{aligned}
|u_{11}|^2-|s_{11}|^2+|u_{12}|^2-|s_{12}|^2 &= 1, \\
|u_{21}|^2-|s_{21}|^2+|u_{22}|^2-|s_{22}|^2 &= 1.
\end{aligned}
\tag{4.13}
$$

Considering Eq. (4.4) in terms of operators $\hat{b}_i$ and $\hat{b}_i^\dagger$ and using the definitions, Eqs. (4.10) and (4.11), we get the Hamiltonian in terms of the operators $\hat{a}_i$ and $\hat{a}_i^\dagger$ as

$$
\begin{aligned}
\hat{H} = K_1 \big[&\gamma_1 \hat{a}_x^\dagger \hat{a}_x + \gamma_1 \hat{a}_x \hat{a}_x^\dagger + \gamma_2 \hat{a}_x^\dagger \hat{a}_x^\dagger + \gamma_2^* \hat{a}_x \hat{a}_x + \gamma_3 \hat{a}_y^\dagger \hat{a}_y + \gamma_3 \hat{a}_y \hat{a}_y^\dagger + \gamma_4 \hat{a}_y^\dagger \hat{a}_y^\dagger + \\
&+ \gamma_4^* \hat{a}_y \hat{a}_y + 2\gamma_5 \hat{a}_x^\dagger \hat{a}_y^\dagger + 2\gamma_5^* \hat{a}_x \hat{a}_y + 2\gamma_6 \hat{a}_x^\dagger \hat{a}_y + 2\gamma_6^* \hat{a}_y^\dagger \hat{a}_x \big] + \\
&+ K_2 \left[\hat{a}_x^\dagger \left(u_{11}^* + s_{11}\right) + \hat{a}_x \left(s_{11}^* + u_{11}\right) + \hat{a}_y^\dagger \left(u_{12}^* + s_{12}\right) + \hat{a}_y \left(s_{12}^* + u_{12}\right)\right],
\end{aligned}
\tag{4.14}
$$

where, for simplicity, we have defined,

$$\gamma_1 := |u_{11}|^2+|s_{11}|^2-u_{11}^* s_{11}^* - u_{11}s_{11} + |u_{21}|^2+|s_{21}|^2-u_{21}^* s_{21}^* - u_{21}s_{21}, \tag{4.15a}$$

$$\gamma_2 := 2u_{11}^* s_{11} - \left(u_{11}^*\right)^2 - s_{11}^2 + 2u_{21}^* s_{21} - \left(u_{21}^*\right)^2 - s_{21}^2, \tag{4.15b}$$

$$\gamma_3 := |u_{12}|^2+|s_{12}|^2-u_{12}^* s_{12}^* - u_{12}s_{12} + |u_{22}|^2+|s_{22}|^2-u_{22}^* s_{22}^* - u_{22}s_{22}, \tag{4.15c}$$

$$\gamma_4 := 2u_{12}^* s_{12} - \left(u_{12}^*\right)^2 - s_{12}^2 + 2u_{22}^* s_{22} - \left(u_{22}^*\right)^2 - s_{22}^2, \tag{4.15d}$$

$$\gamma_5 := u_{11}^* s_{12} + s_{11}u_{12}^* - u_{11}^* u_{12}^* - s_{11}^* s_{12}^* + u_{21}^* s_{22} + s_{21}u_{22}^* - u_{21}^* u_{22}^* - s_{21}^* s_{22}^*, \tag{4.15e}$$

$$\gamma_6 := u_{11}^* u_{12} + s_{11}s_{12}^* - u_{11}^* s_{12}^* - s_{11}^* u_{12}^* + u_{21}^* u_{22} + s_{21}s_{22}^* - u_{21}^* s_{22}^* - s_{21}^* u_{22}^*. \tag{4.15f}$$

Comparing the Hamiltonian in Eq. (4.14) to the one in Eq. (4.9), we can immediately set the conditions for the γ_i's

$$\gamma_1 = 1, \tag{4.16a}$$

$$\gamma_2 = -1, \tag{4.16b}$$

$$\gamma_3 = 1, \tag{4.16c}$$

$$\gamma_4 = -1, \tag{4.16d}$$

$$\gamma_5 = 0, \tag{4.16e}$$

$$\gamma_6 = \mathrm{i}\frac{\eta}{4m\hbar^{\frac{2}{3}}K_1} := i\eta \mathrm{c}, \mathrm{c} \in \mathbb{R}. \tag{4.16f}$$

Furthermore, comparing the terms that are linear in the $\hat{a}$ operators, we get two additional equations for the u and s parameters,

$$u_{11}^* + s_{11} = 1, \tag{4.17a}$$

$$u_{12}^* + s_{12} = 0. \tag{4.17b}$$

In total we now have 16 variables and a total of 16 distinct equations constraining the values of this variables. Hence, this system of equations has either a single solution or none. It is found that this system has no solution for $\eta \neq 0$, which can be verified using well known Mathematica or MatLab procedures. Therefore, it is not possible to describe, as expected, the noncommutative Hamiltonian as a mixture of eigenstates of the commutative Hamiltonian, and so it is a completely different problem. Once again we stress that this result is only valid at first order in both noncommutative parameters. However, it is reassuring to confirm that, at least at this level, noncommutativity is indeed a completely different problem than the commutative one.

4.2 Equivalence Principle

Having verified that the noncommutative Hamiltonian of the GQW is in fact a different problem than the commutative one, we can try to examine the issue of the noncommutative Equivalence Principle. We have seen that the only parameter having an effect on the eigenstates and eigenvalues is η, as the θ factor can be absorbed by a phase factor in the wave function of the system. The WEP states that, locally, any gravitational field is equivalent to an accelerated reference frame. This is one of the basic tenets of General Relativity and holds with great accuracy (see e.g. Ref. [36], chapter 22, for a review of the experimental status of relativity). In standard QM, for the GQW, this can be verified to hold in a quite simple way. In the context of NCQM we will show how it can be verified in what follows next. For this purpose we consider the noncommutative GQW Schrödinger equation,

$$\hat{H}_g^{NC}\Psi = \left[\frac{1}{2m}\left(\hat{\pi}_x^2 + \hat{\pi}_y^2\right) + mg\hat{Q}_x\right]\Psi = E\Psi \tag{4.18}$$

and applying the Darboux transformation to write it in terms of the commutative variables, that is, Eq. (4.8):

$$\left[\frac{1}{2m}\left(\hat{p}_x^2+\hat{p}_y^2\right)+mg\hat{x}+\frac{\eta}{2m\hbar}(\hat{x}\hat{p}_y-\hat{y}\hat{p}_x)+\frac{\eta^2}{8m\hbar^2}\left(\hat{x}^2+\hat{y}^2\right)\right]\Psi=\mathrm{i}\hbar\frac{\partial\Psi}{\partial t}, \quad (4.19)$$

where we have considered the time dependent problem as we have to use a change of coordinates evolving in time. We now consider the noncommutative free particle equation:

$$\left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)-\frac{\mathrm{i}\eta}{2m}\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)+\frac{\eta^2}{8m\hbar^2}\left(x^2+y^2\right)\right]\Psi=\mathrm{i}\hbar\frac{\partial\Psi}{\partial t}, \quad (4.20)$$

and introduce a change of coordinates defined as

$$x'=x-\sigma(t) \quad (4.21a)$$

$$y'=y \quad (4.21b)$$

In order for the WEP to be preserved we require that

$$\hat{H}_g^{NC}(\hat{\boldsymbol{x}},\hat{\boldsymbol{p}})\Psi(x,y)=\hat{H}_{free}^{NC}(\hat{\boldsymbol{x'}},\hat{\boldsymbol{p'}})\Psi'(x',y'), \quad (4.22)$$

where $\hat{H}_g^{NC}$ is the noncommutative GQW Hamiltonian and $\hat{H}_{free}^{NC}$ is the noncommutative Hamiltonian of a free particle and $\Psi'(x',y')=\mathrm{e}^{\mathrm{i}\phi(x',y')}\Psi(x',y')$, so that the eigenfunctions are the same, but by a phase. Starting from the free particle Hamiltonian we write it in terms of an accelerated reference frame coordinates, and thus,

$$\frac{\partial}{\partial x'}\Psi(x',y')=\frac{\partial}{\partial x}\Psi(x,y), \quad (4.23a)$$

$$\frac{\partial}{\partial y'}\Psi(x',y')=\frac{\partial}{\partial y}\Psi(x,y), \quad (4.23b)$$

$$\frac{\partial}{\partial t}\Psi(x',y')=\left(\frac{\partial}{\partial t}-\frac{\mathrm{d}\sigma(t)}{\mathrm{d}t}\frac{\partial}{\partial x}\right)\Psi(x,y). \quad (4.23c)$$

Hence, combining Eqs. (4.21) and (4.23), the right-hand side of Eq. (4.20) becomes:

$$\left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)-\frac{\mathrm{i}\eta}{2m}\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)-\frac{\mathrm{i}\eta}{2m}\sigma(t)\frac{\partial}{\partial y}+\frac{\eta^2}{8m\hbar^2}\left(x^2+y^2\right)+\right.$$
$$\left.\frac{\eta^2}{8m\hbar^2}\left(-2x\sigma(t)+\sigma^2(t)\right)\right]\Psi'(x,y)=\mathrm{i}\hbar\left(\frac{\partial}{\partial t}-\frac{\mathrm{d}\sigma(t)}{\mathrm{d}t}\frac{\partial}{\partial x}\right)\Psi'(x,y). \quad (4.24)$$

In order to check if Eq. (4.22) is consistent we must either compute the phase ϕ or prove that there is no wave function which holds for the mentioned relation. For this we consider the relation between Ψ and Ψ' and compute the action of the operators on the wave function $\Psi'(x',y')=\mathrm{e}^{\mathrm{i}\phi(x',y')}\Psi(x',y')$. The obtained result is as follows:

$$\left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)-\frac{\mathrm{i}\eta}{2m}\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)+\frac{\eta^2}{8m\hbar^2}\left(x^2+y^2\right)\right]\Psi'+$$
$$+\left[-\frac{\mathrm{i}\hbar^2}{2m}\frac{\partial^2\phi}{\partial x^2}+\frac{\hbar^2}{2m}\frac{\partial\phi}{\partial x}^2-\frac{\mathrm{i}\hbar^2}{2m}\frac{\partial^2\phi}{\partial y^2}+\frac{\hbar^2}{2m}\frac{\partial\phi}{\partial y}^2+\frac{\eta}{2m}y\frac{\partial\phi}{\partial x}-\frac{\eta}{2m}x\frac{\partial\phi}{\partial y}+\right.$$
$$\left.+\frac{\eta}{2m}\sigma(t)\frac{\partial\phi}{\partial y}-\frac{\eta^2}{4m\hbar^2}x\sigma(t)+\frac{\eta^2}{4\hbar^2}\sigma^2(t)+\hbar\frac{\partial\phi}{\partial t}+\hbar\frac{\mathrm{d}\sigma}{\mathrm{d}t}\frac{\partial\sigma}{\partial x}\right]\Psi'+$$
$$+\left[-\frac{\mathrm{i}\hbar^2}{2m}\frac{\partial\phi}{\partial x}+\mathrm{i}\hbar\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right]\frac{\partial\Psi'}{\partial x}+\left[-\frac{\mathrm{i}\hbar^2}{m}\frac{\partial\phi}{\partial y}-\frac{\mathrm{i}\eta}{2m}\sigma(t)\right]\frac{\partial\Psi'}{\partial t}=\mathrm{i}\hbar\frac{\partial\Psi'}{\partial t}. \quad (4.25)$$

Now, for the purpose of retrieving the noncommutative GQW we must compare both Schrödinger equations to set constraints on the form of the phase ϕ. Imposing that the term multiplying the derivative of Ψ' vanishes, we get:

$$\frac{\partial\phi}{\partial x}=\frac{m}{\hbar}\frac{\mathrm{d}\sigma}{\mathrm{d}t}, \quad (4.26)$$

which implies, taking into account the fact that σ only depends on time, that:

$$\phi=\frac{m}{\hbar}\frac{\mathrm{d}\sigma}{\mathrm{d}t}x+f(y,t). \quad (4.27)$$

Considering that the last term on the left-hand side of Eq. (4.25) must vanish, and Eq. (4.27), it follows that

$$\frac{\hbar^2}{m}\frac{\partial f}{\partial y} = -\frac{\eta}{2m}\sigma(t) \Rightarrow f(y,t) = -\frac{\eta}{2\hbar^2}\sigma(t)y + \mu(t); \tag{4.28}$$

replacing this result into the second term of Eq. (4.25) and comparing with the Hamiltonian, Eq. (4.19), yields

$$m\frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2}x + \nu(t) = mgx \tag{4.29}$$

where $\nu(t)$ is the sum of all time dependent terms and can be made to vanish through a suitable choice of the function $\mu(t)$. There is only one non-vanishing remaining term and in order to Eq. (4.22) to hold we must impose that

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2} = g \Rightarrow \sigma(t) = \sigma_0 + vt + \frac{1}{2}gt^2 \tag{4.30}$$

Thus, we can see that Eq. (4.22) holds as far as

$$x' = x + \sigma_0 + vt + \frac{1}{2}gt^2 \tag{4.31}$$

which corresponds to an accelerated reference frame. The WEP is then verified to hold for NCQM at least as long as we consider that the noncommutative parameters are isotropic. Hence, bounds on the WEP turn out to be limits on the isotropy of the NC parameters.

Finally, the phase difference between the wave functions Φ and Φ' is given by:

$$\Psi = \mathrm{e}^{\mathrm{i}\left(\frac{m}{\hbar}\frac{\mathrm{d}\sigma}{\mathrm{d}t}x - \frac{\eta}{2\hbar^2}\sigma(t)y + \mu(t)\right)}\Psi' \tag{4.32}$$

and, as it has been analysed in Ref. [37], this does not give rise to any physically meaningful effect.

4.3 Anisotropic noncommutativity

As we have seen in the last subsection, the WEP holds in NCQM, unless NC parameters are anisotropic, i.e. $\eta_{xy} \neq \eta_{xz}$. In what follows we use the bounds on the WEP to constrain the difference between components of the η matrix. The ensued discussion is similar to the one carried out in Ref. [34] in the context of the entropic gravity proposal [38]. The noncommutative Hamiltonian for the GQW is given by Eq. (4.8). In order to find the eigenstates for this problem we use perturbation theory up to first order in η, which is sufficient to obtain differences in the energy spectrum for different directions of the gravitational field. For this purpose we define

$$\hat{H}^{NC} = \hat{H}_0^{NC} + \hat{V}, \tag{4.33}$$

where we consider $\hat{V}$ a perturbation to the exactly soluble Hamiltonian $\hat{H}_0^{NC}$, defined by

$$\hat{H}_0^{NC} := \frac{\hat{p}_x^2}{2m} + \frac{\hat{p}_y^2}{2m} + mg\hat{x}, \tag{4.34a}$$

$$\hat{V} := \frac{\eta}{2m\hbar}\left(\hat{y}\hat{p}_x - \hat{x}\hat{p}_y\right) + \frac{\eta^2}{8m\hbar^2}\left(\hat{x}^2 + \hat{y}^2\right). \tag{4.34b}$$

Since we are only interested in the corrections of order η, we can disregard the second term in $\hat{V}$. The soluble Hamiltonian is that of a free particle in the y direction and that of the GQW in the x direction. Solutions to these problems are well-known and are given by (e.g. Ref. [31])

$$\Psi_{nk}(x,y) = A_n Ai\left(\left(\frac{2m^2 g}{\hbar^2}\right)^{1/3}\left(x - \frac{E_n}{mg}\right)\right)\chi(y), \tag{4.35}$$

where $Ai(z)$ is the Airy function, $\chi(y)$ is the solution for the free particle, and E_n and A_n are the energy eigenvalues in the x direction and the normalization factor for the Airy function, given, respectively, by,

$$E_n = -\left(\frac{mg^2\hbar^2}{2}\right)^{1/3}\alpha_n, \tag{4.36}$$

$$A_n = \left[\left(\frac{\hbar^2}{2m^2 g}\right)^{1/3}\int_{\alpha_n}^{+\infty} \mathrm{d}z Ai^2(z)\right]^{-1/2}, \tag{4.37}$$

where α_n are the zeros of the Airy function. The energy eigenvalues in the y direction are given by,

$$E_y = \frac{\hbar^2 k^2}{2m}, \tag{4.38}$$

where k is the momentum of the particle. The change in energy is given by the expectation value of the operator $\hat{V}$ in a general state given by Eq. (4.35) and, the leading order perturbation to the energy of the system in any state, is given by,

$$\Delta E_n = \langle \Psi_{nk} | \hat{V} | \Psi_{nk} \rangle = \frac{\eta k}{2m} \left[\left(\frac{2m^2 g}{\hbar^2} \right)^{-2/3} \mathrm{I}_1^{(n)} + \frac{E_n}{mg} \right], \tag{4.39}$$

where it was defined,

$$\mathrm{I}_1^{(n)} := \int_{\alpha_n}^{+\infty} \mathrm{d}z Ai(z) z^n Ai(z). \tag{4.40}$$

It must be noted that we computed the energy eigenvalues for the case of a two dimensional Hamiltonian in the xy plane, so we can write,

$$E_{nk}^{xy} = -\left(\frac{mg^2\hbar^2}{2} \right)^{1/3} \alpha_n + \frac{\hbar^2 k^2}{2m} + \frac{\eta_{xy} k}{2m} \left[\left(\frac{2m^2 g}{\hbar^2} \right)^{-2/3} \mathrm{I}_1^{(n)} + \frac{E_n}{mg} \right]. \tag{4.41}$$

Thus an anisotropy in the momentum space breaks the Equivalence Principle.

Consider now the NC GQW for a particle moving along the y direction with a gravitational field in the x direction and the same equation for a particle traveling along the x direction with a gravitational field in the z direction. Assuming that the test particles have the same momentum in the direction in which they are free, hence:

$$mx(g_x - g_z) = \frac{k}{2m} \left[\left(\frac{2m^2 g}{\hbar^2} \right)^{-2/3} \mathrm{I}_1^{(n)} + \frac{E_n}{mg} \right] (\eta_{xy} - \eta_{yz}), \tag{4.42}$$

where x is the position of the test particle, and it was considered that both particles have the same position relative to their reference frame, therefore using $m(xg_x - zg_z) = mx(g_x - g_z)$. Thus, using the bound on the WEP for two different directions (see e.g. Ref. [39]):

$$\frac{\Delta a}{a} := \frac{|a_1 - a_2|}{a} \lesssim 10^{-13}, \tag{4.43}$$

plus data from Ref. [33] , namely that $k = 1.03 \times 10^8 \ m^{-1}$ and $x = 12.2 \ \mu m$ for the eigenstate of the lowest energy and $g = 9.80665 \ m/s^2$, Eq. (4.42) yields:

$$\frac{\Delta g}{g} = 1.4 \times 10^{60} \Delta\eta. \tag{4.44}$$

Applying the bound from Eq. (4.43) to Eq. (4.44), the bound for $\Delta\eta$ is computed to be:

$$\Delta\eta \lesssim 10^{-73}\,\mathrm{kg}^2\mathrm{m}^2\mathrm{s}^{-2}, \tag{4.45}$$

which bounds the noncommutative momentum anisotropy in a quite stringent way. In natural units:

$$\sqrt{\Delta\eta} \lesssim 10^{-10}\ \mathrm{eV}. \tag{4.46}$$

When comparing this result to the bounds on η from NCGQW [9], $\sqrt{\eta} \lesssim 10^{-3}$ eV (for the lowest energy state) it is clear that the anisotropy must be several orders of magnitude smaller.

Chapter 5

Lorentz invariance

The final considerations in this work are concerned with Lorentz symmetry. Although being a rather heuristic approach, it allows us to set constraints on η. This symmetry is a fundamental cornerstone of all known physical theories. Thus, it is natural to consider experimental bounds on this invariance to constrain non-commutativity which explicitly violates Lorentz symmetry. A major tool for these tests is the relativistic dispersion relation,

$$E^2 = p^2c^2 + m^2c^4. \tag{5.1}$$

This relation is tested with great accuracy at very high energies. Indeed, ultra-high energy cosmic rays allow for constraining this relationship for an extra quadratic term on the energy to the 1.7×10^{-25} level [40]. This estimate is confirmed through direct measurements by the Auger Collaboration [41]. Thus, considering a correction of the form,

$$E^2 = p^2c^2 + m^2c^4 + \alpha E^2, \tag{5.2}$$

with $\beta\eta^2 = \alpha E^2$, $\beta \sim 1$, at the 1.7×10^{-25} level [40], it is possible to constrain the η parameter, that is:

$$\eta \leqslant (1.7 \times 10^{-25})E^2, \tag{5.3}$$

hence for ultra-high energy cosmic rays, with $E \sim 10^{20}$ eV, we can establish that $\sqrt{\eta} \leqslant 4.1 \times 10^7$ eV, which is not at all a very stringent upper bound. A much more constraining bound can be set through low-energy tests of Lorentz symmetry.

Indeed, assuming limits arising from the nuclear Zeeman levels, one can establish that $\eta \leqslant 10^{-22}E^2$, which for $E \sim$ MeV [42], implies that $\sqrt{\eta} \leqslant 10^{-11}$ MeV $\simeq 10^{-5}$ eV. This result is competitive with the most stringent bound on η, namely $\sqrt{\eta} \leqslant 2 \times 10^{-6}$ eV [30], obtained from the hydrogen hyperfine transition, the most accurate experimental result ever obtained.

A more thorough treatment of the deformation of the dispersion relation, Eq. (5.2) should be possible by constructing a full relativistic NCQM theory, yet it lies beyond the scope of this work.

Chapter 6

Conclusions and discussion of results

In this work we have addressed several issues on NCQM. Gauge invariance of the electromagnetic field is verified to hold only if the parameter θ vanishes, which is consistent with previous work for fermionic fields [30]. This result implies that, for abelian gauge theories, spatial directions do commute and noncommutative effects are expected only for the momenta.

Also, we have compared the GQW Hamiltonian in the context of NCQM with the Hamiltonian for the same problem in QM. Using the Fock space formalism with creation and annihilation operators, we found no evidence for a connection between this two problems at first order in the parameter η. This shows that NCQM poses a different problem from QM at least in the context of GQW. Following this result, we studied the WEP in the noncommutative scenario. It is concluded that this principle holds for NCQM in the sense that an accelerated frame of reference is locally equivalent to a gravitational field, as long as noncommutativity is isotropic. If an anisotropy is introduced in the noncommutative parameters, using data from Refs. [33, 39], we set a bound on the anisotropy of the η parameter, $\sqrt{\Delta\eta} \lesssim 10^{-10}$ eV. It is then clear that the anisotropy of the noncommutative momentum parameter is many orders of magnitude smaller than the NC parameter itself. This result also states that the existence of a preferential observer to whom the spatial x,y and z directions are well defined is limited to the same degree as the anisotropy factor.

Additionally, the breaking of Lorentz symmetry is examined in the context of NCQM. Assuming a violation of the relativistic dispersion relation proportional to E^2, bounds from ultra-high energy cosmic rays (see Refs. [40, 41])

imply that $\sqrt{\eta} \leq 4.1 \times 10^7$ eV. Considering instead bounds arising from nuclear Zeeman levels, one can obtain that $\sqrt{\eta} \leq 10^{-5}$ eV, which is competitive with bounds arising from the hydrogen hyperfine transition $\sqrt{\eta} \leq 2 \times 10^{-6}$ eV [30], the most stringent bound ever obtained.

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