Fermion zero-energy modes and fractional fermion numbers in a fractional vortex-fermion model

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Abstract. Topological materials have attracted much attention from both physicists and mathematicians recently. Topological properties are closely related to the fermion number (index) of Dirac fermions. The fermion number is given by the η invariant introduced by Atiyah, Padoti and Singer. We discuss a system of Dirac fermions interacting with a vortex and a kink. This system will be realized as a layered material of superconductors and topological insulators, where the Dirac fermion exists on the surface of the topological insulator. The fermion number is fractionalized and the fermion zero-energy excitation mode emerges when Dirac fermions interact with vortices and kinks. Our discussion includes the case where there is a half-flux quantum vortex associated with a kink in a magnetic field in a bilayer superconductor. A normalizable single-valued fermion zero-energy mode does not exist in the core of the half-flux quantum vortex.

1. Introduction

Recently, topological materials have been studied intensively. New interesting topological properties will be found in the study of quantum systems from the viewpoint of topology. The index of Dirac operators plays an important role in topological systems[1]. Dirac fermions appear and play an significant role in many topological materials such as topological insulators[2, 3, 4], topological superconductors[5], graphene[6, 7, 8] and also Kondo systems[9, 10, 11, 12]. The fermion number is related to the η invariant introduced by Atiyah, Padoti and Singer[13, 14, 15, 16]. New excitation modes will appear when fermions interact with soliton-like objects such as domain walls, vortices, kinks and monopoles[17, 18, 19, 20, 21]. There also exist zero-energy bosonic modes on solitons[20, 22, 23]. Thus both bosonic and fermionic zero-energy modes will emerge in the presence of solitons. These exotic quantum states carry fermionic quantum numbers that can be fractional[24, 25, 26, 27].

This may raise a question what is quantization? In superconductors, the magnetic flux is quantized as integer times the unit quantum flux ϕ_0. There are, however, exceptions when superconductors have multi components or form some geometric structure. A fractional-flux quantum vortex has been observed in Nb thin film superconducting bilayers recently[28]. If a bilayer system including superconductors and a topological insulator is synthesized, the Dirac fermion on the surface of the topological insulator may cause interesting phenomena. This subject is formulated as a vortex-fermion model in two or three space dimensions.

We investigate a vortex-fermion system in this paper. The model is given by the Dirac Hamiltonian interacting with the $U(1)$ gauge field and the scalar field with a soliton-like

structure. We consider the fermion zero-energy mode and the fermion number in this system.

2. Fermion zero-energy mode and vortices

We consider a layered structure formulated with superconductors and a topological insulator. We assume that there are Dirac fermions on the surface of the topological insulator and they interact with vortices. A superconducting bilayer mimics a two-band superconductor and the formation of a fractional-flux quantum vortex is possible[28]. Let us consider Dirac fermions in (1+2) dimensions. The Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}\gamma^{\mu}(i\partial_{\mu} - qA_{\mu})\psi - \frac{1}{2}ig\phi\bar{\psi}\psi^{c} + \frac{1}{2}ig^{*}\phi^{*}\bar{\psi}^{c}\psi, \tag{1}$$

where ψ is a two-component spinor and q is the coupling to the gauge field. A_{μ} is the abelian gauge field and $F_{\mu\nu}$ is the field strength given by $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$. Usually we take $q = e$ or $q = 2e$. ψ^{c} is the charge conjugate spinor given as $\psi^{c} = C\bar{\psi}^{T}$ where C is the charge conjugation matrix and T indicates the transposition. g is the coupling constant. Dirac matrices are chosen as

$$\gamma^{0} = \sigma_{3}, \quad \gamma^{1} = i\sigma_{2}, \quad \gamma^{2} = -i\sigma_{1}, \tag{2}$$

and

$$C = i\gamma^{0}\gamma^{2} = i\sigma_{2}. \tag{3}$$

We assume that $A_{0} = 0$ and

$$A^{i}(\mathbf{r}) = \epsilon^{ij}\hat{\mathbf{r}}_{j}\frac{1}{2e}A(r), \tag{4}$$

where $\mathbf{r} = (x, y)$, $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$, $r = |\mathbf{r}|$. The scalar field ϕ indicates the gap function for which we use the form

$$\phi(\mathbf{r}) = e^{iQ\theta}f(r), \tag{5}$$

where Q is the vorticity and $f(r)$ is a function of the radial direction variable r. In this paper Q could take a non-integer value although Q takes an integer value in general. We assume the asymptotic behaviors for $f(r)$ and $A(r)$ as follows.

$$\begin{aligned} f(r) \quad &\to \quad f_{\infty} \quad (r \to \infty) \tag{6} \\ &\to \quad f_{0}r^{|Q|} \quad (r \to 0) \tag{7} \\ A(r) \quad &\to \quad -Q/r \quad (r \to \infty) \tag{8} \\ &\to \quad 0 \quad (r \to 0). \tag{9} \end{aligned}$$

The magnetic flux is given by

$$-\int d^{2}x F_{12} = \int dx dy F_{xy} = \frac{\pi}{e}Q, \tag{10}$$

where $F_{xy} = \partial_{x}A_{y} - \partial_{y}A_{x}$ with $A_{x} = A^{1}$ and $A_{y} = A^{2}$. We use the unit $\hbar = c = 1$.

The equation of motion for ψ is

$$i\partial_{t}\psi = \alpha^{j}\left(\frac{1}{i}\partial_{j} - qA^{j}\right) - g\phi\sigma^{2}\psi^{*}, \tag{11}$$

where $\alpha^{1} = \sigma^{1}$ and $\alpha^{2} = \sigma^{2}$. Then the zero-energy equation for ψ is written as

$$\alpha^{j}(-i\partial_{j} - A^{j})\psi - g\phi\sigma^{2}\psi^{*} = 0. \tag{12}$$

We put $D_j = \partial_j - ieA^j$. Since $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$D_1 + iD_2 = e^{i\theta}\left(\partial_r + i\frac{1}{r}\partial_\theta - \frac{q}{2e}A(r)\right), \qquad (13)$$

$$D_1 - iD_2 = e^{-i\theta}\left(\partial_r - i\frac{1}{r}\partial_\theta - \frac{q}{2e}A(r)\right). \qquad (14)$$

A solution ψ is written in the form

$$\psi = \begin{pmatrix} e^K\chi_1 \\ e^{-K}\chi_2 \end{pmatrix}, \qquad (15)$$

where

$$K = \frac{q}{2e}\int_0^r dr' A(r'). \qquad (16)$$

The equations for χ_1 and χ_2 are

$$e^{i\theta}\left(\partial_r + \frac{i}{r}\partial_\theta\right)\chi_1 + gfe^{iQ\theta}\chi_1^* = 0, \qquad (17)$$

$$e^{i\theta}\left(\partial_r - \frac{i}{r}\partial_\theta\right)\chi_2 - gfe^{iQ\theta}\chi_2^* = 0. \qquad (18)$$

$$(19)$$

We examine the case of half-flux vortex with $Q = 1/2$. In this case we have a solution

$$\chi_1 = h(r)e^{-i\theta/4}, \qquad (20)$$

and $\chi_2 = 0$. For this ansatz we obtain

$$h(r) = r^{-\frac{1}{4}}\exp\left(-\int_0^r dr' gf(r')\right). \qquad (21)$$

This solution has a singularity at $r \sim 0$ but can be normalized. This solution, however, is not accepted because χ_1 is not a single-valued function. In the system with a half-flux quantum vortex, a wave function should be a single-valued or two-valued function[29]. For a unit-flux quantum vortex, we have a normalizable single-valued solution

$$\chi_{1;Q=1} = \exp\left(-\int_0^r dr' gf(r')\right). \qquad (22)$$

In general, a solution for $Q > 0$ may be written as[17]

$$\chi_1 = p(r)e^{im\theta} + q(r)e^{i(Q-1-m)\theta}, \qquad (23)$$

and $\chi_2 = 0$. When Q is an integer, there are Q normalizable solutions for

$$Q - 1 \geq m \geq 0, \quad m \in \mathbf{Z}. \qquad (24)$$

This is written as $Q > m \geq 0$. When Q is a half-integer, m must be also a half-integer so that the wave function is a two-valued function. In this case we must have

$$[Q] > m > -\frac{1}{2}, \quad 2m \in \mathbf{Z}, \quad 2Q \in \mathbf{Z}, \qquad (25)$$

where $[Q]$ indicates the integer part of Q (Gauss symbol). Thus there is no normalizable and two-valued solution of the zero-energy modes for $Q = 1/2$. For $Q = 3/2$ we have $m = 0, 1/2$. For negative vorticity $Q < 0$, we replace Q by $|Q|$.

Although there is a fermion zero-energy mode for the unit-flux vortex, this zero-energy mode disappears for the half-flux vortex. Let us consider the case where a unit-flux vortex is divided into two half-flux quantum vortices. Does a fermion zero-energy mode in the unit-flux vortex disappear in this process? A zero-energy mode remains in the kink connecting two half-flux vortices. The number of fermion zero-energy modes would be conserved.

3. Fractional fermion number and vortices

Let us investigate the fermion number in a vortex-fermion system. The model is given by the (1+2)-dimensional Dirac Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\gamma^\mu\partial_\mu - q\gamma^\mu A_\mu - m)\psi, \tag{26}$$

where A_μ is the abelian gauge field and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We consider the static solution so that we set $A_0 = 0$. The magnetic flux Φ is defined by

$$\Phi = -\frac{1}{2}\int d^2x \epsilon^{ij}F_{ij} = \int dxdy F_{xy}. \tag{27}$$

The Dirac Hamiltonian is in the form as

$$H = \alpha^j(-i\partial_j - qA^j) + \beta m, \tag{28}$$

where $\alpha^j = \gamma^0\gamma^j$ ($j = 1, 2$) and $\beta = \gamma^0$. α^k and β satisfy the algebra

$$\{\alpha^k, \alpha^i\} = \delta^{ki}, \quad \{\alpha^k, \beta\} = 0. \tag{29}$$

We use the representation $\alpha^1 = \sigma^1$, $\alpha^2 = \sigma^2$ and $\beta = \sigma^3$. Then the Hamiltonian is

$$H = \begin{pmatrix} m & D \\ D^\dagger & -m \end{pmatrix}, \tag{30}$$

where

$$D = -i\frac{\partial}{\partial x} - qA_x - \frac{\partial}{\partial y} + iqA_y. \tag{31}$$

The fermion number N is defined as

$$N = \int d^2x : \psi^\dagger(\mathbf{x})\psi(\mathbf{x}) := \frac{1}{2}\int d^2x[\psi^\dagger(\mathbf{x}), \psi(\mathbf{x})], \tag{32}$$

for $\mathbf{x} = (x^1, x^2) = (x, y)$ where $: \cdots :$ indicates the normal ordering. N is related to the η invariant defined by

$$\eta_H(s) = \sum_\lambda \text{sign}(\lambda)|\lambda|^{-s}, \tag{33}$$

where λ's are eigenvalues of H. By introducing the spectral density $\rho_H(\lambda)$, $\eta_H(s)$ is represented as[30]

$$\eta_H(s) = \int_{-\infty}^{\infty} d\lambda \rho_H(\lambda)\text{sign}(\lambda)|\lambda|^{-s}. \tag{34}$$

The fermion number N for H is given by

$$N = -\frac{1}{2}\eta_H(0). \tag{35}$$

In the massless limit $m \to +0$, the η invariant is related to the index of H. We put

$$\slashed{D} = \begin{pmatrix} 0 & D \\ D^\dagger & 0 \end{pmatrix}. \tag{36}$$

The index of $\slashed{D}$ is defined by

$$\mathrm{Index}\slashed{D} = \mathrm{Tr}_{\slashed{D}\psi=0}\sigma_3. \tag{37}$$

This definition leads to

$$\mathrm{Index}\slashed{D} = \mathrm{dimKer}D^\dagger - \mathrm{dimKer}D. \tag{38}$$

By introducing the cutoff, the index is represented as

$$\mathrm{Index}\slashed{D} = \lim_{\Lambda\to\infty} \mathrm{Tr}\sigma_3 e^{-\slashed{D}^2/\Lambda^2}. \tag{39}$$

In two-space dimensions (static case), the index of the Dirac Hamiltonian is evaluated as

$$\mathrm{Index}\slashed{D} = \frac{q}{2\pi}\int d^2x F_{xy} = \frac{q}{2\pi}\Phi. \tag{40}$$

There is the relation[30, 31]

$$\eta_{\slashed{D}}(0) = -\mathrm{Index}\slashed{D}. \tag{41}$$

Then the fermion number in the massless limit is given as

$$N = \frac{q}{4\pi}\Phi. \tag{42}$$

This result is also obtained by explicit calculations using the trace formula for the Dirac Hamiltonian[31]. When Φ is $-Q$ times the unit flux quantum $\phi_0 = h/(2|e|)$ in a superconductor ($e < 0$) as $\Phi = -Q\phi_0$, we have

$$\mathrm{Index}\slashed{D} = \frac{q}{2e}Q. \tag{43}$$

When Q is an integer in the conventional case, we require that the index $\mathrm{Index}\slashed{D}$ is an integer since this index is given by the difference of dimensions of vector spaces. This results in $q = 2e$. With this choice we have

$$\mathrm{Index}\slashed{D} = Q, \tag{44}$$

$$N = \frac{1}{2}Q. \tag{45}$$

The sign of these formulas is not important because the sign of the flux depends upon the direction of applied magnetic field. A fractional-flux vortex leads to the fractional Dirac index. When the unit-flux vortex is divided into several fractional-flux vortices, each fractional vortex can carry the fractional index. For finite mass m, N is evaluated as

$$N = \frac{q}{4\pi}\frac{m}{|m|}\Phi = -\frac{q}{8\pi}\frac{m}{|m|}\int d^2x \epsilon^{ij}F_{ij}. \tag{46}$$

By introducing the fermion current j^μ, N is given by

$$N = \int d^2x\, j^0. \tag{47}$$

Since $\delta S/\delta A_\mu = -q\langle\bar{\psi}\gamma^\mu\psi\rangle = -q\langle j^\mu\rangle$ for the action S, this suggests that the additional effective action is formulated as

$$\Delta S = \text{sign}(m)\frac{q^2}{16\pi}\epsilon^{\mu\nu\sigma}\int d^3x\, F_{\mu\nu}A_\sigma. \tag{48}$$

This indicates that the Chern-Simons term is induced on the surface of a junction of a superconductor and a topological insulator.

4. Fermion number and kinks

A fermion zero-energy mode exists when Dirac fermions interact with the scalar field with the kink structure[32]. This is closely related to the existence of non-zero fermion number. Let us consider the (1+2)-dimensional model with coupling to the kink given as

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\gamma^\mu\partial_\mu - q\gamma^\mu A_\mu - \varphi_1 - \varphi_2\gamma^1)\psi, \tag{49}$$

where φ_1 and φ_2 are scalar fields. We assume that there is a kink outside the region where the vortex exists. The kink is a one-dimensional object that depends on one variable x or y. Then we may have two contribution from the kink and anomalous term (the index of Dirac operator) to the fermion number N:

$$N = N_{anomaly} + N_{kink}. \tag{50}$$

N_{kink} is given by the formula

$$N_{kink} = -\frac{1}{2\pi}\left(\tan^{-1}\left(\frac{\varphi_2(\infty)}{\varphi_1(\infty)}\right) - \tan^{-1}\left(\frac{\varphi_2(-\infty)}{\varphi_1(-\infty)}\right)\right), \tag{51}$$

which is the generalization of the Goldstone-Wilczek index. $N_{anomaly}$ is given by Index$\not{D}$. Let us assume that φ_1 is constant giving the mass term. Then we have

$$N = \text{sign}(m)\frac{q}{4\pi}\Phi + N_{kink}. \tag{52}$$

We adopt the asymptotic behavior for $\varphi_2(x)$ as follows:

$$\varphi_2(x) \to v \quad \text{as } x \to \infty, \tag{53}$$
$$\varphi_2(x) \to -v \quad \text{as } x \to -\infty. \tag{54}$$

Then, in the limit $m \to 0$ for $v > 0$, N is given as

$$N = \text{sign}(m)\left[\frac{q}{4\pi}\Phi - \frac{1}{2}\right]. \tag{55}$$

For $q = 2e$, this shows

$$N = -\text{sign}(m)\frac{1}{2}(Q+1). \tag{56}$$

For $v < 0$, N becomes

$$N = -\text{sign}(m)\frac{1}{2}(Q-1). \tag{57}$$

N is represented as a sum of two contributions where one is from the vortex and the other comes from the kink. This is similar to the calculation of the topological index (skyrmion number)[29]. This equation for N holds for $Q = 1/2$ or $Q = -1/2$. When Q is an integer, there is no kink and thus N_{kink} vanishes in a real system of superconductors. The non-vanishing N_{kink} indicates the existence of fermion zero-energy modes.

5. Summary

We have investigated the fermion zero-energy excitation mode and the fermion number in a vortex-fermion system. We considered a vortex-fermion with magnetic vortices and Dirac fermion in (1+2)-dimensional space-time. This kind of systems can be realized in a junction of superconductors and topological insulators. The existence of fractional-flux quantum vortices has been reported in a superconducting bilayer. We have shown that there is no fermion zero mode in a vortex with fractional vorticity less than unity since wave function becomes a multi-valued function. We have also evaluated the fermion number in a system of fractional-flux quantum vortices, kinks and Dirac fermions.

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